Algebra & Number Theory

Volume 18
2024
No. 6

msp

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Antoine Chambert-Loir
Université Paris-Diderot
France

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Jason P. Bell	University of Waterloo, Canada
Bhargav Bhatt	University of Michigan, USA
Frank Calegari	University of Chicago, USA
J-L. Colliot-Thélène	CNRS, Université Paris-Saclay, France
Brian D. Conrad	Stanford University, USA
Samit Dasgupta	Duke University, USA
Hélène Esnault	Freie Universität Berlin, Germany
Gavril Farkas	Humboldt Universität zu Berlin, Germany
Sergey Fomin	University of Michigan, USA
Edward Frenkel	University of California, Berkeley, USA
Wee Teck Gan	National University of Singapore
Andrew Granville	Université de Montréal, Canada
Ben J. Green	University of Oxford, UK
Christopher Hacon	University of Utah, USA
Roger Heath-Brown	Oxford University, UK
János Kollár	Princeton University, USA
Michael J. Larsen	Indiana University Bloomington, USA
Philippe Michel	École Polytechnique Fédérale de Lausanne
Martin Olsson	University of California, Berkeley, USA
Irena Peeva	Cornell University, USA
Jonathan Pila	University of Oxford, UK
Anand Pillay	University of Notre Dame, USA
Bjorn Poonen	Massachusetts Institute of Technology, USA
Victor Reiner	University of Minnesota, USA
Peter Sarnak	Princeton University, USA
Michael Singer	North Carolina State University, USA
Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Shunsuke Takagi	University of Tokyo, Japan
Pham Huu Tiep	Rutgers University, USA
Ravi Vakil	Stanford University, USA
Akshay Venkatesh	Institute for Advanced Study, USA
Melanie Matchett Wood	Harvard University, USA
Shou-Wu Zhang	Princeton University, USA

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2024 is US $525/year for the electronic version, and $770/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2024 Mathematical Sciences Publishers

ALGEBRA AND NUMBER THEORY 18:6 (2024)
https://doi.org/10.2140/ant.2024.18.1039

Refined height pairing

Bruno Kahn
Appendix by Qing Liu

For a d-dimensional regular proper variety X over the function field of a smooth variety B over a field k and for $i \geq 0$, we define a subgroup $\mathrm{CH}^i(X)^{(0)}$ of $\mathrm{CH}^i(X)$ and construct a "refined height pairing"

$$\mathrm{CH}^i(X)^{(0)} \times \mathrm{CH}^{d+1-i}(X)^{(0)} \to \mathrm{CH}^1(B)$$

in the category of abelian groups up to isogeny. For $i = 1, d$, $\mathrm{CH}^i(X)^{(0)}$ is the group of cycles numerically equivalent to 0. This pairing relates to pairings defined by P. Schneider and A. Beilinson if B is a curve, to a refined height defined by L. Moret-Bailly when X is an abelian variety, and to a pairing with values in $H^2(B_{\bar{k}}, \mathbb{Q}_l(1))$ defined by D. Rössler and T. Szamuely in general. We study it in detail when $i = 1$.

Introduction

Let X be a regular proper (for example, smooth projective) variety of dimension d over a field K, finitely generated of transcendence degree δ over a subfield k. Suppose given a smooth (separated) k-scheme of finite type B, with function field K. For $i \in [0, d]$, write $\mathrm{CH}^i(X)$ for the i-th Chow group of X. In this paper, we define a subgroup $\mathrm{CH}^i(X)^{[0]}$ and a "refined height pairing"

$$\mathrm{CH}^i(X)^{[0]} \times \mathrm{CH}^{d+1-i}(X)^{[0]} \to \mathrm{CH}^1(B) \tag{1}$$

in the category $\mathbf{Ab} \otimes \mathbb{Z}[1/p]$ of abelian groups up to p-isogeny: this category is recalled in Section 4C. Here p is the exponential characteristic of k, so nothing is inverted in characteristic 0; the only reason to invert it in nonzero characteristic is a lack of resolution of singularities; see Section 4A.

MSC2020: primary 14C17; secondary 14E15.
Keywords: intersection theory, alteration theory, category theory.

© 2024 The Authors, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

If B is a smooth projective curve and we compose (1) with the degree map, we get a $\mathbb{Z}[1/p]$-valued pairing (with values in $p^{-s}\mathbb{Z}$ for some integer $s \geq 0$), which relates to the one constructed by Beilinson [1987, §1]. Beilinson [1987, p. 5] asked what happens when $\operatorname{trdeg}(K/k) > 1$: (1) gives one answer to this question.

The quotient $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^{[0]}$ is finitely generated. When we vary (X, i), $\mathrm{CH}^i(X)^{[0]}$ defines an adequate equivalence relation for smooth projective K-varieties, which a priori depends on the choice of B. Its saturation $\mathrm{CH}^i(X)^{(0)}$ lies between the subgroups $\mathrm{CH}^i_{\mathrm{alg}}(X)$ and $\mathrm{CH}^i_{\mathrm{num}}(X)$ of algebraically and numerically trivial cycles, hence equals $\mathrm{CH}^i_{\mathrm{num}}(X)$ when $i = 1, d$. We conjecture that this holds for all i, and prove it in further special cases (Theorem 5.5(ii)). One can show that it would follow in general from the Tate conjecture, or the Hodge conjecture in characteristic 0, for cycles of codimension $< i$, although we don't include a proof here. More generally, one might hope that Lemma 1.1 below induces pairings in $\mathbf{Ab} \otimes \mathbb{Q}$

$$F^n\mathrm{CH}^i(X) \times F^n\mathrm{CH}^{d+n-i}(X) \to \mathrm{CH}^n(B), \quad i \geq 0,$$

where $F^*\mathrm{CH}^*(X)$ is the conjectural Bloch–Beilinson–Murre filtration [Jannsen 1994], the case $n = 0$ (resp. 1) being the intersection pairing (resp. (1)).

Works following Néron's seminal paper [1965] have much relied on l-adic cohomology to analyse or define height pairings (because of the cohomological definition of Hasse–Weil L-functions): for $\delta = 1$, this is the case in [Schneider 1982] ($i = 1$, X an abelian variety), [Bloch 1984] and [Beĭlinson 1987]. This is also the case in the work of Damian Rössler and Tamás Szamuely [2022], which is the direct inspiration of this one: they construct a pairing

$$\mathrm{CH}^i_l(X) \times \mathrm{CH}^{d+1-i}_l(X) \to H^2_{\text{ét}}(B_{\bar{k}}, \mathbb{Q}_l(1)), \tag{2}$$

where l is a prime number invertible in k and $\mathrm{CH}^i_l(X)$ denotes cycles homologically equivalent to 0 with respect to l-adic cohomology. By contrast, our approach here is completely cycle-theoretic and very close in spirit to Moret-Bailly's geometric height [1985, chapitre III, définition 3.2]; it relies on Fulton's marvellous theory of Gysin maps [1984, Chapters 6 and 8]. This gives a different flavour to the definitions because numerical and homological equivalence have rather opposite functoriality under specialisation, as described in detail by Grothendieck in [SGA 6 1971, 7.9 and 7.13]. See Remark 2.7.

Comparing various definitions of height pairings is a highly nontrivial issue, which is solved only in a few cases: for example, as far as I know those defined by Bloch [1984] and Beilinson [1987] have still not been checked to agree. Schneider [1982] compares an l-adic height pairing [loc. cit., p. 298] with the Néron–Tate height by comparing each to an intermediate Yoneda pairing [loc. cit., p. 502]

$$H^0(B, \mathcal{A}^0) \times \mathrm{Ext}^1_B(\mathcal{A}^0, \mathbb{G}_m) \to \mathrm{CH}^1(B), \tag{3}$$

where $\mathcal{A}^0$ is the connected component of the identity of the Néron model $\mathcal{A}$ of the abelian variety A ($= X$ here).

In Proposition 2.11, I show that (1) and (2) are compatible (at least in characteristic 0) on a common subgroup $\mathrm{CH}^*_{\mathrm{B},l}(X)$ of $\mathrm{CH}^*_l(X)$ and $\mathrm{CH}^*(X)^{[0]}$ via the cycle class map $\mathrm{Pic}(B) \to H^2_{\text{ét}}(B_{\bar{k}}, \mathbb{Q}_l(1))$: this is what Rössler and Szamuely [2022, Proposition 6.1] had checked in the special case where X/K has a *smooth* model, by using a variant of Proposition 2.8 here. In Theorem 5.10, I show that (1) is the opposite of Silverman's refined height pairing [1994, Theorem III.9.5(b)] in the classical case of an elliptic curve X over the function field of a smooth projective curve B over an algebraically closed field k.

Another case where a compatibility should not be hard to show is that of [Moret-Bailly 1985].

Note that (1) is finer than (2) inasmuch as it takes homologically trivial cycles on B into account. This extra structure is presumably arithmetically significant; it is studied in Section 6E in the case $d = 1$, B projective.

It may seem disturbing that (1) is essentially integral, while the classical height pairing is usually rational: this may be "explained" by (3) which is integral but takes values on the subgroup of finite index $\mathcal{A}^0(B) \subseteq A(K)$. In this spirit, I show in Remarks 5.13(a) that in the elliptic curve case mentioned above, $\mathrm{CH}^1(X)^{[0]}$ contains $\mathcal{N}^0(B)$ as a subgroup of finite index, where $\mathcal{N}^0$ is the identity component of the Néron model of X.

The *raison d'être* of [Beĭlinson 1987; Bloch 1984] was to refine the conjectures of Tate [1965] on the orders of poles of zeta functions at integers by describing special values at these integers, when K is a global field. Thus one might like to extend (1) to the case where B is regular and flat over $\mathbb{Z}$. I consider this as beyond the scope of this article for two reasons:

- The present method fails in this case even if one is given a regular projective model $f : \mathcal{X} \to B$ of X, because Fulton's techniques do not define an intersection product on $\mathcal{X}$, except when $\delta = 1$ and f is smooth [1984, p. 397]. One does get an intersection product with $\mathbb{Q}$ coefficients, by using either K-theory as in [Gillet and Soulé 1987, 8.3], or alterations and deformation to the normal cone as in Andreas Weber's thesis [2015, Corollary 4.2.3 and Theorem 4.3.3]; it is possible that the present approach may be adapted by using one of these products.
- However, the main point in characteristic 0 is to involve archimedean places to get a complete height pairing whose determinant has a chance to describe the special values as mentioned above: this is what was done successfully in [Bloch 1984; Beĭlinson 1987] when $\delta = 1$. In higher dimensions, one probably would have to use something like Arakelov intersection theory (see [Rössler and Szamuely 2022, Conjecture 7.1] for a conjectural statement).

I leave these issues to the interested readers. Rather, I hope to show here that height pairings in the style of (1) also raise interesting geometric questions. These are discussed in Section 6, which is closely related to [Kahn 2014, Question 7.6].

Contents. Up to Section 4F, we assume k perfect; this assumption is removed in the said subsection. In Definition 2.2, we introduce subgroups $\mathrm{CH}^i(\mathcal{X})^0$ of *admissible cycles* in the Chow groups of a k-model $f : \mathcal{X} \to B$ of $f' : X \to \operatorname{Spec} K$, with $\mathcal{X}$ smooth; when B is projective, $\mathrm{CH}^i(\mathcal{X})^0$ contains numerically trivial cycles (Proposition 2.5) and in general it contains locally homologically trivial cycles in the sense of Beilinson [1987, 1.2] (Proposition 2.6). From the intersection pairing on $\mathcal{X}$, pushed forward to $\mathrm{CH}^1(B)$, we then get, thanks to Proposition 2.8, a height pairing $\langle\ ,\ \rangle_f$ defined on the groups $\mathrm{CH}^i(X)^0_f := \mathrm{Im}(\mathrm{CH}^i(\mathcal{X})^0 \to \mathrm{CH}^i(X))$ (2-9). This is a pairing of genuine abelian groups. We prove in Propositions 3.6 and 3.8 that the $\mathrm{CH}^i(X)^0_f$ and $\langle\ ,\ \rangle_f$ are independent of f and compatible with the action of correspondences, and in Proposition 3.9 that they behave well with respect to base change. The group $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0$ is finitely generated (Proposition 3.11).

If we are in characteristic 0, the construction is finished since X always admits a smooth model by resolution of singularities (Proposition 4.1). In characteristic $p > 0$, there turns out to be quite a bit of work to get a pairing in general after suitably inverting p, by using Gabber's refinement of de Jong's theorem: the general height pairing (4-1) is defined in Theorem 4.14; as said above, it makes sense in the category $\mathbf{Ab} \otimes \mathbb{Z}[1/p]$. Functoriality and base change extend to this pairing (Theorem 4.14).

In Section 5, we investigate Conjecture 5.1: $\mathrm{CH}^i(X)^{[0]}$ is of finite index in $\mathrm{CH}^i_{\mathrm{num}}(X)$, the group of cycles numerically equivalent to 0 (the inclusion is always true by Lemma 4.3(d)); we prove it for $i = 1, d$ in Theorem 5.6(b) (see Theorem 5.5(ii) for other cases). In Section 5C, we also relate (1) to the classical Néron–Tate height pairing in the case where X is an elliptic curve and B is a smooth projective curve.

In Section 6, we study the height pairing (2-9) in the basic case $i = 1$. If B is projective, it leads to a coarser pairing (6-2) between the Lang–Néron groups $\mathrm{LN}(\mathrm{Pic}^0_X, K/k)$ and $\mathrm{LN}(\mathrm{Alb}_X, K/k)$ with values in $N^1(B)$, codimension 1 cycles modulo numerical equivalence (Theorem 6.2). When $\delta = 1$, a version of this pairing involving an ample divisor is negative definite (Theorem 6.6): one should compare this with a result of Shioda [1999] when $d = 1$. See also Theorem 6.6 for a conjectural statement when $\delta > 1$. We finally get an intriguing homomorphism from $\mathrm{LN}(\mathrm{Pic}^0_X, K/k)$ to homomorphisms between certain abelian varieties in (6-6).

Notation and conventions. We try and follow Fulton's notation [1984] as much as possible. In particular, given a morphism of k-schemes $f : X \to Y$, we write γ_f for the associated graph morphism $X \to X \times_k Y$ and δ_X for γ_{1_X}; if f admits refined Gysin morphisms as in [loc. cit., Chapters 6 and 8], we write them $f^!$ and sometimes use the notation f^* for ordinary Gysin morphisms.

We usually abbreviate the notation $\times_k$ (fibre product over k) to $\times$, and re-establish it when it may be confused with other fibre products.

We shall encounter k-schemes essentially of finite type, being of finite type over some localisation of B. We shall sometimes commit the abuse of treating them as if they were of finite type: for example, call them smooth even if they really are essentially smooth, and take (refined) Gysin morphisms associated to morphisms between them even if these morphisms are not of finite type. This is easily justified by the fact that Chow groups commute with inverse limits of open immersions [Bloch 2010, Lemma IA.1].

1. An elementary reduction

1A. ***Intersection on regular K-schemes.*** Let K be a field. If $\mathrm{char}\, K = 0$, every regular K-scheme X, separated of finite type, is smooth, so the intersection theory of [Fulton 1984, Chapter 8] applies. Here we point out that this is also true in characteristic $p > 0$: it will be needed in and after Section 4B.

We may assume K to be finitely generated over its (perfect) subfield $k = \mathbb{F}_p$, and X (regular) to be irreducible of dimension d. We may find a smooth connected separated k-scheme B of finite type with generic point $\eta = \operatorname{Spec} K$, and a dominant morphism $f : \mathcal{X} \to B$ with $\mathcal{X}$ k-smooth, of generic fibre X. We have the intersection pairing of [Fulton 1984, §8.1]: for $i, r \geq 0$,

$$\mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}^{d+r-i}(\mathcal{X}) \xrightarrow{\cdot} \mathrm{CH}^{d+r}(\mathcal{X}), \tag{1-1}$$

which commutes with base change by [Fulton 1984, Proposition 6.6(c) and 8.3(a)]. Then (1-1) induces an intersection product on X by passing to the limit. If $f_1 : \mathcal{X}_1 \to B_1$ is another choice, then B and B_1 share a common open subset with isomorphic fibres, so this intersection product is independent of the choice of (B, f).

Suppose moreover X and f proper. Composing (1-1) with f_*, we get a pairing

$$\mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}^{d+r-i}(\mathcal{X}) \xrightarrow{\langle\ ,\ \rangle} \mathrm{CH}^r(B). \tag{1-2}$$

For the same reason, numerical equivalence makes sense on X via (1-2), and does not depend on any choice.

1B. *The set-up.* Let now k be any perfect field; we place ourselves in the situation $(B, \mathcal{X}, f)$ of Section 1A with f proper, and let $f' : X \to \eta$ be the generic fibre of f. In particular, the observations of Section 1A apply to X.

For a subscheme Z of B, write $\mathcal{X}_Z = f^{-1}(Z)$, $\iota : \mathcal{X}_Z \hookrightarrow \mathcal{X}$ for the corresponding immersion and $f_Z : \mathcal{X}_Z \to Z$ for the projection induced by f. We extend these notations to pull-backs by a morphism $Z \to B$ when there is no ambiguity in the context.

Lemma 1.1. *Suppose that* $\operatorname{codim}_B Z > r$. *Then* (1-2) *factors through a pairing*

$$\mathrm{CH}^i(\mathcal{X} - \mathcal{X}_Z) \times \mathrm{CH}^{d+r-i}(\mathcal{X} - \mathcal{X}_Z) \xrightarrow{\langle\ ,\ \rangle} \mathrm{CH}^r(B).$$

Proof. We have

$$\mathrm{CH}^r(B) \xrightarrow{\sim} \mathrm{CH}^r(B - Z). \qquad \square$$

We shall use the case $r = 1$ of this lemma in the rest of this paper.

Remarks 1.2. (a) Let $\mathcal{Z}$ be the locus of nonsmoothness of f. If f' is smooth, $f(\mathcal{Z})$ is a proper closed subset of B, hence contains only finitely many points of $B^{(1)}$, the set of codimension 1 points of B.

(b) If $\delta = 1$, any proper surjective morphism φ from an irreducible k-variety V to B is flat [Hartshorne 1977, Chapter II, Proposition 9.7]; in general, this is true after base-changing to the local scheme at any point $b \in B^{(1)}$. If $F \subset V$ is the (closed) locus of nonflatness of φ, the closed subset $\varphi(F)$ is therefore of codimension ≥ 2 in B. This shows that one may reduce to φ flat by removing a closed subset of codimension ≥ 2 from B. This technique may be applied to f if necessary; a variant will be used in the proof of Proposition 3.11.

Let $\mathrm{CH}^i_{\mathrm{num}}(X)$ denote the subgroup of $\mathrm{CH}^i(X)$ formed of cycles numerically equivalent to 0; write j for the inclusion $X \hookrightarrow \mathcal{X}$.

Lemma 1.3. *For $\alpha \in \mathrm{CH}^i(\mathcal{X})$, the following are equivalent*:

(1) $j^*\alpha \in \mathrm{CH}^i_{\mathrm{num}}(X)$;

(2) *for any* $\beta \in \mathrm{CH}^{d-i}(\mathcal{X})$, $f_*(\alpha \cdot \beta) = 0$.

Proof. We have (2) $\Rightarrow$ (1) because of the surjectivity of j^* and the formula

$$\jmath^* f_*(\alpha \cdot \beta) = f'_* j^*(\alpha \cdot \beta) = f'_*(j^*\alpha \cdot j^*\beta) \tag{1-3}$$

[Fulton 1984, Proposition 1.7 and 8.3(a)], where $\jmath : \eta \hookrightarrow B$ is the inclusion, and (1) $\Rightarrow$ (2) because of (1-3) and the injectivity of $\jmath^* : \mathrm{CH}^0(B) \to \mathrm{CH}^0(\eta)$. □

2. The refined height pairing

We keep the set-up of Section 1B.

2A. *Review of Fulton's refined Gysin morphisms.* Let $f : X \to Y$ be a morphism of algebraic k-schemes, of constant dimensions d_X and d_Y for simplicity, and let $d = d_Y - d_X$. In certain cases, Fulton associates to f "refined Gysin morphisms"

$$f^! : \mathrm{CH}_*(Y') \to \mathrm{CH}_{*-d}(X \times_Y Y')$$

for any Y-scheme Y'; these morphisms are compatible with push-forward, pull-back and intersection products in the sense of [Fulton 1984, Definition 17.1]. Such collections of morphisms are called *orientations* in [loc. cit., §17.4]. Orientable morphisms are

- flat morphisms [loc. cit., Theorem 1.7],
- regular embeddings [loc. cit., §§6.2, 6.4],
- more generally, l.c.i. morphisms [loc. cit., §6.5],
- morphisms to a smooth Y [loc. cit., Definition 8.1.2].

The definitions of $f^!$ agree when f is of several of these forms at the same time, e.g., [loc. cit., Proposition 8.1.2]. The assignment $f \mapsto f^!$ is functorial in certain cases, many of which are summarised in [loc. cit., Example 17.4.6].

Since it is difficult to find a unified statement of all these compatibilities in [Fulton 1984], we shall strive to give precise references for all those we use; the above reminder should only be viewed as a guide to the reader.

We shall very often use the following situation, that we record as a lemma.

Lemma 2.1. *Let*

$$\begin{tikzcd} S' \arrow[r, "f'"] \arrow[d, "g'"'] & T' \arrow[d, "g"'] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

be a Cartesian square of k-schemes, where g is proper and f is an l.c.i. morphism. Then:

(a) *One has*

$$f^! g_* = g'_* f^!$$

as homomorphisms from $\mathrm{CH}_*(T')$ *to* $\mathrm{CH}_*(S)$.

(b) *If f' is also an l.c.i. morphism, of same codimension, then $f^! = f'^!$.*

(c) *If f and g are two composable l.c.i. morphisms, then $(g \circ f)^! = f^! \circ g^!$.*

Proof. This follows from [Fulton 1984, Theorem 6.6(c)]. □

2B. *Admissible cycles.* Let $b \in B^{(1)}$; write $Z = \overline{\{b\}}$. Recall the cap-product [Fulton 1984, p. 131]

$$\cdot_\iota : \mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}_l(\mathcal{X}_Z) \to \mathrm{CH}_{l-i}(\mathcal{X}_Z), \quad (\alpha, \beta) \mapsto \gamma_\iota^!(\beta \times \alpha),$$

where ι is the closed immersion $\mathcal{X}_Z \hookrightarrow \mathcal{X}$.

Take $l = \delta + i - 1$. Composing with $(f_Z)_*$, we get a pairing

$$\langle\ ,\ \rangle_b : \mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z) \to \mathrm{CH}_{\delta-1}(Z) = \mathrm{CH}^0(Z) = \mathbb{Z}, \quad \langle \alpha, \beta \rangle_b = (f_Z)_*(\alpha \cdot_\iota \beta). \tag{2-1}$$

We record two useful formulas:

$$\alpha \cdot \iota_* \beta = \iota_*(\alpha \cdot_\iota \beta) \in \mathrm{CH}_{l-i}(\mathcal{X}), \tag{2-2}$$

which follows from Lemma 2.1 applied to the Cartesian diagram

$$\begin{tikzcd} \mathcal{X}_Z \arrow[r, "\gamma_\iota"] \arrow[d, "\iota"'] & \mathcal{X}_Z \times \mathcal{X} \arrow[d, "\iota \times 1"'] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

of regular embeddings of codimension $d + \delta$. Hence

$$f_*(\alpha \cdot \iota_* \beta) = f_* \iota_*(\alpha \cdot_\iota \beta) = \iota'_* \langle \alpha, \beta \rangle_b, \tag{2-3}$$

where ι' is the closed immersion $Z \hookrightarrow B$.

Definition 2.2. With the above notation, we set

$$\mathrm{CH}^i(\mathcal{X})^0_b = \{\alpha \in \mathrm{CH}^i(\mathcal{X}) \mid j^*\alpha \in \mathrm{CH}^i_{\mathrm{num}}(X) \quad \text{and} \quad \langle \alpha, \beta \rangle_b = 0 \quad \text{for all } \beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)\}$$

for $b \in B^{(1)}$, and

$$\mathrm{CH}^i(\mathcal{X})^0 = \bigcap_{b \in B^{(1)}} \mathrm{CH}^i(\mathcal{X})^0_b.$$

We call the cycles in $\mathrm{CH}^i(\mathcal{X})^0$ *admissible*.

Even if it is not apparent anymore, this definition was inspired by [Bloch 1984, Assumption 2; Beĭlinson 1987, 1.2].

Remarks 2.3. (a) One should be careful that $\mathrm{CH}^i(\mathcal{X})^0$ does not contain $\operatorname{Ker} j^*$ in general. For example, let $B = \boldsymbol{A}^1 = \operatorname{Spec} k[t]$ and let $\mathcal{X}$ be the hypersurface in $B \times \mathbb{P}^2$ with (partly) homogeneous equation $tX_0^2 = X_1X_2$. Then the pull-back of the curve $(t = X_1 = 0)$, viewed as a codimension 1 cycle on $\mathcal{X}$, to the curve $(t = X_2 = 0)$, is the point $(0, (1:0:0))$ which is not numerically equivalent to 0. On the other hand, if f is smooth above $\operatorname{Spec} \mathcal{O}_{B,b}$ for a $b \in B^{(1)}$, then any element of $\operatorname{Ker} j^*$ vanishes when restricted to $\mathcal{X}_b$ thanks to [Fulton 1984, §20.3]. So this caveat only involves finitely many exceptional b.

(b) The pairing (2-1) makes sense for any $b \in B$ (replacing $\mathrm{CH}_{\delta+i-1}(Z)$ by $\mathrm{CH}_{\delta+i-r}(Z)$ if $b \in B^{(r)}$), and defines an equivalence relation $\alpha \equiv_b 0$ if $\langle \alpha, \beta \rangle_b = 0$ for any $\beta \in \mathrm{CH}_{\delta+i-r}(Z)$. One can show that $\alpha \equiv_{b'} 0 \Rightarrow \alpha \equiv_b 0$ if b' is a specialisation of b; in particular, the condition $j^*\alpha \in \mathrm{CH}^i_{\mathrm{num}}(X)$ is superfluous in the definition of $\mathrm{CH}^i(\mathcal{X})^0_b$, thanks to Lemma 1.3. We shall not use these facts in the present paper, so the rather long proof is omitted (see [Kahn 2023]).

(c) Let $b \in B^{(1)}$. Suppose that all the irreducible components $\mathcal{X}_b^\lambda$ of $\mathcal{X}_b$ are of dimension d and smooth over $k(b)$. Then it is easy to see that $\alpha \equiv_b 0$ if and only if $\kappa_\lambda^! \alpha \in \mathrm{CH}^i_{\mathrm{num}}(\mathcal{X}_b^\lambda)$ for all λ, where $\kappa_\lambda : \mathcal{X}_b^\lambda \hookrightarrow \mathcal{X}$ is the inclusion. Our initial approach to the refined height pairing was based on such models; they are not necessary anymore.

We obviously have

Lemma 2.4. *The quotient* $\mathrm{CH}^i(\mathcal{X})/\mathrm{CH}^i(\mathcal{X})^0$ *is torsion-free.* □

2C. *Comparison with numerical and homological equivalence.*

Proposition 2.5. *If B is projective* (*hence $\mathcal{X}$ is k-proper*), *we have* $\mathrm{CH}^i_{\mathrm{num}}(\mathcal{X}) \subseteq \mathrm{CH}^i(\mathcal{X})^0$.

Proof. Let $\alpha \in \mathrm{CH}^i_{\mathrm{num}}(\mathcal{X})$: we want to show that $\alpha \in \mathrm{CH}^i(\mathcal{X})^0$. Let first $\beta \in \mathrm{CH}^{d-i}(\mathcal{X}) = \mathrm{CH}_{\delta+i}(\mathcal{X})$. Choose a 0-cycle $z \in \mathrm{CH}_0(B)$ of nonzero degree. Then

$$0 = \deg(\alpha \cdot \beta \cdot f^*z) = \deg(f_*(\alpha \cdot \beta) \cdot z) = f_*(\alpha \cdot \beta)\deg(z);$$

hence $f_*(\alpha \cdot \beta) = 0$, and we conclude that $j^*\alpha \in \mathrm{CH}^i_{\mathrm{num}}(X)$ by Lemma 1.3.

Let now $b \in B^{(1)}$, and $Z = \overline{\{b\}}$ as above. Let $\beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)$. We have this time

$$0 = f_*(\alpha \cdot \iota_*\beta \cdot f^*z) = f_*(\alpha \cdot \iota_*\beta) \cdot z$$

for any $z \in \mathrm{CH}_1(B) = \mathrm{CH}^{\delta-1}(B)$, i.e., $f_*(\alpha \cdot \iota_*\beta) = \iota'_*\langle \alpha, \beta \rangle_b \in \mathrm{CH}^1_{\mathrm{num}}(B)$ (see (2-3)). But

$$\iota'_* : \mathbb{Z} = \mathrm{CH}^0(Z) \to \mathrm{CH}^1(B)/\mathrm{CH}^1_{\mathrm{num}}(B)$$

is injective since Z, as an irreducible divisor on a smooth projective variety, is not numerically equivalent to 0 (compare [Debarre 2001, Chapter I, Theorem 1.21]). Therefore $\langle\alpha, \beta\rangle_b = 0$, as requested. □

Let now l be a prime number invertible in k. We have a composition

$$\mathrm{CH}^i(\mathcal{X}) \to H^{2i}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_l(i)) \to H^0(B_{\bar{k}}, R^{2i} f_*\mathbb{Q}_l(i)), \tag{2-4}$$

where the first map is the (geometric) cycle class map. Write $\mathrm{CH}^i_l(\mathcal{X})$ (resp. $\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$ for the kernel of the first map (resp. of their composition): the latter group is introduced by analogy to [Beĭlinson 1987, 1.2], which is the special case $\delta = 1$, k algebraically closed. We obviously have $\mathrm{CH}^i_l(\mathcal{X}) \subseteq \mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$.

The following is parallel to Proposition 2.5, without assuming B projective. It will be used in Proposition 2.11 and in Remarks 5.4(a) and 3.12.

Proposition 2.6. *At least in characteristic* 0, $\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l} \subseteq \mathrm{CH}^i(\mathcal{X})^0$.

Proof. Let $\alpha \in \mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$. Then α vanishes in $H^0(K\bar{k}, R^{2i} f_*\mathbb{Q}_l(i)) = H^{2i}(X \otimes_k \bar{k}, \mathbb{Q}_l(i))$, hence a fortiori in $H^{2i}(X \otimes_K \overline{K}, \mathbb{Q}_l(i))$: this means that $j^*\alpha$ is l-adically homologically equivalent to 0, hence also numerically equivalent to 0. This part of the proof works in all characteristics.

We now give the sequel of the proof in characteristic 0: to oversimplify, it follows by functoriality from the fact that the cycle class map is injective in codimension 0 (sic). (So this argument is geometrically cheaper than the one for Proposition 2.5.)

We may assume k finitely generated and choose an embedding of k in $\mathbb{C}$. By Artin's comparison theorem,

$$\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l} = \mathrm{Ker}\big(\mathrm{CH}^i(\mathcal{X}) \to H^0_B(B_{\mathbb{C}}, R^{2i} f_*\mathbb{Q}(i))\big),$$

where H_B denotes Betti (or analytic) cohomology. Let $b \in B^{(1)}$, and let Z, ι, β be as in Definition 2.2. To show that $\langle\alpha, \beta\rangle_b = 0$ in $\mathrm{CH}^0(Z) \xrightarrow{\sim} \mathrm{CH}^0(Z_{\mathbb{C}})$, we may assume $k = \mathbb{C}$ and drop all Tate twists.

In [Fulton 1984, Chapter 19], a cycle class map cl is defined for Chow groups of complex, possibly singular, varieties, with values in their Borel–Moore homology and we have the formula

$$\mathrm{cl}(\alpha \cdot_\iota \beta) = \iota'^*(\mathrm{cl}(\alpha)) \cap \mathrm{cl}(\beta) \in H_{2\delta-2}(\mathcal{X}_Z) \tag{2-5}$$

[Fulton 1984, Proposition 19.2], where ι' is the closed immersion $Z \hookrightarrow B$ as in the previous proof, hence

$$\mathrm{cl}(\langle\alpha, \beta\rangle_b) = (f_Z)_*(\iota^*(\mathrm{cl}(\alpha)) \cap \mathrm{cl}(\beta)) \in H_{2\delta-2}(Z) \tag{2-6}$$

since cl commutes with push-forwards, by definition and [Fulton 1984, Lemma 19.1.2].

It now suffices to show that the right hand side of (2-6) vanishes since $\mathrm{CH}_{\delta-1}(Z) \to H_{2\delta-2}(Z)$ is injective, as one sees by reducing to Z smooth by removing from it a proper closed subset. For this, it suffices to show that the pairing

$$H^{2i}(\mathcal{X}) \times H_{2\delta-2+2i}(\mathcal{X}_Z) \to H_{2\delta-2}(Z), \tag{2-7}$$

given by $(x, y) \mapsto (f_Z)_*(\iota^* x \cap y)$, factors through $H^0(B, R^{2i} f_*\mathbb{Q}) \times H_{2\delta-2+2i}(\mathcal{X}_Z)$.

We switch by Poincaré duality from the Borel–Moore homology of $\mathcal{X}_Z$ (resp. Z) to the cohomology of the smooth variety $\mathcal{X}$ (resp. B) with supports in $\mathcal{X}_Z$ (resp. in Z). Then (2-7) becomes the composition

$$H^{2i}(\mathcal{X}) \times H^{2d+2-2i}_{\mathcal{X}_Z}(\mathcal{X}) \xrightarrow{\cap} H^{2d+2}_{\mathcal{X}_Z}(\mathcal{X}) \xrightarrow{f_*} H^2_Z(B), \tag{2-8}$$

where $\cap$ is the usual cap-product. The (global) trace map f_* factors as a composition

$$H^{2d+2}_{\mathcal{X}_Z}(\mathcal{X}) \to H^0_Z(B, R^{2d+2} f_*\mathbb{Q}) \xrightarrow{(\mathrm{Tr}_f)_*} H^2_Z(B)$$

where Tr_f is the local trace map in étale cohomology for the proper morphism f. Thus, (2-8) factor through the map

$$H^{2i}(\mathcal{X}) \times H^{2d+2-2i}_{\mathcal{X}_Z}(\mathcal{X}) \to H^0(B, R^{2i} f_*\mathbb{Q}) \times H^0_Z(B, R^{2d+2-i} f_*\mathbb{Q})$$

as requested.

In positive characteristic, the leap of faith is that (2-5) and (2-6) hold for the cycle class maps defined in l-adic Borel–Moore homology [Laumon 1976, §6]. The commutation with push-forwards causes no problem, and (2-5) indeed appears in [Laumon 1976, Theorem (7.2)], except that the extraordinary cap-product $\cdot_\iota$ (defined in [Verdier 1976, 2.1.1] using intersection multiplicities) should be shown to agree with Fulton's. (This is suggested in the notes and references of [Fulton 1984, Chapter 19]; see also [loc. cit., p. 382].)[1]

This being accepted, the same argument goes through. □

Remark 2.7. As a referee pointed out, there is an important conceptual difference between $\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$ and $\mathrm{CH}^i(\mathcal{X})^0$: by the smooth and proper base change, we have the equality

$$\mathrm{Ker}\big(H^{2i}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(i)) \to H^{2i}(X_{\bar k}, \mathbb{Q}_l(i))\big) = \mathrm{Ker}\big(H^{2i}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(i)) \to H^0(U_{\bar k}, R^{2i} f_*\mathbb{Q}_l(i))\big)$$

for any open subset $U \subseteq B$ over which f is smooth. Thus, the condition $\alpha \in \mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$ for $\alpha \in \mathrm{CH}^i(\mathcal{X})$ only has to be checked at the generic fibre and at the "bad fibres" of f. This contrasts with the case of $\mathrm{CH}^i(\mathcal{X})^0$, see Remarks 2.3(a). See also Remarks 5.4 further down.

2D. *Global height pairing.*

The following proposition is the key point of this paper.

Proposition 2.8. *Let $\alpha \in \mathrm{CH}^i(\mathcal{X})^0$. If $\beta \in \mathrm{CH}^{d+1-i}(\mathcal{X})$ and $j^*\beta = 0$, then $f_*(\alpha \cdot \beta) = 0$ in $\mathrm{CH}^1(B)$.*

Proof. By [Fulton 1984, Proposition 1.8], write $\beta = \iota_*\beta'$ with $\beta' \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)$ for some proper closed subset $Z \subset B$, where $\iota : \mathcal{X}_Z \hookrightarrow \mathcal{X}$ is the inclusion. We may assume that β' is the class of an irreducible cycle, hence take Z irreducible. If $\mathrm{codim}_B Z > 1$, the result follows from Lemma 1.1. If $Z = \overline{\{b\}}$ for $b \in B^{(1)}$, the conclusion follows from (2-3). □

The proof of the following lemma is in the same spirit, so we include it here. It will be used in the proof of Proposition 3.9(ii).

[1] Using Olsson's theorem [2015, Theorem 2.34] that the cycle class maps commute with refined Gysin homomorphisms, it would suffice to show the identity $\gamma_\iota^!(x \times y) = \iota^* y \cap x$ in Borel–Moore l-adic homology.

Lemma 2.9. *Let $b_1, \dots, b_n$ be a finite set of points on $B^{(1)}$ and let $Z = \overline{\{b_1, \dots, b_n\}}$. Then one has $(f_Z)_*(\alpha \cdot_\iota \beta) = 0$ for any $\alpha \in \mathrm{CH}^i(\mathcal{X})^0$ and any $\beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)$, where ι is the closed immersion $\mathcal{X}_Z \hookrightarrow \mathcal{X}$.*

Proof. We may assume that β is the class of an irreducible cycle β'; then β' is supported on $\mathcal{X}_{Z_r}$ for some r, where $Z_r = \overline{\{b_r\}}$. Let $\kappa : \mathcal{X}_{Z_r} \hookrightarrow \mathcal{X}_Z$ be the corresponding closed immersion, and let $\iota_r = \iota\kappa$: by applying again Lemma 2.1 to the obvious Cartesian square involving κ, we get the identity

$$\alpha \cdot_\iota \kappa_*\beta' = \kappa_*(\alpha \cdot_{\iota_r} \beta')$$

etc. □

Definition 2.10. Let $\mathrm{CH}^i(X)^0_f$ be the image of $\mathrm{CH}^i(\mathcal{X})^0$ in $\mathrm{CH}^i(X)$. By Proposition 2.8, (1-2) induces a pairing

$$\mathrm{CH}^i(\mathcal{X})^0 \times \mathrm{CH}^{d+1-i}(X) \to \mathrm{CH}^1(B)$$

hence, swapping i with $d+1-i$, a "height" pairing

$$\langle\ ,\ \rangle_f : \mathrm{CH}^i(X)^0_f \times \mathrm{CH}^{d+1-i}(X)^0_f \to \mathrm{CH}^1(B). \tag{2-9}$$

We shall see in the next section (Propositions 3.6 and 3.8) that neither $\mathrm{CH}^i(X)^0_f$ nor $\langle\ ,\ \rangle_f$ depends in the choice of f.

2E. *Comparison with the pairing of Rössler–Szamuely.*

Proposition 2.11. *The pairing (2-9) is compatible with the pairing (2) of the introduction on the subgroups $\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$ and $\mathrm{CH}^{d+1-i}(\mathcal{X})^0_{\mathrm{B},l}$ of Proposition 2.6.*

Proof. Using cup-product and push-forward in l-adic cohomology,

$$H^{2i}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(i)) \otimes H^{2(d+1-i)}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(d+1-i)) \xrightarrow{\cup} H^{2(d+1)}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(d+1)) \xrightarrow{f_*} H^2(B_{\bar k}, \mathbb{Q}_l(1)), \tag{2-10}$$

we get from (2-4) a pairing

$$\mathrm{CH}^i(\mathcal{X}) \otimes \mathrm{CH}^{d+1-i}(\mathcal{X}) \to H^2(B_{\bar k}, \mathbb{Q}_l(1)) \tag{2-11}$$

which is evidently compatible with (1-2) (for $r = 1$). On the other hand, the Leray spectral sequence

$$H^r(B_{\bar k}, R^s f_*\mathbb{Q}_l(i)) \Rightarrow H^{r+s}(\mathcal{X}_{\bar k}, \mathbb{Q}_l(i)) \tag{2-12}$$

yields Abel–Jacobi maps

$$\mathrm{AJ}^i_B : \mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l} \to H^1(B_{\bar k}, R^{2i-1} f_*\mathbb{Q}_l(i)). \tag{2-13}$$

We have a pairing parallel to (2-10),

$$H^1(B_{\bar k}, R^{2i-1} f_*\mathbb{Q}_l(i)) \otimes H^1(B_{\bar k}, R^{2(d-i+1)-1} f_*\mathbb{Q}_l(d-i+1)) \\ \xrightarrow{\cup} H^2(B_{\bar k}, R^{2d} f_*\mathbb{Q}_l(i)) \xrightarrow{\mathrm{Tr}_f} H^2(B_{\bar k}, \mathbb{Q}_l(1)), \tag{2-14}$$

which is compatible with the former via (2-12). This implies that the restriction of (2-11) to

$$\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l} \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})^0_{\mathrm{B},l}$$

is *compatible with* (2-9) via Proposition 2.6, i.e., that the diagram

$$\begin{array}{ccc} \mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l} \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})^0_{\mathrm{B},l} & \longrightarrow & H^2(B_{\bar{k}}, \mathbb{Q}_l(1)) \\ \big\downarrow & & \big\uparrow \\ \mathrm{CH}^i(\mathcal{X})^0 \otimes \mathrm{CH}^{d+1-i}(\mathcal{X})^0 & \longrightarrow & \mathrm{CH}^1(B) \end{array} \tag{2-15}$$

commutes.

On the other hand, the height pairing of [Rössler and Szamuely 2022] is defined on

$$\mathrm{CH}^i_l(X) \otimes \mathrm{CH}^{d+1-i}_l(X),$$

also with values in $H^2(B_{\bar{k}}, \mathbb{Q}_l(1))$. More precisely, by [loc. cit., Proposition 2.3], if $\alpha \in \mathrm{CH}^i_l(X)$, $j : U \hookrightarrow B$ is an open subset over which f is smooth and α_U is a lift of α to $\mathrm{CH}^i(\mathcal{X}_U)$, then $\mathrm{AJ}^i_U(\alpha_U) \in H^1(U_{\bar{k}}, R^{2i-1}(f_U)_*\mathbb{Q}_l(i))$ lies in the subgroup $H^{1-\delta}(B_{\bar{k}}, j_{!*}R^{2i-1}(f_U)_*\mathbb{Q}_l(i))$ [loc. cit., Proposition 2.1], and the height pairing of Rössler and Szamuely is defined by (2-14) on these subgroups. Let $\mathcal{F} = R^{2i-1}(f_U)_*\mathbb{Q}_l(i) = j^*R^{2i-1}f_*\mathbb{Q}_l(i)$. Since $j^*j_{!*}\mathcal{F} = \mathcal{F}$ [Beĭlinson et al. 1982, remarque 1.4.14.1], the image of $H^1(B_{\bar{k}}, R^{2i-1}f_*\mathbb{Q}_l(i))$ in $H^1(U_{\bar{k}}, R^{2i-1}(f_U)_*\mathbb{Q}_l(i))$ is contained in $H^{1-\delta}(B_{\bar{k}}, j_{!*}R^{2i-1}(f_U)_*\mathbb{Q}_l(i))$. □

3. Independence from the (smooth) model

3A. ***Review of the Corti–Hanamura category.*** A morphism $f : \mathcal{X} \to B$ as in Section 1 defines an object in the Corti–Hanamura category $\mathrm{CH}\mathcal{C}(B)$ of [Corti and Hanamura 2000, Definition 2.8].[2] Given two such objects $f_i : \mathcal{X}_i \to B$ $(i = 1, 2)$, morphisms in $\mathrm{CH}\mathcal{C}(B)$ are defined by relative correspondences

$$\mathrm{CH}\mathcal{C}(B)(\mathcal{X}_1, \mathcal{X}_2) = \mathrm{CH}_{\dim \mathcal{X}_2}(\mathcal{X}_1 \times_B \mathcal{X}_2) = \mathrm{CH}^{\dim X_1}(\mathcal{X}_1 \times_B \mathcal{X}_2),$$

where X_1 is the generic fibre of $\mathcal{X}_1$.

If $f_3 : \mathcal{X}_3 \to B$ is a third object, the composition of two such correspondences $u : \mathcal{X}_1 \to \mathcal{X}_2$ and $v : \mathcal{X}_2 \to \mathcal{X}_3$ is defined as

$$v \bullet u = (p^{1,2,3}_{1,3})_* \delta^!_2 (u \times_k v), \tag{3-1}$$

where $\delta^!$ is the refined Gysin morphism from [Fulton 1984, §6.2] associated to the (regular immersion) diagonal $\delta_2 : \mathcal{X}_2 \to \mathcal{X}_2 \times_k \mathcal{X}_2$ in the (augmented) Cartesian square

$$\begin{array}{ccccc} (\mathcal{X}_1 \times_B \mathcal{X}_3) & \xleftarrow{p^{1,2,3}_{1,3}} & \mathcal{X}_1 \times_B \mathcal{X}_2 \times_B \mathcal{X}_3 & \xrightarrow{\Delta} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times_k (\mathcal{X}_2 \times_B \mathcal{X}_3) \\ & & \big\downarrow {\scriptstyle p^{1,2,3}_2} & & \big\downarrow {\scriptstyle p^{1,2}_2 \times p^{2,3}_1} \\ & & \mathcal{X}_2 & \xrightarrow{\delta_2} & \mathcal{X}_2 \times_k \mathcal{X}_2 \end{array} \tag{3-2}$$

[2] Except that f is assumed projective in [Corti and Hanamura 2000]; proper is sufficient to apply its formalism.

and the notation for the projections is self-evident.

As usual, one can generalise this to "graded correspondences"

$$\mathrm{CH}\mathcal{C}(B)_r(\mathcal{X}_1,\mathcal{X}_2)=\mathrm{CH}_{\dim\mathcal{X}_2-r}(\mathcal{X}_1\times_B\mathcal{X}_2)=\mathrm{CH}^{\dim X_1+r}(\mathcal{X}_1\times_B\mathcal{X}_2)$$

and reduce these graded correspondences to ordinary ones if one wishes, by using the projective bundle formula [Fulton 1984, Theorem 3.3(b)].

Since Δ is also a regular immersion of the same codimension as δ (namely, $\dim\mathcal{X}_2$), we may apply Lemma 2.1(b) which gives

$$\delta_2^!(v\times_k u)=\Delta^!(v\times_k u). \tag{3-3}$$

If the f_i are smooth, we also have a "classical" composition of correspondences à la Deninger–Murre [1991]:

$$v\circ u=(p_{1,3}^{1,2,3})_*\big((p_{2,3}^{1,2,3})^*v\cdot(p_{1,2}^{1,2,3})^*u\big).$$

Lemma 3.1. (a) *In the above case,* $v\circ u=v\bullet u$.

(b) *The category* $\mathrm{CH}\mathcal{C}(B)$ *is contravariant for smooth k-morphisms* $\varphi: C\to B$.

(c) *The pro-open immersion j defines a functor to the category of Chow correspondences over K from the full subcategory of* $\mathrm{CH}\mathcal{C}(B)$ *consisting of those* $f:\mathcal{X}\to B$ *whose generic fibre is smooth.*

Proof. (a) We use (3-3). We have the Cartesian square

$$\begin{array}{ccc}
\mathcal{X}_1\times_B\mathcal{X}_2\times_B\mathcal{X}_3 & \xrightarrow{\Delta_1} & (\mathcal{X}_1\times_B\mathcal{X}_2\times_B\mathcal{X}_3)\times_k(\mathcal{X}_1\times_B\mathcal{X}_2\times_B\mathcal{X}_3)\\
\| \big\downarrow & & \big\downarrow p_{2,3}^{1,2,3}\times p_{1,2}^{1,2,3}\\
\mathcal{X}_1\times_B\mathcal{X}_2\times_B\mathcal{X}_3 & \xrightarrow{\Delta} & (\mathcal{X}_2\times_B\mathcal{X}_3)\times_k(\mathcal{X}_1\times_B\mathcal{X}_2)
\end{array}$$

in which all morphisms are l.c.i. morphisms, hence

$$\Delta^!(v\times_k u)=\Delta_1^!(p_{2,3}^{1,2,3}\times p_{1,2}^{1,2,3})^!(v\times_k u)$$

by Lemma 2.1(c),

$$(p_{2,3}^{1,2,3}\times p_{1,2}^{1,2,3})^!(v\times_k u)=(p_{2,3}^{1,2,3}\times p_{1,2}^{1,2,3})^*(v\times_k u)=(p_{2,3}^{1,2,3})^*v\times_k(p_{1,2}^{1,2,3})^*u$$

by [Fulton 1984, Proposition 6.6(b)], and finally

$$\Delta_1^!\big((p_{2,3}^{1,2,3})^*v\times_k(p_{1,2}^{1,2,3})^*u\big)=(p_{2,3}^{1,2,3})^*v\cdot(p_{1,2}^{1,2,3})^*u$$

by definition of the intersection product on smooth varieties [Fulton 1984, p. 131].

(b) The statement means that φ defines a functor $\varphi^*:\mathrm{CH}\mathcal{C}(B)\to\mathrm{CH}\mathcal{C}(C)$, given by fibre product. It is defined on objects by the smoothness of φ, and on morphisms because smooth morphisms are flat. To check that it respects composition involves chasing in the Cartesian cube obtained by pulling back the square of B-schemes (3-2) along the morphism $C\times_B C\to B$, and then further pulling back along the diagonal $\delta': C\to C\times_B C$; this latter operation is unnecessary if C is an open subset of B. The first step involves [Fulton 1984, Proposition 6.6] as in the proof of (a), to take care of the flat l.c.i morphisms $C \times_B (\mathcal{X}_i \times_B \mathcal{X}_j) \to \mathcal{X}_i \times_B \mathcal{X}_j$; the second step uses the fact that δ' is a regular immersion.

(c) This follows from (a), (b) and [Bloch 2010, Lemma IA.1], since $U \times_B \mathcal{X}$ is smooth over U for a suitable open subset U of B for $\mathcal{X}$ as in the statement. □

Remark 3.2. The associativity of the composition • is not proven in [Corti and Hanamura 2000]. It will not be used here and is left to the reader. See nevertheless Remark 3.4.

As a special case of (3-1), take $\mathcal{X}_3 = B$: we get pairings

$$\mathrm{CH}^{\dim X_2+r}(\mathcal{X}_1 \times_B \mathcal{X}_2) \otimes \mathrm{CH}^i(\mathcal{X}_2) \to \mathrm{CH}^{i+r}(\mathcal{X}_1), \quad (\psi, \alpha) \mapsto \psi^*\alpha := (p_1)_*\delta_2^!(\psi \times_k \alpha)$$

compatible via j^* with the usual action of correspondences over K, by Lemma 3.1(c). For clarity, we repeat (3-1) in this special case:

$$\begin{array}{ccccc}
\mathcal{X}_1 & \xleftarrow{p_1} & \mathcal{X}_1 \times_B \mathcal{X}_2 & \xrightarrow{\gamma_{p_2}} & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times_k \mathcal{X}_2 \\
 & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2 \times 1} \\
 & & \mathcal{X}_2 & \xrightarrow{\delta_2} & \mathcal{X}_2 \times_k \mathcal{X}_2
\end{array} \tag{3-4}$$

where γ_{p_2} is the graph of $p_2 := p_2^{1,2}$.

We also write ψ_* for $({}^t\psi)^*$.

As an even more special case, when $\mathcal{X}_1 = B$: writing β rather than ψ, we recover the pairing (1-2)

$$\langle \alpha, \beta \rangle = (f_2)_*(\alpha \cdot \beta) = (f_2)_*\delta_2^!(\alpha \times_k \beta) = \beta^*\alpha \in \mathrm{CH}^*(B). \tag{3-5}$$

Lemma 3.3. *Let* $(\alpha, \beta) \in \mathrm{CH}^i(\mathcal{X}_1) \times \mathrm{CH}^{d_1-i+1}(\mathcal{X}_2)$ *and* $\psi \in \mathrm{CH}^{d_2}(\mathcal{X}_1 \times_B \mathcal{X}_2)$. *Then*

$$\langle \psi^*\alpha, \beta \rangle = \langle \alpha, \psi_*\beta \rangle.$$

Proof. For clarity, write δ_i for the diagonal map $\mathcal{X}_i \to \mathcal{X}_i \times_k \mathcal{X}_i$. As in the proof of Proposition 3.6, let p_i be the projection $\mathcal{X}_1 \times_B \mathcal{X}_2 \to \mathcal{X}_i$. Developing, the identity to be proven is

$$(f_1)_*((p_1)_*\delta_2^!(\psi \times \alpha) \cdot \beta) = (f_2)_*(\alpha \cdot (p_2)_*\delta_1^!({}^t\psi \times \beta)). \tag{3-6}$$

Let $\lambda = \delta_2^!(\psi \times \alpha)$. We have

$$(p_1)_*\lambda \cdot \beta = \delta_1^!((p_1)_*\lambda \times \beta) = \delta_1^!(p_1 \times 1)_*(\lambda \times \beta) = (p_1)_*\delta_1^!(\lambda \times \beta)$$

by Lemma 2.1(a). Similarly, if $\lambda' = \delta_1^!({}^t\psi \times \beta)$ and $\lambda'' := \delta_1^!(\psi \times \beta)$:

$$\alpha \cdot (p_2)_*\lambda' = (p_2)_*\delta_2^!(\alpha \times \lambda') = (p_2)_*\delta_2^!(\lambda'' \times \alpha).$$

Since $f_1 p_1 = f_2 p_2$, to show (3-6) it suffices to show that

$$\delta_1^!(\lambda \times \beta) = \delta_2^!(\lambda'' \times \alpha).$$

We now observe that since $\mathcal{X}_2$ is smooth, γ_{p_2} is also a regular embedding in (3-4), hence $\delta_2^! = \gamma_{p_2}^*$ (nonrefined Gysin map) by Lemma 2.1(b) (see also (3-3)); similarly, $\delta_1^! = \gamma_{p_1}^*$. The expression $\gamma_{p_i}^*(x \times y)$ is also written $x \cdot_{p_i} y$ in [Fulton 1984, Definition 8.1.1] (cf. proof of Proposition 2.8). The formula to be proven therefore becomes

$$(\psi \cdot_{p_2} \alpha) \cdot_{p_1} \beta = (\psi \cdot_{p_1} \beta) \cdot_{p_2} \alpha$$

which is [Fulton 1984, Proposition 8.1.1(b)]. □

Remark 3.4. There is a much more conceptual proof by interpreting both sides as compositions of correspondences: we then have

$$\langle \psi^* \alpha, \beta \rangle = (\alpha \bullet \psi) \bullet \beta = \alpha \bullet (\psi \bullet \beta) = \langle \alpha, \psi_* \beta \rangle$$

by the associativity of $\bullet$.

3B. *Independence from the model and functoriality.*

Lemma 3.5. *Let $b \in B^{(1)}$ and $Z = \overline{\{b\}}$ as usual. For $\psi \in \mathrm{CH}^*(\mathcal{X}_1 \times_B \mathcal{X}_2)$ and $\beta \in \mathrm{CH}_*(\mathcal{X}_{1,Z})$, let*

$$\psi_! \beta = (p_{2,Z})_* \delta_1^!(\psi \times \beta) \in \mathrm{CH}_*(\mathcal{X}_{2,Z}).$$

Then:

(a) $(\iota_2)_* \psi_! \beta = \psi_*((\iota_1)_* \beta)$.

(b) *For any $\alpha \in \mathrm{CH}^i(\mathcal{X}_2)$, $\beta \in \mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z)$ and $\psi \in \mathrm{CH}^{d_2}(\mathcal{X}_1 \times_B \mathcal{X}_2)$, we have $\langle \alpha, \psi_! \beta \rangle_b = \langle \psi^* \alpha, \beta \rangle_b$.*

Proof. (a) Let us first draw the diagram of Cartesian squares underlying the coming computation:

$$\begin{tikzcd}
\mathcal{X}_{2,Z} \arrow[d, "\iota_2"'] & \mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} \arrow[l, "p_{2,Z}"'] \arrow[r, "(\kappa, p_{1,Z})"] \arrow[d, "\kappa"'] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} \arrow[d, "1 \times \iota_1"'] \\
\mathcal{X}_2 & \mathcal{X}_1 \times_B \mathcal{X}_2 \arrow[l, "p_2"'] \arrow[r, "\gamma_{p_1}"] \arrow[d, "p_1"'] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 \arrow[d, "p_1 \times 1"'] \\
& \mathcal{X}_1 \arrow[r, "\delta_1"] & \mathcal{X}_1 \times \mathcal{X}_1
\end{tikzcd} \tag{3-7}$$

It already explains the use of $\delta_1^!$ in the definition of $\psi_!$. Now

$$\begin{aligned}
(\iota_2)_* \psi_! \beta = (\iota_2)_* (p_{2,Z})_* \delta_1^!(\psi \times \beta) &= (p_2)_* \kappa_* \delta_1^!(\psi \times \beta) \\
&= (p_2)_* \delta_1^!(1 \times \iota_1)_*(\psi \times \beta) = (p_2)_* \delta_1^!(\psi \times (\iota_1)_* \beta) \\
&=: ({}^t\psi)^*((\iota_1)_* \beta) =: \psi_*((\iota_1)_* \beta),
\end{aligned}$$

where the third equality follows as usual from Lemma 2.1(a).

(b) First

$$\begin{aligned}\alpha \cdot_{\iota_2} \psi_! \beta := \gamma_{\iota_2}^!((p_{2,Z})_* \delta_1^!(\psi \times \beta) \times \alpha) &\overset{\text{(a)}}{=} \gamma_{\iota_2}^!((p_{2,Z})_* \gamma_{p_1}^!(\psi \times \beta) \times \alpha)\\ &\overset{\text{(b)}}{=} (p_{2,Z})_* \gamma_{\iota_2}^!(\gamma_{p_1} \times 1)^!(\psi \times \beta \times \alpha)\\ &\overset{\text{(c)}}{=} (p_{2,Z})_* \gamma_{p_2}^!(\gamma_{p_1} \times 1)^!(\psi \times \beta \times \alpha) \overset{\text{(d)}}{=} (p_{2,Z})_* \gamma_\lambda^!(\psi \times \beta \times \alpha),\end{aligned}$$

where λ is the regular embedding $\mathcal{X}_1 \times_B \mathcal{X}_2 \hookrightarrow \mathcal{X}_1 \times \mathcal{X}_2$, so that γ_λ is the composition of the bottom row in the diagram of Cartesian squares

$$\begin{tikzcd}
\mathcal{X}_{2,Z} \arrow[r, "\gamma_{\iota_2}"] & \mathcal{X}_{2,Z} \times \mathcal{X}_2 & \\
\mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} \arrow[u, "p_{2,Z}"] \arrow[r, "\gamma_{p_{2,Z}\iota_2}"] \arrow[d, "\kappa"] & (\mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z}) \times \mathcal{X}_2 \arrow[u, "p_{2,Z} \times 1"] \arrow[r, "(\kappa, p_{1,Z}) \times 1"] \arrow[d, "\kappa \times 1"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} \times \mathcal{X}_2 \arrow[d, "1 \times \iota_1 \times 1"] \\
\mathcal{X}_1 \times_B \mathcal{X}_2 \arrow[r, "\gamma_{p_2}"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 \arrow[r, "\gamma_{p_1} \times 1"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 \times \mathcal{X}_2
\end{tikzcd} \tag{3-8}$$

Here (a) follows from Lemma 2.1(b) applied to (3-7), (b) from Lemma 2.1(a), (c) from Lemma 2.1(b) again (applied twice), and (d) from Lemma 2.1(c).

Next

$$\psi^* \alpha \cdot_{\iota_1} \beta := \gamma_{\iota_1}^!(\beta \times (p_1)_* \delta_2^!(\psi \times \alpha)) \overset{\text{(a)}}{=} \gamma_{\iota_1}^!(\beta \times (p_1)_* \gamma_{p_2}^!(\psi \times \alpha)) \overset{\text{(b)}}{=} (p_{1,Z})_* \gamma_{\iota_1}^!(1 \times \gamma_{p_2})^!(\beta \times \psi \times \alpha),$$

where (a) follows from Lemma 2.1(b) applied to (3-4) and (b) follows from Lemma 2.1(a) applied to the Cartesian square

$$\begin{tikzcd}
\mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} \arrow[r, "(p_{1,Z}, \kappa)"] \arrow[d, "p_{1,Z}"] & \mathcal{X}_{1,Z} \times (\mathcal{X}_1 \times_B \mathcal{X}_2) \arrow[d, "1 \times p_1"] \\
\mathcal{X}_{1,Z} \arrow[r, "\gamma_{\iota_1}"] & \mathcal{X}_{1,Z} \times \mathcal{X}_1
\end{tikzcd}$$

Since $f_{1,Z} p_{1,Z} = f_{2,Z} p_{2,Z}$, we are left to prove the equality

$$\gamma_\lambda^!(\psi \times \beta \times \alpha) = \gamma_{\iota_1}^!(1 \times \gamma_{p_2})^!(\beta \times \psi \times \alpha).$$

For this we draw the diagram of Cartesian squares, similar to (3-8):

$$\begin{tikzcd}
\mathcal{X}_{1,Z} \arrow[r, "{}^t\gamma_{\iota_1}"] & \mathcal{X}_1 \times \mathcal{X}_{1,Z} & \\
\mathcal{X}_{1,Z} \times_Z \mathcal{X}_{2,Z} \arrow[u, "p_{1,Z}"] \arrow[r, "(\kappa, p_{1,Z})"] \arrow[d, "\kappa"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_{1,Z} \arrow[u, "p_1 \times 1"] \arrow[r, "\gamma_{p_2} \times 1_{\mathcal{X}_{1,Z}}"] \arrow[d, "1 \times \iota_1"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 \times \mathcal{X}_{1,Z} \arrow[d, "1 \times 1 \times \iota_1"] \\
\mathcal{X}_1 \times_B \mathcal{X}_2 \arrow[r, "\gamma_{p_1}"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_1 \arrow[r, "\gamma_{p_2} \times 1_{\mathcal{X}_1}"] & (\mathcal{X}_1 \times_B \mathcal{X}_2) \times \mathcal{X}_2 \times \mathcal{X}_1
\end{tikzcd}$$

Here the composition of the bottom row is γ_λ, up to permuting $\mathcal{X}_1$ and $\mathcal{X}_2$. By Lemma 2.1(b), $({}^t\gamma_{\iota_1})^!$ and $\gamma_{p_1}^!$ both compute the refined Gysin map corresponding to the arrow $(\kappa, p_{1,Z})$, and also $(\gamma_{p_2} \times 1_{\mathcal{X}_{1,Z}})^! = (\gamma_{p_2} \times 1_{\mathcal{X}_1})^!$; we conclude by applying Lemma 2.1(c) to the bottom row once again. □

Proposition 3.6. *Let $f_1 : \mathcal{X}_1 \to B$, $f_2 : \mathcal{X}_2 \to B$ be two proper morphisms with generic fibres X_1, X_2 of dimensions d_1, d_2, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are smooth; let $r \in \mathbb{Z}$ and let $\gamma \in \mathrm{CH}^{d_2+r}(X_1 \times_K X_2)$ be a Chow correspondence of degree r. Then*

$$\gamma^* \mathrm{CH}^i(X_2)^0_{f_2} \subseteq \mathrm{CH}^{i+r}(X_1)^0_{f_1} \tag{3-9}$$

for any $i \geq 0$. In particular,

(i) *if $r = 0$, we also have $\gamma_* \mathrm{CH}_i(X_1)^0_{f_2} \subseteq \mathrm{CH}_i(X_2)^0_{f_1}$;*

(ii) *the group $\mathrm{CH}^i(X)^0_f$ does not depend on f.*

Proof. First, (i) (resp. (ii)) follows from (3-9) by considering ${}^t\gamma$ (resp. by taking $X_1 = X_2 = X$, $\gamma = \Delta_X$). To prove (3-9), we may assume that γ is the class of an integral cycle $\Gamma \subset X_1 \times_K X_2$.

Let $j_i : X_i \hookrightarrow \mathcal{X}_i$ be the corresponding immersions, and ψ be the closure of Γ in $\mathcal{X}_1 \times_B \mathcal{X}_2$. By Lemma 3.1(c),

$$\gamma^* \circ j_2^* = j_1^* \circ \psi^*, \tag{3-10}$$

and it suffices to show that $\psi^*\alpha \in \mathrm{CH}^{i+r}(\mathcal{X}_1)^0$ for any $\alpha \in \mathrm{CH}^i(\mathcal{X}_2)^0$. Formula (3-10) shows that $j_1^*(\psi^*\alpha) \in \mathrm{CH}^{i+r}_{\mathrm{num}}(X_1)$; the other condition follows from Lemma 3.5(b). □

Remark 3.7. If B is projective, Lemma 3.5(a) is sufficient for the proof of Proposition 3.6 by using (2-3), as in the proof of Proposition 2.5.

Proposition 3.8. *The pairing (2-9) does not depend on the choice of f (we drop f from its notation from now on). Moreover, in the situation of Proposition 3.6 with $r = 0$, we have the identity*

$$\langle \gamma^*\alpha, \beta \rangle = \langle \alpha, \gamma_*\beta \rangle \tag{3-11}$$

for $(\alpha, \beta) \in \mathrm{CH}^i(X_2)^0 \times \mathrm{CH}_{i-1}(X_1)^0$.

Proof. As in the proof of Proposition 3.6, the first claim follows from the second by taking $X_1 = X_2 = X$, $\gamma = \Delta_X$. For the second claim, we take γ and ψ as in the proof of Proposition 3.6. Then (3-11) follows from Lemma 3.3 applied to lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β in $\mathrm{CH}^i(\mathcal{X}_2)^0$ and $\mathrm{CH}^{d_1-i+1}(\mathcal{X}_1)^0$, respectively. □

3C. *Base change.*

Proposition 3.9. *Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\bar{g}} & B_2 \end{array} \tag{3-12}$$

where f_1, f_2 satisfy the hypotheses of Section 1, $\bar{g}$ is finite surjective and g proper; we assume that the diagram of generic fibres,

$$\begin{tikzcd} X_1 \arrow[r, "g'"] \arrow[d, "f_1'"'] & X_2 \arrow[d, "f_2'"] \\ \eta_1 \arrow[r, "\bar{g}'"] & \eta_2 \end{tikzcd}$$

is Cartesian (in particular, g is generically finite). Then, for all $i \geq 0$, one has:

(i) $g^* \operatorname{CH}^i(\mathcal{X}_2)^0 \subseteq \operatorname{CH}^i(\mathcal{X}_1)^0$, *hence* $g'^* \operatorname{CH}^i(X_2)^0 \subseteq \operatorname{CH}^i(X_1)^0$.

(ii) $g_* \operatorname{CH}^i(\mathcal{X}_1)^0 \subseteq \operatorname{CH}^i(\mathcal{X}_2)^0$, *hence* $g'_* \operatorname{CH}^i(X_1)^0 \subseteq \operatorname{CH}^i(X_2)^0$.

(iii) $(g^*)^{-1} \operatorname{CH}^i(\mathcal{X}_1)^0 = \operatorname{CH}^i(\mathcal{X}_2)^0$.

(iv) *One has the identities*

$$\bar{g}_*\langle g'^*\alpha, \beta'\rangle = \langle \alpha, g'_*\beta'\rangle, \tag{3-13}$$
$$\langle g'^*\alpha, g'^*\beta\rangle = \bar{g}^*\langle \alpha, \beta\rangle, \tag{3-14}$$

for any $i \geq 0$ and any $(\alpha, \beta, \beta') \in \operatorname{CH}^i(X_2)^0 \times \operatorname{CH}^{d+1-i}(X_2)^0 \times \operatorname{CH}^{d+1-i}(X_1)^0$.

Proof. (i) Write $j_i : X_i \hookrightarrow \mathcal{X}_i$ for the inclusions. Let $\alpha \in \operatorname{CH}^i(\mathcal{X}_2)^0$: then $j_1^* g^*\alpha = g'^* j_2^*\alpha \in \operatorname{CH}^i_{\text{num}}(X_1)$. Next, let $b \in B_1^{(1)}$ and $Z = \overline{\{b\}}$. Let $\beta \in \operatorname{CH}_{\delta+i-1}(\mathcal{X}_{1,Z})$, $f_{1,Z} : \mathcal{X}_{1,Z} \to Z$ be the restriction of f_1 and $\iota_1 : \mathcal{X}_{1,Z} \hookrightarrow \mathcal{X}_1$ be the closed immersion: we need to prove that $(f_{1,Z})_*(g^*\alpha \cdot_{\iota_1} \beta) = 0$. Let $T = \bar{g}(Z)$ and $\bar{h} : Z \to T$ be the (finite surjective) projection: it suffices to show that $\bar{h}_*(f_{1,Z})_*(g^*\alpha \cdot_{\iota_1} \beta) = 0 \in \operatorname{CH}^0(T)$. This follows from the computation

$$\begin{aligned} 0 &\overset{\text{(a)}}{=} (f_{2,T})_*(\alpha \cdot_{\iota_2} h_*\beta) = (f_{2,T})_*\gamma^!_{\iota_2}(h_*\beta \times \alpha) \\ &\overset{\text{(b)}}{=} (f_{2,T})_* h_* \gamma^!_{g\iota_1}(\beta \times \alpha) = \bar{h}_*(f_{1,Z})_*\gamma^!_{g\iota_1}(\beta \times \alpha) \\ &\overset{\text{(c)}}{=} \bar{h}_*(f_{1,Z})_*\gamma^!_{\iota_1}(1 \times g)^!(\beta \times \alpha) = \bar{h}_*(f_{1,Z})_*(g^*\alpha \cdot_{\iota_2} \beta), \end{aligned}$$

where $h : \mathcal{X}_{1,Z} \to \mathcal{X}_{2,T}$ is the restriction of g and ι_2 is the inclusion $\mathcal{X}_{2,T} \hookrightarrow \mathcal{X}_2$, in which (a) is by hypothesis, (b) follows from Lemma 2.1, and (c) follows from [Fulton 1984, Proposition 8.1.1(a)] (see comment in [op. cit., mid p. 134]).

(ii) The inclusion $j_2^* g_* \operatorname{CH}^i(\mathcal{X}_1)^0 \subseteq \operatorname{CH}^i_{\text{num}}(X_2)$ is obtained this time from the identity $j_2^* g_* = g'_* j_1^*$. Next, let $b \in B_2^{(1)}$ $Z = \overline{\{b\}}$ and $\iota_2 : \mathcal{X}_{2,Z} \hookrightarrow \mathcal{X}_2$, $f_{2,Z} : \mathcal{X}_{2,Z} \to Z$ be the inclusion and the projection. Let $\alpha \in \operatorname{CH}^i(\mathcal{X}_1)^0$ and $\beta \in \operatorname{CH}_{\delta+i-1}(\mathcal{X}_{2,Z})$: we need to prove that $(f_{2,Z})_*(g_*\alpha \cdot_{\iota_2} \beta) = 0 \in \operatorname{CH}^0(Z)$.

Let $T = \bar{g}^{-1}(Z)$. Then $\mathcal{X}_{1,T} \xrightarrow{\sim} \mathcal{X}_1 \times_{\mathcal{X}_2} \mathcal{X}_{2,Z}$; hence refined Gysin morphisms

$$g^! : \operatorname{CH}_j(\mathcal{X}_{2,Z}) \to \operatorname{CH}_j(\mathcal{X}_{1,T}).$$

By Lemma 2.9, we have $(f_{1,T})_*(\alpha \cdot_{\iota_1} g^!\beta) = 0$, where ι_1 is the inclusion $\mathcal{X}_{1,T} \hookrightarrow \mathcal{X}_1$. The commutative square

$$\begin{tikzcd} \mathcal{X}_{1,T} \arrow[r, "h"] \arrow[d, "f_{1,T}"'] & \mathcal{X}_{2,Z} \arrow[d, "f_{2,Z}"'] \\ T \arrow[r, "\bar{h}"] & Z \end{tikzcd}$$

where h and $\bar{h}$ are the restrictions of g and $\bar{g}$, gives the identity of push-forwards

$$\bar{h}_*(f_{1,T})_* = (f_{2,Z})_* h_*.$$

Therefore, it suffices to prove the identity (projection formula)

$$g_*\alpha \cdot_{\iota_2} \beta = h_*(\alpha \cdot_{\iota_1} g^!\beta). \tag{3-15}$$

For this, consider the commutative diagram of Cartesian squares

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "\delta_{\mathcal{X}_1}"] & \mathcal{X}_1 \times \mathcal{X}_1 \arrow[r, "g\times 1"] & \mathcal{X}_2 \times \mathcal{X}_1 \\ \mathcal{X}_{1,T} \arrow[u, "\iota_1"] \arrow[r, "\gamma_{\iota_1}"] \arrow[d, "h"'] & \mathcal{X}_{1,T} \times \mathcal{X}_1 \arrow[u, "\iota_1\times 1"] \arrow[r, "h\times 1"] & \mathcal{X}_{2,Z} \times \mathcal{X}_1 \arrow[u, "\iota_2\times 1"] \arrow[d, "1\times g"'] \\ \mathcal{X}_{2,Z} \arrow[rr, "\gamma_{\iota_2}"] \arrow[d, "\iota_2"'] & & \mathcal{X}_{2,Z} \times \mathcal{X}_2 \arrow[d, "\iota_2\times 1"'] \\ \mathcal{X}_2 \arrow[rr, "\delta_{\mathcal{X}_2}"] & & \mathcal{X}_2 \times \mathcal{X}_2 \end{tikzcd}$$

Applying Lemma 2.1 to the two bottom squares yields first

$$g_*\alpha \cdot_{\iota_2} \beta := \gamma_{\iota_2}^!(1\times g)_*(\beta\times\alpha) = h_*\gamma_{\iota_2}^!(\beta\times\alpha) = h_*\delta_{\mathcal{X}_2}^!(\beta\times\alpha).$$

We are now left to show the identity

$$\delta_{\mathcal{X}_2}^!(\beta\times\alpha) = \alpha \cdot_{\iota_1} g^!\beta := \gamma_{\iota_1}^!(g\times 1)^!(\beta\times\alpha),$$

where the right hand side stems from the top part of the diagram (with vertical arrows pointing upwards). But $\gamma_{\iota_1}^!(g\times 1)^!(\beta\times\alpha) = \delta_{\mathcal{X}_1}^!(g\times 1)^!(\beta\times\alpha)$ and $\delta_{\mathcal{X}_1}^!(g\times 1)^!(\beta\times\alpha) = [(g\times 1)\delta_{\mathcal{X}_1}]^!(\beta\times\alpha) = {}^t\gamma_g^!(\beta\times\alpha)$, both by Lemma 2.1. Here, ${}^t\gamma$ denotes the transpose of a graph (graph composed with the switch of factors). Finally, ${}^t\gamma_g^!(\beta\times\alpha) = \delta_{\mathcal{X}_2}^!(\beta\times\alpha)$ by applying once again Lemma 2.1(b) to the diagram of

Cartesian squares

$$\begin{tikzcd} \mathcal{X}_{1,T} \arrow[r, "(h,\iota_1)"] \arrow[d, "\iota_1"'] & \mathcal{X}_{2,Z} \times \mathcal{X}_1 \arrow[d, "\iota_2\times 1"'] \\ \mathcal{X}_1 \arrow[r, "{}^t\gamma_g"] \arrow[d, "g"'] & \mathcal{X}_2 \times \mathcal{X}_1 \arrow[d, "1\times g"'] \\ \mathcal{X}_2 \arrow[r, "\delta_{\mathcal{X}_2}"] & \mathcal{X}_2 \times \mathcal{X}_2 \end{tikzcd}$$

(iii) This follows from (i) and (ii) by the projection formula $g_* g^* = \deg(g)$ (generic degree), and Lemma 2.4.

(iv) This follows from the special case $B_1 = B_2$, $\bar{g} = 1_B$ in (i) or (ii).

In (iv), the identities can be checked on the level of $\mathcal{X}_1$ and $\mathcal{X}_2$. The first, (3-13), is an easy consequence of the projection formula. Let us prove (3-14). The diagram of Cartesian squares

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "(g,f_1)^B"] \arrow[d, "||"'] & \mathcal{X}_2 \times_{B_2} B_1 \arrow[r, "1\times_{B_2}\bar{g}"] \arrow[d, "\mathrm{inj}"'] & \mathcal{X}_2 \arrow[d, "(1,f_2)"'] \\ \mathcal{X}_1 \arrow[r, "(g,f_1)"] & \mathcal{X}_2 \times_k B_1 \arrow[r, "1\times\bar{g}"] & \mathcal{X}_2 \times_k B_2 \end{tikzcd}$$

together with Lemma 2.1 (b), and (c) gives a factorisation of g^* into a composition of refined Gysin morphisms

$$g^* = (g, f_1)^!(1 \times \bar{g})^!. \tag{3-16}$$

Next, [Fulton 1984, Example 8.1.7] applied to the left square with $x = [\mathcal{X}_1]$ and $y = (1 \times \bar{g})^! z$ for some $z \in \mathrm{CH}^*(\mathcal{X}_2)$ yields via [Fulton 1984, Proposition 8.1.2(b)] the identity

$$(g, f_1)^B_* (g, f_1)^! y = (g, f_1)^B_* [\mathcal{X}_1] \cdot y = y; \tag{3-17}$$

indeed, $(g, f_1)^B$ maps $\mathcal{X}_1$ birationally onto an irreducible component of $\mathcal{X}_2 \times_{B_2} B_1$, and the other irreducible components have support away from η_2, hence have smaller dimensions. Taking $z = \alpha \cdot \beta$ for $(\alpha, \beta) \in \mathrm{CH}^i(\mathcal{X}_2)^0 \times \mathrm{CH}^{d+1-i}(\mathcal{X}_2)^0$, we get

$$\begin{aligned} \langle g^*\alpha, g^*\beta\rangle = (f_1)_*(g^*\alpha \cdot g^*\beta) &= (f_1)_* g^*(\alpha \cdot \beta) \\ &\overset{(3\text{-}16)}{=} (f_2 \times 1)_*(g, f_1)^B_*(g, f_1)^!(1 \times \bar{g})^!(\alpha \cdot \beta) \\ &\overset{(3\text{-}17)}{=} (f_2 \times 1)_*(1 \times \bar{g})^!(\alpha \cdot \beta) \\ &= \bar{g}^*(f_2)_*(\alpha \cdot \beta) = \bar{g}^*\langle \alpha, \beta\rangle, \end{aligned}$$

where the last but one equality follows once again from Lemma 2.1. This readily implies (3-14). □

Remark 3.10. In Proposition 3.9, suppose that $\bar{g}$ is only an alteration. I cannot prove (i). On the other hand, (ii) holds with the same proof, as well as (iv) for $(\alpha, \beta) \in \mathrm{CH}^i(X_2)^0 \times \mathrm{CH}^{d+1-i}(X_2)^0$ such that

$(g^*\alpha, g^*\beta) \in \mathrm{CH}^i(X_1)^0 \times \mathrm{CH}^{d+1-i}(X_1)^0$. This is not very important, in view of Remarks 1.2(b) (see proof of Proposition 3.11).

3D. *Structure of* $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0$.

Proposition 3.11. *The groups* $\mathrm{CH}^i(\mathcal{X})/\mathrm{CH}^i(\mathcal{X})^0$ *and* $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0$ *are finitely generated.*

Proof. It suffices to show the first claim. We proceed in several steps.

(1) Suppose B' is an open subset of B, let $\mathcal{X}' = \mathcal{X} \times_B B'$ and let $\lambda : \mathcal{X}' \to \mathcal{X}$ be the corresponding open immersion. Then $\mathrm{CH}^i(\mathcal{X})^0 \subseteq (\lambda^*)^{-1} \mathrm{CH}^i(\mathcal{X}')^0$, with equality if $B - B'$ has codimension ≥ 2. Therefore the claim for $\mathcal{X}$ implies the claim for $\mathcal{X}'$, and conversely in the latter case.

(2) B is projective: this follows from Proposition 2.5.

(3) In general, let $\bar{B}$ be a compactification of B and $\bar{\mathcal{X}} \xrightarrow{\bar{f}} \bar{B}$ a projective morphism extending f (in the sense that $\mathcal{X} = \bar{\mathcal{X}} \times_{\bar{B}} B$).

- By [de Jong 1996, Theorem 4.1], alter $\bar{B}$ into a smooth projective k-variety $\bar{B}_1$.
- Let $K_1 = k(\bar{B}_1)$ (a finite extension of K), and let $\bar{\mathcal{X}}'$ be the closure of $X \otimes_K K_1$ in $\bar{\mathcal{X}} \times_{\bar{B}} \bar{B}_1$. Again by [de Jong 1996, Theorem 4.1], alter $\bar{\mathcal{X}}'$ into a smooth projective k-variety $\bar{\mathcal{X}}_1$. We are now in the situation of (2).
- Let $B_1 = B \times_{\bar{B}} \bar{B}_1$ and $\mathcal{X}_1 = B_1 \times_{\bar{B}_1} \bar{\mathcal{X}}_1$.
- By Remarks 1.2(b), the alteration $B_1 \to B$ becomes flat, hence finite, after removing from B a closed subset F of codimension ≥ 2. Let $B' = B - F$ and $\mathcal{X}'$, B_1', $\mathcal{X}_1'$ be the corresponding base changes of $\mathcal{X}$, B_1 and $\mathcal{X}_1$.

By (2), the claim is true for $\bar{\mathcal{X}}_1$; therefore it is also true for $\mathcal{X}_1'$ by (1). By Proposition 3.9 (i), (ii), the projection $\mathcal{X}_1' \to \mathcal{X}'$ induces maps between $\mathrm{CH}^i(\mathcal{X}')/\mathrm{CH}^i(\mathcal{X}')^0$ and $\mathrm{CH}^i(\mathcal{X}_1')/\mathrm{CH}^i(\mathcal{X}_1')^0$, whose composition is multiplication by $[K_1 : K]$. Since $\mathrm{CH}^i(\mathcal{X}')/\mathrm{CH}^i(\mathcal{X}')^0$ is torsion-free by Lemma 2.4, it is finitely generated, and so is $\mathrm{CH}^i(\mathcal{X})/\mathrm{CH}^i(\mathcal{X})^0$ by reapplying (1). □

Remark 3.12. Proposition 2.6 gives a more direct proof of Proposition 3.11 in characteristic 0, by the comparison theorem between Betti and l-adic cohomology.

3E. ***A vanishing result.*** Let l be a prime number invertible in k. For any smooth k-variety V, there are cycle class maps with values in Jannsen's continuous étale cohomology

$$\mathrm{cl}^i : \mathrm{CH}^i(V) \to H^{2i}_{\mathrm{cont}}(V, \mathbb{Z}_l(i))$$

which are compatible with pull-backs, push-forwards and products [Jannsen 1988, (3.25) and (6.14)].[3]

[3] Strangely, [Jannsen 1988, (3.25)] only mentions push-forwards for closed immersions, but the case of a general proper morphism is proven in the same way.

Lemma 3.13. *Suppose k finitely generated. Then the composition of* cl^1 *with the projection*

$$H^2_{\mathrm{cont}}(V,\mathbb{Z}_l(1)) \to H^2_{\mathrm{cont}}(V,\mathbb{Z}_l(1))/H^2_{\mathrm{cont}}(k,\mathbb{Z}_l(1))$$

has finite kernel.

Proof. By construction of cl^i, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^1(V) \arrow[r, "\alpha"] \arrow[d, "\mathrm{cl}^1"'] & \mathrm{CH}^1(V)^\wedge \arrow[d, "(\mathrm{cl}^1)^\wedge"'] \\
H^2_{\mathrm{cont}}(V,\mathbb{Z}_l(1)) \arrow[r] & \varprojlim H^2(V,\mu_l^n)
\end{tikzcd}$$

where the bottom map is part of the Milnor exact sequence of [Jannsen 1988, (3.16)] and $\mathrm{CH}^1(V)^\wedge$ is the l-adic completion of $\mathrm{CH}^1(V)$. The Kummer exact sequences imply the injectivity of $(\mathrm{cl}^1)^\wedge$. Since k is finitely generated, $\mathrm{CH}^1(V)$ is a finitely generated abelian group, which implies that α has finite kernel of order prime to l. Hence the same holds for cl^1.

On the other hand, the choice of a 0-cycle of nonzero degree on V (e.g., a closed point), plus transfer, provide a map $\rho : H^2_{\mathrm{cont}}(V,\mathbb{Z}_l(1)) \to H^2_{\mathrm{cont}}(k,\mathbb{Z}_l(1))$ such that the composition

$$H^2_{\mathrm{cont}}(k,\mathbb{Z}_l(1)) \to H^2_{\mathrm{cont}}(V,\mathbb{Z}_l(1)) \xrightarrow{\rho} H^2_{\mathrm{cont}}(k,\mathbb{Z}_l(1))$$

is multiplication by some integer $m > 0$. Since $\mathrm{CH}^1(k) = 0$, the naturality of the cycle class map implies that $\rho \circ \mathrm{cl}^1 = 0$. Hence the lemma. □

The following proposition will be used in the proof of Proposition 6.8.

Proposition 3.14. *Let $(\alpha,\beta) \in \mathrm{CH}^i(\mathcal{X}) \times \mathrm{CH}^{d+1-i}(\mathcal{X})$. Consider the pairing (1-2). As in Section 2C, let $\mathrm{CH}^i_l(\mathcal{X})$ be the kernel of the* **geometric** *cycle class map. If $(\alpha,\beta) \in \mathrm{CH}^i_l(\mathcal{X}) \times \mathrm{CH}^{d+1-i}_l(\mathcal{X})$, then $\langle\alpha,\beta\rangle$ is torsion.*

Proof. We may assume k to be the perfect closure of a finitely generated field. We use the spectral sequences of [Jannsen 1988, Theorem (3.3)]

$$E_2^{p,q} = H^p_{\mathrm{cont}}\big(k, H^q(V_{\bar k},\mathbb{Z}_l(n))\big) \Rightarrow H^{p+q}_{\mathrm{cont}}(V,\mathbb{Z}_l(n)).$$

They are compatible with the action of correspondences, in particular with products and push-forwards. Thus, if $F^\bullet H_{\mathrm{cont}}$ is the filtration on H_{cont} induced by the spectral sequence, we have

$$\begin{aligned}
\mathrm{cl}^1(f_*(\alpha\cdot\beta)) &= f_*\mathrm{cl}^{d+1}(\alpha\cdot\beta) \\
&= f_*\big(\mathrm{cl}^i(\alpha)\cup \mathrm{cl}^{d+1-i}(\beta)\big) \in F^2H^2_{\mathrm{cont}}(B,\mathbb{Z}_l(1)) = \mathrm{Im}\big(H^2_{\mathrm{cont}}(k,\mathbb{Z}_l(1)) \to H^2_{\mathrm{cont}}(B,\mathbb{Z}_l(1))\big)
\end{aligned}$$

if $(\alpha,\beta) \in \mathrm{CH}^i_l(\mathcal{X}) \times \mathrm{CH}^{d+1-i}_l(\mathcal{X})$. We conclude by Lemma 3.13. □

Question 3.15. When B is projective, can one prove Proposition 3.14 with CH_l replaced by $\mathrm{CH}_{\mathrm{num}}$, without assuming the standard conjectures?

3F. ***Local height pairing.*** In this context, there is not much to say. Let f be as in Section 1. Let $C_1 \in \mathcal{Z}^i(X)$, $C_2 \in \mathcal{Z}^{d+1-i}(X)$ be two integral cycles with disjoint supports. Let $\mathcal{C}_i$ be the closure of C_i in $\mathcal{X}$; then $\mathcal{C}_1 \times_{\mathcal{X}} \mathcal{C}_2$ has support in $\mathcal{X}_Z$ for some proper closed subset Z of B, whence a refined intersection product [Fulton 1984, §8.1],

$$\mathcal{C}_1 \cdot \mathcal{C}_2 \in \mathrm{CH}_{\delta-1}(\mathcal{X}_Z).$$

Given the isomorphism

$$\mathrm{CH}_{\delta-1}(Z) \xrightarrow{\sim} \bigoplus_{b \in Z \cap B^{(1)}} \mathbb{Z},$$

the class $(f_Z)_*(\mathcal{C}_1 \cdot \mathcal{C}_2)$ defines a divisor on B, whose class in $\mathrm{Pic}(B) = \mathrm{CH}^1(B)$ is obviously $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ (cf. [Beĭlinson 1987, Lemma 2.0.1]). One may then extend by bilinearity and get an expression of $\langle\ ,\ \rangle$ as the class of a divisor.

We leave it to the interested reader to refine Lemma 3.3 to this local height pairing in the style of [Bloch 1984, (A.2)].

4. Extension to the general case

Let X be regular, connected and proper over K. In the previous section, we defined subgroups $\mathrm{CH}^i(X)^0 \subset \mathrm{CH}^i(X)$ and pairings (1) assuming the existence of a k-smooth model $\mathcal{X}$ of X, proper over B.

4A. ***Characteristic* 0.**

Proposition 4.1. *Assuming resolution of singularities à la Hironaka, a smooth model always exists. This is the case in particular if* $\operatorname{char} k = 0$, *or if* $d + \delta \leq 3$ [Cossart and Piltant 2009].

Proof. Start from an integral proper model $f : \mathcal{X} \to B$ of X/K. Let $U \subseteq \mathcal{X}$ be the regular locus of $\mathcal{X}/k$: it is open [EGA IV$_2$ 1965, corollaire 6.12.6] and since X is regular, we have $X \subset U$. By hypothesis, we may find $\mathcal{X}_1$ regular over k and a projective morphism $\pi : \mathcal{X}_1 \to \mathcal{X}$ such that $\pi_{|\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism. Then the immersion $X \hookrightarrow \mathcal{X}$ lifts to $X \hookrightarrow \mathcal{X}_1$, and $\mathcal{X}_1$ is the desired smooth model of X (since k is assumed to be perfect). □

4B. ***Positive characteristic.*** Here we cannot directly use de Jong's theorem [1996] to replace Hironaka resolution, because there is no control in this theorem on the centre of the alteration. Instead we must proceed more carefully.

Definition 4.2. Let X be an integral proper K-scheme.

(a) X is *good* (relatively to B) if it admits a k-regular proper model $\mathcal{X} \xrightarrow{f} B$. (In particular, X is then regular.)

(b) A K-morphism $\pi : X_1 \to X$ is *admissible* if X_1 is good.

(c) We set

$$\mathrm{CH}^i(X)^0 = \{\alpha \in \mathrm{CH}^i(X) \mid \pi^*\alpha \in \mathrm{CH}^i(X_1)^0\ \forall \pi \text{ admissible}\}.$$

Lemma 4.3. (a) *For any X as in Definition 4.2, admissible alterations exist; in particular* $\mathrm{CH}^i(X)^0 \neq \varnothing$.

(b) *If X is good,* $\mathrm{CH}^i(X)^0$ *agrees with the subgroup of Proposition 3.6*(ii).

(c) *Given two admissible morphisms $\pi_i : X_i \to X$, there exists an admissible morphism $\pi_3 : X_3 \to X$ factoring through π_1 and π_2.*

(d) *If X is regular, we have* $\mathrm{CH}^i(X)^0 \subseteq \mathrm{CH}^i_{\mathrm{num}}(X)$.

Proof. (a) This follows from [de Jong 1996, Theorem 4.1] applied to a (not necessarily smooth) model.

(b) This follows from Proposition 3.6.

(c) Let $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2$ be B-proper models of X, X_1 and X_2. Taking the graphs of the rational maps $\pi_i : \mathcal{X}_i \dashrightarrow \mathcal{X}$, we may assume these to be B-morphisms. Applying [de Jong 1996, Theorem 4.1] again to an irreducible component of $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ dominant over B, we get a k-smooth $\mathcal{X}$-scheme $\mathcal{X}_3$, projective over B and mapping to $\mathcal{X}_1$ and $\mathcal{X}_2$, whose generic fibre X_3 maps to X_1 and X_2 over X.

(d) Let $\alpha \in \mathrm{CH}^i(X)^0$ and $\beta \in \mathrm{CH}^{d-i}(X)$. Choose an admissible π. Writing $[\ ,\ ]$ for the intersection product, we have $[\pi_\eta^*\alpha, \pi_\eta^*\beta] = 0$ by definition of $\mathrm{CH}^i(X_1)^0$, hence $[\alpha, \beta] = 0$. □

To go further, we need to invert p in characteristic p; this is the object of the next subsections.

4C. *The category* Ab ⊗ *R, where R is a subring of* $\mathbb{Q}$. (See also [Barbieri-Viale and Kahn 2016, Appendix B].) This category has two equivalent descriptions:

- It is the localisation of the category **Ab** of abelian groups with respect to the Serre subcategory of abelian groups killed by some integer invertible in A; in particular, $\mathbf{Ab} \otimes R$ is abelian and the localisation functor $\mathbf{Ab} \to \mathbf{Ab} \otimes R$ is exact.
- It has the same objects as **Ab**, while morphisms are those of **Ab** tensored with A.

If $R = \mathbb{Z}[1/p]$, we shall abbreviate $\mathbf{Ab} \otimes R$ to $\mathbf{Ab}[1/p]$.

Lemma 4.4. *The tensor product of* **Ab** *induces a tensor structure on* $\mathbf{Ab} \otimes R$, *still denoted by* $\otimes$.

(This allows us to talk of a "pairing in $\mathbf{Ab} \otimes R$".)

Proof. It suffices to show that, if $f \in \mathbf{Ab}(A, B)$ becomes invertible in $\mathbf{Ab} \otimes R$ (i.e., $\operatorname{Ker} f$, $\operatorname{Coker} f$ have p-power exponent), the same holds for $f \otimes 1_C$ for any $C \in \mathbf{Ab}$. By considering the image of f, we may treat separately the cases where f is injective and f is surjective. Both hold because, if $G \in \mathbf{Ab}$ has p-power exponent, so do $G \otimes C$ and $\operatorname{Tor}(G, C)$ for any $C \in \mathbf{Ab}$. □

Remarks 4.5. (a) Let A, B be two abelian groups. By definition, a morphism in $(\mathbf{Ab} \otimes R)(A, B) = \varinjlim_{N \neq 0} \mathbf{Ab}(A, B)$ is represented by a pair (φ, N) with $\varphi : A \to B$ and N an integer invertible in R; two pairs (φ_1, N_1) and (φ_2, N_2) are equivalent if there exist two such integers d_1, d_2 such that $d_1 N_1 = d_2 N_2 =: N_3$ and $(d_1\varphi_1, N_3) = (d_2\varphi_2, N_3)$. We get a well-defined homomorphism

$$\rho : (\mathbf{Ab} \otimes R)(A, B) \to \mathbf{Ab}(A, B \otimes R)$$

by sending a pair (φ, N) to $\psi; = N^{-1}\varphi$; its image is contained in the subgroup formed of those homomorphisms $\psi : A \to B \otimes R$ such that $\psi(A) \subseteq N^{-1}\bar{B}$ for some $N \neq 0$, with $\bar{B} = B/\text{torsion}$. If B is torsion-free, ρ is injective with the above image.

(b) In any category, the commutativity of a diagram (i.e., the equality of two arrows) is equivalent to the commutativity of a family of diagrams of sets, thanks to Yoneda's lemma. In the category of modules over a ring R, one can test such commutativity on elements, because the R-module R is a generator.

In the sequel, we shall extend identities such as (3-11), (3-13) and (3-14) to $\mathbf{Ab} \otimes R$. However this category is not Grothendieck (note that abelian groups with finite exponent are not closed under infinite direct sums), so reasoning with "elements" is abusive. Writing out the above identities as commutative diagrams in $\mathbf{Ab}$ is straightforward, but cumbersome. (For example, (3-11) means that two homomorphisms from $\mathrm{CH}^{d_2}(X_1 \times_K X_2) \otimes \mathrm{CH}^i(X_2)^0 \otimes \mathrm{CH}_{i-1}(X_1)^0$ to $\mathrm{CH}^1(B)$ agree.) We shall therefore sometimes make the abuse of talking of such identities in $\mathbf{Ab} \otimes R$ when we mean the corresponding commutative diagrams.

In Theorem 4.14, we shall use a local-to-global result for these localisations (Corollary 4.8 below).

Theorem 4.6. *Let H be a module over an integral domain R with quotient field Q. Suppose given, for each maximal ideal $\mathfrak{m} \subset R$, an element $f_\mathfrak{m} \in H_\mathfrak{m}$, all of which become equal in $Q \otimes_R H$. Then there exists at most one element $f \in H$ which becomes equal to $f_\mathfrak{m}$ in $H_\mathfrak{m}$ for every $\mathfrak{m}$; f exists provided*

(i) *H is torsion free, or*

(ii) *R is Noetherian and $S = \mathrm{Supp}(M_{\text{tors}})$ is a finite set of maximal ideals.*

(Counterexample without Hypothesis (ii): $R = \mathbb{Z}$, $H = \bigoplus_\mathfrak{m} \mathbb{Z}/\mathfrak{m}$, $f_\mathfrak{m} = 1_\mathfrak{m}$.)

Proof. Uniqueness. Let f, f' verifying the condition. Then f and f' become equal in $H_\mathfrak{m}$ for all $\mathfrak{m}$. This means that, for every $\mathfrak{m}$, there exists $M_\mathfrak{m} \in R - \mathfrak{m}$ such that $M_\mathfrak{m}(f - f') = 0$. Since the $M_\mathfrak{m}$ generate R as an ideal, we get $f = f'$.

Existence. We may write $f_\mathfrak{m} = r_\mathfrak{m}^{-1}\tilde{f}_\mathfrak{m}$ with $\tilde{f}_\mathfrak{m} \in H$ and $r_\mathfrak{m} \in R - \mathfrak{m}$; again, the $r_\mathfrak{m}$ generate the unit ideal of R. In case (i), if $g \in Q \otimes_R H$ is the common value of the $f_\mathfrak{m}$, then $r_\mathfrak{m} g \in H$ for all $\mathfrak{m}$; if $(a_\mathfrak{m})$ is a family of elements of R with finite support such that $\sum a_\mathfrak{m} r_\mathfrak{m} = 1$, then $g = \sum a_\mathfrak{m} r_\mathfrak{m} g \in H$.

In case (ii), write $T = H_{\text{tors}}$ for notational simplicity. Considering H/T, we find f_0 such that $1_\mathfrak{m} \otimes f_0 - f_\mathfrak{m}$ is torsion for all $\mathfrak{m}$, hence is 0 for $\mathfrak{m} \notin S$.

Claim 4.7. *The monomorphism $T \mapsto \prod_{\mathfrak{m}\in S} T_\mathfrak{m}$ is surjective.*

Proof. For each $\mathfrak{m} \in S$, let $T^\mathfrak{m} = \mathrm{Ker}(T \to \prod_{\mathfrak{m}'\neq\mathfrak{m}} T_{\mathfrak{m}'})$: we must show that $T = \sum T^\mathfrak{m}$. Let $t \in T$; by assumption, the radical of $\mathrm{Ann}(t)$ (the annihilator of t) is of the form $\prod_{\mathfrak{m}\in S'} \mathfrak{m}$ for a subset S' of S. By [Bourbaki 1985, IV.2.5, proposition 9], $R(t) = R/\mathrm{Ann}(t)$ is Artinian, hence $R(t) \xrightarrow{\sim} \prod_{\mathfrak{m}\in S'} R(t)_\mathfrak{m}$ [loc. cit., corollaire 1]; equivalently, $Rt \xrightarrow{\sim} \prod_{\mathfrak{m}\in S'} (Rt)_\mathfrak{m}$, which shows that $t \in \sum T^\mathfrak{m}$. □

Coming back to the proof of case (ii), the claim yields an element $t \in T$ such that $t_\mathfrak{m} = 1_\mathfrak{m} \otimes f_0 - f_\mathfrak{m}$ for all $\mathfrak{m} \in S$; then $f = f_0 - t$ yields the desired element. □

Corollary 4.8. *Let $A, B \in \mathbf{Ab}$ and R be a subring of $\mathbb{Q}$. Suppose given, for each prime number l not invertible in R, a morphism $f_l : A \to B$ in $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$, all of which become equal in $\mathbf{Ab} \otimes \mathbb{Q}$. Then there exists at most one morphism $f : A \to B$ in $\mathbf{Ab} \otimes R$ which becomes equal to f_l in $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$ for every l; f exists provided B is l-torsion free for almost all l not invertible in R.*

Proof. Apply Theorem 4.6 to $H = \mathrm{Hom}(A, B) \otimes R$, noting that the hypothesis on B implies the hypothesis on H. □

4D. *p-covers.*

Definition 4.9. Let X be an integral proper K-scheme. A *p-cover* of X is a finite family $(\pi_l : X_l \to X)$, indexed by prime numbers $l \neq p$ and such that

(i) for each l, π_l is an admissible alteration of generic degree d_l prime to l;

(ii) $\gcd_l(d_l)$ is a power of p.

Proposition 4.10. (a) *p-covers exist.*

(b) *Given two p-covers (π_l), (π_l'), there exists a third p-cover (π_l'') such that, for each l, π_l'' factors through π_l and π_l'.*

(c) *Given a p-cover (π_l) and an admissible morphism $f_1 : X_1 \to X$, there exists a p-cover $(\pi_{1,l})$ of X_1 such that the composition $X_{1,l} \to X_1 \to X$ factors through X_l for each l.*

Proof. (a) We use Gabber's refinement of de Jong's alteration theorem [Illusie and Temkin 2014, Theorem 2.1]: given a model $\mathcal{X}$ of X and a prime number $l \neq p$, we may find an alteration $\mathcal{X}_l \to \mathcal{X}$ with $\mathcal{X}_l$ regular (hence smooth over k) and of generic degree d_l prime to l; the induced alteration $\pi_l : X_l \to X$ is then admissible of generic degree d_l. Considering the other prime divisors of d_l different from p, we may find a finite number of l and π_l such that the gcd of the d_l is a power of p.

(b) and (c) These are proven similarly to (a). □

4E. *The refined height pairing (characteristic p).*

Definition 4.11. We set

$$\mathrm{CH}^i(X)^{[0]} = \{\alpha \in \mathrm{CH}^i(X) \mid \exists s \geq 0 : p^s\alpha \in \mathrm{CH}^i(X)^0\}.$$

Proposition 4.12. (a) *If X is regular, $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0$ is an extension of a finitely generated abelian group by a torsion group of p-power exponent, and $\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^{[0]}$ is finitely generated with prime-to-p torsion.*

(b) *Let (π_l) be a p-cover of X, and let $\alpha \in \mathrm{CH}^i(X)$. Then $\alpha \in \mathrm{CH}^i(X)^{[0]}$ if and only if $\pi_l^*\alpha \in \mathrm{CH}^i(X_l)^{[0]}$ for each l.*

(c) *Propositions 3.6 and 3.9* (i), (ii), (iii) *extend to all regular X after replacing $\mathrm{CH}^i(X)^0$ by $\mathrm{CH}^i(X)^{[0]}$.*

Proof. (a) Given a p-cover (π_l), since $(\pi_l)_*\pi_l^*$ is multiplication by d_l for each l, $\operatorname{Ker}(\mathrm{CH}^i(X)/\mathrm{CH}^i(X)^0 \to \prod_l \mathrm{CH}^i(X_l)/\mathrm{CH}^i(X_l)^0)$ is killed by a power of p, say p^s, and the first claim follows from Proposition 3.11. The second follows by definition of $\mathrm{CH}^i(X)^{[0]}$.

(b) The condition is necessary by definition; the converse follows from Proposition 4.10 (c), as in (a).

(c) Let X_1, X_2, γ be as in Proposition 3.6. To prove (3-9), we must show that $\pi^*\gamma^*\mathrm{CH}^i(X_2)^0 \subseteq \mathrm{CH}^{i+r}(X_1')^0$ for any admissible $\pi : X_1' \to X_1$; replacing γ by $\gamma \circ \pi$, we may assume that X_1 is good and $\pi = 1_{X_1}$. Choose a p-cover (π_l) of X_2. For $\alpha \in \mathrm{CH}^i(X_2)^{[0]}$, we have $\pi_l^*\alpha \in \mathrm{CH}^i(X_l)^{[0]}$, hence

$$d_l\gamma^*\alpha = (\gamma^*(\pi_l)_*)\pi_l^*\alpha \in \mathrm{CH}^{i+r}(X_1)^{[0]}$$

for all l thanks to Proposition 3.6, hence $p^s\gamma^*\alpha \in \mathrm{CH}^{i+r}(X_1)^{[0]}$ and finally $\gamma^*\alpha \in \mathrm{CH}^{i+r}(X_1)^{[0]}$ as desired. The cases in Proposition 3.9 are treated similarly. □

Lemma 4.13. *Let $\pi : X_1 \to X$ be an admissible alteration, of generic degree d prime to l, where $l \neq p$. Then the morphism in $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$*

$$\langle\ ,\ \rangle_{(l)} : \mathrm{CH}^i(X)^{[0]} \otimes \mathrm{CH}^{d+1-i}(X)^{[0]} \xrightarrow{(\pi^*\otimes\pi^*)} \mathrm{CH}^i(X_1)^{[0]} \otimes \mathrm{CH}^{d+1-i}(X_1)^{[0]} \xrightarrow{d^{-1}\langle\ ,\ \rangle} \mathrm{CH}^1(B)$$

does not depend on the choice of π, and coincides with $\langle\ ,\ \rangle$ if X is good. For two prime numbers $l, l' \neq p$, we have $\langle\ ,\ \rangle_{(l)} = \langle\ ,\ \rangle_{(l')}$ in $\mathbf{Ab} \otimes \otimes \mathbb{Q}$.

Proof. Let $\pi' : X_1' \to X$ another such alteration, with generic degree d'. By Proposition 4.10(c) applied to an irreducible component of $X_1 \times X X_1'$ dominating X, we can find admissible alterations $X_1'' \xrightarrow{\rho} X_1$, $X_1'' \xrightarrow{\rho'} X_1'$ of generic degrees δ, δ' such that $\pi\rho = \pi'\rho'$, hence $\delta d = \delta' d'$. Using elements to clarify the argument, we have for $(\alpha, \beta) \in \mathrm{CH}^i(X)^{[0]} \times \mathrm{CH}^{d+1-i}(X)^{[0]}$,

$$d^{-1}\langle\pi^*\alpha, \pi^*\beta\rangle = d^{-1}\delta^{-1}\langle\rho^*\pi^*\alpha, \rho^*\pi^*\beta\rangle = d'^{-1}\delta'^{-1}\langle\rho'^*\pi'^*\alpha, \rho'^*\pi'^*\beta\rangle = d'^{-1}\langle\pi'^*\alpha, \pi'^*\beta\rangle,$$

where we used (3-11) and the identities $\rho_*\rho^* = \delta$, $\rho'_*\rho'^* = \delta'$. The second claim follows by taking $\pi = 1_X$. For the third claim, we argue similarly by using an admissible alteration covering two admissible alterations of generic degrees prime to l and l'. □

Theorem 4.14. (a) *There exists a unique pairing*

$$\langle\ ,\ \rangle : \mathrm{CH}^i(X)^{[0]} \otimes \mathrm{CH}^{d+1-i}(X)^{[0]} \to \mathrm{CH}^1(B) \tag{4-1}$$

in $\mathbf{Ab}[1/p]$ which coincides with $\langle\ ,\ \rangle_{(l)}$ in $\mathbf{Ab} \otimes \mathbb{Z}_{(l)}$ for each l.

(b) *The identities of Propositions 3.8 (see Remarks 4.5*(b)*) and 3.9*(iv) *extend to these pairings.*

Proof. (a) Suppose first that k is the perfect closure of a field k_0 finitely generated over $\mathbb{F}_p$, and that $B = B_0 \otimes_{k_0} k$ for some smooth k_0-variety B_0. Then $\mathrm{CH}^1(B_0)$ is a finitely generated abelian group [Kahn 2006], and $\mathrm{CH}^1(B_0) \otimes \mathbb{Z}[1/p]$ does not change under purely inseparable extensions; in particular, $\mathrm{CH}^1(B) \otimes \mathbb{Z}[1/p]$ has finite torsion and a fortiori verifies the hypothesis of Corollary 4.8. The result then follows from this theorem and Lemma 4.13.

In general, the situation is defined over such a subfield of k, so reduces to the first case.

(b) Let X_1, X_2 be (proper) regular, and let $\gamma \in \mathrm{CH}^{\dim X_2}(X_1 \times_K X_2)$. We need to prove the analogue of (3-11),

$$\langle\ ,\ \rangle_1 \circ \gamma^* \otimes 1 = \langle\ ,\ \rangle_2 \circ 1 \otimes \gamma_*,$$

where $\langle\ ,\ \rangle_i$ is the height pairing of X_i. By the uniqueness statement of Corollary 4.8, it suffices to prove this identity after localising at l for all $l \neq p$. Let $\pi_i : X_{i,l} \to X_i$ $(i = 1, 2)$ be two admissible alterations of generic degrees d_i prime to l, and let $\gamma_l = \pi_2^* \circ \gamma \circ (\pi_1)_* \in \mathrm{CH}^{\dim X_2}(X_{1,l} \times_K X_{2,l})$, so that $d_2\gamma \circ (\pi_1)_* = (\pi_2)_*\gamma_l$ and $\gamma_l \circ \pi_1^* = d_1\pi_2^* \circ \gamma$. By Lemma 4.13, we have, with obvious notation,

$$\begin{aligned}\langle\ ,\ \rangle_1 \circ \gamma^* \otimes 1 &= d_1^{-1}\langle\ ,\ \rangle_{1,l} \circ \pi_1^*\gamma^* \otimes \pi_1^* = d_1^{-1}d_2^{-1}\langle\ ,\ \rangle_{1,l} \circ \gamma_l^*\pi_2^* \otimes \pi_1^* \\ &\overset{\text{(a)}}{=} d_1^{-1}d_2^{-1}\langle\ ,\ \rangle_{2,l} \circ \pi_2^* \otimes (\gamma_l)_*\pi_1^* = d_2^{-1}\langle\ ,\ \rangle_{2,l} \circ \pi_2^* \otimes \pi_2^*\gamma_* \\ &= \langle\ ,\ \rangle_2 \circ 1 \otimes \gamma_*,\end{aligned}$$

where (a) used (3-11) for γ_l.

The identity of Proposition 3.9(iv) is extended in similar fashion. □

We shall use the following fact in the proof of Theorem 6.2:

Example 4.15. Suppose that X is an abelian variety. For $a \in X(K)$, write τ_a for the translation by a. It yields a self-correspondence of degree 0 still denoted by τ_a, and we have the obvious formula ${}^t\tau_a = \tau_{-a}$. This yields the identity (see Remarks 4.5(b))

$$\langle \tau_a^*\alpha, \beta\rangle = \langle \alpha, \tau_{-a}^*\beta\rangle$$

for $(\alpha, \beta) \in \mathrm{CH}^i(X)^{[0]} \times \mathrm{CH}^{d+1-i}(X)^{[0]}$.

Remark 4.16. The functoriality of Proposition 4.12(c) means that the subgroups $\mathrm{CH}^i(X)^{[0]}$, for varying X and i, define an *adequate equivalence relation* on algebraic cycles with integral coefficients on smooth projective K-varieties. This adequate relation a priori depends on the choice of B, but see Conjecture 5.1 and Remarks 5.4 below.

4F. ***Extension to imperfect fields.*** Let X, K, B be as in the introduction, but relax the assumption that k is perfect; specifically, we assume k imperfect of characteristic p. Write k^p (resp. K^p, B^p, X^p for the perfect closure of k (resp. for $K \otimes_k k^p$, $B \otimes_k k^p$, $X \otimes_K K^p$).

We define $\mathrm{CH}^i(X)^{[0]}$ as the inverse image of $\mathrm{CH}^i(X^p)^{[0]}$ under the pull-back morphism $\mathrm{CH}^i(X) \to \mathrm{CH}^i(X^p)$. We claim that the pairing (4-1) for X^p induces a similar pairing for X, with the same properties.

Since the homomorphism $\lambda : \mathrm{CH}^1(B) \to \mathrm{CH}^i(B^p)$ has p-primary torsion kernel and cokernel, this is trivial if we accept to replace $\mathrm{CH}^1(B)$ by $\mathrm{CH}^1(B) \otimes \mathbb{Z}[1/p]$ (note that $\operatorname{Ker}\lambda$ and $\operatorname{Coker}\lambda$ do not have finite exponent, so λ is not an isomorphism in $\mathbf{Ab}[1/p]$). We can avoid this, however, by observing that all constructions involved in constructing (4-1) for X^p and proving its properties are defined over some finite subextension of k^p/k.

5. Homologically and algebraically trivial cycles

From now on, we write

$$\mathrm{CH}^i(X)^{(0)} = \{\alpha \in \mathrm{CH}^i(X) \mid \exists n \neq 0 : n\alpha \in \mathrm{CH}^i(X)^0\}$$

for the saturation of $\mathrm{CH}^i(X)^0$. We have the inclusion

$$\mathrm{CH}^i(X)^{(0)} \subseteq \mathrm{CH}^i_{\mathrm{num}}(X) \tag{5-1}$$

by Lemma 4.3(d) and the fact that $\mathrm{CH}^i(X)/\mathrm{CH}^i_{\mathrm{num}}(X)$ is torsion-free.

5A. *Conjectures.* The following is a numerical analogue to [Beĭlinson 1987, Conjecture 2.2.5].

Conjecture 5.1. *The inclusion (5-1) is an equality.*

Let the index l denote homological equivalence for l-adic cohomology, $l \neq \operatorname{char} k$. Conjecture 5.1 implies

Conjecture 5.2. *One has the inclusion* $\mathrm{CH}^i_l(X) \subseteq \mathrm{CH}^i(X)^{(0)}$.

Conversely, Conjecture 5.2 implies Conjecture 5.1 under Grothendieck's standard Conjecture D, by Proposition 3.11 (and Proposition 4.12(a) in characteristic p).

Proposition 5.3. *Conjecture 5.2 is true if X admits a model $f : \mathcal{X} \to B$ with f smooth.*

Proof. This follows from the smooth and proper base change theorem (see Remark 2.7). □

Remarks 5.4. (a) More generally, Proposition 2.6 shows that $\mathrm{CH}^i(X)^0$ contains the image of $\mathrm{CH}^i(\mathcal{X})^0_{\mathrm{B},l}$ for any model $f : \mathcal{X} \to B$ of X with $\mathcal{X}$ smooth.

(b) Suppose X smooth (not just regular). For clarity, let us write $\mathrm{CH}^i(X)^{(0)}_B$ to mark the dependence of $\mathrm{CH}^i(X)^{(0)}$ on the model B. If U is an open subset of B, we obviously have $\mathrm{CH}^i(X)^{(0)}_B \subseteq \mathrm{CH}^i(X)^{(0)}_U$, and this direct system is *essentially constant* by Proposition 4.12(b). For U small enough, Proposition 5.3 thus gives inclusions

$$\mathrm{CH}^i_l(X) \subseteq \mathrm{CH}^i(X)^{(0)}_U \subseteq \mathrm{CH}^i_{\mathrm{num}}(X),$$

where the middle group does not change when U gets smaller (note that equality on the right is *not* clear: see Remark 2.7). In view of Remark 4.16, this defines a new adequate equivalence on smooth projective K-varieties, this time independent of the choice of B (and which conjecturally agrees with numerical equivalence).

Theorem 5.5. *Conjecture 5.1 is true in the following cases*:

(i) $i = 1, d$.

(ii) $\operatorname{char} K = 0$, *$f$ is smooth and*

- *either* $i \in \{2, d-1\}$,

- *or X is "of abelian type" (i.e., its homological motive is isomorphic to a direct summand of the motive of an abelian variety).*

Proof. For (i), see Theorem 5.6(b) below. For (ii), homological and numerical equivalences agree in the said cases by Lieberman [1968]. Therefore, the statement follows from Proposition 5.3. □

5B. *Algebraic equivalence.*

Theorem 5.6. *(a) One has $\mathrm{CH}^i_{\mathrm{alg}}(X) \subseteq \mathrm{CH}^i(X)^{(0)}$.*

(b) Conjecture 5.1 is true for $i = 1, d$.

Of course, (b) follows from (a) (using Matsusaka's theorem [1957] in the case $i = 1$).

To prove (a), we first reduce to the case where X has a smooth model $\mathcal{X}$ as in Section 4: this is automatic if $\operatorname{char} k = 0$ by Proposition 4.1, and if $\operatorname{char} k > 0$ we first reduce to k perfect as in Section 4F, then we can use Proposition 4.10(a) and a transfer argument.

We now give ourselves a model $f : \mathcal{X} \to B$ of X with $\mathcal{X}$ smooth. The proof is in two steps.

Step 1. Assume $d = 1$ and two sections $\tilde{c}_0, \tilde{c}_1$ of f are given. Let c_0, c_1 be their generic fibres and $\alpha = [c_0] - [c_1]$.

Lemma 5.7. *There exists an integer $N > 0$ such that $N\alpha \in \mathrm{CH}^1(X)^0$.*

Proof. Let $\tilde{\alpha} = [\tilde{c}_0(B)] - [\tilde{c}_1(B)] \in \mathrm{CH}^1(\mathcal{X})$. Then $j^*\tilde{\alpha} \in \mathrm{CH}^1_{\mathrm{alg}}(X) \subseteq \mathrm{CH}^1_{\mathrm{num}}(X)$. We now need to find $N > 0$ and $\xi \in \operatorname{Ker} j^*$ such that $N\tilde{\alpha} + \xi \in \mathrm{CH}^1(\mathcal{X})^0$. We shall look for ξ in the form

$$\xi = \sum_{b \in B^{(1)}} (\iota_b)_* \xi_b,$$

where $\iota_b : \mathcal{X}_{Z_b} \hookrightarrow \mathcal{X}$ is the inclusion (with $Z_b = \overline{\{b\}}$ as usual) and each ξ_b is a linear combination of classes of irreducible δ-dimensional components $\mathcal{X}^\lambda_{Z_b}$ of $\mathcal{X}_{Z_b}$ (almost all ξ_b will be 0). For this, I claim that the method of [Silverman 1994, III.8] extends to this case:

The first thing to check is that the hypothesis of [loc. cit., Proposition III.8.3] is verified, namely that $\langle \tilde{\alpha}, [\mathcal{X}_{Z_b}] \rangle_b = 0$ for all $b \in B^{(1)}$. For simplicity, write Z and ι instead of Z_b and ι_b. Up to removing a proper closed subset from Z, we may assume it smooth. In the Cartesian square of the diagram

$$\begin{tikzcd}
 & Z \arrow[r, "g_i=(d_i,\iota')"] \arrow[d, "d_i"'] & \mathcal{X}_Z \times B \arrow[d, "1\times\tilde{c}_i"] \\
Z & \mathcal{X}_Z \arrow[l, "f_Z"'] \arrow[r, "\gamma_\iota"] & \mathcal{X}_Z \times \mathcal{X}
\end{tikzcd}$$

where $d_i = (\tilde{c}_i)_{|Z}$ and ι' is the inclusion $Z \hookrightarrow B$, the top horizontal map g_i is a regular embedding of codimension $\delta + 1$ as the composite of the two regular embeddings

$$Z \xrightarrow{\delta} Z \times Z \xrightarrow{d_i \times \iota'} \mathcal{X}_Z \times B.$$

Here we use that the embedding d_i is regular [EGA IV$_4$ 1967, proposition 19.1.1]. Then

$$\begin{aligned}\langle[\tilde{c}_i(B)],[\mathcal{X}_Z]\rangle_b &= (f_Z)_*\gamma_\iota^!([\mathcal{X}_Z]\times(\tilde{c}_i)_*[B]) = (f_Z)_*\gamma_\iota^!((1\times\tilde{c}_i)_*[\mathcal{X}_Z\times B])\\ &\overset{(a)}{=} (f_Z)_*(d_i)_*\gamma_i^![\mathcal{X}_Z\times B] = \gamma_i^![\mathcal{X}_Z\times B]\\ &\overset{(b)}{=} g_i^![\mathcal{X}_Z^\lambda\times B] \overset{(c)}{=} \delta^!(d_i^![\mathcal{X}_Z]\times\iota'^![B]) = [Z],\end{aligned}$$

where (a) (resp. (b), (c)) is once again Lemma 2.1(a) (resp. (b), (c)).

Now

$$\mathrm{CH}_{\delta+i-1}(\mathcal{X}_Z) = \mathrm{CH}_\delta(\mathcal{X}_Z) \xleftarrow{\sim} \bigoplus_\lambda \mathbb{Z}[\mathcal{X}_Z^\lambda],$$

where the $\mathcal{X}_Z^\lambda$ are the irreducible components of $\mathcal{X}_Z$ of dimension δ: this follows from [Fulton 1984, Example 1.8.1] by induction on the number of components. The second thing to observe is that the statement and proof of [Silverman 1994, Proposition III.8.2] apply verbatim, namely that the quadratic form $\alpha \mapsto \langle\iota_*\alpha,\alpha\rangle_b$ on $\mathrm{CH}_\delta(\mathcal{X}_Z)$ is negative, with kernel generated by $[\mathcal{X}_Z]$. Indeed, this is a local computation so we can consider the fibre of $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_{B,b}$ and simply apply the said proposition. (The fact that $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_B$, which is used in its proof, follows from the fact that X is geometrically connected since it has rational points, and that B is normal.)

We can now find N and ξ just as in [Silverman 1994, Proposition III.8.3]. □

Step 2. The general case. Let $\alpha \in \mathrm{CH}^i_{\mathrm{alg}}(X)$. By [Achter et al. 2019, Lemma 3.8], there exist an integer $s \geq 0$, a smooth projective K-curve C, two rational points $c_0, c_1 \in C(K)$ and an element $y \in \mathrm{CH}^i(C\times X)$ such that $p^s\alpha = (c_0^* - c_1^*)y$ (recall that p is the exponential characteristic of k).

Lemma 5.8 (Q. Liu; cf. [Liu and Tong 2016]). *There exists a closed subset $F \subset B$ of codimension > 1 such that C lifts to a regular proper $(B-F)$-scheme $\mathcal{C}$ and the c_i lift to sections $\tilde{c}_i$ of $\mathcal{C}\to B-F$.*

Proof. See Proposition A.1 of the Appendix. □

By Step 1 and Lemma 5.8, there exists an integer $N > 0$ such that $N([c_1]-[c_1]) \in \mathrm{CH}^1(C)^0$. Then $Np^s\alpha = N(c_0^*-c_1^*)y = y^*N([c_0]-[c_1]) \in \mathrm{CH}^i(X)^0$ by Proposition 3.6, where y is considered as a correspondence (Weil–Bloch trick). This concludes the proof of Theorem 5.6.

Remark 5.9. There is a statement parallel to Theorem 5.6 in [Beĭlinson 1987, Lemma 2.2.2(b)], with a similar proof.

5C. *Example: elliptic curves.* In Step 1 of the proof of Theorem 5.6, suppose $\delta = 1$, B projective and that X is an elliptic curve. Applying $\deg : \mathrm{CH}^1(B) \to \mathbb{Z}$, we get an integral pairing $\langle\ ,\ \rangle$ on the finite index subgroup $\mathrm{CH}^1(X)^0 \cap X(K)$ of $\mathrm{CH}^1_{\mathrm{alg}}(X) = \mathrm{Pic}^0(X) = X(K)$; this pairing coincides with the Néron–Tate height pairing by the description in [Silverman 1994, Theorem III.9.3].

Theorem 5.10. *Assume k algebraically closed. Then*:

(a) $\mathrm{CH}^1(X)^0 \cap X(K)$ *contains the subset denoted by* $X(K)_0$ *in* [Silverman 1994, Remark III.9.4.2].

(b) *If $\mathcal{X}$ is a minimal model, $X(K)_0$ is a subgroup and the pairing*

$$X(K)_0 \times X(K)_0 \to \operatorname{Pic}(B) \tag{5-2}$$

of [loc. cit., Theorem III.9.5(b)] *equals* $-\langle\ ,\ \rangle$.

Proof. (a) Let $P \in X(K)$. As in Lemma 5.7, write $\tilde{P} : B \to \mathcal{X}$ for the section of f extending P (here its existence as a morphism is automatic since $\delta = 1$, by the valuative criterion of properness). What is written $[\tilde{P}(B)] = \tilde{P}_*[B]$ in its proof of is denoted by (P) in [Silverman 1994]. Since

$$(P)\cdot[\mathcal{X}_b] = \tilde{P}_*[B]\cdot f^*b = [B]\cdot\tilde{P}^* f^* b = \deg(b) = 1$$

(projection formula), and the intersection numbers of (P) with the components of $\mathcal{X}_b$ are ≥ 0, this implies that P meets exactly one component of $\mathcal{X}_b$, with multiplicity 1.

By definition, $X(K)_0$ is the set of P such that (P) meets the same component of $\mathcal{X}_b$ as (0) for all $b \in B^{(1)}$. Equivalently, $\deg\big(((P)-(0))\cdot[\mathcal{X}_b^\lambda]\big) = 0$ for all b and all such components. By (2-3), this degree is none else than $\langle (P)-(0), \mathcal{X}_b^\lambda\rangle_b$, so we get that

$$P \in X(K)_0 \Rightarrow (P)-(0) \in \mathrm{CH}^1(\mathcal{X})^0 \Rightarrow P - 0 \in \mathrm{CH}^1(X)^0.$$

(b) What we use here is that

$$\tilde{P} = \tau_P \circ \tilde{0} \tag{5-3}$$

for all $P \in X(K)$, where τ_P is the translation by P [Silverman 1994, Proposition III.9.1]. This already implies that $X(K)_0$ is a subgroup of $X(K)$.

We start with a convenient description of (5-2) by reformulating part (a) of [Silverman 1994, Theorem III.9.5]. For $P, Q \in X(K)$, we have $j^*((P+Q)-(P)-(Q)+(0)) = 0$ in $\operatorname{Pic}^0(X)$; the sequence

$$0 \to \operatorname{Pic}(B) \xrightarrow{f^*} \operatorname{Pic}(\mathcal{X}) \xrightarrow{j^*} \operatorname{Pic}(X) \to 0$$

is exact except at $\operatorname{Pic}(\mathcal{X})$ where its homology is given by $\bigoplus_b \mathrm{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$ (see [Kahn 2009, 3.2(a)]). If now $P, Q \in X(K)_0$, then

$$(P+Q)-(P)-(Q)+(0) = ((P+Q)-(0)) - ((P)-(0)) - ((Q)-(0)) \in \mathrm{CH}^1(\mathcal{X})^0,$$

which implies that its homology class is 0 by the nondegeneracy of the intersection pairings on $\mathrm{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$. Thus $(P+Q)-(P)-(Q)+(0) = f^*[P,Q]$ for a unique $[P,Q] \in \operatorname{Pic}(B)$. In particular,

$$[P,Q] = \tilde{R}^*((P+Q)-(P)-(Q)+(0)) \quad \text{for all } R \in X(K). \tag{5-4}$$

For convenience, we now write

$$P * Q = f_*((P)\cdot(Q)) \in \operatorname{Pic}(B)$$

for $P, Q \in X(K)$.

Lemma 5.11. *We have the identities $P * Q = \tilde{Q}^*(P)$ and $P * Q = 0 * (P - Q)$.*

Proof. For the first identity,

$$f_*((P)\cdot(Q)) = f_*(\tilde P_*[B]\cdot \tilde Q_*[P]) = f_*(\tilde Q_*\tilde Q^*\tilde P_*[B]) = \tilde Q^*(P)$$

by the projection formula. For the second one,

$$\tilde Q^*(P) = \tilde 0^*\tau_Q^*(\tau_P)_*\tilde 0_*[B] = \tilde 0^*(\tau_Q)_*^{-1}(\tau_P)_*\tilde 0_*[B] = \tilde 0^*(\tau_{P-Q})_*\tilde 0_*[B] = \tilde 0^*(P-Q). \qquad \square$$

Remark 5.12. Since $P*Q=Q*P$, we also get the intriguing identity $0*P=0*(-P)$.

To prove the claim of Theorem 5.10(b), we now apply (5-4) with $R=Q$:

$$\begin{aligned}[P,Q] &= \tilde Q^*((P+Q)-(P)-(Q)+(0))\\ &= (P+Q)*Q-P*Q-Q*Q+0*Q\\ &= 0*P-P*Q-0*0+0*Q = P*0-P*Q-0*0+0*Q = -\langle P,Q\rangle. \qquad \square\end{aligned}$$

Remarks 5.13. (a) We have $X(K)_0=\mathcal{N}^0(B)$, where $\mathcal{N}^0$ is the identity component of the Néron model $\mathcal{N}$ of X. Indeed, $\mathcal{N}$ is isomorphic to the smooth locus $\mathcal{X}_{\mathrm{sm}}$ of $\mathcal{X}$ [Artin 1986, Proposition 1.15] and $\mathcal{N}^0$ contains the 0-section (0). For any $P\in X(K)$, $(P)\subset\mathcal{X}_{\mathrm{sm}}$ (see end of Step 2 in the proof of Proposition A.1), and $(P)\in\mathcal{N}^0$ if and only if $P\in X(K)_0$ since $\mathcal{N}_b^0$ is the identity component of $\mathcal{N}_b$ for all $b\in B^{(1)}$ by definition of $\mathcal{N}^0$.

(b) Suppose that $P-0\in \mathrm{CH}^1(X)^0$. We can find a fibral divisor ξ such that $(P)-(0)-\xi$ is orthogonal to all fibral divisors (as in the proof of Lemma 5.7, with $N=1$), and this divisor is unique modulo $\mathrm{Im}\, f^*$ by [Silverman 1994, Proposition III.8.3]. By [loc. cit., Lemma III.9.4] (or by (a)), $X(K)/X(K)_0 = X(K)/\mathcal{N}^0(B)$ is finite, so the class of ξ is torsion in each $\mathrm{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b]$. Thus $\mathrm{CH}^1(X)^0/\mathcal{N}^0(B)\hookrightarrow \bigoplus_{b\in B^{(1)}}(\mathrm{CH}_1(\mathcal{X}_b)/[\mathcal{X}_b])_{\mathrm{tors}}$.

6. The pairing in codimension 1

In this section, we assume X projective and geometrically irreducible. Recall that $\delta=\mathrm{trdeg}(K/k)=\dim B$. We shall study the height pairing (4-1) for $i=1$, in $\mathbf{Ab}\otimes\mathbb{Q}$; note that $\mathrm{CH}^i(X)^{(0)}=\mathrm{CH}^i_{\mathrm{num}}(X)$ for $i=1,d$ by Theorem 5.6.

6A. ***A general result.*** We write $T(X)\subset \mathrm{CH}^d_{\mathrm{num}}(X)=\mathrm{CH}^d(X)_0$ for the Albanese kernel. For an abelian K-variety A, write $\mathrm{Tr}_{K/k}A$ for its K/k-trace and

$$\mathrm{LN}(A,K/k)=A(K)/(\mathrm{Tr}_{K/k}A)(k)$$

for its *Lang–Néron group*: it is finitely generated by the Lang–Néron theorem [1959]. We shall need the following classical fact:

Lemma 6.1. *The Albanese map $a_X:\mathrm{CH}^d(X)_0\to \mathrm{Alb}_X(K)$ has a cokernel of finite exponent.*

Proof. This could be deduced from [Kahn 2021, Proposition A.1]; here is a different and more direct proof. Choose a smooth irreducible multiple hyperplane section of dimension 1 $i : C \hookrightarrow X$. By the usual transfer argument, we may assume that X has a rational point lying on C. Then a_C is bijective. By [Murre 1990, Lemma 2.3], the composition

$$\mathrm{Pic}^0_X \xrightarrow{i^*} \mathrm{Pic}^0_C = \mathrm{Alb}_C \xrightarrow{i_*} \mathrm{Alb}_X \tag{6-1}$$

is an isogeny, hence $\mathrm{Coker}\, i_*(K)$ has finite exponent and so does its quotient $\mathrm{Coker}\, a_X$. □

Theorem 6.2. (a) *The pairing* $\langle\ ,\ \rangle$ *vanishes on* $\mathrm{CH}^1_{\mathrm{num}}(X) \times T(X)$.

(b) *This induces a pairing* (*in* $\mathbf{Ab} \otimes \mathbb{Q}$)

$$\langle\ ,\ \rangle : \mathrm{Pic}^0(X) \times \mathrm{Alb}_X(K) \to \mathrm{CH}^1(B).$$

(*c*) *Suppose* B *projective. Composing this pairing with the projection* $\mathrm{CH}^1(B) \to N^1(B)$ (*where* $N^1(B)$ *is the group of cycles of codimension* 1 *modulo numerical equivalence*) *induces a pairing*

$$\langle\ ,\ \rangle_{\mathrm{num}} : \mathrm{LN}(\mathrm{Pic}^0_X, K/k) \times \mathrm{LN}(\mathrm{Alb}_X, K/k) \to N^1(B). \tag{6-2}$$

Proof. (a) Up to extending scalars to the perfect closure of k, we may assume k perfect. Let L/K be a finite extension. Let B_L be the normalisation of B in L; up to removing from B a closed subset F of codimension ≥ 2 and from B_L the inverse image of F (which does not affect $\mathrm{CH}^1(B)$ or $\mathrm{CH}^1(B_L)$), we may assume B_L smooth. In $\mathbf{Ab}[1/p]$, the map $\mathrm{CH}^1(B) \to \mathrm{CH}^1(B_L)$ is a monomorphism (transfer argument). In view of the functoriality in Theorem 4.14(b), to prove the vanishing we may thus increase scalars as much as we wish. In particular, we may assume that $X(K) \neq \varnothing$.

Let $x \in X(K)$ and let $a : X \to \mathrm{Alb}_X$ be the corresponding Albanese map. Then a induces an isomorphism $a^* : \mathrm{Pic}^0(\mathrm{Alb}_X) \xrightarrow{\sim} \mathrm{Pic}^0(X)$, and $a_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Alb}_X)$ sends $T(X)$ into $T(\mathrm{Alb}_X)$. Still by functoriality, we are reduced to the case $X = \mathrm{Alb}_X =: A$.

The sequel is inspired by Néron's proof of [Néron 1965, Proposition 7]. In order to reason with elements, pick a representative of $\langle\ ,\ \rangle$ in $\mathbf{Ab}$ as in Remarks 4.5(a). Let $\beta \in T(A)$, and let $\overline{K}$ be an algebraic closure of K. In $T(A_{\overline{K}})$, we may write $\beta_{\overline{K}} = \sum_i([a_i+b_i]-[a_i]-[b_i]+[0])$, with $a_i, b_i \in A(\overline{K})$. Choose L/K finite such that all the a_i are rational over L. As above, we may extend scalars from K to L, and thus reduce to $\beta = [a+b]-[a]-[b]+[0]$ for $a, b \in A(K)$. The vanishing now follows from Example 4.15 and the theorem of the square [Mumford 2008, II.6, Corollary 4].

(b) This follows immediately from (a) and Lemma 6.1, which implies that $\mathrm{CH}^d(X)_0/T(X) \to \mathrm{Alb}_X(K)$ is an isomorphism in $\mathbf{Ab} \otimes \mathbb{Q}$.

(c) We may assume k algebraically closed; then the claim follows from the divisibility of $Y(k)$ for an abelian k-variety Y and the finite generation of $N^1(B)$. □

6B. ***Another conjecture.*** For the needs of Theorem 6.6 below, we introduce a new conjecture. From now on, B is projective as in Theorem 6.2(c).

Let R be a discrete valuation ring with quotient field K and residue field E. Suppose that an abelian K-variety A has good reduction with respect to R; then its Néron model $\mathcal{A}$ is an abelian scheme over $\operatorname{Spec} R$, whose special fibre A_s is an abelian E-variety. We have a specialisation homomorphism

$$A(K) = \mathcal{A}(R) \to A_s(E). \tag{6-3}$$

Suppose now that R contains k. The notion of K/k-trace readily extends to a notion of R/k-trace for abelian R-schemes; viewing these traces as right adjoints shows that

- $\operatorname{Tr}_{R/k} \mathcal{A}$ exists and equals $\operatorname{Tr}_{K/k} A$;
- the 'special fibre' functor yields a canonical morphism $\operatorname{Tr}_{K/k} A \to \operatorname{Tr}_{E/k} A_s$.

It follows that (6-3) induces a homomorphism of Lang–Néron groups

$$\operatorname{LN}(A, K/k) \to \operatorname{LN}(A_s, E/k). \tag{6-4}$$

Conjecture 6.3. *Assume that A has semistable reduction at every point of $B^{(1)}$, and that $\delta > 1$. For any projective embedding $B \hookrightarrow \mathbb{P}^N$, there exists a smooth, geometrically connected hyperplane section h of B such that A has good reduction at h and the kernel of (6-4) is finite, with $E = k(h)$.*

Suppose A constant. Then (6-4) may be rewritten as

$$\operatorname{Hom}_k(\operatorname{Alb}_B, A) \to \operatorname{Hom}_k(\operatorname{Alb}_h, A),$$

and Conjecture 6.3 follows from the surjectivity of $\operatorname{Alb}_h \to \operatorname{Alb}_B$ (see (6-1)). This gives some evidence for this conjecture.

Remark 6.4. Perhaps the hypotheses of Conjecture 6.3 are too weak.[4] In any case, we only need it in the special case $A = \operatorname{Pic}^0_X$, when X satisfies the conclusion of Lemma 6.5 (or any suitable variant of it); it may be easier to prove in such a case.

6C. *A technical lemma.* This lemma will be needed in the proofs of Theorem 6.6 and Proposition 6.8 below.

Lemma 6.5. *Suppose that $d = 1$. Then there exists an alteration $\tilde{B} \to B$, with $\tilde{B}$ smooth, such that $X \otimes_K k(\tilde{B})$ has a projective model $f : \mathcal{X} \to \tilde{B}$ where $\mathcal{X}$ is smooth over k and, for all $b \in \tilde{B}^{(1)}$, the irreducible components of $\mathcal{X}_b$ are smooth over $k(b)$.*

Proof. Start from a projective embedding $X \hookrightarrow \mathbb{P}^N_K$ and consider its closure $\mathcal{X}_0$ in $\mathbb{P}^N_B$. In the following reasoning using results of [de Jong 1997], we always take the group G appearing there equal to 1. By [de Jong 1997, Theorem 5.9] (or just [de Jong 1997, Theorem 2.4 and Lemma 5.7]), we may (projectively) alter $f_0 : \mathcal{X}_0 \to B$ into $f_1 : \mathcal{X}_1 \to B_1$ so that f_1 is a projective quasisplit semistable curve in the sense of [de Jong 1997, section after Lemma 5.6]. This condition is stable under base change, hence, by the reasoning at the end of the proof of [de Jong 1996, Theorem 5.13], we may alter B_1 into B_2 so that B_2 is

[4] Added in proof: A version of this conjecture has now been proven: see Bruno Kahn and Long Liu, *A specialisation theorem for Lang–Néron groups*, in preparation.

smooth and $f_2 : \mathcal{X}_2 := \mathcal{X}_1 \times_{B_1} B_2 \to B_2$ verifies the hypotheses of [de Jong 1997, Proposition 5.11] (note that varieties over a field verify [de Jong 1997, (5.12.1)] by [de Jong 1996, Theorem 4.1]); in particular, $\tilde{B} := B_2$ is smooth. Next, the beginning of the proof of [de Jong 1997, Proposition 5.11] yields a modification $\pi : \mathcal{X}_3 \to \mathcal{X}_2$ such that the singular locus Σ of $\mathcal{X}_3$ is smooth of codimension ≥ 3 and $f_3 : \mathcal{X}_3 \to \tilde{B}$ is still a quasisplit semistable curve. The end of this proof then yields a desingularisation $\mathcal{X}_4$ of $\mathcal{X}_3$ by blowing up the components of Σ. Since they lie over points of codimension ≥ 2 in $\tilde{B}$, this does not affect the fibres of f_3 at points of codimension 1, so $f_4 : \mathcal{X}_4 \to \tilde{B}$ is "quasisplit semistable in codimension 1".

We are left to desingularise the singular components of $(\mathcal{X}_4)_b$ for all $b \in \tilde{B}^{(1)}$. Let D be such a component, and let x be a singular point of D. Note that x does not lie on any other component, since all singular points of $(\mathcal{X}_4)_b$ are quadratic by the "semistable" condition. By the "quasisplit" one, the completion of $\mathcal{O}_{\mathcal{X}_4,x}$ is isomorphic to $k[[u, v, t_1, \dots, t_\delta]]/(uv - t_1)$, where t_1 is a local equation of D (compare [de Jong 1996, 2.16]). The ideal of x is (u, v, t_1). Blowing up this ideal retains the regularity of $\mathcal{X}_4$, separates the two branches of D at x (making its strict transform regular at the two corresponding points) and adds a smooth irreducible exceptional divisor. We have therefore decreased by 1 the total number of singular points of the irreducible components of $(\mathcal{X}_4)_b$. Since only finitely many b are involved, we end the process after a finite number of steps. □

6D. *A negativity theorem.*

Theorem 6.6. *Let $L \in \operatorname{Pic}(X)$ and $\ell \in \operatorname{Pic}(B) - \{0\}$. Consider the quadratic form*

$$q = q(X, B, L, \ell) : \operatorname{LN}(\operatorname{Pic}^0_X, K/k) \ni \alpha \mapsto \deg\big(\langle \alpha, L^{d-1}\alpha\rangle_{\text{num}} \cdot \ell^{\delta-1}\big)$$

obtained from the pairing of Theorem 6.2(c). If L is ample and $\delta = 1$ (hence $\ell^{\delta-1} = 1$), then $q(X, B, L, \ell)$ is negative definite (in particular, nondegenerate). If Conjecture 6.3 holds for Pic^0_X when $d = 1$ and in the situation of Lemma 6.5, this extends to $\delta > 1$ for ℓ ample.

Remark 6.7. As pointed out in Remarks 4.5(b), the notation using elements is abusive in $\mathbf{Ab} \otimes \mathbb{Q}$. Theorem 6.6 could be converted into an arrow-theoretic statement; similarly, the notion "negative definite" for a quadratic form with values in $\mathbb{Z}$ is unambiguous in $\mathbf{Ab} \otimes \mathbb{Q}$, by using Remarks 4.5(a).

However, converting the proof below into arrow-theoretic notation would be cumbersome at best. Since the source and target of the quadratic form q are finitely generated abelian groups, we can tensor everything with $\mathbb{Q}$ (i.e., apply the natural functor from $\mathbf{Ab} \otimes \mathbb{Q}$ to $\mathbb{Q}$-vector spaces) without losing information, and reason with honest elements. This is what we do in this proof.

Proof. (a) We first reduce to $d = 1$ as follows. Suppose $d > 1$. We may assume L very ample. Let $i : C \hookrightarrow X$ be a smooth irreducible curve given by successive hyperplane sections from the projective embedding determined by L. By the functoriality of Theorem 4.14, we have

$$\langle i^*\alpha, i^*\alpha\rangle_{\text{num}} = \langle \alpha, i_* i^*\alpha\rangle_{\text{num}} = \langle \alpha, L^{d-1} \cdot \alpha\rangle_{\text{num}};$$

hence $q(X, B, L, \ell)(\alpha) = q(C, B, i^*L, \ell)(i^*\alpha)$. By the isogeny (6-1), $\operatorname{LN}(\operatorname{Pic}^0_X, K/k) \to \operatorname{LN}(\operatorname{Pic}^0_C, K/k)$ is mono in $\mathbf{Ab} \otimes \mathbb{Q}$.

We now assume $d = 1$.

(b) We reduce to the situation of Lemma 6.5. Let $\mathcal{X} \xrightarrow{f} \tilde{B}$ be as in Lemma 6.5. Since $\pi : \tilde{B} \to B$ is projective, pick a very ample divisor $\mathcal{L}$ relative to π. By [EGA II 1961, proposition 4.4.10(ii)], $\mathcal{L} + n\pi^*\ell$ is then very ample (relative to $\tilde{B} \to \operatorname{Spec} k$) for all $n \gg 0$. Let $\alpha \in \mathrm{LN}(\mathrm{Pic}^0_X, K/k) - \{0\}$. Assuming the theorem true over $\tilde{B}$, we have

$$\deg\big(\langle \pi^*\alpha, \pi^*L^{d-1}\pi^*\alpha\rangle_{\mathrm{num}} \cdot (\mathcal{L} + n\pi^*\ell)^{\delta-1}\big) < 0$$

for all $n \gg 0$. This is a polynomial in n, with dominant term

$$\deg\big(\langle \pi^*\alpha, \pi^*L^{d-1}\pi^*\alpha\rangle_{\mathrm{num}} \cdot \pi^*\ell^{\delta-1}\big) = \deg\big(\langle \alpha, L^{d-1}\alpha\rangle_{\mathrm{num}} \cdot \ell^{\delta-1}\big)$$

by (3-14), which must be negative.

We now assume that we are in the situation of Lemma 6.5.

(c) Assume $\delta = 1$. Observe that the pairing (1-2), composed with the degree, is then the intersection pairing. By the Hodge index theorem, this pairing has signature $(1, \rho - 1)$ where $\rho = \operatorname{rk} N^1(\mathcal{X})$. Since $N^1(\mathcal{X})^0$ is the orthogonal of the isotropic vector f^*t for $t \in N^1(B) - \{0\}$, the restriction of the intersection pairing to this subspace is negative with kernel generated by f^*t. Since f^*t also generates the kernel of $N^1(\mathcal{X})^0 \to \mathrm{LN}(\mathrm{Pic}^0_X, K/k)$, the quadratic form q is negative definite, as requested.

(d) Assume finally $\delta > 1$. Similarly to (a), we may assume ℓ very ample. We may also assume k algebraically closed (in particular, infinite). Let $Z \subset B$ be the locus of nonsmoothness of f. In the family of hyperplane sections of B relative to the projective embedding given by ℓ, only finitely many may be contained in Z, therefore we can pick a smooth hyperplane section $h \not\subset Z$. By induction, there exists a smooth ample curve $i : \Gamma \subset B$ determined by ℓ such that the generic fibre $X(E)$ of $\mathcal{X}_\Gamma = f^{-1}(\Gamma)$ is smooth over $E = k(\Gamma)$.

Write $I : \mathcal{X}_\Gamma \hookrightarrow \mathcal{X}$, $g : \mathcal{X}_\Gamma \to \Gamma$ for the two corresponding projections. For $\tilde{\alpha} \in \mathrm{CH}^1(\mathcal{X})$, we have

$$\langle \tilde{\alpha}, \tilde{\alpha}\rangle \cdot \ell^{\delta-1} = i_* i^* f_*(\tilde{\alpha}^2) = i_* g_* I^!(\tilde{\alpha}^2).$$

Since $\deg_B \circ i_* = \deg_\Gamma$, it is enough to compute $g_* I^!(\tilde{\alpha}^2)$.

Choose a resolution of singularities $\pi : \mathcal{Y} \to \mathcal{X}_\Gamma$ of the surface $\mathcal{X}_\Gamma$; let $\tilde{I} = I \circ \pi$ and $\tilde{g} = g \circ \pi$. The same reasoning as in the proof of Proposition 3.9(iv) yields the identity $I^! = \pi_* \tilde{I}^*$, hence

$$g_* I^!(\tilde{\alpha}^2) = \tilde{g}_* \tilde{I}^*(\tilde{\alpha}^2) = \tilde{g}_*(\tilde{I}^*\tilde{\alpha})^2.$$

Now there exists a finite extension E'/E with smooth projective k-model Γ', and a semistable model $\mathcal{Y}'$ of $X(E) \otimes_E E'$ over Γ' mapping to $\mathcal{Y}$ by a morphism φ. If $d = [E' : E]$, we therefore have

$$(\tilde{g} \circ \varphi)_*(\tilde{I} \circ \varphi)^*(\tilde{\alpha})^2 = d\,\tilde{g}_*(\tilde{I}^*\tilde{\alpha})^2.$$

Under Conjecture 6.3, Γ may be chosen such that the map induced by $\tilde{I}^*$

$$\mathrm{LN}(\mathrm{Pic}^0_X, K/k) \to \mathrm{LN}(\mathrm{Pic}^0_{X(E)}, E/k)$$

has finite kernel, and our reduction to $\delta = 1$ is complete. □

6E. ***Another pairing.*** Here we assume B projective; we write $A = \mathrm{Tr}_{K/k} \mathrm{Pic}^0_X$ and $P = \mathrm{Pic}^0_B$.

Proposition 6.8. *Suppose $d = 1$. In the pairing of Theorem 6.2(b), we have $\langle A(k), A(k)\rangle \subseteq \mathrm{Pic}^0(B)\{p\}$ in $\mathbf{Ab} \otimes \mathbb{Q}$, where p is the exponential characteristic of k. This induces a pairing in $\mathbf{Ab} \otimes \mathbb{Q}$*

$$\mathrm{LN}(\mathrm{Pic}^0_X, K/k) \times A(k) \to P(k)/P(k)\{p\}. \tag{6-5}$$

Proof. We may first reduce to k perfect and then pass to a finite extension of K, hence reduce to the existence of a smooth model $\mathcal{X}$ (e.g., as in Lemma 6.5). By [Kahn 2009, 3.2(a)], we have

$$j^* \mathrm{Pic}^0(\mathcal{X}) = A(k).$$

By Proposition 3.14, $\langle \mathrm{Pic}^0(\mathcal{X}), \mathrm{Pic}^0(\mathcal{X})\rangle$ is p-primary torsion, hence the claim. □

Question 6.9. Does (6-5) extend to arbitrary d, replacing $A(k)$ by $\mathrm{Tr}_{K/k} \mathrm{Alb}_X(k)$?

Let $E = k(A)$. Using [Milne 1986, Theorem 3.1], we deduce from (6-5) a pairing

$$\mathrm{LN}(\mathrm{Pic}^0_X, K/k) \times \mathrm{Mor}_k(A, A) \to \mathrm{Mor}_k(A, P)/\mathrm{Mor}_k(A, P)\{p\}.$$

Evaluating on the identity $1_A \in \mathrm{Mor}_k(A, A)$, we get a homomorphism

$$\mathrm{LN}(\mathrm{Pic}^0_X, K/k) \to \mathrm{Mor}_k(A, P)/\mathrm{Mor}_k(A, P)\{p\}$$

and using the canonical isomorphism $\mathrm{Mor}_k(A, P) \simeq P(k) \oplus \mathrm{Hom}_k(A, P)$, a final homomorphism

$$\mathrm{LN}(\mathrm{Pic}^0_X, K/k) \to \mathrm{Hom}(\mathrm{Tr}_{K/k} \mathrm{Pic}^0_X, \mathrm{Pic}^0_B) \tag{6-6}$$

because the right hand group is torsion-free. It is an exercise to check that, evaluating this homomorphism on k-points, we get back (6-5) (improved).

If $B = \mathbb{P}^1$ or $\mathrm{Tr}_{K/k} \mathrm{Pic}^0_X = 0$, the right hand side is 0 while the left hand side is nonzero in general. Yet we may ask:

Question 6.10. When is (6-6) surjective (in $\mathbf{Ab} \otimes \mathbb{Q}$)?

Appendix: Extending rational points to sections

by Qing Liu

Proposition A.1. *Let B be a noetherian connected regular excellent scheme. Let C be a connected projective regular curve over the function field K of B. Let $c_1, \dots, c_n \in C(K)$. Then there exist an open subset $U \subseteq B$ with $\mathrm{codim}(B \setminus U, B) \geq 2$ and a proper scheme $\mathcal{C} \to U$, with $\mathcal{C}$ regular, containing the c_i such that the latter extend to sections of $\mathcal{C} \to U$.*

Step 1. We extend C to a projective regular scheme $\mathcal{C}_0$ over some "big" open subset U_0 of B.

First we extend C to an integral projective scheme $f : \mathcal{X} \to B$ (taking for instance the scheme-theoretical closure of C in a suitable $\mathbb{P}^n_B$). Let $\mathcal{X}_{\mathrm{sing}}$ be the closed subset of the singular points of $\mathcal{X}$. Then $V := B \setminus f(\mathcal{X}_{\mathrm{sing}})$ is a dense open subset of B such that $\mathcal{X}_V$ is regular.

Let $b_1, \ldots, b_m$ be the codimension 1 points of B inside of $B \setminus V$. We are going to extend $\mathcal{X}_V$ above an open subset U_0 of B containing all the b_j. For each $j \le m$, we have a relative integral curve $\mathcal{X} \times_B \operatorname{Spec} \mathcal{O}_{B,b_j}$ over the discrete valuation ring $\mathcal{O}_{B,b_j}$ with regular generic fibre C. As B is excellent, there exists a resolution of singularities

$$\mathcal{X}'_j \to \mathcal{X} \times_B \operatorname{Spec} \mathcal{O}_{B,b_j} \to \operatorname{Spec} \mathcal{O}_{B,b_j}.$$

Each $\mathcal{X}'_j$ is a projective regular curve over $\operatorname{Spec} \mathcal{O}_{B,b_j}$ and extends to a projective regular curve $\mathcal{X}_j$ over some open neighbourhood $V_j \ni b_j$. Shrinking the (finitely many) V_j if necessary, we can suppose that for all $j, \ell \le m$, $\mathcal{X}_j$ and $\mathcal{X}_\ell$ coincide over $V_j \cap V_\ell$ and that $\mathcal{X}_j$ coincides with $\mathcal{X}_V$ over $V \cap V_j$. Let U_0 be the union of V and the V_j and let $\mathcal{C}_0 \to U_0$ be obtained by glueing $\mathcal{X}_V$ and the $\mathcal{X}_j$. Then $\mathcal{C}_0$ is regular, proper over U_0 $\big($by the fpqc descent $V \coprod \big(\coprod_{1 \le i \le m} V_j\big) \to U_0$ of properness; see [EGA IV$_2$ 1965, proposition 2.7.1(vii)]$\big)$, and $\operatorname{codim}(B \setminus U_0, B) \ge 2$.

Step 2. For all $i \le n$ we let $P_i \subseteq \mathcal{C}_0$ be the scheme-theoretical closure of $\{c_i\}$. Then $P_i \to U_0$ is proper birational, hence is an isomorphism away from a closed subset $Z_i \subset U_0$ of codimension at least 2. To finish we let $U := U_0 \setminus \big(\bigcup_{1 \le i \le n} Z_i\big)$ and let $\mathcal{C} = (\mathcal{C}_0)_U \to U$. (As U and $\mathcal{C}$ are regular, the section $(P_i)_U$ of $\mathcal{C} \to U$ is contained in the smooth locus of $\mathcal{C}$ [Bosch et al. 1990, 3.1, Proposition 2 and following paragraph].)

Acknowledgements

What triggered me to start this research was a talk by Tamás Szamuely at the June 2019 Oberwolfach workshop on algebraic K-theory, where he explained a preliminary version of [Rössler and Szamuely 2022]; I thank him and Damian Rössler for several exchanges during the preparation of this work. Part of this theory was developed while I was visiting Jilali Assim in Meknès in March 2020; I would like to thank him and Université Moulay Ismail for their hospitality and excellent working conditions. I thank Qing Liu for accepting to write the Appendix and for helping me with references in EGA, Marc Hindry for a discussion around Remarks 5.13(a) and Tamás Szamuely for his help with the end of the proof of Proposition 2.11.

It is common to thank the referees for helpful remarks. Here, I would like to stress my appreciation for exceptionally lucid and helpful reports from one of them, who not only spotted gaps and mistakes in some of my initial proofs, but also provided insights which eventually led to a much more direct construction of the refined height pairing, making it more integral and completely avoiding my initial recourse to semistable models (see Remarks 2.3(c)).

References

[Achter et al. 2019] J. D. Achter, S. Casalaina-Martin, and C. Vial, "Parameter spaces for algebraic equivalence", *Int. Math. Res. Not.* **2019**:6 (2019), 1863–1893. MR Zbl

[Artin 1986] M. Artin, "Néron models", pp. 213–230 in *Arithmetic geometry*, edited by G. Cornell and J. H. Silverman, Springer, 1986. MR Zbl

[Barbieri-Viale and Kahn 2016] L. Barbieri-Viale and B. Kahn, *On the derived category of 1-motives*, Astérisque **381**, Société Mathématique de France, 2016. MR Zbl

[Beĭlinson 1987] A. A. Beĭlinson, "Height pairing between algebraic cycles", pp. 1–25 in *K-theory, arithmetic and geometry* (Moscow, 1984–1986), edited by Y. I. Manin, Lecture Notes in Math. **1289**, Springer, 1987. MR Zbl

[Beĭlinson et al. 1982] A. A. Beĭlinson, J. Bernstein, and P. Deligne, "Faisceaux pervers", pp. 5–171 in *Analysis and topology on singular spaces, I* (Luminy, 1981), Astérisque **100**, Société Mathématique de France, 1982. MR Zbl

[Bloch 1984] S. Bloch, "Height pairings for algebraic cycles", *J. Pure Appl. Algebra* **34**:2-3 (1984), 119–145. MR Zbl

[Bloch 2010] S. Bloch, *Lectures on algebraic cycles*, 2nd ed., New Mathematical Monographs **16**, Cambridge University Press, 2010. MR Zbl

[Bosch et al. 1990] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Math. (3) **21**, Springer, 1990. MR Zbl

[Bourbaki 1985] N. Bourbaki, *Algèbre commutative, chapitres 1 à 4*, Masson, 1985. Zbl

[Corti and Hanamura 2000] A. Corti and M. Hanamura, "Motivic decomposition and intersection Chow groups, I", *Duke Math. J.* **103**:3 (2000), 459–522. MR Zbl

[Cossart and Piltant 2009] V. Cossart and O. Piltant, "Resolution of singularities of threefolds in positive characteristic, II", *J. Algebra* **321**:7 (2009), 1836–1976. MR Zbl

[Debarre 2001] O. Debarre, *Higher-dimensional algebraic geometry*, Springer, 2001. MR Zbl

[Deninger and Murre 1991] C. Deninger and J. Murre, "Motivic decomposition of abelian schemes and the Fourier transform", *J. Reine Angew. Math.* **422** (1991), 201–219. MR Zbl

[EGA II 1961] A. Grothendieck, "Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes", *Inst. Hautes Études Sci. Publ. Math.* **8** (1961), 5–222. MR Zbl

[EGA IV_2 1965] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II", *Inst. Hautes Études Sci. Publ. Math.* **24** (1965), 5–231. MR Zbl

[EGA IV_4 1967] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, IV", *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 5–361. MR Zbl

[Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Math. (3) **2**, Springer, 1984. MR Zbl

[Gillet and Soulé 1987] H. Gillet and C. Soulé, "Intersection theory using Adams operations", *Invent. Math.* **90**:2 (1987), 243–277. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, 1977. MR Zbl

[Illusie and Temkin 2014] L. Illusie and M. Temkin, "Gabber's modification theorem (log smooth case)", exposé X in *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, edited by L. Illusie et al., Astérisque **363–364**, Société Mathématique de France, 2014. Zbl

[Jannsen 1988] U. Jannsen, "Continuous étale cohomology", *Math. Ann.* **280**:2 (1988), 207–245. MR Zbl

[Jannsen 1994] U. Jannsen, "Motivic sheaves and filtrations on Chow groups", pp. 245–302 in *Motives, I* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55.1**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[de Jong 1996] A. J. de Jong, "Smoothness, semi-stability and alterations", *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93. MR Zbl

[de Jong 1997] A. J. de Jong, "Families of curves and alterations", *Ann. Inst. Fourier (Grenoble)* **47**:2 (1997), 599–621. MR Zbl

[Kahn 2006] B. Kahn, "Sur le groupe des classes d'un schéma arithmétique", *Bull. Soc. Math. France* **134**:3 (2006), 395–415. MR Zbl

[Kahn 2009] B. Kahn, "Démonstration géométrique du théorème de Lang–Néron et formules de Shioda–Tate", pp. 149–155 in *Motives and algebraic cycles*, edited by R. de Jeu and J. D. Lewis, Fields Inst. Commun. **56**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[Kahn 2014] B. Kahn, "A motivic formula for the L-function of an abelian variety over a function field", preprint, 2014. arXiv 1401.6847

[Kahn 2021] B. Kahn, "Albanese kernels and Griffiths groups", *Tunis. J. Math.* **3**:3 (2021), 589–656. MR Zbl

[Kahn 2023] B. Kahn, "Pointwise intersection products", preprint, IMJ-PRG, 2023, available at https://webusers.imj-prg.fr/~bruno.kahn/preprints/height-pairing-pointwise.pdf. Zbl

[Lang and Néron 1959] S. Lang and A. Néron, "Rational points of abelian varieties over function fields", *Amer. J. Math.* **81** (1959), 95–118. MR Zbl

[Laumon 1976] G. Laumon, "Homologie étale", pp. 163–188 in *Séminaire de géométrie analytique* (Paris, 1974–75), edited by A. Douady and J.-L. Verdier, Astérisque **36–37**, Société Mathématique de France, 1976. MR Zbl

[Lieberman 1968] D. I. Lieberman, "Numerical and homological equivalence of algebraic cycles on Hodge manifolds", *Amer. J. Math.* **90** (1968), 366–374. MR Zbl

[Liu and Tong 2016] Q. Liu and J. Tong, "Néron models of algebraic curves", *Trans. Amer. Math. Soc.* **368**:10 (2016), 7019–7043. MR Zbl

[Matsusaka 1957] T. Matsusaka, "The criteria for algebraic equivalence and the torsion group", *Amer. J. Math.* **79** (1957), 53–66. MR Zbl

[Milne 1986] J. S. Milne, "Abelian varieties", pp. 103–150 in *Arithmetic geometry*, edited by G. Cornell and J. H. Silverman, Springer, 1986. MR Zbl

[Moret-Bailly 1985] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque **129**, 1985. MR Zbl

[Mumford 2008] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics **5**, Hindustan Book Agency, New Delhi, 2008. MR Zbl

[Murre 1990] J. P. Murre, "On the motive of an algebraic surface", *J. Reine Angew. Math.* **409** (1990), 190–204. MR Zbl

[Néron 1965] A. Néron, "Quasi-fonctions et hauteurs sur les variétés abéliennes", *Ann. of Math.* (2) **82** (1965), 249–331. MR Zbl

[Olsson 2015] M. Olsson, "Borel–Moore homology, Riemann–Roch transformations, and local terms", *Adv. Math.* **273** (2015), 56–123. MR Zbl

[Rössler and Szamuely 2022] D. Rössler and T. Szamuely, "A generalization of Beilinson's geometric height pairing", *Doc. Math.* **27** (2022), 1671–1692. MR Zbl

[Schneider 1982] P. Schneider, "Zur Vermutung von Birch und Swinnerton-Dyer über globalen Funktionenkörpern", *Math. Ann.* **260**:4 (1982), 495–510. MR Zbl

[SGA 6 1971] A. Grothendieck, "Problemes ouverts en theorie des intersections", exposé XIV, pp. 667–689 in *Théorie des intersections et théorème de Riemann–Roch* (Séminaire de Géométrie Algébrique du Bois Marie 1966–1967), edited by A. Grothendieck et al., Lecture Notes in Math. **225**, Springer, 1971. MR Zbl

[Shioda 1999] T. Shioda, "Mordell–Weil lattices for higher genus fibration over a curve", pp. 359–373 in *New trends in algebraic geometry* (Warwick, 1996), edited by K. Hulek et al., London Math. Soc. Lecture Note Ser. **264**, Cambridge University Press, 1999. MR Zbl

[Silverman 1994] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer, 1994. MR Zbl

[Tate 1965] J. T. Tate, "Algebraic cycles and poles of zeta functions", pp. 93–110 in *Arithmetical algebraic geometry* (West Lafayette, IN, 1963), edited by O. F. G. Schilling, Harper & Row, New York, 1965. MR Zbl

[Verdier 1976] J.-L. Verdier, "Le théorème de Riemann–Roch pour les variétés algébriques éventuellement singulières (d'après P. Baum, W. Fulton et R. MacPherson)", pp. 5–20 in *Séminaire de géométrie analytique* (Paris, 1974–75), edited by A. Douady and J.-L. Verdier, Astérisque **36–37**, Société Mathématique de France, 1976. MR Zbl

[Weber 2015] A. Weber, *Intersection theory on regular schemes via alterations and deformation to the normal cone*, Ph.D. thesis, Universität Regensburg, 2015. Zbl

Communicated by Jean-Louis Colliot-Thélène
Received 2021-03-25 Revised 2023-06-11 Accepted 2023-09-03

bruno.kahn@imj-prg.fr *CNRS, IMJ-PRG, Sorbonne Université and Université Paris Cité, Paris, France*

qing.liu@math.u-bordeaux.fr *Institut de Mathématiques de Bordeaux, Université de Bordeaux, CNRS, Talence, France*

ALGEBRA AND NUMBER THEORY 18:6 (2024)
https://doi.org/10.2140/ant.2024.18.1081

Balmer spectra and Drinfeld centers

Kent B. Vashaw

The Balmer spectrum of a monoidal triangulated category is an important geometric construction which is closely related to the problem of classifying thick tensor ideals. We prove that the forgetful functor from the Drinfeld center of a finite tensor category $\boldsymbol{C}$ to $\boldsymbol{C}$ extends to a monoidal triangulated functor between their corresponding stable categories, and induces a continuous map between their Balmer spectra. We give conditions under which it is injective, surjective, or a homeomorphism. We apply this general theory to prove that Balmer spectra associated to finite-dimensional cosemisimple quasitriangular Hopf algebras (in particular, group algebras in characteristic dividing the order of the group) coincide with the Balmer spectra associated to their Drinfeld doubles, and that the thick ideals of both categories are in bijection. An analogous theorem is proven for certain Benson–Witherspoon smash coproduct Hopf algebras, which are not quasitriangular in general.

Introduction

Tensor triangular geometry, initiated by Balmer [2005; 2010], has proven to be a useful prism through which modular representation theory, algebraic geometry, commutative algebra, algebraic topology, and homotopy theory may all be studied (for a few examples, see [Balmer and Sanders 2017; Boe et al. 2017a; 2017b; Matsui and Takahashi 2017; Balmer 2020]). The uniting feature is the existence, in each case, of a braided monoidal triangulated category; the braiding condition implies that there is a natural isomorphism

$$X \otimes Y \cong Y \otimes X$$

for all objects X and Y. A noncommutative analogue of Balmer's theory (that is, one with no assumption of a braiding) was initiated and explored in [Buan et al. 2007; Nakano et al. 2022a; 2022b], motivated by the abundance of examples of nonbraided monoidal triangulated categories arising in representation theory. This theory defines a topological space, called the *Balmer spectrum*, for any monoidal triangulated category $\boldsymbol{T}$. This space is denoted $\operatorname{Spc} \boldsymbol{T}$, and is defined as the collection of prime ideals of $\boldsymbol{T}$, reflecting the usual notion of prime spectrum from ring theory.

Nonbraided monoidal triangulated categories arise naturally as the stable categories of finite tensor categories. Broadly speaking, if $\boldsymbol{C}$ is a finite tensor category, then the stable category of $\boldsymbol{C}$, denoted $\operatorname{st}(\boldsymbol{C})$, is the category obtained by factoring out the projective objects of $\boldsymbol{C}$. One motivation for factoring

This research was supported in part by a Board of Regents LSU fellowship, an Arthur K. Barton Superior Graduate Student Scholarship in Mathematics from LSU, NSF grant DMS-1901830, and NSF Postdoctoral Fellowship DMS-2103272.
MSC2020: 16T05, 18G65, 18G80, 18M05, 18M15.
Keywords: Hopf algebra, stable module category, tensor triangulated category, thick ideal.

© 2024 The Author, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

out projectives comes from the theory of support varieties, where the support variety of an object only distinguishes an object up to direct sums with projective objects. The stable category is a monoidal triangulated category, where the monoidal product of $\mathsf{st}(\boldsymbol{C})$ is an extension of the monoidal product of $\boldsymbol{C}$.

An important tool in the study of tensor categories is the Drinfeld center, a categorical analogue of the center of a ring; it is a generalization of the quantum or Drinfeld double construction for Hopf algebras, originally introduced by Drinfeld [1987]. For any tensor category $\boldsymbol{C}$, its Drinfeld center $\mathsf{Z}(\boldsymbol{C})$ is a braided tensor category equipped with a functor $F : \mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$. The Drinfeld center satisfies the universal property: if $G : \boldsymbol{D} \to \boldsymbol{C}$ is a strict tensor functor between strict tensor categories, and $\boldsymbol{D}$ is braided, such that G is bijective on objects and surjective on morphisms, then there exists a strict tensor functor $H : \boldsymbol{D} \to \mathsf{Z}(\boldsymbol{C})$

$$\begin{tikzcd} \boldsymbol{C} & \\ \mathsf{Z}(\boldsymbol{C}) \arrow[u, "F"] & \boldsymbol{D} \arrow[ul, "G"'] \arrow[l, dashed, "H"] \end{tikzcd}$$

with $F \circ H = G$.

If $\boldsymbol{C}$ is abelian, then $\mathsf{Z}(\boldsymbol{C})$ is automatically abelian as well. We do not see an analogue for this argument in the triangulated case: if $\boldsymbol{T}$ is triangulated, it does not seem to follow immediately that $\mathsf{Z}(\boldsymbol{T})$ is triangulated. This is a reflection of the fact that the morphism given in the extension axiom for triangulated categories is not necessarily unique.

However, if $\boldsymbol{C}$ is a finite tensor category, one can form its stable category $\mathsf{st}(\boldsymbol{C})$ on one hand; on the other hand, $\mathsf{Z}(\boldsymbol{C})$ is again a finite tensor category, and one can form its stable category $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. The natural question that arises is, therefore: how are the Balmer spectra between these two categories connected?

This question is of particular interest because Balmer spectra are related intimately with cohomological support varieties (as in [Bergh et al. 2021]); for example, under a particular homological condition, the projectivization of the spectrum of the cohomology ring of the small quantum groups $u_\zeta(\mathfrak{b})$ of Borel subalgebras at roots of unity (as computed in [Ginzburg and Kumar 1993; Bendel et al. 2014]) identifies with the Balmer spectrum of its stable category [Nakano et al. 2022a], which can be used to show that the support varieties for the small quantum Borel possess the tensor product property [Nakano et al. 2022b; Negron and Pevtsova 2023]. In many specific cases, for instance see [Friedlander and Negron 2018; Negron 2021; Negron and Plavnik 2022], the cohomology of Drinfeld doubles has been studied, and its relationship to the cohomology of the original finite tensor category explored.

Additionally, this project will provide tools to aid in thick ideal classification problems. Balmer spectra, which are defined as the collection of prime ideals of the category, are intimately related to these problems, since every thick ideal of a rigid monoidal triangulated category is equal to an intersection of prime ideals. Classifications of thick ideals in various settings have been undertaken in many different settings, for instance in various categories arising from

(1) commutative algebra and algebraic geometry [Hopkins 1987; Thomason 1997; Matsui and Takahashi 2017];

(2) Lie superalgebras [Boe et al. 2017a];

(3) finite groups and finite group schemes [Benson et al. 1997; Friedlander and Pevtsova 2007];

(4) tilting modules for quantum groups and algebraic groups in positive characteristic [Ostrik 1997; Achar et al. 2019];

(5) Hopf algebras which are not necessarily commutative, cocommutative, or even quasitriangular [Benson and Witherspoon 2014; Boe et al. 2017a; Nakano et al. 2022a; 2022b].

There are examples, for instance the small quantum groups of Borel subalgebras $u_\zeta(\mathfrak{b})$ at roots of unity, where, as mentioned above, the Balmer spectrum and thick ideals are known for its stable module category; however, it is an open question to classify the Balmer spectrum and thick ideals for the stable category of its Drinfeld center, that is, the stable module category of $u_\zeta(\mathfrak{g}) \otimes \Bbbk T$, where $\Bbbk T$ is the group algebra of the group of generators K_i for $u_\zeta(\mathfrak{g})$. This motivates our central question, to reiterate: what relationship exists between the Balmer spectra of $\mathsf{st}(\boldsymbol{C})$ and $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$?

We answer this question by the following approach.

In Section 1, we give a brief background on tensor triangular geometry, compactly generated triangulated categories, stable categories and finite tensor categories, support data, and Drinfeld centers, and establish notation.

Next, in Section 2, we consider directly the relationship between the Balmer spectra $\operatorname{Spc}\mathsf{st}(\boldsymbol{C})$ and $\operatorname{Spc}\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. Since the prime ideals of the Balmer spectrum of a nonbraided monoidal triangulated category are a categorical analogue of the prime ideals in a noncommutative ring, we are motivated by prime ideal contraction, that is, the statement that if $\mathfrak{p}$ is a prime ideal of a noncommutative ring R, then $\mathfrak{p} \cap Z(R)$ is a prime ideal in $Z(R)$, the center of R. For general background on prime ideals for noncommutative rings, see [Goodearl and Warfield 2004, Chapter 3]. Finding a categorical analogue to prime ideal contraction is complicated by the fact that we work with $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ rather than $\mathsf{Z}(\mathsf{st}(\boldsymbol{C}))$; the latter is equipped with a forgetful functor $\mathsf{Z}(\mathsf{st}(\boldsymbol{C})) \to \mathsf{st}(\boldsymbol{C})$, but, as noted above, is not necessarily triangulated. Nevertheless, we verify that the forgetful functor $F : \mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$ extends to a functor $\overline{F} : \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{st}(\boldsymbol{C})$.

Reflecting the analogous property for rings, Balmer spectra of braided monoidal triangulated categories are functorial; but in the nonbraided situation, a monoidal triangulated functor does not necessarily induce a continuous map between Balmer spectra. However, we show that $\overline{F}$ does induce a continuous map, and we obtain an analogue of prime ideal contraction. This is summarized by the following:

Theorem A (See Propositions 2.1.2 and 2.1.3). *Let $\boldsymbol{C}$ be a finite tensor category. There exists a monoidal triangulated functor $\overline{F} : \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{st}(\boldsymbol{C})$ extending the forgetful functor $F : \mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$, which induces a continuous map $f : \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, defined by*

$$f : \boldsymbol{P} \mapsto \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{P}\},$$

for $\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C})$.

To study the image of the map f, we utilize the machinery of localization and colocalization functors. To apply these functors, one must work in the setting of compactly generated triangulated categories. For us, the role of compactly generated monoidal triangulated category will be filled by the stable category of the indization of $\boldsymbol{C}$; this category will be referred to as $\mathsf{St}(\boldsymbol{C})$, and it contains $\mathsf{st}(\boldsymbol{C})$ as a triangulated subcategory. For the details of this setting, see Section 1.2. It is straightforward that the functor $\overline{F}$ extends to a functor $\mathsf{St}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{St}(\boldsymbol{C})$; denote this extension again by $\overline{F}$. We are then able to use the kernel of this functor to describe the image of f.

Theorem B (See Proposition 2.4.1). *Denote by* $\boldsymbol{K}$ *the kernel of* $\overline{F} : \mathsf{St}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{St}(\boldsymbol{C})$, *and*

$$f : \operatorname{Spc} \mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$$

the continuous map induced by $\overline{F}$ *as above. Then there are containments*

$$\left\{\boldsymbol{P} \in \operatorname{Spc} \mathsf{st}(\boldsymbol{C}) : \boldsymbol{P} \supseteq \boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\right\} \supseteq \operatorname{im} f \supseteq \left\{\boldsymbol{P} \in \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \operatorname{Loc}(\boldsymbol{P}) \supseteq \boldsymbol{K}\right\}.$$

Here, $\operatorname{Loc}(\boldsymbol{P})$ *refers to the localizing subcategory* (*meaning triangulated and closed under set-indexed coproducts*) *of* $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ *generated by* $\boldsymbol{P}$.

This implies that if $\boldsymbol{C}$ satisfies the following property, then f is surjective:

$$\text{for } X \text{ in } \operatorname{Ind}(\mathsf{Z}(\boldsymbol{C})), \text{ if } F(X) \text{ is projective, then so is } X. \tag{$*$}$$

Additionally, if $\boldsymbol{C}$ is a braided tensor category to begin with, then we prove that f is injective. This leads to the following theorem.

Theorem C (See Theorem 2.5.1). *Let* $\boldsymbol{C}$ *be a finite braided tensor category satisfying property* $(*)$. *Then* f *is a homeomorphism* $\operatorname{Spc} \mathsf{st}(\boldsymbol{C}) \xrightarrow{\cong} \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, *and there is a bijection between the thick ideals of* $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ *and the thick ideals of* $\mathsf{st}(\boldsymbol{C})$, *given by*

$$\boldsymbol{I} \mapsto \langle \overline{F}(X) : X \in \boldsymbol{I} \rangle$$

for a thick ideal $\boldsymbol{I}$ *of* $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.

In Section 3, we illustrate the theory with concrete examples. We first consider $\boldsymbol{C}$ to be alternately $\mathsf{mod}(\Bbbk G)$ and $\mathsf{mod}((\Bbbk[G]^{\mathrm{cop}})$, for G a finite group and $\Bbbk$ an algebraically closed field of characteristic p dividing the order of G, where $\Bbbk G$ denotes the group algebra of G and $\Bbbk[G]$ denotes the dual group algebra to $\Bbbk G$. Of these two examples, the first satisfies property $(*)$ and the second does not. This allows us to classify the Balmer spectrum and classify the thick ideals for $\mathsf{stmod}(D(\Bbbk G))$, where $D(\Bbbk G)$ is the Drinfeld double of the group algebra $\Bbbk G$. We are then able to generalize this example in the following way.

Theorem D (See Propositions 3.1.2, 3.2.5, and Theorem 3.3.4). *For the following classes of Hopf algebras* H, *the Balmer spectrum of* $\mathsf{stmod}(D(H))$ *is homeomorphic via the map* f *to the Balmer spectrum of* $\mathsf{stmod}(H)$, *and the thick ideals of the two categories are in bijection*,

(1) *finite-dimensional cosemisimple quasitriangular Hopf algebras* (*e.g.*, *group algebras of finite groups* G *in characteristic dividing the order of* G);

(2) *Benson–Witherspoon smash coproducts* $(\Bbbk[G]\#\Bbbk L)^*$, *where* G *and* L *are finite groups with dual group algebra and group algebra* $\Bbbk[G]$ *and* $\Bbbk L$ *respectively*, $\Bbbk$ *an algebraically closed field of characteristic* p *dividing the order of* G *and not dividing the order of* L, *such that* L *acts by group automorphisms on* G.

1. Preliminaries

1.1. ***Tensor triangular geometry.*** We will recall some of the background of noncommutative tensor triangular geometry. Following the terminology of [Nakano et al. 2022a; 2022b], a *monoidal triangulated category* $\boldsymbol{T}$ is a category such that the following conditions hold:

(1) $\boldsymbol{T}$ is triangulated: it is an additive category equipped with an additive autoequivalence $\Sigma : \boldsymbol{T} \to \boldsymbol{T}$, called the *shift functor*, and a collection of distinguished triangles

$$A \to B \to C \to \Sigma A,$$

subject to the usual axioms (see [Happel 1988; Neeman 2001]).

(2) $\boldsymbol{T}$ is monoidal: it is equipped with a monoidal product $\otimes$ and unit $\mathbf{1}_{\boldsymbol{T}}$, subject to the usual associativity and unit axioms (see [Kassel 1995; Bakalov and Kirillov 2001; Etingof et al. 2015]).

(3) The triangulated and monoidal structures on $\boldsymbol{T}$ are compatible: for any object A of $\boldsymbol{T}$, the functors $A\otimes -$ and $-\otimes A$ are triangulated functors. In other words, there exists a natural isomorphism $\Sigma(A)\otimes B \cong \Sigma(A\otimes B) \cong A\otimes \Sigma(B)$, such that if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle, then for any object D, the triangles

$$D\otimes A \to D\otimes B \to D\otimes C \to \Sigma(D\otimes A)$$

and

$$A\otimes D \to B\otimes D \to C\otimes D \to \Sigma(A\otimes D)$$

are distinguished.

Remark 1.1.1. In the terminology of [Balmer 2005; 2010], a *tensor triangulated category* is a monoidal triangulated category such that the monoidal product is symmetric. Note that contrary to the definition of tensor category as in [Etingof et al. 2015], a tensor triangulated category is not required to have duals.

We will recall the definition of the Balmer spectrum of a monoidal triangulated category $\boldsymbol{T}$, as in [Buan et al. 2007; Nakano et al. 2022a].

(1) A (*two-sided*) *thick ideal* $\boldsymbol{I}$ of $\boldsymbol{T}$ is a full subcategory such that the following hold.

(a) $\boldsymbol{I}$ is triangulated: it is closed under Σ and Σ^{-1}, and if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle, then if any two of A, B, and C are in $\boldsymbol{I}$, then so is the third.

(b) $\boldsymbol{I}$ is thick: if $A \oplus B$ is in $\boldsymbol{I}$, then so are A and B.

(c) $\boldsymbol{I}$ is an ideal: if $A \in \boldsymbol{I}$, then so are $A \otimes B$ and $B \otimes A$, for any object B.

The collection of thick ideals of $\boldsymbol{T}$ will be denoted by ThickId($\boldsymbol{T}$), and the thick ideal generated by a collection of objects $\mathcal{T}$ will be denoted $\langle \mathcal{T} \rangle$.

(2) A thick ideal $\boldsymbol{P}$ of $\boldsymbol{T}$ is called *prime* if for all thick ideals $\boldsymbol{I}$ and $\boldsymbol{J}$ of $\boldsymbol{T}$, a containment $\boldsymbol{I} \otimes \boldsymbol{J} \subseteq \boldsymbol{P}$ implies either $\boldsymbol{I}$ or $\boldsymbol{J} \subseteq \boldsymbol{P}$; equivalently, $\boldsymbol{P}$ is prime if and only if for all objects A and B of $\boldsymbol{T}$, a containment $A \otimes \boldsymbol{T} \otimes B \subseteq \boldsymbol{P}$ implies either A or B is in $\boldsymbol{P}$ (see [Nakano et al. 2022a, Theorem 3.2.2]). Here $\boldsymbol{I} \otimes \boldsymbol{J}$ refers to the collection of objects $\{A \otimes B : A \in \boldsymbol{I},\ B \in \boldsymbol{J}\}$, and $A \otimes \boldsymbol{T} \otimes B$ refers to the collection of objects $\{A \otimes C \otimes B : C \in \boldsymbol{T}\}$ for A and B in $\boldsymbol{T}$.

(3) A thick ideal $\boldsymbol{P}$ of $\boldsymbol{T}$ is called *completely prime* if $A \otimes B \in \boldsymbol{P}$ implies either A or $B \in \boldsymbol{P}$, for all objects A and B of $\boldsymbol{T}$.

(4) The *Balmer spectrum* of $\boldsymbol{T}$, denoted $\operatorname{Spc} \boldsymbol{T}$, is the collection of prime ideals of $\boldsymbol{T}$ under the Zariski topology, where closed sets are defined as the sets

$$V_{\boldsymbol{T}}(\mathcal{T}) = \{\boldsymbol{P} \in \operatorname{Spc} \boldsymbol{T} : \mathcal{T} \cap \boldsymbol{P} = \varnothing\}$$

for all collections of objects $\mathcal{T}$ of $\boldsymbol{T}$.

(5) An arbitrary open set of $\operatorname{Spc} \boldsymbol{T}$, that is, the complement of a closed set $V_{\boldsymbol{T}}(\mathcal{T})$ for some collection $\mathcal{T}$ of objects of $\boldsymbol{T}$, will be denoted

$$U_{\boldsymbol{T}}(\mathcal{T}) := \operatorname{Spc} \boldsymbol{T} \backslash V_{\boldsymbol{T}}(\mathcal{T}) = \{\boldsymbol{P} \in \operatorname{Spc} \boldsymbol{T} : \mathcal{T} \cap \boldsymbol{P} \neq \varnothing\}.$$

Note that every completely prime ideal is prime. If $\boldsymbol{T}$ is a braided category then every prime ideal is completely prime, and so in that case the two notions coincide.

Remark 1.1.2. We emphasize that this choice of topology on the Balmer spectrum does not match what one might expect, by the analogy to ring theory. This reflects the fact that in natural examples when the Balmer spectrum of a monoidal triangular category $\boldsymbol{T}$ is realized concretely as the Proj or Spec of a commutative ring R, the bijection between prime ideals of $\boldsymbol{T}$ and the (homogeneous) prime ideals of R is containment-reversing. See Example 1.4.2 below for concrete examples.

Remark 1.1.3. While we have only defined the Balmer spectrum as a topological space, Balmer's original definition [2005, Section 6] gives Spc the additional structure of a ringed space (which is actually locally ringed, by [Balmer 2010, Corollary 6.6]). Many of the classification theorems for Balmer spectra prove existence of isomorphisms of ringed spaces, rather than just homeomorphisms of topological spaces. However, the ringed space structures will not play a role in this paper, so we omit the precise definition.

We recall one topological property of the Balmer spectrum, for reference. This was proven by Balmer [2005, Corollary 2.17].

Theorem 1.1.4. *Let $\boldsymbol{T}$ be a braided monoidal triangulated category. Then* $\operatorname{Spc} \boldsymbol{T}$ *is Noetherian if and only if every closed subset of* $\operatorname{Spc} \boldsymbol{T}$ *is of the form $V_{\boldsymbol{T}}(A)$, for some object A of $\boldsymbol{T}$.*

Remark 1.1.5. In fact, if $\boldsymbol{T}$ is braided, or if there is an object of $\boldsymbol{T}$ which generates $\boldsymbol{T}$ as a thick subcategory, then $\operatorname{Spc}\boldsymbol{T}$ is a spectral topological space. In other words, $\operatorname{Spc}\boldsymbol{T}$ is T_0, quasicompact, the quasicompact open sets form an open basis, and every nonempty irreducible closed subset has a generic point. It is a theorem of Hochster that this implies $\operatorname{Spc}\boldsymbol{T}$ is homeomorphic to the prime spectrum of a commutative ring [Hochster 1969, Theorem 6].

Suppose $\boldsymbol{T}$ is rigid, in other words, every object is dualizable (see [Etingof et al. 2015, Section 2.10]). We then obtain the following facts, which we recall for reference. Both follow directly from the fact that if A is dualizable with dual A^*, then A is a direct summand of $A \otimes A^* \otimes A$.

Proposition 1.1.6. *Let $\boldsymbol{T}$ be a rigid monoidal triangulated category. Let A be an object of $\boldsymbol{T}$ with dual A^*. Then*

(1) $\langle A\rangle = \langle A^*\rangle$, *and*

(2) *every thick two-sided ideal $\boldsymbol{I}$ of $\boldsymbol{T}$ is semiprime*, *i.e.*, *it is the intersection*

$$\boldsymbol{I} = \bigcap_{\boldsymbol{P}\in\operatorname{Spc}\boldsymbol{T},\ \boldsymbol{I}\subseteq\boldsymbol{P}} \boldsymbol{P}$$

of prime ideals over itself. Equivalently, for every ideal $\boldsymbol{I}$ of $\boldsymbol{T}$, if the set of objects $A\otimes\boldsymbol{T}\otimes A\subseteq\boldsymbol{I}$ for some object A in $\boldsymbol{T}$, then $A\in\boldsymbol{I}$, where $A\otimes\boldsymbol{T}\otimes A$ refers to the collection $\{A\otimes B\otimes A : B\in\boldsymbol{T}\}$.

For the details of the proofs, see [Nakano et al. 2022a, Lemma 5.1.1; 2022b, Proposition 4.1.1].

1.2. ***Compactly generated triangulated categories.*** A powerful result in the theory of triangulated categories is Brown representability, which ensures the existence of adjoints to certain triangulated functors [Neeman 2001, Chapter 8]. However, in order to apply these results, one must work in the setting of compactly generated triangulated categories. We recall the definition now.

(1) An object C in a triangulated category $\boldsymbol{T}$ is *compact* if the functor $\operatorname{Hom}_{\boldsymbol{T}}(C,-)$ commutes with arbitrary set-indexed coproducts. If $\boldsymbol{T}$ is a triangulated category, then $\boldsymbol{T}^c$ will denote the subcategory of compact objects.

(2) A *localizing subcategory* of a triangulated category is a triangulated subcategory which is also closed under taking set-indexed coproducts. The smallest localizing category containing a collection $\mathcal{T}$ of objects will be denoted $\operatorname{Loc}(\mathcal{T})$ and will be referred to as the *localizing category generated by* $\mathcal{T}$.

(3) A *compactly generated triangulated category* is a triangulated category $\boldsymbol{T}$ which contains arbitrary set-indexed coproducts such that $\operatorname{Loc}(\boldsymbol{T}^c)=\boldsymbol{T}$.

Note that any localizing subcategory $\boldsymbol{I}$ of $\boldsymbol{T}$ is thick by a version of the Eilenberg swindle: if $A\oplus B$ is in $\boldsymbol{I}$, then we have a distinguished triangle

$$(A\oplus B)^{\oplus\mathbb{N}} \to (A\oplus B)^{\oplus\mathbb{N}} \to A \to \Sigma(A\oplus B)^{\oplus\mathbb{N}},$$

where the first map sends the i-th copy of B in the first object to the i-th copy of B in the second object, and sends the i-th copy of A in the first object to the $(i+1)$-th copy of A in the second object. Since $\boldsymbol{I}$ is localizing, the first and second objects are in $\boldsymbol{I}$, and hence A is in $\boldsymbol{I}$ as well. For additional background on compactly generated triangulated categories; see [Benson et al. 2012, Section 1.3.9].

The following theorem, due to Rickard [1997], is the primary technical reason we need to move to the compactly generated setting. For details; see [Boe et al. 2017a, Theorems 3.1.1, 3.1.2; Benson et al. 2008, Section 3; 2012, Section 2].

Theorem 1.2.1. *Let $\boldsymbol{T}$ be a compactly generated triangulated category. Given a thick subcategory $\boldsymbol{S}$ of $\boldsymbol{T}^c$, there exist functors $\Gamma_{\boldsymbol{S}}$ and $L_{\boldsymbol{S}}$ from $\boldsymbol{T} \to \boldsymbol{T}$, which gives for every object M of $\boldsymbol{T}$ a distinguished triangle*

$$\Gamma_{\boldsymbol{S}}(M) \to M \to L_{\boldsymbol{S}}(M) \to \Sigma(\Gamma_{\boldsymbol{S}}(M)),$$

such that

(1) *$L_{\boldsymbol{S}}$ and $\Gamma_{\boldsymbol{S}}$ are unique up to isomorphism,*

(2) *$\Gamma_{\boldsymbol{S}}(M)$ is in* $\mathrm{Loc}(\boldsymbol{S})$,

(3) *$L_{\boldsymbol{S}}(M)$ is in* $\mathrm{Loc}(\boldsymbol{S})^{\perp}$, *that is, there are no nonzero maps from* $\mathrm{Loc}(\boldsymbol{S}) \to L_{\boldsymbol{S}}(M)$, *and*

(4) *$M \in \mathrm{Loc}(\boldsymbol{S})$ if and only if $\Gamma_{\boldsymbol{S}}(M) \cong M$, or, equivalently, $L_{\boldsymbol{S}}(M) \cong 0$.*

The functors $\Gamma_{\boldsymbol{S}}$ and $L_{\boldsymbol{S}}$ are called *colocalizing* and *localizing* functors, respectively. They are constructed by first taking a Verdier quotient of $\boldsymbol{T}$ by $\mathrm{Loc}(\boldsymbol{S})$, that is, forming a category where all morphisms with cones in $\mathrm{Loc}(\boldsymbol{S})$ are formally inverted, which one may do using the calculus of roofs. This quotient is a triangulated category where the objects isomorphic to 0 are precisely those from $\mathrm{Loc}(\boldsymbol{S})$, and Brown representability guarantees that there are right adjoint functors $i^!$ and j_* to the inclusion $i_* : \mathrm{Loc}(\boldsymbol{S}) \to \boldsymbol{T}$ and quotient $j^* : \boldsymbol{T} \to \boldsymbol{T} / \mathrm{Loc}(\boldsymbol{S})$ functors, giving a diagram

$$\mathrm{Loc}(\boldsymbol{S}) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \boldsymbol{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \boldsymbol{T} / \mathrm{Loc}(\boldsymbol{S}).$$

The functor $L_{\boldsymbol{S}}$ is then defined as $j_* \circ j^*$, and $\Gamma_{\boldsymbol{S}}$ is defined as $i_* \circ i^!$. For the details of the categorical localization and Verdier quotient, as well as additional details on the formation of the localization and colocalization functors, see [Neeman 2001, Section 2.1, Theorem 8.4.4; Krause 2010; Stevenson 2018, Section 3].

1.3. *Stable categories and finite tensor categories.* The monoidal triangulated categories that are the primary focus of this paper arise as stable categories. We first recall the construction of the stable category of any quasi-Frobenius category. Recall that a quasi-Frobenius category is an abelian category with enough projectives, such that projective and injective objects coincide. For any quasi-Frobenius category $\boldsymbol{C}$, one may form the stable category $\mathrm{st}(\boldsymbol{C})$, which is triangulated (see [Happel 1988, Chapter I]). The stable category is constructed by factoring out the projective objects of $\boldsymbol{C}$. In more detail, let $\mathrm{PHom}_{\boldsymbol{C}}(A, B)$ consist of the morphisms $f : A \to B$ in $\boldsymbol{C}$ such that f factors through a projective object. The stable category $\mathrm{st}(\boldsymbol{C})$ is the category where

(1) objects are the same as the objects of $\boldsymbol{C}$;

(2) morphisms $A \to B$ are defined to be $\mathrm{Hom}_{\boldsymbol{C}}(A, B)/\mathrm{PHom}_{\boldsymbol{C}}(A, B)$.

There is a straightforward functor $G : \boldsymbol{C} \to \mathrm{st}(\boldsymbol{C})$ sending objects to themselves and morphisms to their image in the quotient.

If P is a projective object of $\boldsymbol{C}$, note that the corresponding object $G(P)$ in $\mathrm{st}(\boldsymbol{C})$ is isomorphic to 0, since id_P factors through a projective; and the converse is also true, since $G(P) \cong 0$ in $\mathrm{st}(\boldsymbol{C})$ implies that the 0-morphism $G(P) \to G(P)$ is equal to $\mathrm{id}_{G(P)}$ in $\mathrm{Hom}_{\mathrm{st}(\boldsymbol{C})}(G(P), G(P))$, in other words, id_P factors through a projective Q in $\boldsymbol{C}$:

$$\begin{tikzcd} P \arrow[rr, "\mathrm{id}_P"] \arrow[dr, hook] & & P \\ & Q \arrow[ur, two heads] & \end{tikzcd}$$

Of course, this implies P is a summand of Q, and so P is projective.

We recall the triangulated structure on $\mathrm{st}(\boldsymbol{C})$, for reference. If A is an object of $\boldsymbol{C}$, denote by $\Omega(A)$ the kernel of the projective cover of A. The functor Ω extends to the stable module category, and this in fact gives an autoequivalence on $\mathrm{st}(\boldsymbol{C})$. The shift Σ is then defined to be $\Sigma(A) = \Omega^{-1}(A)$. For any short exact sequence of objects in $\boldsymbol{C}$, say

$$0 \to A \to B \to C \to 0,$$

there exists a triangle

$$A \to B \to C \to \Sigma A$$

in $\mathrm{st}(\boldsymbol{C})$; the distinguished triangles of $\mathrm{st}(\boldsymbol{C})$ are then defined to be all triangles which are isomorphic to triangles of this form.

We now specialize to the case that $\boldsymbol{C}$ is a finite tensor category. Recall that a finite tensor category (following the notation given in [Etingof and Ostrik 2004; Etingof et al. 2015]) consists of a monoidal category $\boldsymbol{C}$ such that

(1) $\boldsymbol{C}$ is abelian and $\Bbbk$-linear for an algebraically closed field $\Bbbk$;

(2) the tensor product $- \otimes -$ is bilinear on spaces of morphisms;

(3) every object of $\boldsymbol{C}$ has finite length;

(4) $\mathrm{Hom}_{\boldsymbol{C}}(\mathbf{1}, \mathbf{1}) \cong \Bbbk$;

(5) for any pair of objects A and B, the vector space $\mathrm{Hom}_{\boldsymbol{C}}(A, B)$ is finite-dimensional over $\Bbbk$;

(6) $\boldsymbol{C}$ has enough projectives;

(7) there are finitely many isomorphism classes of simple objects of $\boldsymbol{C}$;

(8) $\boldsymbol{C}$ is rigid, i.e., every object has a left and a right dual.

The prototypical example of a finite tensor category is the category of finite-dimensional modules of a finite-dimensional Hopf algebra H.

Notation 1.3.1. Denote the category of finite-dimensional modules of an algebra H by $\mathsf{mod}(H)$. Denote the category of all (not necessarily finite-dimensional) modules of H by $\mathsf{Mod}(H)$.

Recall that if $\boldsymbol{C}$ is a finite tensor category, it is a consequence that the tensor product is biexact [Etingof et al. 2015, Proposition 4.2.1]. Additionally, every finite tensor category is quasi-Frobenius [Etingof et al. 2015, Proposition 6.1.3]. The stable category $\mathsf{st}(\boldsymbol{C})$ inherits a monoidal product directly from $\boldsymbol{C}$: we define $G(A) \otimes G(B) := G(A \otimes B)$, and similarly for morphisms $f : A \to B$ and $g : C \to D$ we define $G(f) \otimes G(g) := G(f \otimes g)$. This is well-defined: if $G(f) = G(\hat{f})$, then $f - \hat{f}$ factors through a projective P, and then $f \otimes g - \hat{f} \otimes g$ factors through $P \otimes D$, which is projective by projectivity of P (see [Etingof et al. 2015, Proposition 4.2.12]).

Although the primary objects of focus in this paper are stable categories $\mathsf{st}(\boldsymbol{C})$ for finite tensor categories $\boldsymbol{C}$, note that $\mathsf{st}(\boldsymbol{C})$ is *not* compactly generated, since in particular it does not contain arbitrary set-indexed coproducts. Thus, in order to apply Theorem 1.2.1, it is necessary to produce a compactly generated monoidal triangulated category which contains $\mathsf{st}(\boldsymbol{C})$ as a monoidal triangulated subcategory. In fact, this is possible, using the Ind-completion (see [Kashiwara and Schapira 2006, Chapter 6]) of $\boldsymbol{C}$:

Theorem 1.3.2. *Let $\boldsymbol{C}$ be a finite tensor category. Then its* Ind-*completion* $\mathrm{Ind}(\boldsymbol{C})$ *is a quasi-Frobenius abelian monoidal category, and its stable category* $\mathsf{st}(\mathrm{Ind}(\boldsymbol{C}))$ *is a compactly generated monoidal triangulated category, with* $\mathsf{st}(\mathrm{Ind}(\boldsymbol{C}))^c \cong \mathsf{st}(\boldsymbol{C})$ *via the stabilization of the natural inclusion functor* $\boldsymbol{C} \to \mathrm{Ind}(\boldsymbol{C})$.

Proof. See [Nakano et al. 2023, Theorem A.0.1]. □

Concretely, the there exists a finite-dimensional algebra A such that $\boldsymbol{C} \cong \mathsf{mod}(A)$, the category of finite-dimensional A-modules [Etingof et al. 2015, page 10]. Then $\mathrm{Ind}(\boldsymbol{C}) \cong \mathsf{Mod}(A)$, the category of all A-modules.

Notation 1.3.3. If $\boldsymbol{C}$ is a finite tensor category, we denote

$$\mathsf{St}(\boldsymbol{C}) := \mathsf{st}(\mathrm{Ind}(\boldsymbol{C})),$$

to avoid crowding the notation.

1.4. *Support data.* Suppose that $\boldsymbol{T}$ is a monoidal triangulated category and S a topological space. We will denote the collection of subsets of S by $\mathcal{X}(S)$, closed subsets of S by $\mathcal{X}_{\mathrm{cl}}(S)$, and specialization-closed subsets of S by $\mathcal{X}_{\mathrm{sp}}(S)$; recall that by definition, a set is specialization-closed if it is a union of closed sets. When the underlying space is clear from context, we will denote these collections by $\mathcal{X}$, $\mathcal{X}_{\mathrm{cl}}$, and $\mathcal{X}_{\mathrm{sp}}$.

Given a monoidal triangulated category $\boldsymbol{T}$ and a topological space S, a *support datum on $\boldsymbol{T}$ with value in S* is a map $\sigma : \boldsymbol{T} \to \mathcal{X}_{\mathrm{cl}}(S)$ satisfying the following axioms:

(1) $\sigma(0) = \varnothing$ and $\sigma(\mathbf{1}) = S$;

(2) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, for all $A, B \in \boldsymbol{T}$;

(3) $\sigma(\Sigma A) = \sigma(A)$, for all $A \in \boldsymbol{T}$;

(4) if $A \to B \to C \to \Sigma A$ is a distinguished triangle, then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$;

(5) $\bigcup_{C\in \boldsymbol{T}} \sigma(A\otimes C\otimes B)=\sigma(A)\cap\sigma(B)$, for all $A, B\in \boldsymbol{T}$.

See [Nakano et al. 2022a, Section 4] for a more in-depth discussion of support data (although note that in that paper, a support datum is permitted to take value in $\mathcal{X}(S)$ rather than $\mathcal{X}_{\mathrm{cl}}(S)$). For any monoidal triangulated category $\boldsymbol{T}$, the map $V_{\boldsymbol{T}}(A)=\{\boldsymbol{P}\in \operatorname{Spc}\boldsymbol{T} : A\notin \boldsymbol{P}\}$ defined above is a support datum $\boldsymbol{T}\to\mathcal{X}_{\mathrm{cl}}(\operatorname{Spc}\boldsymbol{T})$, since by definition, $V_{\boldsymbol{T}}(A)$ is a closed set in $\operatorname{Spc}\boldsymbol{T}$. We will refer to this support datum as the *Balmer support*. Indeed, the Balmer support satisfies a universal property in the category of support data, see [Nakano et al. 2022a, Theorem 4.2.2].

Theorem 1.4.1. *If $\sigma:\boldsymbol{T}\to\mathcal{X}_{\mathrm{cl}}(S)$ is a support datum with value in S, then there exists a unique continuous map*

$$S\xrightarrow{f}\operatorname{Spc}\boldsymbol{T}$$

such that $\sigma(A)=f^{-1}(V(A))$ for all $A\in\boldsymbol{T}$.

For any support datum σ, we have a map

$$\Phi_\sigma(\mathcal{T}):=\bigcup_{A\in\mathcal{T}}\sigma(A),$$

where $\mathcal{T}$ is any collection of objects of $\boldsymbol{T}$. If σ takes values in $\mathcal{X}_{\mathrm{cl}}$, then by definition Φ_σ takes values in $\mathcal{X}_{\mathrm{sp}}$. The map Φ_σ in fact only depends on thick ideals rather than arbitrary subsets, since by [Nakano et al. 2022a, Lemma 4.3.2] we have $\Phi_\sigma(\mathcal{T})=\Phi_\sigma(\langle\mathcal{T}\rangle)$.

For a support datum σ, we have a second map $\Theta_\sigma:\mathcal{X}_{\mathrm{sp}}\to\operatorname{ThickId}(\boldsymbol{T})$ defined by

$$\Theta_\sigma(S'):=\{A\in\boldsymbol{T}:\sigma(A)\subseteq S'\}$$

for any specialization-closed subset S' of S. For any specialization closed set S', the collection $\Theta_\sigma(S')$ is a thick ideal of $\boldsymbol{T}$. Hence, we have the following collection of maps, given a support datum σ on $\boldsymbol{T}$:

$$\operatorname{ThickId}(\boldsymbol{T})\underset{\Theta_\sigma}{\overset{\Phi_\sigma}{\rightleftarrows}}\mathcal{X}_{\mathrm{sp}}.$$

Classifications of thick ideals are obtained in many cases (see [Balmer 2005; 2010; Boe et al. 2017a; 2017b; Nakano et al. 2022a; 2022b] for examples) by constructing a support datum for which these maps are bijective and inverse to each other. In that case, the support datum σ is called *classifying*. For rigid braided monoidal triangulated categories $\boldsymbol{T}$, the Balmer support $V_{\boldsymbol{T}}$ is always classifying [Balmer 2005, Theorem 4.10].

Example 1.4.2. For a finite group scheme G, the *cohomological support* is the map

$$\mathsf{stmod}(G)\to\mathcal{X}_{\mathrm{cl}}(\operatorname{Proj}\mathrm{H}^\bullet(G,\Bbbk))$$

defined by

$$M\mapsto\{\mathfrak{p}\in\operatorname{Proj}\mathrm{H}^\bullet(G,\Bbbk):\mathfrak{p}\text{ contains }I(M)\},$$

where $I(M)$ is the annihilator of $\bigoplus_{i\geq0}\operatorname{Ext}^i_G(M,M)$ in $\mathrm{H}^\bullet(G,\Bbbk):=\bigoplus_{i\geq0}\operatorname{Ext}^i_G(\mathbf{1},\mathbf{1})$ under the action induced by the functor $M\otimes-$ [Benson 1998, Section 5.7]. Cohomological support is a support datum; the most nontrivial property is (5), referred to as the *tensor product property*, and was proven by Friedlander and Pevtsova [2007]. It is a theorem that for finite group schemes, the cohomological support is classifying, and the map $f : \operatorname{Proj} H^\bullet(G, \Bbbk) \to \operatorname{Spc} \mathsf{stmod}(G)$ is a homeomorphism [Benson et al. 1997; Balmer 2005; Friedlander and Pevtsova 2007]. Cohomological support exists for arbitrary finite tensor categories [Bergh et al. 2021], but is not known to be classifying in general, see [Nakano et al. 2023, Conjecture E].

1.5. ***The Drinfeld center.*** Let $\boldsymbol{C}$ be a strict monoidal category. Then the *Drinfeld center* or *center* of $\boldsymbol{C}$, which we will denote by $\mathsf{Z}(\boldsymbol{C})$, is defined as the following braided monoidal category.

(1) Objects are pairs (A, γ) where A is an object of $\boldsymbol{C}$ and γ is a natural isomorphism $\gamma_B : B \otimes A \xrightarrow{\cong} A \otimes B$ for all $B \in \boldsymbol{C}$, satisfying the diagram

$$\begin{tikzcd} B \otimes C \otimes A \arrow[r, "\mathrm{id}_B \otimes \gamma_C"] \arrow[rr, bend left, "\gamma_{B\otimes C}"] & B \otimes A \otimes C \arrow[r, "\gamma_B \otimes \mathrm{id}_C"] & A \otimes B \otimes C \end{tikzcd}$$

for all B and C. Such a natural isomorphism γ is called a *half-braiding* of A.

(2) Morphisms $(A, \gamma) \to (A', \gamma')$ are morphisms $f : A \to A'$ such that for all B, the diagram

$$\begin{tikzcd} B \otimes A \arrow[r, "\mathrm{id}_B \otimes f"] \arrow[d, "\gamma_B"] & B \otimes A' \arrow[d, "\gamma'_B"] \\ A \otimes B \arrow[r, "f \otimes \mathrm{id}_B"] & A' \otimes B \end{tikzcd}$$

commutes.

(3) The monoidal product $(A, \gamma) \otimes (A', \gamma')$ is defined as $(A \otimes A', \tilde{\gamma})$ where $\tilde{\gamma}$ is defined as

$$\begin{tikzcd} B \otimes A \otimes A' \arrow[r, "\gamma_B \otimes \mathrm{id}_{A'}"] \arrow[d, "\tilde{\gamma}_B"] & A \otimes B \otimes A' \arrow[dl, "\mathrm{id}_A \otimes \gamma'_B"] \\ A \otimes A' \otimes B & \end{tikzcd}$$

(4) The braiding $c_{(A,\gamma),(A',\gamma')} : (A, \gamma) \otimes (A', \gamma') \xrightarrow{\cong} (A', \gamma') \otimes (A, \gamma)$ is defined as γ'_A. The map γ'_A being a valid map in $\mathsf{Z}(\boldsymbol{C})$ amounts to checking the commutativity of the diagram

$$\begin{tikzcd} B \otimes A \otimes A' \arrow[r, "\mathrm{id}_B \otimes \gamma'_A"] \arrow[d, "\gamma_B \otimes \mathrm{id}_{A'}"] & B \otimes A' \otimes A \arrow[d, "\gamma'_B \otimes \mathrm{id}_A"] \\ A \otimes B \otimes A' \arrow[d, "\mathrm{id}_A \otimes \gamma'_B"] & A' \otimes B \otimes A \arrow[d, "\mathrm{id}_{A'} \otimes \gamma_B"] \\ A \otimes A' \otimes B \arrow[r, "\gamma'_A \otimes \mathrm{id}_B"] & A' \otimes A \otimes B \end{tikzcd}$$

This diagram commutes by the naturality of γ', since it can be rewritten, using the defining diagram for γ', as

$$\begin{array}{ccc} B \otimes A \otimes A' & \xrightarrow{\gamma'_{B\otimes A}} & A' \otimes B \otimes A \\ \downarrow{\scriptstyle \gamma_B \otimes \mathrm{id}_{A'}} & & \downarrow{\scriptstyle \mathrm{id}_{A'} \otimes \gamma_B} \\ A \otimes B \otimes A' & \xrightarrow{\gamma'_{A\otimes B}} & A' \otimes A \otimes B \end{array}$$

We will denote by $F : \mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$ the forgetful functor sending $(A, \gamma) \mapsto A$.

If H is a Hopf algebra and $\boldsymbol{C}$ is the category of H-modules, it is well-known that the Drinfeld center $\mathsf{Z}(\boldsymbol{C})$ of $\boldsymbol{C}$ is equivalent to the category of modules of $D(H)$ the Drinfeld (or quantum) double of H. For the details of Drinfeld doubles, see [Montgomery 1993, Section 10.3], [Chari and Pressley 1994, Section 4.2.D], [Kassel 1995, Section IX.4], or [Etingof et al. 2015, Section 7.14]. The Drinfeld double $D(H)$ is isomorphic as a vector space to $(H^{\mathrm{op}})^* \otimes H$, and contains both H and $(H^{\mathrm{op}})^*$ as Hopf subalgebras. Here if H is a Hopf algebra with multiplication μ, unit η, comultiplication Δ, counit ϵ, and antipode S, then $(H^{\mathrm{op}})^*$ is the Hopf algebra with multiplication Δ^*, unit ϵ^*, comultiplication $(\mu^{\mathrm{op}})^*$, counit η^*, and antipode $(S^{-1})^*$.

The following result of Etingof and Ostrik [2004] will be important in extending the forgetful functor $\mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$ to the stable categories.

Proposition 1.5.1. *If $\boldsymbol{C}$ is a finite tensor category, then its Drinfeld center $\mathsf{Z}(\boldsymbol{C})$ is a finite tensor category, and the forgetful functor F is exact and sends projective objects to projective objects.*

The fact that F preserves projectivity is a generalization of the classical Nichols–Zoeller theorem [1989] for Hopf algebras, which states that a finite-dimensional Hopf algebra is free as a module over any Hopf subalgebra.

2. Drinfeld Centers and Balmer Spectra

In this section, we prove general results relating the Balmer spectrum of $\mathsf{st}(\boldsymbol{C})$ to the Balmer spectrum of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, under the assumption that $\boldsymbol{C}$ is an arbitrary finite tensor category.

2.1. ***Construction of a continuous map between Balmer spectra.*** Recall the stable categories defined in Section 1.3. For the rest of this section, let $\boldsymbol{C}$ be a finite tensor category, $\mathsf{st}(\boldsymbol{C})$ its stable category, $\mathsf{Z}(\boldsymbol{C})$ its Drinfeld center, $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ the stable category of its Drinfeld center (which may be formed by Proposition 1.5.1), and $\mathsf{St}(\boldsymbol{C})$ and $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ the respective "big" stable categories, recall Notation 1.3.3. We have a forgetful functor $F : \mathsf{Z}(\boldsymbol{C}) \to \boldsymbol{C}$, and we have functors $G : \boldsymbol{C} \to \mathsf{st}(\boldsymbol{C})$ and $H : \mathsf{Z}(\boldsymbol{C}) \to \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. The functor F extends to a functor $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C})) \to \mathrm{Ind}(\boldsymbol{C})$, which we again denote by F, by [Kashiwara and Schapira 2006, Proposition 6.1.9]. We have the respective Balmer support data associated to $\mathsf{st}(\boldsymbol{C})$ and $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$,

$$V_{\mathsf{st}\,\boldsymbol{C}} : \mathsf{st}(\boldsymbol{C}) \to \mathcal{X}_{\mathrm{cl}}(\mathrm{Spc}\,\mathsf{st}(\boldsymbol{C}))$$

and

$$V_{\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))} : \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \to \mathcal{X}_{\mathrm{cl}}\big(\mathrm{Spc}\,\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big),$$

defined in their respective categories by sending

$$A \mapsto \{\text{primes not containing } A\}.$$

Notation 2.1.1. For readability, when $\boldsymbol{C}$ is a finite tensor category we will denote $V_{\boldsymbol{C}} := V_{\mathsf{st}\,\boldsymbol{C}}$ and $V_{\mathsf{Z}} := V_{\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}$. The corresponding maps Φ (recalling the construction from Section 1.4) associated to these support data will similarly be denoted $\Phi_{\boldsymbol{C}}$ and Φ_{Z}, respectively. We will similarly denote open sets in the Balmer spectrum on these respective categories by $U_{\boldsymbol{C}} := U_{\mathsf{st}(\boldsymbol{C})}$ and $U_{\mathsf{Z}} := U_{\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}$, recall the notation from Section 1.1.

The following proposition is probably well-known to experts, but we record it for completeness.

Proposition 2.1.2. *There is a functor* $\overline{F} : \mathsf{St}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{St}(\boldsymbol{C})$ *which extends the forgetful functor* F*, i.e., the diagram of functors*

$$\begin{tikzcd} \mathrm{Ind}(\boldsymbol{C}) \arrow[r, "G"] & \mathsf{St}(\boldsymbol{C}) \\ \mathrm{Ind}(\mathsf{Z}(\boldsymbol{C})) \arrow[u, "F"] \arrow[r, "H"] & \mathsf{St}(\mathsf{Z}(\boldsymbol{C})) \arrow[u, dashed, "\overline{F}"'] \end{tikzcd}$$

commutes. This functor $\overline{F}$ *is monoidal and triangulated.*

Proof. Since the objects of $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ are the in bijection with those of $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$, the functor $\overline{F}$ is well-defined on objects, namely by defining

$$\overline{F}(H(X)) := G(F(X)).$$

Let $f : X \to Y$ be a morphism in $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$. Then for $\overline{F}(H(f)) := GF(f)$ to be well-defined, we need $GF(g) = 0$ for each morphism g which factors through a projective in $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$. In other words, we need $F(g)$ to factor through a projective in $\mathrm{Ind}(\boldsymbol{C})$. Hence, to define $\overline{F}$, it is enough to know that $G \circ F$ sends all projective objects of $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$ to 0, which is true by Proposition 1.5.1.

Let $H(X) \in \mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ an arbitrary object, where $X \in \mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$. Then $\Sigma H(X)$ is defined as $H(Z)$, such that there exists a short exact sequence

$$0 \to X \to P \to Z \to 0$$

in $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$, where P is a projective object in $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$. The object $\Sigma H(Z)$ is well-defined in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$, by Schanuel's lemma. Since F is exact and sends projectives to projectives,

$$0 \to F(X) \to F(P) \to F(Z) \to 0$$

is an exact sequence in $\boldsymbol{C}$ with $F(P)$ projective; therefore, $\Sigma(GF(X)) \cong GF(Z)$ in $\mathsf{st}(\boldsymbol{C})$, and so we have $\overline{F}(\Sigma X) \cong \Sigma \overline{F}(X)$.

Now, let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$. Then it is isomorphic to a triangle of the form

$$H(X') \to H(Y') \to H(Z') \to \Sigma H(X')$$

for some short exact sequence

$$0 \to X' \to Y' \to Z' \to 0$$

in $\mathrm{Ind}(\mathsf{Z}(\boldsymbol{C}))$. Since F is exact, and G sends exact sequences to triangles, we have that the composition GF is exact and hence

$$\overline{F}H(X') \to \overline{F}H(Y') \to \overline{F}H(Z') \to \Sigma\overline{F}H(X')$$

is a triangle in $\mathsf{St}(\boldsymbol{C})$. Therefore,

$$\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma\overline{F}(X)$$

is a triangle as well, and so $\overline{F}$ is a triangulated functor. $\square$

For braided tensor triangulated categories, the Balmer spectrum Spc is functorial, as Balmer [2005, Proposition 3.6] has shown. This is a categorical reflection the ring-theoretic fact that Spec is functorial for commutative rings. On the other hand, for noncommutative rings, Spec is not a functor (for an in-depth exploration of the extent of the failure of functoriality of Spec for noncommutative rings, see [Reyes 2012]). It is not surprising, then, that for generic monoidal triangulated categories, the Balmer spectrum is also not functorial; in other words, a monoidal triangulated functor between monoidal triangulated categories does not necessarily induce a map between their Balmer spectra.

However, reflecting the classical prime ideal contraction for noncommutative rings, the forgetful functor $\overline{F}$ does induce a map between the Balmer spectra of $\mathsf{st}(\boldsymbol{C})$ and $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.

Proposition 2.1.3. *The functor $\overline{F}$ induces a continuous map* $\operatorname{Spc}\mathsf{st}(\boldsymbol{C}) \xrightarrow{f} \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, *defined by*

$$f(\boldsymbol{P}) := \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{P}\}.$$

Proof. Let $\boldsymbol{P}$ be a prime ideal of $\mathsf{st}(\boldsymbol{C})$. We must first show that $f(\boldsymbol{P})$ is a prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.

We first check that $f(\boldsymbol{P})$ is a thick ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. This necessitates checking four properties:

Triangulated. Suppose $\Sigma X \in f(\boldsymbol{P})$, in other words, $\overline{F}(\Sigma X) \in \boldsymbol{P}$. Since $\overline{F}$ is triangulated, this is true if and only if $\Sigma\overline{F}(X) \in \boldsymbol{P}$, which is true if and only if $\overline{F}(X) \in \boldsymbol{P}$, in other words, $X \in f(\boldsymbol{P})$. Now, suppose

$$X \to Y \to Z \to \Sigma X$$

is a distinguished triangle with X and Y in $f(\boldsymbol{P})$. This means that $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\boldsymbol{P}$. Since $\overline{F}$ is triangulated, the triangle

$$\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma\overline{F}(X)$$

is distinguished in $\mathsf{st}(\boldsymbol{C})$. Now since the first two objects are in $\boldsymbol{P}$, so is $\overline{F}(Z)$, and so $Z \in f(\boldsymbol{P})$.

Thick. If $X \oplus Y$ is in $f(\boldsymbol{P})$, then $\overline{F}(X \oplus Y) \in \boldsymbol{P}$; $\overline{F}$ is an additive functor, and so $\overline{F}(X) \oplus \overline{F}(Y) \in \boldsymbol{P}$. This implies that both $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\boldsymbol{P}$, and so X and Y are both in $f(\boldsymbol{P})$.

Ideal. Suppose $X \in f(\boldsymbol{P})$ and $Y \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. Since $\overline{F}$ is monoidal, we have $\overline{F}(X \otimes Y) \cong \overline{F}(X) \otimes \overline{F}(Y)$. Since $\overline{F}(X) \in \boldsymbol{P}$, so is $\overline{F}(X) \otimes \overline{F}(Y)$, and thus $\overline{F}(X \otimes Y) \in \boldsymbol{P}$ as well. Hence $X \otimes Y \in f(\boldsymbol{P})$. The symmetric argument shows that $Y \otimes X$ is in $f(\boldsymbol{P})$ as well, so $f(\boldsymbol{P})$ is a two-sided ideal.

Prime. Let $A \otimes B \in f(\boldsymbol{P})$. Then $\overline{F}(A) \otimes \overline{F}(B) \in \boldsymbol{P}$. But $\overline{F}(A)$ and $\overline{F}(B)$ commute with every object of $\mathsf{st}(\boldsymbol{C})$: by the ideal property of $\boldsymbol{P}$, we have

$$\mathsf{st}(\boldsymbol{C}) \otimes \overline{F}(A) \otimes \overline{F}(B) \subseteq \boldsymbol{P} \Rightarrow \overline{F}(A) \otimes \mathsf{st}(\boldsymbol{C}) \otimes \overline{F}(B) \subseteq \boldsymbol{P} \Rightarrow \overline{F}(A) \text{ or } \overline{F}(B) \in \boldsymbol{P},$$

with the last step following by primeness of $\boldsymbol{P}$. This implies that either A or B is in $f(\boldsymbol{P})$, which means that $f(\boldsymbol{P})$ is prime.

We can also check directly that f is continuous: an arbitrary closed set of $\operatorname{Spc}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$ is of the form $V_{\mathsf{Z}}(\mathcal{T}) = \{\boldsymbol{P} \in \operatorname{Spc}(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))) : \mathcal{T} \cap \boldsymbol{P} = \varnothing\}$ (recalling Notation 2.1.1) for some collection of objects $\mathcal{T}$ of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. Then

$$\begin{aligned} f^{-1}(V_{\mathsf{Z}}(\mathcal{T})) &= \{\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) : \mathcal{T} \cap \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{P}\} = \varnothing\} \\ &= \{\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) : \overline{F}(\mathcal{T}) \cap \boldsymbol{P} = \varnothing\} \\ &= V_{\boldsymbol{C}}(\overline{F}(\mathcal{T})), \end{aligned}$$

where by $\overline{F}(\mathcal{T})$ we mean the collection $\{\overline{F}(X) : X \in \mathcal{T}\}$. □

Remark 2.1.4. Recall the construction of the Drinfeld double from Section 1.5. If R is a finite-dimensional Hopf algebra, then $\mathsf{Z}(\mathsf{mod}(R)) \cong \mathsf{mod}(D(R))$. In this case, $D(R) \cong D((R^{\mathrm{op}})^*)$, and so we have two functors,

$$\begin{tikzcd} \mathsf{mod}(R) & & \mathsf{mod}((R^{\mathrm{op}})^*) \\ & \mathsf{Z}(\mathsf{mod}(R)) \cong \mathsf{mod}(D(R)) \cong \mathsf{Z}\big(\mathsf{mod}((R^{\mathrm{op}})^*)\big) \arrow[ul, "F_R"] \arrow[ur, "F_{(R^{\mathrm{op}})^*}"'] & \end{tikzcd}$$

which then give two maps between Balmer spectra,

$$\begin{tikzcd} \operatorname{Spc}(\mathsf{stmod}(R)) \arrow[dr, "f_R"'] & & \operatorname{Spc}\big(\mathsf{stmod}((R^{\mathrm{op}})^*)\big) \arrow[dl, "f_{(R^{\mathrm{op}})^*}"] \\ & \operatorname{Spc}\big(\mathsf{stmod}(D(R))\big) & \end{tikzcd}$$

2.2. ***A support data interpretation.*** We can interpret the map f in the context of support data (recalling the definition from Section 1.4), by first defining a new support datum given as the composition of the functor $\overline{F}$ with the Balmer support $V_{\boldsymbol{C}}$ on $\mathsf{st}(\boldsymbol{C})$.

Proposition 2.2.1. *Define a map $W : \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \to \mathcal{X}_{\mathrm{cl}}(\operatorname{Spc}\mathsf{st}(\boldsymbol{C}))$ by*

$$W(X) := V_{\boldsymbol{C}}(\overline{F}(X)) = \{\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) : \overline{F}(X) \notin \boldsymbol{P}\}.$$

This map is a support datum.

Proof. The first four conditions follow directly from the facts that $\overline{F}$ is a triangulated functor and $V_{\boldsymbol{C}}$ is itself a support datum, since

(1) $\overline{F}(0_{\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}) = 0_{\mathsf{st}(\boldsymbol{C})}$,

(2) $\overline{F}(X \oplus Y) = \overline{F}(X) \oplus \overline{F}(Y)$,

(3) $\overline{F}(\Sigma X) \cong \Sigma \overline{F}(X)$,

(4) and if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle, then so is $\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma \overline{F}(X)$.

To check the last condition, we need to show that

$$\bigcup_{Z \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))} W(X \otimes Z \otimes Y) = W(X) \cap W(Y).$$

By the ideal condition, if $\boldsymbol{P}$ is a prime ideal which does not contain $\overline{F}(X) \otimes \overline{F}(Z) \otimes \overline{F}(Y)$ for some object Z, then it must also not contain $\overline{F}(X)$ or $\overline{F}(Y)$. Hence,

$$\bigcup_{Z \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))} W(X \otimes Z \otimes Y) \subseteq W(X) \cap W(Y)$$

is automatic.

For the reverse containment, suppose $\boldsymbol{P}$ is a prime ideal which does not contain $\overline{F}(X)$ or $\overline{F}(Y)$. By the prime condition, that means $\boldsymbol{P}$ does not contain the entire collection of objects $\overline{F}(X) \otimes \mathsf{st}(\boldsymbol{C}) \otimes \overline{F}(Y)$. But since $\overline{F}(X)$ and $\overline{F}(Y)$ commute up to isomorphism with all elements of $\mathsf{st}(\boldsymbol{C})$, if $\overline{F}(X) \otimes \overline{F}(Y) \in \boldsymbol{P}$, that would imply there is a containment $\overline{F}(X) \otimes \overline{F}(Y) \otimes \mathsf{st}(\boldsymbol{C}) \subseteq \boldsymbol{P}$, which would then imply

$$\overline{F}(X) \otimes \mathsf{st}(\boldsymbol{C}) \otimes \overline{F}(Y) \subseteq \boldsymbol{P},$$

a contradiction. Hence, $\boldsymbol{P} \in W(X \otimes Y)$, and we have the claimed equality. □

By the universal property of the Balmer spectrum as in Theorem 1.4.1, the support datum W induces a continuous map $\operatorname{Spc} \mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. This map is defined as

$$\boldsymbol{P} \mapsto \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \boldsymbol{P} \notin W(X)\}$$

by [Nakano et al. 2022a, Theorem 4.2.2]. One may observe that this map is the same as the map defined in Proposition 2.1.3. We have the following diagram, which commutes by definition:

$$\begin{tikzcd}
\mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \arrow[rr, "W"] \arrow[dr, "\overline{F}"'] & & \mathcal{X}_{\mathrm{cl}}\big(\operatorname{Spc}(\mathsf{st}(\boldsymbol{C}))\big) \\
& \mathsf{st}(\boldsymbol{C}) \arrow[ur, "V_{\boldsymbol{C}}"'] &
\end{tikzcd}$$

On the level of ideals, we have the following induced maps, recall Φ and Θ associated to a support datum as constructed in Section 1.4:

$$\begin{tikzcd}
\mathrm{ThickId}(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))) \arrow[rr, "\Phi_W"] \arrow[dr, "\Psi"] & & \mathcal{X}_{\mathrm{sp}}(\mathrm{Spc}(\mathsf{st}(\boldsymbol{C}))) \arrow[ll, bend right, "\Theta_W"'] \arrow[dl, bend left, "\Theta_{\boldsymbol{C}}"] \\
& \mathrm{ThickId}(\mathsf{st}(\boldsymbol{C})) \arrow[ur, "\Phi_{\boldsymbol{C}}"'] \arrow[ul, bend left, "\Lambda"] &
\end{tikzcd}$$

Here, for thick ideals $\boldsymbol{I}$ of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ and $\boldsymbol{J}$ of $\mathsf{st}(\boldsymbol{C})$, the maps Ψ and Λ are defined by

$$\Psi : \boldsymbol{I} \mapsto \langle \overline{F}(\boldsymbol{I}) \rangle, \quad \Lambda : \boldsymbol{J} \mapsto \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{J}\}.$$

By definition, the inner and outer triangles commute: in other words, $\Phi_W = \Phi_{\boldsymbol{C}} \circ \Psi$, and $\Theta_W = \Lambda \circ \Theta_{\boldsymbol{C}}$.

2.3. ***Recovering ideals from their supports.*** In [Nakano et al. 2022a, Theorem 6.2.1], conditions were given under which an arbitrary support datum $\sigma : \boldsymbol{T} \to \mathcal{X}(S)$ has the property that Φ_σ is a left, right, and two-sided inverse to Θ_σ. If Φ_σ is a left inverse to Θ_σ, this means that all thick ideals can be recovered from their supports; when Φ_σ and Θ_σ are a mutually inverse bijection, the ideals are completely classified by the topological space S. Since the support datum $W(-)$ defined above might not satisfy conditions under which every ideal may be recovered from their support (see Section 3 for examples), in this section we discuss precisely which ideals can be recovered in this way; this allows us to describe the image of the map f defined above.

We now introduce some terminology, which will be useful for our reconstruction theory.

Notation 2.3.1. When the finite tensor category $\boldsymbol{C}$ is clear by context, we will denote by $\boldsymbol{K}$ the kernel of the functor $\overline{F} : \mathsf{St}(\mathsf{Z}(\boldsymbol{C})) \to \mathsf{St}(\boldsymbol{C})$.

An equivalent characterization of the kernel of $\overline{F}$ can be given by

$$\boldsymbol{K} = \big\{H(X) : X \in \mathrm{Ind}(\mathsf{Z}(\boldsymbol{C})) \text{ such that } F(X) \text{ is projective in } \mathrm{Ind}(\boldsymbol{C})\big\}.$$

This follows from the fact that the objects of $\mathsf{St}(\boldsymbol{C})$ isomorphic to 0 correspond precisely to the projective objects of $\mathrm{Ind}(\boldsymbol{C})$, as we saw in Section 1.3.

Lemma 2.3.2. *The kernel of $\overline{F}$ is a thick localizing ideal of* $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$.

Proof. Since $\overline{F}$ is a monoidal triangulated functor, it is straightforward to verify that the collection of objects X such that $\overline{F}(X) \cong 0$ is closed under taking cones, shifts, direct summands, and by tensoring on the left or right by arbitrary objects of $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$. The functor F commutes with arbitrary coproducts by [Kashiwara and Schapira 2006, Proposition 6.1.9], and so the kernel of $\overline{F}$ is closed under arbitrary coproducts, i.e., $\boldsymbol{K}$ is localizing. □

Lemma 2.3.3. *An object $A \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ satisfies $W(A) = \varnothing$ if and only if $A \in \boldsymbol{K}$.*

Proof. First, note that if $A \in \boldsymbol{K}$, then by definition $\overline{F}(A) \cong 0$, and so

$$W(A) = V_{\boldsymbol{C}}(0) = \{\boldsymbol{P} \in \operatorname{Spc}(\mathsf{st}(\boldsymbol{C})) : 0 \notin \boldsymbol{P}\} = \varnothing.$$

For the other direction, recall that by the rigidity of $\boldsymbol{C}$, all thick ideals of $\mathsf{st}(\boldsymbol{C})$ are semiprime, i.e., intersections of prime ideals, by Proposition 1.1.6. This implies in particular that the ideal $\langle 0 \rangle$ is semiprime; in other words, the only object contained in all prime ideals of $\mathsf{st}(\boldsymbol{C})$ is 0. By definition, this means that if X is an object of $\mathsf{st}(\boldsymbol{C})$ such that $V_{\boldsymbol{C}}(X) = \varnothing$, then $X \cong 0$. Hence, we have

$$\varnothing = W(A) = V_{\boldsymbol{C}}(\overline{F}(A)) \Rightarrow \overline{F}(A) \cong 0 \Rightarrow A \in \boldsymbol{K}. \qquad \square$$

Using the localization and colocalization functors defined in Section 1.2, we are now able to prove the following, which is the critical step in determining which ideals can be recovered from their W-support and determining the image of the map $f : \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ defined in Proposition 2.1.3.

Theorem 2.3.4. *Let $\boldsymbol{I}$ be a thick ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ such that $\operatorname{Loc}(\boldsymbol{I})$ contains $\boldsymbol{K}$. Suppose that X is an object of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ such that $\overline{F}(X) \in \langle \overline{F}(\boldsymbol{I}) \rangle$, that is, the thick ideal of $\mathsf{st}(\boldsymbol{C})$ generated by all $\overline{F}(Y)$ for $Y \in \boldsymbol{I}$. Then X is in $\boldsymbol{I}$.*

Proof. By Theorem 1.2.1, we have a distinguished triangle

$$\Gamma_{\boldsymbol{I}}(X) \to X \to L_{\boldsymbol{I}}(X) \to \Sigma\Gamma_{\boldsymbol{I}}(X)$$

in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$, using the localization and colocalization functors associated to the thick ideal $\boldsymbol{I}$. We know that there are no morphisms from $\boldsymbol{I}$ to $L_{\boldsymbol{I}}(X)$; in other words, if $Y \in \boldsymbol{I}$ and Z is any compact object in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$, then

$$0 = \operatorname{Hom}_{\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))}(Z \otimes Y, L_{\boldsymbol{I}}(X)) \cong \operatorname{Hom}_{\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))}(Z, L_{\boldsymbol{I}}(X) \otimes Y^*).$$

Since this holds for all compact objects Z, this implies that $L_{\boldsymbol{I}}(X) \otimes Y^* \cong 0$. Since all compact objects are rigid, and by Proposition 1.1.6 all thick ideals are closed under taking duals, we have $L_{\boldsymbol{I}}(X) \otimes Y \cong 0$ for all $Y \in \boldsymbol{I}$. Since $\overline{F}$ is a monoidal functor, this additionally implies that

$$\overline{F}(L_{\boldsymbol{I}}(X)) \otimes \overline{F}(Y) \cong 0$$

in $\mathsf{St}(\boldsymbol{C})$, for all $Y \in \boldsymbol{I}$.

Now, consider the thick ideal $\langle \overline{F}(\boldsymbol{I}) \rangle$. This is formed successively by taking shifts, cones, direct summands, and tensor products with arbitrary elements of $\mathsf{st}(\boldsymbol{C})$, starting from the collection of objects of the form $\overline{F}(Y)$ for $Y \in \boldsymbol{I}$. This allows us to conclude inductively that $\overline{F}(L_{\boldsymbol{I}}(X)) \otimes A \cong 0$ for all A in $\langle \overline{F}(\boldsymbol{I}) \rangle$, since inductively each step by which we construct $\langle \overline{F}(\boldsymbol{I}) \rangle$ preserves the property that tensoring with $\overline{F}(L_{\boldsymbol{I}}(X))$ gives 0. To be more explicit, if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle in $\mathsf{st}(\boldsymbol{C})$ such that $A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong B \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$, then it is straightforward that additionally $C \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$ as well. Similarly, if $A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$, then $\Sigma(A) \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong$

$\Sigma(A \otimes \overline{F}(L_{\boldsymbol{I}}(X))) \cong \Sigma 0 \cong 0$. Furthermore, if we have $(A \oplus B) \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$, then we also have $A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0 \cong B \otimes \overline{F}(L_{\boldsymbol{I}}(X))$. Lastly, if $A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$ and B is an arbitrary object in $\mathrm{st}(\boldsymbol{C})$, then $A \otimes B \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \otimes B \cong 0$ as well, using the commutativity of $\overline{F}(L_{\boldsymbol{I}}(X))$.

To reiterate, the upshot of the previous paragraph is that $A \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0$ for all $A \in \langle \overline{F}(\boldsymbol{I}) \rangle$. But by assumption, we have $\overline{F}(X) \in \langle \overline{F}(\boldsymbol{I}) \rangle$. Hence,

$$\overline{F}(X \otimes L_{\boldsymbol{I}}(X)) \cong \overline{F}(X) \otimes \overline{F}(L_{\boldsymbol{I}}(X)) \cong 0.$$

Therefore, $X \otimes L_{\boldsymbol{I}}(X)$ is an object in $\boldsymbol{K}$, the collection of objects of $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ mapped to 0 by $\overline{F}$. By assumption, $\mathrm{Loc}(\boldsymbol{I})$ contains $\boldsymbol{K}$, and so $X \otimes L_{\boldsymbol{I}}(X) \in \mathrm{Loc}(\boldsymbol{I})$.

Now, consider the distinguished triangle obtained by tensoring the triangle

$$\Gamma_{\boldsymbol{I}}(X) \to X \to L_{\boldsymbol{I}}(X) \to \Sigma \Gamma_{\boldsymbol{I}}(X)$$

by X: this gives us

$$X \otimes \Gamma_{\boldsymbol{I}}(X) \to X \otimes X \to X \otimes L_{\boldsymbol{I}}(X) \to \Sigma X \otimes \Gamma_{\boldsymbol{I}}(X).$$

We have just finished showing that the third object of this triangle is in $\mathrm{Loc}(\boldsymbol{I})$. The first object is in $\mathrm{Loc}(\boldsymbol{I})$ as well, by Theorem 1.2.1. Since $\mathrm{Loc}(\boldsymbol{I})$ is triangulated, this implies $X \otimes X$ is in $\mathrm{Loc}(\boldsymbol{I})$. But by [Neeman 1992, Lemma 2.2], since $\boldsymbol{I}$ is a thick subcategory of compact objects, the compact objects in $\mathrm{Loc}(\boldsymbol{I})$ are precisely the objects of $\boldsymbol{I}$. Thus, $X \otimes X \in \boldsymbol{I}$, and by semiprimeness of $\boldsymbol{I}$ (Proposition 1.1.6) so is X; this completes the proof. □

We can now give a condition under which an ideal $\boldsymbol{I}$ can be recovered from its support $\Phi_W(\boldsymbol{I})$.

Corollary 2.3.5. *Let $\boldsymbol{I}$ be an ideal such that* $\mathrm{Loc}(\boldsymbol{I})$ *contains $\boldsymbol{K}$. Then $\Theta_W \circ \Phi_W(\boldsymbol{I}) = \boldsymbol{I}$.*

Proof. By definition,

$$\begin{aligned}
\Theta_W \circ \Phi_W(\boldsymbol{I}) &= \Theta_W(\Phi_{\boldsymbol{C}}(\overline{F}(\boldsymbol{I}))) \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : W(X) \subseteq \Phi_{\boldsymbol{C}}(\overline{F}(\boldsymbol{I}))\} \\
&= \left\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : V_{\boldsymbol{C}}(\overline{F}(X)) \subseteq \Phi_{\boldsymbol{C}}(\langle \overline{F}(\boldsymbol{I}) \rangle)\right\} \\
&= \left\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \forall \boldsymbol{P} \in \operatorname{Spc} \mathrm{st}(\boldsymbol{C}) \text{with } \overline{F}(X) \notin \boldsymbol{P}, \langle \overline{F}(\boldsymbol{I}) \rangle \not\subseteq \boldsymbol{P}\right\} \\
&= \left\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \forall \boldsymbol{P} \in \operatorname{Spc} \mathrm{st}(\boldsymbol{C}) \text{ with } \langle \overline{F}(\boldsymbol{I}) \rangle \subseteq \boldsymbol{P}, \overline{F}(X) \in \boldsymbol{P}\right\} \\
&= \left\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \textstyle\bigcap_{\boldsymbol{P} \in \operatorname{Spc} \mathrm{st}(\boldsymbol{C}), \langle \overline{F}(\boldsymbol{I}) \rangle \subseteq \boldsymbol{P}} \boldsymbol{P}\right\} \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \langle \overline{F}(\boldsymbol{I}) \rangle\}.
\end{aligned}$$

The last equality follows from Proposition 1.1.6. The corollary now follows directly from Theorem 2.3.4. □

2.4. *The image of prime ideal contraction.* We now describe the relationship of the image of the map f to the kernel $\boldsymbol{K}$ of $\overline{F}$.

Proposition 2.4.1. *Let $\boldsymbol{C}$ be a finite tensor category.*

(1) *If $\boldsymbol{P}$ is in the image of the map $f : \operatorname{Spc} \mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, then $\boldsymbol{P}$ contains $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, the kernel of $\overline{F}$ restricted to compact objects.*

(2) *If $\boldsymbol{P}$ is a prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ such that $\operatorname{Loc}(\boldsymbol{P})$ contains $\boldsymbol{K}$, then $\boldsymbol{P}$ is in the image of f.*

Proof. For (1), if $\boldsymbol{Q}$ is a prime ideal of $\mathsf{st}(\boldsymbol{C})$, then $f(\boldsymbol{Q})$ contains $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, which are by definition the finite-dimensional objects X such that $\overline{F}(X) \cong 0$: if X is in $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ and $\overline{F}(X) \cong 0$, then

$$X \in \{Y \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(Y) \in \boldsymbol{Q}\} = f(\boldsymbol{Q}),$$

since 0 is in every prime ideal of $\mathsf{st}(\boldsymbol{C})$.

Part (2) is an application of both Theorem 2.3.4 and [Nakano et al. 2022a, Theorem 3.2.3]. Let $\boldsymbol{P}$ be a prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ such that $\operatorname{Loc}(\boldsymbol{P})$ contains $\boldsymbol{K}$. Consider the following two collections of objects in $\mathsf{st}(\boldsymbol{C})$:

(1) The ideal $\boldsymbol{I} := \langle \overline{F}(X) : X \in \boldsymbol{P}\rangle$ of $\mathsf{st}(\boldsymbol{C})$.

(2) The collection $\mathcal{M} := \{\overline{F}(Y) : Y \notin \boldsymbol{P}\}$ of objects in $\mathsf{st}(\boldsymbol{C})$.

We first claim that these two collections of objects are disjoint. If $\overline{F}(Y) \in \boldsymbol{I}$ then $Y \in \Theta_W(\Phi_W(\boldsymbol{P}))$, implying that $Y \in \boldsymbol{P}$ by Corollary 2.3.5. This means that in particular, if $\overline{F}(X) \cong \overline{F}(Y)$, then either both X and Y are in $\boldsymbol{P}$, or neither are, and so $\boldsymbol{I}$ and $\mathcal{M}$ are indeed disjoint.

Since $\boldsymbol{P}$ is a proper ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, it follows that $\mathcal{M}$ is nonempty, and thus $\boldsymbol{I}$ is a proper ideal of $\mathsf{st}(\boldsymbol{C})$. We claim that $\mathcal{M}$ is a multiplicative subset. Suppose $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\mathcal{M}$. Then if $\overline{F}(X) \otimes \overline{F}(Y) \cong \overline{F}(X \otimes Y)$ was not in $\mathcal{M}$, this would imply that $X \otimes Y \in \boldsymbol{P}$; by the prime condition of $\boldsymbol{P}$, either X or Y (without loss of generality, say Y) would then be in $\boldsymbol{P}$. This is a contradiction, since $\overline{F}(Y) \in \mathcal{M}$ implies $Y \notin \boldsymbol{P}$, which is a consequence of the observation above that $\boldsymbol{I}$ and $\mathcal{M}$ are disjoint.

By [Nakano et al. 2022a, Theorem 3.2.3], given a disjoint pair consisting of a multiplicative subset and a proper ideal of any monoidal triangulated category (in this case, $\mathsf{st}(\boldsymbol{C})$), there exists a prime ideal $\boldsymbol{Q}$ of $\mathsf{st}(\boldsymbol{C})$ such that $\boldsymbol{Q} \cap \mathcal{M} = \varnothing$ and $\boldsymbol{I} \subseteq \boldsymbol{Q}$. We have

$$f(\boldsymbol{Q}) = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{Q}\},$$

and then since $\boldsymbol{I} \subseteq \boldsymbol{Q}$, it is automatic that $\boldsymbol{P} \subseteq f(\boldsymbol{Q})$; and since $\boldsymbol{Q}$ is disjoint from $\mathcal{M}$, in fact $\boldsymbol{P} = f(\boldsymbol{Q})$. Thus, f surjects onto the collection of prime ideals $\boldsymbol{P}$ such that $\operatorname{Loc}(\boldsymbol{P})$ contains $\boldsymbol{K}$, which completes the proof. □

By Proposition 2.4.1, we have inclusions of the following subsets of $\operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$:

$$\{\boldsymbol{P} : \boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \subseteq \boldsymbol{P}\} \supseteq \operatorname{im} f \supseteq \{\boldsymbol{P} : \boldsymbol{K} \subseteq \operatorname{Loc}(\boldsymbol{P})\}. \tag{2.4.1}$$

We note the following lemma, which is a special case of [Benson et al. 2012, Proposition 1.47].

Lemma 2.4.2. *The following are equivalent.*

(1) *The kernel $\boldsymbol{K}$ of $\overline{F}$ is generated as a localizing category (recalling Section 1.2) by the set $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.*

(2) *For every nonzero X in $\boldsymbol{K}$, there exists a compact object Y in $\boldsymbol{K}$ which has some nonzero map $Y \to X$ in* $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$.

In particular, to prove that (2) $\Rightarrow$ (1), one simply observes that if $X \in \boldsymbol{K}$, then the distinguished triangle

$$\Gamma_{\boldsymbol{K}\cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}X \to X \to L_{\boldsymbol{K}\cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}X \to \Sigma\Gamma_{\boldsymbol{K}\cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}X$$

given by Theorem 1.2.1 implies that $L_{\boldsymbol{K}\cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}X \in \boldsymbol{K}$. But by definition, it is in the perpendicular space to $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, which by the assumption of (2) means that it is 0. Hence $\Gamma_{\boldsymbol{K}\cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}X \cong X$, that is, X is in $\mathrm{Loc}\big(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$.

If these conditions are satisfied, then we can sharpen (2.4.1), as well as Corollary 2.3.5.

Corollary 2.4.3. *Suppose the kernel $\boldsymbol{K}$ of $\bar{F}$ satisfies the equivalent conditions of Lemma 2.4.2.*

(1) *The image of f is precisely the collection of prime ideals of* $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ *which contain* $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, *that is, the collection of objects X in* $\mathsf{st}(\boldsymbol{C})$ *such that* $\bar{F}(X) \cong 0$.

(2) *A thick ideal $\boldsymbol{I}$ of* $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ *satisfies* $\Theta_W \circ \Phi_W(\boldsymbol{I}) = \boldsymbol{I}$ *if and only if $\boldsymbol{I}$ contains* $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.

Proof. Suppose $\mathrm{Loc}(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))) = \boldsymbol{K}$. For (1), let $\boldsymbol{P}$ be a prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ containing $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. Then $\mathrm{Loc}(\boldsymbol{P})$ contains $\mathrm{Loc}(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))) = \boldsymbol{K}$. Hence the collection of inequalities of (2.4.1) becomes an equality, and we are done.

For (2), similarly, we have by Corollary 2.3.5 that if $\mathrm{Loc}(\boldsymbol{I})$ contains $\boldsymbol{K}$, then $\Theta_W \circ \Phi_W(\boldsymbol{I}) = \boldsymbol{I}$. Since $\boldsymbol{K} = \mathrm{Loc}(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})))$, we have $\boldsymbol{K} \subseteq \mathrm{Loc}(\boldsymbol{I})$ if and only if there is containment $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \subseteq \boldsymbol{I}$. For the other direction, we note that for any ideal $\boldsymbol{I}$, we have $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \subseteq \Theta_W \circ \Phi_W(\boldsymbol{I})$, and so any thick ideal satisfying $\Theta_W \circ \Phi_W(\boldsymbol{I}) = \boldsymbol{I}$ must have $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) \subseteq \boldsymbol{I}$ as well. $\square$

Remark 2.4.4. Corollary 2.4.3 implies that if $\boldsymbol{C}$ satisfies the conditions of Lemma 2.4.2, then the image of $f : \mathrm{Spc}\,\mathsf{st}(\boldsymbol{C}) \to \mathrm{Spc}\,\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ is automatically the complement of a specialization-closed set, since we have

$$\mathrm{im}(f) = \{\boldsymbol{P} \in \mathrm{Spc}\,\mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \boldsymbol{P} \supseteq \boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\} = \mathrm{Spc}\,\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\backslash\big(\Phi_{\mathsf{Z}}\big(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)\big).$$

In other words, the image of f can be written as an intersection of open sets. If $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ is generated (as a thick ideal) by a finite collection of objects, say $\{X_i\}_{i=1}^n$, then it follows that $\mathrm{im}(f)$ is in fact an open subset of $\mathrm{Spc}\,\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, namely

$$\mathrm{im}(f) = U_{\mathsf{Z}}(X_1 \oplus \cdots \oplus X_n)$$

(recall the notation of U_{Z} from Section 1.1 and Notation 2.1.1).

Remark 2.4.5. In the situation of Corollary 2.4.3(2), we have Corollary 2.3.5 sharpened from a one-way implication to a two-way implication. We note on the other hand that if the conditions of Lemma 2.4.2 are not satisfied, then Corollary 2.3.5 can never be an if-and-only-if, for the following reason. The collection of objects $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ is itself a thick ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, since it is in particular the kernel of the monoidal triangulated functor $\overline{F}$ restricted to compact objects. But now note that

$$\begin{aligned}\Theta_W \circ \Phi_W(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))) &= \big\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : W(X) \subseteq \Phi_W\big(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)\big\} \\ &= \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : W(X) \subseteq \varnothing\} \\ &= \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \cong 0\} \\ &= \boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C})).\end{aligned}$$

Here the first equality is by the definition of Θ_W, the second and third equalities are by Lemma 2.3.3 and the definition of Φ_W, and the last equality by the definition of the kernel $\boldsymbol{K}$. In other words, the thick ideal $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ can be recovered from its support. But plainly, since we are assuming the conditions of Lemma 2.4.2 are not satisfied, we have

$$\mathrm{Loc}\big(\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big) \not\supseteq \boldsymbol{K},$$

and so Corollary 2.3.5 cannot be sharpened to an if-and-only-if statement.

2.5. ***Conditions under which f is injective, surjective, or a homeomorphism.*** We now give conditions under which Φ_W and Θ_W are inverses, and f is surjective, injective, and a homeomorphism.

Theorem 2.5.1. *Let $\boldsymbol{C}$ be a finite tensor category.*

(1) *The following conditions are equivalent.*

(a) *For all $X \in \boldsymbol{K}$, there exists an isomorphism $X \cong 0$ in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$.*

(b) *The map f is surjective and $\boldsymbol{K}$ is generated as a localizing category by its subcategory $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$.*

(c) *As maps* $\mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big) \to \mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$, *we have* $\Lambda \circ \Psi = \mathrm{id}$.

(d) *As maps* $\mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big) \to \mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$, *we have* $\Theta_W \circ \Phi_W = \mathrm{id}$.

(2) *If $\boldsymbol{C}$ is braided, then the following hold.*

(a) *The map f is injective.*

(b) *As maps* $\mathrm{ThickId}(\mathsf{st}(\boldsymbol{C})) \to \mathrm{ThickId}(\mathsf{st}(\boldsymbol{C}))$, *we have* $\Psi \circ \Lambda = \mathrm{id}$.

(c) *If additionally* $\mathrm{Spc}\,\mathsf{st}(\boldsymbol{C})$ *is topologically Noetherian, then* $\Phi_W \circ \Theta_W = \mathrm{id}$.

(3) *If $X \cong 0$ in $\mathsf{St}(\mathsf{Z}(\boldsymbol{C}))$ for all $X \in \boldsymbol{K}$ and $\boldsymbol{C}$ is braided, then the following hold.*

(a) *The map f is a homeomorphism.*

(b) *The maps Ψ and Λ define mutually inverse bijections between* $\mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$ *and* $\mathrm{ThickId}(\mathsf{st}(\boldsymbol{C}))$.

(c) *If additionally* $\mathrm{Spc}\,\mathsf{st}(\boldsymbol{C})$ *is topologically Noetherian, then Φ_W and Θ_W are mutually inverse bijections between* $\mathrm{ThickId}\big(\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))\big)$ *and* $\mathcal{X}_{\mathrm{sp}}\big(\mathrm{Spc}(\mathsf{st}(\boldsymbol{C}))\big)$.

Proof. Suppose (1a) holds, and so $\boldsymbol{K}$ consists only of objects isomorphic to 0, in other words, for all objects $X \in \mathsf{Z}(\boldsymbol{C})$,

$$F(X) \text{ is projective in } \boldsymbol{C} \Leftrightarrow X \text{ is projective in } \mathsf{Z}(\boldsymbol{C}).$$

In particular this means that $\boldsymbol{K}$ is generated by $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, since all objects of $\boldsymbol{K}$ are isomorphic to 0. Then (1c) follows from Theorem 2.3.4, and the conditions (1b) and (1d) follow directly from Corollary 2.4.3.

Now, suppose (1b) is satisfied. By Proposition 2.4.1, this means that every prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ contains $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$. But since every ideal is semiprime, the zero ideal is equal to the intersection of all primes of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, and so $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ is contained in the zero ideal. Since $\boldsymbol{K}$ is generated by $\boldsymbol{K} \cap \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, i.e., the zero ideal, this implies that (1a) holds.

Note that

$$\Lambda(\Psi(\langle 0_{\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))}\rangle)) = \Lambda(\langle 0_{\mathsf{st}(\boldsymbol{C})}\rangle) = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \langle 0_{\mathsf{st}(\boldsymbol{C})}\rangle\} = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \cong 0\} = \boldsymbol{K}.$$

Hence (1c) implies (1a).

Lastly, suppose condition (1d) holds. This implies by Corollary 2.3.5 that $\boldsymbol{K} \subseteq \mathrm{Loc}(\boldsymbol{I})$ for every thick ideal $\boldsymbol{I}$; in particular, this means that $\boldsymbol{K}$ is contained in the localizing category generated by 0, which consists only of objects isomorphic to 0. Hence, (1a) holds.

To show (2), first note that if $\boldsymbol{C}$ is braided with a braiding γ, then $\overline{F}$ is essentially surjective, since for any object X in $\boldsymbol{C}$, the pair (X, γ_X) is an object of $\mathsf{Z}(\boldsymbol{C})$ and $\overline{F}$ sends $H(X, \gamma_X)$ to $G(X)$. Now, we note that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are prime ideals of $\mathsf{st}(\boldsymbol{C})$, then

$$\begin{gathered} f(\boldsymbol{P}) = f(\boldsymbol{Q}) \\ \Updownarrow \\ \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{P}\} = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{Q}\} \\ \Updownarrow \\ \text{for all } X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})), \quad \overline{F}(X) \in \boldsymbol{P} \Leftrightarrow \overline{F}(X) \in \boldsymbol{Q} \\ \Updownarrow \\ \text{for all } Y \in \mathsf{st}(\boldsymbol{C}), \quad Y \in \boldsymbol{P} \Leftrightarrow Y \in \boldsymbol{Q} \\ \Updownarrow \\ \boldsymbol{P} = \boldsymbol{Q}. \end{gathered}$$

Hence, if $\boldsymbol{C}$ is braided then (2a) follows.

Condition (2b) also follows directly from the fact that $\overline{F}$ is essentially surjective.

For (2c), recall that by Theorem 1.1.4, $\mathrm{Spc}(\mathsf{st}(\boldsymbol{C}))$ is Noetherian if and only if every closed set is of the form $V_{\boldsymbol{C}}(A)$ for some object $A \in \mathsf{st}(\boldsymbol{C})$. If S is a specialization-closed set in $\mathrm{Spc}(\mathsf{st}(\boldsymbol{C}))$, then by definition

$$\Phi_W(\Theta_W(S)) = \Phi_W\big(\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : W(X) \subseteq S\}\big) = \bigcup_{X \in \Theta_W(S)} W(X) \subseteq S.$$

For the other direction, we can write S as a union of closed sets, say $S = \bigcup_{i \in I} S_i$, and by the Noetherianity of $\operatorname{Spc}(\mathsf{st}(\boldsymbol{C}))$, there exist objects A_i of $\mathsf{st}(\boldsymbol{C})$ such that $S_i = V_{\boldsymbol{C}}(A_i)$. Since $\overline{F}$ is essentially surjective, we can pick $X_i \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ with $\overline{F}(X_i) = A_i$. Since

$$W(X_i) = V_{\boldsymbol{C}}(A_i) = S_i \subseteq S,$$

we have by definition each X_i is in $\Theta_W(S)$. Therefore,

$$\Phi_W(\Theta_W(S)) \supseteq \bigcup_{i \in I} W(X_i) = \bigcup_{i \in I} S_i = S.$$

Thus $S = \Phi_W(\Theta_W(S))$.

Suppose the assumptions of (3). Then (3b) and (3c) follow immediately from parts (1) and (2). To show (3a), it is enough to show that f is a closed map, by (1a) and (2a). Take an arbitrary closed set $V_{\boldsymbol{C}}(\mathcal{T})$ in $\operatorname{Spc}\mathsf{st}(\boldsymbol{C})$. We claim that the image of $V_{\boldsymbol{C}}(\mathcal{T})$ under f is precisely $V_{\mathsf{Z}}(\widehat{\mathcal{T}})$, where

$$\widehat{\mathcal{T}} = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \mathcal{T}\}.$$

For the first direction, suppose $\boldsymbol{P} \in V_{\boldsymbol{C}}(\mathcal{T})$, in other words, $\boldsymbol{P} \cap \mathcal{T} = \varnothing$. Since $f(\boldsymbol{P}) = \{X : \overline{F}(X) \in \boldsymbol{P}\}$, this implies that for all $X \in f(\boldsymbol{P})$, we have $X \notin \widehat{\mathcal{T}}$. Therefore $f(\boldsymbol{P}) \cap \widehat{\mathcal{T}} = \varnothing$, and so $f(\boldsymbol{P}) \in V_{\mathsf{Z}}(\widehat{\mathcal{T}})$. This shows $f(V_{\boldsymbol{C}}(\mathcal{T})) \subseteq V_{\mathsf{Z}}(\widehat{\mathcal{T}})$.

For the other containment, suppose $\boldsymbol{Q}$ is a prime ideal of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ in $V_{\mathsf{Z}}(\widehat{\mathcal{T}})$. Then $\overline{F}(X) \notin \mathcal{T}$ for all $X \in \boldsymbol{Q}$. Since f is surjective, we can pick $\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C})$ with $f(\boldsymbol{P}) = \boldsymbol{Q}$, and for all $\overline{F}(X) \in \boldsymbol{P}$, we must have $\overline{F}(X) \notin \mathcal{T}$. Since $\overline{F}$ is essentially surjective, this implies $A \notin \mathcal{T}$ for all $A \in \boldsymbol{P}$, and so $\boldsymbol{P} \cap \mathcal{T} = \varnothing$, i.e., $\boldsymbol{P} \in V_{\boldsymbol{C}}(\mathcal{T})$. This shows the other containment $f(V_{\boldsymbol{C}}(\mathcal{T})) \supseteq V_{\mathsf{Z}}(\widehat{\mathcal{T}})$, and so we have equality.

Hence, f sends the closed set $V_{\boldsymbol{C}}(\mathcal{T})$ to the closed set $V_{\mathsf{Z}}(\widehat{\mathcal{T}})$, and so it is a continuous, bijective, closed map, and therefore a homeomorphism. □

3. Applications

The time has come for concrete applications of our theory.

3.1. ***Group algebras and dual group algebras.*** Let G be a finite group, $\Bbbk$ be an algebraically closed field of characteristic p which divides the order of G, and $\Bbbk G$ the group algebra of G over $\Bbbk$. Let $\boldsymbol{C} = \mathsf{mod}(\Bbbk G)$, a finite tensor category. The Drinfeld double $D(\Bbbk G)$ is a Hopf algebra containing $\Bbbk G$ and $(\Bbbk G^{\mathrm{op}})^*$ as Hopf subalgebras. We will denote the dual of the group algebra by $\Bbbk[G]$, and in that case we can write $(\Bbbk G^{\mathrm{op}})^* = \Bbbk[G]^{\mathrm{cop}}$. The collection

$$\{p_g h : g, h \in G\}$$

is a $\Bbbk$-basis of $D(\Bbbk G)$, where the elements $\{p_g : g \in G\}$ refer to the basis of $\Bbbk[G]^{\mathrm{cop}}$ dual to the standard basis of $\Bbbk G$. The multiplication is determined by the relations

$$hp_g = p_{hgh^{-1}}h,$$

see for instance [Kassel 1995, Section IX.4.3].

Lemma 3.1.1. *Let G and $\Bbbk$ be as above and $F : \mathsf{Mod}(D(\Bbbk G)) \to \mathsf{Mod}(\Bbbk G)$ the forgetful functor. Then if $F(P)$ is projective as a $\Bbbk G$-module, then P is projective as a $D(\Bbbk G)$-module.*

Proof. A module for $D(\Bbbk G)$ is a $\Bbbk G$ module M which is also a G-graded vector space, such that if $m \in M$ is a homogeneous element of degree g, then $h.m$ is homogeneous of degree hgh^{-1}. Suppose we have a short exact sequence

$$0 \to A \to B \xrightarrow{t} C \to 0$$

of $D(\Bbbk G)$-modules such that

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a split short exact sequence of G-modules. We claim that the original sequence splits as $D(\Bbbk G)$-modules. Pick a homogeneous basis $\{c_i\}$ of C under the G-grading, where c_i has degree g_i. Now pick a splitting $s : C \to B$. Define $\hat{s}(c_i) = p_{g_i}s(c_i)$. This map is homogeneous with respect to the G-grading, and it is still a G-module map:

$$g\hat{s}(c_i) = gp_{g_i}s(c_i) = p_{gg_ig^{-1}}gs(c_i) = p_{gg_ig^{-1}}s(gc_i) = \hat{s}(gc_i).$$

Since on the basis $\{c_i\}$ we have

$$t \circ \hat{s}(c_i) = t(p_{g_i}.s(c_i)) = p_{g_i}.ts(c_i) = p_{g_i}.c_i = c_i,$$

we have that $\hat{s}$ is a splitting of $D(\Bbbk G)$-modules.

Now, to prove the original claim, suppose $F(P)$ is projective as a G-module. Since F is exact, this means that for every short exact sequence

$$0 \to A \to B \to P \to 0$$

in $D(H)$-modules, the sequence

$$0 \to F(A) \to F(B) \to F(P) \to 0$$

is split as G-modules. Therefore, the original sequences are all split, and so P is projective. $\square$

We recall that by [Balmer 2005, Corollary 5.10], $\mathrm{Spc}\,\mathsf{stmod}(\Bbbk G) \cong \mathrm{Proj}\,\mathrm{H}^\bullet(G, \Bbbk)$, where $\mathrm{H}^\bullet(G, \Bbbk) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\Bbbk G}(\Bbbk, \Bbbk)$ is the cohomology ring of G (recall Example 1.4.2).

Proposition 3.1.2. *Let G, $\Bbbk$, and $\mathrm{H}^\bullet(G, \Bbbk)$ be as above.*

(1) *The map $f : \mathrm{Spc}\,\mathsf{stmod}(\Bbbk G) \to \mathrm{Spc}\,\mathsf{stmod}(D(\Bbbk G))$ is a homeomorphism, and so*

$$\mathrm{Spc}\,\mathsf{stmod}(D(\Bbbk G)) \cong \mathrm{Spc}\,\mathsf{stmod}(\Bbbk G) \cong \mathrm{Proj}\,\mathrm{H}^\bullet(G, \Bbbk).$$

(2) *Thick ideals of* $\mathsf{stmod}(D(\Bbbk G))$ *are in bijection with specialization-closed sets in* $\operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$, *which are in bijection with thick ideals of* $\mathsf{stmod}(\Bbbk G)$, *via the maps*

$$\operatorname{ThickId}\big(\mathsf{stmod}(D(\Bbbk G))\big) \underset{\Theta_W}{\overset{\Phi_W}{\rightleftarrows}} \mathcal{X}_{\mathrm{sp}}(\operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)) \underset{\Phi_{\Bbbk G}}{\overset{\Theta_{\Bbbk G}}{\rightleftarrows}} \operatorname{ThickId}(\mathsf{stmod}(\Bbbk G)).$$

Proof. Since $\Bbbk G$ is cocommutative, $\mathsf{mod}(\Bbbk G)$ is braided symmetric. By Lemma 3.1.1, we have $X \cong 0$ in $\mathsf{StMod}(D(H))$ for all $X \in \boldsymbol{K}$, and so we are in the situation given of Theorem 2.5.1(3). Additionally, since cohomology rings of groups are finitely generated (for instance by the more general result of [Friedlander and Suslin 1997], in which finite generation of cohomology rings for finite-dimensional cocommutative Hopf algebras in positive characteristic was proven), we know that $\operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$ is a Noetherian topological space. Using Balmer's classification of thick ideals [2005, Theorem 4.10], the thick ideals of $\mathsf{stmod}(\Bbbk G)$ are in bijection with specialization-closed sets in $\operatorname{Spc} \mathsf{stmod}(\Bbbk G)$. The rest of the theorem now follows directly as an application of Theorem 2.5.1. □

Now, note that since $\Bbbk[G]^{\mathrm{cop}}$ is a semisimple algebra, $\mathsf{stmod}(\Bbbk[G]^{\mathrm{cop}})$ consists only of the zero object, up to isomorphism, and so $\operatorname{Spc}(\mathsf{stmod}(\Bbbk[G]^{\mathrm{cop}}))$ is the empty set. Thus, the diagram from Remark 2.1.4 becomes

$$\begin{tikzcd} \operatorname{Spc} \mathsf{stmod}(\Bbbk G) \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "\cong"'] & \operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk) & \operatorname{Spc} \mathsf{stmod}(\Bbbk[G]^{\mathrm{cop}}) = \varnothing \arrow[dl] \\ & \operatorname{Spc} \mathsf{stmod}(D(\Bbbk G)) & \end{tikzcd}$$

3.2. ***Cosemisimple Hopf algebras.*** In fact, we are able to generalize Lemma 3.1.1 and Proposition 3.1.2 from the group algebra case to the case certain finite-dimensional cosemisimple Hopf algebras. Recall that a finite-dimensional Hopf algebra is called *cosemisimple* if its Hopf dual is semisimple, as an algebra. There has been significant interest in the algebraic properties of cosemisimple Hopf algebras in the past few decades; see, e.g., [Larson and Radford 1988a; 1988b; Etingof and Gelaki 1998; Chirvasitu 2014; Chirvasitu et al. 2019].

We first record the following straightforward lemma.

Lemma 3.2.1. *Let H be a finite-dimensional Hopf algebra such that $\mathbf{1}_{D(H)}$ is a direct summand of $D(H) \otimes_H \mathbf{1}_H$ as $D(H)$-modules, and let $F : \mathsf{Mod}(D(H)) \to \mathsf{Mod}(H)$ be the forgetful functor. Then $F(P)$ is projective in* $\mathsf{Mod}(H)$ *if and only if P is projective in* $\mathsf{Mod}(D(H))$.

Proof. The functor $D(H) \otimes_H -$ is a left adjoint to the forgetful functor F. Since F is exact, if Q is a projective H-module then

$$\operatorname{Hom}_H(Q, F(-)) \cong \operatorname{Hom}_{D(H)}(D(H) \otimes_H Q, -)$$

is an exact functor (recalling that projectives are also injective), and so $D(H) \otimes_H -$ preserves projectivity. Therefore, if P is a $D(H)$-module such that $F(P)$ is projective, then $D(H) \otimes_H F(P)$ is a projective $D(H)$-module. But then, we have

$$D(H) \otimes_H F(P) \cong D(H) \otimes_H (\mathbf{1}_H \otimes_\Bbbk F(P)) \cong (D(H) \otimes_H \mathbf{1}_H) \otimes_\Bbbk P,$$

where the last isomorphism here can be seen from, e.g., [Garland and Lepowsky 1976, Proposition 1.7] and the remark following it, which notes that although the proposition is stated for certain universal enveloping algebras, in fact the proof uses only the Hopf algebra structure, and so the result holds for arbitrary Hopf algebras. Note that it holds not just for finite-dimensional modules, but for arbitrary modules, which we need since in this case P may be infinite-dimensional.

Now, since $\mathbf{1}_{D(H)}$ is a summand of $D(H) \otimes_H \mathbf{1}_H$, we have that $P \cong \mathbf{1}_{D(H)} \otimes_{\Bbbk} P$ is a direct summand of $(D(H) \otimes_H \mathbf{1}_H) \otimes_{\Bbbk} P$, which is a projective $D(H)$-module, and hence P is projective as well, and the claim is proven. □

Recall that a Hopf algebra (or, more generally, a tensor category) is called *unimodular* if its spaces of left and right integrals coincide (see [Montgomery 1993, Section 2.1; Etingof et al. 2015, Section 6.5]). Unimodular Hopf algebras are of particular interest due to their use in constructing Hennings–Kauffman–Radford invariants for 3-manifolds [Kauffman and Radford 1995; Hennings 1996]. In light of Shimizu's result [2017, Theorem 4.10] on unimodular finite tensor categories, if H satisfies the conditions of Lemma 3.2.1 — that is, if $\mathbf{1}_{D(H)}$ is a direct summand of $D(H) \otimes_H \mathbf{1}_H$ — then H must be unimodular. The converse is not true; the dual of a finite group algebra is unimodular [Shimizu 2017, Corollary 5.5], but $\mathbf{1}_{D(\Bbbk[G])}$ is not a direct summand of $D(\Bbbk[G]) \otimes_{\Bbbk G} \mathbf{1}_{\Bbbk[G]}$ (since $F(\mathbf{1}_{D(\Bbbk[G])}) = \mathbf{1}_{\Bbbk[G]}$ is projective and $\mathbf{1}_{D(\Bbbk[G])}$ is not).

Corollary 3.2.2. *Let H be a finite-dimensional unimodular cosemisimple Hopf algebra with Drinfeld double $D(H)$ and forgetful functor $F : \mathrm{Mod}(D(H)) \to \mathrm{Mod}(H)$. Then $F(P)$ is projective as an H-module if and only if P is projective as a $D(H)$-module.*

Proof. This follows from Lemma 3.2.1 and the proof of [Etingof et al. 2015, Proposition 7.18.15]. In the course of the proof of the latter, it is shown that if H is unimodular and cosemisimple, then $\mathbf{1}_{D(H)}$ is a direct summand of $D(H) \otimes_H \mathbf{1}_H$ as $D(H)$-modules (note that here, we are reversing the roles of H and H^* given in their proof). Although this proposition assumes a stronger condition– that H itself is also semisimple– this assumption is not used for the part of the proof by which $D(H) \otimes_H \mathbf{1}_H$ has $\mathbf{1}_{D(H)}$ as a summand. By Lemma 3.2.1, the corollary follows. □

Remark 3.2.3. The condition that H is unimodular in Corollary 3.2.2 is not too restrictive. It is a long-standing conjecture of Kaplansky [1975] that finite-dimensional cosemisimple Hopf algebras are involutory (i.e., the square of the antipode is the identity). In view of results of Larson [1971, Corollary 4.2], a weaker form of the Kaplansky conjecture is that all finite-dimensional cosemisimple Hopf algebras are unimodular [Aljadeff et al. 2002, Remark 3.9]. This conjecture is still open.

Corollary 3.2.2 and Theorem 2.5.1 now immediately imply the following.

Proposition 3.2.4. *Let H be a finite-dimensional unimodular cosemisimple Hopf algebra. Then the map $f : \mathrm{Spc}\,\mathrm{stmod}(H) \to \mathrm{Spc}\,\mathrm{stmod}(D(H))$ constructed in Section 2.1 is surjective, and the maps $\Lambda \circ \Psi$ and $\Theta_W \circ \Phi_W$ (as in Section 2.2) are each the identity, as maps from the collection of thick ideals of $\mathrm{stmod}(D(H))$ to itself.*

Gelaki [1997, Theorem 1.3.6] has shown that every quasitriangular cosemisimple Hopf algebra is unimodular. Hence, again by Corollary 3.2.2 and Theorem 2.5.1, we conclude:

Proposition 3.2.5. *Let H be a finite-dimensional quasitriangular cosemisimple Hopf algebra.*

(1) *The map f constructed in Section 2.1 is a homeomorphism*

$$\operatorname{Spc}\mathsf{stmod}(H) \xrightarrow{\cong} \operatorname{Spc}\mathsf{stmod}(D(H)),$$

and the maps Ψ and Λ as in Section 2.2 give inverse bijections between the thick ideals of $\mathsf{stmod}(H)$ *and* $\mathsf{stmod}(D(H))$.

(2) *If* $\operatorname{Spc}\mathsf{stmod}(H)$ *is topologically Noetherian, then the Φ_W and Θ_W constructed in Section 2.2 are inverse maps, and so we have the following bijections of thick ideals*:

$$\operatorname{ThickId}\bigl(\mathsf{stmod}(D(H))\bigr) \underset{\Theta_W}{\overset{\Phi_W}{\rightleftarrows}} \mathcal{X}_{\mathrm{sp}}\bigl(\operatorname{Spc}(\mathsf{stmod}(H))\bigr) \underset{\Phi_H}{\overset{\Theta_H}{\rightleftarrows}} \operatorname{ThickId}(\mathsf{stmod}(H)).$$

Of course, if H itself is also semisimple, then Propositions 3.2.4 and 3.2.5 are not particularly illuminating, since this implies that $D(H)$ is also semisimple, and then the Balmer spectra of $\mathsf{stmod}(H)$ and $\mathsf{stmod}(D(H))$ are both $\varnothing$. It is a classical theorem of Larson and Radford [1988a] that in characteristic 0, all cosemisimple finite-dimensional Hopf algebras are also semisimple. Hence, Propositions 3.2.4 and 3.2.5 only provide interesting examples in positive characteristic.

3.3. ***Benson–Witherspoon smash coproduct Hopf algebras.*** We will now consider the Benson–Witherspoon smash coproducts which were originally studied in [Benson and Witherspoon 2014], with generalizations studied in [Montgomery et al. 2016; Plavnik and Witherspoon 2018]; their Balmer spectra and thick ideals were classified in [Nakano et al. 2022a]. We recall the general construction of these algebras. Let G and L be finite groups, such that L acts on G by group automorphisms, and let $\Bbbk$ be an algebraically closed field of characteristic dividing the order of G. We then define $H_{G,L}$ to be the Hopf algebra dual of the smash product $\Bbbk[G]\#\Bbbk L$, where $\Bbbk[G]$ is the coordinate ring of G, and $\Bbbk L$ is the group algebra of L.

As an algebra, $H_{G,L}$ is isomorphic to $\Bbbk G\otimes\Bbbk[L]$. We will denote by $\{p_x : x\in L\}$ the standard dual basis for $\Bbbk[L]$, as in Section 3.1. Denote by e the identity element of L. The additional Hopf algebra structures of comultiplication, counit, and antipode on A are defined by

$$\Delta(g\otimes p_x)=\sum_{y\in L}(g\otimes p_y)\otimes(y^{-1}.g\otimes p_{y^{-1}x}),\quad \epsilon(g\otimes p_x)=\delta_{x,1},\quad S(g\otimes p_x)=x^{-1}.(g^{-1})\otimes p_{x^{-1}},$$

for all $g\in G$ and $x\in L$.

Since as an algebra $H_{G,L}\cong\Bbbk G\otimes\Bbbk[L]$, an $H_{G,L}$-module is the same as a G-module with an L-grading, such that the action of G preserves the L-grading. That is, every $H_{G,L}$-module M may be decomposed

$$M\cong\bigoplus_{x\in L}M_x\otimes\Bbbk_x,$$

where M_x is a G-module, and $\Bbbk_x$ is the 1-dimensional $\Bbbk[L]$-module on which p_x acts as the identity, and p_y acts as 0 for $y \neq x$ (in other words, the $\Bbbk[L]$-module corresponding to a L-graded vector space of one dimension where every element is homogeneous of degree x). The $H_{G,L}$-action on the component $M_x \otimes \Bbbk_x$ is defined by letting $\Bbbk G$ act on the first tensorand, and $\Bbbk[L]$ act on the second.

Using the definition of the coproduct on $H_{G,L}$, Benson and Witherspoon [2014, Theorem 2.1] compute the formula for the tensor product of $H_{G,L}$-modules:

$$(M_x \otimes \Bbbk_x) \otimes (N_y \otimes \Bbbk_y) = (M_x \otimes {}^x N_y) \otimes \Bbbk_{xy},$$

for any $\Bbbk G$-modules M_x and N_y, and for all $x, y \in L$, where the module ${}^x N_y$ is defined as the twist of the module N_y by the action of x. Namely, this is the $\Bbbk G$-module which is equal to N_y as a vector space, and if we write $g \cdot v$ for the action of G on the original module N_y, then the new action $*$ of G on ${}^x N_y$ is defined $g * v = (x^{-1} g) \cdot v$.

Proposition 3.3.1. *Let $H_{G,L}$ be the Benson–Witherspoon smash coproduct Hopf algebra as defined above, $\boldsymbol{C}$ the category $\mathsf{mod}(H_{G,L})$, and $\mathsf{Z}(\boldsymbol{C})$ the category $\mathsf{mod}(D(H_{G,L}))$ for the Drinfeld double $D(H_{G,L})$ of $H_{G,L}$.*

(1) *The continuous map $f : \operatorname{Spc} \mathsf{st}(\boldsymbol{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ constructed in Section 2.1 is injective.*

(2) *The map $\Psi \circ \Lambda$ constructed in Section 2.2 is equal to the identity, as a map*

$$\operatorname{ThickId}(\mathsf{st}(\boldsymbol{C})) \to \operatorname{ThickId}(\mathsf{st}(\boldsymbol{C})).$$

(3) *The map $\Phi_W \circ \Theta_W$ constructed in Section 2.2 is equal to the identity, as a map*

$$\mathcal{X}_{\mathrm{sp}}(\operatorname{Spc} \mathsf{st}(\boldsymbol{C})) \to \mathcal{X}_{\mathrm{sp}}(\operatorname{Spc} \mathsf{st}(\boldsymbol{C})).$$

Remark 3.3.2. We note that if $\boldsymbol{C}$ was braided, then Proposition 3.3.1 would follow directly from Theorem 2.5.1. However, in general, $H_{G,L}$ is not a quasitriangular Hopf algebra, i.e., the category of $H_{G,L}$-modules is not braided.

Proposition 3.3.1 will be proven by first showing the following intermediary lemma.

Lemma 3.3.3. *Suppose $\boldsymbol{I}$ and $\boldsymbol{J}$ are thick ideals of $\mathsf{st}(\boldsymbol{C})$ such that*

$$\{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{I}\} = \{X \in \mathsf{st}(\mathsf{Z}(\boldsymbol{C})) : \overline{F}(X) \in \boldsymbol{J}\}.$$

Then $\boldsymbol{I} = \boldsymbol{J}$. In particular, if M is an object of $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$, then there exists an object $\widehat{M}$ which is in the image of $\overline{F}$, and given any thick ideal $\boldsymbol{I}$, the object M is in $\boldsymbol{I}$ if and only if $\widehat{M}$ is in $\boldsymbol{I}$.

Proof. Suppose $\boldsymbol{I}$ and $\boldsymbol{J}$ are thick ideals satisfying the condition above. Since $\boldsymbol{I}$ and $\boldsymbol{J}$ are thick, it is enough to show that the indecomposable objects in $\boldsymbol{I}$ are equal to the indecomposable objects in $\boldsymbol{J}$. Suppose $M_x \otimes \Bbbk_x$ is an object in $\boldsymbol{I}$. Then the module

$$(M_x \otimes \Bbbk_x) \otimes (\Bbbk \otimes \Bbbk_{x^{-1}}) \cong M_x \otimes \Bbbk_e$$

is in $\boldsymbol{I}$. We also then have

$$(\Bbbk \otimes \Bbbk_y) \otimes (M_x \otimes \Bbbk_e) \otimes (\Bbbk \otimes \Bbbk_{y^{-1}}) \cong {}^y M_x \otimes \Bbbk_e$$

is an object of $\boldsymbol{I}$ as well. The ideal $\boldsymbol{I}$ then contains the direct sum

$$\widehat{M} := \bigoplus_{y \in H} {}^y M_x \otimes \Bbbk_e.$$

We claim that $\widehat{M}$ is in the image of $\bar{F}$; in other words, $\widehat{M}$ has a half-braiding which allows it to be lifted to the Drinfeld center. To see this, consider an $H_{G,L}$-module $N_z \otimes \Bbbk_z$. We observe that

$$\widehat{M} \otimes (N_z \otimes \Bbbk_z) \cong \bigoplus_{y \in L} ({}^y M_x \otimes N_z) \otimes \Bbbk_z, \qquad (N_z \otimes \Bbbk_z) \otimes \widehat{M} \cong \bigoplus_{y \in L} (N_z \otimes {}^{zy} M_x) \otimes \Bbbk_z.$$

Since $\Bbbk G$ is itself cocommutative (and thus ${}^y M_x \otimes N_z \cong N_z \otimes {}^y M_x$ in a natural way), this formula can be used to observe a natural isomorphism $\widehat{M} \otimes - \cong - \otimes \widehat{M}$. This isomorphism satisfies the half-braiding condition, and so $\widehat{M}$ is in the image of $\bar{F}$.

Since $\boldsymbol{I}$ and $\boldsymbol{J}$ are assumed to agree on their intersections with the image of $\bar{F}$, we can conclude that $\widehat{M}$ is in $\boldsymbol{J}$ as well. But then its summand $M_x \otimes \Bbbk_e$, and hence

$$(M_x \otimes \Bbbk_e) \otimes (\Bbbk \otimes \Bbbk_x) \cong M_x \otimes \Bbbk_x,$$

is also an object of $\boldsymbol{J}$. Note that we have proven generally that $M_x \otimes \Bbbk_x$ is in any thick ideal if and only if $\widehat{M}$, as constructed above, is in that ideal. Thus, the objects of $\boldsymbol{I}$ are a subset of the objects of $\boldsymbol{J}$, and by symmetry the ideals are equal. □

We can now prove Proposition 3.3.1, as a consequence of Lemma 3.3.3:

Proof. The map f is defined by

$$f(\boldsymbol{P}) = \{X \in \mathrm{st}(\mathrm{Z}(\boldsymbol{C})) : \bar{F}(X) \in \boldsymbol{P}\}$$

for a given prime ideal $\boldsymbol{P}$ in $\operatorname{Spc} \mathrm{st}(\boldsymbol{C})$. But Lemma 3.3.3 has shown that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two prime ideals with $f(\boldsymbol{P}) = f(\boldsymbol{Q})$, then since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are more generally examples of thick ideals, we have $\boldsymbol{P} = \boldsymbol{Q}$. Hence, f is injective, showing (1).

For (2), let S be an arbitrary specialization-closed set in $\operatorname{Spc} \mathrm{st}(\boldsymbol{C})$, in other words, a (possibly infinite) union $S = \bigcup_{i \in I} S_i$ where each S_i is a closed set. Recall that by construction, it is automatic that $\Phi_W(\Theta_W(S)) \subseteq S$ (the details are included above in the proof of Theorem 2.5.1).

To show the opposite containment, we note that by the classification of thick ideals and Balmer spectrum of $\mathrm{st}(\boldsymbol{C})$ as given in [Nakano et al. 2022a], $\operatorname{Spc} \mathrm{st}(\boldsymbol{C})$ is a Noetherian topological space. We claim that this implies that every closed set in $\operatorname{Spc} \mathrm{st}(\boldsymbol{C})$ has the form $V_{\boldsymbol{C}}(M)$, for some object M of $\mathrm{st}(\boldsymbol{C})$, just as in the commutative setting Theorem 1.1.4, using Lemma 3.3.3 as a substitute for the commutativity of the tensor product. Let $V_{\boldsymbol{C}}(\mathcal{T})$ be an arbitrary closed set in $\operatorname{Spc} \mathrm{st}(\boldsymbol{C})$, for some collection $\mathcal{T}$ of objects in

$\mathsf{st}(\boldsymbol{C})$. Then the complement of $V_{\boldsymbol{C}}(\mathcal{T})$ is by definition

$$U_{\boldsymbol{C}}(\mathcal{T}) = \{\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) : \boldsymbol{P} \cap \mathcal{T} \neq \varnothing\},$$

and has an open cover

$$U_{\boldsymbol{C}}(\mathcal{T}) = \bigcup_{A \in \mathcal{T}} U_{\boldsymbol{C}}(A) = \bigcup_{A \in \mathcal{T}} \{\boldsymbol{P} \in \operatorname{Spc}\mathsf{st}(\boldsymbol{C}) : A \in \boldsymbol{P}\}.$$

By Noetherianity, this set is compact, and hence has a finite subcover

$$U_{\boldsymbol{C}}(\mathcal{T}) = \bigcup_{A \in \mathcal{T}'} U_{\boldsymbol{C}}(A),$$

where $\mathcal{T}' \subseteq \mathcal{T}$ is some finite collection of objects. Enumerate the objects of $\mathcal{T}'$ by $A_1, \ldots, A_n$. Choose $\hat{A}_1, \ldots, \hat{A}_n$ as constructed in Lemma 3.3.3: they are in the image of $\bar{F}$, and for any thick ideal $\boldsymbol{I}$, we have $A_j \in \boldsymbol{I}$ if and only if $\hat{A}_j \in \boldsymbol{I}$. Using this property, it is clear that $V_{\boldsymbol{C}}(A_j) = V_{\boldsymbol{C}}(\hat{A}_j)$ for all j. Now we claim that

$$U_{\boldsymbol{C}}(\mathcal{T}) = U_{\boldsymbol{C}}(A_1) \cup \cdots \cup U_{\boldsymbol{C}}(A_n) = U_{\boldsymbol{C}}(\hat{A}_1) \cup \cdots \cup U_{\boldsymbol{C}}(\hat{A}_n) = U_{\boldsymbol{C}}(\hat{A}_1 \otimes \cdots \otimes \hat{A}_n).$$

The last equality (more specifically, the containment $\supseteq$) uses the fact that each $\hat{A}_j$ is in the image of $\bar{F}$, and hence commutes with all objects of $\mathsf{st}(\boldsymbol{C})$ up to isomorphism, since this implies that

$$\hat{A}_1 \otimes \cdots \otimes \hat{A}_n \in \boldsymbol{P} \Rightarrow A_j \in \boldsymbol{P} \text{ for some } j.$$

Our claim is now shown: every closed set in $\operatorname{Spc}\mathsf{st}(\boldsymbol{C})$ is of the form $V_{\boldsymbol{C}}(A)$ for some object A.

In particular, each of the closed sets S_i, for $i \in I$, can be written as $V_{\boldsymbol{C}}(M_i)$ for some object $M_i \in \mathsf{st}(\boldsymbol{C})$. As above, we can replace M_i by $\widehat{M}_i$, which is in the image of $\bar{F}$, i.e., we can pick an object X_i in $\mathsf{st}(\mathsf{Z}(\boldsymbol{C}))$ with $\bar{F}(X_i) = \widehat{M}_i$. Since

$$W(X_i) = V_{\boldsymbol{C}}(\bar{F}(X_i)) = V_{\boldsymbol{C}}(\widehat{M}_i) = V_{\boldsymbol{C}}(M_i) = S_i \subseteq S,$$

we have $X_i \in \Theta_W(S)$ by definition. Hence, we now have

$$\Phi_W(\Theta_W(S)) \supseteq \bigcup_{i \in I} W(X_i) = \bigcup_{i \in I} S_i = S.$$

Since we have both containments, we can conclude that $\Phi_W(\Theta_W(S)) = S$ for any specialization-closed set S in $\operatorname{Spc}\mathsf{st}(\boldsymbol{C})$. □

We also note that if p does not divide the order of L, then we can apply the results of the previous section to obtain:

Theorem 3.3.4. *Let G, L, $\Bbbk$, $H_{G,L}$, $\boldsymbol{C} = \mathsf{mod}(H_{G,L})$, and $\mathsf{Z}(\boldsymbol{C}) = \mathsf{mod}(D(H_{G,L}))$ be as above, and assume additionally that p does not divide the order of L. Then we have the following.*

(1) *The map f constructed in Section 2.1 is a homeomorphism*

$$\operatorname{Spc}\mathsf{st}(\boldsymbol{C}) \xrightarrow{\cong} \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\boldsymbol{C})).$$

(2) *The maps* Φ_W *and* Θ_W *constructed in Section 2.2 are mutually inverse*, *and so we have the following bijections of thick ideals*:

$$\mathrm{ThickId}\big(\mathrm{stmod}(D(H_{G,L}))\big) \underset{\Theta_W}{\overset{\Phi_W}{\rightleftarrows}} \mathcal{X}_{\mathrm{sp}}\big(\mathrm{Spc}(\mathrm{stmod}(H_{G,L}))\big) \underset{\Phi_{H_{G,L}}}{\overset{\Theta_{H_{G,L}}}{\rightleftarrows}} \mathrm{ThickId}(\mathrm{stmod}(H_{G,L})).$$

Proof. First, note that $H_{G,L}$ is cosemisimple: its dual is the smash product $\Bbbk[G]\#\Bbbk L$. Since p does not divide the order of L, the group algebra $\Bbbk L$ is semisimple, and by [Cohen and Fishman 1986, Theorem 6], as the smash product of two semisimple algebras, $\Bbbk[G]\#\Bbbk L$ is semisimple as well.

Next, we claim that $H_{G,L}$ is unimodular. This can be observed directly, by noting that the element

$$h := \Big(\sum_{g\in G} g\Big) \otimes p_1$$

is both a left and a right integral in $H_{G,L}$.

By application of Propositions 3.3.1 and 3.2.4, f is bijective and the maps Φ_W and Θ_W are inverse bijections. To conclude the proof, we must just prove that f is closed, and hence a homeomorphism. This follows similarly to the proof of Theorem 2.5.1(3a), except that we must again use Lemma 3.3.3 as a substitute for commutativity of the tensor product. Let $V_{\boldsymbol{C}}(M)$ an arbitrary closed set, and, just as before, we may assume (by replacing M with $\widehat{M}$ as in Lemma 3.3.3 if need be) that M is in the image of $\overline{F}$, and so we can pick $X \in \mathsf{Z}(\mathrm{st}(\boldsymbol{C}))$ with $\overline{F}(X) = M$. We now have

$$f(V_{\boldsymbol{C}}(M)) = \{f(\boldsymbol{P}) : \boldsymbol{P} \in \mathrm{Spc}\,\mathrm{st}(\boldsymbol{C}),\, M \notin \boldsymbol{P}\} = \{\boldsymbol{Q} \in \mathrm{Spc}\,\mathrm{st}(\mathsf{Z}(\boldsymbol{C})) : X \notin \boldsymbol{Q}\} = V_{\mathsf{Z}}(X).$$

The second equality follows from the fact that f is bijective. Hence, f is closed, and the theorem is complete. □

Acknowledgements

The author would like to thank Daniel Nakano and Milen Yakimov for many useful discussions, and also to thank Pavel Etingof, Siu-Hung Ng, Victor Ostrik, and Sean Sanford for providing valuable comments used to improve this paper. We also thank the anonymous referee for their careful reading of the paper and helpful suggestions.

References

[Achar et al. 2019] P. N. Achar, W. Hardesty, and S. Riche, "On the Humphreys conjecture on support varieties of tilting modules", *Transform. Groups* **24**:3 (2019), 597–657. MR Zbl

[Aljadeff et al. 2002] E. Aljadeff, P. Etingof, S. Gelaki, and D. Nikshych, "On twisting of finite-dimensional Hopf algebras", *J. Algebra* **256**:2 (2002), 484–501. MR Zbl

[Bakalov and Kirillov 2001] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series **21**, American Mathematical Society, Providence, RI, 2001. MR

[Balmer 2005] P. Balmer, "The spectrum of prime ideals in tensor triangulated categories", *J. Reine Angew. Math.* **588** (2005), 149–168. MR Zbl

[Balmer 2010] P. Balmer, "Spectra, spectra, spectra — tensor triangular spectra versus Zariski spectra of endomorphism rings", *Algebr. Geom. Topol.* **10**:3 (2010), 1521–1563. MR Zbl

[Balmer 2020] P. Balmer, "A guide to tensor-triangular classification", pp. 145–162 in *Handbook of homotopy theory*, edited by H. Miller, CRC Press, Boca Raton, FL, 2020. MR Zbl

[Balmer and Sanders 2017] P. Balmer and B. Sanders, "The spectrum of the equivariant stable homotopy category of a finite group", *Invent. Math.* **208**:1 (2017), 283–326. MR Zbl

[Bendel et al. 2014] C. P. Bendel, D. K. Nakano, B. J. Parshall, and C. Pillen, *Cohomology for quantum groups via the geometry of the nullcone*, Mem. Amer. Math. Soc. **1077**, 2014. MR Zbl

[Benson 1998] D. J. Benson, *Representations and cohomology*, *II*: *Cohomology of groups and modules*, 2nd ed., Cambridge Studies in Advanced Mathematics **31**, Cambridge University Press, 1998. MR Zbl

[Benson and Witherspoon 2014] D. Benson and S. Witherspoon, "Examples of support varieties for Hopf algebras with noncommutative tensor products", *Arch. Math.* (*Basel*) **102**:6 (2014), 513–520. MR Zbl

[Benson et al. 1997] D. J. Benson, J. F. Carlson, and J. Rickard, "Thick subcategories of the stable module category", *Fund. Math.* **153**:1 (1997), 59–80. MR Zbl

[Benson et al. 2008] D. Benson, S. B. Iyengar, and H. Krause, "Local cohomology and support for triangulated categories", *Ann. Sci. Éc. Norm. Supér.* (4) **41**:4 (2008), 573–619. MR Zbl

[Benson et al. 2012] D. J. Benson, S. Iyengar, and H. Krause, *Representations of finite groups: local cohomology and support*, Oberwolfach Seminars **43**, Springer, 2012. MR Zbl

[Bergh et al. 2021] P. A. Bergh, J. Y. Plavnik, and S. Witherspoon, "Support varieties for finite tensor categories: complexity, realization, and connectedness", *J. Pure Appl. Algebra* **225**:9 (2021), art. id. 106705. MR Zbl

[Boe et al. 2017a] B. D. Boe, J. R. Kujawa, and D. K. Nakano, "Tensor triangular geometry for classical Lie superalgebras", *Adv. Math.* **314** (2017), 228–277. MR Zbl

[Boe et al. 2017b] B. D. Boe, J. R. Kujawa, and D. K. Nakano, "Tensor triangular geometry for quantum groups", preprint, 2017. arXiv 1702.01289

[Buan et al. 2007] A. B. Buan, H. Krause, and Ø. Solberg, "Support varieties: an ideal approach", *Homology Homotopy Appl.* **9**:1 (2007), 45–74. MR Zbl

[Chari and Pressley 1994] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994. MR Zbl

[Chirvasitu 2014] A. Chirvasitu, "Cosemisimple Hopf algebras are faithfully flat over Hopf subalgebras", *Algebra Number Theory* **8**:5 (2014), 1179–1199. MR Zbl

[Chirvasitu et al. 2019] A. Chirvasitu, C. Walton, and X. Wang, "Gelfand–Kirillov dimension of cosemisimple Hopf algebras", *Proc. Amer. Math. Soc.* **147**:11 (2019), 4665–4672. MR Zbl

[Cohen and Fishman 1986] M. Cohen and D. Fishman, "Hopf algebra actions", *J. Algebra* **100**:2 (1986), 363–379. MR Zbl

[Drinfeld 1987] V. G. Drinfeld, "Quantum groups", pp. 798–820 in *Proceedings of the International Congress of Mathematicians* (Berkeley, CA, 1986), vol. 1, edited by A. M. Gleason, American Mathematical Society, Providence, RI, 1987. MR Zbl

[Etingof and Gelaki 1998] P. Etingof and S. Gelaki, "On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic", *Internat. Math. Res. Notices* **16** (1998), 851–864. MR Zbl

[Etingof and Ostrik 2004] P. Etingof and V. Ostrik, "Finite tensor categories", *Mosc. Math. J.* **4**:3 (2004), 627–654, 782–783. MR Zbl

[Etingof et al. 2015] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs **205**, American Mathematical Society, Providence, RI, 2015. MR Zbl

[Friedlander and Negron 2018] E. Friedlander and C. Negron, "Cohomology for Drinfeld doubles of some infinitesimal group schemes", *Algebra Number Theory* **12**:5 (2018), 1281–1309. MR Zbl

[Friedlander and Pevtsova 2007] E. M. Friedlander and J. Pevtsova, "Π-supports for modules for finite group schemes", *Duke Math. J.* **139**:2 (2007), 317–368. MR Zbl

[Friedlander and Suslin 1997] E. M. Friedlander and A. Suslin, "Cohomology of finite group schemes over a field", *Invent. Math.* **127**:2 (1997), 209–270. MR Zbl

[Garland and Lepowsky 1976] H. Garland and J. Lepowsky, "Lie algebra homology and the Macdonald–Kac formulas", *Invent. Math.* **34**:1 (1976), 37–76. MR Zbl

[Gelaki 1997] S. Gelaki, "Quantum groups of dimension pq^2", *Israel J. Math.* **102** (1997), 227–267. MR Zbl

[Ginzburg and Kumar 1993] V. Ginzburg and S. Kumar, "Cohomology of quantum groups at roots of unity", *Duke Math. J.* **69**:1 (1993), 179–198. MR Zbl

[Goodearl and Warfield 2004] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., London Mathematical Society Student Texts **61**, Cambridge University Press, 2004. MR Zbl

[Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series **119**, Cambridge University Press, 1988. MR Zbl

[Hennings 1996] M. Hennings, "Invariants of links and 3-manifolds obtained from Hopf algebras", *J. London Math. Soc.* (2) **54**:3 (1996), 594–624. MR Zbl

[Hochster 1969] M. Hochster, "Prime ideal structure in commutative rings", *Trans. Amer. Math. Soc.* **142** (1969), 43–60. MR Zbl

[Hopkins 1987] M. J. Hopkins, "Global methods in homotopy theory", pp. 73–96 in *Homotopy theory* (Durham, 1985), edited by E. Rees and J. D. S. Jones, London Math. Soc. Lecture Note Ser. **117**, Cambridge University Press, 1987. MR Zbl

[Kaplansky 1975] I. Kaplansky, *Bialgebras*, University of Chicago, 1975. MR Zbl

[Kashiwara and Schapira 2006] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundl. Math. Wissen. **332**, Springer, 2006. MR Zbl

[Kassel 1995] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer, 1995. MR Zbl

[Kauffman and Radford 1995] L. H. Kauffman and D. E. Radford, "Invariants of 3-manifolds derived from finite-dimensional Hopf algebras", *J. Knot Theory Ramifications* **4**:1 (1995), 131–162. MR Zbl

[Krause 2010] H. Krause, "Localization theory for triangulated categories", pp. 161–235 in *Triangulated categories*, edited by T. Holm et al., London Math. Soc. Lecture Note Ser. **375**, Cambridge University Press, 2010. MR Zbl

[Larson 1971] R. G. Larson, "Characters of Hopf algebras", *J. Algebra* **17** (1971), 352–368. MR Zbl

[Larson and Radford 1988a] R. G. Larson and D. E. Radford, "Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple", *J. Algebra* **117**:2 (1988), 267–289. MR Zbl

[Larson and Radford 1988b] R. G. Larson and D. E. Radford, "Semisimple cosemisimple Hopf algebras", *Amer. J. Math.* **110**:1 (1988), 187–195. MR Zbl

[Matsui and Takahashi 2017] H. Matsui and R. Takahashi, "Thick tensor ideals of right bounded derived categories", *Algebra Number Theory* **11**:7 (2017), 1677–1738. MR Zbl

[Montgomery 1993] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics **82**, American Mathematical Society, Providence, RI, 1993. MR Zbl

[Montgomery et al. 2016] S. Montgomery, M. D. Vega, and S. Witherspoon, "Hopf automorphisms and twisted extensions", *J. Algebra Appl.* **15**:6 (2016), art. id. 1650103. MR Zbl

[Nakano et al. 2022a] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry", *Amer. J. Math.* **144**:6 (2022), 1681–1724. MR Zbl

[Nakano et al. 2022b] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry and the tensor product property for support maps", *Int. Math. Res. Not.* **2022**:22 (2022), 17766–17796. MR Zbl

[Nakano et al. 2023] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "On the spectrum and support theory of a finite tensor category", *Math. Ann.* (online publication November 2023).

[Neeman 1992] A. Neeman, "The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel", *Ann. Sci. École Norm. Sup.* (4) **25**:5 (1992), 547–566. MR Zbl

[Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, 2001. MR Zbl

[Negron 2021] C. Negron, "Finite generation of cohomology for Drinfeld doubles of finite group schemes", *Selecta Math.* (*N.S.*) **27**:2 (2021), art. id. 26. MR Zbl

[Negron and Pevtsova 2023] C. Negron and J. Pevtsova, "Support for integrable Hopf algebras via noncommutative hypersurfaces", *Int. Math. Res. Not.* **2023**:3 (2023), 1882–1958. MR Zbl

[Negron and Plavnik 2022] C. Negron and J. Plavnik, "Cohomology of finite tensor categories: duality and Drinfeld centers", *Trans. Amer. Math. Soc.* **375**:3 (2022), 2069–2112. MR Zbl

[Nichols and Zoeller 1989] W. D. Nichols and M. B. Zoeller, "A Hopf algebra freeness theorem", *Amer. J. Math.* **111**:2 (1989), 381–385. MR Zbl

[Ostrik 1997] V. Ostrik, "Tensor ideals in the category of tilting modules", *Transform. Groups* **2**:3 (1997), 279–287. MR Zbl

[Plavnik and Witherspoon 2018] J. Y. Plavnik and S. Witherspoon, "Tensor products and support varieties for some noncocommutative Hopf algebras", *Algebr. Represent. Theory* **21**:2 (2018), 259–276. MR Zbl

[Reyes 2012] M. L. Reyes, "Obstructing extensions of the functor Spec to noncommutative rings", *Israel J. Math.* **192**:2 (2012), 667–698. MR Zbl

[Rickard 1997] J. Rickard, "Idempotent modules in the stable category", *J. London Math. Soc.* (2) **56**:1 (1997), 149–170. MR Zbl

[Shimizu 2017] K. Shimizu, "On unimodular finite tensor categories", *Int. Math. Res. Not.* **2017**:1 (2017), 277–322. MR Zbl

[Stevenson 2018] G. Stevenson, "A tour of support theory for triangulated categories through tensor triangular geometry", pp. 63–101 in *Building bridges between algebra and topology*, edited by D. Herbera et al., Springer, 2018. MR Zbl

[Thomason 1997] R. W. Thomason, "The classification of triangulated subcategories", *Compositio Math.* **105**:1 (1997), 1–27. MR

Communicated by Jason P. Bell
Received 2021-12-01 Revised 2023-06-05 Accepted 2023-09-03

kentv@mit.edu Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

ALGEBRA AND NUMBER THEORY 18:6 (2024)
https://doi.org/10.2140/ant.2024.18.1117

On the p-adic interpolation of unitary Friedberg–Jacquet periods

Andrew Graham

We establish functoriality of higher Coleman theory for certain unitary Shimura varieties and use this to construct a p-adic analytic function interpolating unitary Friedberg–Jacquet periods.

1. Introduction

The conjecture of Bloch–Kato describes a precise relation between special values of L-functions attached to geometric Galois representations and the dimension of the associated Bloch–Kato Selmer group (which can be seen as a generalization of the free part of the Mordell–Weil group for an abelian variety). One of the key tools in establishing cases of this conjecture is an Euler system — a collection of group cohomology classes for the Galois representation which, under a “nonvanishing criterion”, impose constraints on the size of the Bloch–Kato Selmer group; for example, see [Rubin 2000; Mazur and Rubin 2004]. The application to the Bloch–Kato conjecture then arises from a relation between this “nonvanishing criterion” and special values of the L-function; such a relation is commonly referred to as an *explicit reciprocity law*.

In the setting where the Galois representation is automorphic, it is often the case that these special L-values can be expressed as an automorphic period for a pair of reductive groups $(\boldsymbol{G}, \boldsymbol{H})$. If $(\boldsymbol{G}, \boldsymbol{H})$

MSC2020: 11F67, 11G18.
Keywords: p-adic L-functions, higher Coleman theory, automorphic cohomology.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY).
Open Access made possible by subscribing institutions via Subscribe to Open.

can be enhanced to a pair of Shimura data, then one can often describe this automorphic period as a pairing in coherent cohomology for the pair of associated Shimura varieties. This provides an arithmetic interpretation of the L-values, which can be related to the Euler system classes via a p-adic L-function (interpolating these automorphic periods and hence the L-values).

This present article describes the construction of a p-adic analytic function which should play the role of this p-adic L-function in an explicit reciprocity law for the anticyclotomic Euler system constructed in [Graham and Shah 2023] (or more precisely, its generalization to CM fields, which will appear in forthcoming work of the author, D. Barrera and C. Williams).[1] The construction crucially uses the recently developed higher Coleman theory of Boxer and Pilloni [2021] and the strategy is similar to the work of Loeffler and Zerbes [2021] and Loeffler, Pilloni, Skinner and Zerbes [Loeffler et al. 2021]. Furthermore, as a key ingredient, we p-adically interpolate the branching laws for representations of GL_{2n} and GL_{2n-1} restricted to $\mathrm{GL}_n \times \mathrm{GL}_n$ and $\mathrm{GL}_{n-1} \times \mathrm{GL}_n$ respectively (see Appendix A), using the fact that these pairs give rise to *spherical varieties*.

Unfortunately, our result is not optimal — there is a missing variable in this p-adic analytic function, which would therefore lead to a suboptimal version of an explicit reciprocity law (similar to the restriction in [Loeffler and Zerbes 2021]). To account for the missing variable, one would need to incorporate the p-adic variation of certain theta operators into the picture. This incorporation will be pursued in future work.

1.0.1. *Unitary Friedberg–Jacquet periods.* The p-adic analytic function we construct interpolates so-called *unitary Friedberg–Jacquet periods* for certain cuspidal automorphic representations of unitary groups, which is a variant of the automorphic periods for general linear groups studied by Friedberg and Jacquet [1993]. Although expected, it is not yet known (in general) whether these unitary Friedberg–Jacquet periods calculate L-values, but there has been a lot of recent work towards showing this; in particular:

- The "relative trace formula approach" in forthcoming work of Jingwei Xiao and Wei Zhang, and the work of Spencer Leslie [2019a; 2019b].
- Applications of the residue method in the work of Pollack, Wan and Zydor [Pollack et al. 2021].
- An approach via theta correspondences in the work of Chen and Gan [2021].

As a consequence of these works, we at least know that if certain values of this p-adic analytic function are nonvanishing then the corresponding (complex) L-values are also nonvanishing (see Corollary C below). We expect that there is an analogous version of Waldspurger's formula in this setting which will express (the square of) these values in terms of the complex L-values, but we do not attempt to establish such an identity in this article. Nevertheless, with these considerations in mind, we will henceforth refer to this p-adic analytic function as a p-adic L-function.

[1] In fact, we also show that these Euler system classes vary in Coleman families.

1.1. ***Statement of the results.*** Let F be a CM field with maximal totally real subfield F^+, and fix an odd rational prime p which splits completely in $F/\mathbb{Q}$. We impose the following assumptions:

Assumption 1.1.1. *We assume that*:

(1) *$F^+ \neq \mathbb{Q}$ and F contains an imaginary quadratic number field $E/\mathbb{Q}$.*

(2) *p does not divide the class number of F.*

Fix an integer $n \geq 1$. Let W be a $2n$-dimensional Hermitian space over F with signature $(1, 2n-1)$ at one place, and signature $(0, 2n)$ at the remaining places. Fix a decomposition $W = W_1 \oplus W_2$ of Hermitian spaces where each factor has dimension n, the signature of W_1 is $(1, n-1)$ at one place and $(0, n)$ at all remaining places, and the signature of W_2 is $(0, n)$ at all places. Let $\boldsymbol{G}$ be the reductive group over $\mathbb{Q}$ of unitary similitudes of W with similitude in $\mathbb{G}_m$. We let $\boldsymbol{H} \subset \boldsymbol{G}$ denote the subgroup preserving the decomposition $W = W_1 \oplus W_2$.

Let π be a discrete series cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$, and let $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ be an algebraic Hecke character which is *anticyclotomic* (i.e., its restriction to $\mathbb{A}_{F^+}^\times$ is trivial). Then for any $\varphi \in \pi$, we can consider the following automorphic period

$$\mathscr{P}_{\pi,\chi}(\varphi) := \int_{[\boldsymbol{H}]} \varphi(h) \cdot \chi\left(\frac{\det h_2}{\det h_1}\right) dh.$$

Here h_i denotes the component of h corresponding to the factor W_i, and $[\boldsymbol{H}] = \boldsymbol{H}(\mathbb{Q}) A_{\boldsymbol{G}}(\mathbb{A}) \backslash \boldsymbol{H}(\mathbb{A})$ with $A_{\boldsymbol{G}}$ denoting the maximal split subtorus of the center of $\boldsymbol{G}$ (which can be shown to lie in $\boldsymbol{H}$). For this to make sense, we also need to assume that the central character of π is trivial on $A_{\boldsymbol{G}}(\mathbb{A})$.

Let $\psi \boxtimes \Pi_0$ denote the (weak) automorphic base-change of π to $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$, as constructed in [Shin 2014]. We have the following conjecture of Xiao–Zhang; see [Chen and Gan 2021, Conjecture 7.4]:

Conjecture A. *With set-up as above, assume that π is tempered. Then there exists $\varphi \in \pi$ such that $\mathscr{P}_{\pi,\chi}(\varphi) \neq 0$ if and only if the following three conditions are satisfied*:

(1) *The standard L-function $L(\Pi_0 \otimes \chi, s)$ is nonvanishing at $s = \frac{1}{2}$.*

(2) *The exterior square L-function $L(\Pi_0, \bigwedge^2, s)$ has a pole at $s = 1$.*

(3) *There exists an irreducible constituent $\pi_0 \subset \pi|_{\boldsymbol{H}_0}$ such that, for every (finite) rational prime ℓ, the Hom-space satisfies*

$$\mathrm{Hom}_{\boldsymbol{H}_0(\mathbb{Q}_\ell)}(\pi_{0,\ell}, \chi^{-1} \circ \nu) \neq 0$$

where $\boldsymbol{H}_0 \subset \boldsymbol{H}$ is the kernel of the similitude character and ν is the character on $\boldsymbol{H}_0$ given by $\nu(h) = \det h_2 / \det h_1$.

Remark 1.1.2. Because we are working with unitary similitudes, this conjecture is presented in a slightly different way to [Chen and Gan 2021, Conjecture 7.4]. However the two statements are equivalent by Remark 8.2.8.

Suppose that π is ramified only at primes which split in $E/\mathbb{Q}$,[2] the base-change Π_0 is cuspidal and π satisfies a "small slope condition" at the prime p (see Assumption 6.1.4). Then, following [Boxer and Pilloni 2021, Section 6.9] and [Loeffler and Zerbes 2021], we show that there exists a unique family $\underline{\pi}$ of automorphic representations, passing though π and defined over a certain open affinoid subspace U of $n[F^+ : \mathbb{Q}]$-dimensional weight space $\mathcal{W}_G$. Here, by family we mean an $\mathcal{O}(U)$-valued system of eigenvalues for a certain collection of Hecke operators (see Definition 6.1.6) — for a classical point $x \in U$, the specialization of the family at x corresponds to a cohomological cuspidal automorphic representation $\underline{\pi}_x$ of $\boldsymbol{G}(\mathbb{A})$ (see Remark 6.2.6).

On the other hand, by Assumption 1.1.1(2), we can construct a family $\underline{\chi}$ of anticyclotomic characters defined over the $([F^+ : \mathbb{Q}] - 1)$-dimensional weight space $\mathcal{W}_H$ parametrizing characters of $(\mathbb{Z}_p^\times)^{[F^+:\mathbb{Q}]-1}$, which passes through the character χ. As above, for a point $x \in \mathcal{W}_H$, we let $\underline{\chi}_x$ denote the specialization of the family at x. The main result of the article is the following:

Theorem B (Corollary 8.2.4). *There exists a Zariski dense subset of classical weights $\Sigma^{\text{int}} \subset U \times \mathcal{W}_H$ and a p-adic analytic function $\mathscr{L}_p = \mathscr{L}_p(\underline{\eta}.\underline{\chi}) \in \mathcal{O}(U \times \mathcal{W}_H)$ which interpolates the periods $\mathscr{P}_{\underline{\pi}_x, \underline{\chi}_x}(\varphi_x)$ for $x \in \Sigma^{\text{int}}$ (where $\varphi_x \in \underline{\pi}_x$ is a certain nonzero choice of automorphic form).*

Combining this with [Chen and Gan 2021, Corollary 7.6] (and the fact that regular algebraic conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ are tempered [Caraiani 2012]), we see that:

Corollary C. *Let $x \in \Sigma^{\text{int}}$ and let $\psi_x \boxtimes \mathrm{BC}(\underline{\pi}_x)$ denote the automorphic base-change of $\underline{\pi}_x$ to a representation of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$. Suppose that $\mathrm{BC}(\underline{\pi}_x)$ is cuspidal. Then*

$$\mathscr{L}_p(x) \neq 0 \Rightarrow L\big(\mathrm{BC}(\underline{\pi}_x) \otimes \underline{\chi}_x, \tfrac{1}{2}\big) \neq 0.$$

The strategy we will use for constructing $\mathscr{L}_p$ consists of three key steps:

(1) Express the automorphic periods $\mathscr{P}_{\underline{\pi}_x, \underline{\chi}_x}(\varphi_x)$ as a cup product in the coherent cohomology of a Shimura variety associated with $\boldsymbol{H}$ involving (the restriction to $\boldsymbol{H}$) of a coherent cohomology class $\underline{\eta}_x$ corresponding to φ_x.

(2) Using higher Coleman theory one can reinterpret (1) in terms of a pairing in coherent cohomology over certain strata in the adic Shimura varieties for $\boldsymbol{G}$ and $\boldsymbol{H}$. In particular, this interpretation is amenable to p-adic interpolation provided that there exist families of cohomology classes $\underline{\eta}$ and $\underline{\chi}$ passing through $\underline{\eta}_x$ and $\underline{\chi}_x$ respectively.

(3) Under the above assumptions, we construct these families $\underline{\eta}$ and $\underline{\chi}$. The p-adic L-function $\mathscr{L}_p$ is then defined as a pairing between the classes $\underline{\eta}$ and $\underline{\chi}$.

Remark 1.1.3. Assumption 1.1.1(1) is imposed throughout the whole article, however assumption (2) is only imposed when showing the existence of certain anticyclotomic algebraic Hecke characters for F (which we expect can be removed by passing to a finite cover of weight space). In fact, it is likely that

[2] In the weakest possible sense, namely there does not exist a maximal special subgroup with nontrivial fixed points on the corresponding local component of π.

assumption (1) is not needed until Section 6 when applying the automorphic base-change results in [Shin 2014].

Remark 1.1.4 (Example 6.1.5). The "small slope condition" at the prime p is implied by (but more general than) a Borel-ordinarity condition on π (i.e., there exists an eigenvalue for the action of a suitably normalized Borel U_p-Hecke operator on π_p which is a p-adic unit).

Remark 1.1.5. To show the existence of the family $\underline{\pi}$ we need to implicitly use the results in [Mok 2015] and [Kaletha et al. 2014] on the endoscopic classification for unitary groups. As far as the author is aware, this work is still conditional on the stabilization of the twisted trace formula for $\boldsymbol{G}_0$ and $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2n}$ and their endoscopy groups.

1.2. *Notation.* Throughout this article, we fix a totally real number field $F^+ \neq \mathbb{Q}$ with a fixed embedding $\tau_0 \colon F^+ \hookrightarrow \mathbb{R}$. We fix a totally imaginary quadratic extension F/F^+ and a CM type Ψ for F, i.e., Ψ is a set of embeddings $F \hookrightarrow \mathbb{C}$ of size $[F^+ : \mathbb{Q}]$, with no two embeddings being equivalent to one another. We denote by τ_0 the element of Ψ which extends the embedding $\tau_0 \colon F^+ \hookrightarrow \mathbb{R}$. Let F^{cl} denote the Galois closure of F. We assume that F contains an imaginary quadratic number field E.

We fix an odd prime p which splits completely in $F/\mathbb{Q}$, and we fix an isomorphism $\iota_p \colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. Under this isomorphism every embedding $\tau \in \Psi$ gives rise to a prime ideal $\mathfrak{p}_\tau$ of F, lying above p.

We also fix the following notation and conventions throughout:

- For any split reductive group G, we let $w_G^{\max}$ denote the element of its Weyl group of maximal length.
- The group law on characters will be written additively, unless specified otherwise.
- Let G be a split reductive group with a fixed parabolic $P \subset G$ and Levi M, and let $T \subset P$ be a maximal torus. Then for any algebraic character κ of T which is M-dominant, we will write

$$\kappa^\vee = -w_M^{\max}\kappa - 2\rho_{nc}$$

  for the *Serre dual* of κ, where ρ_{nc} is the half-sum of positive roots not lying in M (with respect to a fixed Borel containing T and contained in P). We will also use the notation $(-)^\vee$ to refer to the Serre dual of a vector bundle on a scheme.
- We will use the terminology *neat* or *sufficiently small* to refer to a compact open subgroup of the finite adelic points of a reductive group satisfying [Graham and Shah 2023, Definition B.6].
- All torsors are right torsors unless specified otherwise.

2. Preliminaries

Let $n \geq 1$ be a positive integer. Let W denote a $2n$-dimensional Hermitian space over F which has signature $(1, 2n-1)$ with respect to the embedding τ_0, and signature $(0, 2n)$ at $\tau \in \Psi - \{\tau_0\}$. Fix a decomposition $W = W_1 \oplus W_2$ of Hermitian spaces, where W_i is a Hermitian space over F of dimension

n with signatures

$$\mathrm{signature}(W_i \otimes_{F,\tau} \mathbb{C}) = \begin{cases} (1, n-1) & \text{if } i = 1 \text{ and } \tau = \tau_0, \\ (0, n) & \text{otherwise.} \end{cases}$$

Denote the Hermitian pairings on W and W_i by $\langle \cdot , \cdot \rangle_W$ and $\langle \cdot , \cdot \rangle_{W_i}$ respectively.

Definition 2.0.1. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ denote the reductive groups over $\mathbb{Q}$ whose values on R-points, for a $\mathbb{Q}$-algebra R, are

$$\boldsymbol{G}(R) = \{g \in \mathrm{GL}(W \otimes_{\mathbb{Q}} R) : \langle g \cdot x, g \cdot y\rangle_W = c(g)\langle x, y\rangle_W \text{ for all } x, y \in W \otimes_{\mathbb{Q}} R \text{ and some } c(g) \in R^\times\},$$

$$\begin{aligned} \boldsymbol{H}(R) = \{g = (g_1, g_2) \in \mathrm{GL}(W_1 \otimes_{\mathbb{Q}} R) \times \mathrm{GL}(W_2 \otimes_{\mathbb{Q}} R) \\ : \langle g_i \cdot x_i, g_i \cdot y_i\rangle_{W_i} = c(g)\langle x_i, y_i\rangle_{W_i}, \text{ for all } x_i, y_i \in W_i \otimes_{\mathbb{Q}} R \text{ and } i = 1, 2, \text{ and some } c(g) \in R^\times\}. \end{aligned}$$

We also let $\boldsymbol{G}_0$ (resp. $\boldsymbol{H}_0$) denote the kernel of the similitude character $c\colon \boldsymbol{G} \to \mathbb{G}_m$ (resp. $c\colon \boldsymbol{H} \to \mathbb{G}_m$). Note that we have natural embeddings

$$\boldsymbol{H}_0 \hookrightarrow \boldsymbol{G}_0, \quad \boldsymbol{H} \hookrightarrow \boldsymbol{G}$$

both of which we will denote by ι.

Remark 2.0.2. If R is an F^{cl}-algebra (with fixed embedding $F^{\mathrm{cl}} \hookrightarrow R$), then we have an identification

$$W \otimes_{\mathbb{Q}} R = \bigoplus_{\tau \in \Psi} (W \otimes_{F,\tau} R \oplus W \otimes_{F,\bar{\tau}} R)$$

where $\tau\colon F \hookrightarrow R$ denotes the embedding obtained from precomposing the fixed embedding $F^{\mathrm{cl}} \hookrightarrow R$ with $\tau\colon F \hookrightarrow F^{\mathrm{cl}}$, and $\bar{\tau}\colon F \hookrightarrow R$ denotes its complex conjugate. Under this identification, one has

$$\boldsymbol{G}_{0,F^{\mathrm{cl}}} = \prod_{\tau \in \Psi} \mathrm{GL}_{2n,F^{\mathrm{cl}}}, \quad \boldsymbol{G}_{F^{\mathrm{cl}}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times \prod_{\tau \in \Psi} \mathrm{GL}_{2n,F^{\mathrm{cl}}}$$

where the latter is described by sending an element $g \in \boldsymbol{G}_{F^{\mathrm{cl}}}(R)$ to $(c(g), g|_{W \otimes_{F,\tau} R})_{\tau \in \Psi}$.

In particular, if p is a prime which splits completely in $F/\mathbb{Q}$ and we have an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, then we obtain a distinguished embedding $F \hookrightarrow \mathbb{Q}_p$ arising from τ_0 (and factoring through F^{cl}) and $\boldsymbol{G}_{\mathbb{Q}_p}$ is identified with $\mathrm{GL}_{1,\mathbb{Q}_p} \times \prod_{\tau \in \Psi} \mathrm{GL}_{2n,\mathbb{Q}_p}$.

Similarly, we have identifications

$$\boldsymbol{H}_{0,F^{\mathrm{cl}}} = \prod_{\tau \in \Psi} (\mathrm{GL}_{n,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}}), \quad \boldsymbol{H}_{F^{\mathrm{cl}}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times \prod_{\tau \in \Psi} (\mathrm{GL}_{n,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}})$$

and the embeddings $\boldsymbol{H}_{0,F^{\mathrm{cl}}} \xrightarrow{\iota} \boldsymbol{G}_{0,F^{\mathrm{cl}}}$ and $\boldsymbol{H}_{F^{\mathrm{cl}}} \xrightarrow{\iota} \boldsymbol{G}_{F^{\mathrm{cl}}}$ map the $\mathrm{GL}_{1,F^{\mathrm{cl}}}$-factor to itself, and for each $\tau \in \Psi$, map $\mathrm{GL}_{n,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}}$ into $\mathrm{GL}_{2n,F^{\mathrm{cl}}}$ block diagonally.

Using the identifications in Remark 2.0.2, we define the following parabolic subgroups:

- Let $B_{\boldsymbol{G}}$ (resp. $B_{\boldsymbol{H}}$) denote the upper-triangular Borel subgroup of $\boldsymbol{G}_{F^{\mathrm{cl}}}$ (resp. $\boldsymbol{H}_{F^{\mathrm{cl}}}$). We let T denote the standard maximal torus inside $B_{\boldsymbol{G}}$ (which also coincides with the standard maximal torus inside $B_{\boldsymbol{H}}$). In particular, elements of T can be described as tuples

$$(x; y_{1,\tau}, \ldots, y_{2n,\tau})_{\tau \in \Psi}$$

corresponding to the diagonal matrix

$$x \times \prod_{\tau \in \Psi} \operatorname{diag}(y_{1,\tau}, \ldots, y_{2n,\tau}) \in \mathrm{GL}_1 \times \prod_{\tau \in \Psi} \mathrm{GL}_{2n} .$$

- Let $P_{\boldsymbol{G}}$ denote the parabolic subgroup of $\boldsymbol{G}_{F^{\mathrm{cl}}}$ containing $B_{\boldsymbol{G}}$ with Levi given by

$$M_{\boldsymbol{G}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times (\mathrm{GL}_{1,F^{\mathrm{cl}}} \times \mathrm{GL}_{2n-1,F^{\mathrm{cl}}}) \times \prod_{\tau \in \Psi - \{\tau_0\}} \mathrm{GL}_{2n,F^{\mathrm{cl}}} .$$

Similarly, we let $P_{\boldsymbol{H}}$ denote the parabolic of $\boldsymbol{H}_{F^{\mathrm{cl}}}$ containing $B_{\boldsymbol{H}}$ with Levi given by

$$M_{\boldsymbol{H}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times (\mathrm{GL}_{1,F^{\mathrm{cl}}} \times \mathrm{GL}_{n-1,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}}) \times \prod_{\tau \in \Psi - \{\tau_0\}} (\mathrm{GL}_{n,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}})$$

so that $P_{\boldsymbol{H}} = P_{\boldsymbol{G}} \cap \boldsymbol{H}_{F^{\mathrm{cl}}}$ and $M_{\boldsymbol{H}} = M_{\boldsymbol{G}} \cap \boldsymbol{H}_{F^{\mathrm{cl}}}$.

- Let $T_0 \subset T$ denote the subtorus given by elements of the form

$$(x; y_{1,\tau}, \ldots, y_{n,\tau}, y_{n,\tau}, \ldots, y_{1,\tau})_{\tau \in \Psi}.$$

We now describe the relevant Weyl groups that will be used throughout this article.

Definition 2.0.3. For $? \in \{\boldsymbol{G}_{F^{\mathrm{cl}}}, \boldsymbol{H}_{F^{\mathrm{cl}}}, M_{\boldsymbol{G}}, M_{\boldsymbol{H}}\}$, let $W_?$ denote the associated Weyl group. Let ${}^M W_{\boldsymbol{G}}$ denote the set of Kostant representatives for the quotient $W_{M_{\boldsymbol{G}}} \backslash W_{\boldsymbol{G}_{F^{\mathrm{cl}}}}$. This set comprises of $2n$ elements

$${}^M W_{\boldsymbol{G}} = \{w_0, \ldots, w_{2n-1}\}$$

where the length of w_i is i. Similarly, ${}^M W_{\boldsymbol{H}}$ (the set of Kostant representatives for $W_{M_{\boldsymbol{H}}} \backslash W_{\boldsymbol{H}_{F^{\mathrm{cl}}}}$) is a set $\{w_0, \ldots, w_{n-1}\}$ where the length of w_i is i. We can (and do) choose representatives for the Weyl elements w_i in $\boldsymbol{G}$ such that the embedding $\boldsymbol{H} \hookrightarrow \boldsymbol{G}$ identifies ${}^M W_{\boldsymbol{H}}$ with the subset of ${}^M W_{\boldsymbol{G}}$ of elements of lengths $0, \ldots, n-1$ (which justifies the notation). In Section 2.4, we will make a specific choice of representative for w_n.

We let $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ denote the abelian group of algebraic characters of T. We also define $X^*(T/T_0) = \mathrm{Hom}(T/T_0, \mathbb{G}_m)$ which can naturally be viewed as a subgroup of $X^*(T)$ (by precomposing with the quotient $T \twoheadrightarrow T/T_0$). We identify elements of $X^*(T)$ with tuples of integers

$$(c_0; c_{1,\tau}, \ldots, c_{2n,\tau})_{\tau \in \Psi}$$

which correspond to the character mapping an element $(x; y_{1,\tau}, \dots, y_{2n,\tau})_{\tau\in\Psi} \in T$ to the quantity

$$x^{c_0} \prod_{\substack{\tau\in\Psi \\ i=1,\dots,2n}} y_{i,\tau}^{c_{i,\tau}}.$$

With this description, elements of $X^*(T/T_0)$ are identified with tuples as above, satisfying $c_0 = 0$ and $c_{i,\tau} + c_{2n+1-i,\tau} = 0$ for all $\tau \in \Psi$ and $i = 1, \dots, 2n$. We let $X^*(T)^+ \subset X^*(T)$ denote the cone of dominant characters, i.e., tuples of integers as above which satisfy $c_{1,\tau} \geq \cdots \geq c_{2n,\tau}$ for all $\tau \in \Psi$, and we set $X^*(T/T_0)^+ = X^*(T/T_0) \cap X^*(T)^+$.

The Weyl group $W_{\boldsymbol{G}}$ naturally acts on $X^*(T)$ by the formula

$$w \cdot \lambda(t) = \lambda(w^{-1}tw), \quad w \in W_{\boldsymbol{G}}, t \in T.$$

In particular the set ${}^M W_{\boldsymbol{G}}$ acts by shuffles, i.e., w_0 acts as the identity, and for $i = 1, \dots, 2n-1$, one has the following description:

$$w_i \cdot (c_0; c_{1,\tau}, \dots, c_{2n,\tau})_{\tau\in\Psi} = (c_0; c_{i+1,\tau_0}, c_{1,\tau_0}, \dots, c_{i,\tau_0}, c_{i+2,\tau_0}, \dots, c_{2n,\tau_0}; c_{1,\tau}, \dots, c_{2n,\tau})_{\tau\in\Psi-\{\tau_0\}}.$$

Definition 2.0.4. Let $\rho \in \frac{1}{2}X^*(T)$ denote the half-sum of the positive roots of $\boldsymbol{G}_F$ with respect to the Borel $B_{\boldsymbol{G}}$. Explicitly, this is given by

$$\rho = \left(0; \tfrac{1}{2}(2n-1), \tfrac{1}{2}(2n-3), \dots, \tfrac{1}{2}(3-2n), \tfrac{1}{2}(1-2n)\right)_{\tau\in\Psi}.$$

Let $\rho_c \in \frac{1}{2}X^*(T)$ (resp. $\rho_{nc} \in \frac{1}{2}X^*(T)$) denote the half-sum of positive roots which lie in $M_{\boldsymbol{G}}$ (resp. do not lie in $M_{\boldsymbol{G}}$). Explicitly, the components of ρ_c (resp. ρ_{nc}) agree with ρ on the GL_1-factor and on the GL_{2n}-factor for $\tau \neq \tau_0$ (resp. are zero on the $\tau \neq \tau_0$ factor), but the τ_0-factors are given

$$(0, n-1, n-2, \dots, 2-n, 1-n) \quad \text{and} \quad \left(\tfrac{1}{2}(2n-1), -\tfrac{1}{2}, -\tfrac{1}{2}, \dots, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

respectively.

We conclude this section by introducing notation for the categories of algebraic representations of $M_{\boldsymbol{G}}$ and $M_{\boldsymbol{H}}$.

Notation 2.0.5. Let $\mathrm{Rep}(M_{\boldsymbol{G}})$ (resp. $\mathrm{Rep}(M_{\boldsymbol{H}})$) denote the category of finite-dimensional algebraic representations of $M_{\boldsymbol{G}}$ (resp. $M_{\boldsymbol{H}}$).

2.1. *Shimura varieties.* We consider the following Shimura data for the groups $\boldsymbol{G}$ and $\boldsymbol{H}$. Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ denote the Deligne torus. Recall from Remark 2.0.2 that we have an identification

$$W \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\tau\in\Psi} (W \otimes_{F,\tau} \mathbb{C} \oplus W \otimes_{F,\bar{\tau}} \mathbb{C}).$$

For an embedding $\tau \colon F \hookrightarrow \mathbb{C}$, each piece $W_\tau := W \otimes_{F,\tau} \mathbb{C}$ comes equipped with a Hermitian pairing by base-extension of $\langle \cdot, \cdot \rangle_W$. We fix a decomposition $W_\tau = W_\tau^+ \oplus W_\tau^-$ into maximal subspaces where the induced pairing is positive (resp. negative) definite. We define the following Hodge structure (of type $\{(-1,0),(0,-1)\}$)

$$W \otimes_{\mathbb{Q}} \mathbb{C} = W^{(-1,0)} \oplus W^{(0,-1)}$$

by imposing that

$$W^{(-1,0)} := \bigoplus_{\tau\in\Psi}(W_\tau^+ \oplus W_{\bar\tau}^-), \quad W^{(0,-1)} := \bigoplus_{\tau\in\Psi}(W_\tau^- \oplus W_{\bar\tau}^+).$$

This defines a homomorphism $h_{\boldsymbol{G}}\colon \mathbb{S} \to \boldsymbol{G}_{\mathbb{R}}$. We have a similar description for $h_{\boldsymbol{H}}$ and we can arrange it in such a way that $h_{\boldsymbol{G}} = \iota \circ h_{\boldsymbol{H}}$.

Let $\mu_{\boldsymbol{G}}$ denote the restriction of $h_{\boldsymbol{G},\mathbb{C}}$ to the first component in the identification $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Then (after possibly conjugating $h_{\boldsymbol{G}}$ by an element of $\boldsymbol{G}(\mathbb{R})$) under the identification in Remark 2.0.2, the cocharacter $\mu_{\boldsymbol{G}}$ is given by

$$\begin{aligned}\mu_{\boldsymbol{G}}\colon \mathbb{G}_{m,\mathbb{C}} &\to \mathrm{GL}_{1,\mathbb{C}} \times \prod_{\tau\in\Psi}\mathrm{GL}_{2n,\mathbb{C}},\\ z &\mapsto z \times \mathrm{diag}(z,1,\dots,1) \times \prod_{\tau\in\Psi-\{\tau_0\}}\mathrm{diag}(1,\dots,1).\end{aligned}$$

In particular, $\mu_{\boldsymbol{G}}$ is defined over F^{cl}. Furthermore, the field of definition of the $\boldsymbol{G}(\mathbb{C})$-conjugacy class of $\mu_{\boldsymbol{G}}$ is F, because of the conditions on the signatures and our assumption that F contains an imaginary quadratic number field. Note that $\mu_{\boldsymbol{G}}$ is of the form $\iota \circ \mu_{\boldsymbol{H}}$ for a cocharacter $\mu_{\boldsymbol{H}}\colon \mathbb{G}_{m,\mathbb{C}} \to \boldsymbol{H}_{\mathbb{C}}$, and this cocharacter coincides with the one obtained from $h_{\boldsymbol{H}}$ similar to above. The field of definition of the $\boldsymbol{H}(\mathbb{C})$-conjugacy class of $\mu_{\boldsymbol{H}}$ is also F, and the cocharacter $\mu_{\boldsymbol{H}}$ is defined over F^{cl}.

Remark 2.1.1. The centralizer of $\mu_{\boldsymbol{G}}$ (resp. $\mu_{\boldsymbol{H}}$) in $\boldsymbol{G}_{F^{\mathrm{cl}}}$ (resp. $\boldsymbol{H}_{F^{\mathrm{cl}}}$) is $M_{\boldsymbol{G}}$ (resp. $M_{\boldsymbol{H}}$).

Lemma 2.1.2. *The data $(\boldsymbol{G}, h_{\boldsymbol{G}})$ and $(\boldsymbol{H}, h_{\boldsymbol{H}})$ define Shimura–Deligne data in the sense of* [Graham and Shah 2023, Appendix B], *and additionally satisfy* (*SD*5). *The datum $(\boldsymbol{G}, h_{\boldsymbol{G}})$ is a Shimura datum in the usual sense. The reflex field for both of these data is F.*

For a neat compact open subgroup $K \subset \boldsymbol{G}(\mathbb{A}_f)$, we let $S_{\boldsymbol{G},K}$ denote the associated Shimura variety over the reflex field F. Similarly, for a neat compact open subgroup $U \subset \boldsymbol{H}(\mathbb{A}_f)$, we let $S_{\boldsymbol{H},U}$ denote the associated Shimura–Deligne variety over the reflex field F (a canonical model exists as the connected component of the PEL-type moduli problem associated with $\boldsymbol{H}$ and $h_{\boldsymbol{H}}$). If $\iota(U) \subset K$, then we have an induced finite unramified morphism

$$\iota\colon S_{\boldsymbol{H},U} \to S_{\boldsymbol{G},K}.$$

We note that $S_{\boldsymbol{H},U}$ and $S_{\boldsymbol{G},K}$ are smooth projective varieties, because we have assumed $F^+ \neq \mathbb{Q}$ (for example, the conditions in [Lan 2013, Section 5.3.3] are satisfied).

Convention 2.1.3. From now on, all of the Shimura–Deligne varieties we consider will be base-changed to F^{cl} (or a field extension of F^{cl}) via the embedding $\tau_0\colon F \hookrightarrow F^{\mathrm{cl}}$, but we will suppress this from the notation.

2.2. ***Automorphic vector bundles.*** In this section, we recall the construction of automorphic vector bundles on $S_{\boldsymbol{G},K}$.

Let $P_{\boldsymbol{G}}^{\mathrm{std}}$ denote the opposite of $P_{\boldsymbol{G}}$ with respect to the torus T, and consider the flag variety $\mathrm{FL}_{\boldsymbol{G}}^{\mathrm{std}} := \boldsymbol{G}/P_{\boldsymbol{G}}^{\mathrm{std}}$. Let $X_{\boldsymbol{G}}$ denote the $\boldsymbol{G}(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to \boldsymbol{G}_{\mathbb{R}}$ containing $h_{\boldsymbol{G}}$, which is a Hermitian symmetric domain. Then we have a holomorphic embedding (the Borel embedding)

$$\beta\colon X_{\boldsymbol{G}} \hookrightarrow \mathrm{FL}_{\boldsymbol{G}}^{\mathrm{std}}(\mathbb{C}).$$

Definition 2.2.1. Let $K \subset \boldsymbol{G}(\mathbb{A}_f)$ be a sufficiently small compact open subgroup. For an algebraic representation V of $P_{\boldsymbol{G}}^{\mathrm{std}}$, let $[V]$ denote the vector bundle on $S_{\boldsymbol{G},K}(\mathbb{C})$ defined as

$$[V] := \boldsymbol{G}(\mathbb{Q})\backslash \beta^*(V) \times \boldsymbol{G}(\mathbb{A}_f)/K$$

where we view V as a $\boldsymbol{G}(\mathbb{C})$-homogeneous vector bundle on $\mathrm{FL}_{\boldsymbol{G}}^{\mathrm{std}}(\mathbb{C})$ in the usual way.

Remark 2.2.2. One can show that $[V]$ descends to an algebraic vector bundle on $S_{\boldsymbol{G},K}$; see [Milne 1990, Section III] for example.

Definition 2.2.3. The association in Definition 2.2.1 defines a functor

$$[-] = [-]_K\colon\ \mathrm{Rep}(M_{\boldsymbol{G}}) \to \mathrm{VB}(S_{\boldsymbol{G},K})$$

by inflating a representation of $M_{\boldsymbol{G}}$ to one of $P_{\boldsymbol{G}}^{\mathrm{std}}$, where $\mathrm{VB}(-)$ denotes the category of vector bundles on a scheme. This functor is compatible with varying K, in the sense that if $g \in \boldsymbol{G}(\mathbb{A}_f)$ and $L \subset g^{-1}Kg$, then $g^*[-]_K = [-]_L$. Here g^* denotes pullback under the map $S_{\boldsymbol{G},L} \to S_{\boldsymbol{G},K}$ induced from right-translation by g.

We have a similar description of automorphic vector bundles over $S_{\boldsymbol{H},U}$ arising from algebraic representations of $M_{\boldsymbol{H}}$, and one has the relation

$$\iota^*[V] = [V|_{M_{\boldsymbol{H}}}]$$

where V is an algebraic representation of $M_{\boldsymbol{G}}$ and $\iota\colon S_{\boldsymbol{H},U} \to S_{\boldsymbol{G},K}$ is the finite unramified morphism at the end of the previous section.

Example 2.2.4 [Boxer and Pilloni 2021, Section 4.2.8]. Let $V_{-2\rho_{nc}}$ denote the irreducible algebraic representation of $M_{\boldsymbol{G}}$ with highest weight $-2\rho_{nc}$ (see Definition 2.0.4). Then $[V_{-2\rho_{nc}}] \cong \Omega^{2n-1}_{S_{\boldsymbol{G},K}}$.

2.3. ***Discrete series representations.*** Let $K_\infty \subset \boldsymbol{G}(\mathbb{R})$ denote the stabilizer of $h_{\boldsymbol{G}}$ under the adjoint action. Explicitly, this has the following description. Upon base-change to $\mathbb{R}$, one has the following identification

$$W \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{\tau \in \Psi} (W \otimes_{F^+,\tau} \mathbb{R})$$

where each summand is a $2n$-dimensional Hermitian space over $\mathbb{C}$. In particular, $W \otimes_{\mathbb{Q}} \mathbb{R}$ is a Hermitian space over $\mathbb{C}$, and the fixed decomposition

$$W \otimes_{\mathbb{Q}} \mathbb{C} = \Big(\bigoplus_{\tau \in \Psi} W_\tau^+ \oplus W_{\bar{\tau}}^+\Big) \oplus \Big(\bigoplus_{\tau \in \Psi} W_\tau^- \oplus W_{\bar{\tau}}^-\Big)$$

descends to a decomposition $W \otimes_{\mathbb{Q}} \mathbb{R} = W^+ \oplus W^-$ into maximal subspaces where the Hermitian pairing is positive (resp. negative) definite. Then K_∞ can be described as the subgroup of $\boldsymbol{G}(\mathbb{R})$ preserving the decomposition $W \otimes_{\mathbb{Q}} \mathbb{R} = W^+ \oplus W^-$. In particular, the complexification of K_∞ is equal to $M_{\boldsymbol{G}}(\mathbb{C})$.

Let H_∞ denote the compact (mod center) Cartan subgroup of K_∞ whose complexification is equal to $T(\mathbb{C})$. Then algebraic characters of H_∞ can be identified with tuples $(c_0; c_{1,\tau}, \dots, c_{2n,\tau}) \in X^*(T)$ satisfying the parity condition

$$c_0 \equiv \sum_{\tau \in \Psi} \sum_{i=1}^{2n} c_{i,\tau} \quad \text{modulo } 2.$$

For any dominant algebraic character λ of H_∞ and $i = 0, \dots, 2n-1$, we set $\xi_i := w_i \cdot (\lambda + \rho)$. Then ξ_i is the Harish-Chandra parameter of a discrete series representation $\pi(\xi_i)$ of $\boldsymbol{G}(\mathbb{R})$ (see [Blasius et al. 1994, Section 3]) and the local L-packet containing this representation is of the form

$$\{\pi(\xi_0), \dots, \pi(\xi_{2n-1})\}.$$

Therefore, discrete series L-packets of $\boldsymbol{G}(\mathbb{R})$ are parametrized by dominant algebraic characters of H_∞. One has a similar description for discrete series L-packets of $\boldsymbol{G}_0(\mathbb{R})$.

Remark 2.3.1. Note that discrete series L-packets of both $\boldsymbol{G}_0(\mathbb{R})$ and $\boldsymbol{G}(\mathbb{R})$ have size $2n$, because K_∞ differs from the maximal compact subgroup of $\boldsymbol{G}_{\mathrm{der}}(\mathbb{R})$ by the center of $\boldsymbol{G}(\mathbb{R})$. In particular, if $\pi(\xi_i)$ is a discrete series representation of $\boldsymbol{G}(\mathbb{R})$ as above, then

$$\pi(\xi_i)|_{\boldsymbol{G}_0(\mathbb{R})} = \pi(\xi_i')$$

where ξ_j' denotes the restriction of ξ_j to $H_\infty \cap \boldsymbol{G}_0(\mathbb{R})$ and $\pi(\xi_j')$ denotes the discrete series representation of $\boldsymbol{G}_0(\mathbb{R})$ with Harish-Chandra parameter ξ_j'.

For convenience, we introduce the following dictionary of weights and parameters. Let λ be a dominant algebraic character of H_∞. Then

(1) (Harish-Chandra parameters) The Harish-Chandra parameters in the L-packet parametrized by λ are given by

$$\xi_i = w_i \cdot (\lambda + \rho)$$

for $i = 0, \dots, 2n-1$.

(2) (Blattner parameters) The Blattner parameters associated with λ are

$$\nu_i = w_i \cdot (\lambda + 2\rho) - 2\rho_c.$$

In particular, the lowest $K_\infty \cap \boldsymbol{G}_0(\mathbb{R})$-type of $\pi(\xi_i')$ has highest weight given by (the restriction to $H_\infty \cap \boldsymbol{G}_0(\mathbb{R})$ of) ν_i. This implies that

$$\dim \operatorname{Hom}_{K_\infty}(\nu_j, \pi(\xi_i)) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

(3) (Vector bundle weights) If we let $\lambda^* = -w_G^{\max}\lambda$, then the vector bundle weights are

$$\kappa_i = w_i \star \lambda^* := w_i \cdot (\lambda^* + \rho) - \rho.$$

In the notation of [Boxer and Pilloni 2021], we have

$$C(\kappa_i)^- = \{w \in W_{\boldsymbol{G}} : w^{-1}(\kappa_i + \rho) \in X^*(T)^+_{\mathbb{Q}}\} = \{w_i\}$$

so we expect the coherent cohomology of $[V_{\kappa_i}]$ to be concentrated in degree $\ell_-(w_i) = 2n - 1 - i$ (at least on small slope parts). Let $\mathfrak{p} = \operatorname{Lie} P_{\boldsymbol{G}}^{\mathrm{std}}$ and $\mathfrak{m} = \operatorname{Lie} M_{\boldsymbol{G}}$, then for $i = 0, \dots, 2n-1$, $\bigwedge^i \mathfrak{p}/\mathfrak{m}$ is an irreducible algebraic representation of $M_{\boldsymbol{G}}$ under the adjoint action. If we let α_i denote the highest weight of this representation, then the vector bundle weights and Blattner parameters are related by the formula:

$$\nu_i = \alpha_i - w_{M_G}^{\max}\kappa_{2n-1-i}.$$

2.4. ***Some important elements.*** Recall that we have identifications

$$\boldsymbol{G}_{F^{\mathrm{cl}}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times \prod_{\tau\in\Psi} \mathrm{GL}_{2n,F^{\mathrm{cl}}} \quad \text{and} \quad \boldsymbol{H}_{F^{\mathrm{cl}}} = \mathrm{GL}_{1,F^{\mathrm{cl}}} \times \prod_{\tau\in\Psi} (\mathrm{GL}_{n,F^{\mathrm{cl}}} \times \mathrm{GL}_{n,F^{\mathrm{cl}}}).$$

In particular $\boldsymbol{G}_{F^{\mathrm{cl}}}$ and $\boldsymbol{H}_{F^{\mathrm{cl}}}$ (and the algebraic subgroups considered throughout this section) have models over $\mathcal{O} := \mathcal{O}_{F^{\mathrm{cl}}}$, which we will denote by the same letters.

Let $w_n \in {}^M W_{\boldsymbol{G}}$ denote the Weyl element of length n. We will now make explicit a choice of representative (which we will also denote w_n) in $\boldsymbol{G}(\mathcal{O})$ which represents the element $w_n \in {}^M W_{\boldsymbol{G}}$.

Definition 2.4.1. Let $w_n = 1 \times \prod_{\tau\in\Psi}(w_n)_\tau \in \boldsymbol{G}(\mathcal{O})$ be the element where $(w_n)_\tau = \mathrm{id}$ for $\tau \neq \tau_0$, and $(w_n)_{\tau_0}$ is the matrix

$$[(w_n)_{\tau_0}]_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (1, n+1), \\ 1 & \text{if } j = i-1, 2 \le i \le n+1, \\ 1 & \text{if } i = j \ge n+2, \\ 0 & \text{otherwise.} \end{cases}$$

The following elements are key to the whole construction in this paper.

Definition 2.4.2. Let $u'_{\tau_0} \in \mathrm{GL}_{2n-1}(\mathcal{O})$ denote the matrix whose (i,j)-th element is

$$(u'_{\tau_0})_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } j = 2n - i, i \le n, \\ 0 & \text{otherwise,} \end{cases}$$

and we let $u_{\tau_0} = 1 \times u'_{\tau_0} \in \mathrm{GL}_1(\mathcal{O}) \times \mathrm{GL}_{2n-1}(\mathcal{O})$. For $\tau \neq \tau_0$, we let $u_\tau \in \mathrm{GL}_{2n}(\mathcal{O})$ denote the block matrix (with block size $(n \times n)$) given by

$$u_\tau = \begin{pmatrix} 1 & \\ w^{\max}_{\mathrm{GL}_n} & 1 \end{pmatrix}$$

where $w^{\max}_{\mathrm{GL}_n}$ denotes the antidiagonal matrix with 1s along the antidiagonal (which represents the longest Weyl element in W_{GL_n}). We let $u \in M_{\boldsymbol{G}}(\mathcal{O})$ be the element $u = 1 \times \prod_{\tau \in \Psi} u_\tau$.

Denote by x_{τ_0} the $(1 \times 2n-1)$-matrix whose first n entries are 1 and the rest are 0. We let $\gamma_{\tau_0} \in \mathrm{GL}_{2n}(\mathcal{O})$ denote the block matrix

$$\gamma_{\tau_0} = u_{\tau_0} \cdot \begin{pmatrix} 1 & x_{\tau_0} \\ & 1 \end{pmatrix}$$

and we set $\gamma_\tau = u_\tau \in \mathrm{GL}_{2n}(\mathcal{O})$ for $\tau \neq \tau_0$. Define $\gamma \in P_{\boldsymbol{G}}(\mathcal{O})$ to be $\gamma := 1 \times \prod_\tau \gamma_\tau$.

Finally, we define $\hat{\gamma} := \gamma \cdot w_n \in \boldsymbol{G}(\mathcal{O})$ (with the specific choice of w_n fixed above).

Here are some key properties of these elements.

Lemma 2.4.3. (1) *The orbit $M_{\boldsymbol{H}} \cdot u \cdot B_{M_{\boldsymbol{G}}}$ is Zariski open in $M_{\boldsymbol{G}}$ (over* Spec $\mathcal{O}$), *where $B_{M_{\boldsymbol{G}}}$ denotes the standard Borel of $M_{\boldsymbol{G}}$.*

(2) *The orbit $\boldsymbol{H} \cdot \hat{\gamma} \cdot B_{\boldsymbol{G}}$ is Zariski open in $\boldsymbol{G}$ (over* Spec $\mathcal{O}$).

Proof. It is enough to check that the stabilizer $M_{\boldsymbol{H}} \cap u B_{M_{\boldsymbol{G}}} u^{-1}$ (resp. $\boldsymbol{H} \cap \hat{\gamma} B_{\boldsymbol{G}} \hat{\gamma}^{-1}$) for the action of $M_{\boldsymbol{H}}$ (resp. $\boldsymbol{H}$) on the flag variety $M_{\boldsymbol{G}}/B_{M_{\boldsymbol{G}}}$ (resp. $\boldsymbol{G}/B_{\boldsymbol{G}}$) has the required dimension. But an explicit calculation shows that

$$\begin{aligned} M_{\boldsymbol{H}} \cap u B_{M_{\boldsymbol{G}}} u^{-1} = \mathrm{GL}_1 \times &\{\mathrm{diag}(x_1, x_2, \dots, x_{n+1}, x_n, \dots, x_2) \in \mathrm{GL}_1 \times \mathrm{GL}_{n-1} \times \mathrm{GL}_n\} \\ &\times \prod_{\tau \neq \tau_0} \{\mathrm{diag}(y_1, \dots, y_n, y_n, \dots, y_1) \in \mathrm{GL}_n \times \mathrm{GL}_n\} \end{aligned}$$

which proves part (1). For part (2), we separate the calculation into three separate cases depending on the decomposition of $\boldsymbol{H}$ and $\boldsymbol{G}$ into general linear groups, namely the GL_1-component, the τ_0-component and the τ-component for $\tau \neq \tau_0$.

There is nothing to check for the GL_1-component, and the $\tau \neq \tau_0$-component follows from the computation as in part (1). So we are left to prove the lemma for the τ_0-component. One can find $X, Z \in \mathrm{GL}_n(\mathcal{O})$, Y an $n \times n$-matrix with entries in $\mathcal{O}$, such that:

- Z is upper triangular.
- $X w^{\max}_{\mathrm{GL}_n} = U$ is block upper triangular and lies in the standard parabolic of GL_n with Levi $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$. Its projection to the Levi is $1 \times w^{\max}_{\mathrm{GL}_{n-1}}$.
- One has the equality

$$\hat{\gamma}_{\tau_0} = \begin{pmatrix} X & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ w^{\max}_{\mathrm{GL}_n} & 1 \end{pmatrix} \begin{pmatrix} 1 & Y \\ & Z \end{pmatrix}.$$

We therefore find that, for $h = (A, B) \in \mathrm{GL}_n \times \mathrm{GL}_n$, $\hat{\gamma}_{\tau_0}^{-1} h \hat{\gamma}_{\tau_0}$ lies in the standard Borel of GL_{2n} if and only if $U^{-1}AU$ (resp. B) is lower (resp. upper triangular) and $B = U^{-1}AU$. This gives the required dimension for the stabilizer. $\square$

2.5. *Level subgroups at p.* Let p be a prime which splits completely in $F/\mathbb{Q}$, and fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. Then, as in Remark 2.0.2, we have identifications

$$G := \boldsymbol{G}_{\mathbb{Q}_p} = \mathrm{GL}_{1,\mathbb{Q}_p} \times \prod_{\tau \in \Psi} \mathrm{GL}_{2n,\mathbb{Q}_p} \quad \text{and} \quad H := \boldsymbol{H}_{\mathbb{Q}_p} = \mathrm{GL}_{1,\mathbb{Q}_p} \times \prod_{\tau \in \Psi} (\mathrm{GL}_{n,\mathbb{Q}_p} \times \mathrm{GL}_{n,\mathbb{Q}_p}).$$

Remark 2.5.1. Note that the choice of $\mathcal{O}$-models in the previous section give rise to $\mathbb{Z}_p$-models of G, H, and the various subgroups under consideration. We will denote these models by the same letters. For various objects attached to $\boldsymbol{G}$ and $\boldsymbol{H}$, we will use nonbold letters to indicate their analogue for the groups G and H. For example, will write M_G for $M_{\boldsymbol{G},\mathbb{Q}_p}$.

We introduce the following level subgroups:

Definition 2.5.2. (1) For $t \geq 1$, let $K^G_{\mathrm{Iw}}(p^t) \subset G(\mathbb{Z}_p)$ denote the depth t upper triangular Iwahori of G, i.e., all elements in $G(\mathbb{Z}_p)$ which land in B_G modulo p^t. We also use the same definition for H.

(2) For $t \geq 1$, we let $K^H_{\diamond}(p^t) \subset H(\mathbb{Q}_p)$ denote the subgroup $H(\mathbb{Q}_p) \cap \hat{\gamma} K^G_{\mathrm{Iw}}(p^t) \hat{\gamma}^{-1}$, where $\hat{\gamma}$ is treated as an element of $G(\mathbb{Z}_p)$.

We have the following:

Lemma 2.5.3. *The subgroup $K^H_{\diamond}(p^t)$ is contained in $K^H_{\mathrm{Iw}}(p^t)$. Furthermore, one has*

$$[K^H_{\diamond}(p^t) : K^H_{\diamond}(p^{t+1})] = [K^G_{\mathrm{Iw}}(p^t) : K^G_{\mathrm{Iw}}(p^{t+1})] = p^{dn(2n-1)}$$

where $d = [F^+ : \mathbb{Q}]$.

Proof. For the first part, the computation for the GL_1-component and $\tau \neq \tau_0$-component follows from the stabilizer computations in Lemma 2.4.3. For the τ_0-component, with notation as in Lemma 2.4.3, we note that if $U^{-1}AU$ lies in the standard maximal torus modulo p^t, then A lies in the depth p^t Iwahori for GL_n, because the Levi component of U is $1 \times w^{\max}_{\mathrm{GL}_{n-1}}$ which normalizes the maximal torus. The index calculation follows from a direct computation using the stabilizer descriptions in Lemma 2.4.3. $\square$

We will choose the level-at-p of our Shimura varieties to be one of these subgroups; therefore we introduce the following notation.

Notation 2.5.4. For a fixed neat compact open subgroup $K^p \subset \boldsymbol{G}(\mathbb{A}^p_f)$, we set $S_{\boldsymbol{G},\mathrm{Iw}}(p^t)$ to be the Shimura variety of level $K^p K^G_{\mathrm{Iw}}(p^t)$. Similarly, for a fixed neat compact open subgroup $U^p \subset \boldsymbol{H}(\mathbb{A}^p_f)$, we let $S_{\boldsymbol{H},\diamond}(p^t)$ and $S_{\boldsymbol{H},\mathrm{Iw}}(p^t)$ denote the Shimura varieties of levels $U^p K^H_{\diamond}(p^t)$ and $U^p K^H_{\mathrm{Iw}}(p^t)$ respectively. If $U^p \subset K^p$, then we have a morphism

$$\hat{\iota} \colon S_{\boldsymbol{H},\diamond}(p^t) \to S_{\boldsymbol{G},\mathrm{Iw}}(p^t)$$

defined as the composition $\hat{\gamma} \circ \iota$.

2.6. *Branching laws.* To be able to construct the relevant pairing in coherent cohomology, we need to understand how representations of $M_{\boldsymbol{G}}$ decompose after restricting them to $M_{\boldsymbol{H}}$. For convenience, we recall that a general element of $M_{\boldsymbol{H}}$ is of the form $(x; y_1, y_2, y_3; z_{1,\tau}, z_{2,\tau})$ where τ runs over $\Psi - \{\tau_0\}$ and

- $x \in \mathrm{GL}_1$,
- $y_1 \in \mathrm{GL}_1$, $y_2 \in \mathrm{GL}_{n-1}$ and $y_3 \in \mathrm{GL}_n$,
- $z_{i,\tau} \in \mathrm{GL}_n$ for $i = 1, 2$.

This description will be useful for describing characters of $M_{\boldsymbol{H}}$.

Proposition 2.6.1. *Let $\lambda = (c_0; c_{1,\tau}, \dots, c_{2n,\tau}) \in X^*(T/T_0)^+$ and $\kappa_n = w_n \star (-w_G^{\max}\lambda) = w_n \star \lambda$ as in Section 2.3. Set $\kappa_n^* = -w_{M_G}^{\max}\kappa_n$ and let $V_{\kappa_n^*}$ denote the irreducible algebraic representation of $M_{\boldsymbol{G}}$ with highest weight κ_n^*. Let $j = (j_\tau)_{\tau \in \Psi - \{\tau_0\}}$ be a tuple of integers satisfying $|j_\tau| \leq c_{n,\tau}$. Then there exists a unique up to scaling vector $v_{\kappa_n}^{[j]} \in V_{\kappa_n^*}$ such that $M_{\boldsymbol{H}}$ acts on $v_{\kappa_n}^{[j]}$ through the character*

$$\begin{aligned} M_{\boldsymbol{H}} &\to \mathbb{G}_m \\ (x; y_1, y_2, y_3; z_{1,\tau}, z_{2,\tau}) &\mapsto y_1^{n+c_{n,\tau_0}} \det y_2^{c_{n,\tau_0}} \det y_3^{-(c_{n,\tau_0}+1)} \prod_{\tau \neq \tau_0} \det z_{1,\tau}^{j_\tau} \det z_{2,\tau}^{-j_\tau}. \end{aligned} \tag{2.6.2}$$

Proof. This follows from [Knapp 2001, Theorem 2.1] (see also Appendix A). ☐

Remark 2.6.3. We fix a specific model of $V_{\kappa_n^*}$ namely the space of algebraic functions $f \colon M_{\boldsymbol{G}} \to \mathbb{A}^1$ which transform as

$$f(gb) = \kappa_n(b) f(g)$$

for all $g \in M_{\boldsymbol{G}}$ and $b \in B_{M_G}$. The action of $m \in M_{\boldsymbol{G}}$ is then given by $(m \cdot f)(g) = f(m^{-1}g)$. Since $M_{\boldsymbol{H}} \cdot u \cdot B_{M_G}$ is Zariski dense in $M_{\boldsymbol{G}}$ (Lemma 2.4.3), we can (and do) normalize $v_{\kappa_n}^{[j]}$ so that its value on u is 1.

Let $\sigma_n^{[j]}$ denote the inverse of the character in (2.6.2). Then after fixing an isomorphism $V_{\kappa_n^*} \cong V_{\kappa_n}^*$, we obtain a $M_{\boldsymbol{H}}$-equivariant linear map

$$V_{\kappa_n} \twoheadrightarrow \sigma_n^{[j]}.$$

We can therefore consider the following F^{cl}-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{alg}} \colon \mathrm{H}^{n-1}(S_{\boldsymbol{G},\mathrm{Iw}}(p), [V_{\kappa_n}]) \times \mathrm{H}^0(S_{\boldsymbol{H},\diamond}(p), [\sigma_n^{[j]}]^\vee) \to F^{\mathrm{cl}}$$

defined as $\langle \eta, \chi \rangle_{\mathrm{alg}} = \mathrm{tr}(\hat{\iota}^* \eta \cup \chi)$, where tr denotes the residue morphism

$$\mathrm{H}^{n-1}(S_{\boldsymbol{H},\diamond}(p), \Omega^{n-1}) \to F^{\mathrm{cl}}.$$

In Section 8, we will show that this recovers twisted unitary Friedberg–Jacquet periods when η (resp. χ) is associated with an automorphic representation of $\boldsymbol{G}(\mathbb{A})$ (resp. automorphic character of $\boldsymbol{H}(\mathbb{A})$). The goal of this paper is to p-adically interpolate this pairing.

3. Functoriality on the flag variety

In this section we consider the functoriality of higher Coleman theory on the level of flag varieties (over $\mathbb{Z}_p$). This section is entirely local; in particular, we use notation and conventions as in Section 2.5 (so G and H denote the integral models in Remark 2.5.1 for $\boldsymbol{G}_{\mathbb{Q}_p}$ and $\boldsymbol{H}_{\mathbb{Q}_p}$ respectively, etc.).

Definition 3.0.1. Let FL_G (resp. FL_H) denote the flag variety $P_G\backslash G$ (resp. $P_H\backslash H$) over $\mathbb{Z}_p$. This can be described as the space of row vectors in $\mathbb{P}^{2n-1}$ (resp. $\mathbb{P}^{n-1}$) with the action of $g \in G$ (resp. $h \in H$) given by

$$[x_0 : \cdots : x_{2n-1}] \star g = [x_0 : \cdots : x_{2n-1}] \cdot {}^t g^{-1} \quad \text{and} \quad [y_0 : \cdots : y_{n-1}] \star h = [y_0 : \cdots : y_{n-1}] \cdot {}^t h^{-1}.$$

The embedding $\mathrm{FL}_H \xrightarrow{\iota} \mathrm{FL}_G$ induced from $H \hookrightarrow G$ is described in coordinates as

$$\iota([y_0 : \cdots : y_{n-1}]) = [y_0 : \cdots : y_{n-1} : 0 : \cdots : 0].$$

We will consider certain stratifications on these flag varieties, and relations between them. Recall that ${}^M W_G$ denotes the set of Kostant representatives for the quotient $W_{M_G}\backslash W_G$, where $W_?$ denotes the Weyl group of ?. This can be described as

$${}^M W_G = \{w_0, \ldots, w_{2n-1}\}$$

where $l(w_i) = i$, and each w_i corresponds to a shuffle and acts on the flag variety FL_G as

$$[x_0 : \cdots : x_{2n-1}] \star w_i = [x_1 : \cdots : x_i : x_0 : x_{i+1} : \cdots : x_{2n-1}]$$

(the element w_0 acts as the identity). We have a similar description for H and, as mentioned in Section 2, we have a map ${}^M W_H \hookrightarrow {}^M W_G$ induced from $H \hookrightarrow G$, preserving the lengths of the Weyl elements.

3.1. ***The Bruhat stratification.*** For either $? = G, H$, we have the following stratification of $\mathrm{FL}_{?,\mathbb{F}_p}$ given by the cells

$$C^?_w = P_?\backslash P_? \cdot w \cdot B_?$$

for $w \in {}^M W_?$. In coordinates, we have that $C^G_{w_i}$ is the orbit of $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ (where the 1 is in the $(i+1)$-th place) under the $\star$-action of B_G. Explicitly, this is described as the collection of tuples

$$[x_0 : \cdots : x_{i-1} : 1 : 0 : \cdots : 0], \quad x_j \in \mathbb{A}^1_{\mathbb{F}_p} \text{ for } j = 0, \ldots, i-1.$$

Each cell $C^G_{w_i}$ has dimension i, and they are ordered as $C^G_{w'} \subset \overline{C^G_w}$ if and only if $l(w') \leq l(w)$. We have a similar description for H.

Definition 3.1.1. For $? = G, H$ and $w \in {}^M W_?$, we set

$$Y^?_w = \bigcup_{l(w') \geq l(w)} C^?_{w'}, \quad X^?_w = \bigcup_{l(w') \leq l(w)} C^?_{w'}.$$

The former is open in $\mathrm{FL}_{?,\mathbb{F}_p}$, the latter is closed, and one has the relation $C^?_w = Y^?_w \cap X^?_w$.

Recall the definition of $\hat{\gamma}$ in Section 2.4, which we view as an element of $G(\mathbb{Z}_p)$. Let $\hat{\iota}\colon \mathrm{FL}_H \to \mathrm{FL}_G$ denote the map given by $P_H \cdot h \mapsto P_G \cdot h\hat{\gamma}$. This map satisfies the following properties:

Lemma 3.1.2. *One has*:

(1) $\hat{\iota}^{-1}(C^G_{w_i}) = \varnothing$ *if* $i < n$.

(2) $\hat{\iota}^{-1}(C^G_{w_n}) = C^H_{\mathrm{id}}$.

Proof. In coordinates, the map $\hat{\iota}$ is given by

$$\hat{\iota}([y_0 : \cdots : y_{n-1}]) = \left[y_1 : y_2 : \cdots : y_{n-1} : 0 : y_0 - \sum_{i=1}^{n-1} y_i, -y_{n-1} : \cdots : -y_1 \right].$$

The result immediately follows from this and the description of $C^G_{w_i}$ in coordinates. □

3.2. *Tubes in the flag variety.* We recall some notation from [Boxer and Pilloni 2021, Section 3.3] and [Loeffler and Zerbes 2021, Section 5.4]. Suppose that $X/\mathbb{Z}_p$ is a finite-type scheme and let

$$\mathcal{X} = X \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$$

denote the associated adic space over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let X_0 denote the special fiber of X over $\mathbb{F}_p$. Then one has a specialization map $\mathrm{sp}\colon \mathcal{X} \to X_0$, and for any locally closed subscheme $U \subset X_0$, we define the tube $]U[\subset \mathcal{X}$ to be the interior of $\mathrm{sp}^{-1}(U)$.

Definition 3.2.1. For $m \in \mathbb{Q}$, let $\mathcal{B}^\circ_m \subset \overline{\mathcal{B}}^\circ_m \subset \mathcal{B}_m \subset \overline{\mathcal{B}}_m$ denote the four flavors of "disc" inside the adic affine line defined as follows:

$$\mathcal{B}_m = \{|\cdot| : |z| \le |p|^m\}, \quad \overline{\mathcal{B}}_m = \bigcap_{m' < m} \mathcal{B}_{m'}, \quad \mathcal{B}^\circ_m = \bigcup_{m' > m} \mathcal{B}_{m'}, \quad \overline{\mathcal{B}}^\circ_m = \{|\cdot| : |z| < |p|^m\}.$$

We let $\mathcal{FL}^G$ and $\mathcal{FL}^H$ denote the adic flag varieties (over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$) associated with FL_G and FL_H. For $? = G, H$, we let $\Phi^\pm$ denote the set of $\pm$-roots with respect to $B_?$, and set $\Phi^{-,M}$ to be the set of negative roots which are not contained in $M_?$. Set $\delta_H = n-1$ and $\delta_G = 2n-1$. Then, for $w \in {}^M W_?$, we set $U_w = C^?_{w_{\delta_?}} \cdot w_{\delta_?}^{-1} w$ which is an open set containing $C^?_w$. Let $\mathcal{U}^{\mathrm{an}}_w$ denote its analytification. Then, following [Boxer and Pilloni 2021, Section 3.3.6], we have an Iwahori decomposition

$$\begin{aligned} \prod_{\alpha \in w^{-1}\Phi^{-,M}} \mathbb{A}^{1,\mathrm{an}} &\xrightarrow{\sim} \mathcal{U}^{\mathrm{an}}_w, \\ (u_\alpha) &\mapsto w \prod_\alpha u_\alpha. \end{aligned} \tag{3.2.2}$$

Definition 3.2.3. Let $m, k \in \mathbb{Q}$ and $w \in {}^M W_?$. We define $]C^?_w[_{m,k}$, $]C^?_w[_{\bar{m},k}$, $]C^?_w[_{m,\bar{k}}$ and $]C^?_w[_{\bar{m},\bar{k}}$ to be the images of

$$\prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \mathcal{B}^\circ_m \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \mathcal{B}_k,$$
$$\prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \overline{\mathcal{B}}^\circ_m \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \mathcal{B}_k,$$
$$\prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \mathcal{B}^\circ_m \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \overline{\mathcal{B}}_k,$$
$$\prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^-} \overline{\mathcal{B}}^\circ_m \times \prod_{\alpha \in (w^{-1}\Phi^{-,M})\cap\Phi^+} \overline{\mathcal{B}}_k,$$

respectively, under the map (3.2.2).

Remark 3.2.4. If $m, k \in \mathbb{Q}_{\geq 0}$ then $]C^?_w[_{m,k} \subset\,]C^?_w[$ with equality if $m = k = 0$. If $m \geq k \geq 0$, then $]C^?_{w_i}[_{m,k}$ is described in coordinates as the subset of tuples

$$[y_0 : \cdots : y_{\delta_?}]$$

satisfying

$$y_j \in \begin{cases} \mathcal{B}_k & \text{if } j < i, \\ 1 + \mathcal{B}^\circ_m & \text{if } j = i, \\ \mathcal{B}^\circ_m & \text{if } j > i. \end{cases}$$

One has a similar description for $]C^?_{w_i}[_{\bar{m},k}$ by replacing $\mathcal{B}^\circ_m$ with $\overline{\mathcal{B}}^\circ_m$, and a similar description for $]C^?_{w_i}[_{m,\bar{k}}$ when $k > 0$ by replacing $\mathcal{B}_k$ with $\overline{\mathcal{B}}_k$; see [Boxer and Pilloni 2021, Section 3.3.10]. In particular, if $i = 0$ (so $w_0 = \mathrm{id}$) then these tubes do not depend on k, so we will drop it from the notation.

We will now make specific choices of tubes which will be relevant for the construction of the p-adic L-function. Throughout, we let m, k, t be integers satisfying

$$0 \leq k \leq m < t, \quad \text{with } m > k \text{ if } k \neq 0. \tag{3.2.5}$$

We also introduce the following stronger condition:

$$m, k, t \text{ as in (3.2.5) with } m > (2n-1)(k+1) \text{ and } t > m + k. \tag{3.2.6}$$

We define some tubes in $\mathtt{FL}^G$ as follows.

Definition 3.2.7. Let m, k, t be as in (3.2.5):

(1) Let $\mathtt{U}^G_0 =]Y^G_{w_n}[$, $\mathtt{Z}^G_0 = \overline{]X^G_{w_n}[}$ and $\mathtt{I}^G_{0,0} = \mathtt{U}^G_0 \cap \mathtt{Z}^G_0$.

(2) We define $\mathtt{I}^G_{m,k} =]C^G_{w_n}[_{\bar{m},k} \cdot K^G_{\mathrm{Iw}}(p^t)$, which is independent of t by the description in [Boxer and Pilloni 2021, Section 3.3.10].

(3) For $k \geq 1$, we define $\mathtt{U}^G_k =]C^G_{w_n}[_{k,k} \cdot K^G_{\mathrm{Iw}}(p^t)$, which is independent of t by the description in [loc. cit.]. Furthermore, we have $\mathtt{I}^G_{m,k} \subset \mathtt{U}^G_k$.

We now define some tubes for H.

Definition 3.2.8. (1) For $m \geq 0$ and $t \geq 1$, one defines

$$\mathtt{Z}_m^H =]C_{\mathrm{id}}^H[_{\bar{m}} \cdot K_{\diamond}^H(p^t)$$

which is equal to $]C_{\mathrm{id}}^H[_{\bar{m}}$ for $t > m$.

(2) For $k \geq 1$ and $t \geq 1$, we define

$$\mathtt{U}_k^H =]C_{\mathrm{id}}^H[_k \cdot K_{\diamond}^H(p^t)$$

which is equal to $]C_{\mathrm{id}}^H[_k$ for $t > k$. For $k = 0$, we define $\mathtt{U}_0^H = \mathtt{FL}^H$.

We obtain the following lemma, essentially by construction:

Lemma 3.2.9. *For m, t, k as in (3.2.5), one has $\mathtt{U}_k^H = \hat{\iota}^{-1}(\mathtt{U}_k^G)$ and $\mathtt{Z}_m^H = \hat{\iota}^{-1}(\mathtt{I}_{m,k}^G)$. Furthermore, there is a Cartesian diagram*

$$\begin{tikzcd} \mathtt{Z}_m^H \arrow[r, hook] \arrow[d] & \mathtt{U}_k^H \arrow[d] \\ \mathtt{I}_{m,k}^G \arrow[r, hook] & \mathtt{U}_k^G \end{tikzcd}$$

with each map a closed embedding.

Proof. The lemma is clear for $(m, k) = (0, 0)$ by Lemma 3.1.2; so assume that $(m, k) \neq (0, 0)$. Then we can express $\mathtt{I}_{m,k}^G$ as the intersection

$$\mathtt{I}_{m,k}^G =]C_{w_n}^G[_{k,k} \cdot K_{\mathrm{Iw}}^G(p^t) \cap]C_{w_n}^G[_{\bar{m},\bar{0}} \cdot K_{\mathrm{Iw}}^G(p^t).$$

Indeed, the group $K_{\mathrm{Iw}}^G(p^t)$ acts continuously and preserves $]C_{w_n}^G[_{\bar{m},0}$, so must also preserve $\overline{]C_{w_n}^G[_{\bar{m},0}} =]C_{w_n}^G[_{\bar{m},\bar{0}}$. One then follows the proof of [Boxer and Pilloni 2021, Lemma 3.3.17].

The above description implies that $\mathtt{I}_{m,k}^G$ is closed in $\mathtt{U}_k^G$. Furthermore, the map $\hat{\iota}$ is a closed embedding of flag varieties, therefore it is enough to check $\mathtt{U}_k^H = \hat{\iota}^{-1}(\mathtt{U}_k^G)$ and $\mathtt{Z}_m^H = \hat{\iota}^{-1}(\mathtt{I}_{m,k}^G)$. But this follows immediately from the explicit description involving coordinates, and the fact that $\hat{\iota}(\mathtt{U}_k^H) \subset]C_{w_n}^G[_{k,k}$ and $\hat{\iota}(\mathtt{Z}_m^H) \subset]C_{w_n}^G[_{\bar{m},k}$ for $(m, k) \neq (0, 0)$. □

4. Pullbacks on adic Shimura varieties

We now transfer the functoriality of the last section to the setting of adic Shimura varieties, via the Hodge–Tate period map. We fix a neat compact open subgroup $K^p \subset \boldsymbol{G}(\mathbb{A}_f^p)$, and let $K = K^p K_p$ for a compact open subgroup $K_p \subset G(\mathbb{Q}_p)$. Let $\mathcal{S}_{G,K} = \mathcal{S}_{G,K}^{\mathrm{an}}$ denote the adic Shimura variety over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) = \mathrm{Spa}(F_{\mathfrak{p}_{\tau_0}}, \mathcal{O}_{F_{\mathfrak{p}_{\tau_0}}})$ associated with $S_{\boldsymbol{G},K}$ (note our assumption $F^+ \neq \mathbb{Q}$ implies that $S_{\boldsymbol{G},K}$ is proper). Similarly, we fix a neat compact open subgroup $U^p \subset \boldsymbol{H}(\mathbb{A}_f^p)$ contained in K^p, and we let $\mathcal{S}_{H,U}$ denote the corresponding adic Shimura variety of level $U = U^p U_p$. If we choose $K_p = K_{\mathrm{Iw}}^G(p^t)$ or $U_p = K_{\mathrm{Iw}}^H(p^t)$, $K_{\diamond}^H(p^t)$ then we will use the notation $\mathcal{S}_{G,\mathrm{Iw}}(p^t)$, $\mathcal{S}_{H,\mathrm{Iw}}(p^t)$ and $\mathcal{S}_{H,\diamond}(p^t)$ respectively.

4.1. ***The Hodge–Tate period map.*** Since $(\boldsymbol{G}, h_{\boldsymbol{G}})$ defines a PEL-type (and hence Hodge-type) Shimura datum, there exists a perfectoid space $\mathcal{S}_{G,K^p}$ over $\mathbb{Q}_p$ which represents the diamond $\varprojlim_{K_p} \mathcal{S}_{G,K}$. In fact, the existence of such a perfectoid space does not require axiom (SV3), i.e., $\boldsymbol{G}^{\mathrm{ad}}(\mathbb{R})$ has no $\mathbb{Q}$-simple factors which are $\mathbb{R}$-anisotropic, provided that one has embedding into a Siegel datum. This leads to the following proposition:

Proposition 4.1.1. *There exists a perfectoid space $\mathcal{S}_{H,U^p}$ over $\mathbb{Q}_p$ which represents the diamond $\varprojlim_{U_p} \mathcal{S}_{H,U}$.*

Proof. Although the set-up is slightly different, this follows the proof of [Scholze 2015, Theorem IV.1.1] verbatim. Note that we do not need a description of the connected components of $S_{\boldsymbol{H},U}$ in terms of Shimura data for the group $\boldsymbol{H}^{\mathrm{der}}$ (this would require (SV3)). □

Both of these perfectoid spaces come equipped with a Hodge–Tate period map into a flag variety associated with the ambient Siegel datum. It is shown in [Caraiani and Scholze 2017] that one can refine this morphism so that its image is contained in a flag variety associated with G or H. In particular, since the same Siegel datum can be chosen for $\boldsymbol{G}$ and $\boldsymbol{H}$ (compatible with the embedding $\iota\colon \boldsymbol{H} \hookrightarrow \boldsymbol{G}$), one has a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_{G,K^p} & \xrightarrow{\pi_{\mathrm{HT},G}} & \mathrm{FL}^G \\ \uparrow & & \uparrow \\ \mathcal{S}_{H,U^p} & \xrightarrow{\pi_{\mathrm{HT},H}} & \mathrm{FL}^H \end{array}$$

where the vertical arrows are the natural ones (induced from ι) and π_{HT} denotes the Hodge–Tate period map. We will often drop the subscripts for π_{HT} when the context is clear. Since $\pi_{\mathrm{HT},G}$ is $G(\mathbb{Q}_p)$-equivariant, the twisted embedding $\hat{\iota}\colon \mathcal{S}_{H,\diamond}(p^t) \to \mathcal{S}_{G,\mathrm{Iw}}(p^t)$ commutes with the twisted morphism

$$\hat{\iota}\colon \mathrm{FL}^H/K^H_{\diamond}(p^t) \to \mathrm{FL}^G/K^G_{\mathrm{Iw}}(p^t),$$
$$xK^H_{\diamond}(p^t) \mapsto \hat{\iota}(x)K^G_{\mathrm{Iw}}(p^t)$$

via the Hodge–Tate period morphisms. This is of course well-defined because $\hat{\gamma}^{-1}K^H_{\diamond}(p^t)\hat{\gamma} \subset K^G_{\mathrm{Iw}}(p^t)$.

4.2. ***Twisting torsors.*** In this section, we describe a general procedure for Tate-twisting proétale torsors and record some properties of this construction. Our choice of convention for twisting below will be consistent with our convention for the torsors on Shimura varieties (namely that they are defined via frames of relative *homology* groups).

Let $L/\mathbb{Q}_p$ be a finite extension and $\mathcal{X}/L$ a smooth adic space. Let $\mathcal{T}^\times \to \mathcal{X}$ denote the proétale $\mathbb{Z}_p^\times$-torsor parametrizing isomorphisms (of proétale sheaves) $\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1)$. The action of $\mathbb{Z}_p^\times$ is given by precomposition, i.e., for $\lambda \in \mathbb{Z}_p^\times$ and $\phi\colon \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1)$, we set

$$\phi \cdot \lambda = \phi(\lambda \cdot -).$$

Let M be a smooth adic group scheme over $\operatorname{Spa} L$ and suppose that we have a homomorphism

$$\mu\colon \mathbb{Z}_p^\times \to M$$

that is central (i.e., its image is contained in the center of M).

Definition 4.2.1. Let $\mathcal{M} \to \mathcal{X}$ be a (right) proétale M-torsor. We define the twist of $\mathcal{M}$ along μ to be

$${}^\mu\mathcal{M} := \mathcal{M} \times^{[\mathbb{Z}_p^\times, \mu]} \mathcal{T}^\times$$

where the right-hand side denotes the quotient of $\mathcal{M} \times_{\mathcal{X}} \mathcal{T}^\times$ by the equivalence relation:

$$(m \cdot \mu(\lambda), \phi) \sim (m, \phi \cdot \lambda^{-1}), \quad \text{for all } m \in \mathcal{M}, \phi \in \mathcal{T}^\times, \lambda \in \mathbb{Z}_p^\times.$$

This defines a proétale M-torsor ${}^\mu\mathcal{M} \to \mathcal{X}$ via the action $(m, \phi) \cdot n = (m \cdot n, \phi)$, for $m \in \mathcal{M}$, $\phi \in \mathcal{T}^\times$ and $n \in M$, because the homomorphism μ is central.

Example 4.2.2. Suppose that $M = \mathbb{G}_m^{\mathrm{an}}$ and $\mu\colon \mathbb{Z}_p^\times \to M$ is the natural inclusion. Let $\mathscr{F}$ be a locally free sheaf of rank one on the proétale site of $\mathcal{X}$. Then $\mathcal{M} := \underline{\operatorname{Isom}}(\hat{\mathcal{O}}_{\mathcal{X}}, \mathscr{F})$ is a proétale M-torsor, and we have a natural identification

$${}^\mu\mathcal{M} = \underline{\operatorname{Isom}}(\hat{\mathcal{O}}_{\mathcal{X}}, \mathscr{F}(-1)).$$

This twisting procedure enjoys the following properties:

Lemma 4.2.3. (1) *The construction ${}^\mu\mathcal{M}$ is functorial in the (right) proétale torsor $\mathcal{M}$.*

(2) *If $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of smooth adic spaces over* $\operatorname{Spa} L$, *then*

$$f^*({}^\mu\mathcal{M}) \cong {}^\mu(f^*\mathcal{M})$$

canonically (i.e., we have a natural isomorphism $f^* \circ {}^\mu(-) \xrightarrow{\sim} {}^\mu(-) \circ f^*$).

(3) *If $N \subset M$ is a smooth subgroup and μ factors through N, then for any proétale N-torsor $\mathcal{N} \to \mathcal{X}$ one has*

$${}^\mu(\mathcal{N} \times^N M) \cong {}^\mu\mathcal{N} \times^N M$$

canonically (i.e., it is natural in $\mathcal{N}$).

Proof. All of these properties follow immediately from tracing through the definitions. □

4.3. *Torsors on adic Shimura varieties.* We would like to recover the construction of the automorphic vector bundles in Section 2.2 via the Hodge–Tate period morphism (which plays the role of the Borel embedding). This is accomplished in [Caraiani and Scholze 2017, Section 2], and we give a brief review of the results. We will describe the construction for the group $\boldsymbol{G}$ only, as the construction for $\boldsymbol{H}$ follows the same argument.

Let $\mathcal{M}_G$ denote the adic generic fiber associated with M_G (the adic generic fiber of its completion along the special fiber) and $\mathcal{M}_G^{\mathrm{an}} = M_G^{\mathrm{an}}$. Let $\mu\colon \mathbb{Z}_p^\times \to \mathcal{M}_G$ denote the (central) homomorphism induced from the Hodge cocharacter μ_G defined in Section 2.1. By the results of [loc. cit.], there exists a proétale

$\mathcal{M}_G^{\mathrm{an}}$-torsor $\mathcal{M}_{G,\mathrm{HT}}^{\mathrm{an}}$ over $\mathcal{S}_{G,K}$ such that its twist ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}^{\mathrm{an}}$ along μ is canonically isomorphic to $M_{G,\mathrm{dR}}$ under analytification.[3] It is shown in [Boxer and Pilloni 2021, Section 4.6] that $\mathcal{M}_{G,\mathrm{HT}}^{\mathrm{an}}$ has an integral structure, namely the proétale $\mathcal{M}_G$-torsor $\mathcal{M}_{G,\mathrm{HT}}$. By Lemma 4.2.3, this defines an integral structure ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}$ on ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}^{\mathrm{an}}$, which is an étale $\mathcal{M}_G$-torsor because the morphism ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}} \to \mathcal{S}_{G,K}$ is surjective on geometric points and smooth (as ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}$ is an open subset of ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}^{\mathrm{an}} = M_{G,\mathrm{dR}}^{\mathrm{an}}$).

On the other hand, if N_G is the unipotent radical of P_G with associated adic generic fiber $\mathcal{N}_G$, then one can consider the (right) $\mathcal{M}_G$-torsor

$$\mathtt{M}^G \colon \mathcal{G}/\mathcal{N}_G \to \mathtt{FL}^G$$

via the morphism $x \mapsto x^{-1}$. These torsors are related in the following way:

Lemma 4.3.1. *The pullback of $\mathcal{M}_{G,\mathrm{HT}}$ to the perfectoid space $\mathcal{S}_{G,K^p}$ is identified with $\pi_{\mathrm{HT}}^{*}\mathtt{M}^G$.*

Proof. Immediate from the proof of [Boxer and Pilloni 2021, Proposition 4.6.3]. □

Recall that we have a twisted morphism $\hat{\iota}\colon \mathcal{S}_{H,\diamond}(p^t) \to \mathcal{S}_{G,\mathrm{Iw}}(p^t)$. Also, recall that the choice of Hodge cocharacters $\mu_{\boldsymbol{G}}$ and $\mu_{\boldsymbol{H}}$ are compatible under the inclusion $\boldsymbol{H} \hookrightarrow \boldsymbol{G}$, therefore the homomorphism μ above factors through $\mathcal{M}_H$. The description in the above lemma gives the following reduction of structure.

Proposition 4.3.2. *One has a reduction of structure of proétale torsors over $\mathcal{S}_{H,\diamond}(p^t)$*

$$\hat{\iota}^{*}\mathcal{M}_{G,\mathrm{HT}} = \mathcal{M}_{H,\mathrm{HT}} \times^{[\mathcal{M}_H,u]} \mathcal{M}_G$$

where the superscript means we view $\mathcal{M}_H$ as a subgroup of $\mathcal{M}_G$ via the embedding $u^{-1}\mathcal{M}_H u \subset \mathcal{M}_G$. In particular, one has a reduction of structure of étale torsors

$$\hat{\iota}^{*}({}^{\mu}\mathcal{M}_{G,\mathrm{HT}}) = {}^{\mu}\mathcal{M}_{H,\mathrm{HT}} \times^{[\mathcal{M}_H,u]} \mathcal{M}_G.$$

Proof. For the first part and via the interpretation in Lemma 4.3.1, it is enough to show that $\hat{\iota}^{*}\mathtt{M}^G = \mathtt{M}^G \times^{[\mathcal{M}_H,u]} \mathcal{M}_G$ on the level of flag varieties. This follows from the following commutative diagram:

$$\begin{tikzcd} \mathcal{H}/\mathcal{N}_H \arrow[r] \arrow[d] & \mathcal{G}/\mathcal{N}_G \arrow[d] \\ \mathtt{FL}^H \arrow[r, "\hat{\iota}"] & \mathtt{FL}^G \end{tikzcd}$$

where the vertical arrows are the torsors $\mathtt{M}^H$ and $\mathtt{M}^G$ and the top horizontal map is given by

$$h\mathcal{N}_H \mapsto \hat{\gamma}^{-1}h\gamma\mathcal{N}_G = \hat{\gamma}^{-1}hu\mathcal{N}_G.$$

where the last equality follows from the fact that γ maps to u under the projection $\mathcal{P}_G \twoheadrightarrow \mathcal{M}_G$.

The last part of the proposition follows from the functoriality properties of twisted torsors in Lemma 4.2.3 and the fact μ is central (so is unaffected by conjugation by u). □

[3] See the paragraph preceding [Caraiani and Scholze 2017, Lemma 2.3.5] for the definition of this torsor (which in the notation of [loc. cit.] would be $\mathcal{M}_{\mathrm{dR}}$).

Remark 4.3.3. One has an alternative reduction of structure as follows. In this remark only, set $U = U^p K^H_\diamondsuit(p^t)$, $K = K^p K^G_{\mathrm{Iw}}(p^t)$ and $K_{\hat{\gamma}} = \hat{\gamma} K \hat{\gamma}^{-1}$, and we will include the level in the notation for $\mathcal{M}_{\mathrm{HT}}$ and M_{dR}. Then we obtain a twisted morphism $\hat{\iota}\colon {}^{\mu}\mathcal{M}^{\mathrm{an}}_{H,\mathrm{HT},U} \to {}^{\mu}\mathcal{M}^{\mathrm{an}}_{G,\mathrm{HT},K}$ defined as the analytification of the composition

$$M_{H,\mathrm{dR},U} \xrightarrow{\iota} M_{G,\mathrm{dR},K_{\hat{\gamma}}} \xrightarrow{\hat{\gamma}} M_{G,\mathrm{dR},K}.$$

This is simply the twist along μ of the morphism of torsors induced from the natural map on the level of flag varieties $\mathcal{H}/\mathcal{N}_H \to \mathcal{G}/\mathcal{N}_G$ sending $h\mathcal{N}_H$ to $\hat{\gamma}^{-1}h\mathcal{N}_G$ (see Appendix B), so in fact preserves the integral structure. This gives a reduction of structure

$$\hat{\iota}^*({}^{\mu}\mathcal{M}_{G,\mathrm{HT},K}) = {}^{\mu}\mathcal{M}_{H,\mathrm{HT},U} \times^{\mathcal{M}_H} \mathcal{M}_G \tag{4.3.4}$$

and we have a commutative diagram

$$\begin{tikzcd}
 & {}^{\mu}\mathcal{M}_{H,\mathrm{HT},U} \times^{\mathcal{M}_H} \mathcal{M}_G \arrow[dd] \\
\hat{\iota}^*({}^{\mu}\mathcal{M}_{G,\mathrm{HT},K}) \arrow[ur] \arrow[dr] & \\
 & {}^{\mu}\mathcal{M}_{H,\mathrm{HT},U} \times^{[\mathcal{M}_H,u]} \mathcal{M}_G
\end{tikzcd}$$

where every map is an isomorphism; the top diagonal map is the reduction of structure in (4.3.4), the bottom diagonal map is the reduction of structure in Proposition 4.3.2, and the vertical map is given by $[x, m] \mapsto [x, u^{-1}mu]$.

The reduction of structure in (4.3.4) will be useful for the comparison with the archimedean setting, whereas the reduction of structure in Proposition 4.3.2 will be useful when we speak about sheaves of distributions in Section 5.

4.4. ***Comparison with the archimedean pairing.*** We can now reinterpret the pairing at the end of Section 2 in the setting of adic Shimura varieties via rigid GAGA. For a representation $V \in \mathrm{Rep}(M_G)$ we let $[V]$ denote the associated bundle on $\mathcal{S}_{G,K}$ using the torsor ${}^{\mu}\mathcal{M}^{\mathrm{an}}_{G,\mathrm{HT}} = M^{\mathrm{an}}_{G,\mathrm{dR}}$; and similarly for H. We place ourselves in the setting of Section 2.6 — in particular, we let $\lambda \in X^*(T/T_0)^+$. Then, after fixing an isomorphism $V_{\kappa_n^*} \cong V_{\kappa_n}^*$ we obtain a M_H-equivariant morphism

$$V_{\kappa_n} \twoheadrightarrow \sigma_n^{[j]} \tag{4.4.1}$$

by pairing with the vector $u^{-1} \cdot v^{[j]}_{\kappa_n}$, where M_H acts on V_{κ_n} via the embedding $u^{-1}M_H u \subset M_G$. Via the reduction of structure in Proposition 4.3.2, this gives a morphism of sheaves

$$\hat{\iota}^*[V_{\kappa_n}] \to [\sigma_n^{[j]}]$$

over $\mathcal{S}_{H,\diamond}(p)$. Using this morphism, we therefore obtain a pairing

$$\langle \cdot , \cdot \rangle_{\mathrm{an}} \colon \mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}]) \times \mathrm{H}^0(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}]^\vee) \to \mathbb{Q}_p$$

defined as $\langle \eta, \chi \rangle_{\mathrm{an}} = \mathrm{tr}(\hat{\iota}^* \eta \cup \chi)$. By the discussion in Remark 4.3.3 and the fact that the analytification of M_{dR} is identified with ${}^{\mu}\mathcal{M}_{\mathrm{HT}}^{\mathrm{an}}$, we obtain the following proposition:

Proposition 4.4.2. *The pairings $\langle \cdot , \cdot \rangle_{\mathrm{alg}}$ and $\langle \cdot , \cdot \rangle_{\mathrm{an}}$ correspond to each other under rigid GAGA, where we have base-changed the former to $\mathbb{Q}_p$ via the embedding $F^{\mathrm{cl}} \hookrightarrow \mathbb{Q}_p$ induced from the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$.*

4.5. *Hecke operators.* We would like to restrict the pairing $\langle \cdot , \cdot \rangle_{\mathrm{an}}$ to one over certain strata in the adic Shimura varieties, without losing any information. To accomplish this, we need to pass to "small-slope" parts of cohomology with respect to the action of certain Hecke operators, which we will now describe.

Let $T^- \subset T(\mathbb{Q}_p)$ denote the submonoid defined as

$$T^- = \{x \in T(\mathbb{Q}_p) : v(\alpha(x)) \leq 0 \text{ for all } \alpha \in \Phi^+\}$$

where Φ^+ is the set of positive roots of G (with respect to B_G) and $v \colon \mathbb{Q}_p^\times \to \mathbb{Z}$ is the p-adic valuation, normalized so that $v(p) = 1$. We let $T^{--} \subset T^-$ be the subset of elements satisfying $v(\alpha(x)) < 0$ for all $\alpha \in \Phi^+$. For $t \geq 1$, We let $\mathcal{H}_{p,t}^-$ denote the algebra $\mathbb{Q}_p[K_{\mathrm{Iw}}^G(p^t) \backslash T^- / K_{\mathrm{Iw}}^G(p^t)]$ with multiplication given by the double coset description in [Boxer and Pilloni 2021, Section 4.2]. This is isomorphic to the algebra $\mathbb{Q}_p[T^-]$ (with the usual definition of multiplication), with an element $x \in T^-$ corresponding to $[K_{\mathrm{Iw}}^G(p^t) x K_{\mathrm{Iw}}^G(p^t)]$.

We fix a specific choice of Hecke operator.

Definition 4.5.1. Let λ be an algebraic character of H_∞ (see Section 2.3) and set $\lambda^* = -w_G^{\max}\lambda$. We let $\mathcal{U}_B'(p^t) \in \mathcal{H}_{p,t}^-$ denote the Hecke operator $\lambda^*(x^{-1})[K_{\mathrm{Iw}}^G(p^t) x K_{\mathrm{Iw}}^G(p^t)]$ where $x \in T^{--}$ is given by

$$x = (1; 1, p, p^2, \ldots, p^{2n-1})_{\tau \in \Psi}.$$

Remark 4.5.2. It will turn out that the action of $\mathcal{U}_B'(p^t)$ on cohomology will be independent of the level, so we will often write $\mathcal{U}_B'$ instead.

Note that a $\mathbb{Q}_p$-algebra homomorphism $\mathcal{H}_{p,t}^- \to \overline{\mathbb{Q}}_p$ is identified with a monoid homomorphism $\theta \colon T^- \to (\overline{\mathbb{Q}}_p, \times)$ via the isomorphism above. We say that θ is *finite-slope* if $\theta(x) \neq 0$ for some $x \in T^{--}$ (in fact, this implies $\theta(x) \neq 0$ for all $x \in T^-$).

Definition 4.5.3. Let M be a Banach $\mathbb{Q}_p$-module (or more generally, a bounded complex of projective Banach $\mathbb{Q}_p$-modules) with an action of a potent compact operator T (see [Boxer and Pilloni 2021, Definition 2.4.13]). Then M has a slope decomposition with respect to (some power of) T and we set

$$M^{\mathrm{fs}} := \mathrm{colim}_h\, M^{\leq h}$$

where the colimit is over $h \in \mathbb{Q}_{\geq 0}$. This is called the finite-slope part of M.

If M carries an action of $\mathcal{H}^-_{p,t}$ such that $[K^G_{\mathrm{Iw}}(p^t)xK^G_{\mathrm{Iw}}(p^t)]$ acts as a potent compact operator for some $x \in T^{--}$, then we denote the finite-slope part by $M^{-,\mathrm{fs}}$ (which is independent of x by [Boxer and Pilloni 2021, Lemma 5.1.7]). Furthermore, $M^{\leq h}$ can be decomposed into generalized eigenspaces for the action of T^-, for any $h \in \mathbb{Q}_{\geq 0}$ (since slope decompositions are unique and $M^{\leq h}$ is finite-dimensional). This will allow us to pass to the "small slope part" of M, in the following sense.

Definition 4.5.4. Let $\lambda \in X^*(T/T_0)^+$. We say that a (monoid) homomorphism $\theta\colon T^- \to \overline{\mathbb{Q}}_p^\times$ is *small slope* (with respect to κ_n) if, for every $w \in {}^M W_G - \{w_n\}$, there exists $x \in T^-$ such that

$$v(\theta(x)) < v((w^{-1} \star \kappa_n)(x)). \tag{4.5.5}$$

If M is as in the paragraph following Definition 4.5.3, then we let $M^{-,\mathrm{ss}(\kappa_n)}$ denote the sum of generalized eigenspaces in $M^{\leq h}$ for which T^- acts through a small slope homomorphism $\theta\colon T^- \to \overline{\mathbb{Q}}_p^\times$ (for any sufficiently large h depending on κ_n). We will write $M^{-,\mathrm{ss}}$ when κ_n is clear from the context.

4.6. ***Restriction to smaller strata.*** We transfer the strata in Section 3.2 to adic Shimura varieties via the Hodge–Tate period map.

Definition 4.6.1. For m, t, k as in (3.2.5), we define

- $\mathcal{U}^G_k(p^t) = \pi^{-1}_{\mathrm{HT},G,t}(\mathtt{U}^G_k)$,
- $\mathcal{I}^G_{m,k}(p^t) = \pi^{-1}_{\mathrm{HT},G,t}(\mathtt{I}^G_{m,k})$,
- $\mathcal{Z}^G_0(p^t) = \pi^{-1}_{\mathrm{HT},G,t}(\mathtt{Z}^G_0)$,
- $\mathcal{Z}^G_m(p^t) = \pi^{-1}_{\mathrm{HT},G,t}(]C^G_{w_n}[_{\bar{m},\bar{0}} \cdot K^G_{\mathrm{Iw}}(p^t))$ for $m \geq 1$.

where $\pi_{\mathrm{HT},G,t}\colon \mathcal{S}_{G,\mathrm{Iw}}(p^t) \to \mathtt{FL}^G/K^G_{\mathrm{Iw}}(p^t)$ is the map (of topological spaces) induced from the Hodge–Tate period map. We will write $\mathcal{U}^G_k$, $\mathcal{I}^G_{m,k}$ and $\mathcal{Z}^G_m$ when t is clear from the context.

Note that, by the Iwahori decompositions in Section 3.2, $\mathcal{U}^G_k(p^t)$ is an open subset of $\mathcal{S}_{G,\mathrm{Iw}}(p^t)$ which is a finite union of quasi-Stein open subsets, and $\mathcal{Z}^G_m(p^t)$ is a closed subset of $\mathcal{S}_{G,\mathrm{Iw}}(p^t)$ whose complement is a finite union of quasi-Stein open subsets. Note that we have

$$\mathcal{I}^G_{m,k}(p^t) = \mathcal{U}^G_k(p^t) \cap \mathcal{Z}^G_m(p^t)$$

so by [Boxer and Pilloni 2021, Lemma 2.5.21], the cohomology complex $R\Gamma_{\mathcal{I}^G_{m,k}}(\mathcal{U}^G_k, [V_{\kappa_n}])$ is represented by a complex in $\mathrm{Pro}_{\mathbb{N}}(\mathcal{K}^{\mathrm{proj}}(\mathrm{Ban}(\mathbb{Q}_p)))$. Furthermore, $R\Gamma_{\mathcal{I}^G_{0,0}}(\mathcal{U}^G_0, [V_{\kappa_n}])$ carries an action of $\mathcal{H}^-_{p,t}$ for which $\mathcal{U}'_B(p^t)$ acts as a potent compact operator; see [loc. cit., Theorem 5.4.3].

Proposition 4.6.2. *For m, k, t in (3.2.6), the complex $R\Gamma_{\mathcal{I}^G_{m,k}}(\mathcal{U}^G_k, [V_{\kappa_n}])$ carries an action of $\mathcal{U}'_B(p^t)^m$ as a potent compact operator, and the natural maps*

$$R\Gamma_{\mathcal{I}^G_{m,k}(p^t)}(\mathcal{U}^G_k(p^t), [V_{\kappa_n}]) \xleftarrow{\mathrm{res}} R\Gamma_{\mathcal{I}^G_{m,0}(p^t)}(\mathcal{U}^G_0(p^t), [V_{\kappa_n}]) \xrightarrow{\mathrm{cores}} R\Gamma_{\mathcal{I}^G_{0,0}(p^t)}(\mathcal{U}^G_0(p^t), [V_{\kappa_n}])$$

are equivariant for $\mathcal{U}'_B(p^t)^m$ and become quasiisomorphisms after passing to finite-slope parts.

Proof. For this proof only, let $K = K^p K^G_{\mathrm{Iw}}(p^t)$ and $K_x = K \cap xKx^{-1}$, where x is the element in Definition 4.5.1. Let T denote the correspondence

$$\mathcal{S}_{G,K} \xleftarrow{p_2} \mathcal{S}_{G,K_x} \xrightarrow{p_1} \mathcal{S}_{G,K}$$

where p_1 is the forgetful map associated with the inclusion $K_x \subset K$, and p_2 is the composition of right-translation by x and the forgetful map associated with the inclusion $x^{-1}K_x x \subset K$. For a subset $\mathcal{W} \subset \mathcal{S}_{G,K}$, we let $T(\mathcal{W}) = p_2 p_1^{-1}(\mathcal{W})$ and $(T^t)(\mathcal{W}) = p_1 p_2^{-1}(\mathcal{W})$. For a nonnegative integer s, we let

$$T^{s+1}(\mathcal{W}) = T(T^s(\mathcal{W})), \quad (T^t)^{s+1}(\mathcal{W}) = (T^t)((T^t)^s(\mathcal{W}))$$

with the convention that $T^0(\mathcal{W}) = (T^t)^0(\mathcal{W}) = \mathcal{W}$.

By [Boxer and Pilloni 2021, Lemmas 3.3.17 and 3.5.10], one has the following inclusions

$$(T^t)^{k+1+m}(\mathcal{Z}^G_0) \cap \mathcal{U}^G_0 \subset \mathcal{Z}^G_m \subset (T^t)^{k+1}(\mathcal{Z}^G_0)$$
$$T^m(\mathcal{U}^G_0) \cap (T^t)^{k+1}(\mathcal{Z}^G_0) \subset \mathcal{U}^G_k \subset \mathcal{U}^G_0$$

so the result follows from [loc. cit., Corollary 5.3.8] (note that the action of x factors through its projection to the τ_0-component on the flag variety, so we can apply the cited lemmas with $\min(x) = 1$ and $\max(x) = 2n - 1$). □

Remark 4.6.3. It does not seem possible to apply [loc. cit., Corollary 5.3.8] for general m, k, t satisfying (3.2.5), and we do not know if there is an alternative way to show that $R\Gamma_{\mathcal{I}^G_{m,k}}(\mathcal{U}^G_k, [V_{\kappa_n}])$ carries an action of a power of $\mathcal{U}'_B$ as a potent compact operator such that the conclusion of Proposition 4.6.2 holds.

We also define strata for $\mathcal{S}_{H,\diamond}(p^t)$.

Definition 4.6.4. Let $\pi_{\mathrm{HT},H,t} : \mathcal{S}_{H,\diamond}(p^t) \to \mathtt{FL}^H / K^H_\diamond(p^t)$ denote the map induced from the Hodge–Tate period map:

- For $m \geq 0$ and $t \geq 1$, we define

$$\mathcal{Z}^H_m(p^t) = \pi^{-1}_{\mathrm{HT},H,t}(\mathtt{Z}^H_m).$$

- For $k \geq 0$ and $t \geq 1$, we define

$$\mathcal{U}^H_k(p^t) = \pi^{-1}_{\mathrm{HT},H,t}(\mathtt{U}^H_k).$$

We will write $\mathcal{Z}^H_m$ and $\mathcal{U}^H_k$ when t is clear from the context.

We now define the relevant cohomology complexes with partial compact support conditions, following [Boxer and Pilloni 2021, Section 5.4].

Definition 4.6.5. Let $\lambda \in X^*(T/T_0)^+$. Then we define

$$R\Gamma^G_{w_n}(\kappa_n)^{-,\mathrm{fs}} := R\Gamma_{\mathcal{I}^G_{0,0}(p)}(\mathcal{U}^G_0(p), [V_{\kappa_n}])^{-,\mathrm{fs}}$$

where $(-)^{-,\mathrm{fs}}$ denotes the finite-slope part with respect to the action of $\mathcal{H}^-_{p,1}$ as in Section 4.5. We denote the cohomology of this complex by $\mathrm{H}^i_{w_n}(\kappa_n)^{-,\mathrm{fs}}$.

We record some important properties.

Theorem 4.6.6. *Let* $\lambda \in X^*(T/T_0)^+$:

(1) (*Change of level*) *Let* m, k, t *be as in* (3.2.6) (*resp.* $m = 0$, $k = 0$ *and* $t \geq 1$). *The trace map*

$$R\Gamma_{\mathcal{I}^G_{m,k}(p^{t+1})}(\mathcal{U}^G_k(p^{t+1}), [V_{\kappa_n}]) \to R\Gamma_{\mathcal{I}^G_{m,k}(p^t)}(\mathcal{U}^G_k(p^t), [V_{\kappa_n}])$$

is $(\mathcal{U}'_B)^m$*-equivariant* (*resp.* T^-*-equivariant*) *and induces a quasiisomorphism on finite-slope parts.*

(2) (*Classicality for small slope*) *The natural maps*

$$R\Gamma_{\mathcal{I}^G_{0,0}(p)}(\mathcal{U}^G_0(p), [V_{\kappa_n}]) \xrightarrow{\text{cores}} R\Gamma(\mathcal{U}^G_0(p), [V_{\kappa_n}]) \xleftarrow{\text{res}} R\Gamma(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])$$

are $\mathcal{H}^-_{p,1}$*-equivariant and induce quasiisomorphisms on small slope parts.*

(3) (*Vanishing for small slope*) *The complex* $R\Gamma(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])^{-,ss}$ *is concentrated in degree* $n-1$.

Proof. Part (1) is an application of [Boxer and Pilloni 2021, Corollary 4.2.16 and Theorem 5.4.14]. Because the Shimura variety is compact, Theorem 6.10.1 implies Conjecture 5.9.2 in [loc. cit.] (i.e., the expected slope bounds hold). Parts (2) and (3) then follow immediately from the small slope versions of Theorems 5.12.3 and 5.12.5 in [loc. cit.]. □

We define similar complexes for $\mathcal{S}_{H,\diamond}(p^t)$, however we do not consider the finite-slope part of these complexes.

Definition 4.6.7. We set

$$R\Gamma^H_{\mathrm{id}}(\mathcal{S}_{H,\diamond}(p^t), \sigma_n^{[j]})^{(-,\dagger)} := \varprojlim_m R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), [\sigma_n^{[j]}])$$

where the transition maps are given by corestriction. If $t = 1$, we simply write $R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)}$ and denote the cohomology of this complex by $\mathrm{H}^i_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)}$.

4.7. *Functoriality.*

The goal of this section is to construct a map

$$R\Gamma^G_{w_n}(\kappa_n)^{-,\mathrm{fs}} \to R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)}$$

which is compatible with pull-back by $\hat{\iota}$ on the usual cohomology.

Definition 4.7.1. Let m, k, t be as in (3.2.5). Then we define a morphism

$$\vartheta_{m,k,t} \colon R\Gamma_{\mathcal{I}^G_{m,k}(p^t)}(\mathcal{U}^G_k(p^t), [V_{\kappa_n}]) \to R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), [\sigma_n^{[j]}])$$

as the composition of the following maps:

- $\hat{\iota}^* \colon R\Gamma_{\mathcal{I}^G_{m,k}(p^t)}(\mathcal{U}^G_k(p^t), [V_{\kappa_n}]) \to R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{U}^H_k(p^t), \hat{\iota}^*[V_{\kappa_n}])$.
- (Excision) $R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{U}^H_k(p^t), \hat{\iota}^*[V_{\kappa_n}]) \xrightarrow{\sim} R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), \hat{\iota}^*[V_{\kappa_n}])$.
- $R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), \hat{\iota}^*[V_{\kappa_n}]) \to R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), [\sigma_n^{[j]}])$.

Where the last map is induced from $V_{\kappa_n} \to \sigma_n^{[j]}$ (as in (4.4.1)). Note that $\hat{\iota}^*$ is well-defined by the Cartesian square in Lemma 3.2.9 and the fact that the strata on the level of flag varieties are independent of t for $t > m$ (see property (2) in [Boxer and Pilloni 2021, Section2.1]). The excision step is well-defined because $\mathcal{Z}_m^H(p^t)$ is closed $\mathcal{S}_{H,\diamond}(p^t)$; see property (3) in [loc. cit.].

Let m, k, t and m', k', t' be triples satisfying (3.2.5), such that $m' \geq m$, $k' \geq k$ and $t' \geq t$. Then the maps in Definition 4.7.1 fit into the following commutative diagram:

$$\begin{tikzcd}
R\Gamma_{\mathcal{I}^G_{m',k'}(p^{t'})}(\mathcal{U}^G_{k'}(p^{t'}), [V_{\kappa_n}]) \arrow[r, "\vartheta_{m',k',t'}"] & R\Gamma_{\mathcal{Z}^H_{m'}(p^{t'})}(\mathcal{S}_{H,\diamond}(p^{t'}), [\sigma_n^{[j]}]) \\
R\Gamma_{\mathcal{I}^G_{m',k}(p^{t'})}(\mathcal{U}^G_{k}(p^{t'}), [V_{\kappa_n}]) \arrow[u, "\mathrm{res}"] \arrow[r, "\vartheta_{m',k,t'}"] \arrow[d, "\mathrm{cores}"'] & R\Gamma_{\mathcal{Z}^H_{m'}(p^{t'})}(\mathcal{S}_{H,\diamond}(p^{t'}), [\sigma_n^{[j]}]) \arrow[u, equal] \arrow[d, "\mathrm{cores}"] \\
R\Gamma_{\mathcal{I}^G_{m,k}(p^{t'})}(\mathcal{U}^G_{k}(p^{t'}), [V_{\kappa_n}]) \arrow[r, "\vartheta_{m,k,t'}"] \arrow[d, "\mathrm{tr}"'] & R\Gamma_{\mathcal{Z}^H_{m}(p^{t'})}(\mathcal{S}_{H,\diamond}(p^{t'}), [\sigma_n^{[j]}]) \arrow[d, "\mathrm{tr}"] \\
R\Gamma_{\mathcal{I}^G_{m,k}(p^{t})}(\mathcal{U}^G_{k}(p^{t}), [V_{\kappa_n}]) \arrow[r, "\vartheta_{m,k,t}"] & R\Gamma_{\mathcal{Z}^H_{m}(p^{t})}(\mathcal{S}_{H,\diamond}(p^{t}), [\sigma_n^{[j]}])
\end{tikzcd}$$

where tr denotes the trace map; see [Boxer and Pilloni 2021, Lemma 2.1.2]. The bottom square is commutative because, by Lemma 2.5.3, we have a Cartesian diagram of Shimura varieties:

$$\begin{tikzcd}
\mathcal{S}_{H,\diamond}(p^{t+1}) \arrow[r, "\hat{\iota}"] \arrow[d] & \mathcal{S}_{G,\mathrm{Iw}}(p^{t+1}) \arrow[d] \\
\mathcal{S}_{H,\diamond}(p^{t}) \arrow[r, "\hat{\iota}"] & \mathcal{S}_{G,\mathrm{Iw}}(p^{t})
\end{tikzcd}$$

for any $t \geq 1$.

Proposition 4.7.2. *One has a well-defined map*

$$R\Gamma^G_{w_n}(\kappa_n)^{-,\mathrm{fs}} \to R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)}$$

defined as the (inverse limit over m of the) composition of

- *the inverse of the trace map followed by the inverse of corestriction*

$$R\Gamma_{\mathcal{I}^G_{0,0}(p)}(\mathcal{U}^G_0(p), [V_{\kappa_n}])^{-,\mathrm{fs}} \xrightarrow{\sim} R\Gamma_{\mathcal{I}^G_{m,0}(p^t)}(\mathcal{U}^G_0(p^t), [V_{\kappa_n}])^{-,\mathrm{fs}}$$

  which makes sense by Proposition 4.6.2 and Theorem 4.6.6,

- *the morphism $\vartheta_{m,0,t}$ and*

- *the trace map*

$$R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{S}_{H,\diamond}(p^t), [\sigma_n^{[j]}]) \to R\Gamma_{\mathcal{Z}^H_m(p)}(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}])$$

for any $m \geq 0$, $t \geq 1$ satisfying $m > 2n-1$ and $t > m+1$ (i.e., the tuple $(m, 0, t)$ satisfies (3.2.6)).

Proof. This map is well-defined by the above commutative diagram and the fact that trace and corestriction commute with each other; see the construction in [Boxer and Pilloni 2021, Lemma 2.1.2]. □

Set $R\Gamma^{H}_{\mathrm{id}}(\sigma_n^{[j],\vee})^{(+,\dagger)} := \varinjlim_m R\Gamma(\mathcal{Z}^H_m(p), [\sigma_n^{[j]}]^\vee)$ with transition maps given by restriction, and denote the cohomology of this complex by $\mathrm{H}^i_{\mathrm{id}}(\sigma_n^{[j],\vee})^{(+,\dagger)}$. By [loc. cit., Theorem 2.7.1] (using the fact that $\mathcal{Z}^H_m$ is the closure of $\mathcal{U}^H_m$), one has a natural pairing between $R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)}$ and $R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j],\vee})^{(+,\dagger)}$ built from the Serre duality pairings, which commutes with the Serre duality pairing between $R\Gamma(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}])$ and $R\Gamma(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}]^\vee)$ via corestriction and restriction on the former and latter complex respectively.[4] Proposition 4.7.2 therefore allows us to define a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle^{-}_{\mathrm{an}} \colon \mathrm{H}^{n-1}_{w_n}(\kappa_n)^{-,\mathrm{fs}} \times \mathrm{H}^0_{\mathrm{id}}(\sigma_n^{[j],\vee})^{(+,\dagger)} \to \mathbb{Q}_p$$

by composing the map in Proposition 4.7.2 with the duality pairing between the $(-,\dagger)$ and $(+,\dagger)$ cohomologies above. Considering classes in the small slope part, we obtain the following result:

Theorem 4.7.3. *Let*

- $\chi \in \mathrm{H}^0(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}]^\vee)$,
- $\eta \in \mathrm{H}^{n-1}_{w_n}(\kappa_n)^{-,\mathrm{ss}} \cong \mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])^{-,ss}$,

and denote by $\mathrm{res}\,\chi$ *the image of* χ *under the restriction map*

$$\mathrm{H}^0(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}]^\vee) \to \mathrm{H}^0_{\mathrm{id}}(\sigma_n^{[j],\vee})^{(+,\dagger)}.$$

Then $\langle \eta, \mathrm{res}\,\chi\rangle^{-}_{\mathrm{an}} = \langle \eta, \chi\rangle_{\mathrm{an}}$.

Proof. Since the embedding $\hat{\iota}\colon \mathcal{S}_{H,\diamond}(p^t) \to \mathcal{S}_{G,\mathrm{Iw}}(p^t)$ factors through $\mathcal{U}^G_0(p^t)$, we obtain the commutative diagram:

$$\begin{tikzcd}
R\Gamma^G_{w_n}(\kappa_n)^{-,\mathrm{fs}} \arrow[r] \arrow[d, "\mathrm{cores}"'] & R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]})^{(-,\dagger)} \arrow[d, "\mathrm{cores}"] \\
R\Gamma(\mathcal{U}^G_0(p), [V_{\kappa_n}]) \arrow[r] & R\Gamma(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}]) \arrow[d, equal] \\
R\Gamma(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}]) \arrow[r] \arrow[u, "\mathrm{res}"] & R\Gamma(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}])
\end{tikzcd}$$

where the top horizontal arrow is as in Proposition 4.7.2, and the bottom two are obtained from composing $\hat{\iota}^*$ with the map of sheaves $\hat{\iota}^*[V_{\kappa_n}] \to [\sigma_n^{[j]}]$. Passing to small slope parts and cohomology gives the result. □

[4]To be more precise, one cannot directly apply [Boxer and Pilloni 2021, Theorem 2.7.1] because $\mathcal{U}^H_m$ is not quasicompact. However one can find quasicompact open subsets $\mathcal{U}'_m$ satisfying $\mathcal{U}^H_{m+1} \subset \mathcal{U}'_m \subset \mathcal{U}^H_m$ and apply the theorem with these strata instead, as this does not affect the cohomology groups in the limit.

5. Locally analytic cohomology

5.1. ***Further reduction of structure.*** We first consider the reduction of structure for $\mathcal{M}_{G,\mathrm{HT}}$. Let U_G, $\bar{U}_G$, U_{M_G} and $\bar{U}_{M_G}$ denote the unipotent radicals of B_G, $\bar{B}_G$, B_{M_G} and $\bar{B}_{M_G}$ respectively. For $k>0$, let $\mathcal{G}^1_{k,k}$ (resp. $\mathcal{M}^1_{G,k,k}$) denote the subgroup of $\mathcal{G}$ (resp. $\mathcal{M}_G$) of elements which reduce to $\mathcal{U}_G$ (resp. $\mathcal{U}_{M_G}$) modulo $p^{k+\varepsilon}$ for all $\varepsilon>0$, and to $\bar{\mathcal{U}}_G$ (resp. $\bar{\mathcal{U}}_{M_G}$) modulo p^k. We have similar definitions for $\mathcal{M}^1_{H,k,k}$ and $\mathcal{H}^1_{k,k}$.

We introduce the following group:

Definition 5.1.1. Let $\mathcal{M}^{\square}_{G,k,k}=\mathcal{M}^1_{G,k,k}\cdot B_{M_G}(\mathbb{Z}_p)$, which is a subgroup of $\mathcal{M}_G$ containing the Iwahori subgroup of $M_G(\mathbb{Z}_p)$ of depth p^t for any $t>k$.

Remark 5.1.2. The homomorphism $\mu\colon \mathbb{Z}_p^{\times}\to\mathcal{M}_G$ factors through the subgroup $\mathcal{M}^{\square}_{G,k,k}$.

Remark 5.1.3. Let $t>k>0$. If we let K_{p,w_n,M_G} equal the projection of $w_nK^G_{\mathrm{Iw}}(p^t)w_n^{-1}\cap\mathcal{P}_G$ to $\mathcal{M}_G$, then the proof of [Boxer and Pilloni 2021, Proposition 4.6.9] shows that K_{p,w_n,M_G} equals the Iwahori subgroup of $M_G(\mathbb{Z}_p)$ of depth p^t (the proposition only treats the case $t=1$, but the proof easily generalizes to arbitrary t). Therefore $\mathcal{M}^{\square}_{G,k,k}=\mathcal{M}^1_{G,k,k}\cdot K_{p,w_n,M_G}=K_{p,w_n,M_G}\cdot\mathcal{M}^1_{G,k,k}$.

For $t>k>0$, let $\mathtt{M}^G_{k,k,t}$ denote the space

$$K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}/(K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}\cap w_n^{-1}\mathcal{N}_Gw_n)\to\mathcal{P}_G\backslash\mathcal{P}_Gw_nK^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}=\mathtt{U}^G_k,$$
$$x\mapsto\mathcal{P}_Gw_nx^{-1},$$

which is a (right) torsor for the group $\mathcal{M}^{\square}_{G,k,k}$ via the embedding $w_n^{-1}\mathcal{M}^{\square}_{G,k,k}w_n\subset K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}$.

Proposition 5.1.4. *Let $t>k>0$. The torsor $\mathcal{M}_{G,\mathrm{HT}}$ has a reduction of structure to a proétale $\mathcal{M}^{\square}_{G,k,k}$-torsor $\mathcal{M}_{G,\mathrm{HT},k,k,t}$ over $\mathcal{U}^G_k(p^t)$. Furthermore, the pullback of $\mathcal{M}_{G,\mathrm{HT},k,k,t}$ to the perfectoid space $\mathcal{S}_{G,K^p}$ is canonically isomorphic to the torsor $\pi^*_{\mathrm{HT}}\mathtt{M}^G_{k,k,t}$.*

Moreover, the twisted torsor ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,k,t}$ defines a reduction of structure of the torsor ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}$ to an étale $\mathcal{M}^{\square}_{G,k,k}$-torsor.

Proof. This is essentially [Boxer and Pilloni 2021, Proposition 4.6.12], but we have conjugated our groups by w_n. Note that we have a commutative diagram:

$$\begin{array}{ccc} K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}/(K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}\cap w_n^{-1}\mathcal{N}_Gw_n) & \longrightarrow & \mathcal{G}/\mathcal{N}_G \\ \downarrow & & \downarrow \\ \mathtt{U}^G_k & \longrightarrow & \mathtt{FL}^G \end{array}$$

where the vertical maps are the torsors $\mathtt{M}^G_{k,k,t}$ and $\mathtt{M}^G$, the bottom map is the natural inclusion and the top map is given by

$$g(K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_k\cap w_n^{-1}\mathcal{N}_Gw_n)\mapsto gw_n^{-1}\mathcal{N}_G.$$

Therefore $\mathtt{M}^G_{k,k,t}$ gives a reduction of structure for $\mathtt{M}^G$, and the proétale torsor $\pi^*_{\mathrm{HT}}\mathtt{M}^G_{k,k,t}$ descends to a proétale torsor $\mathcal{M}_{G,\mathrm{HT},k,k,t}$ over $\mathcal{U}^G_k(p^t)$ because it is a $K^G_{\mathrm{Iw}}(p^t)$-invariant open subset of the proétale torsor $\pi^*_{\mathrm{HT}}\mathtt{M}^G$ (which we already know descends). The last part follows from the fact that μ factors through $\mathcal{M}^{\square}_{G,k,k}$, Lemma 4.2.3(3), and because ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,k,t} \to \mathcal{U}^G_k(p^t)$ is surjective on geometric points and smooth (as ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,k,t}$ is an open subset of the étale torsor ${}^{\mu}\mathcal{M}_{G,\mathrm{HT}}$). □

We now discuss the reduction of structure for $\mathcal{M}_{H,\mathrm{HT}}$. Consider the following subtori of T consisting of elements $(x; y_{1,\tau}, \dots, y_{2n,\tau})$ satisfying the following relations:

- $T^{\clubsuit} \subset T$ is the subtorus given by the relations $y_{i,\tau_0} = y_{2n+2-i,\tau_0}$ for $i = 2, \dots, 2n$, and $y_{i,\tau} = y_{2n+1-i,\tau}$ for all $i = 1, \dots, 2n$ and $\tau \neq \tau_0$.
- $T^{\diamondsuit} \subset T^{\clubsuit}$ is the subtorus with the additional relation that $y_{1,\tau_0} = y_{n+1,\tau_0}$.

We begin with the following lemma:

Lemma 5.1.5. *Let $\mathrm{Iw}_{M_G}(p^t) \subset M_G(\mathbb{Z}_p)$ denote the Iwahori subgroup of depth p^t, and let $M^H_{\diamondsuit}(p^t)$ denote the projection of $K^H_{\diamondsuit}(p^t) \cap \mathcal{P}_H$ to $\mathcal{M}_H$. Then:*

(1) *$M^H_{\diamondsuit}(p^t)$ is the subgroup of $M_H(\mathbb{Z}_p)$ of all elements which land in $T^{\diamondsuit}$ modulo p^t.*

(2) *$M^H_{\clubsuit}(p^t) := u\,\mathrm{Iw}_{M_G}(p^t)u^{-1} \cap \mathcal{M}_H$ is the subgroup of $M_H(\mathbb{Z}_p)$ of all elements which land in $T^{\clubsuit}$ modulo p^t. It is contained in the projection of $K^H_{\mathrm{Iw}}(p^t) \cap \mathcal{P}_H$ to $\mathcal{M}_H$.*

In particular, one has $M^H_{\diamondsuit}(p^t) \subset M^H_{\clubsuit}(p^t)$.

Proof. By the proof of Lemma 2.4.3, we see that

$$h = x \times \begin{pmatrix} y_{1,\tau} & \\ & y_{2,\tau} \end{pmatrix} \in H(\mathbb{Z}_p)$$

lies in $K^H_{\diamondsuit}(p^t)$ if and only if:

- For all $\tau \neq \tau_0$, the block diagonal matrix $(y_{1,\tau}, y_{2,\tau})$ lies in the τ-component of $T^{\diamondsuit}$ modulo p^t.
- The elements $U^{-1}y_{1,\tau_0}U$ and y_{2,τ_0} are lower-triangular and upper-triangular modulo p^t respectively, where U is a $(n \times n)$ matrix lying the standard parabolic of GL_n with Levi $\mathrm{GL}_1 \times \mathrm{GL}_{n-1}$, whose projection to the Levi equals
$$1 \times w^{\max}_{\mathrm{GL}_{n-1}}.$$
- The elements $U^{-1}y_{1,\tau_0}U$ and y_2 are congruent to each other modulo p^t.

From these properties, one then immediately obtains part (1). Part (2) follows from the stabilizer computations in Lemma 2.4.3. It is contained in the projection of $K^H_{\mathrm{Iw}}(p^t) \cap \mathcal{P}_H$ to $\mathcal{M}_H$ because $T^{\clubsuit}$ is contained in B_H. □

For $t \geq 1$ and $k > 0$, we let $\mathcal{M}^{\diamondsuit}_{H,k,k,t} = M^H_{\diamondsuit}(p^t)\mathcal{M}^1_{H,k,k}$ and $\mathcal{M}^{\clubsuit}_{H,k,k,t} = M^H_{\clubsuit}(p^t)\mathcal{M}^1_{H,k,k}$. Both of these are groups by [Boxer and Pilloni 2021, Lemma 3.3.15] (because $K^H_{\diamondsuit}(p^t) \subset K^H_{\mathrm{Iw}}(p^t)$). If $t > k$, then these groups don't depend on t; explicitly, we have

$$\mathcal{M}^{\diamondsuit}_{H,k,k,t} = \mathcal{M}^{\diamondsuit}_{H,k,k} := T^{\diamondsuit}(\mathbb{Z}_p)\mathcal{M}^1_{H,k,k}, \quad \mathcal{M}^{\clubsuit}_{H,k,k,t} = \mathcal{M}^{\clubsuit}_{H,k,k} := T^{\clubsuit}(\mathbb{Z}_p)\mathcal{M}^1_{H,k,k}.$$

Furthermore, we have $u^{-1}\mathcal{M}^{\diamondsuit}_{H,k,k}u \subset u^{-1}\mathcal{M}^{\clubsuit}_{H,k,k}u \subset \mathcal{M}^{\square}_{G,k,k}$.

Remark 5.1.6. The homomorphism $\mu\colon \mathbb{Z}_p^{\times} \to \mathcal{M}_H$ induced from the Hodge cocharacter $\mu_{\boldsymbol{H}}$ factors through $\mathcal{M}^{\clubsuit}_{H,k,k,t}$ for any $t \geq 1$ and $k > 0$. It doesn't factor through $\mathcal{M}^{\diamondsuit}_{H,k,k,t}$, although the latter group is useful for discussing the reduction of structure below.

As above, we introduce the following space $\mathtt{M}^H_{k,k,t}$ (for $k > 0$ and $t \geq 1$):

$$K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k}/(K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k} \cap \mathcal{N}_H) \to \mathcal{P}_H \backslash \mathcal{P}_H K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k} = \mathtt{U}^H_k,$$
$$x \mapsto \mathcal{P}_H x^{-1},$$

which is a (right) torsor for the group $\mathcal{M}^{\diamondsuit}_{H,k,k,t}$ via the embedding $\mathcal{M}^{\diamondsuit}_{H,k,k,t} \subset K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k}$.

Proposition 5.1.7. *Let $t \geq 1$ and $k > 0$:*

1. *Then the torsor $\mathcal{M}_{H,\mathrm{HT}}$ has a reduction of structure to a proétale $\mathcal{M}^{\diamondsuit}_{H,k,k,t}$-torsor $\mathcal{M}'_{H,\mathrm{HT},k,k,t}$ over $\mathcal{U}^H_k(p^t)$. Furthermore, the pullback of $\mathcal{M}'_{H,\mathrm{HT},k,k,t}$ to the perfectoid space $\mathcal{S}_{H,U^p}$ is canonically isomorphic to $\pi^*_{\mathrm{HT}}\mathtt{M}^H_{k,k,t}$.*
2. *If we define $\mathcal{M}_{H,\mathrm{HT},k,k,t}$ as the pushout $\mathcal{M}'_{H,\mathrm{HT},k,k,t} \times^{\mathcal{M}^{\diamondsuit}_{H,k,k,t}} \mathcal{M}^{\clubsuit}_{H,k,k,t}$, then the proétale $\mathcal{M}^{\clubsuit}_{H,k,k,t}$-torsor $\mathcal{M}_{H,\mathrm{HT},k,k,t}$ (resp. étale $\mathcal{M}^{\clubsuit}_{H,k,k,t}$-torsor ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,k,t}$) provides a reduction of structure of the torsor $\mathcal{M}_{H,\mathrm{HT}}$ (resp. ${}^{\mu}\mathcal{M}_{H,\mathrm{HT}}$).*

Proof. For the first part, this follows from a similar argument in Proposition 5.1.4. Note that the proof of [Boxer and Pilloni 2021, Proposition 4.6.12] also applies in this situation, even though $K^H_{\diamondsuit}(p^t)$ is not of the form in the statement of [loc. cit.].

The second part follows immediately from the inclusions

$$\mathcal{M}^{\diamondsuit}_{H,k,k,t} \subset \mathcal{M}^{\clubsuit}_{H,k,k,t} \subset \mathcal{M}_H,$$

the functoriality properties in Lemma 4.2.3, and the fact that ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,k,t} \to \mathcal{U}^H_k(p^t)$ is smooth and surjective on geometric points. □

We have the following proposition which relates the torsors for G and H.

Proposition 5.1.8. *Let $t > k > 0$. One has a reduction of structure of étale torsors*

$$\hat{\iota}^*({}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,k,t}) = {}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,k,t} \times^{[\mathcal{M}^{\clubsuit}_{H,k,k},u]} \mathcal{M}^{\square}_{G,k,k}$$

where $\hat{\iota}$ denotes the embedding $\mathcal{U}^H_k(p^t) \hookrightarrow \mathcal{U}^G_k(p^t)$.

Proof. We first show that we have a reduction of structure

$$\hat{\iota}^*\mathcal{M}_{G,\mathrm{HT},k,k,t} = \mathcal{M}'_{H,\mathrm{HT},k,k,t} \times^{[\mathcal{M}^{\diamondsuit}_{H,k,k},u]} \mathcal{M}^{\square}_{G,k,k}. \tag{5.1.9}$$

It is enough to show the analogous statement for the torsors $\mathtt{M}^G_{k,k,t}$ and $\mathtt{M}^H_{k,k,t}$. In this case, we have a commutative diagram:

$$\begin{tikzcd} K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k}/(K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k}\cap\mathcal{N}_H) \arrow[r] \arrow[d] & K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}/(K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}\cap w_n^{-1}\mathcal{N}_G w_n) \arrow[d] \\ \mathtt{U}^H_k \arrow[r, "\hat{\iota}"] & \mathtt{U}^G_k \end{tikzcd}$$

where the vertical maps are the torsors $\mathtt{M}^H_{k,k,t}$ and $\mathtt{M}^G_{k,k,t}$, and the top map is induced from the map $K^H_{\diamondsuit}(p^t)\mathcal{H}^1_{k,k} \to K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}$ given by $h \mapsto \hat{\gamma}^{-1}h\hat{\gamma}$. Note that this diagram is commutative because $\gamma \in P_G$.

Since $\mathtt{M}^G_{k,k,t}$ is a torsor for the group $\mathcal{M}^{\square}_{G,k,k}$ via the conjugated embedding $w_n^{-1}\mathcal{M}^{\square}_{G,k,k}w_n \subset K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_{k,k}$, and the projection of γ to M_G is equal to u, (5.1.9) follows.

Since $u^{-1}\mathcal{M}^{\diamondsuit}_{H,k,k}u \subset u^{-1}\mathcal{M}^{\clubsuit}_{H,k,k}u \subset \mathcal{M}^{\square}_{G,k,k}$, we also obtain the reduction of structure

$$\hat{\iota}^*\mathcal{M}_{G,\mathrm{HT},k,k,t} = \mathcal{M}_{H,\mathrm{HT},k,k,t} \times^{[\mathcal{M}^{\clubsuit}_{H,k,k},u]} \mathcal{M}^{\square}_{G,k,k}$$

and we can twist this along μ by Lemma 4.2.3 (and the fact μ is central, so unaffected by conjugation by u). $\square$

Remark 5.1.10. If $t' \geq t$ and $k' \geq k$, then the torsors ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k',k',t'}$ and ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k',k',t'}$ provide a reduction of structure for the pullbacks of ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,k,t}$ and ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,k,t}$ along the trace/inclusion maps $\mathcal{U}^G_{k'}(p^{t'}) \to \mathcal{U}^G_k(p^t)$ and $\mathcal{U}^H_{k'}(p^{t'}) \to \mathcal{U}^H_k(p^t)$ respectively; see [Boxer and Pilloni 2021, Proposition 4.6.14].

Remark 5.1.11. Let $k > 0$, and let $\mathcal{M}^1_{G,k}$ (resp. $\mathcal{M}^1_{H,k}$) denote the normal affinoid subgroup of $\mathcal{M}_G$ (resp. $\mathcal{M}_H$) consisting of elements which reduce to the identity modulo p^k. We set

$$\mathcal{M}^{\square}_{G,k} = \mathcal{M}^1_{G,k}B_{M_G}(\mathbb{Z}_p), \quad \mathcal{M}^{\clubsuit}_{H,k} = \mathcal{M}^1_{H,k}T^{\clubsuit}(\mathbb{Z}_p), \quad \mathcal{M}^{\clubsuit}_{H,k,t} = \mathcal{M}^1_{H,k}M^H_{\clubsuit}(p^t)$$

for $k, t \geq 1$. All of these groups are *open affinoid* analytic subgroups of $\mathcal{M}_?$, where $? = G, H$ according to the subscript.

To be able to apply the results in [loc. cit., Section 6], it will be more convenient to work with the following torsors, obtained as the pushouts

$$\mathcal{M}_{G,\mathrm{HT},k,t} := \mathcal{M}_{G,\mathrm{HT},k,k,t} \times^{\mathcal{M}^{\square}_{G,k,k}} \mathcal{M}^{\square}_{G,k} \quad \text{and} \quad \mathcal{M}_{H,\mathrm{HT},k,t} := \mathcal{M}_{H,\mathrm{HT},k,k,t} \times^{\mathcal{M}^{\clubsuit}_{H,k,k,t}} \mathcal{M}^{\clubsuit}_{H,k,t}.$$

In particular, we can twist these torsors along μ and the torsors ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,t}$ and ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,t}$ are étale torsors by Lemma 4.2.3. The analogous compatibility for varying k and t as in Remark 5.1.10 still continues to hold for these torsors, and we have an analogue of Proposition 5.1.8, namely one has a reduction of structure of étale torsors

$$\hat{\iota}^*({}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,t}) = {}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,t} \times^{[\mathcal{M}^{\clubsuit}_{H,k},u]} \mathcal{M}^{\square}_{G,k}$$

whenever $t > k > 0$.

5.2. *Weight spaces.* For an integer $r \in \mathbb{Q}_{>0}$ let $\mathcal{T}_r^1$ denote the subgroup of $\mathcal{T}$ of elements which reduce to the identity modulo p^r. Recall that for a Tate algebra (A, A^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$, a character

$$\lambda \colon T(\mathbb{Z}_p) \to (A^+)^\times$$

is r-analytic if it extends to an analytic A-valued function on $T(\mathbb{Z}_p)\mathcal{T}_r^1 \subset T^{\mathrm{ad}}$.

Definition 5.2.1. Let (A, A^+) be a Tate algebra above. We let $X^*(T; A)$ denote the space of all characters

$$\lambda \colon T(\mathbb{Z}_p) \to (A^+)^\times$$

which are r-analytic, for some $r \in \mathbb{Q}_{>0}$. We let $X^*(T/T_0; A) \subset X^*(T; A)$ be the subspace of all characters which are trivial on $T_0(\mathbb{Z}_p)$.

Remark 5.2.2. Note that there is a Weyl action on $X^*(T; A)$ by the usual formulae. Furthermore, even though the half sum of positive roots doesn't strictly give an element of this space, the $\star$-action of the Weyl group also still makes sense.

Remark 5.2.3. The functor $(A, A^+) \mapsto X^*(T/T_0; A)$ is representable by a group adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, which we will denote by $\mathcal{W}_G$.

Definition 5.2.4. For $i = 1, \dots, n$ and $\tau \in \Psi$, let $\lambda_{i,\tau} \in X^*(T/T_0)^+$ be the character which is trivial outside the τ-component, and in the τ-component is given by the tuple

$$(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$$

where there are i lots of 1s and -1s.

These characters give a generating set for $X^*(T/T_0; A)$ in the following sense.

Lemma 5.2.5. *Let $\lambda \in X^*(T/T_0; A)$ be an r-analytic character. Then there exist unique r-analytic characters $\xi_{i,\tau} \colon \mathbb{Z}_p^\times \to (A^+)^\times$, for $i = 1, \dots, n$ and $\tau \in \Psi$, such that*

$$\lambda = \sum_{i=1}^{n} \sum_{\tau \in \Psi} \xi_{i,\tau} \circ \lambda_{i,\tau}$$

where the group structure on $X^*(T/T_0; A)$ is written additively.

Proof. Any such character λ is a (unique) product of r-analytic characters $\alpha_{i,\tau} \colon \mathbb{Z}_p^\times \to (A^+)^\times$ where $i = 1, \dots, 2n$ and $\tau \in \Psi$, where $\alpha_{i,\tau}$ is determined by where it sends $y_{i,\tau}$. Since λ is trivial on T_0, we have $\alpha_{i,\tau} = -\alpha_{2n+1-i,\tau}$ for all $i = 1, \dots, 2n$ and $\tau \in \Psi$. One then defines

$$\xi_{i,\tau} = \begin{cases} \alpha_{i,\tau} - \alpha_{i+1,\tau} & \text{for } i = 1, \dots, n-1, \\ \alpha_{n,\tau} & \text{for } i = n. \end{cases}$$

Uniqueness is a simple check. $\square$

Remark 5.2.6. The above lemma implies that $\mathcal{W}_G$ is a finite disjoint union of $n[F^+ : \mathbb{Q}]$-dimensional open unit polydiscs.

Let S denote the torus $\prod_{\tau \neq \tau_0} \mathbb{G}_m$, and for a Tate algebra (A, A^+), let $X^*(S; A)$ denote the space of locally analytic characters $S(\mathbb{Z}_p) \to (A^+)^\times$. A general element of $X^*(S; A)$ is a tuple $\beta = (\beta_\tau)_{\tau \neq \tau_0}$, where $\beta_\tau : \mathbb{Z}_p^\times \to (A^+)^\times$ are locally analytic. The functor $(A, A^+) \mapsto X^*(S; A)$ is representable by a $[F^+ : \mathbb{Q}] - 1$-dimensional group adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ denoted $\mathcal{W}_H$.

Definition 5.2.7. Let $X_0^*(T \times S; A) = X^*(T/T_0; A) \times X^*(S; A)$. The functor $(A, A^+) \mapsto X_0^*(T \times S; A)$ is then represented by $\mathcal{W} := \mathcal{W}_G \times \mathcal{W}_H$.

5.3. *Analytic and distribution modules.* We now define the relevant analytic and distribution modules. We introduce some notation:

Notation 5.3.1. For $\lambda \in X^*(T/T_0; A)$, we set $\kappa_n(\lambda) = w_n \star (-w_G^{\max}\lambda)$. We also define $\kappa_n(\lambda)^* = -w_{M_G}^{\max}\kappa_n(\lambda)$.

Definition 5.3.2. Let $\lambda \in X^*(T/T_0; A)$ be an r_0-analytic character, for some $r_0 \in \mathbb{Z}_{>0}$. Set $S = \mathrm{Spa}(A, A^+)$. Then for any $r \geq r_0$, we define

$$\begin{aligned} V^{r-\mathrm{an}}_{G,\kappa_n(\lambda)^*} &= \mathrm{anInd}^{\mathcal{M}^{\square}_{G,r}}_{\mathcal{M}^{\square}_{G,r} \cap \mathcal{B}_{M_G}} (w^{\max}_{M_G}\kappa_n(\lambda)^*) \\ &:= \{ f : (\mathcal{M}^{\square}_{G,r})_S \to \mathbb{A}^{1,\mathrm{an}}_S \\ &\qquad : f(mb) = (w^{\max}_{M_G}\kappa_n(\lambda)^*)(b^{-1}) f(m) \text{ for all } b \in (\mathcal{M}^{\square}_{G,r} \cap \mathcal{B}_{M_G})_S \text{ and } m \in (\mathcal{M}^{\square}_{G,r})_S \} \end{aligned}$$

as in [Boxer and Pilloni 2021, Section 6.2.4]. This carries actions of $(\mathcal{M}^{\square}_{G,r})_S$ and $T^{M,+}$ by the formulae in [loc. cit.], where $T^{M,+} \subset T(\mathbb{Q}_p)$ denotes the submonoid of elements $t \in T(\mathbb{Q}_p)$ which satisfy $t B_{M_G}(\mathbb{Z}_p) t^{-1} \subset B_{M_G}(\mathbb{Z}_p)$. Note that $V^{r-\mathrm{an}}_{G,\kappa_n(\lambda)^*} \subset V^{r'-\mathrm{an}}_{G,\kappa_n(\lambda)^*}$ for $r' \geq r$, where the inclusion is given by restricting a function to $(\mathcal{M}^{\square}_{G,r'})_S$.

We write $\tilde{D}^{r-\mathrm{an}}_{G,\kappa_n(\lambda)}$ for the continuous A-dual of $V^{r-\mathrm{an}}_{G,\kappa_n(\lambda)^*}$, which carries actions of $(\mathcal{M}^{\square}_{G,r})_S$ and $T^{M,-} = (T^{M,+})^{-1}$ in the usual way. This is a Banach A-module but in general, it is not necessarily projective. To remedy this, one introduces the open subgroup

$$\mathcal{M}^{\square,\circ}_{G,r} = \mathcal{M}^{1,\circ}_{G,r} B_{M_G}(\mathbb{Z}_p)$$

where $\mathcal{M}^{1,\circ}_{G,r} \subset \mathcal{M}^{1}_{G,r}$ denotes the open subgroup of elements $m \equiv 1$ modulo $p^{r+\varepsilon}$ for some $\varepsilon > 0$. Note that this subgroup contains $\mathcal{M}^{\square}_{G,r+1}$. In [loc. cit., Section 6.2.20], the authors introduce a modification of the space of analytic functions $V^{\circ,r-\mathrm{an}}_{G,\kappa_n(\lambda)^*}$ using this open subgroup, and one has a $(\mathcal{M}^{\square,\circ}_{G,r}, T^{M,+})$-equivariant morphism $V^{r-\mathrm{an}}_{G,\kappa_n(\lambda)^*} \to V^{\circ,r-\mathrm{an}}_{G,\kappa_n(\lambda)^*}$ with dense image. One defines the space of r-analytic distributions $D^{r-\mathrm{an}}_{G,\kappa_n(\lambda)}$ to be the continuous A-dual of $V^{\circ,r-\mathrm{an}}_{G,\kappa_n(\lambda)^*}$, which is a projective Banach A-module. One has a $(\mathcal{M}^{\square,\circ}_{G,r}, T^{M,-})$-equivariant morphism $D^{r-\mathrm{an}}_{G,\kappa_n(\lambda)} \to \tilde{D}^{r-\mathrm{an}}_{G,\kappa_n(\lambda)}$ with dense image.

We also introduce the following characters:

Definition 5.3.3. Let $(\lambda, \beta) \in X_0^*(T \times S; A)$ be an r_0-analytic character. Set $S = \mathrm{Spa}(A, A^+)$. Then for any $r \geq r_0$, we let $\sigma_n^{[\beta]}(\lambda)\colon (\mathcal{M}_{H,r,1}^{\clubsuit})_S \to \mathbb{G}_{m,S}^{\mathrm{an}}$ be the analytic character given by

$$(x; y_1, y_2, y_3; z_{1,\tau}, z_{2,\tau})_{\tau \neq \tau_0} \mapsto y_1^{-n-\xi_{n,\tau_0}} \det{}^{-\xi_{n,\tau_0}} \det y_3^{\xi_{n,\tau_0}+1} \prod_{\tau \neq \tau_0} \det z_{1,\tau}^{-\beta_\tau} \det z_{2,\tau}^{\beta_\tau}$$

where $\xi_{i,\tau}$ are the characters associated with λ as in Lemma 5.2.5.

We obtain the following "branching law in families", which is an analytic version of Proposition 2.6.1. As the proof of this theorem is rather technical (and involves significantly changing the notation), we provide the proof in Appendix A.

Theorem 5.3.4. *Let (A, A^+) be a Tate algebra over $(\mathbb{Q}_p, \mathbb{Z}_p)$ and $(\lambda, \beta) \in X_0^*(T \times S; A)$ which is r_0-analytic for some $r_0 \in \mathbb{Z}_{>0}$. Then, for any $r \in \mathbb{Z}$ such that $r \geq r_0$, there exists a nonzero vector $x_n^{[\beta]}(\lambda) \in V_{G,\kappa_n(\lambda)^*}^{r-\mathrm{an}}$ satisfying*:

(1) *The group $\mathcal{M}_{H,r}^{\clubsuit}$ acts on $x_n^{[\beta]}(\lambda)$ through the inverse of the character $\sigma_n^{[\beta]}(\lambda)$, via the embedding $u^{-1}\mathcal{M}_{H,r}^{\clubsuit}u \subset \mathcal{M}_{G,r}^{\square}$.*

(2) *If (B, B^+) denotes another Tate algebra with a morphism $(A, A^+) \to (B, B^+)$, and $(\lambda', \beta') \in X_0^*(T \times S; B)$ denotes the composition of (λ, β) with this morphism, then the image of $x_n^{[\beta]}(\lambda)$ under the natural map*

$$V_{G,\kappa_n(\lambda)^*}^{r-\mathrm{an}} \to V_{G,\kappa_n(\lambda')^*}^{r-\mathrm{an}}$$

is equal to $x_n^{[\beta']}(\lambda')$.

(3) *If $(\lambda, j) \in X^*(T/T_0)^+ \times X^*(S)$ is a pair of algebraic characters satisfying $0 \leq j_\tau \leq c_{n,\tau}$ for all $\tau \neq \tau_0$, then $x_n^{[\beta]}(\lambda)$ equals the image of $u^{-1} \cdot v_{\kappa_n}^{[j]}$ under the natural map*

$$V_{\kappa_n^*} \to V_{G,\kappa_n(\lambda)^*}^{r-\mathrm{an}}.$$

Here any undefined notation is as in Proposition 2.6.1.

(4) *The vector $x_n^{[\beta]}(\lambda)$ does not depend on the radius of analyticity*; *see Theorem A.5.10*(4).

Proof. This follows from Theorem A.5.10, noting that the character $\kappa_n(\lambda)^*$ satisfies the conditions in Lemma A.5.6 (because λ is trivial on $T_0(\mathbb{Z}_p)$), and this character specializes to a M_G-dominant character in $\mathcal{C}$ whenever $\lambda \in X^*(T/T_0)^+$. $\square$

Remark 5.3.5. Note that if $(\lambda, j) \in X^*(T/T_0)^+ \times X^*(S)$ is a pair of algebraic characters as in Theorem 5.3.4(3), then (after fixing an isomorphism $V_{\kappa_n} \cong V_{\kappa_n^*}^*$) we have a commutative diagram:

$$\begin{tikzcd} D_{G,\kappa_n(\lambda)}^{r-\mathrm{an}} \arrow[d] \arrow[dr] & \\ V_{\kappa_n} \arrow[r] & \sigma_n^{[j]} \end{tikzcd}$$

where the vertical map is the dual of the map in Theorem 5.3.4(3) restricted to $D^{r-\mathrm{an}}_{G,\kappa_n(\lambda)}$, the bottom map is pairing with the vector $u^{-1}\cdot v^{[j]}_{\kappa_n}$, and the diagonal map is evaluation at $x_n^{[\beta]}(\lambda)$. All of the maps are equivariant for the action of $\mathcal{M}^{\clubsuit}_{H,r+1}$ via the embedding $u^{-1}\mathcal{M}^{\clubsuit}_{H,r+1}u\subset\mathcal{M}^{\square}_{G,r+1}\subset\mathcal{M}^{\square,\circ}_{G,r}$.

5.4. ***Locally analytic cohomology.*** Let $(\underline{\lambda},\beta)\in X^*_0(T\times S;A)$ be an r_0-analytic character, and let $t>k>r_0$ be integers. Let ${}^{\mu}\mathcal{M}^{\circ}_{G,\mathrm{HT},k-1,t}$ denote the pushout of ${}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,t}$ along the inclusion $\mathcal{M}^{\square}_{G,k}\subset\mathcal{M}^{\square,\circ}_{G,k-1}$, and consider the base-extension of the torsor

$$\pi\times 1\colon {}^{\mu}\mathcal{M}^{\circ}_{G,\mathrm{HT},k-1,t}\times\mathrm{Spa}(A,A^+)\to\mathcal{U}^G_k(p^t)\times\mathrm{Spa}(A,A^+).$$

We define $[V^{\circ,(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})^*}]$ to be the subsheaf of $(\pi\times 1)_*\mathcal{O}_{{}^{\mu}\mathcal{M}^{\circ}_{G,\mathrm{HT},k-1,t}\times\mathrm{Spa}(A,A^+)}$ of bounded sections which transform as $f(mb)=(w^{\max}_{M_G}\kappa_n(\lambda)^*)(b^{-1})f(m)$ for every $b\in\mathcal{M}^{\square,\circ}_{G,k-1}\cap\mathcal{B}_{M_G}$. This defines a sheaf of topological modules over $\mathcal{U}^G_k(p^t)$ locally modeled on $V^{\circ,(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})^*}$ by the same proof as [Boxer and Pilloni 2021, Proposition 6.3.3]. We define $[D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})}]$ to be the continuous dual of $[V^{\circ,(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})^*}]$ which is a locally projective Banach sheaf locally modeled on the representation $D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})}$.

Remark 5.4.1. The sheaf $[D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})}]$ can alternatively be described as

$$((\pi\times 1)_*\mathcal{O}_{{}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,t}\times\mathrm{Spa}(A,A^+)}\hat{\otimes}D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})})^{\mathcal{M}^{\square}_{G,k}}$$

where the invariants are via the (left) diagonal action and

$$\pi\times 1\colon {}^{\mu}\mathcal{M}_{G,\mathrm{HT},k,t}\times\mathrm{Spa}(A,A^+)\to\mathcal{U}^G_k(p^t)\times\mathrm{Spa}(A,A^+)$$

denotes the structural map.

Let $t>m>k>r_0$ satisfy (3.2.6). We can therefore form the cohomology

$$R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\underline{\lambda}))^{-,\mathrm{fs}}:=R\Gamma_{\mathcal{I}^G_{m,k}(p^t)}(\mathcal{U}^G_k(p^t),[D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})}])^{-,\mathrm{fs}}$$

where the finite-slope part is with respect to a certain power of $\mathcal{U}'_B(p^t)$ (by [Boxer and Pilloni 2021, Theorem 6.4.3] and a similar calculation in the proof of Proposition 4.6.2). This definition is independent of the choice of (m,k,t) by [loc. cit., Theorems 6.4.5 and 6.4.8]. If one has a continuous morphism $(A,A^+)\to(\mathbb{Q}_p,\mathbb{Z}_p)$ such that the composition of this morphism with $\underline{\lambda}$ (denoted λ) lies in $X^*(T/T_0)^+$, then one has a natural specialization map $R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\underline{\lambda}))^{-,\mathrm{fs}}\to R\Gamma^G_{w_n}(\kappa_n(\lambda))^{-,\mathrm{fs}}$ (after fixing an isomorphism $V_{\kappa_n(\lambda)}\cong V^*_{\kappa_n(\lambda)^*}$) arising from the map $D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\lambda)}\to V_{\kappa_n(\lambda)}$. Furthermore, if $(A,A^+)=(\mathbb{Q}_p,\mathbb{Z}_p)$ then this specialization map is an isomorphism on small slope parts; [loc. cit., Corollary 6.8.4] using the improved slope bounds implied by [loc. cit., Theorem 6.10.1] because the Shimura variety is compact.

Similarly, we can also form the cohomology complexes

$$R\Gamma^H_{\mathrm{id},\mathrm{an}}(\mathcal{S}_{H,\diamond}(p^t),\sigma_n^{[\beta]}(\underline{\lambda}))^{(-,\dagger)}:=\varprojlim_m R\Gamma_{\mathcal{Z}^H_m(p^t)}(\mathcal{U}^H_k(p^t),[\sigma_n^{[\beta]}(\underline{\lambda})]),$$

$$R\Gamma^H_{\mathrm{id},\mathrm{an}}(\mathcal{S}_{H,\diamond}(p^t),\sigma_n^{[\beta]}(\underline{\lambda})^\vee)^{(+,\dagger)}:=\varinjlim_m R\Gamma(\mathcal{Z}^H_m(p^t),[\sigma_n^{[\beta]}(\underline{\lambda})]^\vee),$$

for $k>r_0$ and $t\geq 1$, where the sheaves are defined using the torsor ${}^{\mu}\mathcal{M}_{H,\mathrm{HT},k,t}$. The first definition is independent of k by excision and Remark 5.1.10. As before, if $t=1$ then we omit the variety from the notation. If $(A,A^+)\to(\mathbb{Q}_p,\mathbb{Z}_p)$ is a continuous homomorphism and the composition of this morphism with $(\underline{\lambda}, \beta)$ (denoted (λ, j)) lies in $X^*(T/T_0)^+ \times X^*(S)$, then we have specialization maps $R\Gamma^H_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\underline{\lambda}))^{(-,\dagger)} \to R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]}(\lambda))^{(-,\dagger)}$ and $R\Gamma^H_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\underline{\lambda})^\vee)^{(+,\dagger)} \to R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]}(\lambda)^\vee)^{(+,\dagger)}$.

Proposition 5.4.2. *Let $(\underline{\lambda}, \beta) \in X_0^*(T \times S; A)$ be an r_0-analytic character. Then we have a well-defined A-linear map*

$$R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\underline{\lambda}))^{-,\mathrm{fs}} \to R\Gamma^H_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\underline{\lambda}))^{(-,\dagger)} \tag{5.4.3}$$

which satisfies:

(1) *If $(A, A^+) \to (B, B^+)$ is a morphism of Tate algebras over $(\mathbb{Q}_p, \mathbb{Z}_p)$, and $(\underline{\lambda}', \beta') \in X_0^*(T \times S; B)$ denotes the induced character, then the morphisms in (5.4.3) for the pairs $(\underline{\lambda}, \beta)$ and $(\underline{\lambda}', \beta')$ are compatible under base-change along the morphism $(A, A^+) \to (B, B^+)$.*

(2) *If $(A, A^+) = (\mathbb{Q}_p, \mathbb{Z}_p)$ and $(\underline{\lambda}, \beta) = (\lambda, j)$ is algebraic as in Theorem 5.3.4(3), then one has a commutative diagram*:

$$\begin{array}{ccc} R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\lambda))^{-,\mathrm{fs}} & \xrightarrow{(5.4.3)} & R\Gamma^H_{\mathrm{id,an}}(\sigma_n^{[j]}(\lambda))^{(-,\dagger)} \\ \downarrow & & \| \\ R\Gamma^G_{w_n}(\kappa_n(\lambda))^{-,\mathrm{fs}} & \longrightarrow & R\Gamma^H_{\mathrm{id}}(\sigma_n^{[j]}(\lambda))^{(-,\dagger)} \end{array}$$

where the bottom map is the one in Proposition 4.7.2.

Proof. This is constructed in a similar way as Proposition 4.7.2, using the morphism of sheaves $\hat{\iota}[D^{(k-1)-\mathrm{an}}_{G,\kappa_n(\underline{\lambda})}] \to [\sigma_n^{[\beta]}(\underline{\lambda})]$ arising from evaluation at the vector $x_n^{[\beta]}(\underline{\lambda})$, i.e., the pullback is constructed using a triple (m, k, t) satisfying (3.2.6) and then one traces down to level $K^H_\diamond(p)$. Parts (1) and (2) follow from the properties of the vector $x_n(\underline{\lambda})$ in Theorem 5.3.4. □

We have a Serre duality pairing between the complexes $R\Gamma^H_{\mathrm{id,an}}(\cdots)^{(-,\dagger)}$ and $R\Gamma^H_{\mathrm{id,an}}(\cdots)^{(+,\dagger)}$ which is compatible with the duality in Section 4.7 via the specialization maps above. Therefore we obtain a pairing

$$\langle\langle \cdot, \cdot \rangle\rangle^-_{\mathrm{an}} \colon \mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\underline{\lambda}))^{-,\mathrm{fs}} \times \mathrm{H}^0_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\underline{\lambda})^\vee)^{(+,\dagger)} \to A$$

which is compatible under change of coefficients. We have the following compatibility with the previously defined pairings.

Corollary 5.4.4. *Let $f \colon (A, A^+) \to (\mathbb{Q}_p, \mathbb{Z}_p)$ be a homomorphism over $(\mathbb{Q}_p, \mathbb{Z}_p)$, and suppose that the character (λ, j), induced from composing $(\underline{\lambda}, \beta)$ with this morphism, is algebraic as in Theorem 5.3.4(3). Then for any*

- $\underline{\eta} \in \mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\underline{\lambda}))^{-,\mathrm{fs}}$,
- $\underline{\chi} \in \mathrm{H}^0_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\underline{\lambda})^\vee)^{(+,\dagger)}$,

one has $f(\langle\langle \underline{\eta}, \underline{\chi} \rangle\rangle^-_{\mathrm{an}}) = \langle \eta, \chi \rangle^-_{\mathrm{an}}$, where η and χ denote the specializations of $\underline{\eta}$ and $\underline{\chi}$ respectively under the morphism f.

Remark 5.4.5. There are analogous constructions of all the various pairings in Sections 4–5 working over a finite extension $L/\mathbb{Q}_p$ and they are related by base-change of coefficients. This will be important in the construction of the p-adic L-function, because we will have to enlarge the field of definition to include the Hecke eigenvalues of the relevant automorphic representation/character.

6. Families of cohomology classes

In this section we show that, under some hypotheses on the ramification of the automorphic representation π, there exists a family of cohomology classes in $\mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))^{-,\mathrm{fs}}$ corresponding to a family of automorphic representations passing through π. This family of cohomology classes will be one half of the input for the pairing $\langle\langle \cdot, \cdot \rangle\rangle^{-}_{\mathrm{an}}$ when constructing the p-adic L-function in Section 8. Recall that we have assumed F contains an imaginary quadratic number field E. This will be important when speaking about automorphic base-change for unitary similitude groups.

6.1. ***Families for the group G.*** Let π be a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ such that π_∞ lies in the discrete series. We impose the following assumptions:

Assumption 6.1.1. *Assume that*:

(1) *The Harish-Chandra parameter of π_∞ is of the form $w_n \cdot (\lambda_\pi + \rho)$ for some $\lambda_\pi \in X^*(T/T_0)^+$ (see Section 2.3).*

(2) *Any weak base-change of π to an automorphic representation of* $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$ *is cuspidal.*[5]

(3) *There exist compact open subgroups $K_p \subset \boldsymbol{G}(\mathbb{Q}_p)$ and $K^p \subset \boldsymbol{G}(\mathbb{A}_f^p)$ with K_p hyperspecial, such that $K = K^p K_p$ is sufficiently small and*

$$\dim_{\mathbb{C}} \pi_f^K = 1.$$

Remark 6.1.2. Under the additional assumptions below, Assumption 6.1.1(3) is not a severe restriction thanks to the local newform theory for general linear groups. More precisely, under Assumption 6.2.1 below, the local component of π at any ramified prime occurs as the local component of its cuspidal base-change to $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$, and is therefore generic. In particular, by [Jacquet et al. 1981], there exists a compact open subgroup $K = K^p K_p$ with K_p hyperspecial, such that $\dim_{\mathbb{C}} \pi_f^K = 1$. If K is neat then Assumption 6.1.1(3) holds, otherwise one can use a similar strategy as in [Loeffler and Zerbes 2021, Remark 3.2.1] to handle more general levels.

Fix a finite set of primes S containing p and all primes where K^p is not a good special maximal compact open subgroup as in Lemma C.0.1. Let $\mathbb{T}^-$ denote the Hecke algebra (over $\mathbb{Q}$) given by

$$\mathbb{T}^- = \mathrm{C}^\infty(K^S \backslash \boldsymbol{G}(\mathbb{A}_f^S)/K^S) \otimes \mathbb{Q}[T^-]$$

[5] Here by weak base-change, we mean an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$ satisfying the conditions in [Shin 2014, Theorem A.1] (the theorem of course shows that such a base-change exists).

where the convolution product for the first factor is with respect to a fixed Haar measure on $\boldsymbol{G}$. We fix a $\mathbb{C}$-algebra homomorphism $\theta_\pi : \mathbb{T}^-_{\mathbb{C}} \to \mathbb{C}$ which is an eigencharacter for the action of $\mathbb{T}^-_{\mathbb{C}}$ on $\pi_f^{K^p K^G_{\mathrm{Iw}}(p)}$. By Assumption 6.1.1(3), this homomorphism has finite-slope at p, so gives rise to a monoid homomorphism $\theta_{\pi,p} : T^- \to \mathbb{C}^\times$. We let I_π denote the kernel of the morphism θ_π.

Lemma 6.1.3. *There exists a number field Φ containing F, such that θ_π is defined over Φ.*

Proof. Let $\psi \boxtimes \Pi_0$ denote the weak base-change of π to $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$. By [Labesse and Schwermer 2019, Theorem 5.2.1], there exists $\pi_0 \subset \pi|_{\boldsymbol{G}_0(\mathbb{Q}_\ell)}$ cuspidal automorphic such that Π_0 is the weak base-change of π_0. Since Π_0 is cuspidal, we have $\mathrm{BC}_\ell(\pi_{0,\ell}) \cong \Pi_{0,\ell}$ for all rational primes ℓ, where BC_ℓ denotes the local (standard) base-change map; see [Liu et al. 2022, Section C.3].

This implies that the homomorphism θ_π matches with the Hecke eigensystem for $\psi \boxtimes \Pi_0$, which is regular algebraic. The result then follows from [Grobner and Raghuram 2014, Proposition 3.4.3] (note that F is taken to be a totally real field in [loc. cit.], but the cited result holds in general via the same proof). □

The above lemma implies that we can view θ_π as a homomorphism valued in any field extension of Φ. For example, if we let L denote the completion of the image of Φ under the fixed isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, then $L/\mathbb{Q}_p$ is a finite extension and we can view θ_π as an L-algebra homomorphism $\mathbb{T}^-_L \to L$. This leads to the following small slope assumption:

Assumption 6.1.4. *We assume that the monoid homomorphism $\theta_{\pi,p} : T^- \to L^\times$ is of small slope (with respect to $\kappa_n = w_n \star (-w_G^{\max}\lambda_\pi)$).*

Example 6.1.5. Let $\lambda_\pi^* = -w_G^{\max} \cdot \lambda_\pi$ (which is in fact equal to λ_π by Assumption 6.1.1(1)). We say that π is Borel ordinary if $\lambda_\pi^*(x)^{-1}\theta_{\pi,p}(x)$ is a p-adic unit, where $x \in T^{--}$ is the element in Definition 4.5.1. As seen below, π contributes to the coherent cohomology of $S_{\boldsymbol{G},\mathrm{Iw}}(p)$, and the slope bounds in [Boxer and Pilloni 2021, Conjecture 5.9.2] hold because the Shimura variety is compact; see [loc. cit., Theorem 6.48]. Therefore, being Borel ordinary in fact implies that the homomorphism $(-\lambda_\pi^*) \cdot \theta_{\pi,p}$ is valued in $\mathcal{O}_L^\times$.

Suppose that π is Borel ordinary. Then we will show that $\theta_{\pi,p}$ is of small slope. For this, it is enough to calculate, for $i \neq n$, the τ_0-component of $\delta_i := w_i^{-1} \star \kappa_n - \lambda_\pi^*$ and show that there exists $x \in T^-$ such that $v(\delta_i(x)) > 0$. For $1 \leq i \leq 2n-1$, let $x_i \in T^-$ be the element which is the identity outside the τ_0-component, and equal to $(1, \dots, 1, p, \dots, p)$ in the τ_0-component (where there are i lots of p). Write $\lambda_\pi = (0; c_{1,\tau}, \dots, c_{2n,\tau})_{\tau \in \Psi}$. We break the analysis into two cases.

Suppose that $i < n$. Then the action of w_i^{-1} only affects the first $i+1$ entries of the τ_0-component of the weight. In this case, we take $x = x_n$ and find that $v(\delta_i(x)) = 2c_{n,\tau_0} + 1 > 0$ because $c_{n,\tau_0} \geq 0$ (Assumption 6.1.1(1)).

Suppose that $i = n + \varepsilon$ for an integer $1 \leq \varepsilon \leq n-1$. Then the last $n - \varepsilon$ entries of the τ_0-component of δ_i are $c_{n-\varepsilon} - c_n + \varepsilon, 0, \dots, 0$ (using the fact that $c_{j,\tau_0} = -c_{2n+1-j,\tau_0}$). We then take $x = x_{n-\varepsilon}$ and conclude that $v(\delta_i(x)) = c_{n-\varepsilon} - c_n + \varepsilon > 0$ because λ_π is dominant.

Recall that we can view $X^*(T/T_0)^+$ as a subset of $\mathcal{W}_G(\mathbb{Q}_p)$ (we will refer to this subset as the classical weights). We now introduce the notion of a family of automorphic representations and cohomology classes.

Definition 6.1.6. By a family $\underline{\pi}$ over an open affinoid $U = \mathrm{Spa}(A, A^+) \subset \mathcal{W}_{G,L}$ containing λ_π, passing through π, we mean an A-algebra homomorphism

$$\theta_{\underline{\pi}} : \mathbb{T}_A^- \to A$$

such that for all but finitely many classical weights $\lambda \in U \cap X^*(T/T_0)^+$, there exists a cuspidal automorphic representation σ such that the specialization of $\theta_{\underline{\pi}}$ at λ is an eigencharacter for the action of $\mathbb{T}_L^-$ on $\sigma^{K^p K^G_{\mathrm{Iw}}(p)}$ (under the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$).

Let $\eta \in \mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])^{-,\mathrm{ss}}$ be an eigenvector for the action of $\mathbb{T}_L^-$ with eigencharacter θ_π. Let $\lambda_A : T(\mathbb{Z}_p) \to (A^+)^\times$ denote the universal character associated with U. If such a family $\underline{\pi}$ exists then, by a family $\underline{\eta}$ of cohomology classes passing through η, we mean an eigenvector $\underline{\eta} \in \mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))^{-,\mathrm{fs}}$ for the action of $\mathbb{T}_A^-$ with eigencharacter $\theta_{\underline{\pi}}$, whose specialization at λ_π equals η under the comparison isomorphism

$$\mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\lambda_\pi))^{-,\mathrm{ss}} \cong \mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])^{-,\mathrm{ss}}.$$

6.2. *Existence of families.* In this section, we introduce some further assumptions on π which ensure the existence of a family passing through π as well as a family of cohomology classes. We begin with the following ramification assumption on the representation π:

Assumption 6.2.1. *Assume that*:

(1) *The set S above contains only primes which split in $E/\mathbb{Q}$, i.e., $K^S = \prod_{\ell \notin S} K_\ell$ where $K_\ell \subset \boldsymbol{G}(\mathbb{Q}_\ell)$ is a good special maximal compact open. We further assume that K_ℓ is hyperspecial if $\boldsymbol{G}_{\mathbb{Q}_\ell}$ is unramified (for $\ell \notin S$).*

(2) *The eigencharacter $\theta_{\pi,p}$ appears with multiplicity one for the action on $\pi_p^{K^G_{\mathrm{Iw}}(p)}$.*

As a consequence of this assumption, we have:

Lemma 6.2.2. *Suppose that π satisfies Assumption 6.2.1 (as well as the assumptions in the previous section). Let σ be a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ such that σ_∞ is cohomological. Suppose that $\sigma_f^K \neq 0$ and $\pi_\ell \cong \sigma_\ell$ for all $\ell \notin S$. Then $\pi_f \cong \sigma_f$.*

Proof. This is an application of Proposition C.0.3. □

We obtain the following corollary:

Corollary 6.2.3. *Let π be as in Lemma 6.2.2 and set $\kappa_n = w_n \star (-w_G^{\max} \cdot \lambda_\pi)$. Then the localized cohomology*

$$\mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])_{I_\pi}$$

is one-dimensional (*over L*).[6]

[6] We are abusing notation slightly — by the localization $(\cdots)_{I_\pi}$ we mean first base-change to L and then localize at I_π (the kernel of the map $\mathbb{T}_L^- \to L$).

Proof. Via the rigid GAGA comparison, this localized cohomology group has the same dimension as

$$\mathrm{H}^{n-1}(S(\mathbb{C}), [V_{\kappa_n}])_{I_\pi}$$

where $S = S_{\boldsymbol{G},\mathrm{Iw}}(p)$ and we are considering its sheaf cohomology with coefficients in $[V_{\kappa_n}]$.

Let $A_{\boldsymbol{G}} \cong \mathbb{G}_m$ denote the maximal split torus inside the center of $\boldsymbol{G}$, and let $A_{\boldsymbol{G}}(\mathbb{R})^\circ$ denote the connected component of the identity in the analytic topology. Let $K_\infty^\circ \subset K_\infty$ denote the maximal compact subgroup, where $K_\infty = A_{\boldsymbol{G}}(\mathbb{R})^\circ K_\infty^\circ$ is as in Section 2.3. Let $\mathfrak{p}$ denote the Lie algebra of the *opposite* of $P_{\boldsymbol{G}}$, and we can write

$$\mathfrak{p} = \mathfrak{p}^\circ \oplus \mathfrak{a}_G$$

where $\mathfrak{a}_G$ is the Lie algebra of A_G and $\mathfrak{p}^\circ = \mathfrak{p} \cap \mathfrak{g}_0$, where $\mathfrak{g}_0$ denotes the Lie algebra of $\boldsymbol{G}_0$.

By [Su 2019], we have the following description

$$\mathrm{H}^{n-1}(S(\mathbb{C}), [V_{\kappa_n}]) = \bigoplus_\sigma (\mathrm{H}^{n-1}_{(\mathfrak{p}^\circ, K_\infty^\circ)}(\sigma_\infty \otimes V_{\kappa_n}) \otimes \sigma_f^{K^p K_{\mathrm{Iw}}^G(p)})^{m(\sigma)} \tag{6.2.4}$$

where the sum runs over all cuspidal automorphic representations σ of $\boldsymbol{G}(\mathbb{A})$ which lie in the discrete spectrum (with multiplicity $m(\sigma)$), and are such that $A_{\boldsymbol{G}}(\mathbb{R})^\circ$ acts trivially on σ_∞. Since $\mathfrak{a}_G$ and $A_{\boldsymbol{G}}(\mathbb{R})^\circ$ act trivially on $\sigma_\infty \otimes V_{\kappa_n}$, we have

$$\mathrm{H}^{n-1}_{(\mathfrak{p}^\circ, K_\infty^\circ)}(\sigma_\infty \otimes V_{\kappa_n}) = \mathrm{H}^{n-1}_{(\mathfrak{p}, K_\infty)}(\sigma_\infty \otimes V_{\kappa_n}).$$

By the Hodge decomposition (see [Lan and Polo 2018] for example) of the singular cohomology $\mathrm{H}^{2n-1}(S(\mathbb{C}), W_{\lambda_\pi})$ with coefficients in the algebraic representation with highest weight λ_π, we see that σ_∞ is cohomological if

$$\mathrm{H}^{n-1}_{(\mathfrak{p}, K_\infty)}(\sigma_\infty \otimes V_{\kappa_n}) \otimes \sigma_f^{K^p K_{\mathrm{Iw}}^G(p)} \neq 0.$$

Furthermore, if this space is nonzero after localizing at I_π, the conditions in Lemma 6.2.2 are satisfied for σ.

Note that if σ satisfies $\sigma_f \cong \pi_f$ then by the strong base-change results in [Mok 2015] and [Kaletha et al. 2014] (and that $A_{\boldsymbol{G}}(\mathbb{R})^\circ$ acts trivially on σ_∞), σ_∞ must lie in the same L-packet for π_∞. By [Blasius et al. 1994, Theorem 3.2.1], if the vector space $\mathrm{H}^{n-1}_{(\mathfrak{p}, K_\infty)}(\sigma_\infty \otimes V_{\kappa_n})$ is nonzero, then we must have $\sigma_\infty \cong \pi_\infty$ and $\mathrm{H}^{n-1}_{(\mathfrak{p}, K_\infty)}(\pi_\infty \otimes V_{\kappa_n})$ is one-dimensional. Therefore, localizing (6.2.4) at the ideal I_π, we see that

$$\mathrm{H}^{n-1}(S(\mathbb{C}), [V_{\kappa_n}])_{I_\pi} = (\mathrm{H}^{n-1}_{(\mathfrak{p}, K_\infty)}(\pi_\infty \otimes V_{\kappa_n}) \otimes \pi_f^{K^p K_{\mathrm{Iw}}^G(p)}[\theta_{\pi,p}])^{m(\pi)}.$$

where $\pi_f^{K^p K_{\mathrm{Iw}}^G(p)}[\theta_{\pi,p}]$ denotes the (generalized) eigenspace for the character $\theta_{\pi,p}$.

By Assumption 6.2.1, we therefore see that the dimension of the cohomology group in the statement of the corollary is equal to $m(\pi)$. Since $m(\pi) > 0$ (by definition), it is enough to show that $m(\pi) \leq 1$. But there is an injective $\boldsymbol{G}_0$-equivariant restriction map

$$L^2_{\mathrm{disc}}(\boldsymbol{G}) \hookrightarrow L^2_{\mathrm{disc}}(\boldsymbol{G}_0)$$

from the discrete spectrum of $\boldsymbol{G}$ to that of $\boldsymbol{G}_0$ (see [Labesse and Schwermer 2019, Theorem 1.1.1]), hence it is enough to show that the multiplicity of any cuspidal automorphic representation in $L^2_{\mathrm{disc}}(\boldsymbol{G}_0)$ is at most 1. But this follows from Arthur's multiplicity formula for unitary groups; see [Chen and Zou 2021]. ☐

Recall that we have classicality isomorphisms on the small slope part

$$R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n)^{-,\mathrm{ss}} \cong R\Gamma^G_{w_n}(\kappa_n)^{-,\mathrm{ss}} \cong R\Gamma(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])^{-,\mathrm{ss}}.$$

Note that the cohomology of the right-hand side vanishes outside degree $n-1$, and since θ_π is of small slope, we see that $R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n)_{I_\pi}$ has cohomology concentrated in degree $n-1$ where it is free of rank one (over L).

The Tor-spectral sequence

$$E_2^{p,q}:\ \mathrm{Tor}^A_{-p}(\mathrm{H}^q_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))^{-,\mathrm{fs}}, \lambda_\pi) \Rightarrow \mathrm{H}^{p+q}_{w_n,\mathrm{an}}(\kappa_n(\lambda_\pi))^{-,\mathrm{fs}}$$

therefore implies that there exists an affinoid $U = \mathrm{Spa}(A, A^+) \subset (\mathcal{W}_G)_L$ containing λ_π, such that

$$R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))_{I_\pi}$$

has cohomology concentrated in degree $n-1$ where it is free of rank one over the stalk of A at λ_π. Here $\lambda_A : T(\mathbb{Z}_p) \to (A^+)^\times$ denotes the universal character (which is trivial on $T_0(\mathbb{Z}_p)$).

The construction in [Boxer and Pilloni 2021, Section 6.9] gives rise to an eigenvariety $\mathcal{E} \to \mathcal{W}_G$ which is locally quasifinite and partially proper, and parametrizes finite-slope Hecke eigensystems appearing in the coherent cohomology of $\mathcal{S}_{G,\mathrm{Iw}}(p)$.[7] In particular, we have coherent sheaves $\widetilde{\mathcal{M}}^{\bullet,-,\mathrm{fs}}_{w_n}$ whose pushforward to $\mathcal{W}_G$ recovers the cohomology groups $\mathrm{H}^\bullet_{w_n,\mathrm{an}}(\cdots)^{-,\mathrm{fs}}$, and the ideal I_π gives a point $x \in \mathcal{E}(L)$. Since $R\Gamma^G_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))_{I_\pi}$ has cohomology concentrated in degree $n-1$ where it is free of rank one over the stalk of A at λ_π, we can (after shrinking U) find an open affinoid neighborhood $V \subset \mathcal{E}_L$ of x such that the induced map $V \to U$ is an isomorphism. In particular, this implies:

Theorem 6.2.5. *Shrinking U if necessary*:

(1) *There exists a unique family $\underline{\pi}$ over U passing through π.*

(2) *The generalized eigenspace $S^{n-1}(\underline{\pi}) \subset \mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\lambda_A))^{-,\mathrm{fs}}$ on which $\mathbb{T}^-_A$ acts through the character $\theta_{\underline{\pi}}$, is a direct summand that is free of rank one over A. In particular, a basis $\underline{\eta}$ of $S^{n-1}(\underline{\pi})$ is a family of cohomology classes passing through a basis η of $\mathrm{H}^{n-1}(\mathcal{S}_{G,\mathrm{Iw}}(p), [V_{\kappa_n}])_{I_\pi}$.*

Proof. The above discussion implies that there exists a character $\theta_{\underline{\pi}}$ specializing to θ_π at λ_π and satisfying (2), so we just need to show that $\theta_{\underline{\pi}}$ defines a unique family. But the fact that $\theta_{\underline{\pi}}$ arises from the eigenvariety $\mathcal{E}$ implies that for any $\lambda \in U \cap X^*(T/T_0)^+$, the specialization of $\theta_{\underline{\pi}}$ is an eigencharacter for

[7] This is not the "full eigenvariety" but rather the pullback of the eigenvariety constructed in [Boxer and Pilloni 2021, Section 6.9] along the closed embedding $\mathcal{W}_G \hookrightarrow \mathcal{W}^{\mathrm{full}}_G$, here $\mathcal{W}^{\mathrm{full}}_G$ is the weight space parametrizing characters of $T(\mathbb{Z}_p)$. Furthermore, including level subgroups which are good special maximal compact open but not hyperspecial does not affect the construction.

the action of $\mathbb{T}_L^-$ on $\mathrm{H}^{n-1}_{w_n,\mathrm{an}}(\kappa_n(\lambda))^{-,\mathrm{fs}}$. Shrinking U if necessary, we can ensure that it is of small slope, so (under the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$) contributes to $\mathrm{H}^{n-1}(S_{\boldsymbol{G},\mathrm{Iw}}(\mathbb{C}), [V_{\kappa_n(\lambda)}])$ *with multiplicity one*. The description in (6.2.4) holds for this cohomology group, and therefore, letting I denote the kernel of the specialization θ of $\theta_{\underline{\pi}}$ at λ, we must have a Hecke-equivariant isomorphism

$$\mathrm{H}^{n-1}(S_{\boldsymbol{G},\mathrm{Iw}}(\mathbb{C}), [V_{\kappa_n(\lambda)}])_I \cong \sigma_f^{K^p K^G_{\mathrm{Iw}}(p)}[\theta_p]$$

for some cuspidal automorphic representation σ, since we know the dimension of the left-hand side is one. □

Remark 6.2.6. We will refer to σ in the above theorem as *the* specialization of $\theta_{\underline{\pi}}$ at λ, even though there will be several automorphic representations σ' which have the same Hecke eigenvalues. Note that, by the Hodge decomposition, σ_∞ is cohomological with respect to the algebraic representation of $\boldsymbol{G}(\mathbb{C})$ with highest weight λ.

7. Families of anticyclotomic characters

In this section we exhibit families of anticyclotomic characters in the coherent cohomology of $\mathcal{S}_{H,\diamond}(p)$.

7.1. *Anticyclotomic characters.* Let $\boldsymbol{R}$ denote the unitary similitude group associated with the Hermitian space $\bigwedge^n_F W_1 \oplus \bigwedge^n_F W_2$ (with common similitude on each factor) where W_1 and W_2 are the Hermitian spaces in Section 2. This can be upgraded to a PEL Shimura datum via the homomorphism $h_{\boldsymbol{R}} := \det \circ h_{\boldsymbol{H}}$ and has Hodge cocharacter $\mu_{\boldsymbol{R}} := \det \circ \mu_{\boldsymbol{H}}$. Here $\det\colon \boldsymbol{H} \to \boldsymbol{R}$ denotes the homomorphism given by $(h_1, h_2) \mapsto (\det h_1, \det h_2)$. By design, one has a morphism of Shimura data $(\boldsymbol{H}, h_{\boldsymbol{H}}) \to (\boldsymbol{R}, h_{\boldsymbol{R}})$. Note that $\mu_{\boldsymbol{R}}$ is central in $\boldsymbol{R}_{F^{\mathrm{cl}}}$, so the associated parabolics and Levi are all equal to

$$\boldsymbol{R}_{F^{\mathrm{cl}}} \cong \mathbb{G}_{m,F^{\mathrm{cl}}} \times \prod_{\tau \in \Psi} (\mathbb{G}_{m,F^{\mathrm{cl}}} \times \mathbb{G}_{m,F^{\mathrm{cl}}}).$$

Let $\mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(1)$ be the restriction of scalars of the unitary group associated with the one-dimensional Hermitian space over F (with respect to F/F^+). Then we have a morphism of algebraic groups

$$\mathcal{N}\colon \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \to \mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(1),$$
$$z \mapsto \bar{z}/z,$$

which is open and surjective on $\mathbb{A}_f$-points. On the other hand, we have a morphism

$$\nu\colon \boldsymbol{H} \xrightarrow{\det} \boldsymbol{R} \to \mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(1)$$

where the second map is given by sending a pair (z_1, z_2) to z_2/z_1.

Notation 7.1.1. Let $\mathfrak{N}$ be the smallest ideal of $\mathcal{O}_F$ such that $\nu(U) \subset \mathcal{N}((\widehat{\mathcal{O}}_{F^+} + \mathfrak{N}\widehat{\mathcal{O}}_F)^\times)$, where $U \subset \boldsymbol{H}(\mathbb{A}_f)$ is the level of $S_{\boldsymbol{H},\diamond}(p)$.

We introduce the following space of anticyclotomic characters:

Definition 7.1.2. Let $\Sigma(\mathfrak{N})$ denote the set of algebraic Hecke characters $\chi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ satisfying:

(1) χ is anticyclotomic, i.e., its restriction to $\mathbb{A}_{F^+}^\times$ is trivial.

(2) The infinity type of χ is $(j,-j)$ for some tuple of integers $j=(j_\tau)_{\tau\in\Psi}$, i.e., for any $z=(z_\tau)_{\tau\in\Psi}\in\prod_{\tau\in\Psi}F_\tau$ one has
$$\chi(z)=\prod_{\tau\in\Psi}z_\tau^{-j_\tau}\overline{z_\tau}^{j_\tau}.$$

(3) The conductor of χ divides the ideal $\mathfrak{N}$.

Remark 7.1.3. Let $\chi\in\Sigma(\mathfrak{N})$. Then, since χ is anticyclotomic, the character χ descends to a unique character
$$\chi'\colon (\mathrm{Res}_{F^+/\mathbb{Q}}\mathrm{U}(1))(\mathbb{Q})\backslash(\mathrm{Res}_{F^+/\mathbb{Q}}\mathrm{U}(1))(\mathbb{A})\to\mathbb{C}^\times$$
satisfying $\chi=\chi'\circ\mathcal{N}$. We consider the character $\bar{\chi}\colon \boldsymbol{R}(\mathbb{Q})\backslash\boldsymbol{R}(\mathbb{A})\to\mathbb{C}^\times$ defined as $\bar{\chi}(z_1,z_2)=\chi'(z_2/z_1)$.

Any character $\chi\in\Sigma(\mathfrak{N})$ has an associated p-adic algebraic Hecke character, denoted $\chi_p\colon \mathbb{A}_F^\times\to\overline{\mathbb{Q}}_p^\times$, by defining
$$\chi_p(x)=\iota_p(\chi_f(x))\prod_{\tau\in\Psi}x_{\mathfrak{p}_\tau}^{-j_\tau}x_{\bar{\mathfrak{p}}_\tau}^{j_\tau}$$
where $\iota_p\colon\mathbb{C}\cong\overline{\mathbb{Q}}_p$ denotes the fixed isomorphism in Section 1.2, and $\mathfrak{p}_\tau$ is the prime ideal corresponding to τ with respect to this isomorphism. We are interested in p-adically interpolating algebraic p-adic characters of the form
$$\chi_{0,p}\prod_{\tau\in\Psi}\chi_{\tau,p}^{m_\tau}$$
where $\chi_0\in\Sigma(\mathfrak{N})$ is an anticyclotomic Dirichlet character, $\chi_\tau\in\Sigma(\mathfrak{N})$ is a fixed anticyclotomic character of infinity type $(1_\tau,-1_\tau)$ (1_τ is the tuple which is nonzero only in the τ-component, where it is equal to 1) and m_τ are integers. Furthermore, we want to interpret such a family as a coherent cohomology class.

The strategy we will use for producing such a family follows three steps:

(1) We will first construct a family of cohomology classes interpolating these characters in the cohomology of a Shimura set associated with the group $\boldsymbol{R}$.

(2) Using the results in Appendix B, we will pull back this construction to the Shimura variety $\mathcal{S}_{H,\diamond}(p)$ via the morphism $\det\colon \boldsymbol{H}\to\boldsymbol{R}$.

(3) Finally, we will construct the family and describe the interpolation property.

7.2. ***Step 1: Classes for the Shimura set.*** Let $C\subset\boldsymbol{R}(\mathbb{A}_f)$ be a sufficiently small compact open subgroup, and let $\chi\in\Sigma(\mathfrak{N})$ be an anticyclotomic character of infinity type $(j,-j)$ such that $\bar{\chi}$ is trivial on C. Let $\Delta:=S_{\boldsymbol{R},C}$ denote the associated Shimura set (over F^{cl}), which satisfies
$$\Delta(\mathbb{C})=\boldsymbol{R}(\mathbb{Q})\backslash\boldsymbol{R}(\mathbb{A}_f)/C.$$

The goal of this section is to associate to $\bar{\chi}$ a class in the coherent cohomology of Δ, and explain how one can raise this class to p-adic powers.

Let $R_{\mathrm{dR}} \to \Delta$ denote the standard principal $\boldsymbol{R}_{F^{\mathrm{cl}}}$-bundle, which satisfies

$$R_{\mathrm{dR}}(\mathbb{C}) = \boldsymbol{R}(\mathbb{Q})\backslash \boldsymbol{R}(\mathbb{C}) \times \boldsymbol{R}(\mathbb{A}_f)/C$$

(via the embedding $F^{\mathrm{cl}} \subset \mathbb{C}$). This bundle has a trivialization in the following way. Fix a set of representatives $\{s_1, \dots, s_r\} \subset \boldsymbol{R}(\mathbb{A}_f)$ for each point of $\Delta(\mathbb{C})$, then we have an identification of torsors

$$\Delta(\mathbb{C}) \times \boldsymbol{R}(\mathbb{C}) = R_{\mathrm{dR}}(\mathbb{C}) \tag{7.2.1}$$

by sending $([s_i], \gamma)$ to $[\gamma, s_i]$. One can show that, for any number field Φ/F^{cl}, this identification descends to an identification $\Delta_\Phi \times \boldsymbol{R}_\Phi = R_{\mathrm{dR},\Phi}$.[8]

Recall that we have a fixed prime $\mathfrak{p}$ of F lying above p (corresponding to the fixed embedding τ_0). We fix a choice of prime $\mathfrak{P}$ of Φ lying above $\mathfrak{p}$, and by passing to completions, we obtain a finite extension $L := \Phi_{\mathfrak{P}}$ of $\mathbb{Q}_p$. Let Δ_L^{an} denote the adic space associated with Δ_L, and let $\mathcal{R}^{\mathrm{an}}_{\mathrm{HT},L} \to \Delta_L^{\mathrm{an}}$ denote the R_L^{an}-torsor parametrizing frames of (the pro-étale sheaf) $\mathcal{V}_{\text{ét}} \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_{\Delta_L^{\mathrm{an}}}$ (respecting certain tensors), where $R = \boldsymbol{R}_{\mathbb{Q}_p}$ and $\mathcal{V}_{\text{ét}}$ is the p-adic local system associated with a faithful representation V of R; see [Caraiani and Scholze 2017, Section 2.3].

Since $\mu_{\boldsymbol{R}}$ is central in $\boldsymbol{R}_{F^{\mathrm{cl}}}$, one has an isomorphism of torsors between the analytification of $R_{\mathrm{dR},L}$ and ${}^{\mu}\mathcal{R}^{\mathrm{an}}_{\mathrm{HT},L}$ (the twist of $\mathcal{R}^{\mathrm{an}}_{\mathrm{HT},L}$ along $\mu_{\boldsymbol{R}}$).

Notation 7.2.2. Consider the open affinoid subgroup

$$\mathcal{R}_{k,L} = \mathcal{O}_L^\times(1+\mathcal{B}_k) \times \prod_{\tau\in\Psi} (\mathcal{O}_L^\times(1+\mathcal{B}_k) \times \mathcal{O}_L^\times(1+\mathcal{B}_k)) \subset R_L^{\mathrm{an}}$$

where $\mathcal{B}_k$ is the "closed disc" (over L) in Section 3.2. We denote a general element of this subgroup by $(x_0, x_{1,\tau}, x_{2,\tau})_{\tau\in\Psi}$.

Corollary 7.2.3. *The above identification induces an identification*

$$\Delta_L^{\mathrm{an}} \times R_L^{\mathrm{an}} = {}^{\mu}\mathcal{R}^{\mathrm{an}}_{\mathrm{HT},L}.$$

It is evident from this identification that one obtains the following reduction of structure

$$\mathcal{R}_{\mathrm{HT},L,k} := \Delta_L^{\mathrm{an}} \times \mathcal{R}_{k,L} \hookrightarrow \Delta_L^{\mathrm{an}} \times R_L^{\mathrm{an}} = \mathcal{R}^{\mathrm{an}}_{\mathrm{HT},L}$$

for any $k \geq 1$. We can (and do) choose the set of representatives $\{s_1, \dots, s_r\}$ such that $s_i \in \boldsymbol{R}(\mathbb{A}_f^p)$.[9] Then we associate with $\bar{\chi}$ the global section

$$R_{\mathrm{dR}}(\mathbb{C}) \to \mathbb{C}$$

[8] One should think of such a choice of representatives as a choice of canonical model for $\Delta(\mathbb{C})$. Of course, canonical models are unique up to unique isomorphism, but for this identification of torsors, it is helpful to fix such a choice.

[9] This is possible because a finite Galois extension can be generated by Frobeniuses outside any finite set of primes.

given by sending $[x, y] \mapsto \xi^{[j]}(x)\bar{\chi}(y)$, where $x \in \boldsymbol{R}(\mathbb{C})$ and $y \in \boldsymbol{R}(\mathbb{A}_f)$, and

$$\xi^{[j]} \colon \boldsymbol{R}(\mathbb{C}) \cong \mathbb{C}^\times \times \prod_{\tau \in \Psi} (\mathbb{C}^\times \times \mathbb{C}^\times) \to \mathbb{C}^\times,$$
$$(x_0, x_{1,\tau}, x_{2,\tau})_{\tau \in \Psi} \mapsto \prod_{\tau \in \Psi} \left(\frac{x_{2,\tau}}{x_{1,\tau}} \right)^{j_\tau}.$$

This global section is well-defined precisely because χ is an algebraic Hecke character of infinity-type $(j, -j)$, and transforms under the action of $\boldsymbol{R}(\mathbb{C})$ by the character $\xi^{[j]}$, so descends to a cohomology class

$$[\chi]_B \in \mathrm{H}^0(\Delta(\mathbb{C}), [\xi^{[j]}]).$$

Via the identification in (7.2.1), the class $[\chi]_B$ coincides with the product of the global section of $\Delta(\mathbb{C})$ taking s_i to $\bar{\chi}(s_i)$, and the global section $\boldsymbol{R}(\mathbb{C}) \xrightarrow{\xi^{[j]}} \mathbb{C}^\times \subset \mathbb{C}$. Since $\bar{\chi}(s_i)$ are elements of some number field, we can find a large enough Φ such that $[\chi]_B$ descends to a global section in $\mathrm{H}^0(\Delta_\Phi, [\xi^{[j]}])$. Via the rigid GAGA comparison, we therefore obtain a global section $[\chi]_{\mathrm{HT}} \in \mathrm{H}^0(\Delta_L^{\mathrm{an}}, [\xi^{[j]}])$ characterized by the global section

$$\Delta_L^{\mathrm{an}} \times R_L^{\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}},$$
$$([s_i], t) \mapsto \bar{\chi}(s_i)\xi^{[j],\mathrm{an}}(t),$$

where we are viewing $\bar{\chi}(s_i)$ as an element of $L^\times$ via the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$.

Lemma 7.2.4. *For any integer $k \geq 1$, the global section $[\chi]_{\mathrm{HT}}$ is described by the morphism*

$$\Delta_L^{\mathrm{an}} \times \mathcal{R}_{k,L} \to \mathbb{A}^{1,\mathrm{an}},$$
$$([s_i], (x_0, x_{1,\tau}, x_{2,\tau})_{\tau \in \Psi}) \mapsto \bar{\chi}(s_i) \prod_{\tau \in \Psi} \left(\frac{x_{2,\tau}}{x_{1,\tau}} \right)^{j_\tau},$$

which is valued in $\mathcal{O}_L^\times(1 + \mathcal{B}_k)$.

Proof. This follows immediately from the fact that $\bar{\chi}(s_i) \in \mathcal{O}_L^\times$ (under the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$). Indeed, because the representatives s_i have been chosen to have no component at p, $\bar{\chi}(s_i)$ is in the image of the (continuous) Galois character $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to L^\times$ associated with χ_p (via class field theory). But Galois groups are compact, so this is valued in $\mathcal{O}_L^\times$. □

The description in Lemma 7.2.4 allows us to raise this cohomology class to p-adic powers, in the following way. Let (A, A^+) be a Tate algebra over $(L, \mathcal{O}_L)$ and let $\beta \colon \mathcal{O}_L^\times \to (A^+)^\times$ be a k-analytic character, i.e., it extends to a pairing

$$\mathcal{O}_L^\times(1 + \mathcal{B}_k) \times_{\mathrm{Spa}(L, \mathcal{O}_L)} \mathrm{Spa}(A, A^+) \to \mathbb{G}_m^{\mathrm{an}}.$$

Then via the torsor $\mathcal{R}_{\mathrm{HT},L,k}$, one obtains an A-Banach sheaf $[\beta \circ \xi^{[j]}]$ and a cohomology class

$$[\chi]_{\mathrm{HT}}^\beta \in \mathrm{H}^0(\Delta_L^{\mathrm{an}}, [\beta \circ \xi^{[j]}])$$

described by the morphism

$$\Delta_L^{\mathrm{an}} \times \mathcal{R}_{k,L} \to \mathbb{A}^{1,\mathrm{an}} \times \mathrm{Spa}(A, A^+),$$
$$([s_i], (x_0, x_{1,\tau}, x_{2,\tau})_{\tau\in\Psi}) \mapsto \beta\left(\bar{\chi}(s_i) \prod_{\tau\in\Psi} \left(\frac{x_{2,\tau}}{x_{1,\tau}}\right)^{j_\tau}\right),$$

which is well-defined by Lemma 7.2.4. This description is independent of the radius of analyticity k.

Remark 7.2.5. If we take $(A, A^+) = (L, \mathcal{O}_L)$ and $\beta(-) = (-)^k$ for some integer k, then $[\chi]_{\mathrm{HT}}^{\beta}$ is equal to the k-fold cup product of $[\chi]_{\mathrm{HT}}$ (which makes sense for negative integers because $[\chi]_{\mathrm{HT}}$ is an invertible section). In particular, under the rigid GAGA comparison (and the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$)

$$\mathrm{H}^0(\Delta_L^{\mathrm{an}}, [\beta \circ \xi^{[j]}]) = \mathrm{H}^0(\Delta_{\mathbb{Q}_p}^{\mathrm{an}}, [\beta \circ \xi^{[j]}]) \otimes_{\mathbb{Q}_p} L \hookrightarrow \mathrm{H}^0(\Delta(\mathbb{C}), [\beta \circ \xi^{[j]}])$$

the class $[\chi]_{\mathrm{HT}}^{\beta}$ is mapped to $[\chi^k]_B$.

7.3. ***Step 2: Pullback to the Shimura variety for $\boldsymbol{H}$.*** Recall that we have a morphism $(\boldsymbol{H}, X_{\boldsymbol{H}}) \to (\boldsymbol{R}, X_{\boldsymbol{R}})$ of Shimura data induced from the homomorphism $\det\colon \boldsymbol{H} \twoheadrightarrow \boldsymbol{R}$. Let $U = U^p K_{\diamond}^{H}(p)$ and let $C = \nu(U)$. By shrinking U^p is necessary, we may assume that C is neat. We therefore obtain a morphism

$$S_{\boldsymbol{H},\diamond}(p) \to S_{\boldsymbol{R},C} := \Delta$$

which we will also denote by det. The fibers of this morphism (after base-changing to a sufficiently large field extension) are disjoint unions of connected components of $S_{\boldsymbol{H},\diamond}(p)$.

Let $H_{\mathrm{dR}} \to S_{\boldsymbol{H},\diamond}(p)$ denote the standard principle $\boldsymbol{H}_{F^{\mathrm{cl}}}$-bundle as in [Milne 1990, Section III.3], which satisfies

$$H_{\mathrm{dR}}(\mathbb{C}) = \boldsymbol{H}(\mathbb{Q})\backslash X_{\boldsymbol{H}} \times \boldsymbol{H}(\mathbb{C}) \times \boldsymbol{H}(\mathbb{A}_f)/K.$$

One has a natural morphism $H_{\mathrm{dR}}(\mathbb{C}) \to R_{\mathrm{dR}}(\mathbb{C})$ induced from the morphism det and, as explained in Section III.4 of [loc. cit.], this descends to a morphism on the canonical models of these standard principle bundles;[10] i.e., we obtain a morphism (of principle bundles) $H_{\mathrm{dR}} \to R_{\mathrm{dR}}$. One can check on complex points that this induces an isomorphism $H_{\mathrm{dR}} \times^{\boldsymbol{H}_{F^{\mathrm{cl}}}} \boldsymbol{R}_{F^{\mathrm{cl}}} \cong \det^* R_{\mathrm{dR}}$, where the pushout is via the morphism $\det\colon \boldsymbol{H}_{F^{\mathrm{cl}}} \to \boldsymbol{R}_{F^{\mathrm{cl}}}$.

On the other hand, the bundle H_{dR} can be expressed as the pushout $P_{H,\mathrm{dR}} \times^{\boldsymbol{P}_H} \boldsymbol{H}_{F^{\mathrm{cl}}}$, and since the morphism $\nu\colon \boldsymbol{P}_H \to \boldsymbol{R}_{F^{\mathrm{cl}}}$ factors through the projection $\boldsymbol{P}_H \twoheadrightarrow \boldsymbol{M}_H$, one obtains an isomorphism

$$M_{H,\mathrm{dR}} \times^{\boldsymbol{M}_H} \boldsymbol{R}_{F^{\mathrm{cl}}} \cong H_{\mathrm{dR}} \times^{\boldsymbol{H}_{F^{\mathrm{cl}}}} \boldsymbol{R}_{F^{\mathrm{cl}}} \cong \det^* R_{\mathrm{dR}}.$$

Passing to the associated adic spaces and using the de Rham–p-adic comparison, one obtains an isomorphism (of R^{an}-torsors)

$$^{\mu}\mathcal{M}_{H,\mathrm{HT}}^{\mathrm{an}} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}} \cong \det^*(^{\mu}\mathcal{R}_{\mathrm{HT}}^{\mathrm{an}}). \tag{7.3.1}$$

[10] Since $(\boldsymbol{H}, X_{\boldsymbol{H}})$ does not satisfy axiom (SD3) in [Graham and Shah 2023, Definition B.16], one has to use the additional property that this Shimura–Deligne datum embeds into a Siegel datum to ensure the existence of a canonical model for H_{dR}.

It will be helpful to reinterpret this isomorphism in terms of flag varieties. We have a commutative diagram:

$$\begin{tikzcd} \mathcal{S}_{H,U^p} \arrow[r, "\pi_{H,\mathrm{HT}}"] \arrow[d] & \mathtt{FL}^H \arrow[d] \\ \mathcal{S}_{R,C^p} \arrow[r, "\pi_{R,\mathrm{HT}}"] & \mathtt{FL}^R \end{tikzcd}$$

where the vertical arrows are induced from the homomorphism det.

Let $\mathtt{R}^{\mathrm{an}}$ and $\mathtt{M}^{H,\mathrm{an}}$ denote the torsors $R^{\mathrm{an}} \to \mathtt{FL}^R$ and $H^{\mathrm{an}}/N_H^{\mathrm{an}} \to \mathtt{FL}^H$ respectively (where both structural maps are given by $x \mapsto x^{-1}$ to ensure that they are right torsors). Note that the torsor $\mathtt{R}^{\mathrm{an}}$ is trivial, so $\det^* \mathtt{R}^{\mathrm{an}}$ is identified with $\mathtt{FL}^G \times R^{\mathrm{an}}$ and we have a canonical isomorphism

$$\mathtt{M}^{H,\mathrm{an}} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}} \cong \det^* \mathtt{R}^{\mathrm{an}}.$$

Since pull-back commutes with colimits (so in particular pushouts) and this is compatible with the $K_\diamondsuit^H(p)$-equivariant structure, this induces an isomorphism

$$\mathcal{M}_{H,\mathrm{HT}}^{\mathrm{an}} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}} = \pi_{H,\mathrm{HT}}^*(\mathtt{M}^{H,\mathrm{an}} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}})/K_\diamondsuit^H(p) \cong \pi_{H,\mathrm{HT}}^*(\det^* \mathtt{R}^{\mathrm{an}})/K_\diamondsuit^H(p) = \det^* \mathcal{R}_{\mathrm{HT}}^{\mathrm{an}}.$$

We can twist this isomorphism along $\mu\colon \mathbb{Z}_p^\times \to M_H^{\mathrm{an}} \xrightarrow{\det} R^{\mathrm{an}}$ (induced from $\mu_{\boldsymbol{R}} = \det \circ \mu_{\boldsymbol{H}}$) to obtain an isomorphism

$$^{\mu}\mathcal{M}_{H,\mathrm{HT}}^{\mathrm{an}} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}} \cong \det^*({}^{\mu}\mathcal{R}_{\mathrm{HT}}^{\mathrm{an}}). \tag{7.3.2}$$

Proposition 7.3.3. *The isomorphisms (7.3.1) and (7.3.2) coincide.*

Proof. With notation as in Appendix B, the isomorphism (7.3.1) (resp. (7.3.2)) is induced from the natural transformation η_{dR} (resp. $\eta_{\text{ét}}$). The result now follows from Corollary B.2.4. □

We obtain the following corollary:

Corollary 7.3.4. *Over $\mathcal{U}_k^H(p)_L$ one has a commutative diagram*:

$$\begin{tikzcd} ^{\mu}\mathcal{M}_{H,\mathrm{HT},L}^{\mathrm{an}} \times^{M_{H,L}^{\mathrm{an}}} R_L^{\mathrm{an}} \arrow[r, "(7.3.1)"] & \det^*({}^{\mu}\mathcal{R}_{\mathrm{HT},L}^{\mathrm{an}}) \\ ^{\mu}\mathcal{M}_{H,\mathrm{HT},k,1,L} \times^{\mathcal{M}_{H,k,1,L}^{\clubsuit}} \mathcal{R}_{k,L} \arrow[r, "\sim"] \arrow[u] & \det^*({}^{\mu}\mathcal{R}_{\mathrm{HT},k,L}) \arrow[u] \end{tikzcd}$$

for any finite extension $L/\mathbb{Q}_p$, where the left-hand map is induced from the reduction of structure in Section 5.1.

Proof. To simplify notation, we will establish the case $L = \mathbb{Q}_p$ only, as the general case follows the exact same argument.

Note that the left-hand vertical map is induced from the morphism $\mathcal{M}_{H,\mathrm{HT},k,k,1} \to \mathcal{M}_{H,\mathrm{HT}}^{\mathrm{an}}$ and pushing out along $\mathcal{M}_{H,k,k,1}^{\clubsuit} \to \mathcal{R}_k$ factors through the affinoid group $\mathcal{M}_{H,k,1}^{\clubsuit}$, so the left hand vertical map does indeed make sense.

Using the fact that the morphism (7.3.1) coincides with (7.3.2) and untwisting along $\mu\colon \mathbb{Z}_p^\times \to \mathcal{M}^{\clubsuit}_{H,k,1} \xrightarrow{\det} \mathcal{R}_k$, we can work on the level of flag varieties. In this setting we have a commutative diagram (because the morphism $\mathcal{M}^{\diamondsuit}_{H,k,k,1} \to \mathcal{R}_k$ extends to a morphism $\mathtt{M}^H_{k,k,1} \to \mathcal{R}_k$):

$$\begin{tikzcd} \mathtt{M}^{H,\mathrm{an}}|_{\mathtt{U}^H_k} \times^{M_H^{\mathrm{an}}} R^{\mathrm{an}} \arrow[r, "\sim"] & \mathtt{U}^H_k \times R^{\mathrm{an}} \\ \mathtt{M}^H_{k,k,1} \times^{\mathcal{M}^{\diamondsuit}_{H,k,k,1}} \mathcal{R}_k \arrow[r, "\sim"] \arrow[u] & \mathtt{U}^H_k \times \mathcal{R}_k \arrow[u] \end{tikzcd}$$

which gives the desired result. □

7.4. *Step 3: Construction of the family.* Fix a collection $\{\chi_\tau : \tau \in \Psi\} \subset \Sigma(\mathfrak{N})$ of anticyclotomic characters, where χ_τ has infinity type $(1_\tau, -1_\tau)$ and let $\chi_0 \in \Sigma(\mathfrak{N})$ be a fixed anticyclotomic Dirichlet character. Let $L'/\mathbb{Q}_p$ be a sufficiently large finite extension containing the fields of definition of $\chi_{\tau,p}$, and let L/L' be finite extension containing the field of definition of $\chi_{0,p}$.

Theorem 7.4.1. *Let (A, A^+) be a Tate algebra over $(L, \mathcal{O}_L)$ and let $(\beta_\tau)_{\tau\in\Psi}$ be a collection of locally analytic characters $\mathcal{O}_{L'}^\times \to (A^+)^\times$. Let $\xi^{[\beta]}\colon \mathcal{R}_{k,L'} \to \mathbb{G}_m^{\mathrm{an}}$ denote the character given by sending $(x_0, x_{1,\tau}, x_{2,\tau})_{\tau\in\Psi}$ to $\prod_\tau \beta_\tau(x_{2,\tau}/x_{1,\tau})$, for any sufficiently large k. Then there exists a class*

$$\underline{\chi} \in \mathrm{H}^0_{\mathrm{id},\mathrm{an}}(\xi^{[\beta]} \circ \det)^{(+,\dagger)} := \varinjlim_m \mathrm{H}^0(\mathcal{Z}^H_m(p), [\xi^{[\beta]} \circ \det])$$

such that:

(1) *If $(A, A^+) = (L, \mathcal{O}_L)$ and β_τ are integers, then $\underline{\chi}$ extends to a class in $\mathrm{H}^0(\mathcal{S}_{H,\diamond}(p)_L, [\xi^{[\beta]} \circ \det])$ whose image under the map (induced from rigid GAGA and the identification $\mathbb{C} \cong \overline{\mathbb{Q}}_p$)*

$$\mathrm{H}^0(\mathcal{S}_{H,\diamond}(p)_L, [\xi^{[\beta]} \circ \det]) = \mathrm{H}^0(\mathcal{S}_{H,\diamond}(p), [\xi^{[\beta]} \circ \det]) \otimes_{\mathbb{Q}_p} L \hookrightarrow \mathrm{H}^0(\mathcal{S}_{H,\diamond}(p)(\mathbb{C}), [\xi^{[\beta]} \circ \det])$$

is equal to $\det^*([\chi_0]_B \cdot \prod_{\tau\in\Psi}[\chi_\tau^{\beta_\tau}]_B)$. In other words, for classical weights this family specializes to the cohomology class representing the automorphic form:

$$\boldsymbol{H}(\mathbb{Q})\backslash \boldsymbol{H}(\mathbb{A}) \to \mathbb{C},$$
$$(h_1, h_2) \mapsto \bar{\chi}_0(\det(h_1, h_2)) \cdot \prod_{\tau\in\Psi} \bar{\chi}_\tau(\det(h_1, h_2))^{\beta_\tau}.$$

(2) *For varying (A, A^+), the constructions of $\underline{\chi}$ are compatible.*

Proof. Recall the definitions of $[\chi_0]_{\mathrm{HT}}$ and $[\chi_\tau]^{\beta_\tau}_{\mathrm{HT}}$ from Section 7.2 (where we view $[\chi_\tau]^{\beta_\tau}_{\mathrm{HT}}$ as a class defined over L). We define

$$\underline{\chi} = \det{}^*[\chi_0]_{\mathrm{HT}} \cdot \prod_{\tau\in\Psi} \det{}^*[\chi_\tau]^{\beta_\tau}_{\mathrm{HT}}.$$

The interpolation property follows from Corollary 7.3.4 and Remark 7.2.5, and it is clear from the definition of $[\cdots]_{\mathrm{HT}}$ that this construction is compatible under base-change. □

8. Construction of the p-adic L-function

In this final section, we construct the p-adic L-function associated with a family of cohomology classes $\underline{\eta}$ and a family $\underline{\chi}$ of anticyclotomic characters. We will end by discussing its relation to unitary Friedberg–Jacquet periods.

8.1. *Definition of the p-adic L-function.* Let π be a cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ satisfying Assumptions 6.1.1, 6.1.4 and 6.2.1. Then the construction in Section 6.2 implies that there exists a unique family $\theta_{\underline{\pi}}$ and family of cohomology classes $\underline{\eta} \in S^{n-1}(\underline{\pi})$ passing through π, defined over a sufficiently small affinoid $U = \mathrm{Spa}(A, A^+) \subset \mathcal{W}_{G,L}$.

For the family of anticyclotomic characters, we make the following assumption:

Assumption 8.1.1. *The class number of F is not divisible by p.*

By this assumption, for every $\tau \in \Psi$, we can fix an anticyclotomic character $\chi_\tau \in \Sigma(\mathfrak{N})$ of infinity type $(1_\tau, -1_\tau)$, such that associated p-adic Hecke character is valued in $\mathbb{Q}_p^\times$; see the discussion in [Collins 2020, Section 4.2], for example. Fix an anticyclotomic Dirichlet character $\chi_0 \in \Sigma(\mathfrak{N})$, and let $V = \mathrm{Spa}(B, B^+) \subset \mathcal{W}_{H,L}$ be an open affinoid subspace with universal character $\lambda_B \colon (\mathbb{Z}_p^\times)^{[F^+:\mathbb{Q}]-1} \to (B^+)^\times$. We can naturally view λ_A and $\beta := \lambda_B$ as characters valued in $A \hat{\otimes} B$. Then the results in Section 7 imply that there exists a family $\underline{\chi} \in \mathrm{H}^0_{\mathrm{id,an}}(\sigma_n^{[\beta]}(\lambda_A)^\vee)^{(+,\dagger)}$ which interpolates (the coherent cohomology classes associated with) the anticyclotomic characters

$$\chi_{(\lambda,j)} := \chi_0 \cdot \chi_{\tau_0}^{-(c_{n,\tau_0}+1)} \cdot \prod_{\tau \neq \tau_0} \chi_\tau^{-j_\tau}$$

where $(\lambda, j) \in X^*(T/T_0)^+ \times X^*(S)^+ \cap U \times V$ with $\lambda = (0; c_{1,\tau}, \dots, c_{2n,\tau})_{\tau \in \Psi}$ and $j = (j_\tau)_{\tau \neq \tau_0}$ satisfying $0 \leq j_\tau \leq c_{n,\tau}$.

Definition 8.1.2. With the set-up as above, we define

$$\mathscr{L}_p(\underline{\eta}, \underline{\chi}) := \langle\langle \underline{\eta}, \underline{\chi} \rangle\rangle^-_{\mathrm{an}} \quad \in \mathcal{O}(U \times V)$$

where the right-hand side is as in Section 5.4.

Remark 8.1.3. Since the pairing $\langle\langle \cdot, \cdot \rangle\rangle^-_{\mathrm{an}}$ is compatible with change of coefficients, the p-adic analytic functions $\mathscr{L}_p(\underline{\eta}, \underline{\chi})$ glue as V varies. Therefore, we can (and do) view

$$\mathscr{L}_p(\underline{\eta}, \underline{\chi}) \in \mathcal{O}(U \times \mathcal{W}_{H,L})$$

which makes sense because the families $\underline{\chi}$ glue for varying V, by Theorem 7.4.1(2) (note that we can choose an open affinoid cover of $\mathcal{W}_H$ such that the universal characters for each open are locally analytic — see [Loeffler and Zerbes 2016, Lemma 4.1.5]).

8.2. ***The interpolation property.*** Keeping with the same set-up as in the previous section, we introduce the following "region of interpolation":

Definition 8.2.1. Let Σ^{int} denote the subset of $X^*(T/T_0)^+ \times X^*(S)^+ \cap (U \times \mathcal{W}_H)(L)$ of all pairs (λ, j) with $\lambda = (0; c_{1,\tau}, \dots, c_{2n,\tau})_{\tau \in \Psi}$ and $j = (j_\tau)_{\tau \neq \tau_0}$ satisfying $0 \leq j_\tau \leq c_{n,\tau}$.

For $(\lambda, j) \in \Sigma^{\text{int}}$ let

$$\eta_\lambda \in \mathrm{H}^{n-1}_{w_n,\text{an}}(\kappa_n(\lambda))^{-,\text{ss}} \cong \mathrm{H}^{n-1}(\mathcal{S}_{G,\text{Iw}}(p), [V_{\kappa_n(\lambda)}])^{-,\text{ss}}$$

denote the specialization of $\underline{\eta}$ at (λ, j), which we can view as an element of $\mathrm{H}^{n-1}(S_{\boldsymbol{G},\text{Iw}}(p)(\mathbb{C}), [V_{\kappa_n(\lambda)}])$ via rigid GAGA and the identification $\iota_p \colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $\mathscr{L}_p(\eta_\lambda, \chi_{(\lambda,j)})$ denote the specialization of $\mathscr{L}_p(\underline{\eta}, \underline{\chi})$ under the map $\mathcal{O}(U \times \mathcal{W}_H) \to L$ induced from (λ, j).

We obtain the following interpolation property for $\mathscr{L}_p(\underline{\eta}, \underline{\chi})$.

Proposition 8.2.2. *After possibly shrinking U around λ_π, for any $(\lambda, j) \in \Sigma^{\text{int}}$ one has*

$$\iota_p^{-1} \mathscr{L}_p(\eta_\lambda, \chi_{(\lambda,j)}) = \langle \eta_\lambda, \nu^*[\chi_{(\lambda,j)}]_B \rangle_{\text{alg}}$$

where $\iota_p \colon \mathbb{C} \cong \overline{\mathbb{Q}}_p$ denotes the fixed isomorphism, and the pairing in the right-hand side has been base-changed to $\mathbb{C}$ (via the embedding $F^{\text{cl}} \hookrightarrow \mathbb{C}$).

Proof. If we let

$$\nu^*[\chi_{(\lambda,j)}]_{\text{HT}} \in \mathrm{H}^0_{\text{id},\text{an}}(\sigma^{[j]}(\lambda)^\vee)^{(+,\dagger)} = \mathrm{H}^0_{\text{id}}(\sigma^{[j]}(\lambda)^\vee)^{(+,\dagger)}$$

denote the specialization of $\underline{\chi}$, then the results in Section 7 imply that $\nu^*[\chi_{(\lambda,j)}]_{\text{HT}}$ is in the image of the restriction map

$$\mathrm{H}^0(\mathcal{S}_{H,\diamond}(p), [\sigma_n^{[j]}(\lambda)]^\vee) \to \mathrm{H}^0_{\text{id}}(\sigma^{[j]}(\lambda)^\vee)^{(+,\dagger)}$$

and its image under the rigid GAGA comparison is equal to $\nu^*[\chi_{(\lambda,j)}]_B$. The result then follows from Corollary 5.4.4, Theorem 4.7.3 and Proposition 4.4.2. □

Remark 8.2.3. The equality in Proposition 8.2.2 depends on a choice of isomorphism $V^*_{\kappa_n(\lambda)^*} \cong V_{\kappa_n(\lambda)}$ over F^{cl}.

Let $[\boldsymbol{H}] = \boldsymbol{H}(\mathbb{Q}) A_{\boldsymbol{G},\boldsymbol{H}}(\mathbb{A}) \backslash \boldsymbol{H}(\mathbb{A})$, where $A_{\boldsymbol{G}}$ denotes the maximal split subtorus of the center of $\boldsymbol{G}$ and $A_{\boldsymbol{G},\boldsymbol{H}} = A_{\boldsymbol{G}} \cap \boldsymbol{H}$ (which in fact equals $A_{\boldsymbol{G}}$). By choosing a Haar measure for $\boldsymbol{H}(\mathbb{Q}) A_{\boldsymbol{G},\boldsymbol{H}}(\mathbb{A})$ and using a fixed Haar measure for $\boldsymbol{H}(\mathbb{A})$, one obtains a measure on the quotient $[\boldsymbol{H}]$ which we will denote by $\bar{d}h$. We also let $[\boldsymbol{H}]' = \boldsymbol{H}(\mathbb{Q}) A_{\boldsymbol{G},\boldsymbol{H}}(\mathbb{R})^\circ \backslash \boldsymbol{H}(\mathbb{A})$ and, similar to above, we have an induced measure $\bar{d}'h$. We choose these measures so they are compatible under the quotient map $[\boldsymbol{H}]' \to [\boldsymbol{H}]$. We also assume that the volume of $U_\infty^\circ U$ with respect to the Haar measure on $\boldsymbol{H}(\mathbb{A})$ is contained in $(F^{\text{cl}})^\times$, where U_∞° is the maximal compact subgroup of $U_\infty = K_\infty \cap \boldsymbol{H}(\mathbb{R})$.

Corollary 8.2.4. *Let $(\lambda, j) \in \Sigma^{\mathrm{int}}$ and σ be the cuspidal automorphic representation of $\boldsymbol{G}(\mathbb{A})$ associated with η_λ (see Section 6.2). Then there exists $G \in \sigma$ such that*

$$\iota_p^{-1}\mathscr{L}_p(\eta_\lambda, \chi_{(\lambda,j)}) \sim_{F^{\mathrm{cl},\times}} (2\pi i)^{-(n-1)} \int_{[\boldsymbol{H}]'} G(h)\cdot\chi_{(\lambda,j)}(\nu(h))\,\bar{d}'h \tag{8.2.5}$$

where $\sim_{F^{\mathrm{cl},\times}}$ means up to a nonzero constant in $F^{\mathrm{cl},\times}$ which only depends on λ and the choice of Haar measures as above.

Furthermore, if the central character of π restricted to $A_{\boldsymbol{G},\boldsymbol{H}}(\mathbb{A})$ is trivial, then we have the relation

$$\iota_p^{-1}\mathscr{L}_p(\eta_\lambda, \chi_{(\lambda,j)}) \sim_{F^{\mathrm{cl},\times}} (2\pi i)^{-(n-1)} \int_{[\boldsymbol{H}]} G(h)\cdot\chi_{(\lambda,j)}(\nu(h))\,\bar{d}h$$

after possibly shrinking U around λ_π.

Proof. By Proposition 8.2.2, it is equivalent to showing that $\langle \eta_\lambda, \nu^*[\chi_{(\lambda,j)}]_B\rangle_{\mathrm{alg}}$ equals the right-hand side of (8.2.5). We will freely use the notation from the proof of Corollary 6.2.3. We first note that we have an morphism

$$\mathrm{Hom}_{K_\infty}(\nu_{n-1}, \sigma_\infty) \to \mathrm{Hom}_{K_\infty}\left(\bigwedge^{n-1}(\mathfrak{p}/\mathfrak{m}), \sigma_\infty \otimes V_{\kappa_n(\lambda)}\right)$$

where notation is as in Section 2.3, given by precomposing with the map of $M_{\boldsymbol{G}}$-representations

$$\bigwedge^{n-1}(\mathfrak{p}/\mathfrak{m}) \otimes V_{\kappa_n(\lambda)^*} \to \nu_{n-1} \tag{8.2.6}$$

(which is uniquely determined up to $\mathbb{C}^\times$) and using a fixed isomorphism

$$V^*_{\kappa_n(\lambda)^*} \cong V_{\kappa_n(\lambda)}. \tag{8.2.7}$$

This induces an isomorphism $\mathrm{Hom}_{K_\infty}(\nu_{n-1}, \sigma_\infty) \cong \mathrm{H}^{n-1}_{(\mathfrak{p},K_\infty)}(\sigma_\infty \otimes V_{\kappa_n(\lambda)})$, and hence we obtain an injective map

$$\mathrm{Hom}_{K_\infty}(\nu_{n-1}, \sigma_\infty) \otimes \sigma_f^{K^p K^G_{\mathrm{Iw}}(p)} \hookrightarrow \mathrm{H}^{n-1}(S_{\boldsymbol{G},\mathrm{Iw}}(p)(\mathbb{C}), [V_{\kappa_n(\lambda)}])$$

whose image is identified with the localization of the right-hand side at the kernel of the specialization of $\theta_{\underline{\pi}}$ at λ.

The representation $\bigwedge^{n-1}(\mathfrak{p}/\mathfrak{m})$ is definable over F^{cl} so we choose the map (8.2.6) to be defined over F^{cl}. We also choose the same isomorphism (8.2.7) as in Proposition 8.2.2, which is defined over F^{cl}. Recall from Proposition 2.6.1 that we have a (unique up to scaling) vector $v^{[j]}_{\kappa_n(\lambda)} \in V_{\kappa_n(\lambda)^*}$ on which $M_{\boldsymbol{H}}$ acts through the character $\sigma_n^{[j]}(\lambda)^{-1}$. Let z be the image of $w \otimes v^{[j]}_{\kappa_n(\lambda)}$ under the map (8.2.6), where w is a choice of highest weight vector of $\bigwedge^{n-1}(\mathfrak{p}/\mathfrak{m})$ defined over F^{cl}. This vector z is nonzero because $\sigma_n^{[j]}(\lambda)^\vee$ appears as a direct factor with multiplicity one in both the codomain and domain of (8.2.6).

Via the above injective map, the class η_λ corresponds to a homomorphism $G_{\eta_\lambda} \otimes \varphi_f$, where $\varphi_f \in \sigma^{K^p K^G_{\mathrm{Iw}}(p)}$. We take G to be $G := \hat{\gamma}\cdot(G_{\eta_\lambda}(z) \otimes \varphi_f) \in \sigma$ (where $\hat{\gamma}$ is viewed as an element of $\boldsymbol{G}(\mathbb{Q}_p) \subset \boldsymbol{G}(\mathbb{A})$).

If we let $\mathfrak{p}_H$ (resp. $\mathfrak{m}_H$) denote the Lie algebra of the opposite of $P_{\boldsymbol{H}}$ (resp. $M_{\boldsymbol{H}}$), then $\bigwedge^{n-1}\mathfrak{p}_H/\mathfrak{m}_H$ is identified with the line spanned by the vector w. By [Su 2019], we have an isomorphism

$$\mathrm{H}^{n-1}(S_{\boldsymbol{H},\diamond}(p)(\mathbb{C}), [\sigma_n^{[j]}(\lambda)]) \cong \mathrm{H}^{n-1}_{(\mathfrak{p}_H,U_\infty)}(C^\infty([\boldsymbol{H}]'/U)^{U_\infty-\mathrm{fin}} \otimes \sigma_n^{[j]}(\lambda))$$

where $U_\infty = K_\infty \cap \boldsymbol{H}(\mathbb{R})$ and $U \subset \boldsymbol{H}(\mathbb{A}_f)$ is the level of the Shimura variety $S_{\boldsymbol{H},\diamond}(p)$. Under this identification, the class $\hat{\imath}^*\eta_\lambda$ is represented by the homomorphism

$$\begin{aligned}\bigwedge^{n-1}\mathfrak{p}_H/\mathfrak{m}_H &\to C^\infty([\boldsymbol{H}]'/U)^{U_\infty-\mathrm{fin}} \otimes \sigma_n^{[j]}(\lambda),\\ w &\mapsto G|_{\boldsymbol{H}}.\end{aligned}$$

The result now follows from [Harris 1990, Proposition 3.8].

For the last part, note that the central character of π restricted to $A_{\boldsymbol{G}}(\mathbb{A})$ is necessarily a Dirichlet character (because π contributes to the coherent cohomology of $S_{\boldsymbol{G},\mathrm{Iw}}(p)$ and the center acts trivially on V_{κ_n}) and is therefore determined by the image of Hecke operators $[K^S a K^S]$ under the map θ_π, for $a \in A_{\boldsymbol{G}}(\mathbb{A}_f^S)$. The image of these operators under θ_π form a discrete subgroup, so we can shrink U if necessary so that the images of these operators under $\theta_{\underline{\pi}}$ are constant (note that one normally normalizes the Hecke operators by the weight, but because our weights are trivial on T_0, this normalization is trivial). Therefore our assumption implies that the central character of σ is trivial on $A_{\boldsymbol{G}}(\mathbb{A})$, so we can descend to $[\boldsymbol{H}]$. □

Remark 8.2.8. If we define $[\boldsymbol{H}_0] = \boldsymbol{H}_0(\mathbb{Q})\backslash\boldsymbol{H}_0(\mathbb{A})$, then $[\boldsymbol{H}]$ is the disjoint union of finitely many translates of $[\boldsymbol{H}_0]$. Therefore the integral (over $[\boldsymbol{H}]$) in Corollary 8.2.4 is nonzero if and only if

$$\int_{[\boldsymbol{H}_0]} G(h)\cdot\chi_{(\lambda,j)}(\nu(h))\,\bar{d}h$$

is nonzero. This latter integral is a so-called *unitary Friedberg–Jacquet period.*

Appendix A: Branching laws

The goal of this appendix is to prove Theorem 5.3.4. The idea is to p-adically interpolate the branching law appearing in Proposition 2.6.1. Since the groups M_G and M_H are products of general linear groups indexed by the CM type Ψ (and an additional "similitude factor"), it will be more convenient to analyze the branching law for each factor.

Unfortunately this means that we will have to use conflicting notation when performing this case-by-case analysis; therefore, we warn the reader that the notation in Sections A.1–A.4 is different from the rest of the article. We have however endeavored to keep the notation uniform throughout these four subsections (e.g., the element u and torus $T^\diamond$ play the same role in the analysis, but change for each group). We hope that this change doesn't cause any confusion.

A.1. ***A preliminary lemma.*** For a split unramified reductive group G over $\mathbb{Z}_p$, let $B_G \subset G$ denote a Borel subgroup and $\bar{B}_G$ its opposite with respect to a fixed maximal torus $T \subset B_G$. Let $U_G \subset B_G$ and $\bar{U}_G \subset \bar{B}_G$ denote the unipotent radicals.

Let $\mathcal{G}$ denote the adic generic fiber of the completion of G along its special fiber, and let G^{an} denote the analytification of $G_{\mathbb{Q}_p}$ (so we have $\mathcal{G} \subset G^{\mathrm{an}}$). We use similar notation for U_G, B_G, etc. For an integer $r \geq 1$, we let $\mathcal{G}_r^1$ denote the subgroup of $\mathcal{G}$ of elements which reduce to the identity modulo p^r. Similarly, for $\mathcal{H} = \mathcal{U}_G, \bar{\mathcal{U}}_G, \mathcal{B}_G, \bar{\mathcal{B}}_G$, let $\mathcal{H}_r^1$ denote the elements in $\mathcal{H}$ which reduce to the identity modulo p^r.

Recall the notation $\mathcal{B}_r^\circ \subset \bar{\mathcal{B}}_r^\circ \subset \mathcal{B}_r \subset \bar{\mathcal{B}}_r$ for the four different "flavors of disc" in Section 3.2.

Lemma A.1.1. *Let $d, r \geq 1$ and Y a $(d \times d)$-matrix with entries in $\mathcal{B}_r^\circ$. Let ξ denote the antidiagonal $(d \times d)$-matrix with 1s along the antidiagonal. Then there exist elements $R \in \mathcal{U}^1_{\mathrm{GL}_d,r}$ and $S \in \mathcal{B}^1_{\mathrm{GL}_d,r}$ such that*

$$\xi + Y = R \cdot \xi \cdot S.$$

Proof. The element $1 + Y\xi^{-1}$ defines an element of the group $\mathcal{GL}^1_{d,r}$. One has an Iwahori decomposition

$$\mathcal{GL}^1_{d,r} = \mathcal{U}^1_{\mathrm{GL}_d,r} \cdot \bar{\mathcal{B}}^1_{\mathrm{GL}_d,r}$$

so there exist elements $R \in \mathcal{U}^1_{\mathrm{GL}_d,r}$ and $S' \in \bar{\mathcal{B}}^1_{\mathrm{GL}_d,r}$ such that $1 + Y\xi^{-1} = RS'$. We then take $S = \xi^{-1}S'\xi$. $\square$

A.2. ***The group*** $\mathbf{GL_{2n-1}}$**.** We first establish the following lemma:

Lemma A.2.1. *Let ξ be the $(n \times n-1)$-matrix whose first row is zero and the bottom $(n-1 \times n-1)$-matrix is the antidiagonal matrix with 1s along the antidiagonal. Let Y be any $(n \times n-1)$-matrix with entries in $\mathcal{B}_r^\circ$. Then there exists $R \in \mathcal{U}^1_{\mathrm{GL}_n,r}$ and $S \in \mathcal{B}^1_{\mathrm{GL}_{n-1},r}$ such that*

$$\xi + Y = R \cdot \xi \cdot S.$$

Proof. We denote the top row of Y by y and the bottom $(n-1 \times n-1)$-matrix by Y'. Let $R' \in \mathcal{U}^1_{\mathrm{GL}_n,r}$ and $S \in \mathcal{B}^1_{\mathrm{GL}_{n-1},r}$ be as in Lemma A.1.1 such that

$$\xi' + Y' = R' \cdot \xi' \cdot S$$

where ξ' is the $(n-1 \times n-1)$ antidiagonal matrix with nonzero entries equal to 1. Then we take

$$R = \begin{pmatrix} 1 & r \\ & R' \end{pmatrix} \in \mathcal{U}^1_{\mathrm{GL}_n,r}$$

where $r = yS^{-1}(\xi')^{-1}$. $\square$

Let $G = \mathrm{GL}_{2n-1}$ and $H = \mathrm{GL}_{n-1} \times \mathrm{GL}_n$ over $\mathbb{Z}_p$. We consider H as a subgroup of G via the block diagonal embedding (where the top left block is of size GL_{n-1}). Fix the standard Borel B_G and torus T in G. Elements of the torus T are given by tuples $(y_1, \dots, y_{2n-1})$ (corresponding to the entries of the diagonal matrix) and we let $T^\diamond \subset T$ denote the subtorus of elements satisfying $y_i = y_{2n-i}$ for all $i = 1, \dots, 2n-1$. For an integer $r \geq 1$, we set $\mathcal{G}_r^\square = \mathcal{G}_r^1 \cdot B_G(\mathbb{Z}_p)$ and $\mathcal{H}_r^\diamond = \mathcal{H}_r^1 \cdot T^\diamond(\mathbb{Z}_p)$.

Let $u \in G(\mathbb{Z}_p)$ denote the block matrix

$$u = \begin{pmatrix} 1 & \\ \xi & 1 \end{pmatrix}$$

where the top right block is of size $(n-1 \times n-1)$ and ξ is as in Lemma A.2.1.

Proposition A.2.2. *One has the following equality*

$$\mathcal{G}_r^{\square} = (u^{-1}\mathcal{H}_r^{\diamond} u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G).$$

Proof. By multiplying by elements of $(\mathcal{G}_r^1 \cap \mathcal{B}_G)B_G(\mathbb{Z}_p)$ on the right, we are reduced to proving the statement

$$\bar{\mathcal{U}}_{G,r}^1 \subset (u^{-1}\mathcal{H}_r^{\diamond} u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G)$$

because one has an Iwahori decomposition $\mathcal{G}_r^1 = \bar{\mathcal{U}}_{G,r}^1 \cdot (\mathcal{G}_r^1 \cap \mathcal{B}_G)$. Let $x \in \bar{\mathcal{U}}_{G,r}^1$ be a general element written as a block matrix

$$x = \begin{pmatrix} x_1 & \\ x_2 & x_3 \end{pmatrix}$$

where the top left (resp. bottom right) block has size $(n-1 \times n-1)$ (resp. $n \times n$). Then

$$h := \begin{pmatrix} x_1 & \\ & x_3 \end{pmatrix}$$

defines an element of $\mathcal{H}_r^1$. Let $\bar{N}$ denote the unipotent radical of the standard opposite parabolic of G with Levi H. Then we have

$$(u^{-1}h^{-1}u) \cdot x \in \bar{\mathcal{N}}_r^1$$

where $\bar{\mathcal{N}}_r^1$ denote the subgroup of $\bar{\mathcal{N}}$ of elements which reduce to the identity modulo p^r. Hence we are reduced to proving $\bar{\mathcal{N}}_r^1 \subset (u^{-1}\mathcal{H}_r^{\diamond} u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G)$. But if

$$\begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix} \in \bar{\mathcal{N}}_r^1$$

is a general element, then we have

$$\begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix} = u^{-1} \begin{pmatrix} S^{-1} & \\ & R \end{pmatrix} u \begin{pmatrix} S & \\ & R^{-1} \end{pmatrix}$$

where R, S are as in Lemma A.2.1. □

Remark A.2.3. The proof of Proposition A.2.2 in fact shows that $\mathcal{G}_r^{\square} = (u^{-1}\mathcal{H}_r^1 u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G)$.

A.3. ***The group*** $\mathbf{GL_1 \times GL_{2n-1}}$**.** We now let $G = \mathrm{GL}_1 \times \mathrm{GL}_{2n-1}$ and $H = \mathrm{GL}_1 \times \mathrm{GL}_{n-1} \times \mathrm{GL}_n$ embedded block diagonally. Define $\mathcal{G}_r^{\square}$ and $\mathcal{H}_r^{\diamond}$ analogously as in the previous section, where now $T^{\diamond}$ is the subtorus of elements $(y_1, \ldots, y_{2n})$ with $y_1 = y_{n+1}$ and $y_i = y_{2n+2-i}$ for all $i = 2, \ldots, 2n$.

We take $u \in G(\mathbb{Z}_p)$ to be the element which is 1 in the GL_1-component, and equal to the element u in the previous section in the GL_{2n-1}-component. Then we obtain the following decomposition:

Proposition A.3.1. *Let $r \geq 1$. Then we have*

$$\mathcal{G}_r^{\square} = (u^{-1}\mathcal{H}_r^{\diamond}u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G).$$

Proof. This follows from Proposition A.2.2 and Remark A.2.3. □

A.4. *The group* GL_{2n}. We now let $G = \mathrm{GL}_{2n}$ and $H = \mathrm{GL}_n \times \mathrm{GL}_n$ embedded block diagonally. We define $\mathcal{G}_r^{\square}$ and $\mathcal{H}_r^{\diamond}$ analogously as in the previous section, but now $T^{\diamond}$ is the subtorus given by elements $(y_1, \dots, y_{2n})$ satisfying $y_i = y_{2n+1-i}$ for all $i = 1, \dots, 2n$.

We let $u \in G(\mathbb{Z}_p)$ denote the block matrix

$$u = \begin{pmatrix} 1 & \\ \xi & 1 \end{pmatrix}$$

where all blocks are of size $(n \times n)$, and ξ is the antidiagonal matrix with nonzero entries equal to 1.

Proposition A.4.1. *Let $r \geq 1$. Then we have*

$$\mathcal{G}_r^{\square} = (u^{-1}\mathcal{H}_r^{\diamond}u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G).$$

Proof. By reasoning as in the proof of Proposition A.2.2, it is enough to show

$$\overline{\mathcal{N}}_r^1 \subset (u^{-1}\mathcal{H}_r^{\diamond}u) \cdot (\mathcal{G}_r^{\square} \cap \mathcal{B}_G)$$

where $\overline{N}$ denotes the unipotent radical of the standard opposite parabolic of G with Levi H. But this follows from the same proof in Proposition A.2.2 using Lemma A.1.1 (with $d = 2n$). □

A.5. *Proof of Theorem 5.3.4.* We now return to the setting of Section 5 (and return to using the notation introduced in the main body of the article). By combining the previous sections, we immediately find that:

Proposition A.5.1. *Let $r \geq 1$. Then one has equalities*

$$\mathcal{M}_{G,r}^{\square} = (u^{-1}\mathcal{M}_{H,r}^{\diamond}u) \cdot (\mathcal{M}_{G,r}^{\square} \cap \mathcal{B}_{M_G}), \quad \mathcal{M}_{G,r}^{\square} = (u^{-1}\mathcal{M}_{H,r}^{\clubsuit}u) \cdot (\mathcal{M}_{G,r}^{\square} \cap \mathcal{B}_{M_G}).$$

Proof. For the first equality, this follows by breaking up the groups into the factors indexed by $\tau \in \Psi$. The factor corresponding to τ_0 follows from Proposition A.2.2, and the factors for $\tau \neq \tau_0$ follow from Proposition A.4.1. There is nothing to check for the extra GL_1-factors in M_G and M_H. The second equality follows from $u^{-1}\mathcal{M}_{H,r}^{\diamond}u \subset u^{-1}\mathcal{M}_{H,r}^{\clubsuit}u \subset \mathcal{M}_{G,r}^{\square}$. □

We now introduce the relevant algebraic weights for representations of M_G. Recall any algebraic character of the torus T can be represented by a tuple

$$\kappa = (\kappa_0; \kappa_{1,\tau}, \dots, \kappa_{2n,\tau})_{\tau \in \Psi}$$

where κ_0 and $\kappa_{i,\tau}$ are integers. By the τ-factor or τ-component of κ, we mean the tuple $(\kappa_{1,\tau}, \dots, \kappa_{2n,\tau})$, and by the GL_1-factor, we mean the integer κ_0. It will be helpful to use this terminology when defining certain characters below.

Definition A.5.2. Let κ be an algebraic character of T as above. We say:

(1) κ is M_G-dominant if

$$\kappa_{2,\tau_0} \geq \cdots \geq \kappa_{2n,\tau_0} \quad \text{and} \quad \kappa_{1,\tau} \geq \cdots \geq \kappa_{2n,\tau}$$

for all $\tau \in \Psi - \{\tau_0\}$.

(2) κ is pure of weight $w \in \mathbb{Z}$ if

$$\kappa_{i,\tau_0} + \kappa_{2n+2-i,\tau_0} = w$$

for all $i = 2, \dots, n$, and $\kappa_{i,\tau} + \kappa_{2n+1-i,\tau} = 0$ for all $i = 1, \dots, 2n$ and $\tau \neq \tau_0$.

The set of characters which are pure (of some weight $w \in \mathbb{Z}$) form a group, and we let $\mathcal{C}$ denote the submonoid of M_G-dominant characters which are pure of weight $w \leq 0$ satisfying $\kappa_{n+1,\tau_0} \leq w$. We will always write the group law for $\mathcal{C}$ additively. We consider the following special elements of $\mathcal{C}$:

- $\mu_0 = (1; 0, \dots, 0)_{\tau \in \Psi}$.
- $\mu_w = (\mu_{w,0}, \mu_{w,1,\tau}, \dots, \mu_{w,2n,\tau})_{\tau \in \Psi}$, where $\mu_{w,0} = \mu_{w,1,\tau_0} = \mu_{w,i,\tau} = 0$ for all $i = 1, \dots, 2n$ and $\tau \neq \tau_0$, and we have

$$\mu_{w,2,\tau_0} = \cdots = \mu_{w,n,\tau_0} = 0, \quad \mu_{w,n+1,\tau_0} = \cdots = \mu_{w,2n,\tau_0} = -1.$$

- μ_{1,τ_0} which is the identity in the GL_1-factor and $\tau \neq \tau_0$ factors, and in the τ_0-factor is given by

$$(1, 0, \dots, 0).$$

- For $i = 2, \dots, n$, we let μ_{i,τ_0} be the character which is the identity in the GL_1-factor and $\tau \neq \tau_0$ factors, and in the τ_0-factor is given by

$$(0, 1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$$

where there are $i - 1$ lots of 1s and -1s.

- We let μ_{n+1,τ_0} be the character which is the identity outside the τ_0-factor, and the τ_0-factor is given by

$$(0, 1, \dots, 1, -1, \dots, -1)$$

where there are $n - 1$ lots of 1 and n lots of -1.

- For $i = 1, \dots, n$ and $\tau \neq \tau_0$, we let $\mu_{i,\tau}$ denote the character which is the identity outside the τ-factor, and at the τ-factor is

$$(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$$

where there are i lots of 1s and -1s.

This collection of characters forms a generating set for $\mathcal{C}$ in the following sense: for any $\kappa \in \mathcal{C}$, there exist unique integers $a_0, a_{1,\tau_0} \in \mathbb{Z}$ and $a_w, a_{i,\tau} \in \mathbb{Z}_{\geq 0}$ for $(i, \tau) \neq (1, \tau_0)$, such that

$$\kappa = a_0\mu_0 + a_w\mu_w + a_{n+1,\tau_0}\mu_{n+1,\tau_0} + \sum_{i=1}^{n}\sum_{\tau\in\Psi} a_{i,\tau}\mu_{i,\tau}.$$

Explicitly, the integers are given by:

- $a_0 = \kappa_0$.
- $a_w = -(\kappa_{2,\tau_0} + \kappa_{2n,\tau_0})$.
- $a_{1,\tau_0} = \kappa_{1,\tau_0}$.
- For $i = 2, \dots, n+1$, one has

$$a_{i,\tau_0} = \begin{cases} \kappa_{i,\tau_0} - \kappa_{i+1,\tau_0} & \text{if } i \leq n-1, \\ \kappa_{n+1,\tau_0} - \kappa_{n+2,\tau_0} & \text{if } i = n, \\ (\kappa_{n,\tau_0} + \kappa_{n+2,\tau_0}) - \kappa_{n+1,\tau_0} & \text{if } i = n+1. \end{cases}$$

- For $i = 1, \dots, n$ and $\tau \neq \tau_0$, one has

$$a_{i,\tau} = \begin{cases} \kappa_{i,\tau} - \kappa_{i+1,\tau} & \text{if } i \leq n-1, \\ \kappa_{n,\tau} & \text{if } i = n. \end{cases}$$

Let $\mathcal{D} = \prod_{\tau\neq\tau_0} \mathbb{Z}_{\geq 0}$ equipped with the monoid structure given by component-wise addition. We will denote elements of $\mathcal{D}$ by tuples $j = (j_\tau)_{\tau\neq\tau_0}$. We let $\mathcal{E} \subset \mathcal{C} \times \mathcal{D}$ be the collection of pairs (κ, j) which satisfy $j_\tau \leq \kappa_{n,\tau}$ for all $\tau \neq \tau_0$. This forms a submonoid of $\mathcal{C} \times \mathcal{D}$. Then $\mathcal{E}$ has a generating set given by the pairs $(\mu_0, 0)$, $(\mu_w, 0)$, $(\mu_{i,\tau}, 0)$, and $(\mu_{n,\tau}, 1_\tau)$ for $\tau \neq \tau_0$, where $1_\tau \in \mathcal{D}$ is the tuple which is zero outside τ and has 1 in the τ-component. More precisely, for any $(\kappa, j) \in \mathcal{E}$, there exist unique integers $a_0, a_{1,\tau_0} \in \mathbb{Z}$, $a_w, a_{i,\tau} \in \mathbb{Z}_{\geq 0}$ for $(i, \tau) \neq (1, \tau_0)$, and $b_\tau \in \mathbb{Z}_{\geq 0}$ for $\tau \neq \tau_0$ such that

$$(\kappa, j) = a_0(\mu_0, 0) + a_w(\mu_w, 0) + a_{n+1,\tau_0}(\mu_{n+1,\tau_0}, 0) + \sum_{i=1}^{n}\sum_{\tau\in\Psi} a_{i,\tau}(\mu_{i,\tau}, 0) + \sum_{\tau\neq\tau_0} b_\tau(\mu_{n,\tau}, 1_\tau).$$

Explicitly, the integers are given by:

- $a_0, a_w, a_{1,\tau_0}, \dots, a_{n+1,\tau_0}$ and $a_{1,\tau}, \dots, a_{n-1,\tau}$ are given by the formulae above.
- For $\tau \neq \tau_0$, one has $a_{n,\tau} = \kappa_{n,\tau} - j_\tau$.
- $b_\tau = j_\tau$.

Definition A.5.3. For any $(\kappa, j) \in \mathcal{E}$, we let $\sigma_\kappa^{[j]}$ denote the character of M_H given by sending a general element $(x; y_1, y_2, y_3; z_{1,\tau}, z_{2,\tau})_{\tau\neq\tau_0}$ to

$$x^{-\kappa_0} y_1^{-\kappa_{1,\tau_0}} \det y_2^{\kappa_{n+1,\tau_0}-w} \det y_3^{-\kappa_{n+1,\tau_0}} \prod_{\tau\neq\tau_0} \det z_{1,\tau}^{-j_\tau} \det z_{2,\tau}^{j_\tau}$$

where $w = \kappa_{2,\tau_0} + \kappa_{2n,\tau_0}$ denotes the weight of κ.

For any $\kappa \in \mathcal{C}$, let V_κ denote the irreducible algebraic representation of M_G with highest weight κ, which can be viewed as the space of algebraic functions $f\colon M_G \to \mathbb{A}^1$ satisfying

$$f(mb) = (w_{M_G}^{\max}\kappa)(b^{-1})f(m)$$

for all $b \in B_{M_G}$. The action of M_G on f is then given by $m \cdot f(n) = f(m^{-1}n)$. We have the following classical branching law:

Theorem A.5.4. *Let $(\kappa, j) \in \mathcal{E}$. Then there exists a unique vector $x_\kappa^{[j]} \in V_\kappa$ such that*:

(1) *$x_\kappa^{[j]}$ is an eigenvector for the action of $u^{-1}M_H u$ with eigencharacter given by the inverse of $\sigma_\kappa^{[j]}$.*

(2) *$x_\kappa^{[j]}(1) = 1$, where we are viewing $x_\kappa^{[j]}\colon M_G \to \mathbb{A}^1$ as an algebraic function.*

(3) *The vectors $x_{\mu_0}^{[0]}$ and $x_{\mu_{1,\tau_0}}^{[0]}$ are invertible in $\mathcal{O}(M_G)$, and we have*

$$x_\kappa^{[j]} = (x_{\mu_0}^{[0]})^{a_0} \cdot (x_{\mu_w}^{[0]})^{a_w} \cdot (x_{\mu_{n+1,\tau_0}}^{[0]})^{a_{n+1,\tau_0}} \cdot \prod_{\substack{i=1,\dots,n \\ \tau\in\Psi}} (x_{\mu_{i,\tau}}^{[0]})^{a_{i,\tau}} \cdot \prod_{\tau\neq\tau_0} (x_{\mu_{n,\tau}}^{[1_\tau]})^{b_\tau}$$

where the product takes place in $\mathcal{O}(M_G)$ and the exponents are the integers above.

Proof. By applying [Knapp 2001, Theorem 2.1] for each general linear factor of M_G,[11] there exists a unique up to scaling (nonzero) vector $x_\kappa^{[j]} \in V_\kappa$ satisfying property (1). Since $u^{-1}M_H u B_{M_G}$ is Zariski open in M_G (Lemma 2.4.3), the vector is nonvanishing on this cell, so we can normalize $x_\kappa^{[j]}$ as in (2) to determine the vector uniquely. The vectors $x_{\mu_0}^{[0]}$ and $x_{\mu_{1,\tau_0}}^{[0]}$ are invertible in $\mathcal{O}(M_G)$ because the corresponding representations V_{μ_0} and $V_{\mu_{1,\tau_0}}$ are one-dimensional. Property (3) then follows immediately from uniqueness, the identity

$$\sigma_\kappa^{[j]} = (\sigma_{\mu_0})^{a_0} \cdot (\sigma_{\mu_w}^{[0]})^{a_w} \cdot (\sigma_{\mu_{n+1,\tau_0}}^{[0]})^{a_{n+1,\tau_0}} \cdot (\sigma_{\mu_{1,\tau_0}}^{[0]})^{a_{1,\tau_0}} \cdot \prod_{\tau\neq\tau_0} (\sigma_{\mu_{n,\tau}}^{[1_\tau]})^{b_\tau}$$

and the fact that $\sigma_{\mu_{i,\tau}}^{[0]}$ is the trivial character for $(i, \tau) \neq (1, \tau_0), (n+1, \tau_0)$. ☐

Remark A.5.5. Note that we introduced some asymmetry here — we could have equally worked with the monoid $\mathcal{D} = \prod_{\tau\neq\tau_0} \mathbb{Z}_{\leq 0}$ (or even more generally, products of $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$) and the monoid $\mathcal{E}$ defined by the equations $-j_\tau \leq \kappa_{n,\tau}$.

To prove Theorem 5.3.4, we will use a p-adic version of the product formula in Theorem A.5.4(3).

Lemma A.5.6. *Let (A, A^+) be a Tate algebra over $(\mathbb{Q}_p, \mathbb{Z}_p)$, and suppose that $\kappa\colon T(\mathbb{Z}_p) \to (A^+)^\times$ is an r-analytic character, for some $r \in \mathbb{Q}_{>0}$, which satisfies*

$$\kappa_{i,\tau_0} + \kappa_{2n+2-i,\tau_0} = \kappa_{j,\tau_0} + \kappa_{2n+2-j,\tau_0}$$

for all $i, j = 2, \dots, n$, and $\kappa_{i,\tau} + \kappa_{2n+1-i,\tau} = 0$ for all $i = 1, \dots, n$ and $\tau \neq \tau_0$. Let $\beta = (\beta_\tau)\colon \prod_{\tau\neq\tau_0} \mathbb{Z}_p^\times \to (A^+)^\times$ be an r-analytic character. Then there exist unique r-analytic characters

[11] Knapp works in the setting of compact unitary groups, but the proof works verbatim for general linear groups.

- $\xi_0, \xi_w, \xi_{i,\tau} \colon \mathbb{Z}_p^\times \to (A^+)^\times$ *for* $i = 1, \dots, n$ *and* $\tau \in \Psi$,
- $\xi_{n+1,\tau_0} \colon \mathbb{Z}_p^\times \to (A^+)^\times$,
- $\Xi_\tau \colon \mathbb{Z}_p^\times \to (A^+)^\times$ *for* $\tau \in \Psi - \{\tau_0\}$

such that

$$(\kappa, \beta) = \xi_0 \circ (\mu_0, 0) + \xi_w \circ (\mu_w, 0) + \xi_{n+1,\tau_0} \circ (\mu_{n+1,\tau_0}, 0) + \sum_{i=1}^{n} \sum_{\tau \in \Psi} \xi_{i,\tau} \circ (\mu_{i,\tau}, 0) + \sum_{\tau \neq \tau_0} \Xi_\tau \circ (\mu_{n,\tau}, 1_\tau)$$

where the group law is written additively.

Proof. We define the r-analytic characters via the same formulae as above, i.e., $\xi_0 = \kappa_0$, $\xi_w = -(\kappa_{2,\tau_0} + \kappa_{2n,\tau_0})$, etc. It is clear that these are uniquely determined. □

We will also need the following lemma:

Lemma A.5.7. *Let* $r \in \mathbb{Z}_{>0}$. *Then for any* $(\kappa, j) \in \mathcal{E}$, *one has*

$$x_\kappa^{[j]}(\mathcal{M}_{G,r}^{\square}) \subset \mathbb{Z}_p^\times(1 + \mathcal{B}_r)$$

where we are viewing $x_\kappa^{[j]}$ *as an analytic function* $M_G^{\mathrm{an}} \to \mathbb{A}^{1,\mathrm{an}}$.

Proof. By Proposition A.5.1, we have $\mathcal{M}_{G,r}^{\square} = (u^{-1}\mathcal{M}_{H,r}^{\clubsuit}u)(\mathcal{M}_{G,r}^{\square} \cap \mathcal{B}_{M_G})$, therefore the transformation properties for $x_\kappa^{[j]}$ imply that

$$x_\kappa^{[j]}(m) = \sigma_\kappa^{[j]}(m_1) \cdot (w_{M_G}^{\max}\kappa)(m_2^{-1})$$

for any $m \in \mathcal{M}_{G,r}^{\square}$ satisfying $m = u^{-1}m_1u \cdot m_2$ for $m_1 \in \mathcal{M}_{H,r}^{\clubsuit}$ and $m_2 \in \mathcal{M}_{G,r}^{\square} \cap \mathcal{B}_{M_G}$. But $\sigma_\kappa^{[j]}$ and $w_{M_G}^{\max}\kappa$ are algebraic characters, so their analytifications map $\mathcal{M}_{H,r}^{\clubsuit}$ and $\mathcal{M}_{G,r}^{\square} \cap \mathcal{B}_{M_G}$ into $\mathbb{Z}_p^\times(1 + \mathcal{B}_r)$, as required. □

Remark A.5.8. The previous lemma implies that for any r-analytic character $\xi \colon \mathbb{Z}_p^\times \to (A^+)^\times$, the composition $\xi \circ x_\kappa^{[j]}$ defines an analytic function

$$\xi \circ x_\kappa^{[j]} \colon \mathcal{M}_{G,r}^{\square} \times \mathrm{Spa}(A, A^+) \to \mathbb{G}_{m,A}^{\mathrm{an}} \subset \mathbb{A}_A^{1,\mathrm{an}}.$$

We now introduce the p-adic vectors. Recall that for an r-analytic weight $\kappa \colon T(\mathbb{Z}_p) \to (A^+)^\times$, we let $V_\kappa^{r-\mathrm{an}}$ denote the r-analytic induction as in Definition 5.3.2.

Definition A.5.9. Let $r \in \mathbb{Z}_{>0}$ and let (κ, β) be a pair of r-analytic characters as in Lemma A.5.6. Then we define

$$x_\kappa^{[\beta]} := (x_{\mu_0}^{[0]})^{\xi_0} \cdot (x_{\mu_w}^{[0]})^{\xi_w} \cdot (x_{\mu_{n+1,\tau_0}}^{[0]})^{\xi_{n+1,\tau_0}} \cdot \prod_{\substack{i=1,\dots,n \\ \tau \in \Psi}} (x_{\mu_{i,\tau}}^{[0]})^{\xi_{i,\tau}} \cdot \prod_{\tau \neq \tau_0} (x_{\mu_{n,\tau}}^{[1_\tau]})^{\Xi_\tau}$$

where the product takes place in $\mathcal{O}(\mathcal{M}_{G,r}^{\square}) \hat{\otimes} A$ and the analytic characters $\xi_{\dots}$ and $\Xi_{\dots}$ are as in Lemma A.5.6. Here we have written $(-)^\xi$ as a shorthand for $\xi \circ (-)$. This defines an element of $V_\kappa^{r-\mathrm{an}}$.

We let $\sigma_\kappa^{[\beta]}$ denote the character

$$\sigma_\kappa^{[\beta]} \colon \mathcal{M}^{\clubsuit}_{H,r} \times \mathrm{Spa}(A, A^+) \to \mathbb{G}^{\mathrm{an}}_{m,A},$$
$$(x; y_1, y_2, y_3, z_{1,\tau}, z_{2,\tau}) \mapsto \kappa_0(x^{-1})\kappa_{1,\tau_0}(y_1^{-1})(\kappa_{n+1,\tau_0}\kappa_{2,\tau_0}^{-1}\kappa_{2n,\tau_0}^{-1})(\det y_2)\kappa_{n+1,\tau_0}^{-1}(\det y_3) \cdot \prod_{\tau \neq \tau_0} \beta_\tau(\det z_{1,\tau}^{-1} \det z_{2,\tau}),$$

which makes sense because κ and β are r-analytic.

Finally, we obtain the following theorem:

Theorem A.5.10. *Let $r \in \mathbb{Z}_{>0}$ and let (κ, β) be a pair of r-analytic characters as in Lemma A.5.6. Then:*

(1) *$x_\kappa^{[\beta]}$ is a (nonzero) eigenvector for the action of $u^{-1}\mathcal{M}^{\clubsuit}_{H,r}u$ with eigencharacter given by the inverse of $\sigma_\kappa^{[\beta]}$.*

(2) *If (B, B^+) is another Tate algebra with a morphism $(A, A^+) \to (B, B^+)$, and (κ', β') denotes the composition of (κ, β) with this morphism, then the image of $x_\kappa^{[\beta]}$ under the natural map*
$$V_\kappa^{r-\mathrm{an}} \to V_{\kappa'}^{r-\mathrm{an}}$$
is equal to $x_{\kappa'}^{[\beta']}$.

(3) *If (κ, β) arises from a pair of algebraic characters $(\kappa, j) \in \mathcal{E}$, then $x_\kappa^{[\beta]}$ is equal to the image of $x_\kappa^{[j]}$ under the natural map*
$$V_\kappa \to V_\kappa^{r-\mathrm{an}}$$
given by restricting (the analytification of) a function $M_G \to \mathbb{A}^1$ to $\mathcal{M}^{\square}_{G,r}$.

(4) *The vector $x_\kappa^{[\beta]}$ does not depend on the radius of analyticity, i.e., if $r' \geq r$ is another integer, then the constructions for $x_\kappa^{[\beta]}$ coincide under the map*
$$V_\kappa^{r-\mathrm{an}} \to V_\kappa^{r'-\mathrm{an}}$$
given by restriction to $\mathcal{M}^{\square}_{G,r'}$.

Proof. Part (1) follows from the fact we have a similar product formula for $\sigma_\kappa^{[\beta]}$ as in the proof of Theorem A.5.4, replacing the coefficients $a_{\dots}$ and $b_{\dots}$ by $\xi_{\dots}$ and $\Xi_{\dots}$.

The remaining properties are clear from construction, using the fact that the characters $\xi_{\dots}$ and $\Xi_{\dots}$ are unique and Theorem A.5.4(3). □

Appendix B: Comparisons in families

In this appendix, we describe the key ingredient needed to compare the coherent cohomology classes associated with algebraic Hecke characters and (algebraic) p-adic Hecke characters. We restrict ourselves to the case of PEL Shimura data which give rise to compact Shimura varieties — more general versions of the functorial properties we describe can be found in [Diao et al. 2023].

B.1. *Canonical constructions.* In this section we let $(\mathscr{G}, X_{\mathscr{G}})$ be a PEL-type Shimura–Deligne datum satisfying (SD5) as in [Graham and Shah 2023, Section B.3]. Suppose that the associated Shimura variety admits a canonical model over the reflex field, which we will denote by F. We fix a rational prime $p > 2$ which is unramified in F and for which $\mathscr{G}_{\mathbb{Q}_p}$ is unramified. We fix a prime $\mathfrak{p}$ of F lying above p.

Let $K \subset \mathscr{G}(\mathbb{A}_f)$ be a neat compact open subgroup. Then the Shimura variety $S_{\mathscr{G},K}$ parametrizes abelian varieties A with PEL structure (corresponding to the PEL-data defining $(\mathscr{G}, X_{\mathscr{G}})$), such that the first relative homology of A is modeled on the defining representation for $\mathscr{G}$. Let $S = S_{\mathscr{G},U}$ and denote the universal abelian variety over S by A. We are interested in the local systems/locally free sheaves obtained from the relative homology of A.

Assumption B.1.1. *We assume that the Shimura variety S is compact.*

Recall that there exist "canonical constructions" ξ_B (resp. ξ_{dR}, resp. $\xi_{\text{ét}}$) which are tensor functors on the category of algebraic representations of $\mathscr{G}$ valued in the category of variations of Hodge structures over $S(\mathbb{C})$ (resp. locally free sheaves on S with an integrable connection, resp. p-adic local systems on S). More precisely, if V is an algebraic representation of $\mathscr{G}$, then:

(1) The variation of Hodge structure $\xi_B(V)$ is constructed from the left $\mathscr{G}(\mathbb{Q})$-torsor

$$X_{\mathscr{G}} \times \mathscr{G}(\mathbb{A}_f)/K \to \mathscr{G}(\mathbb{Q})\backslash X_{\mathscr{G}} \times \mathscr{G}(\mathbb{A}_f)/K = S(\mathbb{C})$$

and the $\mathscr{G}(\mathbb{Q})$-representation V; see [Caraiani and Scholze 2017, Section 2.3] for example.

(2) The locally free sheaf $\xi_{\mathrm{dR}}(V)$ arises from the $\mathscr{G}_F$-torsor

$$\mathscr{G}_{\mathrm{dR}} \to S$$

(the standard principal bundle) and the algebraic representation V_F of $\mathscr{G}_F$; see [Milne 1990, Section III.3].

(3) The p-adic local system $\xi_{\text{ét}}(V)$ can be constructed by choosing a $\mathscr{G}(\mathbb{Z}_p)$-stable lattice $T \subset V_{\mathbb{Q}_p}$ and using the pro-system of torsors

$$S_{\mathscr{G},K'} \to S_{\mathscr{G},K}$$

for $K' \subset K$; see [Graham and Shah 2023, Section 4]. One can also interpret this in terms of the perfectoid Shimura variety (see Section B.3 below).

Notation B.1.2. Let V be an algebraic representation of $\mathscr{G}$. We write $\mathcal{V}_B$, $\mathcal{V}_{\mathrm{dR}}$ and $\mathcal{V}_{\text{ét}}$ for $\xi_B(V)$, $\xi_{\mathrm{dR}}(V)$ and $\xi_{\text{ét}}(V)$ respectively.

Remark B.1.3. The above functors are normalized so that $\mathcal{W}_?$ equals the first relative homology of A/S with respect to the relevant cohomology theory, for $? = B, \mathrm{dR}, \text{ét}$, where W denotes the defining representation of $\mathscr{G}$.

We have several comparisons between these sheaves/local systems:

(1) (Betti–p-adic, [SGA 4_3 1973, Exposé xi]) Since S is smooth, one has a morphism of sites $\beta\colon S_{\mathrm{cl}} \to S_{\text{ét}}$ from the site of étale coverings of $S(\mathbb{C})$ to the étale site of S. Then for any algebraic representation V of $\mathscr{G}$, one has

$$\mathcal{V}_{\text{ét}} \cong \beta_*(\mathcal{V}_B \otimes_{\mathbb{Q}} \mathbb{Q}_p).$$

Indeed, one has a similar map of sites for A, whose pushforward is exact and commutes with pushforward along $A \to S$ (in the analytic and étale topologies).

(2) (Betti–de Rham) For an algebraic representation V of $\mathscr{G}$, one has a comparison isomorphism

$$\mathcal{V}_{\mathrm{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S(\mathbb{C})} \cong \mathcal{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{S(\mathbb{C})}.$$

(3) (de Rham–p-adic, [Caraiani and Scholze 2017, Section 2.2]) Let $L/F_{\mathfrak{p}}$ be a finite extension and let $A^{\mathrm{an}} \to S^{\mathrm{an}}$ be the morphism of adic spaces associated with $A_L \to S_L$. Then for any algebraic representation V of $\mathscr{G}$, one has an isomorphism

$$\mathcal{V}^{\mathrm{an}}_{\mathrm{dR},L} \otimes_{\mathcal{O}_{S^{\mathrm{an}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S^{\mathrm{an}}} \cong \mathcal{V}^{\mathrm{an}}_{\text{ét},L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},S^{\mathrm{an}}}$$

of sheaves on the pro-étale site of S^{an} compatible with filtrations and connections. Here $(-)^{\mathrm{an}}$ means pull-back to the associated adic space.

More precisely, one has the above comparisons for $\mathcal{W}_?$ and the work of Ancona [2015] and Torzewski [2020] shows that all of these "canonical constructions" factor through a functor valued in relative Chow motives over S, so the comparisons can be extended to all algebraic representations. In particular, since the comparisons above are functorial with respect to algebraic operations (e.g., correspondences on A), the above comparisons are also functorial in the algebraic representation V.

B.2. *Functoriality.* Let $(\mathscr{G}_1, X_1)$ and $(\mathscr{G}_2, X_2)$ be two PEL-type Shimura–Deligne data (with a common reflex field F) as in the previous subsection, including Assumption B.1.1. Suppose that we have a homomorphism $f\colon \mathscr{G}_1 \to \mathscr{G}_2$ inducing a morphism of Shimura data (and arising from a morphism of PEL data). Let $K_1 \subset \mathscr{G}_1(\mathbb{A}_f)$ be a neat compact open subgroup and $K_2 \subset \mathscr{G}_2(\mathbb{A}_f)$ a neat compact open subgroup containing $f(K_1)$. Let $S_i = S_{\mathscr{G}_i, K_i}$ for $i = 1, 2$.

The morphism f induces a map of torsors

$$\begin{tikzcd} X_1 \times \mathscr{G}_1(\mathbb{A}_f)/K_1 \arrow[r] \arrow[d] & X_2 \times \mathscr{G}_2(\mathbb{A}_f)/K_2 \arrow[d] \\ S_1(\mathbb{C}) \arrow[r] & S_2(\mathbb{C}) \end{tikzcd}$$

and hence a natural isomorphism $\eta_B\colon \xi_{1,B} \circ f^* \to f^* \circ \xi_{2,B}$, where we have use the notation $\xi_{i,B}$ to emphasize which Shimura variety and group the construction refers to.

Similarly, the morphism f induces morphisms of (finite étale) torsors $S_{\mathscr{G}_1,K_1'} \to S_{\mathscr{G}_2,K_2'}$ over $S_1 \to S_2$, for any $K_1' \subset K_1$ and $f(K_1') \subset K_2' \subset K_2$. These are compatible with varying K_1' and K_2', so induce a natural isomorphism $\eta_{\text{ét}} \colon \xi_{1,\text{ét}} \circ f^* \to f^* \circ \xi_{2,\text{ét}}$.

Lemma B.2.1. *The Betti–p-adic comparison identifies the natural isomorphisms $\eta_B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\eta_{\acute{e}t}$.*

Proof. Let V be an algebraic representation of $\mathscr{G}_1$. Then it is well-known that one can construct $\xi_B(V)$ either by considering V as a left $\mathscr{G}_1(\mathbb{Q})$-module (as above) or by viewing V as a right K_1-module (with no left $\mathscr{G}_1(\mathbb{Q})$-action) and setting

$$\xi_B(V) = \mathscr{G}_1(\mathbb{Q})\backslash X_1 \times \mathscr{G}_1(\mathbb{A}_f) \times V/K_1.$$

In particular, choosing a $\mathscr{G}_1(\mathbb{Z}_p)$-stable lattice $T \subset V_{\mathbb{Q}_p}$, one easily sees that the two constructions $\xi_B(V) \otimes \mathbb{Q}_p$ and $\xi_{\text{ét}}(V)$ are identified under the Betti–p-adic comparison. Similar calculations apply for the group $\mathscr{G}_2$. □

We also obtain a natural isomorphism involving the functor ξ_{dR} as follows. Since f induces a morphism of Shimura data, by theory of canonical models for standard principal bundles (see [Milne 1990, Section III.4]), one obtains a morphism of torsors $\mathscr{G}_{1,\text{dR}} \to \mathscr{G}_{2,\text{dR}}$ which induces the desired natural isomorphism $\eta_{\text{dR}} \colon \xi_{1,\text{dR}} \circ f^* \to f^* \circ \xi_{2,\text{dR}}$. Pulling this back to $\mathbb{C}$, this morphism of torsors is identified with the morphism

$$\mathscr{G}_{1,\text{dR}}(\mathbb{C}) = \mathscr{G}_1(\mathbb{Q})\backslash X_1 \times \mathscr{G}_1(\mathbb{C}) \times \mathscr{G}_1(\mathbb{A}_f)/K_1 \to \mathscr{G}_2(\mathbb{Q})\backslash X_2 \times \mathscr{G}_2(\mathbb{C}) \times \mathscr{G}_2(\mathbb{A}_f)/K_2 = \mathscr{G}_{2,\text{dR}}(\mathbb{C})$$

sending $[x, g, g']$ to $[f(x), f(g), f(g')]$. But $\mathscr{G}_{i,\text{dR}}(\mathbb{C})$ is the pushout of the torsor $X_i \times \mathscr{G}_i(\mathbb{A}_f)/K_i$ along the map $\mathscr{G}_i(\mathbb{Q}) \to \mathscr{G}_i(\mathbb{C})$, and it is clear that this morphism of torsors is induced from the one above. In other words, the Betti–de Rham comparison identifies $\eta_{\text{dR}} \otimes_{\mathcal{O}_S} \mathcal{O}_{S(\mathbb{C})}$ and $\eta_B \otimes_{\mathbb{Q}} \mathcal{O}_{S(\mathbb{C})}$.

Proposition B.2.2. *The de Rham–p-adic comparison identifies the natural isomorphisms $\eta_{\text{dR}}^{\text{an}} \otimes_{\mathcal{O}_{S^{\text{an}}}} \mathcal{O}\mathbb{B}_{\text{dR},S^{\text{an}}}$ and $\eta_{\acute{e}t}^{\text{an}} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\text{dR},S^{\text{an}}}$.*

Proof. Essentially this follows because η_B is induced from an (absolute) Hodge cycle for a certain abelian variety, which is known to be de Rham [Blasius 1994].

Let W_2 denote the defining representation of $\mathscr{G}_2$. Since we already know η_B, η_{dR}, $\eta_{\text{ét}}$ are natural isomorphisms of additive tensor functors (respecting this structure and duals), and every representation $\mathscr{G}_2$ is a direct summand of tensor products of W_2 and W_2^*, it enough to check that $\eta_{\text{dR}}^{\text{an}}(W_2) \otimes_{\mathcal{O}_{S^{\text{an}}}} \mathcal{O}\mathbb{B}_{\text{dR},S^{\text{an}}} = \eta_{\text{ét}}^{\text{an}}(W_2) \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\text{dR},S^{\text{an}}}$. Fix a presentation

$$W_2 = e\left(\bigoplus_{i=1}^{k} W_1^{\otimes a_i} \otimes (W_1^*)^{\otimes b_i}\right)$$

for some positive integers a_i, b_i and idempotent e. Since $\xi_{1,B}$, $\xi_{1,\text{dR}}$ and $\xi_{1,\text{ét}}$ factor through a functor valued in relative Chow motives, we obtain idempotents e_B, e_{dR}, $e_{\text{ét}}$ in the respective target categories which are all compatible under the comparison isomorphisms.

Let A_1 and A_2 denote the universal abelian varieties over S_1 and S_2, and let f^*A_2 denote the pullback to S_1. For $? = B$, dR, ét, the isomorphisms $\eta_?(W_2)$ are described by isomorphisms

$$\xi_{1,?}(W_2) \cong e_?\left(\bigoplus_{i=1}^{k} \mathcal{W}_{1,?}^{\otimes a_i} \otimes (\mathcal{W}_{1,?}^{\vee})^{\otimes b_i}\right) \xrightarrow{\sim} \mathcal{H}_1^?(f^*A_2/S_1) \cong f^*\mathcal{H}_1^?(A_2/S_2)$$

where $\mathcal{H}_1^?(\cdots)$ denotes first relative homology of the appropriate cohomology theory and the last isomorphism is proper base-change.

We just need to check the middle isomorphism is compatible under the de Rham–étale comparison isomorphism. It is enough to check this at points of S_1 which are defined over number fields (see the proof of [Caraiani and Scholze 2017, Proposition 2.3.9]) — let $\theta_?$ denote the middle isomorphism specialized at such a point. By above, we know that θ_B and θ_{dR} are compatible under the Betti–de Rham comparison, and that θ_B and $\theta_{\text{ét}}$ are compatible under the Betti–étale comparison. The result now follows from the fact that θ_B can be represented as a Hodge class (by using the polarization and Künneth formula) for an abelian variety constructed from copies of A_1 and f^*A_2. Indeed, by [Deligne et al. 1982] it is an absolute Hodge class whose de Rham realization is defined over the field of definition of the point (by the paragraph preceding the proposition). By [Blasius 1994], this Hodge class is de Rham, which precisely means that θ_{dR} and $\theta_{\text{ét}}$ are compatible under the de Rham–étale comparison, as required. □

Remark B.2.3. One can show that the pushout $\mathscr{G}_{1,\mathrm{dR}} \times^{\mathscr{G}_1} \mathscr{G}_2$ is identified with frames of $\mathcal{H}_1^{\mathrm{dR}}(f^*A_2/S_1)$ preserving a collection of Hodge tensors coming from a choice of $\mathscr{G}_1$-equivariant embedding $W_2^{\otimes} \subset W_1^{\otimes}$ and the isomorphism θ_{dR} above. The isomorphism $\mathscr{G}_{1,\mathrm{dR}} \times^{\mathscr{G}_1} \mathscr{G}_2 \to f^*\mathscr{G}_{2,\mathrm{dR}}$ is then induced from the proper base-change isomorphism $\mathcal{H}_1^{\mathrm{dR}}(f^*A_2/S_1) \cong f^*\mathcal{H}_1^{\mathrm{dR}}(A_2/S_2)$, which matches the Hodge tensors. A similar description also holds for the étale and Betti constructions.

Let L/F be a finite extension and let $\mu_i : \mathbb{G}_{m,L} \to \mathscr{G}_{i,L}$ be a choice of Hodge cocharacter for the Shimura datum $(\mathscr{G}_i, X_i)$, for $i = 1, 2$. We assume that $\mu_2 = f \circ \mu_1$. Fix a prime $\mathfrak{P}$ of L lying above $\mathfrak{p}$, and we base-change the Shimura varieties S_1 and S_2 to $L_{\mathfrak{P}}$ (but omit this from the notation). For $i = 1, 2$ and over $L_{\mathfrak{P}}$, we have two parabolics $\mathscr{P}_i^{\mathrm{std}}$ and $\mathscr{P}_i$, with common Levi $\mathscr{M}_i$, associated with μ_i. We have proétale torsors over S_i given by

$$\mathcal{P}_{i,\mathrm{dR}}^{\mathrm{an}}(U) := \{\hat{\mathcal{O}}_{S_i} \otimes W_i|_U \xrightarrow{\sim} \mathcal{W}_{i,\mathrm{dR}}^{\mathrm{an}} \otimes_{\mathcal{O}_{S_i}} \hat{\mathcal{O}}_{S_i}|_U : \text{preserving Hodge filtration and Hodge tensors}\},$$
$$\mathcal{P}_{i,\mathrm{HT}}^{\mathrm{an}}(U) := \{\hat{\mathcal{O}}_{S_i} \otimes W_i|_U \xrightarrow{\sim} \mathcal{W}_{i,\text{ét}}^{\mathrm{an}} \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_{S_i}|_U : \text{preserving Hodge–Tate filtration and Hodge tensors}\},$$

where W_i is the defining representation of $\mathscr{G}_i$. These are $\mathscr{P}_i^{\mathrm{std}}$ and $\mathscr{P}_i$ torsors respectively. We denote by $\mathcal{M}_{i,\mathrm{dR}}^{\mathrm{an}}$ and $\mathcal{M}_{i,\mathrm{HT}}^{\mathrm{an}}$ their pushouts to $\mathscr{M}_i$. Then the results of [Caraiani and Scholze 2017] imply that $\mathcal{M}_{i,\mathrm{dR}}^{\mathrm{an}} \cong {}^{\mu}\mathcal{M}_{i,\mathrm{HT}}^{\mathrm{an}}$, where the twist is along μ_i as in Section 4.2. Note that

$$f(\mathscr{P}_1^{\mathrm{std}}) \subset \mathscr{P}_2^{\mathrm{std}}, \quad f(\mathscr{P}_1) \subset \mathscr{P}_2 \quad \text{and} \quad f(\mathscr{M}_1) \subset \mathscr{M}_2$$

by the assumption that $\mu_2 = f \circ \mu_1$.

Corollary B.2.4. *We have a commutative diagram of torsors*:

$$\begin{tikzcd} \mathcal{M}^{\mathrm{an}}_{1,\mathrm{dR}} \times^{\mathscr{M}_1} \mathscr{M}_2 \arrow[r] \arrow[d] & f^*\mathcal{M}^{\mathrm{an}}_{2,\mathrm{dR}} \arrow[d] \\ {}^{\mu}\mathcal{M}^{\mathrm{an}}_{1,\mathrm{HT}} \times^{\mathscr{M}_1} \mathscr{M}_2 \arrow[r] & f^*({}^{\mu}\mathcal{M}^{\mathrm{an}}_{2,\mathrm{HT}}) \end{tikzcd}$$

where the horizontal arrows are induced from the natural transformations $\eta^{\mathrm{an}}_{\mathrm{dR}}$ and $\eta^{\mathrm{an}}_{ét}$, and the vertical arrows are induced from the isomorphism of de Rham and twisted Hodge–Tate torsors above.

Proof. Let $\pi: A_2 \to S_2$ denote the universal abelian variety and $f^{-1}\pi: f^*A_2 \to S_1$ its pullback under f. To simplify notation, set $\mathcal{E} := \mathcal{H}^1_{\mathrm{dR}}(A_2/S_2)$, $\mathcal{E}' := \mathcal{H}^1_{\mathrm{dR}}(f^*A_2/S_1)$, $\mathbb{L} := R^1\pi_*\hat{\mathbb{Z}}_{p,A_2}$ and $\mathbb{L}' := R^1(f^{-1}\pi)_*\hat{\mathbb{Z}}_{p,f^*A_2}$. By Proposition B.2.2 and Remark B.2.3, we know that the following diagram commutes:

$$\begin{tikzcd} f^*(\mathcal{E} \otimes_{\mathcal{O}_{S_2}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_2}) \arrow[r, equal] \arrow[d] & f^*\mathcal{E} \otimes_{\mathcal{O}_{S_1}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1} \arrow[r, "\sim"] & \mathcal{E}' \otimes_{\mathcal{O}_{S_1}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1} \arrow[d] \\ f^*(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_2}) \arrow[r, equal] & f^*\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1} \arrow[r, "\sim"] & \mathbb{L}' \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1} \end{tikzcd}$$

where the horizontal arrows are the proper base-change isomorphisms and the vertical arrows (which are isomorphisms) arise from the comparisons of relative p-adic Hodge theory. The module $\mathcal{E} \otimes_{\mathcal{O}_{S_2}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_2}$ is an $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$-module with an integrable connection, so satisfies

$$\mathcal{E} \otimes_{\mathcal{O}_{S_2}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_2} = \mathbb{M}_0 \otimes_{\mathbb{B}^+_{\mathrm{dR},S_2}} \mathcal{O}\mathbb{B}_{\mathrm{dR},\mathrm{S}_2}$$

where $\mathbb{M}_0 = (\mathcal{E} \otimes_{\mathcal{O}_{S_2}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_2})^{\nabla=0}$; see [Scholze 2013, Theorem 7.2]. Hence

$$f^*(\mathcal{E} \otimes_{\mathcal{O}_{S_2}} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_2})^{\nabla=0} = (f^*\mathcal{E} \otimes_{\mathcal{O}_{S_1}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1})^{\nabla=0} = f^*\mathbb{M}_0.$$

Set $\mathbb{M}'_0 = (\mathcal{E}' \otimes_{\mathcal{O}_{S_1}} \mathcal{O}\mathbb{B}_{\mathrm{dR},S_1})^{\nabla=0}$, $\mathbb{M} = \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_2}$ and $\mathbb{M}' = \mathbb{L}' \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}^+_{\mathrm{dR},S_1}$. Since the base-change maps are compatible with structures, they induce isomorphisms

$$f^*\mathbb{M}_0 \xrightarrow{\sim} \mathbb{M}'_0, \tag{B.2.5}$$

$$f^*\mathbb{M} \xrightarrow{\sim} \mathbb{M}', \tag{B.2.6}$$

and we have a commutative diagram:

$$\begin{tikzcd} f^*\mathbb{M}_0 \otimes_{\mathbb{B}^+_{\mathrm{dR},S_1}} \mathbb{B}_{\mathrm{dR},S_1} \arrow[r, "(\mathrm{B.2.5})\otimes 1"] \arrow[d] & \mathbb{M}'_0 \otimes_{\mathbb{B}^+_{\mathrm{dR},S_1}} \mathbb{B}_{\mathrm{dR},S_1} \arrow[d] \\ f^*\mathbb{M} \otimes_{\mathbb{B}^+_{\mathrm{dR},S_1}} \mathbb{B}_{\mathrm{dR},S_1} \arrow[r, "(\mathrm{B.2.6})\otimes 1"] & \mathbb{M}' \otimes_{\mathbb{B}^+_{\mathrm{dR},S_1}} \mathbb{B}_{\mathrm{dR},S_1} \end{tikzcd}$$

where the vertical arrows are as in [Caraiani and Scholze 2017, Proposition 2.2.3]. In particular, considering the relative Hodge filtration as in [loc. cit.] and passing to gradeds, we have

$$\begin{tikzcd} f^* \operatorname{gr}^j \mathcal{E} \arrow[r] \arrow[d] & \operatorname{gr}^j \mathcal{E}' \arrow[d] \\ f^*(\operatorname{gr}_j(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathcal{O}}_{S_2})(j)) \arrow[r] & \operatorname{gr}_j(\mathbb{L}' \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathcal{O}}_{S_1})(j) \end{tikzcd}$$

for all $j \geq 0$. Note that the pullback f^* preserves the relevant filtrations because each graded piece is locally free. This last commutative diagram (or more precisely its dual version) describes the compatibility we desire in the statement of the corollary. Indeed, the isomorphism θ_{dR} in the proof of Proposition B.2.2 preserves Hodge filtrations, and by a similar argument above, one can show $\theta_{\text{ét}}$ preserves relative Hodge–Tate filtrations. Therefore the pushouts $\mathcal{P}^{\mathrm{an}}_{1,\mathrm{dR}} \times^{\mathscr{P}_1^{\mathrm{std}}} \mathscr{P}_2^{\mathrm{std}}$ and $\mathcal{P}^{\mathrm{an}}_{1,\mathrm{HT}} \times^{\mathscr{P}_1} \mathscr{P}_2$ can be described as frames of $\mathcal{H}_1^{\mathrm{dR}}(f^*A_2/S_1)$ and $\mathcal{H}_1^{\text{ét}}(f^*A_2/S_1)$ respectively, preserving Hodge tensors and filtrations. □

B.3. *Perfectoid Shimura varieties.* Continuing with the set-up as in the previous subsection, assume that K_i is of the form $K_i^p \times K_{i,p} \subset \mathscr{G}_i(\mathbb{A}_f^p) \times \mathscr{G}_i(\mathbb{Q}_p)$ for $i = 1, 2$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the adic spaces over $F_{\mathfrak{p}}$ associated with S_1 and S_2, and let $\mathcal{S}_{i,\infty}$ denote the perfectoid Shimura variety (of tame level K_i^p), as constructed in [Scholze 2015]. Then [Caraiani and Scholze 2017, Theorem 1.10], implies that we have a commutative diagram

$$\begin{tikzcd} \mathcal{S}_{1,\infty} \arrow[r, "\pi_{\mathrm{HT},1}"] \arrow[d] & \mathrm{FL}_1 \arrow[d] \\ \mathcal{S}_{2,\infty} \arrow[r, "\pi_{\mathrm{HT},2}"] & \mathrm{FL}_2 \end{tikzcd}$$

where FL_i denotes the adic flag variety associated with the Shimura datum $(\mathscr{G}_i, X_i)$ and $\pi_{\mathrm{HT},i}$ is the corresponding Hodge–Tate period morphism. Both of the vertical maps are induced from f.

For $i = 1, 2$ consider the torsor $\mathrm{FL}_i \times \mathscr{G}_i(\mathbb{Q}_p)$ with the right action of $\mathscr{G}_i(\mathbb{Q}_p)$ given by $(x, g) \cdot g' = (xg', (g')^{-1}g)$. We then obtain a torsor

$$\pi_{\mathrm{HT}}^*(\mathrm{FL}_i \times \mathscr{G}_i(\mathbb{Q}_p))/K_{i,p} = \mathcal{S}_{i,\infty} \times^{K_{i,p}} \mathscr{G}_i(\mathbb{Q}_p)$$

over $\mathcal{S}_i$. By the description of $\xi_{\text{ét}}$ as above, and the fact that $\mathcal{S}_{i,\infty}$ is essentially the limit $\varprojlim_{K'_{i,p}} \mathcal{S}_{\mathscr{G}_i, K_i^p K'_{i,p}}$, this torsor encodes $\xi_{\text{ét}}^{\mathrm{an}}$.

We have a natural map of torsors $\mathrm{FL}_1 \times \mathscr{G}_1(\mathbb{Q}_p) \to \mathrm{FL}_2 \times \mathscr{G}_2(\mathbb{Q}_p)$ induced from f, which is compatible with the equivariant structure. Pulling back along the Hodge–Tate period morphism and descending, we obtain a natural transformation $\eta' : \xi_{1,\text{ét}} \circ f^* \to f^* \circ \xi_{2,\text{ét}}$.

Lemma B.3.1. *The natural transformations $\eta_{\acute{e}t}^{\mathrm{an}}$ and η' coincide.*

Proof. This follows from the above commutative diagram and (on the level of topological spaces) the map $\mathcal{S}_{1,\infty} \to \mathcal{S}_{2,\infty}$ is the inverse limit of (the analytification of) maps $S_{\mathscr{G}_1, K'_1} \to S_{\mathscr{G}_2, K'_2}$. □

Appendix C: Unitary base change

In this appendix, we describe how the results on endoscopic classification of unitary groups in [Mok 2015] and [Kaletha et al. 2014] imply a certain strong multiplicity one theorem for automorphic representations of $\boldsymbol{G}(\mathbb{A})$. Note that these cited papers are conditional on the stabilization of the trace formula for unitary groups. Throughout, we let $\boldsymbol{G}$ and $\boldsymbol{G}_0$ be as in Section 2, and we write U for the unitary group over F^+ associated with W (so $\boldsymbol{G}_0 = \mathrm{Res}_{F^+/\mathbb{Q}} U$). As usual, we assume that F contains an imaginary quadratic number field E.

Lemma C.0.1. *Let ℓ be any (finite) rational prime. Then there exists a good special maximal compact open subgroup $K \subset \boldsymbol{G}(\mathbb{Q}_\ell)$ (as in [Mínguez 2011, Section 2.1]) such that the intersection $K \cap \boldsymbol{G}_0(\mathbb{Q}_\ell) \subset \boldsymbol{G}_0(\mathbb{Q}_\ell)$ is a good special maximal compact open subgroup. Furthermore, if $\boldsymbol{G}_{\mathbb{Q}_\ell}$ is unramified, we can arrange it so that both K and $K \cap \boldsymbol{G}_0(\mathbb{Q}_\ell)$ are hyperspecial.*

Proof. This follows from the fact that:

- $\boldsymbol{G}_0$ and $\boldsymbol{G}$ have the same adjoint group.
- The induced map from the Kottwitz group of $\boldsymbol{G}_0$ to that of $\boldsymbol{G}$ is injective. More precisely, the Kottwitz group of the former is $\mathbb{Z}/2\mathbb{Z}$, of the latter is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the induced map is inclusion into the second factor.

One then applies the description of all parahoric subgroups as in [Pappas and Rapoport 2008]. □

Let ℓ be a finite rational prime. Then the results of [Mok 2015; Kaletha et al. 2014] imply that there exists a local (standard) base change map BC_ℓ from irreducible admissible representations of $\boldsymbol{G}_0(\mathbb{Q}_\ell) \cong \prod_{v \mid \ell} U(F_v^+)$ to irreducible admissible representations of $\boldsymbol{G}_0(\mathbb{Q}_\ell \otimes_{\mathbb{Q}} E) \cong U(\mathbb{Q}_\ell \otimes_{\mathbb{Q}} F) \cong \prod_{w \mid \ell} \mathrm{GL}_{2n}(F_w)$ (this can be defined unconditionally if all primes above ℓ split in F/F^+, or if the group and representation are both unramified).

Lemma C.0.2. *Let ℓ be an odd rational prime. Let $K \subset \boldsymbol{G}(\mathbb{Q}_\ell)$ be a good special maximal compact open subgroup as in Lemma C.0.1, and let π and σ be irreducible admissible unitary representations of $\boldsymbol{G}(\mathbb{Q}_\ell)$. Suppose that:*

- *There exist irreducible admissible unitary representations π_0 and σ_0 of $\boldsymbol{G}_0(\mathbb{Q}_\ell)$ such that $\pi_0 \subset \pi|_{\boldsymbol{G}_0(\mathbb{Q}_\ell)}$, $\sigma_0 \subset \sigma|_{\boldsymbol{G}_0(\mathbb{Q}_\ell)}$ and $\mathrm{BC}_\ell(\pi_0) \cong \mathrm{BC}_\ell(\sigma_0)$.*
- *Both $\pi^K \neq 0$ and $\sigma^K \neq 0$.*

Then $\pi \cong \sigma$ or $\pi \cong \sigma \otimes \omega$, where ω is the quadratic character associated with $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell/\mathbb{Q}_\ell$. In particular, if ℓ ramifies or splits in $E/\mathbb{Q}$, then $\pi \cong \sigma$.

Proof. Suppose for the moment that π and σ share an irreducible constituent under the action of $\boldsymbol{G}_0(\mathbb{Q}_\ell)$. Then [Labesse and Schwermer 2019, Proposition 4.1.3] implies that $\pi \cong \sigma \otimes \chi$ for some character of the quotient $\boldsymbol{G}_0(\mathbb{Q}_\ell)\boldsymbol{Z}_{\boldsymbol{G}}(\mathbb{Q}_\ell)\backslash \boldsymbol{G}(\mathbb{Q}_\ell)$. But this quotient is contained in $N((E \otimes \mathbb{Q}_\ell)^\times)\backslash \mathbb{Q}_\ell^\times$ via the similitude character, where N denotes the norm map, hence χ is either the trivial character or the quadratic character ω. If ℓ is split then $\omega = 1$, otherwise if ℓ is ramified, then ω is ramified. But since π and σ are K-spherical, they cannot be isomorphic via a ramified twist. The latter is true because the image of the $\mathbb{Q}_\ell$-points of a Levi of a minimal parabolic in $\boldsymbol{G}_{\mathbb{Q}_\ell}$ under the similitude map contains $\mathbb{Z}_\ell^\times$ (by the structure of *even-dimensional* unitary groups in [Mínguez 2011, Example 3.2] and that any nontrivial quadratic form in two or more variables represents every element of $\mathbb{F}_\ell^\times$), and the fact that the intersection of the Levi with a good maximal special subgroup is the unique maximal compact open subgroup; see [loc. cit., Section 2.1].

We now show that π and σ share an irreducible constituent. Since $\pi^K \neq 0$, there exists an irreducible constituent $\pi_0' \subset \pi|_{\boldsymbol{G}_0(\mathbb{Q}_\ell)}$ which has nontrivial invariants under $K_0 := K \cap \boldsymbol{G}_0(\mathbb{Q}_\ell)$. Since K_0 is a good special maximal compact open subgroup, π_0' has a set of associated Satake parameters which is determined from the Satake parameters for π. Hence π_0 and π_0' have the same set of Satake parameters (but are spherical for different choices of special subgroups). This implies that $\mathrm{BC}_\ell(\pi_0) \cong \mathrm{BC}_\ell(\pi_0')$. By a similar argument for σ, we may replace π_0 and σ_0 by π_0' and σ_0' respectively. Now we note that the base-change map on Langlands/Arthur parameters is injective [Mok 2015, Section 2.2] hence π_0' and σ_0' have the same Satake parameters, as required. □

Now we discuss a global application of this lemma. Let S be a finite set of rational primes which split in $E/\mathbb{Q}$. Let $K = K^S \times K_S \subset \boldsymbol{G}(\mathbb{A}_f^S) \times \boldsymbol{G}(\mathbb{A}_S)$ be a compact open subgroup such that $K^S = \prod_{\ell \notin S} K_\ell$ with each $K_\ell \subset \boldsymbol{G}(\mathbb{Q}_\ell)$ a good special maximal compact open subgroup, which is hyperspecial if $\boldsymbol{G}_{\mathbb{Q}_\ell}$ is unramified. Let T denote a cofinite set of rational primes containing 2 and all primes which are inert in $E/\mathbb{Q}$.

Proposition C.0.3. *Let π and σ be cuspidal automorphic representations of $\boldsymbol{G}(\mathbb{A})$ such that π_∞ and σ_∞ are cohomological and $\pi_f^K \neq 0$ and $\sigma_f^K \neq 0$. Suppose that $\pi_\ell \cong \sigma_\ell$ for all $\ell \in T - (S \cap T)$. Also, suppose that the weak base-change of π to $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_{2n}(\mathbb{A}_F)$ is cuspidal. Then $\pi_f \cong \sigma_f$.*

Proof. Since π_∞ and σ_∞ are cohomological, they admit weak base-changes by [Shin 2014]. These weak base-changes must be isomorphic by our assumptions and strong multiplicity one, and hence also cuspidal by assumption. By [Labesse and Schwermer 2019, Theorem 5.2.1], we can find cuspidal automorphic representations π_0 and σ_0 of $\boldsymbol{G}_0(\mathbb{A})$ such that $\pi_0 \subset \pi|_{\boldsymbol{G}_0(\mathbb{A})}$ and $\sigma_0 \subset \sigma|_{\boldsymbol{G}_0(\mathbb{A})}$. By compatibility of base-change for unitary and unitary similitude groups, the weak-base changes of π_0 and σ_0 are isomorphic (and cuspidal). Call the common representation Π_0. By the theory of endoscopy (see [Liu et al. 2022, Proposition C.3.1(2)]), we actually have the stronger compatibility $\mathrm{BC}_\ell(\pi_{0,\ell}) \cong \Pi_{0,\ell} \cong \mathrm{BC}_\ell(\sigma_{0,\ell})$ for all rational primes ℓ. Then:

(1) If $\ell \in S$, then the weak base-changes of π and σ are locally isomorphic at ℓ [Shin 2014, Theorem A.1(2)]. Since local base-change is injective at these primes, we have $\pi_\ell \cong \sigma_\ell$.

(2) If $\ell \notin S \cup \{2\}$ and ramifies or splits in $E/\mathbb{Q}$, then we have $\pi_\ell \cong \sigma_\ell$ by Lemma C.0.2.

(3) If $\ell \notin S$ and is inert in $E/\mathbb{Q}$, or $\ell = 2$, then $\ell \in T - (S \cap T)$ and $\pi_\ell \cong \sigma_\ell$ by assumption.

This completes the proof. □

Acknowledgements

The author would like to thank Daniel Barrera Salazar, George Boxer, David Loeffler, Vincent Pilloni, Juan Esteban Rodríguez Camargo, Chris Williams and Sarah Zerbes for helpful comments and discussions. The author is also grateful to the anonymous referee for their feedback and pointing out a significant error in an earlier version of this article. This work was supported in part by ERC-2018-COG-818856-HiCoShiVa.

References

[Ancona 2015] G. Ancona, "Décomposition de motifs abéliens", *Manuscripta Math.* **146**:3-4 (2015), 307–328. MR Zbl

[Blasius 1994] D. Blasius, "A p-adic property of Hodge classes on abelian varieties", pp. 293–308 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55, Part 2**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Blasius et al. 1994] D. Blasius, M. Harris, and D. Ramakrishnan, "Coherent cohomology, limits of discrete series, and Galois conjugation", *Duke Math. J.* **73**:3 (1994), 647–685. MR Zbl

[Boxer and Pilloni 2021] G. Boxer and V. Pilloni, "Higher Coleman theory", preprint, 2021. Zbl arXiv 2110.10251

[Caraiani 2012] A. Caraiani, "Local-global compatibility and the action of monodromy on nearby cycles", *Duke Math. J.* **161**:12 (2012), 2311–2413. MR Zbl

[Caraiani and Scholze 2017] A. Caraiani and P. Scholze, "On the generic part of the cohomology of compact unitary Shimura varieties", *Ann. of Math.* (2) **186**:3 (2017), 649–766. MR Zbl

[Chen and Gan 2021] R. Chen and W. T. Gan, "Unitary Friedberg–Jacquet periods", preprint, 2021. Zbl arXiv 2108.04064

[Chen and Zou 2021] R. Chen and J. Zou, "Arthur's multiplicity formula for even orthogonal and unitary groups", preprint, 2021. Zbl arXiv 2103.07956v1

[Collins 2020] D. J. Collins, "Anticyclotomic p-adic L-functions and Ichino's formula", *Ann. Math. Qué.* **44**:1 (2020), 27–89. MR Zbl

[Deligne et al. 1982] P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, *Hodge cycles*, *motives*, *and Shimura varieties*, Lecture Notes in Mathematics **900**, Springer, 1982. MR Zbl

[Diao et al. 2023] H. Diao, K.-W. Lan, R. Liu, and X. Zhu, "Logarithmic Riemann–Hilbert correspondences for rigid varieties", *J. Amer. Math. Soc.* **36**:2 (2023), 483–562. MR Zbl

[Friedberg and Jacquet 1993] S. Friedberg and H. Jacquet, "Linear periods", *J. Reine Angew. Math.* **443** (1993), 91–139. MR Zbl

[Graham and Shah 2023] A. Graham and S. W. A. Shah, "Anticyclotomic Euler systems for unitary groups", *Proc. Lond. Math. Soc.* (3) **127**:6 (2023), 1577–1680. MR Zbl

[Grobner and Raghuram 2014] H. Grobner and A. Raghuram, "On the arithmetic of Shalika models and the critical values of L-functions for GL_{2n}", *Amer. J. Math.* **136**:3 (2014), 675–728. MR Zbl

[Harris 1990] M. Harris, "Automorphic forms of $\overline{\partial}$-cohomology type as coherent cohomology classes", *J. Differential Geom.* **32**:1 (1990), 1–63. MR Zbl

[Jacquet et al. 1981] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, "Conducteur des représentations du groupe linéaire", *Math. Ann.* **256**:2 (1981), 199–214. MR Zbl

[Kaletha et al. 2014] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, "Endoscopic classification of representations: Inner forms of unitary groups", preprint, 2014. Zbl arXiv 1409.3731

[Knapp 2001] A. W. Knapp, "Branching theorems for compact symmetric spaces", *Represent. Theory* **5** (2001), 404–436. MR Zbl

[Labesse and Schwermer 2019] J.-P. Labesse and J. Schwermer, "Central morphisms and cuspidal automorphic representations", *J. Number Theory* **205** (2019), 170–193. MR Zbl

[Lan 2013] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series **36**, Princeton University Press, 2013. MR Zbl

[Lan and Polo 2018] K.-W. Lan and P. Polo, "Dual BGG complexes for automorphic bundles", *Math. Res. Lett.* **25**:1 (2018), 85–141. MR Zbl

[Leslie 2019a] S. Leslie, "The endoscopic fundamental lemma for unitary Friedberg–Jacquet periods", preprint, 2019. Zbl arXiv 1911.07907

[Leslie 2019b] S. Leslie, "Endoscopy for unitary symmetric spaces", preprint, 2019. Zbl arXiv 1910.09685

[Liu et al. 2022] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, "On the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives", *Invent. Math.* **228**:1 (2022), 107–375. MR Zbl

[Loeffler and Zerbes 2016] D. Loeffler and S. L. Zerbes, "Rankin–Eisenstein classes in Coleman families", *Res. Math. Sci.* **3** (2016), art. it. 29. MR Zbl

[Loeffler and Zerbes 2021] D. Loeffler and S. L. Zerbes, "On the Bloch–Kato conjecture for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$", preprint, 2021. Zbl arXiv 2106.14511

[Loeffler et al. 2021] D. Loeffler, V. Pilloni, C. Skinner, and S. L. Zerbes, "Higher Hida theory and p-adic L-functions for GSp_4", *Duke Math. J.* **170**:18 (2021), 4033–4121. MR Zbl

[Mazur and Rubin 2004] B. Mazur and K. Rubin, "Kolyvagin systems", 799 (2004), viii+96. MR Zbl

[Milne 1990] J. S. Milne, "Canonical models of (mixed) Shimura varieties and automorphic vector bundles", pp. 283–414 in *Automorphic forms, Shimura varieties, and L-functions, I* (Ann Arbor, MI, 1988), edited by L. Clozel and J. S. Milne, Perspect. Math. **10**, Academic Press, Boston, 1990. MR Zbl

[Mínguez 2011] A. Mínguez, "Unramified representations of unitary groups", pp. 389–410 in *On the stabilization of the trace formula*, edited by L. Clozel et al., Stab. Trace Formula Shimura Var. Arith. Appl. **1**, International Press, Somerville, MA, 2011. MR Zbl

[Mok 2015] C. P. Mok, "Endoscopic classification of representations of quasi-split unitary groups", 1108 (2015), vi+248. MR Zbl

[Pappas and Rapoport 2008] G. Pappas and M. Rapoport, "Twisted loop groups and their affine flag varieties", *Adv. Math.* **219**:1 (2008), 118–198. MR Zbl

[Pollack et al. 2021] A. Pollack, C. Wan, and M. Zydor, "On the residue method for period integrals", *Duke Math. J.* **170**:7 (2021), 1457–1515. MR Zbl

[Rubin 2000] K. Rubin, *Euler systems*, Annals of Mathematics Studies **147**, Princeton University Press, 2000. MR Zbl

[Scholze 2013] P. Scholze, "p-adic Hodge theory for rigid-analytic varieties", *Forum Math. Pi* **1** (2013), e1, 77. MR Zbl

[Scholze 2015] P. Scholze, "On torsion in the cohomology of locally symmetric varieties", *Ann. of Math.* (2) **182**:3 (2015), 945–1066. MR Zbl

[SGA 4_3 1973] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 3: Exposés IX–XIX* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Math. **305**, Springer, 1973. MR Zbl

[Shin 2014] S. W. Shin, "On the cohomological base change for unitary similitude groups", (2014). Appendix to W. Goldring, "Galois representations associated to holomorphic limits of discrete series", *Compos. Math.* **150**:2 (2014), 191–228. MR Zbl

[Su 2019] J. Su, *Coherent cohomology of Shimura varieties and automorphic forms*, Ph.D. thesis, Princeton University, 2019, available at https://www.proquest.com/docview/2272840927. MR Zbl

[Torzewski 2020] A. Torzewski, "Functoriality of motivic lifts of the canonical construction", *Manuscripta Math.* **163**:1-2 (2020), 27–56. MR Zbl

Communicated by Shou-Wu Zhang
Received 2021-12-17 Revised 2022-07-19 Accepted 2023-09-03

agraham@mpim-bonn.mpg.de *MPIM-Bonn, Bonn, Germany*

ALGEBRA AND NUMBER THEORY 18:6 (2024)
https://doi.org/10.2140/ant.2024.18.1189

Enumeration of conjugacy classes in affine groups

Jason Fulman and Robert M. Guralnick

Dedicated to Pham Huu Tiep on the occasion of his 60th birthday

We study the conjugacy classes of the classical affine groups. We derive generating functions for the number of classes analogous to formulas of Wall and the authors for the classical groups. We use these to get good upper bounds for the number of classes. These naturally come up as difficult cases in the study of the noncoprime $k(GV)$ problem of Brauer.

1. Introduction

Let G be the group of affine transformations of a vector space V over a finite field. In this paper we derive generating functions for the number of conjugacy classes in this group and in the analogs for the other classical groups. For finite classical groups (not their affine versions), such generating functions were mostly obtained by Wall [1963] (see also [Fulman and Guralnick 2012] for orthogonal and symplectic groups in even characteristic). Besides the natural motivation for considering this, this is one of the most difficult cases in the noncoprime $k(GV)$ problem introduced by Brauer to obtain results about characters. This asks for bounds on the number of conjugacy classes $k(H)$, where H is a group with a normal abelian subgroup V. One of the major results in this area, based on work of many authors over a long period, is that $k(H) \leq |V|$ if V is its own centralizer in H and $\gcd(|H/V|, |V|) = 1$. In fact there is an entire book devoted to this topic [Schmid 2007]. It turns out if we weaken this assumption, the result is no longer true but it still is close. One critical case is when $L = H/V$ acts irreducibly on V (see [Guralnick and Tiep 2005] for reductions and for connections with representation theory). See [Guralnick and Maróti 2013; Guralnick and Tiep 2005; Keller 2006; Robinson 2004] for background and other results. One would like to prove that $k(H) < c|V|$ for some absolute constant c (under suitable hypotheses). Another motivation for studying this is the relationship with the conjugacy classes of the largest maximal parabolic subgroup of the classical groups. See [Nakada and Shinoda 1990] for the case of GL.

In [Guralnick and Tiep 2005], the focus was on the important case when L is close to simple and the same bound was proved in almost all cases studied. One of the main cases left open was the case that V is the natural module for a classical group L. It turns out that again aside from the case of $\mathrm{AGL}(n, q)$,

Fulman was partially funded by a Simons Foundation Grant 400528. Guralnick was partially supported by the NSF grant DMS-1901595 and Simons Foundation Fellowship 609771. We thank the referee for helpful comments.
MSC2020: 05A15, 20C99.
Keywords: affine groups, number of conjugacy classes, generating function.

© 2024 The Authors, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

the bound generally holds. We show that $q^n \leq k(\mathrm{AGL}(n, q)) < (q^{n+1} - 1)/(q - 1) < 2q^n$ and obtain explicit and useful bounds in the analogs for other classical groups.

Variations on this theme and some other small families that were not considered in [Guralnick and Tiep 2005] will be studied in a sequel.

The paper is organized as follows.

Section 2 gives some preliminaries which are fundamental to our two approaches for calculating exact generating functions for $k(\mathrm{AGL})$, $k(\mathrm{AGU})$ and $k(\mathrm{ASp})$ and $k(\mathrm{AO})$. The first approach writes $k(AG)$ as a weighted sum over conjugacy classes of G. We work this out for all cases except for the famously difficult cases of characteristic two symplectic and orthogonal groups. Our second approach enumerates irreducible representations instead of conjugacy classes. This allows us to calculate $k(AG)$ recursively, and has the additional benefit of working in both odd and even characteristic.

Section 3 treats $k(\mathrm{AGL}(n, q))$, and also $k(AH)$, where H is a group between $\mathrm{GL}(n, q)$ and $\mathrm{SL}(n, q)$. Section 4 treats $k(\mathrm{AGU}(n, q))$ and $k(AH)$, where H is a group between $\mathrm{GU}(n, q)$ and $\mathrm{SU}(n, q)$. Section 5 treats the case $\mathrm{ASp}(2n, q)$. Section 6 treats $\mathrm{AO}(n, q)$.

We dedicate this paper to Pham Huu Tiep, our friend and colleague, on the occasion of his 60th birthday. We note that he has done substantial work on the noncoprime $k(GV)$ problem; see [Guralnick and Tiep 2005].

2. Preliminaries

Let G be a finite group and let k be a finite field with A a finite dimensional kG-module. Then we consider the group $H = AG$, the semidirect product of the normal subgroup A and G. We say that H is the corresponding affine group. We will usually take A to be irreducible (and by replacing k by $\mathrm{End}_G(A)$, we can assume that A is absolutely irreducible).

Our first approach, which we call the *orbit* approach, expresses $k(AG)$ as a weighted sum over conjugacy classes of G. To describe this, let $[g, A]$ denote $(I - g)A$, where I is the identity map. The number of orbits of the centralizer $C_G(g)$ on $A/[g, A]$ depends only on the conjugacy class C of g, and we denote it by $o(C)$. If g and x are elements of a group G, then we let $x^g = g^{-1}xg$.

Lemma 2.1. *Let G and A be as above. Then*

$$k(AG) = \sum_C o(C),$$

where the sum is over all conjugacy classes C of G.

Proof. Let $g \in C$ with C a conjugacy class of G. We need to show that the number of conjugacy classes of elements $h \in AG$ such that h is conjugate to some element of gA is the number of orbits of $C_G(g)$ on $A/[g, A]$.

Suppose that $h = ga$. Suppose that gc is conjugate to ga. Note that

$$\{(ga)^b \mid b \in A\} = ga[g, A].$$

Thus if $a, c \in A$, ga and gc are conjugate in H if and only if $a[g, A]$ and $c[g, A]$ are in the same $C_G(g)$ orbit on $A/[g, A]$, whence the result. □

In this paper we find (for all cases except even characteristic symplectic and orthogonal groups) exact formulas for $o(C)$, which may be of independent interest. We then use these formulas, together with generating functions for $k(G)$, to find exact generating functions for $k(AG)$.

Our second approach, which we call the *character* approach, counts irreducible representations instead of conjugacy classes. This leads to recursive expressions for $k(AG)$. Together with known generating functions for $k(G)$, this enables us to obtain exact generating functions for $k(AG)$. One nice feature of the character approach is that it works in both odd and even characteristic.

Crucial to the character approach is the next lemma, which is a well known elementary exercise.

Lemma 2.2. *Let G be a finite group and V a finite G-module. Let $J = VG$ be the semidirect product. Let Δ be a set of G-orbit representatives on the set of irreducible characters of V. Then*

$$k(J) = \sum_{\delta\in\Delta} k(G_\delta),$$

where G_δ is the stabilizer of the character δ in G.

Proof. Let W be an irreducible $\mathbb{C}J$-module. Let δ be a character of V that occurs in W and set W_δ to be the δ eigenspace of V. Note that δ is unique up to G-conjugacy and that the stabilizer of W_δ in J is precisely $J_\delta = VG_\delta$. Thus, G_δ acts irreducibly on W_δ. Conversely given any irreducible G_δ-module U, we can extend it to a J_δ module by having V act via δ. Then inducing U from J_δ to J gives an irreducible J-module. Thus, we see that $k(J) = \sum_{\delta\in\Delta} k(G_\delta)$ as required. □

The following lemma is Euler's pentagonal number theorem (see for instance page 11 of [Andrews 1976]).

Lemma 2.3. *For $q > 1$,*

$$\begin{aligned}\prod_{i\geq 1}\left(1-\frac{1}{q^i}\right) &= 1+\sum_{n=1}^{\infty}(-1)^n\left(q^{-n(3n-1)/2}+q^{-n(3n+1)/2}\right)\\ &= 1-q^{-1}-q^{-2}+q^{-5}+q^{-7}-q^{-12}-q^{-15}+\cdots.\end{aligned}$$

A few times in this paper quantities which can be easily re-expressed in terms of the infinite product $\prod_{i=1}^{\infty}(1-1/q^i)$ will arise, and Lemma 2.3 gives arbitrarily accurate upper and lower bounds on these products. Hence we will state bounds like

$$\prod_{i=1}^{\infty}\left(1+\frac{1}{2^i}\right) = \prod_{i=1}^{\infty}\frac{\left(1-\frac{1}{4^i}\right)}{\left(1-\frac{1}{2^i}\right)} \leq 2.4$$

without explicitly mentioning Euler's pentagonal number theorem on each occasion.

We also use the following well-known lemma (see for instance [Odlyzko 1995]).

Lemma 2.4. *Suppose that $f(u)$ is analytic for $|u| < R$. Let $M(r)$ denote the maximum of $|f|$ restricted to the circle $|u| = r$. Then for any $0 < r < R$, the coefficient of u^n in $f(u)$ has absolute value at most $M(r)/r^n$.*

As a final bit of notation, we let $|\lambda|$ denote the size of a partition λ.

3. AGL and related groups

Section 3A uses the orbit approach to calculate the generating function for $k(\mathrm{AGL}(n, q))$. Section 3B uses the character approach to calculate the generating function for $k(\mathrm{AGL}(n, q))$ and related groups. Section 3C uses these generating functions to obtain bounds on $k(\mathrm{AGL}(n, q))$ and related groups.

3A. *Orbit approach to k(AGL).* We use Lemma 2.1 to determine a generating function for the numbers $k(\mathrm{AGL}(n, q))$.

The following lemma calculates $o(C)$ for a conjugacy class C of $\mathrm{GL}(n, q)$. This formula involves the number of distinct part sizes of a partition λ, which we denote by $d(\lambda)$. For example if λ has 5 parts of size 4, 3 parts of size 2, and 4 parts of size 1, then $d(\lambda) = 3$. If λ is the empty partition, then $d(\lambda) = 0$.

Lemma 3.1. *Let C be a conjugacy class of $\mathrm{GL}(n, q)$, and let $\lambda_{z-1}(C)$ be the partition corresponding to the eigenvalue 1 in the rational canonical form of an element of C. Then*

$$o(C) = d(\lambda_{z-1}(C)) + 1.$$

Proof. Let V be the natural module for $\mathrm{GL}(n, q)$. Let $g \in C$ and let $C(g)$ denote the centralizer of g in $\mathrm{GL}(V)$. Write $V = V_1 \oplus V_2$ where $V_2 = \ker(g - I)^n$. Note that $[g, V] = V_1 \oplus V_2/[g, V_2]$ and the centralizer of g preserves this decomposition. Thus, we may assume that $V = V_2$, i.e., we may assume that g is unipotent.

Now write $V = V_1 \oplus \cdots \oplus V_m$, where $g\,|\,V_i$ has all Jordan blocks of size i. We only consider the nonzero V_i. So $d_i = \dim V_i/[g, V_i]$ is the number of Jordan blocks of size i. It is well known that the centralizer of g induces the full $\mathrm{GL}(d_i, q)$ and in particular any two nonzero elements of $V_i/[g, V_i]$ are in the same $C(g)$ orbit.

Consider gv with $v = v_1 + \cdots + v_m$ with $v_i \in V_i$. Note that if $h \in C(g)$, then $hV_i \subset V_1 \oplus \cdots \oplus V_i + [g, V]$. Thus, two elements in V which are in the same $C(g)$-orbit module $[g, V]$ must have the same highest nonzero (modulo $[g, V]$) term. Conversely, we need to show that any two such vectors are in the same orbit and indeed are in the orbit of v_j with $v_j \in V_j \setminus [g, V_j]$. By induction, we may assume that $j = m$. Note that there exists $h \in C(g)$ so that h is trivial on $V/\sum_{e<m} V_e$ and $hv_m - v_m$ is an arbitrary element in $\bigoplus_{e<m} V_e/[g, V_e]$. Thus, we see that v and v_m are in the same orbit. Since $C(g)$ induces $\mathrm{GL}(d_m, q)$ on $V_m/[g, V_m]$ we see that orbit representatives for $C(g)$ on $V[g, V]$ are 0 and one vector $w_i \in V_i$ for each nonzero V_i. The result follows. □

The following interesting identity will be helpful.

Lemma 3.2.
$$\sum_{\lambda}[d(\lambda)+1]u^{|\lambda|}=\frac{1}{1-u}\prod_{i\geq 1}\frac{1}{1-u^i}.$$

Proof. Clearly
$$\sum_{\lambda}q^{d(\lambda)}u^{|\lambda|}=\prod_{i\geq 1}\left(1+\frac{qu^i}{1-u^i}\right).$$
Differentiate this equation with respect to q and then set $q=1$. The left hand side becomes
$$\sum_{\lambda}d(\lambda)u^{|\lambda|}.$$
By the product rule, the right hand side becomes
$$\sum_{i\geq 1}\frac{u^i}{1-u^i}\prod_{j\neq i}\left(1+\frac{u^j}{1-u^j}\right)=\sum_{i\geq 1}\frac{u^i}{1-u^i}\prod_{j\neq i}\left(\frac{1}{1-u^j}\right)=\left(\sum_{i\geq 1}u^i\right)\prod_{j\geq 1}\frac{1}{1-u^j}.$$
Thus
$$\sum_{\lambda}d(\lambda)u^{|\lambda|}=\left(\sum_{i\geq 1}u^i\right)\prod_{j\geq 1}\frac{1}{1-u^j}. \tag{1}$$
Since
$$\sum_{\lambda}u^{|\lambda|}=\prod_{j\geq 1}\frac{1}{1-u^j},$$
it follows from (1) that
$$\sum_{\lambda}[d(\lambda)+1]u^{|\lambda|}=\left(\sum_{i\geq 0}u^i\right)\prod_{j\geq 1}\frac{1}{1-u^j}=\frac{1}{1-u}\prod_{j\geq 1}\frac{1}{1-u^j},$$
as claimed. □

In what follows, for $d\geq 1$, we let $N(q;d)$ denote the number of monic irreducible polynomials $\phi(z)$ of degree d over F_q for which $\phi(0)\neq 0$, that is monic irreducible polynomials other than z.

The following well known identity (see for example Theorem 3.25 of [Lidl and Niederreiter 1994]) will be useful.

Lemma 3.3.
$$\prod_{d\geq 1}(1-u^d)^{-N(q;d)}=\frac{1-u}{1-qu}.$$

Theorem 3.4 derives a generating function for the number of conjugacy classes in $\mathrm{AGL}(n,q)$.

Theorem 3.4.
$$1+\sum_{n\geq 1}k(\mathrm{AGL}(n,q))u^n=\frac{1}{1-u}\prod_{i\geq 1}\frac{1-u^i}{1-qu^i}.$$

Proof. By Lemma 2.1,
$$1+\sum_{n\geq 1}k(\mathrm{AGL}(n,q))u^n=1+\sum_{n\geq 1}u^n\sum_{C}o(C),$$
where the sum is over all conjugacy classes C of $\mathrm{GL}(n,q)$.

Since conjugacy classes of $\mathrm{GL}(n,q)$ correspond to rational canonical forms, it follows from the previous equation and Lemma 3.1 that

$$1+\sum_{n\geq 1}k(\mathrm{AGL}(n,q))u^n = \Big(\sum_{\lambda}[d(\lambda)+1]u^{|\lambda|}\Big)\Big(\sum_{\lambda}u^{|\lambda|}\Big)^{N(q;1)-1}\prod_{d\geq 2}\Big(\sum_{\lambda}u^{d|\lambda|}\Big)^{N(q;d)}$$
$$= \Big(\sum_{\lambda}[d(\lambda)+1]u^{|\lambda|}\Big)\prod_{i\geq 1}\Big(\frac{1}{1-u^i}\Big)^{N(q;1)-1}\prod_{d\geq 2}\prod_{i\geq 1}\Big(\frac{1}{1-u^{di}}\Big)^{N(q,d)}.$$

By Lemma 3.2 this is equal to

$$\frac{1}{1-u}\prod_{d\geq 1}\prod_{i\geq 1}\Big(\frac{1}{1-u^{di}}\Big)^{N(q;d)} = \frac{1}{1-u}\prod_{i\geq 1}\prod_{d\geq 1}\Big(\frac{1}{1-u^{di}}\Big)^{N(q;d)}.$$

Applying Lemma 3.3, this simplifies to

$$\frac{1}{1-u}\prod_{i\geq 1}\frac{1-u^i}{1-qu^i},$$

as claimed. □

3B. *Character approach to $k(\mathrm{AGL})$ and related groups.* We apply Lemma 2.2. Note that if δ is the trivial character, then $G_\delta = G$. We recall the case of $G=\mathrm{GL}(n,q)$ with V the natural module. The group J is usually denoted as $\mathrm{AGL}(n,q)$ the affine general linear group. Note that in this case $|\Delta|=2$. Note that the stabilizer of a nontrivial linear character is isomorphic to $\mathrm{AGL}(n-1,q)$ and so:

Lemma 3.5. $k(\mathrm{AGL}(n,q)) = k(\mathrm{GL}(n,q)) + k(\mathrm{AGL}(n-1,q)) = 1+\sum_{m=1}^{n} k(\mathrm{GL}(m,q)).$

As a corollary, we get another proof of Theorem 3.4.

Proof. Lemma 3.5 implies that

$$1+\sum_{n\geq 1}k(\mathrm{AGL}(n,q))u^n = \frac{1}{1-u}\Big(1+\sum_{n\geq 1}k(\mathrm{GL}(n,q))u^n\Big).$$

The result now follows from Macdonald's theorem [1981]

$$1+\sum_{n\geq 1}k(\mathrm{GL}(n,q))u^n = \prod_{i\geq 1}\frac{1-u^i}{1-qu^i}.$$ □

Lemma 3.6. *Fix q and let $n\geq 2$. Let $\mathrm{SL}(n,q)\leq H = H(n,q)\leq \mathrm{GL}(n,q)$ with $e=[H:\mathrm{SL}(n,q)]$.*

(1) $k(AH) = k(H) + k(AH(n-1,q))$.

(2) $k(AH) = (q-1)/e + \sum_{i=1}^{n} k(H(i,q))$.

Proof. The first statement follows exactly as in the proof of the case of $\mathrm{GL}(n,q)$. Note that $k(AH(1,q)) = e+(q-1)/e = k(H(1,q)) + (q-1)/e$.

So iterating, we see that

$$k(AH) = k(AH(1,q)) + \sum_{j=2}^{n} k(H(j,q)) = (q-1)/e + \sum_{i=1}^{n} k(H(i,q)). \qquad \square$$

3C. *Bounds on* k(AGL) *and related groups.* There is an interesting corollary of Theorem 3.4. If $f(u) = \sum_{n\geq 0} f(n)u^n$ and $g(u) = \sum_{n\geq 0} g(n)u^n$, we use the notation $f \gg g$ to mean that $f(n) \geq g(n)$ for all $n \geq 0$.

Corollary 3.7. *$k(\mathrm{AGL}(1,q)) = q$ and for $n \geq 2$,*

$$q^n < k(\mathrm{AGL}(n,q)) < 2q^n.$$

Proof. By Theorem 3.4, the fact that $q^n \leq k(\mathrm{AGL}(n,q))$ is equivalent to the statement that

$$\frac{1}{1-u}\prod_{i\geq 1}\frac{1-u^i}{1-qu^i} \gg \frac{1}{1-uq}.$$

Now notice that

$$\frac{1}{1-u}\prod_{i\geq 1}\frac{1-u^i}{1-qu^i} = \frac{1}{1-uq}\prod_{i\geq 2}\frac{1-u^i}{1-qu^i} \gg \frac{1}{1-uq},$$

where the last step follows since $(1-u^i)/(1-qu^i) \gg 1$. In fact this argument shows that the strict inequality $q^n < k(\mathrm{AGL}(n,q))$ holds for $n \geq 2$, since the coefficient of u^i in $(1-u^i)/(1-qu^i)$ is positive.

For a second proof that $q^n \leq k(\mathrm{AGL}(n,q))$ with strict inequality if $n \geq 2$, note that $k(\mathrm{GL}(n,q))$ is at least $q^n - q^{n-1}$ and indeed is strictly greater for $n > 1$, since there are $q^n - q^{n-1}$ semisimple classes (i.e., different characteristic polynomials) and for $n > 1$, there are unipotent classes as well. Now use the fact (Lemma 3.5) that

$$k(\mathrm{AGL}(n,q)) = 1 + \sum_{m=1}^{n} k(\mathrm{GL}(m,q)).$$

For the upper bound, we know from [Maslen and Rockmore 1997] that $k(\mathrm{GL}(m,q)) < q^m$ for all m. So again by Lemma 3.5,

$$k(\mathrm{AGL}(n,q)) \leq q^n + q^{n-1} + \cdots + 1 < 2q^n. \qquad \square$$

Finally, we give a result for AH where H is between GL and SL.

Theorem 3.8. *Fix q and let $\mathrm{SL}(n,q) \leq H = H(n,q) \leq \mathrm{GL}(n,q)$ with $e = [H : \mathrm{SL}(n,q)] < q-1$. Then $k(AH) < q^n$ except for $k(\mathrm{ASL}(1,q)) = q$ and $k(\mathrm{ASL}(2,3)) = 10$.*

Proof. Suppose that $n = 1$. Then as noted in Lemma 3.6, $k(AH(1,q)) = e + (q-1)/e$. Now if $e + (q-1)/e \geq q$, then $e^2 - 1 \geq q(e-1)$. So either $e - 1 = 0$ or $e + 1 \geq q$. But $e < q - 1$ so the only remaining possibility is $n = 1,\ e = 1$, as claimed.

Now we suppose that $n \geq 2$. From [Fulman and Guralnick 2012], $k(H) \leq e \cdot k(SL(n,q))$. So from Lemma 3.6,

$$k(AH) \leq \frac{q-1}{e} + e[k(\mathrm{SL}(1,q)) + \cdots + k(\mathrm{SL}(n,q))].$$

From [Fulman and Guralnick 2012], $k(\mathrm{SL}(j,q)) \leq 2.5q^{j-1}$. Thus

$$k(AH) \leq \frac{q-1}{e} + 2.5e\frac{q^n-1}{q-1}.$$

We claim that if $(q-1)/e \geq 3$, then

$$\frac{q-1}{e} + 2.5e\frac{q^n-1}{q-1} \leq q^n.$$

Indeed, if $(q-1)/e \geq 3$, then

$$\frac{q-1}{e} + 2.5e\frac{q^n-1}{q-1} \leq \frac{q-1}{e} + (q^n-1)\frac{2.5}{3}.$$

Since $(q-1)/e \geq 3$, we have that $q \geq 4$, and it is easy to check that if $q \geq 4$, then

$$\frac{q-1}{e} + (q^n-1)\frac{2.5}{3} \leq q^n.$$

Since $e < q-1$, the remaining case is that $(q-1)/e = 2$. Since $(q-1)/e$ is even, we can assume that q is odd. Then by Proposition 3.8 of [Fulman and Guralnick 2012],

$$k(H) = \begin{cases} \frac{1}{2}k(\mathrm{GL}(n,q)) & \text{if } n \text{ is odd,} \\ \frac{1}{2}k(\mathrm{GL}(n,q)) + \frac{3}{2}k(\mathrm{GL}(n/2,q)) & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Using the fact that $k(\mathrm{GL}(j,q)) < q^j$ and Lemma 3.6, one easily checks that if $q \geq 5$, then $k(AH) \leq q^n$. Similarly if $q = 3$ (so $e = 1$ and $H = \mathrm{SL}$), it is not hard to see that $k(\mathrm{ASL}(2,3)) = 10$ and that $k(\mathrm{ASL}(n,3)) < 3^n$ otherwise. □

4. AGU and related groups

Section 4A uses the orbit approach to calculate the generating function for $k(\mathrm{AGU}(n,q))$. Section 4B uses the character approach to calculate the generating function for $k(\mathrm{AGU}(n,q))$. Section 4C uses this generating function to obtain bounds on the number of conjugacy classes of $\mathrm{AGU}(n,q)$ and related groups.

4A. *Orbit approach to k*(AGU). This section uses the orbit approach to calculate the generating function for $k(\mathrm{AGU}(n,q))$.

The following theorem calculates $o(C)$ for a conjugacy class C of $\mathrm{GU}(n,q)$. This only involves $\lambda_{z-1}(C)$, the partition corresponding to the eigenvalue 1 in the rational canonical form of the conjugacy class C. As in the GL case, let $d(\lambda)$ be the number of distinct parts of the partition λ. In what follows we also let $b(\lambda)$ denote the number of part sizes of λ which have multiplicity exactly 1.

Theorem 4.1. *Let C be a conjugacy class of* $\mathrm{GU}(n, q)$. *Then*

$$o(C) = 1 + q \cdot d(\lambda_{z-1}(C)) - b(\lambda_{z-1}(C)).$$

Proof. It suffices to assume that C consists of unipotent elements and so corresponds to a partition λ. The proof is similar to the case of GL.

Now write $V = V_1 \oplus \cdots \oplus V_m$ where $g|V_i$ has all Jordan blocks of size i. We only consider the nonzero V_i. So $d_i = \dim V_i/[g, V_i]$ is the number of Jordan blocks of size i. It is well known that the centralizer of g induces the full $\mathrm{GU}(d_i, q)$ and so there are q orbits of the form gv with $0 \neq v \in V_i$ for $d_i > 1$ and $q - 1$ orbits if $d_i = 1$ (there are no nontrivial vectors of norm 0 if $d_i = 1$).

Note that if $h \in C(g)$, then $hV_i \subset V_1 \oplus \cdots \oplus V_i + [g, V]$. Thus, two elements in V which are in the same $C(g)$-orbit module $[g, V]$ must have the same highest nonzero (modulo $[g, V]$) term. Conversely, we need to show that any two such vectors are in the same orbit and indeed are in the orbit of v_j with $v_j \in V_j \setminus [g, V_j]$. By induction, we may assume that $j = m$. Note that there exists $h \in C(g)$ so that h is trivial on $V/\sum_{e<m} V_e$ and $hv_m - v_m$ is an arbitrary element in $\bigoplus_{e<m} V_e/[g, V_e]$. Thus, we see that the v and v_m are in the same orbit. The number of orbits for the nontrivial v_m is q or $q - 1$ as above. The result follows. □

The following combinatorial lemma will also be helpful.

Lemma 4.2. (1) *The generating function for the number of unipotent classes of* $\mathrm{GU}(n, q)$ *is*

$$\sum_\lambda u^{|\lambda|}.$$

This is equal to

$$\prod_i \frac{1}{1 - u^i}.$$

(2) *The generating function*

$$\sum_\lambda d(\lambda) u^{|\lambda|}$$

is equal to

$$\frac{u}{1 - u} \prod_i \frac{1}{1 - u^i}.$$

(3) *The generating function*

$$\sum_\lambda b(\lambda) u^{|\lambda|}$$

is equal to

$$\frac{u}{1 - u^2} \prod_i \frac{1}{1 - u^i}.$$

Proof. The first part is just the well known generating function for the partition function. The second part is in the proof of Lemma 3.2.

For the third assertion, note that

$$\sum_{\lambda} x^{b(\lambda)} u^{|\lambda|}$$

is equal to

$$\prod_i (1 + xu^i + u^{2i} + u^{3i} + \cdots).$$

Differentiating with respect to x and setting $x = 1$ gives that

$$\sum_{\lambda} b(\lambda) u^{|\lambda|}$$

is equal to

$$\sum_i u^i \prod_{j \neq i} (1 + u^j + u^{2j} + u^{3j} + \cdots) = \sum_i u^i \prod_{j \neq i} \frac{1}{1-u^j} = \sum_i u^i (1-u^i) \prod_j \frac{1}{1-u^j} = \frac{u}{1-u^2} \prod_j \frac{1}{1-u^j},$$

as claimed. □

Theorem 4.3 gives an exact generating function for $k(\mathrm{AGU}(n,q))$.

Theorem 4.3. *$k(\mathrm{AGU}(n,q))$ is equal to the coefficient of u^n in*

$$\prod_i \frac{1+u^i}{1-qu^i} \cdot \left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right).$$

Proof. By Lemma 2.1 and Theorem 4.1, $k(\mathrm{AGU}(n,q))$ is equal to $T_1 + T_2 - T_3$, where T_1 is $k(\mathrm{GU}(n,q))$, and T_2, T_3 are the following sums over conjugacy classes C of $\mathrm{GU}(n,q)$:

$$T_2 = q \sum_C d(\lambda_{z-1}(C)), \quad T_3 = \sum_C b(\lambda_{z-1}(C)).$$

From Wall [1963], T_1 is the coefficient of u^n in

$$\prod_i \frac{1+u^i}{1-qu^i}.$$

To compute the generating function of T_2, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 4.2, and multiply it by the weighted sum over unipotent classes in part (2) of Lemma 4.2. We conclude that T_2 is the coefficient of u^n in

$$\frac{qu}{1-u} \prod_i \frac{1+u^i}{1-qu^i}.$$

To compute the generating function of T_3, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 4.2, and multiply it by the weighted sum over unipotent classes in part (3) of Lemma 4.2. We conclude that T_3 is the coefficient of u^n in

$$\frac{u}{1-u^2}\prod_i \frac{1+u^i}{1-qu^i}.$$

Putting the pieces together, we conclude that $k(\mathrm{AGU}(n,q))$ is the coefficient of u^n in

$$\prod_i \frac{1+u^i}{1-qu^i}\cdot\left(1+\frac{qu}{1-u}-\frac{u}{1-u^2}\right),$$

which simplifies to the desired result. $\square$

4B. ***Character approach to k(AGU).*** We use Lemma 2.2 to find a recursion for $k(\mathrm{AGU})$. Then we use this to compute the generating function for $k(\mathrm{AGU})$, giving another proof of Theorem 4.3.

Recall that if H is a finite group and p is a prime, then $O_p(H)$ is the (unique) maximal normal p-subgroup of H.

Lemma 4.4.

$$\begin{aligned} k(\mathrm{AGU}(n,q)) = k(\mathrm{GU}(n,q)) &+ (q-1)k(\mathrm{GU}(n-1,q)) \\ &+ k(\mathrm{AGU}(n-2,q)) + (q-1)k(\mathrm{GU}(n-2,q)). \end{aligned}$$

Proof. We use the convention that $\mathrm{GU}(0,q)$ and $\mathrm{AGU}(0,q)$ are trivial groups and that $\mathrm{GU}(-1,q)$ and $\mathrm{AGU}(-1,q)$ are the empty set. We can identify the natural module and the character group of the module because the module is self dual viewed over the field of q-elements.

Note that $\mathrm{AGU}(1,q)$ is a semidirect product of an elementary abelian group of order q^2 and $\mathrm{GU}(1,q)$ which is cyclic of order $q+1$. Thus, it follows that $k(\mathrm{AGU}(1,q)) = k(\mathrm{GU}(1,q)) + (q-1)$ as claimed. If $n=2$, we note that $\mathrm{GU}(2,q)$ has precisely q nontrivial orbits on the natural module. The stabilizer of a nondegenerate vector is $\mathrm{GU}(1,q)$ and the stabilizer of a totally singular vector is elementary abelian of order q and again we see the result holds.

Now suppose that $n \geq 3$. Thus, we see that there are $q-1$ orbits with stabilizer isomorphic to $\mathrm{GU}(n-1,q)$ (corresponding to vectors with a given nonzero norm) and the stabilizer H of a singular vector. Note that H has a center Z of order q and $H/Z \cong \mathrm{AGU}(n-2,q)$. Also note that any irreducible character of $U = O_p(H)$ that is nontrivial on Z has dimension q^{n-2} and corresponds to one of the $q-1$ nontrivial 1-dimensional characters on Z. Moreover each of these representations extends to a representation of H (this can be seen by considering the normalizer of U in the full linear group). Fix a nontrivial linear character of Z and an irreducible module W of H that affords this linear representation. It follows by Clifford theory [Curtis and Reiner 1962, 51.7] that any irreducible representation of H nontrivial on Z is of the form $W \otimes W'$ where W' is an irreducible H/U-module. Since there are $q-1$ nontrivial central characters of U and there are $k(\mathrm{GU}(n-2,q))$ choices for W', the result follows. $\square$

We now give a second proof of Theorem 4.3.

Proof. Let $k_n = k(\mathrm{GU}(n,q))$ and let $a_n = k(\mathrm{AGU}(n,q))$. Then Lemma 4.4 gives

$$a_n = k_n + (q-1)k_{n-1} + (q-1)k_{n-2} + a_{n-2}. \tag{3}$$

Let

$$K(u) = 1 + \sum_{n \geq 1} k_n u^n, \quad A(u) = 1 + \sum_{n \geq 1} a_n u^n.$$

Multiplying (3) by u^n and summing over $n \geq 1$ gives that

$$A(u) - 1 = K(u) - 1 + (q-1)uK(u) + (q-1)u^2 K(u) + u^2 A(u).$$

Solving for $A(u)$, one obtains that

$$A(u) = K(u)\left(\frac{1 + u(q-1) + u^2(q-1)}{1-u^2}\right) = K(u)\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right).$$

From Wall [1963],

$$K(u) = \prod_i \frac{1+u^i}{1-qu^i},$$

and the theorem follows. □

4C. *Bounds for* AGU *and related groups.* As a corollary, we obtain the following result.

Corollary 4.5. $$k(\mathrm{AGU}(n,q)) \leq 20q^n.$$

Proof. From Theorem 4.3 , $k(\mathrm{AGU}(n,q))$ is equal to the coefficient of u^n in

$$\prod_i \frac{1-u^i}{1-qu^i} \prod_i \frac{1+u^i}{1-u^i}\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right).$$

Now all coefficients of powers of u in

$$\prod_i \frac{1+u^i}{1-u^i}\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right)$$

are nonnegative. It follows that $k(\mathrm{AGU}(n,q))$ is at most

$$\sum_{m=0}^{n} \text{Coef. } u^{n-m} \text{ in } \prod_i \frac{1-u^i}{1-qu^i} \text{Coef. } u^m \text{ in } \prod_i \frac{1+u^i}{1-u^i}\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right).$$

Now $\prod_i \frac{1-u^i}{1-qu^i}$ is the generating function for the number of conjugacy classes of $\mathrm{GL}(n,q)$. By [Maslen and Rockmore 1997], $k(\mathrm{GL}(n,q))$ is at most q^n. Hence the coefficient of u^{n-m} in it is at most q^{n-m}. It follows that $k(\mathrm{AGU}(n,q))$ is at most

$$q^n \sum_{m=0}^{n} \frac{1}{q^m}\left(\text{Coef. } u^m \text{ in } \prod_i \frac{1+u^i}{1-u^i}\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right)\right).$$

Since the coefficients of u^m in

$$\prod_i \frac{1+u^i}{1-u^i}\left(1 + \frac{qu^2 + (q-1)u}{1-u^2}\right)$$

are nonnegative, it follows that $k(\mathrm{AGU}(n,q))$ is at most

$$q^n \sum_{m=0}^{\infty} \frac{1}{q^m}\left(\text{Coef. } u^m \text{ in } \prod_i \frac{1+u^i}{1-u^i}\left(1+\frac{qu^2+(q-1)u}{1-u^2}\right)\right),$$

which (set $u=1/q$) is equal to

$$q^n \prod_i \frac{(1+1/q^i)}{(1-1/q^i)} \cdot \left(1+\frac{1}{1-1/q^2}\right).$$

The term

$$\prod_i \frac{(1+1/q^i)}{(1-1/q^i)} \cdot \left(1+\frac{1}{1-1/q^2}\right)$$

is visibly maximized among prime powers q when $q=2$, when it is at most 20 (we used the remark after Lemma 2.3 to bound the infinite product). □

Corollary 4.6. $$k(\mathrm{AGU}(n,q)) \le q^{2n}.$$

Proof. By the preceding result, this holds if $20 \le q^n$. So we only need to check the cases $n=1$, or $n=2$, $q=2,3,4$ or $n=3$, $q=2$ or $n=4$, $q=2$. From the generating function (Theorem 4.3), $k(\mathrm{AGU}(1,q))=2q$, and the other finite number of cases are computed easily from the generating function and seen to be at most q^{2n}. □

We can also use the previous results to get bounds for the groups between $\mathrm{ASU}(n,q)$ and $\mathrm{AGU}(n,q)$. Since $\mathrm{SL}(2,q) \cong \mathrm{SU}(2,q)$, we assume that $n \ge 3$. With more effort one can get much better bounds as we did in the case of $\mathrm{SL}(n,q)$. We just obtain the bound required for the $k(GV)$ problem.

Corollary 4.7. *Let $n \ge 3$. Let $\mathrm{ASU}(n,q) \le H \le \mathrm{AGU}(n,q)$. Then $k(H) \le q^{2n}$.*

Proof. Let $G=\mathrm{AGU}(n,q)$. Since $[G:H] \le q+1$, $k(H) \le k(G)(q+1) \le 20q^n(q+1)$. This is at most q^{2n} unless $q=2$ with $n \le 5$ or $q=3$ or 4 and $n=3$. These cases all follow using the exact values of $k(G)$ (obtained from our generating function) in the bound $k(H) \le k(G)(q+1)$, except for the cases $q=2$, $n=3,4$. One computes (either using a recursion similar to Lemma 4.4 and exact values of $k(\mathrm{SU})$ in [Macdonald 1981], or by Magma) that $k(\mathrm{ASU}(3,2))=24$ and $k(\mathrm{ASU}(4,2))=49$, completing the proof. □

5. ASp

Section 5A uses the orbit approach to calculate the generating function for $k(\mathrm{ASp}(2n,q))$, assuming that the characteristic is odd. Section 5B uses the character approach to calculate the generating function for $k(\mathrm{ASp}(2n,q))$ in both odd and even characteristic. Section 5C uses these generating functions to obtain bounds on $k(\mathrm{ASp}(2n,q))$.

5A. ***Orbit approach to k(ASp), odd characteristic.*** This section treats the affine symplectic groups. We only work in odd characteristic. In this case the conjugacy class of a unipotent element is determined by its Jordan form (over the algebraic closure) and it is much more complicated to deal with the characteristic 2 case. Since our character approach works in characteristic 2, we will not pursue the direct approach in that case. So for this section, let q be odd.

The following theorem calculates $o(C)$ for a conjugacy class C of $\mathrm{Sp}(2n, q)$. This only involves the unipotent part of the class C. Recall that the conjugacy class of a unipotent element is determined (over the algebraic closure) by a partition of $2n$ with a_i parts of i. Moreover, a_i is even if i is odd. Over a finite field, we attach a sign ϵ_i for each even i with $a_i \neq 0$ and this gives a description of all the unipotent conjugacy classes (see [Liebeck and Seitz 2012] for details). We let $\lambda^{\pm}_{z-1}(C)$ denote this signed partition for the unipotent part of the class C.

Theorem 5.1. *Suppose that the characteristic is odd. Let C be a conjugacy class of* $\mathrm{Sp}(2n, q)$*. Let a_i be the number of parts of $\lambda^{\pm}_{z-1}(C)$ of size i. Then $o(C)$ is equal to*

$$1 + \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} 1 + \sum_{\substack{i \text{ even} \\ a_i \neq 0}} f_i,$$

where

$$f_i = \begin{cases} q & \text{if } a_i > 2 \text{ (independently of the sign)}, \\ q & \text{if } a_i = 2 \text{ and the sign is } +, \\ (q-1) & \text{if } a_i = 2 \text{ and the sign is } -, \\ (q-1)/2 & \text{if } a_i = 1 \text{ (independently of the sign)}. \end{cases} \tag{4}$$

Proof. The proof is similar to the case of GL and GU and reduces to the case of unipotent elements. So assume that C is a unipotent class. Let $g \in C$. Write V as an orthogonal direct sum of spaces V_i where g has a_i Jordan blocks of size i on V_i. As in the previous cases, one can show that gv is either conjugate to g or for some i, g is conjugate to gv_i where $v_i \in V_i \setminus [g, V_i]$.

By [Liebeck and Seitz 2012], we see that there is a subgroup of $C(g)$ acting as $\mathrm{Sp}(a_i, q)$ for i odd or $\mathrm{O}^{\epsilon_i}(a_i, q)$ if i is even acting naturally on $V_i/[g, V_i]$. Thus, the number of classes of the form gv_i with $vI \in V_i \setminus [g, V_i]$ is 1 if i is odd and f_i as given above if i is even. □

The following combinatorial lemma will also be helpful.

Lemma 5.2. *Suppose that the characteristic is odd.*

(1) *The generating function for the number of unipotent classes of the groups* $\mathrm{Sp}(2n, q)$ *is*

$$\sum_{\lambda^{\pm}} u^{|\lambda^{\pm}|/2}.$$

This is equal to

$$\prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(\frac{1+u^i}{1-u^i} \right).$$

(2) *The generating function*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} 1$$

is equal to

$$\frac{u}{1-u^2} \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(\frac{1+u^i}{1-u^i} \right).$$

(3) *Let f_j be as in Theorem 5.1. The generating function*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} f_j$$

is equal to

$$\left(\frac{(q-1)u}{1-u} + \frac{u^2}{1-u^2} \right) \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(\frac{1+u^i}{1-u^i} \right).$$

Proof. For the first part, the unipotent conjugacy classes of $\mathrm{Sp}(2n, q)$ correspond to signed partitions $\lambda^\pm$ of size $2n$. Clearly the generating function for such partitions is equal to

$$\prod_{i \text{ odd}} (1 + u^i + u^{2i} + \cdots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \cdots)$$

which is equal to

$$\prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(1 + \frac{2u^i}{1-u^i} \right) = \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(\frac{1+u^i}{1-u^i} \right).$$

For the second part, first note that arguing as in the first part, one has that

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} 1$$

is equal to

$$\begin{aligned}
\sum_{j \text{ odd}} \sum_{\substack{\lambda^\pm \\ a_j \neq 0}} u^{|\lambda^\pm|/2} &= \sum_{j \text{ odd}} (u^j + u^{2j} + \cdots) \prod_{\substack{i \text{ odd} \\ i \neq j}} (1 + u^i + u^{2i} + \cdots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \cdots) \\
&= \sum_{j \text{ odd}} u^j \prod_{i \text{ odd}} (1 + u^i + u^{2i} + \cdots) \prod_{i \text{ even}} (1 + 2u^{i/2} + 2u^{2i/2} + \cdots) \\
&= \frac{u}{1-u^2} \prod_{i \text{ odd}} \frac{1}{1-u^i} \prod_i \left(\frac{1+u^i}{1-u^i} \right).
\end{aligned}$$

For the third part,

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|/2} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} f_j$$

is equal to

$$\sum_{j\text{ even}}\bigl(2u^{j/2}\tfrac{1}{2}(q-1)+u^{2j/2}(q+q-1)+2q(u^{3j/2}+u^{4j/2}+\cdots)\bigr)\prod_{i\text{ odd}}(1+u^i+u^{2i}+\cdots)\prod_{\substack{i\text{ even}\\ i\neq j}}\frac{1+u^{i/2}}{1-u^{i/2}}.$$

This is equal to

$$\sum_{j\text{ even}}\frac{1-u^{j/2}}{1+u^{j/2}}\bigl(u^{j/2}(q-1)+u^{2j/2}(2q-1)+2q(u^{3j/2}+u^{4j/2}+\cdots)\bigr)\prod_{i\text{ odd}}\frac{1}{1-u^i}\prod_i\frac{1+u^i}{1-u^i}.$$

Now clearly

$$\sum_{j\text{ even}}\frac{1-u^{j/2}}{1+u^{j/2}}\bigl(u^{j/2}(q-1)+u^{2j/2}(2q-1)+2q(u^{3j/2}+u^{4j/2}+\cdots)\bigr)$$

is equal to

$$\begin{aligned}\sum_j\frac{1-u^j}{1+u^j}\bigl(u^j(q-1)+u^{2j}(2q-1)+2q(u^{3j}+u^{4j}+\cdots)\bigr)&=\sum_j\frac{1}{1+u^j}(qu^j-u^j+qu^{2j}+u^{3j})\\&=\sum_j(qu^j+u^{2j}-u^j)\\&=\frac{(q-1)u}{1-u}+\frac{u^2}{1-u^2},\end{aligned}$$

and the third part of the lemma follows. □

Theorem 5.3. *In odd characteristic, $k(\mathrm{ASp}(2n,q))$ is equal to the coefficient of u^n in*

$$\prod_i\frac{(1+u^i)^4}{1-qu^i}\Bigl(1+\frac{qu}{1-u}\Bigr).$$

Proof. By Lemma 2.1 and Theorem 5.1, $k(\mathrm{ASp}(2n,q))$ is equal to $T_1+T_2+T_3$, where T_1 is $k(\mathrm{Sp}(2n,q))$, and T_2, T_3 are the following sums over conjugacy classes C of $\mathrm{Sp}(2n,q)$:

$$T_2=\sum_C\sum_{\substack{i\text{ odd}\\ a_i\neq 0}}1,\qquad T_3=\sum_C\sum_{\substack{i\text{ even}\\ a_i\neq 0}}f_i.$$

From Wall [1963], T_1 is the coefficient of u^n in

$$\prod_i\frac{(1+u^i)^4}{1-qu^i}.$$

To compute the generating function of T_2, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 5.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (2) of Lemma 5.2. We conclude that T_2 is the coefficient of u^n in

$$\frac{u}{1-u^2}\prod_i \frac{(1+u^i)^4}{1-qu^i}.$$

To compute the generating function of T_3, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 5.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (3) of Lemma 5.2. We conclude that T_3 is the coefficient of u^n in

$$\left(\frac{(q-1)u}{1-u}+\frac{u^2}{1-u^2}\right)\prod_i \frac{(1+u^i)^4}{1-qu^i}.$$

Since

$$1+\frac{u}{1-u^2}+\frac{(q-1)u}{1-u}+\frac{u^2}{1-u^2}=1+\frac{qu}{1-u},$$

the proof of the theorem is complete. □

5B. ***Character approach to k(ASp(2n, q)), any characteristic.*** We apply Lemma 2.2, as in the other cases.

To begin we treat the case of odd characteristic.

Lemma 5.4. *Let q be odd and $G = \mathrm{Sp}(2n,q)$. Then*

$$k(AG)=k(\mathrm{Sp}(2n,q))+k(\mathrm{ASp}(2n-2,q))+(q-1)k(\mathrm{Sp}(2n-2,q)).$$

Proof. We take $\mathrm{ASp}(0,q)$ and $\mathrm{Sp}(0,q)$ to be the trivial group. If $n=1$, then $G=\mathrm{SL}(2,q)$. It is straightforward to see that $k(\mathrm{SL}(2,q))=q+4$ and that $k(\mathrm{ASL}(2,q))=2q+4$ and so the formula holds.

So suppose that $n\geq 2$. Let V be the natural module for G. Note that in this case G acts transitively on the nontrivial characters of V and the stabilizer of such a character is the stabilizer H of a vector in $\mathrm{Sp}(2n,q)$. Let $U=O_p(H)$ and let $Z=Z(H)$. Then $H/Z\cong \mathrm{ASp}(2n-2,q)$. If an irreducible character of H does not vanish on Z, then there are $q-1$ possibilities (depending on the restriction to Z) and arguing as in the unitary case, we see that the number of such characters of H is $(q-1)k(\mathrm{Sp}(2n-2,q))$. This gives $k(\mathrm{ASp}(2n,q))=k(\mathrm{Sp}(2n,q))+k(\mathrm{ASp}(2n-2,q))+(q-1)k(\mathrm{Sp}(2n-2,q))$ as desired. □

We use this recursion to give another proof of the generating function for $k(\mathrm{Sp}(2n,q))$ in odd characteristic.

Second proof of Theorem 5.3. Let $k_n=k(\mathrm{Sp}(2n,q))$ and let $a_n=k(\mathrm{ASp}(2n,q))$. Lemma 5.4 gives that

$$a_n=k_n+(q-1)k_{n-1}+a_{n-1}. \tag{5}$$

Let

$$K(u)=1+\sum_{n\geq 1}k_nu^n,\qquad A(u)=1+\sum_{n\geq 1}a_nu^n.$$

Multiplying (5) by u^n and summing over $n \geq 1$ gives that

$$A(u) - 1 = K(u) - 1 + (q-1)uK(u) + uA(u).$$

Solving for $A(u)$ gives

$$A(u) = K(u)\left(\frac{1+u(q-1)}{1-u}\right) = K(u)\left(1 + \frac{qu}{1-u}\right).$$

From Wall [1963],

$$K(u) = \prod_i \frac{(1+u^i)^4}{1-qu^i},$$

and the result follows. □

In even characteristic, the unipotent radical is abelian but not irreducible. So let $G = \mathrm{Sp}(2n, q)$ with q even. Let BG denote the semidirect product WG, where W is the $2n+1$ dimensional indecomposable module with G having a one dimensional fixed space W_0 and $W/W_0 \cong V$.

Note that the G-orbits of characters of B consist of the trivial character, one orbit of nontrivial characters with W_0 contained in the kernel and $2(q-1)$ orbits of characters which are nontrivial on W_0. The stabilizer of a character in the second orbit is isomorphic to $B\,\mathrm{Sp}(2n-2, q)$ while in the final case the stabilizers are $\mathrm{O}^{\pm}(2n, q)$ (with $q-1$ of each type). This gives the following:

Lemma 5.5. *Let q be even.*

(1) $k(B\,\mathrm{Sp}(2n, q)) = k(\mathrm{ASp}(2n, q)) + (q-1)\big(k(\mathrm{O}^+(2n, q)) + k(\mathrm{O}^-(2n, q))\big)$

(2) $k(\mathrm{ASp}(2n, q)) = k(\mathrm{Sp}(2n, q)) + k(B\,\mathrm{Sp}(2n-2, q))$.

The next lemma follows immediately from the previous lemma. We use the convention that $\mathrm{ASp}(0, q)$ and $\mathrm{O}^+(0, q)$ are the trivial groups and that $\mathrm{O}^-(0, q)$ is the empty set. So

$$k(\mathrm{ASp}(0, q)) = 1, \quad k(\mathrm{O}^+(0, q)) = 1, \quad \text{and} \quad k(O^-(0, q)) = 0.$$

Lemma 5.6. *For all $n \geq 1$,*

$$k(\mathrm{ASp}(2n, q)) = k(\mathrm{Sp}(2n, q)) + k(\mathrm{ASp}(2n-2, q)) + (q-1)\big[k(\mathrm{O}^+(2n-2, q)) + k(\mathrm{O}^-(2n-2, q))\big].$$

Now we obtain the generating function for $k(\mathrm{ASp}(2n, q))$ in even characteristic.

Theorem 5.7. *In even characteristic, $k(\mathrm{ASp}(2n, q))$ is equal to the coefficient of u^n in*

$$\frac{1}{1-u} \prod_i \frac{1+u^i}{1-qu^i} \left[\prod_i \frac{1}{(1-u^{4i-2})^2} + (q-1)u \prod_i (1+u^{2i-1})^2\right].$$

Proof. We define three generating functions:

$$K_{\mathrm{Sp}}(u) = 1 + \sum_{n\geq 1} k(\mathrm{Sp}(2n,q))u^n,$$

$$K_{\mathrm{O}}(u) = 1 + \sum_{n\geq 1} [k(\mathrm{O}^+(2n,q)) + k(\mathrm{O}^-(2n,q))]u^n,$$

$$A(u) = 1 + \sum_{n\geq 1} k(\mathrm{ASp}(2n,q))u^n.$$

Multiplying the recursion from Lemma 5.6 by u^n and summing over $n \geq 1$ gives that

$$A(u) - 1 = K_{\mathrm{Sp}}(u) - 1 + uA(u) + (q-1)uK_{\mathrm{O}}(u).$$

Thus

$$A(u) = \frac{K_{\mathrm{Sp}}(u) + (q-1)uK_{\mathrm{O}}(u)}{1-u}.$$

From Theorems 3.13 and Theorem 3.21 of [Fulman and Guralnick 2012], elementary manipulations, give that

$$K_{\mathrm{Sp}}(u) = \prod_i \frac{1+u^i}{1-qu^i} \prod_i \frac{1}{(1-u^{4i-2})^2}, \quad K_{\mathrm{O}}(u) = \prod_i \frac{1+u^i}{1-qu^i} \prod_i (1+u^{2i-1})^2,$$

and the result follows. □

5C. *Bounds on* k(ASp($2n, q$)). As a corollary, we obtain the following results.

Corollary 5.8. *In odd characteristic*, $k(\mathrm{ASp}(2n,q)) \leq 27q^n$.

Proof. From Theorem 5.3, $k(\mathrm{ASp}(2n,q))$ is the coefficient of u^n in

$$\prod_i \frac{1-u^i}{1-qu^i} \prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right).$$

Now all coefficients of powers of u in

$$\prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right)$$

are nonnegative. It follows that $k(\mathrm{ASp}(2n,q))$ is at most

$$\sum_{m=0}^{n}\left(\text{Coef. } u^{n-m} \text{ in } \prod_i \frac{1-u^i}{1-qu^i}\right)\left(\text{Coef. } u^m \text{ in } \prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right)\right).$$

Now $\prod_i (1-u^i)/(1-qu^i)$ is the generating function for the number of conjugacy classes in $\mathrm{GL}(n,q)$. By [Maslen and Rockmore 1997], $k(\mathrm{GL}(n,q))$ is at most q^n. Hence the coefficient of u^{n-m} in it is at most q^{n-m}. It follows that $k(\mathrm{ASp}(2n,q))$ is at most

$$q^n \sum_{m=0}^{n} \frac{1}{q^m}\left(\text{Coef. } u^m \text{ in } \prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right)\right).$$

Since the coefficients of u^m in

$$\prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right)$$

are nonnegative, it follows that $k(\mathrm{ASp}(2n,q))$ is at most

$$q^n \sum_{m=0}^{\infty} \frac{1}{q^m}\left(\text{Coef. } u^m \text{ in } \prod_i \frac{(1+u^i)^4}{1-u^i}\left(1+\frac{qu}{1-u}\right)\right),$$

which is equal to

$$q^n \prod_i \frac{(1+1/q^i)^4}{1-1/q^i}\left(1+\frac{1}{1-1/q}\right).$$

The term

$$\prod_i \frac{(1+1/q^i)^4}{1-1/q^i}\left(1+\frac{1}{1-1/q}\right)$$

is visibly maximized among odd prime powers q when $q=3$, when it is at most 27 $\big($we bounded the infinite product $\prod_i (1+1/q^i)^4/(1-1/q^i)$ using the remark after Lemma 2.3$\big)$. □

Corollary 5.9. *In odd characteristic,*

$$k(\mathrm{ASp}(2n,q)) \le q^{2n},$$

except for $k(\mathrm{ASp}(2,3))=10$.

Proof. From the previous result, $k(\mathrm{ASp}(2n,q)) \le 27q^n$. This immediately implies that $k(\mathrm{ASp}(2n,q)) \le q^{2n}$ except possibly for $\mathrm{ASp}(2,q)$, $\mathrm{ASp}(4,3)$, or $\mathrm{ASp}(4,5)$.

From our generating function for $k(\mathrm{ASp}(2n,q))$ (Theorem 5.3), we see that $k(\mathrm{ASp}(4,3))=58$, $k(\mathrm{ASp}(4,5))=110$, and $k(\mathrm{ASp}(2,q))=2q+4$, and the result follows. □

Next we move to even characteristic.

Corollary 5.10. *In even characteristic,* $k(\mathrm{ASp}(2n,q)) \le 56q^n$.

Proof. We rewrite the generating function for $k(\mathrm{ASp}(2n,q))$ in Theorem 5.7 as

$$\prod_i \frac{1-u^i}{1-qu^i}\frac{1}{1-u}\prod_i \frac{1+u^i}{1-u^i}\left[\prod_i \frac{1}{(1-u^{4i-2})^2}+(q-1)u\prod_i (1+u^{2i-1})^2\right].$$

Now arguing exactly as in the odd characteristic case (Corollary 5.8), one sees that $k(\mathrm{ASp}(2n,q))$ is at most

$$q^n \cdot \frac{1}{1-1/q}\prod_i \frac{1+1/q^i}{1-1/q^i}\left[\prod_i \frac{1}{(1-1/q^{4i-2})^2}+(1-1/q)\prod_i (1+1/q^{2i-1})^2\right],$$

and the result follows. □

Next we classify when $k(\mathrm{ASp}(2n,q)) \le q^{2n}$.

Corollary 5.11. *In even characteristic,*

$$k(\mathrm{ASp}(2n,q)) \le q^{2n},$$

except for $k(\mathrm{ASp}(2,2)) = 5,\ k(\mathrm{ASp}(4,2)) = 21,\ k(\mathrm{ASp}(6,2)) = 67.$

Proof. From the previous result, $k(\mathrm{ASp}(2n,q)) \le 56q^n$. This immediately implies that $k(\mathrm{ASp}(2n,q)) \le q^{2n}$ except possibly for $q = 2,\ 1 \le n \le 5$, or $q = 4,\ n = 1, 2$ or $q = 8,\ n = 1$. For these q, n values one calculates $k(\mathrm{ASp}(2n,q))$ from the generating function in Theorem 5.7, and the result follows. □

6. Orthogonal Groups

Section 6A uses the orbit approach to calculate the generating function for $k(\mathrm{AO})$ when the characteristic is odd. Section 6B uses the character approach to calculate the generating function of $k(\mathrm{AO})$ in any characteristic. To be more precise, we actually derive two generating functions, one for $k(\mathrm{AO}^+) + k(\mathrm{AO}^-)$ and one for $k(\mathrm{AO}^+) - k(\mathrm{AO}^-)$. Clearly this is equivalent to deriving generating functions for $k(\mathrm{AO}^+)$ and $k(\mathrm{AO}^-)$.

Section 6C derives some bounds on $k(\mathrm{AO})$.

6A. *Orbit approach for k(AO), odd characteristic.* For the orbit approach we assume the characteristic is odd. It is somewhat more convenient to work in orthogonal groups than the special orthogonal group (there is essentially no difference in the result below for SO). The conjugacy class of a unipotent element in $\mathrm{O}^\epsilon(m,q)$ gives rise to a partition of m with a_i pieces of size i. Moreover, a_i is even for i even. This determines the conjugacy class over the algebraic closure. Over the finite field, we attach a sign ϵ_i for each odd i with a_i nonzero and this determines the class (see [Liebeck and Seitz 2012]). We let $\lambda^{\pm}_{z-1}(C)$ denote this signed partition corresponding to the unipotent part of a conjugacy class C.

The proof of the next result is essentially identical to the case of symplectic groups and so we omit the details (and we can also use the character theory approach below).

Theorem 6.1. *Suppose that the characteristic is odd. Let C be a conjugacy class of $\mathrm{O}^\epsilon(n,q)$. Let a_i be the number of parts of $\lambda^{\pm}_{z-1}(C)$ of size i. Then $o(C)$ is equal to*

$$1 + \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1 + \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i,$$

where

$$f_i = \begin{cases} q & \text{if } a_i > 2 \text{ (independently of the sign)}, \\ q & \text{if } a_i = 2 \text{ and the sign is } +, \\ (q-1) & \text{if } a_i = 2 \text{ and the sign is } -, \\ (q-1)/2 & \text{if } a_i = 1 \text{ (independently of the sign)}. \end{cases} \tag{6}$$

The following combinatorial lemma will also be helpful.

Lemma 6.2. *Suppose that the characteristic is odd.*

(1) *The generating function for the number of unipotent classes of the groups* $\mathrm{O}(n,q)$ *is*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|}.$$

This is equal to

$$\prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left(\frac{1+u^i}{1-u^i}\right).$$

(2) *The generating function*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} 1$$

is equal to

$$\frac{u^4}{1-u^4} \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left(\frac{1+u^i}{1-u^i}\right).$$

(3) *Let* f_i *be as in Theorem 6.1. Then*

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} f_j$$

is equal to

$$\left(\frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4}\right) \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left(\frac{1+u^i}{1-u^i}\right).$$

Proof. For the first part, the unipotent conjugacy classes of the groups $\mathrm{O}(n,q)$ correspond to signed partitions $\lambda^\pm$ of size n. The generating function for such partitions is clearly equal to

$$\prod_{i \text{ odd}} (1+2u^i+2u^{2i}+\cdots) \prod_{i \text{ even}} (1+u^{2i}+u^{4i}+\cdots),$$

which is equal to

$$\prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \frac{1+u^i}{1-u^i}.$$

For the second part, first note that arguing as in the first part, one has that

$$\sum_{\lambda^\pm} u^{|\lambda^\pm|} \sum_{\substack{j \text{ even} \\ a_j \neq 0}} 1$$

is equal to

$$\begin{aligned}
\sum_{j \text{ even}} \sum_{\substack{\lambda^\pm \\ a_j \neq 0}} u^{|\lambda^\pm|} &= \sum_{j \text{ even}} (u^{2j}+u^{4j}+\cdots) \prod_{\substack{i \text{ even} \\ i \neq j}} (1+u^{2i}+u^{4i}+\cdots) \prod_{i \text{ odd}} (1+2u^i+2u^{2i}+\cdots) \\
&= \sum_{j \text{ even}} u^{2j} \prod_{i \text{ even}} (1+u^{2i}+u^{4i}+\cdots) \prod_{i \text{ odd}} (1+2u^i+2u^{2i}+\cdots) \\
&= \frac{u^4}{1-u^4} \prod_i \frac{1}{1-u^{4i}} \prod_{i \text{ odd}} \left(\frac{1+u^i}{1-u^i}\right).
\end{aligned}$$

For the third part,

$$\sum_{\lambda^{\pm}} u^{|\lambda^{\pm}|} \sum_{\substack{j \text{ odd} \\ a_j \neq 0}} f_j$$

is equal to

$$\sum_{j \text{ odd}} \big(2u^j \tfrac{1}{2}(q-1) + u^{2j}(q+q-1) + 2q(u^{3j}+u^{4j}+\cdots)\big) \prod_{\substack{i \neq j \\ i \text{ odd}}} \left(\frac{1+u^i}{1-u^i}\right) \prod_{i \text{ even}} (1+u^{2i}+u^{4i}+\cdots).$$

This is equal to

$$\sum_{j \text{ odd}} \frac{1-u^j}{1+u^j} \big(u^j(q-1) + u^{2j}(2q-1) + 2q(u^{3j}+u^{4j}+\cdots)\big) \prod_{i \text{ odd}} \left(\frac{1+u^i}{1-u^i}\right) \prod_i \frac{1}{1-u^{4i}}.$$

Now, as in the proof of part (3) of Lemma 5.2,

$$\sum_{j \text{ odd}} \frac{1-u^j}{1+u^j} \big(u^j(q-1) + u^{2j}(2q-1) + 2q(u^{3j}+u^{4j}+\cdots)\big)$$

simplifies to

$$\frac{(q-1)u}{1-u^2} + \frac{u^2}{1-u^4},$$

and the result follows. □

As a corollary, we derive a generating function for $k(\mathrm{AO}^+) + k(\mathrm{AO}^-)$.

Theorem 6.3. *In odd characteristic,*

$$1 + \sum_{n \geq 1} u^n \big[k(\mathrm{AO}^+(n,q)) + k(\mathrm{AO}^-(n,q))\big]$$

is equal to

$$\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}} \cdot \left(1 + \frac{u^2+(q-1)u}{1-u^2}\right).$$

Proof. By Lemma 2.1 and Theorem 6.1,

$$k(\mathrm{AO}^+(n,q)) + k(\mathrm{AO}^-(n,q))$$

is equal to $T_1+T_2+T_3$, where T_1 is $k(\mathrm{O}^+(n,q)) + k(\mathrm{O}^-(n,q))$, and T_2, T_3 are the following sums over conjugacy classes C of $\mathrm{O}^+(n,q)$ and $\mathrm{O}^-(n,q)$:

$$T_2 = \sum_C \sum_{\substack{i \text{ even} \\ a_i \neq 0}} 1, \qquad T_3 = \sum_C \sum_{\substack{i \text{ odd} \\ a_i \neq 0}} f_i.$$

From [Wall 1963], T_1 is the coefficient of u^n in

$$\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

To compute the generating function for T_2, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 6.2, and multiply it by the generating function for the weighted sum over unipotent classes in part (2) of Lemma 6.2. We conclude that T_2 is the coefficient of u^n in

$$\frac{u^4}{1-u^4}\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

To compute the generating function for T_3, we take Wall's generating function for T_1, divide it by the generating function for unipotent conjugacy classes in part (1) of Lemma 6.2 and multiply it by the generating function for the weighted sum over unipotent classes in part (3) of Lemma 6.2. We conclude that T_3 is the coefficient of u^n in

$$\left(\frac{(q-1)u}{1-u^2}+\frac{u^2}{1-u^4}\right)\prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}.$$

Since

$$1+\frac{u^4}{1-u^4}+\frac{(q-1)u}{1-u^2}+\frac{u^2}{1-u^4}=1+\frac{u^2+(q-1)u}{1-u^2},$$

the result follows. □

Next, we derive a generating function for $k(\mathrm{AO}^+)-k(\mathrm{AO}^-)$.

Theorem 6.4. *In odd characteristic,*

$$1+\sum_{n\geq 1} u^n\big[k(\mathrm{AO}^+(n,q))-k(\mathrm{AO}^-(n,q))\big]$$

is equal to

$$\frac{1}{1-u^2}\prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}.$$

Proof. By Lemma 2.1 and Theorem 6.1,

$$k(\mathrm{AO}^+(n,q))-k(\mathrm{AO}^-(n,q))$$

is equal to $T_1+T_2+T_3$, where T_1 is $k(\mathrm{O}^+(n,q))-k(\mathrm{O}^-(n,q))$,

$$T_2=\sum_{C^+}\sum_{\substack{i \text{ even}\\ a_i\neq 0}} 1-\sum_{C^-}\sum_{\substack{i \text{ even}\\ a_i\neq 0}} 1, \quad T_3=\sum_{C^+}\sum_{\substack{i \text{ odd}\\ a_i\neq 0}} f_i-\sum_{C^-}\sum_{\substack{i \text{ odd}\\ a_i\neq 0}} f_i.$$

Here C^+ ranges over conjugacy classes of $\mathrm{O}^+(n,q)$, and C^- ranges over conjugacy classes of $\mathrm{O}^-(n,q)$.

From [Wall 1963], T_1 is the coefficient of u^n in

$$\prod_i \frac{1-u^{4i-2}}{1-qu^{4i}}.$$

To compute the generating function of T_2, we take the generating function for T_1, multiply it by $\prod_i(1-u^{4i})$ (which corresponds to removing the unipotent part). Then to add in the weighted unipotent part, one multiplies by

$$\sum_{j \text{ even}} (u^{2j}+u^{4j}+\cdots) \prod_{\substack{i\neq j\\ i \text{ even}}} (1+u^{2i}+u^{4i}+\cdots),$$

which is equal to

$$\frac{u^4}{1-u^4}\prod_i \frac{1}{1-u^{4i}}.$$

We conclude that T_2 is the coefficient of u^n in

$$\frac{u^4}{1-u^4}\prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}.$$

To compute the generating function of T_3, we take the generating function for T_1, multiply it by $\prod_i(1-u^{4i})$ (which corresponds to removing the unipotent part). Then to add in the weighted unipotent part, one multiplies by

$$\sum_{j \text{ odd}} u^{2j} \prod_{i \text{ even}} (1+u^{2i}+u^{4i}+\cdots).$$

Note that the terms involving f_i canceled out (except for the $a_i=2$ case). The upshot is that the generating function for T_3 is

$$\frac{u^2}{1-u^4}\prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}.$$

Since

$$1+\frac{u^4}{1-u^4}+\frac{u^2}{1-u^4}=\frac{1}{1-u^2},$$

the proof is complete. □

6B. *Character approach for k(AO), any characteristic.* Next we consider orthogonal groups. In this case, the natural module V can be identified with its character group and the nontrivial G-orbits correspond to nonzero vectors of V of a given norm.

First consider the case $G=\mathrm{O}^\epsilon(n,q)$ with q odd. The stabilizers are thus $\mathrm{AO}^\epsilon(m-2,q)$ (for an isotropic vector) and $(q-1)/2$ copies each of $\mathrm{O}^+(n-2,q)$ and $\mathrm{O}^-(n-2,q)$. Note that we use the convention that $\mathrm{O}^\epsilon(0,q)$ and $\mathrm{AO}^\epsilon(0,q)$ are empty if $\epsilon=-$ and are the trivial group if $\epsilon=+$. Similarly, $\mathrm{AO}^\epsilon(-1,q)$ is the empty set. And as in earlier cases, the trivial group has one conjugacy class and the empty set has zero conjugacy classes. This yields the following result.

Lemma 6.5. *Let q be odd and $n\geq 1$. Then*

$$k(\mathrm{AO}^\epsilon(n,q))=k(\mathrm{O}^\epsilon(n,q))+k(\mathrm{AO}^\epsilon(n-2,q))+(q-1)\big(k(\mathrm{O}^+(n-1,q))+k(\mathrm{O}^-(n-1,q))\big)/2.$$

As a corollary, we obtain a second proof of Theorems 6.3 and 6.4.

Second proof of Theorem 6.3. Define

$$K_{\mathrm{O}}(u) = 1 + \sum_{n\geq 1} u^n[k(\mathrm{O}^+(n,q)) + k(\mathrm{O}^-(n,q))],$$
$$A_{\mathrm{O}}(u) = 1 + \sum_{n\geq 1} u^n[k(\mathrm{AO}^+(n,q)) + k(\mathrm{AO}^-(n,q))].$$

By the above recursion, we have that for all n,

$$k(\mathrm{AO}^+(n,q)) = k(\mathrm{O}^+(n,q)) + k(\mathrm{AO}^+(n-2,q)) + \tfrac{1}{2}(q-1)[k(\mathrm{O}^+(n-1,q)) + k(\mathrm{O}^-(n-1,q))],$$
$$k(\mathrm{AO}^-(n,q)) = k(\mathrm{O}^-(n,q)) + k(\mathrm{AO}^-(n-2,q)) + \tfrac{1}{2}(q-1)[k(\mathrm{O}^+(n-1,q)) + k(\mathrm{O}^-(n-1,q))].$$

Adding these two equations gives

$$\begin{aligned} &k(\mathrm{AO}^+(n,q)) + k(\mathrm{AO}^-(n,q)) \\ &\quad = k(\mathrm{O}^+(n,q)) + k(\mathrm{O}^-(n,q)) + k(\mathrm{AO}^+(n-2,q)) + k(\mathrm{AO}^-(n-2,q)) \\ &\qquad + (q-1)[k(\mathrm{O}^+(n-1,q)) + k(\mathrm{O}^-(n-1,q))]. \end{aligned}$$

Multiplying this by u^n and summing over $n \geq 0$ gives that

$$A_{\mathrm{O}}(u) = K_{\mathrm{O}}(u) + u^2 A_{\mathrm{O}}(u) + u(q-1)K_{\mathrm{O}}(u).$$

Thus

$$A_{\mathrm{O}}(u) = \frac{K_{\mathrm{O}}(u)}{1-u^2}(1+u(q-1)).$$

The result now follows from Wall's formula

$$K_{\mathrm{O}}(u) = \prod_i \frac{(1+u^{2i-1})^4}{1-qu^{2i}}. \qquad \square$$

Second proof of Theorem 6.4. Let

$$D(u) = 1 + \sum_{n\geq 1} u^n[k(\mathrm{O}^+(n,q)) - k(\mathrm{O}^-(n,q))],$$
$$B(u) = 1 + \sum_{n\geq 1} u^n[k(\mathrm{AO}^+(n,q)) - k(\mathrm{AO}^-(n,q))].$$

From Lemmas 6.5, we have that

$$k(\mathrm{AO}^+(n,q)) - k(\mathrm{AO}^-(n,q)) = k(\mathrm{O}^+(n,q)) - k(\mathrm{O}^-(n,q)) + k(\mathrm{AO}^+(n-2,q)) - k(\mathrm{AO}^-(n-2,q).$$

Multiplying this equation by u^n and summing over all $n \geq 0$ gives that

$$B(u) = D(u) + u^2 B(u).$$

Thus $B(u) = D(u)/(1-u^2)$, and the result follows from Wall's formula

$$D(u) = \prod_i \frac{(1-u^{4i-2})}{1-qu^{4i}}. \qquad \square$$

Finally we turn to characteristic 2. In this case odd dimensional orthogonal groups are isomorphic to symplectic groups, so we need only consider the even dimensional case. So consider $G = \mathrm{O}^{\epsilon}(n, q)$ with q and n both even. The argument is similar. The only difference is that the stabilizer of a vector of nonzero norm in $\mathrm{AO}^{\epsilon}(n, q)$ is $\mathrm{Sp}(n-2, q) \times \mathbb{Z}/2$ and so:

Lemma 6.6. *Let q be even and $n \geq 2$ be even. Then $k(\mathrm{AO}^{\epsilon}(n, q))$ is equal to*

$$k(\mathrm{O}^{\epsilon}(n, q)) + k(\mathrm{AO}^{\epsilon}(n-2, q)) + 2(q-1)\big(k(\mathrm{Sp}(n-2, q))\big).$$

For $n=2$ we used the convention that $k(\mathrm{AO}^{+}(0, q)) = 1$ and $k(\mathrm{Sp}(0, q)) = 1$, and that $k(\mathrm{AO}^{-}(0, q)) = 0$.

Next using Lemma 6.6 (and generating functions for $k(\mathrm{Sp})$ and $k(\mathrm{O})$) we derive generating functions for $k(\mathrm{AO}^{\pm}(2n, q))$ in even characteristic.

Theorem 6.7. *Let q be even. Then $k(\mathrm{AO}^{+}(2n, q)) + k(\mathrm{AO}^{-}(2n, q))$ is the coefficient of u^n in*

$$\frac{1}{1-u}\big(K_{\mathrm{O}}(u) + 4(q-1)uK_{\mathrm{Sp}}(u)\big),$$

where

$$K_{\mathrm{O}}(u) = \prod_{i \geq 1} \frac{(1+u^i)(1+u^{2i-1})^2}{1-qu^i}, \qquad K_{\mathrm{Sp}}(u) = \prod_{i \geq 1} \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)(1-qu^i)}.$$

Proof. Define generating functions,

$$K_{\mathrm{Sp}}(u) = 1 + \sum_{n \geq 1} k(\mathrm{Sp}(2n, q))u^n,$$

$$K_{\mathrm{O}}(u) = 1 + \sum_{n \geq 1} [k(\mathrm{O}^{+}(2n, q)) + k(\mathrm{O}^{-}(2n, q))]u^n,$$

$$A_{\mathrm{O}}(u) = 1 + \sum_{n \geq 1} [k(\mathrm{AO}^{+}(2n, q)) + k(\mathrm{AO}^{-}(2n, q))]u^n.$$

Now take the recursions for $k(\mathrm{AO}^{+}(2n, q))$ and $k(\mathrm{AO}^{-}(2n, q))$ in Lemma 6.6, multiply them by u^n and sum over all $n \geq 0$. We conclude that

$$A_{\mathrm{O}}(u) = K_{\mathrm{O}}(u) + uA_{\mathrm{O}}(u) + 4(q-1)uK_{\mathrm{Sp}}(u).$$

Thus

$$A_{\mathrm{O}}(u) = \frac{1}{1-u}\big(K_{\mathrm{O}}(u) + 4(q-1)uK_{\mathrm{Sp}}(u)\big).$$

From [Fulman and Guralnick 2012],

$$K_{\mathrm{O}}(u) = \prod_{i} \frac{(1+u^i)(1+u^{2i-1})^2}{(1-qu^i)}, \qquad K_{\mathrm{Sp}}(u) = \prod_{i} \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)(1-qu^i)}. \qquad \square$$

Theorem 6.8. *Let q be even. Then $k(\mathrm{AO}^{+}(2n, q)) - k(\mathrm{AO}^{-}(2n, q))$ is the coefficient of u^n in*

$$\frac{1}{1-u} \prod_{i \geq 1} \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

Proof. Define generating functions

$$D(u)=1+\sum_{n\geq 1}u^n[k(\mathrm{O}^+(2n,q))-k(\mathrm{O}^-(2n,q))],$$
$$B(u)=1+\sum_{n\geq 1}u^n[k(\mathrm{AO}^+(2n,q))-k(\mathrm{AO}^-(2n,q))].$$

Multiply the recursions for $k(\mathrm{AO}^+(2n,q))$ and $k(\mathrm{AO}^-(2n,q))$ in Lemma 6.6 by u^n, sum over all $n\geq 0$, and subtract to obtain

$$B(u)=D(u)+uB(u).$$

Using Wall's formula [1963] for $D(u)$, we conclude that

$$B(u)=\frac{1}{1-u}D(u)=\frac{1}{1-u}\prod_i\frac{1-u^{2i-1}}{1-qu^{2i}}. \qquad \square$$

6C. *Bounds on k*(AO). This section derives bounds on $k(\mathrm{AO})$.

We begin with the case of odd characteristic and even dimension.

Corollary 6.9. *Let q be odd. Then $k(\mathrm{AO}^{\pm}(2n,q))\leq 29q^n$.*

Proof. From Theorem 6.3,

$$k(\mathrm{AO}^+(2n,q))+k(\mathrm{AO}^-(2n,q))$$

is the coefficient of u^{2n} in

$$\prod_i\frac{(1+u^{2i-1})^4}{1-qu^{2i}}\cdot\left(1+\frac{u^2+(q-1)u}{1-u^2}\right).$$

Rewrite this as

$$\prod_i\frac{1-u^{2i}}{1-qu^{2i}}\prod_i\frac{(1+u^{2i-1})^4}{1-u^{2i}}\cdot\left(1+\frac{u^2+(q-1)u}{1-u^2}\right).$$

As in the symplectic case, the coefficient of u^{2n-2m} in $\prod_i(1-u^{2i})/(1-qu^{2i})$ is at most q^{n-m}. Thus

$$k(\mathrm{AO}^+(2n,q))+k(\mathrm{AO}^-(2n,q))$$

is at most

$$q^n\sum_{m\geq 0}\frac{1}{q^m}\text{Coef. }u^{2m}\text{ in }\prod_i\frac{(1+u^{2i-1})^4}{1-u^{2i}}\cdot\left(1+\frac{u^2+(q-1)u}{1-u^2}\right),$$

which is equal to $q^n/2$ multiplied by

$$\prod_i\frac{(1+u^{2i-1})^4}{1-u^{2i}}\cdot\left(1+\frac{u^2+(q-1)u}{1-u^2}\right)+\prod_i\frac{(1-u^{2i-1})^4}{1-u^{2i}}\cdot\left(1+\frac{u^2-(q-1)u}{1-u^2}\right)$$

evaluated at $u=1/\sqrt{q}$. Since $q\geq 3$, we conclude that

$$k(\mathrm{AO}^+(2n,q))+k(\mathrm{AO}^-(2n,q))\leq 53q^n.$$

From Theorem 6.4,

$$k(\mathrm{AO}^+(2n,q)) - k(\mathrm{AO}^-(2n,q))$$

is the coefficient of u^n in

$$\frac{1}{1-u}\prod_i \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

This is analytic for $|u| < \frac{1}{q}+\epsilon$, so Lemmas 2.4 and 2.3 imply an upper bound of

$$q^n \frac{1}{1-1/q}\prod_i \frac{1+1/q^{2i-1}}{1-1/q^{2i-1}} \le 3.3q^n.$$

Combining the results of the previous two paragraphs proves the corollary, as $(53+3.3)/2 \le 29$. □

Corollary 6.10. *Let q be odd. Then $k(\mathrm{AO}^{\pm}(2n,q)) \le q^{2n}$.*

Proof. The result follows from the previous corollary whenever $29q^n \le q^{2n}$. So we only need to check the cases $n=1$, or $n=2,\ q=3,5$, or $n=3,\ q=3$. These cases are easily checked from our generating function for $k(\mathrm{AO}^{\pm}(2n,q))$. □

Next we treat the case of odd dimensional groups in odd characteristic. In this case, the upper bound is not of the form constant times q^{rank}. This is because every element in the classical group has eigenvalue 1.

Corollary 6.11. *Let q be odd. Then $k(\mathrm{AO}(2n+1,q)) \le 20q^{n+1}$.*

Proof. We prove this by induction on n. By our earlier recursion,

$$k(\mathrm{AO}(2n+1,q)) = k(\mathrm{O}(2n+1,q)) + k(\mathrm{AO}(2n-1,q)) + \tfrac{1}{2}(q-1)[k(\mathrm{O}^+(2n,q)) + k(\mathrm{O}^-(2n,q))].$$

By [Fulman and Guralnick 2012],

$$k(\mathrm{O}(2n+1,q)) \le 14.2q^n$$

and

$$k(\mathrm{O}^+(2n,q)) + k(\mathrm{O}^-(2n,q)) \le 16.3q^n.$$

Thus

$$k(\mathrm{AO}(2n+1,q)) \le k(\mathrm{AO}(2n-1,q)) + 14.2q^n + 8.2q^{n+1}.$$

By induction, $k(\mathrm{AO}(2n-1,q)) \le 20q^n$, so the result follows since

$$20q^n + 14.2q^n + 8.2q^{n+1} \le 20q^{n+1}$$

for $q \ge 3$. □

Corollary 6.12. *Let q be odd. Then $k(\mathrm{AO}(2n+1,q)) \le q^{2n+1}$.*

Proof. By the previous corollary, the result holds if $20 \le q^n$. So we need only check the cases $n=0$, $n=1$, or $n=2,\ q=3$. The generating function (Theorem 6.3) implies that $k(\mathrm{AO}(1,q)) = (q+3)/2$ and $k(\mathrm{AO}(3,q)) = (q^2+10q+5)/2$, and shows that the exact value of $k(\mathrm{AO}(5,3))$ is less than 243. □

Next we turn to the case of even characteristic.

Corollary 6.13. *Let q be even. Then $k(\mathrm{AO}^{\pm}(2n,q)) \le 60q^n$.*

Proof. From Theorem 6.7, $k(\mathrm{AO}^{+}(2n,q)) + k(\mathrm{AO}^{-}(2n,q))$ is equal to the coefficient of u^n in

$$\prod_i \frac{1-u^i}{1-qu^i}\frac{1}{1-u}\left[\prod_i \frac{(1+u^i)(1+u^{2i-1})^2}{(1-u^i)} + 4(q-1)u\prod_i \frac{(1-u^{4i})}{(1-u^{4i-2})(1-u^i)^2}\right].$$

Arguing as for the symplectic groups, this is at most q^n multiplied by

$$\frac{1}{1-1/q}\left[\prod_i \frac{(1+1/q^i)(1+1/q^{2i-1})^2}{(1-1/q^i)} + \frac{4(q-1)}{q}\prod_i \frac{(1-1/q^{4i})}{(1-1/q^{4i-2})(1-1/q^i)^2}\right],$$

which is at most $111.6q^n$ since $q \ge 2$.

From Theorem 6.8, $k(\mathrm{AO}^{+}(2n,q)) - k(\mathrm{AO}^{-}(2n,q))$ is equal to the coefficient of u^n in

$$\frac{1}{1-u}\prod_i \frac{1-u^{2i-1}}{1-qu^{2i}}.$$

Since this is analytic for $|u| < q^{-1}+\epsilon$, Lemma 2.4 gives that $k(\mathrm{AO}^{+}(2n,q)) - k(\mathrm{AO}^{-}(2n,q))$ is at most

$$q^n\frac{1}{1-1/q}\prod_i \frac{1+1/q^{2i-1}}{1-1/q^{2i-1}} \le 8.4q^n.$$

The corollary now follows since $(111.6+8.4)/2 = 60$. ☐

Corollary 6.14. *Let q be even. Then $k(\mathrm{AO}^{\pm}(2n,q)) \le q^{2n}$ except for*

$$k(\mathrm{AO}^{+}(2,2)) = 5, \quad k(\mathrm{AO}^{-}(2,2)) = 5, \quad k(\mathrm{AO}^{+}(4,2)) = 20,$$
$$k(\mathrm{AO}^{-}(4,2)) = 18, \quad \textit{and} \quad k(\mathrm{AO}^{-}(6,2)) = 65.$$

Proof. By the previous corollary, $k(\mathrm{AO}^{\pm}(2n,q)) \le q^{2n}$ if $60 \le q^n$. So we need only check the cases $n=1$ or $q=2,\ 2\le n\le 5$, or $q=4,\ n=2$. So the only infinite family of cases to check is when $n=1$, in which case the generating function gives $k(\mathrm{AO}^{\pm}(2,q)) = 5q/2$. The remaining finite number of cases can be checked immediately from the generating function. ☐

References

[Andrews 1976] G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications **Vol. 2**, Addison-Wesley, 1976. MR Zbl

[Curtis and Reiner 1962] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **XI**, Interscience, 1962. MR Zbl

[Fulman and Guralnick 2012] J. Fulman and R. Guralnick, "Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements", *Trans. Amer. Math. Soc.* **364**:6 (2012), 3023–3070. MR Zbl

[Guralnick and Maróti 2013] R. M. Guralnick and A. Maróti, "On the non-coprime $k(GV)$-problem", *J. Algebra* **385** (2013), 80–101. MR

[Guralnick and Tiep 2005] R. M. Guralnick and P. H. Tiep, "The non-coprime $k(GV)$ problem", *J. Algebra* **293**:1 (2005), 185–242. MR

[Keller 2006] T. M. Keller, "Fixed conjugacy classes of normal subgroups and the $k(GV)$-problem", *J. Algebra* **305**:1 (2006), 457–486. MR Zbl

[Lidl and Niederreiter 1994] R. Lidl and H. Niederreiter, *Introduction to finite fields and their applications*, 1st ed., Cambridge University Press, 1994. MR Zbl

[Liebeck and Seitz 2012] M. W. Liebeck and G. M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs **180**, American Mathematical Society, Providence, RI, 2012. MR Zbl

[Macdonald 1981] I. G. Macdonald, "Numbers of conjugacy classes in some finite classical groups", *Bull. Austral. Math. Soc.* **23**:1 (1981), 23–48. MR Zbl

[Maslen and Rockmore 1997] D. K. Maslen and D. N. Rockmore, "Separation of variables and the computation of Fourier transforms on finite groups, I", *J. Amer. Math. Soc.* **10**:1 (1997), 169–214. MR Zbl

[Nakada and Shinoda 1990] Y. Nakada and K.-i. Shinoda, "The characters of a maximal parabolic subgroup of $\mathrm{GL}_n(\boldsymbol{F}_q)$", *Tokyo J. Math.* **13**:2 (1990), 289–300. MR Zbl

[Odlyzko 1995] A. M. Odlyzko, "Asymptotic enumeration methods", pp. 1063–1229, Chapter 22 in *Handbook of combinatorics*, vol. 2, edited by R. L. Graham et al., Elsevier, 1995. MR Zbl

[Robinson 2004] G. R. Robinson, "Bounding numbers and heights of characters in p-constrained groups", pp. 307–317 in *Finite groups* 2003, edited by C. Y. Ho et al., de Gruyter, Berlin, 2004. MR Zbl

[Schmid 2007] P. Schmid, *The solution of the $k(GV)$ problem*, ICP Advanced Texts in Mathematics **4**, Imperial College Press, London, 2007. MR Zbl

[Wall 1963] G. E. Wall, "On the conjugacy classes in the unitary, symplectic and orthogonal groups", *J. Austral. Math. Soc.* **3** (1963), 1–62. MR Zbl

Communicated by Michael J. Larsen
Received 2022-01-25 Revised 2023-01-10 Accepted 2023-09-03

fulman@usc.edu Department of Mathematics, University of Southern California, Los Angeles, United States

guralnic@usc.edu Department of Mathematics, University of Southern California, Los Angeles, United States

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at the ANT website.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in *ANT* are usually in English, but articles written in other languages are welcome.

Length There is no a priori limit on the length of an *ANT* article, but *ANT* considers long articles only if the significance-to-length ratio is appropriate. Very long manuscripts might be more suitable elsewhere as a memoir instead of a journal article.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Algebra & Number Theory

Volume 18 No. 6 2024