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VANISHING VISCOSITY PLANE PARALLEL CHANNEL FLOW AND RELATED SINGULAR PERTURBATION PROBLEMS

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We study a special class of solutions to the three-dimensional Navier–Stokes equations $\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu$, with no-slip boundary condition, on a domain of the form $\Omega = \{(x, y, z) : 0 \le z \le 1\}$, dealing with velocity fields of the form $u^\nu(t, x, y, z) = (v^\nu(t, z), w^\nu(t, x, z), 0)$, describing plane-parallel channel flows. We establish results on convergence $u^\nu \to u^0$ as $\nu \to 0$, where u^0 solves the associated Euler equations. These results go well beyond previously established L^2-norm convergence, and provide a much more detailed picture of the nature of this convergence. Carrying out this analysis also leads naturally to consideration of related singular perturbation problems on bounded domains.

1. Introduction

We look at a special class of solutions to the three-dimensional Navier–Stokes equations on a region $\Omega \subset \mathbb{R}^3$ with boundary:

$$\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu + F, \quad \operatorname{div} u^\nu = 0, \tag{1.0.1}$$

with no-slip boundary data

$$u^\nu(t, q) = B(t, q), \quad q \in \partial\Omega, \tag{1.0.2}$$

given $B(t, q)$ a vector field tangent to $\partial\Omega$. This class consists of what are called plane parallel channel flows. They involve a domain of the form

$$\Omega = \{(x, y, z) : 0 \le z \le 1\}, \tag{1.0.3}$$

velocity fields of the form

$$u^\nu(t, x, y, z) = (v^\nu(t, z), w^\nu(t, x, z), 0), \tag{1.0.4}$$

and external forces of the form

$$F = (f(t, z), g(t, x, z), 0). \tag{1.0.5}$$

This class is mentioned by X. Wang [2001] as a class to which his main theorem on $L^2(\Omega)$-convergence as $\nu \to 0$ (itself a refinement of earlier work of T. Kato [1984]) applies.

There is substantial motivation to obtain a much more detailed picture of the behavior as $\nu \to 0$, including convergence in much stronger topologies, especially away from the boundary, if the initial data and forces satisfy appropriate smoothness hypotheses, and also an analysis of the boundary layer

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on which the solution can make an abrupt transition. The goal of this paper is to establish such stronger results for this class of fluid flows, and to explore some related singular perturbation problems that arise in the course of the analysis.

To begin the analysis, we note that if u^ν has the form (1.0.4) then $\operatorname{div} u^\nu = 0$ and

$$\nabla_{u^\nu} u^\nu = (0, v^\nu(t,z)\partial_x w^\nu(t,x,z), 0), \tag{1.0.6}$$

and hence

$$\operatorname{div} \nabla_{u^\nu} u^\nu = 0. \tag{1.0.7}$$

Thus we can take $p^\nu \equiv 0$ in (1.0.1) and rewrite the system (1.0.1) as

$$\begin{aligned} \frac{\partial v^\nu}{\partial t} &= \nu \frac{\partial^2 v^\nu}{\partial z^2} + f(t,z), \\ \frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} &= \nu \Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2} \Big) + g(t,x,z). \end{aligned} \tag{1.0.8}$$

(Note: The equations stated on p. 228 of [Wang 2001] have two misprints.) The boundary conditions take the form

$$\begin{aligned} v^\nu(t,z) &= a(t,z), \qquad z = 0, 1, \\ w^\nu(t,x,z) &= b(t,x,z), \quad z = 0, 1. \end{aligned} \tag{1.0.9}$$

We take initial data independent of ν:

$$\begin{aligned} v^\nu(0,z) &= V(z), \\ w^\nu(0,x,z) &= W(x,z). \end{aligned} \tag{1.0.10}$$

One wants to establish convergence of u^ν to u^0, the solution to the Euler equation

$$\partial_t u^0 + \nabla_{u^0} u^0 + \nabla p^0 = F, \quad \operatorname{div} u^0 = 0, \tag{1.0.11}$$

with boundary condition

$$u^0(t,p) \parallel \partial\Omega, \tag{1.0.12}$$

for $p \in \partial\Omega$, and initial condition

$$u^0(0,x,y,z) = (V(z), W(x,z), 0). \tag{1.0.13}$$

We have

$$u^0(t,x,y,z) = (v^0(t,z), w^0(t,x,z), 0), \tag{1.0.14}$$

satisfying

$$\frac{\partial v^0}{\partial t} = f(t,z), \qquad \frac{\partial w^0}{\partial t} + v^0 \frac{\partial w^0}{\partial x} = g(t,x,z). \tag{1.0.15}$$

Initial data are as in (1.0.10).

We begin the analysis of the convergence of v^ν to v^0 and of w^ν to w^0 in Chapter 2. For simplicity we take vanishing forces and boundary velocity. We also take functions to be periodic in x and work on

$$\overline{\mathcal{O}} = \{(x,z) : x \in \mathbb{R}/\mathbb{Z},\ z \in [0,1]\}. \tag{1.0.16}$$

In Section 2.1 we take the particular case $V \equiv 1$ in (1.0.10) and in Section 2.2 we consider general initial velocities of the form (1.0.10). We see that while the convergence of v^ν to v^0 has a simple nature, with a boundary layer phenomenon easily treatable via the method of images, the nature of the convergence of w^ν to w^0 is much more subtle. One tool we use to analyze w^ν is to compare it with the solution to the analogue of the second equation in (1.0.8) with v^ν replaced by $V(z)$. To state the strategy more abstractly, we analyze the solution to

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X_\nu w^\nu, \quad w^\nu\big|_{\mathbb{R}\times\partial\mathbb{O}} = 0, \tag{1.0.17}$$

where $\Delta = \partial_x^2 + \partial_z^2$ and $X_\nu = v^\nu(t,z)\partial_x$, by considering the solution to

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X w^\nu + g^\nu, \quad w^\nu\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \tag{1.0.18}$$

where $X = V(z)\partial_x$ and $g^\nu = (X - X_\nu)w^\nu$. To tackle (1.0.17), we use Duhamel's formula, which gives

$$w^\nu(t) = e^{t(\nu\Delta - X)}W + \int_0^t e^{(t-s)(\nu\Delta - X)} g^\nu(s)\,ds. \tag{1.0.19}$$

This leads to some successful estimates, produced in §Section 2.1–2.2, on the difference $R^\nu(t,x,z) = w^\nu(t) - e^{t(\nu\Delta - X)}W$. We show that for each $p \in [1,\infty),\ t \in (0,T]$,

$$\|R^\nu(t,\cdot)\|_{L^p(\mathbb{O})} \le C_p \nu^{1/2p} t^{1+1/2p}, \tag{1.0.20}$$

and that, as $\nu \to 0$,

$$R^\nu(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T],\ (x,z,\nu) \in \mathbb{O}_\eta, \tag{1.0.21}$$

where $\mathbb{O}_\eta = \{(x,z,\nu) : \operatorname{dist}((x,z),\partial\mathbb{O}) \ge \eta(\nu)\}$, for each $\eta(\nu)$ satisfying $\eta(\nu)/\nu^{1/2} \to \infty$ as $\nu \to 0$.

Thus much information about w^ν is revealed by the behavior of $e^{t(\nu\Delta - X)}W$. In case $V \equiv 1$, the operators X and Δ commute, and the behavior of $e^{t(\nu\Delta - X)}W = e^{-tX}e^{t\nu\Delta}W$ is also quite accessible via the method of images. For general $V(z)$, the behavior of $e^{t(\nu\Delta - X)}$ requires further study.

Chapter 3 is devoted to the study of $e^{t(\nu\Delta - X)}$. It is natural to work in a more general setting than in Chapter 2. In place of (1.0.16), we take $\overline{\mathbb{O}}$ to be a compact Riemannian manifold with smooth boundary, with Laplace-Beltrami operator Δ, and we take a smooth vector field X on $\overline{\mathbb{O}}$ satisfying

$$X \parallel \partial\mathbb{O}, \quad \operatorname{div} X = 0. \tag{1.0.22}$$

We obtain convergence results

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f \tag{1.0.23}$$

as $\nu \to 0$, in a number of function spaces, including L^q-Sobolev spaces $H^{\sigma,q}(\mathbb{O})$, for $q \in [2,\infty),\ \sigma \in [0, 1/q)$, and also spaces

$$\mathcal{V}^k(\mathbb{O}) = \{f \in L^2(\mathbb{O}) : Y_1 \cdots Y_j f \in L^2(\mathbb{O}),\ \forall\, j \le k,\ Y_\ell \in \mathfrak{X}^1\}, \tag{1.0.24}$$

where $\mathfrak{X}^1$ consists of smooth vector fields on $\overline{\mathbb{O}}$ that are tangent to $\partial\mathbb{O}$.

We also produce a layer potential analysis of $e^{t(\nu\Delta - X)}f$, which provides a detailed picture of the boundary layer behavior as $\nu \to 0$. To do this, we find it convenient to work with

$$v^\nu(t) = e^{tX} e^{t(\nu\Delta - X)} f. \tag{1.0.25}$$

One of the main results is given in Proposition 3.7.4, that for $I = [0, T]$, $\delta > 0$,

$$\|v^\nu - (f - 2\mathfrak{D}^0_\nu f^b)\|_{L^\infty(I\times\mathbb{O})} \le C(I)\nu^{1/2}\|f\|_{C^{1,\delta}(\overline{\mathbb{O}})}, \tag{1.0.26}$$

where $f^b(t, y) = \chi_{\mathbb{R}^+}(t) f(y)$ and $\mathfrak{D}^0_\nu$ is a certain layer potential operator:

$$\mathfrak{D}^0_\nu f^b(t, x) = \nu \int_0^t \int_{\partial\mathbb{O}} f(y) \frac{\partial H_0}{\partial n_{s,y}}(\nu, s, t, x, y)\, dS_s(y)\, ds. \tag{1.0.27}$$

See Section 3.7 for more details, including the definitions of $dS_s(y)$, $\partial/\partial n_{s,y}$, and the Gaussian-type integral kernel $H_0(\nu, s, t, x, y)$.

In Chapter 4 we again consider solutions to (1.0.17). Here we work on a compact Riemannian manifold with boundary $\overline{\mathbb{O}}$ as in Chapter 3. We take X_ν to be a family of time dependent vector fields, suitably generalizing the class $X_\nu = v^\nu(t, z)\partial_x$ that arose in Chapter 2, converging to X in a similar way as $v^\nu(t, z)\partial_x$ converges to $V(z)\partial_x$. The main results are given in Propositions 4.2.1–4.2.4. We obtain convergence results

$$w^\nu(t) \to e^{-tX} f \tag{1.0.28}$$

as $\nu \to 0$, in $\mathcal{V}^k(\mathbb{O})$, and in $L^p(\mathbb{O})$, for $1 \le p < \infty$. Analogues of (1.0.19) play a role in the analysis, and we make strong use of results of Chapter 3.

In Chapter 5 we return to the specific setting of plane parallel channel flow and draw further conclusions about the convergence of v^ν to v^0 and of w^ν to w^0. We extend the scope of Chapter 2 by allowing for some nonzero boundary velocity, arising from rigidly translating the flat boundary faces. We take boundary data $B(t, q)$ of the form

$$B(t, x, z) = (\alpha_j(t), \beta_j(t), 0), \qquad z = j \in \{0, 1\}, \tag{1.0.29}$$

and allow $\alpha_j(t)$ and $\beta_j(t)$ to be fairly rough. We start with the special case $(\alpha_j(t), 0, 0)$, giving motions of the boundary parallel to the x-axis.

The spaces $\mathcal{V}^k(\mathbb{O})$ in (1.0.24) are special cases of "weighted b-Sobolev spaces," introduced and studied in [Melrose 1993]. In Appendix A we discuss this point and use it to establish some complex interpolation results for these spaces, which are of use in Sections 3.3 and 4.2.

This paper is a companion to [Lopes Filho et al. 2007], whose goal was to give a precise analysis of the convergence of the solution of the Navier–Stokes equation, as the vorticity tends to zero, to a steady solution of the Euler equation for 2D circularly symmetric flow in a disk or annulus, sharpening L^2 analyses done in [Matsui 1994], [Bona and Wu 2002], and [Lopes Filho et al. 2008].

2. First results on plane parallel channel flows

Here we start our investigation of the convergence of v^ν and w^ν as $\nu \to 0$, when these functions are solutions to (1.0.8) (with $f = g = 0$ and vanishing boundary condition). The main result of this chapter is the estimate (2.2.11) on

$$w^\nu(t,x,z) - e^{t(\nu\Delta - X)}W(x,z), \tag{2.0.1}$$

together with some of its consequences. To carry on, we need to understand the second term in (2.0.1). This motivates the work of Chapter 3.

2.1. ***Particular case.*** Let us take $f \equiv g \equiv 0$ in (1.0.8) and in (1.0.15), and

$$V \equiv 1, \quad W = W(x,z) \tag{2.1.1}$$

in (1.0.10). Consequently we have

$$v^0(t,z) \equiv 1, \quad w^0(t,x,z) = W(x-t,z) \tag{2.1.2}$$

as the solution to the Euler equations. Let us also take $a \equiv b \equiv 0$ in (1.0.9), i.e., boundary conditions

$$v^\nu(t,z) = w^\nu(t,x,z) = 0, \quad z = 0, 1. \tag{2.1.3}$$

Consequently, for the solution $(v^\nu, w^\nu, 0)$ to the Navier–Stokes equation, we have first of all that

$$v^\nu(t,z) = e^{t\nu A}v_0(z) = e^{t\nu A}1(z), \tag{2.1.4}$$

where A is the self-adjoint operator on $L^2([0,1])$ defined by

$$\mathfrak{D}(A) = H^2([0,1]) \cap H_0^1([0,1]), \quad A = \partial_z^2 \text{ on } \mathfrak{D}(A). \tag{2.1.5}$$

One can analyze (2.1.4) via the method of images to get a good picture of the boundary layer near $z = 0,\ 1$. Then the equation for w^ν becomes

$$\frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} = \nu\Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2}\Big), \tag{2.1.6}$$

with initial condition given in (2.1.1) and boundary condition given in (2.1.3).

Let us assume $W(x,z)$ in (2.1.1) is smooth and periodic of period 1 in x, so

$$W \in C^\infty(\overline{\mathbb{O}}), \quad \overline{\mathbb{O}} = \{(x,z) : x \in \mathbb{R}/\mathbb{Z},\ z \in [0,1]\}. \tag{2.1.7}$$

Elementary estimates imply

$$\|w^\nu(t)\|_{L^p(\mathbb{O})} \le \|W\|_{L^p(\mathbb{O})}, \quad 1 \le p \le \infty. \tag{2.1.8}$$

Note that for $k \in \mathbb{Z}^+$,

$$w_k^\nu = \partial_x^k w^\nu \tag{2.1.9}$$

satisfies

$$\frac{\partial w_k^\nu}{\partial t} + v^\nu \frac{\partial w_k^\nu}{\partial x} = \nu\Delta w_k^\nu, \tag{2.1.10}$$

where we have set

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{2.1.11}$$

Also

$$w_k^\nu(t,x,z) = 0, \quad z = 0,\ 1. \tag{2.1.12}$$

Hence, parallel to (2.1.8), we have

$$\|w_k^\nu(t)\|_{L^p(\mathcal{O})} \le \|\partial_x^k W\|_{L^p(\mathcal{O})}, \quad 1 \le p \le \infty. \tag{2.1.13}$$

To obtain a finer analysis of $w^\nu(t,x,z)$, let us rewrite (2.1.6) as

$$\frac{\partial w^\nu}{\partial t} = -\partial_x w^\nu + \nu \Delta w^\nu + (1 - v^\nu)\frac{\partial w^\nu}{\partial x}. \tag{2.1.14}$$

Then Duhamel's formula gives

$$w^\nu(t,x,z) = e^{t(\nu\Delta - \partial_x)} W(x,z) + \int_0^t e^{(t-s)(\nu\Delta-\partial_x)} \Big[(1 - v^\nu(s,z)) \frac{\partial w^\nu}{\partial x}(s,x,z)\Big]\, ds. \tag{2.1.15}$$

Here Δ stands for the self adjoint operator given by (2.1.11), with

$$\mathcal{D}(\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \tag{2.1.16}$$

Note that $e^{t\nu\Delta}$ and $e^{-t\partial_x}$ are commuting semigroups, with $e^{-t\partial_x} f(x,z) = f(x-t,z)$. Hence we have

$$w^\nu(t,x,z) = e^{t\nu\Delta} W(x-t,z) + \int_0^t e^{(t-s)\nu\Delta} \big[(1 - v^\nu(s,z)) w_1^\nu(s, x-t+s, z)\big]\, ds, \tag{2.1.17}$$

where, as in (2.1.9), we have $w_1^\nu = \partial_x w^\nu$. Let us write (2.1.16) as

$$w^\nu(t,x,z) = e^{t\nu\Delta} W(x-t,z) + R^\nu(t,x,z). \tag{2.1.18}$$

By the method of images (or otherwise) we have a clear picture of the first term on the right side of (2.1.18). Let us estimate the remainder, $R^\nu(t,x,z)$. By (2.1.13) and the positivity of $e^{(t-s)\nu\Delta}$, we have

$$|R^\nu(t,x,z)| \le C \int_0^t e^{(t-s)\nu\Delta} |1 - v^\nu(s,z)|\, ds, \tag{2.1.19}$$

since $\partial_x W \in L^\infty(\mathcal{O})$. The analysis of (2.1.4) via the method of images gives

$$|1 - v^\nu(s,z)| \le C_T \varphi\big((s\nu)^{-1/2}\delta(z)\big), \tag{2.1.20}$$

for $s \in [0,T]$, where $\delta(z) = \mathrm{dist}(z, \{0,1\})$ and $\varphi(\zeta)$ is rapidly decreasing as $\zeta \to \infty$. Hence, for $p \in [1,\infty)$,

$$\|R^\nu(t,\cdot)\|_{L^p(\mathcal{O})} \le C \int_0^t \Big(\int_0^1 |1 - v^\nu(s,z)|^p\, dz \Big)^{1/p} ds \le C_p \nu^{1/2p} t^{1+1/2p}. \tag{2.1.21}$$

Furthermore we have, as $\nu \to 0$,

$$R^\nu(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T],\ \delta(z) \ge \delta_0, \tag{2.1.22}$$

given $\delta_0 > 0$. Indeed, given $\eta(\nu)$ such that

$$\frac{\eta(\nu)}{\nu^{1/2}} \to \infty \quad \text{as} \quad \nu \to 0, \tag{2.1.23}$$

and

$$\mathcal{O}_\eta = \{(x,z,\nu) : x \in \mathbb{R}/\mathbb{Z},\ \delta(z) \ge \eta(\nu)\}, \tag{2.1.24}$$

we have

$$R^\nu(t,x,z) \to 0, \quad \text{uniformly for } t \in [0,T], \ (x,z,\nu) \in \mathbb{O}_\eta. \tag{2.1.25}$$

However, (2.1.15)–(2.1.19) do not reveal the fine structure of $w^\nu(t,x,z)$ on the boundary layer. Some other approach will be required for this.

2.2. *More general case.* As in Section 2.1, we take $f \equiv g \equiv 0$ in (1.0.8), but now we extend (2.1.1) to the more general case

$$v^\nu(0,z) = V(z) \in C^\infty(I), \quad w^\nu(0,x,z) = W(x,z) \in C^\infty(\overline{\mathbb{O}}), \tag{2.2.1}$$

with $\overline{\mathbb{O}}$ as in (2.1.7). Then (2.1.2) is modified to

$$v^0(t,z) = V(z), \quad w^0(t,x,z) = W(x - tV(z), z). \tag{2.2.2}$$

We retain the boundary conditions (2.1.3), i.e.,

$$v^\nu(t,z) = w^\nu(t,x,z) = 0, \quad z = 0, 1. \tag{2.2.3}$$

Thus, in place of (2.1.4), we have

$$v^\nu(t,z) = e^{t\nu A} V(z), \tag{2.2.4}$$

again with A as in (2.1.5). With these modifications, one still has the Equation (2.1.6) for w^ν. We continue to have the estimates (2.1.8) on $\|w^\nu(t)\|_{L^p(\mathbb{O})}$. We also have the estimates (2.1.13) on $\|w_k^\nu(t)\|_{L^p}$, where $w_k^\nu = \partial_x^k w^\nu$.

To obtain a finer analysis of $w^\nu(t,x,z)$, we use the following modification of (2.1.14):

$$\frac{\partial w^\nu}{\partial t} = -V(z)\partial_x w^\nu + \nu \Delta w^\nu + (V - v^\nu)\frac{\partial w^\nu}{\partial x}. \tag{2.2.5}$$

Then Duhamel's formula gives the following variant of (2.1.15):

$$w^\nu(t,x,z) = e^{t(\nu\Delta - V\partial_x)} W(x,z) + \int_0^t e^{(t-s)(\nu\Delta - V\partial_x)} \Big[(V - v^\nu(s)) \frac{\partial w^\nu}{\partial x}(s) \Big]\, ds. \tag{2.2.6}$$

Here $\nu\Delta - V\partial_x$ generates a contraction semigroup on $L^2(\mathbb{O})$ with domain

$$\mathfrak{D}(\nu\Delta - V\partial_x) = H_0^1(\mathbb{O}) \cap H^2(\mathbb{O}). \tag{2.2.7}$$

It also generates a contraction semigroup on $L^p(\mathbb{O})$ for $1 \le p \le \infty$, strongly continuous for $p \in [1,\infty)$, but not for $p = \infty$. We mention that the Trotter product formula—for which see [Trotter 1959] or [Taylor 1996, Chapter 11, Appendix A]—holds here. Given $p \in [1,\infty)$ and $f \in L^p(\mathbb{O})$, we have

$$e^{t(\nu\Delta - V\partial_x)} f = \lim_{n\to\infty} \big(e^{(t/n)\nu\Delta} e^{-(t/n)V\partial_x} \big)^n f, \quad \text{in } L^p\text{-norm}. \tag{2.2.8}$$

Of course,

$$e^{-sV\partial_x} f(x,z) = f(x - sV(z), z). \tag{2.2.9}$$

To proceed, we have, parallel to (2.1.18)–(2.1.19),

$$w^\nu(t,x,z) = e^{t(\nu\Delta - V\partial_x)} W(x,z) + R^\nu(t,x,z), \tag{2.2.10}$$

with

$$|R^\nu(t,x,z)| \le C\int_0^t e^{(t-s)(\nu\Delta - V\partial_x)}|V - v^\nu(s)|\,ds = C\int_0^t e^{(t-s)\nu\Delta}|V(z) - v^\nu(s,z)|\,ds, \tag{2.2.11}$$

since $\partial_x W \in L^\infty(\mathbb{O})$. Again, to get this, one uses the estimate (2.1.13) with $k=1$, and the positivity of $e^{(t-s)(\nu\Delta - V\partial_x)}$. For the last identity in (2.2.11), one uses the fact that $V(z) - v^\nu(s,z)$ is independent of x. Once we have (2.2.11), we can again apply the method of images to estimate

$$|V(z) - v^\nu(s,z)| \le C_T\varphi\big((s\nu)^{-1/2}\delta(z)\big), \tag{2.2.12}$$

as in (2.1.20), except now we have only $\varphi(\zeta) \le C(1+\zeta^2)^{-1}$. This is enough for the estimates (2.1.21)–(2.1.25) on $R^\nu(t,x,z)$ continue to hold.

In the current setting, the term $e^{t(\nu\Delta - V\partial_x)}W$ requires a more vigorous investigation for general smooth $V(z)$ on $[0,1]$ than it did in the case $V\equiv 1$, considered in Section 2.1. We want to establish results of the form

$$e^{t(\nu\Delta - X)}f \to e^{-tX}f, \quad \text{as } \nu \to 0, \tag{2.2.13}$$

in L^p-norm, for all $f \in L^p(\mathbb{O})$, where

$$X = V(z)\partial_x. \tag{2.2.14}$$

We also want to investigate such convergence in other function spaces. We will obtain such results, in a more general context, in the chapters that follow.

3. Analysis of solutions to $u_t = \nu\Delta u - Xu$

We examine the solution operator $e^{t(\nu\Delta - X)}f = u(t)$, given by

$$\frac{\partial u}{\partial t} = \nu\Delta u - Xu, \quad u(0) = f, \quad u(t,x) = 0 \text{ for } x \in \partial\mathbb{O}. \tag{3.0.1}$$

We work in a more general context than in Section 2.2. Assume $\overline{\mathbb{O}}$ is a compact Riemannian manifold, with smooth boundary $\partial\mathbb{O}$, and with Laplace-Beltrami operator Δ, and X is a smooth, real vector field on $\overline{\mathbb{O}}$, satisfying

$$X \parallel \partial\mathbb{O}, \quad \operatorname{div} X = 0. \tag{3.0.2}$$

Under such hypotheses, for each $\nu \in (0,\infty)$, $e^{t(\nu\Delta - X)}$ is a strongly continuous contraction semigroup on $L^p(\mathbb{O})$ for each $p\in[1,\infty)$. Furthermore, the Trotter product formula holds; given $p\in[1,\infty)$, $f\in L^p(\mathbb{O})$,

$$e^{t(\nu\Delta - X)}f = \lim_{n\to\infty}\Big(e^{(t/n)\nu\Delta}e^{-(t/n)X}\Big)^n f, \quad \text{in } L^p\text{-norm}. \tag{3.0.3}$$

Our goal is to obtain precise results on convergence

$$e^{t(\nu\Delta - X)}f \to e^{-tX}f, \tag{3.0.4}$$

as $\nu \searrow 0$. In particular, we establish convergence in a variety of function spaces. In Section 3.1 we establish such convergence in the L^q-Sobolev space $H^{s,q}(\mathbb{O})$ for $q \in [2,\infty)$ and $s \in [0, 1/q)$. In

Section 3.2 we study local convergence. For this, it is convenient to work with

$$v^\nu(t) = e^{tX} e^{t(\nu\Delta - X)} f, \tag{3.0.5}$$

which solves

$$\frac{\partial v^\nu}{\partial t} = \nu L(t) v^\nu, \quad v^\nu(0) = f, \tag{3.0.6}$$

with boundary condition $v^\nu = 0$ on $\mathbb{R}^+ \times \partial\mathbb{O}$, where $L(t)$ is the smooth family of strongly elliptic differential operators given by $L(t) = e^{tX}\Delta e^{-tX}$. Given $\Omega_1 \Subset \Omega_0 \subset\subset \mathbb{O}$, we show that if $f \in L^2(\mathbb{O})$ and $f \in H^k(\Omega_0)$, then $v^\nu(t) \to f$ in $H^k(\Omega_1)$. In Section 3.3 we establish convergence in the space

$$\mathcal{V}^k(\mathbb{O}) = \{f \in L^2(\mathbb{O}) : Y_1 \cdots Y_j f \in L^2(\mathbb{O}),\ \forall\, j \le k,\ Y_\ell \in \mathfrak{X}^1\}, \tag{3.0.7}$$

where $\mathfrak{X}^1$ consists of all smooth vector fields on $\overline{\mathbb{O}}$ that are tangent to $\partial\mathbb{O}$. In Section 3.4 we show that the Laplace operator, with Dirichlet boundary condition, generates a holomorphic semigroup on $\mathcal{V}^k(\mathbb{O})$. This result is peripheral to the other results of this chapter, but it will prove useful in Section 4.1.

In Section 3.5 we extend the results of Section 3.1 to convergence in $H^{\sigma,q}$ for all $q \in [2,\infty)$, $\sigma \ge 0$, in case $\mathbb{O}$ is replaced by a compact manifold without boundary, M. These results are relatively easy, since it is only the presence of a boundary that causes a problem. They are recorded here to lay a foundation for the work in §Section 3.6–3.7. Section 3.6 is devoted to constructing a parametrix for the solution of (3.0.6) on $\mathbb{R}^+ \times M$, valid uniformly for $\nu \in (0,1]$, and with increased precision as $\nu \searrow 0$. The construction here is parallel to, but somewhat more elaborate than the construction of a parametrix for the heat equation $(\partial_t - \Delta)u = 0$ on $\mathbb{R}^+ \times M$, yielding short time asymptotics. The parametrix constructed in Section 3.6 is used in Section 3.7 to produce a layer potential attack on solutions to (3.0.6) on $\mathbb{R}^+ \times \mathbb{O}$, yielding sharp results on convergence in (3.0.4), including a picture of the boundary layer behavior.

3.1. ***L^q-Sobolev estimates on $e^{t(\nu\Delta - X)}$.*** This section is devoted to L^q-Sobolev estimates. To begin, take $q = 2$. We have, for each $\nu > 0$,

$$\mathfrak{D}(\nu\Delta - X) = \{f \in H^2(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0\}, \tag{3.1.1}$$

$$\mathfrak{D}((\nu\Delta - X)^2) = \{f \in H^4(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0,\ \nu\Delta f - Xf|_{\partial\mathbb{O}} = 0\}, \tag{3.1.2}$$

and, for $k \ge 3$,

$$\mathfrak{D}((\nu\Delta - X)^k) = \{f \in H^{2k}(\mathbb{O}) : f|_{\partial\mathbb{O}} = 0,\ (\nu\Delta - X)^j f|_{\partial\mathbb{O}} = 0 \text{ for } j < k\}. \tag{3.1.3}$$

Comparison with analogous formulas for $\mathfrak{D}(\Delta^k)$ yields the following.

Proposition 3.1.1. *We have, for each* $\nu > 0$,

$$\mathfrak{D}((\nu\Delta - X)^k) = \mathfrak{D}(\Delta^k), \quad \textit{for } k = 1, 2. \tag{3.1.4}$$

Proof. The case $k = 1$ is immediate from (3.1.1). As for $k = 2$, note that if $f \in H^4(\mathbb{O})$ and $f|_{\partial\mathbb{O}} = 0$, then also $Xf|_{\partial\mathbb{O}} = 0$ (since $X \parallel \partial\mathbb{O}$), and hence $\Delta f|_{\partial\mathbb{O}} = 0 \Leftrightarrow (\nu\Delta - X)f|_{\partial\mathbb{O}} = 0$. □

As stated in Section 2.2, we want to establish results of the form

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f, \quad \text{as } \nu \to 0, \text{ in } L^p\text{-norm}, \tag{3.1.5}$$

for all $f \in L^p(\mathcal{O})$, $p \in [1, \infty)$. Since we know $e^{t(\nu\Delta - X)}$ is a contraction semigroup on $L^p(\mathcal{O})$, if we can establish (3.1.5) for f in a dense linear subspace $\mathcal{V}$ of $L^p(\mathcal{O})$, we will have it for all $f \in L^p(\mathcal{O})$. This is the approach we will take for $p \in [1, 2]$, using

$$\mathcal{V} = \mathcal{D}(\Delta^2) = \mathcal{D}((\nu\Delta - X)^2), \quad \text{given by (3.1.2).} \tag{3.1.6}$$

Given such f, $u(t) = e^{t(\nu\Delta - X)} f$ satisfies

$$\frac{\partial u}{\partial t} = -Xu + \nu\Delta u, \quad u(0) = f, \tag{3.1.7}$$

and belongs to $C([0, \infty), \mathcal{D}(\Delta^2)) \cap C^1([0, \infty), \mathcal{D}(\Delta))$. Duhamel's formula yields

$$u(t) = e^{-tX} f + \nu \int_0^t e^{-(t-s)X} \Delta u(s)\, ds. \tag{3.1.8}$$

Thus

$$\|e^{t(\nu\Delta - X)} f - e^{-tX} f\|_{L^p} \le \nu \int_0^t \|\Delta u(s)\|_{L^p}\, ds, \tag{3.1.9}$$

so we have (3.1.5) whenever we can obtain a favorable estimate on the right side of (3.1.9). The following lemma provides a key, first for $p = 2$.

Lemma 3.1.2. *Take $f \in \mathcal{V}$, given by (3.1.6), and set $u(t) = e^{t(\nu\Delta - X)} f$, with $\nu > 0$. Then there exists $K \in (0, \infty)$, independent of ν, such that*

$$\|\Delta u(t)\|_{L^2}^2 \le e^{2Kt} \|\Delta f\|_{L^2}^2. \tag{3.1.10}$$

Proof. We have

$$\begin{aligned} \frac{d}{dt} \|\Delta u(t)\|_{L^2}^2 &= 2\,\mathrm{Re}\,(\Delta \partial_t u, \Delta u)_{L^2} = 2\,\mathrm{Re}\,(\nu\Delta^2 u, \Delta u)_{L^2} - 2\,\mathrm{Re}\,(\Delta X u, \Delta u)_{L^2} \\ &\le -2\,\mathrm{Re}\,(\Delta X u, \Delta u)_{L^2} = -2\,\mathrm{Re}\,(X\Delta u, \Delta u)_{L^2} - 2\,\mathrm{Re}\,([\Delta, X]u, \Delta u)_{L^2} \\ &\le 2K \|\Delta u\|_{L^2}^2, \end{aligned} \tag{3.1.11}$$

with K independent of ν. The last estimate holds because

$$g \in \mathcal{D}(\Delta) \implies |(Xg, g)_{L^2}| \le K_1 \|g\|_{L^2}^2, \tag{3.1.12}$$

and

$$\begin{aligned} &u(t) \in \mathcal{D}(\Delta^2) \Rightarrow [\Delta, X]u(t) \in L^2(\mathcal{O}) \text{ and} \\ &\qquad \|[\Delta, X]u(t)\|_{L^2} \le \widetilde{K}_2 \|u(t)\|_{H^2} \le K_2 \|\Delta u(t)\|_{L^2}. \end{aligned} \tag{3.1.13}$$

The asserted estimate (3.1.10) follows. ☐

Proposition 3.1.3. *Given $p \in [1, \infty)$ and $f \in L^p(\mathcal{O})$, we have (3.1.5), with convergence in L^p-norm.*

Proof. For $p \in [1, 2]$, this follows from the operator bound $\|e^{t(\nu\Delta - X)}\|_{\mathcal{L}(L^p)} \le 1$, the denseness of $\mathcal{V}$ in $L^p(\mathcal{O})$, and the application of (3.1.10) to (3.1.9), which gives convergence in L^2-norm, and a fortiori in L^p-norm, for each $f \in \mathcal{V}$.

Suppose now that $p \in (2, \infty)$, with dual exponent $p' \in (1, 2)$. All considerations above apply with X replaced by $-X$, so we have

$$e^{t(\nu\Delta + X)} g \to e^{tX} g, \quad \text{as } \nu \to 0, \tag{3.1.14}$$

in $L^{p'}$-norm, for each $g \in L^{p'}$. This implies that for each $f \in L^p(\mathcal{O})$, convergence in (3.1.5) holds in the weak* topology of $L^p(\mathcal{O})$. Now, since e^{-tX} is an *isometry* on $L^p(\mathcal{O})$, we have

$$\|e^{-tX} f\|_{L^p} \geq \limsup_{\nu \to 0} \|e^{t(\nu\Delta - X)} f\|_{L^p}, \tag{3.1.15}$$

for each $f \in L^p(\mathcal{O})$. Since $L^p(\mathcal{O})$ is a *uniformly convex* Banach space for such p, this yields L^p-norm convergence in (3.1.5). □

To continue, we have from (3.1.10) the estimate

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta)} \leq e^{Kt} \|f\|_{\mathcal{D}(\Delta)}, \tag{3.1.16}$$

first for each $f \in \mathcal{V}$, hence for each $f \in \mathcal{D}(\Delta)$. Interpolation with the L^2- estimate then yields

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}((-\Delta)^{s/2})} \leq e^{Kt} \|f\|_{\mathcal{D}((-\Delta)^{s/2})}, \tag{3.1.17}$$

for each $s \in [0, 2]$, $f \in \mathcal{D}((-\Delta)^{s/2})$. Now

$$\mathcal{D}((-\Delta)^{s/2}) = H^s(\mathcal{O}), \quad \text{for } s \in \left[0, \tfrac{1}{2}\right), \tag{3.1.18}$$

so we have

$$\|e^{t(\nu\Delta - X)} f\|_{H^s(\mathcal{O})} \leq C e^{Kt} \|f\|_{H^s(\mathcal{O})}, \quad s \in \left[0, \tfrac{1}{2}\right), \tag{3.1.19}$$

where the factor of C might arise due to the choice of H^s-norm; the important fact is that C and K are independent of $\nu \in (0, \infty)$. We can interpolate the estimate (3.1.19) with

$$\|e^{t(\nu\Delta - X)} f\|_{L^p(\mathcal{O})} \leq \|f\|_{L^p(\mathcal{O})}, \quad 1 \leq p < \infty. \tag{3.1.20}$$

Using

$$[H^s(\mathcal{O}), L^p(\mathcal{O})]_\theta = H^{(1-\theta)s, q(\theta)}(\mathcal{O}), \quad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{3.1.21}$$

we have

$$\|e^{t(\nu\Delta - X)} f\|_{H^{\sigma,q}(\mathcal{O})} \leq C_{\sigma,q}\, e^{Kt} \|f\|_{H^{\sigma,q}(\mathcal{O})}, \tag{3.1.22}$$

valid for

$$2 \leq q < \infty, \quad \sigma q \in [0, 1). \tag{3.1.23}$$

We mention that similar arguments give analogous operator bounds on e^{-tX}, and also on e^{tX}.

Remark. In the absence of further compatibility conditions between X and Δ, one does *not* have

$$e^{-tX} : \mathcal{D}(\Delta^2) \to \mathcal{D}(\Delta^2). \tag{3.1.24}$$

Hence, typically, for $f \in \mathcal{D}(\Delta^2)$,

$$\sup_{\nu \in (0,1]} \|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta^2)} = \infty. \tag{3.1.25}$$

In some cases one does have (3.1.24), for example when X and Δ commute. In such a case, $e^{t(\nu\Delta - X)} = e^{\nu t\Delta} e^{-tX}$. It is our goal here to analyze $e^{t(\nu\Delta - X)}$ when one does not have this extra compatibility.

From (3.1.22), we have the following convergence result.

Proposition 3.1.4. *Let q, σ satisfy (3.1.23). Then, for each $t \in (0, \infty)$,*

$$f \in H^{\sigma,q}(\mathcal{O}) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.1.26}$$

in $H^{\sigma,q}$-norm.

Proof. Given $f \in H^{\sigma,q}(\mathcal{O})$, (3.1.22) implies $\{e^{t(\nu\Delta-X)} f : \nu \in (0,1]\}$ is bounded in $H^{\sigma,q}(\mathcal{O})$, for each $t \in (0, \infty)$, so there are weak* limit points. But Proposition 3.1.3 yields convergence to $e^{-tX} f$ in L^q-norm, so $e^{-tX} f$ is the only possible weak* limit point. Norm convergence in $H^{\tau,q}(\mathcal{O})$, for each $\tau < \sigma$, then follows from the compactness of the inclusion $H^{\sigma,q}(\mathcal{O}) \hookrightarrow H^{\tau,q}(\mathcal{O})$. Now we can pick $\sigma' > \sigma$ so that $\sigma' q < 1$, and take $f_k \in H^{\sigma',q}(\mathcal{O})$ so that $f_k \to f$ in $H^{\sigma,q}$-norm. We deduce from the argument just made that as $\nu \to 0$, $e^{t(\nu\Delta-X)} f_k \to e^{-tX} f_k$ in $H^{\sigma,q}$-norm, for each k. Application of (3.1.22) with f replaced by $f - f_k$ then finishes the proof. □

We move on to some convergence results for classes of data f that vanish on $\partial\mathcal{O}$.

Proposition 3.1.5. *For each $t \in (0, \infty)$,*

$$f \in \mathcal{D}(\Delta) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f \tag{3.1.27}$$

weak* in $\mathcal{D}(\Delta) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$, hence in H^s-norm for each $s < 2$.

Proof. Lemma 3.1.2 gives $\{e^{t(\nu\Delta-X)} f : \nu \in (0,1]\}$ bounded in $\mathcal{D}(\Delta)$ for each $f \in \mathcal{V}$, hence for each $f \in \mathcal{D}(\Delta)$, as noted in (3.1.16). Since we have convergence to $e^{-tX} f$ in L^2-norm, the weak* convergence in $\mathcal{D}(\Delta)$ follows. The norm convergence in $H^s(\mathcal{O})$ for each $s < 2$ then follows from compactness of the inclusion $H^2(\mathcal{O}) \hookrightarrow H^s(\mathcal{O})$. □

Proposition 3.1.6. *Let $C_b(\overline{\mathcal{O}}) = \{f \in C(\overline{\mathcal{O}}) : f|_{\partial\mathcal{O}} = 0\}$. Then for each $t \in (0, \infty)$,*

$$f \in C_b(\overline{\mathcal{O}}) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.1.28}$$

in the supremum norm, provided $\dim \mathcal{O} \le 3$.

Proof. For $\dim \mathcal{O} \le 3$, $\mathcal{D}(\Delta) \subset C(\overline{\mathcal{O}})$, and it is dense in $C_b(\overline{\mathcal{O}})$. Since $e^{t(\nu\Delta-X)}$ is a contraction on $C_b(\overline{\mathcal{O}})$, a standard argument yields (3.1.28) from (3.1.27). □

If the hypothesis in (3.1.28) is weakened to $f \in C(\overline{\mathcal{O}})$, results obtained above yield convergence, weak* in $L^\infty(\mathcal{O})$, but of course one does not have L^∞-norm convergence if f does not vanish on $\partial\mathcal{O}$. In Section 3.2 we will show that convergence does hold uniformly on compact subsets of $\mathcal{O}$.

3.2. *Local regularity and convergence results for $e^{t(\nu\Delta-X)}$.* Given a function f on $\mathcal{O}$, consider

$$v(t) = e^{tX} e^{t(\nu\Delta-X)} f. \tag{3.2.1}$$

We have

$$\frac{\partial v}{\partial t} = e^{tX}[X + \nu\Delta - X] e^{t(\nu\Delta-X)} f = \nu e^{tX} \Delta e^{t(\nu\Delta-X)} f. \tag{3.2.2}$$

Now

$$L(t) = e^{tX} \Delta e^{-tX} \tag{3.2.3}$$

is a one-parameter family of strongly elliptic differential operators on $\mathcal{O}$, depending smoothly on t, and (3.2.2) yields

$$\frac{\partial v}{\partial t} = \nu L(t) e^{tX} e^{t(\nu\Delta - X)} f, \tag{3.2.4}$$

so $v(t)$ is uniquely characterized by

$$\frac{\partial v}{\partial t} = \nu L(t) v, \quad v(0) = f, \quad v\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0. \tag{3.2.5}$$

We now prove the following local regularity result.

Proposition 3.2.1. *Let $f \in L^2(\mathcal{O})$ and assume Ω_j are smoothly bounded domains satisfying $\Omega_1 \Subset \Omega_0 \Subset \mathcal{O}$. Assume $k \in \mathbb{N}$ and $f \in H^k(\Omega_0)$. Then the solution $v = v^\nu$ to (3.2.5) belongs to $C([0,\infty), H^k(\Omega_1))$, and for each $T \in (0,\infty)$ we have*

$$\|v^\nu(t)\|^2_{H^k(\Omega_1)} + c_{Tk}\nu \int_0^t \|v^\nu(s)\|^2_{H^{k+1}(\Omega_1)}\, ds \le C_{Tk}\big(\|f\|^2_{H^k(\Omega_0)} + \|f\|^2_{L^2(\mathcal{O})}\big), \quad 0 \le t \le T, \tag{3.2.6}$$

with $c_{Tk},\ C_{Tk} \in (0,\infty)$, independent of $\nu \in \mathbb{R}^+$.

Proof. To start, note that

$$\frac{d}{dt}\|v\|^2_{L^2} = 2\nu(L(t)v, v) le - C\nu\|\nabla v\|^2_{L^2} + C'\nu\|v\|^2_{L^2}; \tag{3.2.7}$$

hence, for $0 \le t \le T$,

$$\|v(t)\|^2_{L^2(\mathcal{O})} + c_{T0}\nu \int_0^t \|\nabla v(s)\|^2_{L^2(\mathcal{O})}\, ds \le C_{T0}\|f\|^2_{L^2(\mathcal{O})}, \tag{3.2.8}$$

which contains (3.2.6) for $k = 0$. To proceed, take $\varphi \in C_0^\infty(\Omega_0)$ such that $\varphi = 1$ on a neighborhood of $\overline{\Omega}_1$. Then $w = \varphi v^\nu$ satisfies

$$\partial_t w = \nu L(t) w + \nu Y(t) v, \quad w(0) = \varphi f, \tag{3.2.9}$$

with

$$Y(t) = [\varphi, L(t)]. \tag{3.2.10}$$

Note that $Y(t)$ is a smooth family of differential operators of order 1. Now pick $m \in \{1, \dots, k\}$. We have, for $\|D^m w\|^2_{L^2} = \sum_{|\alpha|\le m} \|D^\alpha w\|^2_{L^2}$,

$$\begin{aligned}\frac{d}{dt}\|D^m w\|^2_{L^2} &= 2\nu(D^m[L(t)w + Y(t)v], D^m w)_{L^2} \\ &= 2\nu(L(t)D^m w, D^m w) + 2\nu([D^m, L(t)]w, D^m w) + 2\nu(D^m Y(t)v, D^m w) \\ &\le -C_1\nu\|D^{m+1}w\|^2_{L^2} + C_2\nu\|D^m w\|^2_{L^2} + C_3\nu\|D^{m-1}Y(t)v\|^2_{L^2}.\end{aligned} \tag{3.2.11}$$

(To get from the second line to the third, integrate by parts to put the term $2\nu(D^m Y(t)v, D^m w)$) in the form $2\nu(D^{m-1}Y(t)v, D^{m+1}w)$.) Hence we obtain, for $t \in [0,T]$,

$$\|D^m w(t)\|^2_{L^2} + c_{Tm}\nu \int_0^t \|D^{m+1}w(s)\|^2_{L^2}\, ds \le C_{Tm}\Big[\|D^m w(0)\|^2_{L^2} + \nu\int_0^t \|D^m v(s)\|^2_{L^2(\Omega_0)}\, ds\Big], \tag{3.2.12}$$

from which (3.2.6) follows inductively. ∎

We can deduce local convergence results from Proposition 3.2.1. Since

$$v^\nu(t) - f = \nu \int_0^t L(s) v(s)\, ds, \tag{3.2.13}$$

we see that under the hypotheses of Proposition 3.2.1,

$$\|v^\nu(t) - f\|_{H^{k-2}(\Omega_1)} \le C\nu^{1/2}\big(\|f\|_{H^k(\Omega_0)} + \|f\|_{L^2(\mathcal{O})}\big). \tag{3.2.14}$$

Interpolation with the bound on $\|v^\nu(t)\|_{H^k(\Omega_1)}$ in (3.2.6) then gives

$$\|v^\nu(t) - f\|_{H^{k-2\theta}(\Omega_1)} \le C\nu^{\theta/2}\big(\|f\|_{H^k(\Omega_0)} + \|f\|_{L^2(\mathcal{O})}\big), \tag{3.2.15}$$

for $\theta \in (0, 1]$. Now if we take $f_j \in L^2(\mathcal{O})$ such that $f_j \in H^{k+1}(\Omega_0)$ and $f_j \to f$ in $L^2(\mathcal{O})$-norm and in $H^k(\Omega_0)$-norm, an argument such as used at the end of the proof of Proposition 3.1.4 gives:

Proposition 3.2.2. *Under the hypotheses of Proposition 3.2.1, as $\nu \to 0$,*

$$v^\nu(t) \to f \ \text{ in } \ H^k(\Omega_1), \tag{3.2.16}$$

for each $t \ge 0$.

We can pass from Proposition 3.2.2 to other local convergence results. Here is one.

Proposition 3.2.3. *Let $f \in C(\overline{\mathcal{O}})$, and take Ω_j as in Proposition 3.2.1. Then the solution v^ν to (3.2.5) satisfies*

$$v^\nu(t) \to f, \ \text{ uniformly on } \ \Omega_1, \tag{3.2.17}$$

as $\nu \to 0$. This holds uniformly in $t \in [0, T]$.

Proof. Take $k > n/2$ ($n = \dim \mathcal{O}$), and take $\varepsilon > 0$. Take $g_\varepsilon \in H^k(\mathcal{O})$ such that $\|f - g_\varepsilon\|_{L^\infty(\mathcal{O})} \le \varepsilon$. Let v_ε^ν satisfy

$$\frac{\partial v_\varepsilon^\nu}{\partial t} = \nu L(t) v_\varepsilon^\nu, \quad v_\varepsilon^\nu(0) = g_\varepsilon, \quad v\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0. \tag{3.2.18}$$

We have, by the maximum principle,

$$\|v_\varepsilon^\nu(t) - v^\nu(t)\|_{L^\infty(\mathcal{O})} \le \|f - g\|_{L^\infty(\mathcal{O})} \le \varepsilon. \tag{3.2.19}$$

Meanwhile, Proposition 3.2.2 gives

$$v_\varepsilon^\nu(t) \to g_\varepsilon \ \text{ in } \ H^k(\Omega_1) \subset C(\overline{\Omega}_1), \tag{3.2.20}$$

as $\nu \to 0$, so (3.2.17) holds. ☐

3.3. *Conormal type estimates on $e^{t(\nu\Delta - X)}$.*

Here we aim to show that $\{e^{t(\nu\Delta - X)} : \nu \in (0, 1]\}$ is a strongly continuous semigroup, with norm bounds independent of $\nu \in (0, 1]$, on spaces of the following form:

$$\mathcal{V}^k(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : Y_1 \cdots Y_j u \in L^2(\mathcal{O}), \ \forall\, j \le k,\, Y_\ell \in \mathfrak{X}^1\}, \tag{3.3.1}$$

for $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, where

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathcal{O}} : Y \parallel \partial\mathcal{O}\}. \tag{3.3.2}$$

See the Remark on page for a discussion of why $\mathcal{V}^k(\mathcal{O})$-norm estimates are called conormal estimates.

Before starting to produce estimates, we develop some notation and preliminary material.

Lemma 3.3.1. *There exists a finite set*

$$\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1 \tag{3.3.3}$$

with the property that each element of $\mathfrak{X}^1$ is a linear combination, with coefficients in $C^\infty(\overline{\mathcal{O}})$, of these vector fields Y_j.

Proof. Routine. □

From here, take Y_j as in (3.3.3), let

$$J = (j_1, \dots, j_k), \tag{3.3.4}$$

and set

$$Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k. \tag{3.3.5}$$

Also set

$$\mathfrak{X}^k = \text{Span}\,\{Z_1 \cdots Z_j : j \le k,\ Z_\ell \in \mathfrak{X}^1\}. \tag{3.3.6}$$

We have

$$\mathfrak{X}^k = \text{Span over } C^\infty(\overline{\mathcal{O}}) \text{ of } \{Y^J : |J| \le k\}, \tag{3.3.7}$$

and

$$\begin{aligned}\mathcal{V}^k(\mathcal{O}) &= \{u \in L^2(\mathcal{O}) : Y^J u \in L^2(\mathcal{O}),\ \forall\, |J| \le k\}\\ &= \{u \in L^2(\mathcal{O}) : Lu \in L^2(\mathcal{O}),\ \forall\, L \in \mathfrak{X}^k\}.\end{aligned} \tag{3.3.8}$$

We have the following square-norm and norm on $\mathcal{V}^k(\mathcal{O})$:

$$N_k^2(u) = \sum_{|J| \le k} \|Y^J u\|_{L^2}^2, \quad N_k(u) = N_k^2(u)^{1/2}. \tag{3.3.9}$$

Also set

$$P_k^2(u) = \sum_{|J| = k} \|Y^J u\|_{L^2}^2. \tag{3.3.10}$$

We now estimate the rate of change of $P_k^2(u(t))$ for

$$u(t) = e^{t(\nu\Delta - X)} f, \quad f \in \mathcal{V}^k(\mathcal{O}), \tag{3.3.11}$$

starting with the case $k = 0$:

$$\frac{d}{dt}\|u\|_{L^2}^2 = 2(u_t, u)_{L^2} = 2\nu(\Delta u, u)_{L^2} - 2(Xu, u)_{L^2} = -2\nu\|\nabla u\|_{L^2}^2, \tag{3.3.12}$$

since, for $t > 0$, $u(t) \in \mathcal{D}((\nu\Delta - X)^m)$ for all m, and hence $u(t) \in H^{2m}(\mathcal{O}) \cap H_0^1(\mathcal{O})$. Moving on to $k = 1$, we have

$$\begin{aligned}\frac{d}{dt}\|Y_j u\|_{L^2}^2 &= 2(Y_j u_t, Y_j u)_{L^2}\\ &= 2\nu(Y_j \Delta u, Y_j u)_{L^2} - 2(Y_j X u, Y_j u)_{L^2}\\ &= 2\nu(\Delta Y_j u, Y_j u)_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2(XY_j u, Y_j u)_{L^2} - 2([Y_j, X]u, Y_j u)L^2\\ &= -2\nu\|\nabla Y_j u\|_{L^2}^2 + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2([Y_j, X]u, Y_j u)_{L^2}.\end{aligned} \tag{3.3.13}$$

Of the three terms in the last line, the first has a clear significance. For the third, we have $[Y_j, X] \in \mathfrak{X}^1$, and hence

$$2([Y_j, X]u, Y_j u)_{L^2} \le C P_1^2(u). \tag{3.3.14}$$

It remains to estimate the second term. For this, write

$$[Y, \Delta] = \sum_\ell A_\ell B_\ell, \tag{3.3.15}$$

with A_ℓ, B_ℓ smooth vector fields on $\overline{\mathcal{O}}$. We have

$$2\nu([Y_j, \Delta]u, Y_j u)_{L^2} = 2\nu \sum_\ell (B_\ell u, A_\ell^* Y_j u)_{L^2} \le \nu\|\nabla Y_j u\|_{L^2}^2 + \nu\|Y_j u\|_{L^2}^2 + K_1\nu\|\nabla u\|_{L^2}^2. \tag{3.3.16}$$

Plugging (3.3.14) and (3.3.16) into (3.3.13) and summing over j gives

$$\frac{d}{dt} P_1^2(u) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u) + M K_1 \nu \|\nabla u\|^2. \tag{3.3.17}$$

The term $M K_1 \nu \|\nabla u\|_{L^2}^2$ is tamed by bringing in (3.3.12), to obtain

$$\frac{d}{dt}\Big(P_1^2(u) + \frac{M K_1}{2} P_0^2(u)\Big) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u). \tag{3.3.18}$$

Proceeding to general k, we take $|J| = k$ and look at

$$\begin{aligned}
\frac{d}{dt}\|Y^J u\|_{L^2}^2 &= 2(Y^J u_t, Y^J u)_{L^2} \\
&= 2\nu(Y^J \Delta u, Y^J u)_{L^2} - 2(Y^J X u, Y^J u)_{L^2} \\
&= 2\nu(\Delta Y^J u, Y^J u)_{L^2} + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2(X Y^J u, Y^J u)_{L^2} - 2([Y^J, X]u, Y^J u)_{L^2} \\
&= -2\nu\|\nabla Y^J u\|_{L^2}^2 + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2([Y^J, X]u, Y^J u)_{L^2}.
\end{aligned} \tag{3.3.19}$$

As with (3.3.13), of the three terms in the last line of (3.3.19), the first has a clear significance. For the third, we have

$$[X, Y^J] = [X, Y_{j_1}]Y_{j_2} \cdots Y_{j_k} + \cdots + Y_{j_1} \cdots Y_{j_{k-1}}[X, Y_{j_k}] \in \mathfrak{X}^k, \tag{3.3.20}$$

and hence

$$|([Y^J, X]u, Y^J u)_{L^2}| \le C_k P_k^2(u). \tag{3.3.21}$$

It remains to estimate the second term in the last line of (3.3.19). For this, write

$$[\Delta, Y^J] = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}}[\Delta, Y_{j_\ell}]Y_{j_{\ell+1}} \cdots Y_{j_k} = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k}, \tag{3.3.22}$$

where $L_{j_\ell} = [\Delta, Y_{j_\ell}]$ is a second order differential operator that annihilates constants. We say a product of k factors

$$Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k} \tag{3.3.23}$$

is of type (k, ℓ), meaning it is a product of k factors, all being vector fields in $\mathfrak{X}^1$ except one, in position ℓ, which is a second order differential operator that annihilates constants. If $\ell \ge 2$, we can write (3.3.23) as

$$Y_{j_1} \cdots Y_{j_{\ell-2}} L_{j_\ell} \cdots Y_{j_k} + Y_{j_1} \cdots Y_{j_{\ell-2}}[Y_{j_{\ell-1}}, L_{j_\ell}] \cdots Y_{j_k}, \tag{3.3.24}$$

a sum of terms of type $(k, \ell-1)$ and of type $(k-1, \ell-1)$. Repeating this process, we convert (3.3.23) into a sum of terms of type $(j, 1)$, for $j \le k$. Hence we have

$$([Y^J, \Delta]u, Y^J u)_{L^2} = \sum_{|I|\le k-1} (L_I Y^I u, Y^J u)_{L^2}, \tag{3.3.25}$$

where the L_I are differential operators of order 2, annihilating constants; hence

$$L_I = \sum_j A_{Ij} B_{Ij}, \tag{3.3.26}$$

where A_{Ij} are first order differential operators and B_{Ij} are vector fields. We then have

$$\begin{aligned} 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} &= 2\nu \sum_{|I|\le k-1} \sum_j (B_{Ij} Y^I u, A_{Ij}^* Y^J u)_{L^2} \\ &\le \tilde{C}\nu \sum_{|I|\le k-1} \|\nabla Y^I u\|_{L^2} \cdot \big(\|\nabla Y^J u\|_{L^2} + \|Y^J u\|_{L^2}\big) \\ &\le \nu\|\nabla Y^J u\|_{L^2}^2 + \nu\|Y^J u\|_{L^2}^2 + C_k\nu \sum_{|I|\le k-1} \|\nabla Y^I u\|_{L^2}^2. \end{aligned} \tag{3.3.27}$$

Inserting (3.3.21) and (3.3.27) into (3.3.19), we get

$$\frac{d}{dt}\|Y^J u\|_{L^2}^2 \le -\nu\|\nabla Y^J u\|_{L^2}^2 + (C_k+\nu)P_k^2(u) + C_k\nu \sum_{|I|\le k-1} \|\nabla Y^I u\|_{L^2}^2, \tag{3.3.28}$$

hence, for $\nu \in (0, 1]$, and with $C_k + 1$ re-notated as C_k,

$$\frac{d}{dt}P_k^2(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|_{L^2}^2 + MC_k P_k^2(u) + MC_k\nu \sum_{|I|\le k-1} \|\nabla Y^I u\|_{L^2}^2. \tag{3.3.29}$$

It follows that there exist $A_{kj} \in (0, \infty)$ and $B_k \in (0, \infty)$ such that if we set

$$\widetilde{N}_k^2(u) = P_k^2(u) + \sum_{j=0}^{k-1} A_{kj} P_j^2(u), \tag{3.3.30}$$

then

$$\frac{d}{dt}\widetilde{N}_k^2(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|_{L^2}^2 + 2B_k \widetilde{N}_k^2(u), \tag{3.3.31}$$

when $u = u(t)$ is given by (3.3.11). In particular, taking

$$\|u\|_{\mathcal{V}^k}^2 = \widetilde{N}_k^2(u), \tag{3.3.32}$$

we obtain

$$\|u(t)\|_{\mathcal{V}^k} \le e^{(t-s)B_k}\|u(s)\|_{\mathcal{V}^k}, \tag{3.3.33}$$

for $0 < s < t < \infty$. The next result will allow us to pass to the limit $s \searrow 0$ for $f \in \mathcal{V}^k$.

Lemma 3.3.2. *For each $k \in \mathbb{Z}^+$, $C_0^\infty(\mathcal{O})$ is dense in $\mathcal{V}^k(\mathcal{O})$.*

Proof. Let $\psi \in C^\infty(\mathbb{R})$ satisfy

$$\begin{aligned} \psi(s) = 0 \quad &\text{for } \ s \le \tfrac{1}{2}, \\ 1 \quad &\text{for } \ s \ge 1, \end{aligned} \tag{3.3.34}$$

and set

$$\varphi_\delta(x)(x) = \psi\bigl(\delta^{-1}\,\mathrm{dist}(x, \partial\mathcal{O})\bigr). \tag{3.3.35}$$

There exists $\delta_0 > 0$ such that $\varphi_\delta \in C_0^\infty(\mathcal{O})$ for $\delta \in (0, \delta_0)$. Given $f \in \mathcal{V}^k(\mathcal{O})$ and $|J| \le k$, we have

$$Y^J(\varphi_\delta f) = \varphi_\delta Y^J f + \sum_{(I_1, I_2)} (Y^{I_1}\varphi_\delta)(Y^{I_2} f), \tag{3.3.36}$$

where (I_1, I_2) runs over the partitions of the ordered set $\{j_1, \dots j_k\}$ into two subsets, such that $|I_1| \ge 1$ (hence $|I_2| \le k-1$). It is clear from (3.3.35) that $\varphi_\delta Y^J f \to Y^J f$ in L^2-norm as $\delta \searrow 0$. Meanwhile $Y^{I_1}\varphi_\delta = 0$ on $\{x \in \mathcal{O} : \mathrm{dist}(x, \partial\mathcal{O}) \ge \delta\}$, and

$$Y_j \in \mathfrak{X}^1 \implies \|Y^{I_1}\varphi_\delta\|_{L^\infty} \le C_{I_1}, \ \text{independent of } \delta \in (0, \delta_0/2), \tag{3.3.37}$$

so the sum over (I_1, I_2) in (3.3.36) tends to 0 in L^2-norm as $\delta \searrow 0$. Hence, whenever $f \in \mathcal{V}^k(\mathcal{O})$,

$$\varphi_\delta f \to f \quad \text{in } \ \mathcal{V}^k\text{-norm}. \tag{3.3.38}$$

From here the density of $C_0^\infty(\mathcal{O})$ in $\mathcal{V}^k(\mathcal{O})$ follows by a standard mollifier argument. □

Since $C_0^\infty(\mathcal{O}) \subset \mathcal{D}((\nu\Delta - X)^m)$ for all m, we have $u \in C^\infty([0, \infty) \times \overline{\mathcal{O}})$ whenever $f \in C_0^\infty(\mathcal{O})$, and hence (3.3.31) holds for $t \ge 0$ and (3.3.33) holds for $s = 0$. That is to say, we have

$$\|e^{t(\nu\Delta - X)} f\|_{\mathcal{V}^k} \le e^{tB_k} \|f\|_{\mathcal{V}^k}, \tag{3.3.39}$$

for all f in the dense linear subspace $C_0^\infty(\mathcal{O})$ of $\mathcal{V}^k(\mathcal{O})$, and hence for all $f \in \mathcal{V}^k$. Also this density implies:

Proposition 3.3.3. *For each $k \in \mathbb{Z}^+$, $\nu > 0$, $e^{t(\nu\Delta - X)}$ is a strongly continuous semigroup on $\mathcal{V}^k(\mathcal{O})$, and (3.3.39) holds for each $f \in \mathcal{V}^k(\mathcal{O})$.*

We emphasize that (3.3.39) holds with B_k independent of $\nu \in (0, 1]$. From here we can obtain convergence results as $\nu \searrow 0$.

Proposition 3.3.4. *In the setting of Proposition 3.3.3,*

$$f \in \mathcal{V}^k(\mathcal{O}) \implies \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \tag{3.3.40}$$

in norm, in $\mathcal{V}^k(\mathcal{O})$.

Proof. The estimate (3.3.39) implies $\{e^{t(\nu\Delta - X)} f : \nu \in (0, 1]\}$ has weak* limit points as $\nu \searrow 0$. By Proposition 3.1.3, (with $p = 2$), $e^{-tX} f$ is the only possible such limit point. This gives convergence in (3.3.40), weak* in $\mathcal{V}^k(\mathcal{O})$. We next aim to improve this to norm convergence. In view of the uniform bounds in (3.3.39), it suffices to establish norm convergence on a dense linear subspace of $\mathcal{V}^k(\mathcal{O})$. Take $f \in C_0^\infty(\mathcal{O})$. We use the complex interpolation identity

$$\mathcal{V}^k(\mathcal{O}) = [L^2(\mathcal{O}), \mathcal{V}^{2k}]_{1/2}. \tag{3.3.41}$$

See Proposition A.1.1 in the Appendix for a proof. This implies

$$\|g\|_{\mathcal{V}^k} \le \|g\|_{L^2}^{1/2}\|g\|_{\mathcal{V}^{2k}}^{1/2}, \tag{3.3.42}$$

for $g \in \mathcal{V}^{2k}(\mathbb{O})$. Hence, for $f \in \mathcal{V}^{2k}(\mathbb{O})$,

$$\big\|(e^{t(\nu\Delta - X)} - e^{-tX})f\big\|_{\mathcal{V}^k} \le \big\|(e^{t(\nu\Delta - X)} - e^{-tX})f\big\|_{L^2}^{1/2}\,\big\|(e^{t(\nu\Delta - X)} - e^{-tX})f\big\|_{\mathcal{V}^{2k}}^{1/2}. \tag{3.3.43}$$

The first factor on the right side tends to zero as $\nu \searrow 0$, by Proposition 3.1.3, and the last factor is uniformly bounded as $\nu \searrow 0$, by (3.3.39), with k replaced by $2k$. This completes the proof. □

Remark. The class of differential operators $\mathfrak{X}^k$, $k \ge 1$, together with multiplications by smooth functions on $\overline{\mathbb{O}}$, is what is called the algebra of totally characteristic differential operators in [Melrose 1981; 1993]. These works also develop a related class of pseudodifferential operators; see also [Melrose 1996] and [Hörmander 1985, §18.3]. The spaces $\mathcal{V}^k(\mathbb{O})$ are special cases of "weighted b-Sobolev spaces," introduced in [Melrose 1993]. This is discussed further in Appendix A.

We briefly comment on why we call $\mathcal{V}^k(\mathbb{O})$-norm estimates "conormal estimates." The term "conormal distribution" was introduced in [Hörmander 1971]. In essence, if M is a smooth manifold, Σ a smooth submanifold and $\mathcal{L}$ a given Banach space of distributions on M (such as $L^2(M)$) and if f and $X_1 \cdots X_k f$ belong to $\mathcal{L}$ for all k and all smooth vector fields X_j on M that are tangent to Σ, then f is said to be conormal distribution with respect to Σ. See also [Hörmander 1985, §18.2] for a detailed treatment.

3.4. ***Holomorphy of the semigroup $e^{\zeta\Delta}$ on $\mathcal{V}^k(\mathbb{O})$.*** As usual, take $\mathfrak{D}(\Delta) = H^2(\mathbb{O}) \cap H_0^1(\mathbb{O})$. The semigroup $e^{\zeta\Delta}$ is a holomorphic semigroup on $L^2(\mathbb{O})$, for $\operatorname{Re}\zeta > 0$. Here we show it has a bound

$$\|e^{\zeta\Delta} f\|_{\mathcal{V}^k} \le e^{B|\zeta|}\|f\|_{\mathcal{V}^k}, \tag{3.4.1}$$

uniformly for ζ in a wedge

$$\mathcal{W}_K = \{t + is : t > 0,\ |s| < Kt\}, \tag{3.4.2}$$

with $B = B(k, K)$. We then derive some useful consequences from this.

To start, take $\theta \in \mathbb{R}$ and set $s = \theta t$ and consider

$$u(t) = e^{t(1+i\theta)\Delta} f, \tag{3.4.3}$$

supressing θ in the notation on the left side of (3.4.3). We have

$$\frac{d}{dt}\|u\|_{L^2}^2 = 2\operatorname{Re}(u_t, u)_{L^2} = 2\operatorname{Re}((1+i\theta)\Delta u, u)_{L^2} = -2\|\nabla u\|_{L^2}^2. \tag{3.4.4}$$

This is the standard result for $\mathcal{V}^0(\mathbb{O}) = L^2(\mathbb{O})$. Moving on to $\mathcal{V}^k(\mathbb{O})$ with $k = 1$, we have

$$\begin{aligned}
\frac{d}{dt}\|Y_j u\|_{L^2}^2 &= 2\operatorname{Re}(Y_j u_t, Y_j u)_{L^2} \\
&= 2\operatorname{Re}(1+i\theta)(Y_j\Delta u, Y_j u)_{L^2} \\
&= 2\operatorname{Re}(1+i\theta)(\Delta Y_j u, Y_j u)_{L^2} + 2\operatorname{Re}(1+i\theta)([Y_j, \Delta]u, Y_j u)_{L^2} \\
&\le -2\|\nabla Y_j u\|_{L^2}^2 + 2\Theta|([Y_j, \Delta]u, Y_j u)_{L^2}|,
\end{aligned} \tag{3.4.5}$$

where we have set $\Theta = \sqrt{1+\theta^2}$. As in (3.3.15)–(3.3.16), we have

$$2\Theta|([Y_j, \Delta]u, Y_j u)_{L^2}| \le \|\nabla Y_j u\|_{L^2}^2 + \|Y_j u\|_{L^2}^2 + K_1\|\nabla u\|_{L^2}^2, \tag{3.4.6}$$

and hence, parallel to (3.3.17),

$$\frac{d}{dt} P_1^2(u) = -\sum_j \|\nabla Y_j u\|_{L^2}^2 + K_2\|\nabla u\|_{L^2}^2. \tag{3.4.7}$$

Then, parallel to (3.3.18), we have

$$\frac{d}{dt}\big(P_1^2(u) + K_2 P_0^2(u)\big) \le -\sum_j \|\nabla Y_j u\|_{L^2}^2, \tag{3.4.8}$$

giving (3.4.1) for $k = 1$, first for $f \in C_0^\infty(\mathcal{O})$, which is dense in $\mathcal{V}^1(\mathcal{O})$, then for general $f \in \mathcal{V}^1(\mathcal{O})$.

The passage to general k proceeds along the same lines, in parallel with estimates done in (3.3.19)–(3.3.31), but with the simplification that X is not involved.

We record some standard but significant consequences of the holomorphy of $e^{\zeta\Delta}$ and the estimates (3.4.1). First,

$$\Big\|\frac{d}{dt} e^{\zeta\Delta} f\Big\|_{\mathcal{V}^k} \le C|\zeta|^{-1} e^{B|\zeta|}\|f\|_{\mathcal{V}^k}, \tag{3.4.9}$$

for $\zeta \in \mathcal{W}_{K/2}$, as follows from the Cauchy integral formula applied to a circle of radius $\sim c|\zeta|$ centered about ζ. This estimate implies

$$\|\Delta e^{t\Delta} f\|_{\mathcal{V}^k} \le \frac{C}{t} e^{Bt}\|f\|_{\mathcal{V}^k}, \tag{3.4.10}$$

for $t > 0$, and hence

$$\|Y^J \Delta e^{t\Delta} f\|_{L^2} \le \frac{C}{t} e^{Bt}\|f\|_{\mathcal{V}^k}, \quad |J| = k. \tag{3.4.11}$$

Using this, we will establish the following.

Proposition 3.4.1. *Take $T_0 \in (0, \infty)$. Then, for $t \in [0, T_0]$, we have*

$$tY^J e^{t\Delta} : \mathcal{V}^k(\mathcal{O}) \to H^2(\mathcal{O}) \ \textit{bounded, for} \ |J| = k. \tag{3.4.12}$$

Proof. We use induction on k. For $k = 0$, (3.4.12) follows from the $k = 0$ case of (3.4.10). To establish (3.4.12) for $k \ge 1$, it suffices to show that

$$t\Delta Y^J e^{t\Delta} : \mathcal{V}^k(\mathcal{O}) \to L^2(\mathcal{O}) \ \text{ is bounded, for } \ |J| = k. \tag{3.4.13}$$

Using (3.3.22)–(3.3.25), we have

$$t\Delta Y^J e^{t\Delta} = tY^J \Delta e^{t\Delta} + t\sum_{|I|\le k-1} L_I Y^I e^{t\Delta}, \tag{3.4.14}$$

where each L_I is a second order differential operator. The bound on the first term on the right side of (3.4.14) in $\mathcal{L}(\mathcal{V}^k(\mathcal{O}), L^2(\mathcal{O}))$ follows from (3.4.11). The bound on the sum over $|I| \le k-1$ follows by the induction hypothesis. This proves (3.4.12). □

We can interpolate the bound

$$\|Y^J e^{t\Delta} f\|_{H^2(\mathcal{O})} \le \frac{C}{t}\|f\|_{\mathcal{V}^k} \tag{3.4.15}$$

with the bound

$$\|Y^J e^{t\Delta} f\|_{L^2(\mathcal{O})} \le C\|f\|_{\mathcal{V}^k}, \tag{3.4.16}$$

valid for $t \in [0, T_0]$ by (3.4.1), using

$$\|F\|_{H^1} \le C\|F\|_{L^2}^{1/2}\|F\|_{H^2}^{1/2}, \tag{3.4.17}$$

to obtain:

Corollary 3.4.2. *In the setting of Proposition 3.4.1,*

$$\|Y^J e^{t\Delta} f\|_{H^1(\mathcal{O})} \le \frac{C}{t^{1/2}}\|f\|_{\mathcal{V}^k}, \quad |J| = k. \tag{3.4.18}$$

Consequently

$$\|e^{t\Delta} f\|_{\mathcal{V}^{k+1}} \le \frac{C}{t^{1/2}}\|f\|_{\mathcal{V}^k}. \tag{3.4.19}$$

3.5. ***Estimates on $e^{t(\nu\Delta-X)}$ in case of empty boundary.*** Here we consider the family of semigroups $e^{t(\nu\Delta-X)}$ acting on functions on M, a compact, n-dimensional, Riemannian manifold without boundary. Again Δ is the Laplace-Beltrami operator. We assume X is a smooth vector field on M. This time we will not assume that $\operatorname{div} X = 0$. We will show that in this setting we have much stronger convergence results than obtained in Section 3.1. Ultimately it will be our goal to use the results obtained here to strengthen the results of Section 3.1.

To begin, let us note that in the current context, (3.1.4) is strengthened to

$$\mathcal{D}((\nu\Delta - X)^k) = \mathcal{D}(\Delta^k) = H^{2k}(M), \quad \forall k \in \mathbb{N}, \tag{3.5.1}$$

whenever $\nu > 0$. Because of this, we can improve Lemma 3.1.2 to the following.

Lemma 3.5.1. *Take $f \in C^\infty(M)$, and set $u(t) = e^{t(\nu\Delta-X)} f$, with $\nu > 0$. For each $k \in \mathbb{Z}^+$, there exists $K = K(k) \in (0, \infty)$, independent of ν, such that*

$$\|(1-\Delta)^k u(t)\|_{L^2}^2 \le e^{2Kt}\|(1-\Delta)^k f\|_{L^2}^2. \tag{3.5.2}$$

Proof. Straightforward analogue of the proof of Lemma 3.1.2. □

Corollary 3.5.2. *We have, for each $k \in \mathbb{Z}^+$,*

$$\|e^{t(\nu\Delta-X)} f\|_{\mathcal{D}(\Delta^k)} \le e^{Kt}\|f\|_{\mathcal{D}(\Delta^k)}, \tag{3.5.3}$$

for each $f \in C^\infty(M)$, hence for each $f \in \mathcal{D}(\Delta^k)$.

Remark. This contrasts with the possibility of (3.1.25), which can occur in case of nonempty boundary.

Note that the maximum principle holds, so, for each $\nu > 0$,

$$\|e^{t(\nu\Delta-X)} f\|_{L^\infty} \le \|f\|_{L^\infty}. \tag{3.5.4}$$

Interpolation with the case $k = 0$ of (3.5.3) implies

$$\|e^{t(\nu\Delta-X)} f\|_{L^p} \le e^{Kt}\|f\|_{L^p}, \tag{3.5.5}$$

for $f \in L^p(M),\ \ p \in [2,\infty)$. We could also get this for $p \in [1,2)$, but we will not take the space to do this. We can further apply interpolation to (3.5.5) and the estimates

$$\|e^{t(\nu\Delta - X)} f\|_{H^{2k}} \le C e^{Kt} \|f\|_{H^{2k}}, \quad k \in \mathbb{Z}^+, \tag{3.5.6}$$

which follow from (3.5.3) and (3.5.1). First, we have

$$\|e^{t(\nu\Delta - X)} f\|_{H^s} \le C e^{Kt} \|f\|_{H^s}, \quad s \in \mathbb{R}^+, \tag{3.5.7}$$

with $C = C_s,\ \ K = K_s$, independent of ν. Then, in place of (3.1.21), we have

$$[H^s(M), L^p(M)]_\theta = H^{(1-\theta)s, q(\theta)}(M), \quad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{3.5.8}$$

and hence

$$\|e^{t(\nu\Delta - X)} f\|_{H^{\sigma,q}(M)} \le C_{\sigma,q} e^{Kt} \|f\|_{H^{\sigma,q}(M)}, \tag{3.5.9}$$

valid for $q \in [2,\infty),\ \ \sigma > 0$.

We next consider convergence results, as $\nu \to 0$. As in (3.1.8), we have for $u(t) = e^{t(\nu\Delta - X)} f$ the identity

$$u(t) = e^{-tX} f + \nu \int_0^t e^{(t-s)X} \Delta u(s)\, ds, \tag{3.5.10}$$

hence

$$\|u(t) - e^{-tX} f\|_{\mathcal{D}(\Delta^k)} \le \nu \int_0^t \|e^{(t-s)X} \Delta u(s)\|_{\mathcal{D}(\Delta^k)}\, ds. \tag{3.5.11}$$

We use (3.5.3) plus the analogous estimate on e^{-tX} to deduce that

$$\|e^{t(\nu\Delta - X)} f - e^{-tX} f\|_{\mathcal{D}(\Delta^k)} \le C\nu \|f\|_{\mathcal{D}(\Delta^{k+1})}, \tag{3.5.12}$$

for $f \in C^\infty(M)$. We hence have

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f \tag{3.5.13}$$

in $\mathcal{D}(\Delta^k)$-norm (hence in H^{2k}-norm), for each $f \in C^\infty(M)$, hence, via (3.5.3), for each $f \in \mathcal{D}(\Delta^k)$. Then, using (3.5.9) and (3.5.4), and standard density arguments, we have:

Proposition 3.5.3. *Given $f \in H^{\sigma,q}(M),\ \sigma \ge 0,\ q \in [2,\infty)$, convergence in (3.5.13) holds in $H^{\sigma,q}$-norm, as $\nu \to 0$. Given $f \in C(M)$, convergence in (3.5.13) holds uniformly, as $\nu \to 0$.*

3.6. *Parametrix for $\partial_t - \nu L(t)$ on $\mathbb{R}^+ \times M$.* As in Section 3.5, let M be a compact, n-dimensional, Riemannian manifold without boundary, with Laplace-Beltrami operator Δ, and let X be a smooth vector field on M. As in Section 3.2, let $L(t) = e^{tX} \Delta e^{-tX}$, so, for $f \in \mathcal{D}'(M)$,

$$v(t) = e^{tX} e^{t(\nu\Delta - X)} f \tag{3.6.1}$$

solves

$$\frac{\partial v}{\partial t} = \nu L(t) v, \quad v(0) = f. \tag{3.6.2}$$

We denote the solution operator by S_ν^t:

$$S_\nu^t = e^{tX} e^{t(\nu\Delta - X)}. \tag{3.6.3}$$

Parallel to results of Section 3.5, we have

$$\|S^t_\nu f\|_{H^{s,p}} \le Ce^{Kt}\|f\|_{H^{s,p}}, \tag{3.6.4}$$

for $f \in H^{s,p}(M)$, with $C = C_{s,p},\ K = K_{s,p}$ independent of $\nu > 0$, given $p \ge 2,\ s \ge 0$. (With a little more work, we could take any $p \in (1,\infty),\ s \in \mathbb{R}$.) Our goal here is to construct a parametrix, revealing the fine structure of S^t_ν as $\nu \to 0$.

Preparatory to beginning this parametrix construction, it is also useful to note that Proposition 3.2.1 continues to hold in the current setting. In particular, given $\Omega_1 \Subset \Omega_0 \subset M,\ k \in \mathbb{N}$,

$$\|S^t_\nu f\|^2_{H^k(\Omega_1)} \le C_{Tk}\big(\|f\|^2_{H^k(\Omega_0)} + \|f\|^2_{L^2(M)}\big), \quad 0 \le t \le T, \tag{3.6.5}$$

with C_{Tk} independent of $\nu > 0$. Applying this and a partition of unity argument, we see it suffices to construct a parametrix for $S^t_\nu f$ when f is supported on a coordinate patch $\Omega \subset M$, and it suffices to analyze this approximation to $S^t_\nu f(x)$ for $(t,x) \in [0,T] \times \Omega$, uniformly in $\nu \in (0,1]$.

On a coordinate patch Ω, we have

$$L(t)u = \sum_{1\le|\alpha|\le 2} L_\alpha(t,x)\partial^\alpha_x. \tag{3.6.6}$$

(Note that $L(t)1 = 0$.) Let us set

$$L_k(t,x,\xi) = \sum_{|\alpha|=k} L_\alpha(t,x)(i\xi)^\alpha, \qquad k = 1,2. \tag{3.6.7}$$

Note that

$$L_2(t,x,\xi) = -G(t,x,\xi) = -\sum_{ij} g^{ij}(t,x)\xi_i\xi_j, \tag{3.6.8}$$

where $(g_{ij}(t,x)) = (g^{ij}(t,x))^{-1}$ is the metric tensor on M, pulled back via the flow generated by X.

We write our approximate solution to (3.6.2) on $\mathbb{R}^+ \times \Omega$ as

$$\mathfrak{S}^t_\nu f(x) = (2\pi)^{-n/2}\int a(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.9}$$

where $\hat{f}(\xi)$ is the Fourier transform of f, given by

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int f(x)e^{-ix\cdot\xi}\,dx,$$

and the *amplitude* $a(\nu,t,x,\xi)$ will take the form of an asymptotic series

$$a(\nu,t,x,\xi) \sim \sum_{j\ge 0} a_j(\nu,t,x,\xi), \tag{3.6.10}$$

whose terms a_j will be constructed below. In outline this construction is similar to that done in [Taylor 1996, Chapter 7, §13], constructing a parametrix for $e^{t\Delta}$ for small t, but here the set-up is more complicated.

We start with the following consequence of the Leibniz identity:

$$\nu L(t)(ae^{ix\cdot\xi}) = \Big[\nu L_2(t,x,\xi)a(\nu,t,x,\xi) + \nu\sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a(\nu,t,x,\xi)\Big]e^{ix\cdot\xi}, \tag{3.6.11}$$

where $B_{2-\ell}(t,x,\xi,D_x)$ is a differential operator of order ℓ, whose coefficients are polynomials of degree $2-\ell$ in ξ, and smooth in (t,x). To satisfy (3.6.2) formally, we require

$$\frac{\partial a}{\partial t} \sim \nu L_2(t,x,\xi)a + \nu \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a, \quad a(\nu,0,x,\xi)=1. \tag{3.6.12}$$

This tells us how to construct the terms a_j. For starters, a_0 is defined by

$$\frac{\partial a_0}{\partial t} = -\nu G(t,x,\xi)a_0, \quad a_0(\nu,0,x,\xi)=1, \tag{3.6.13}$$

so

$$a_0(\nu,t,x,\xi) = e^{-\nu t H(t,x,\xi)}, \quad H(t,x,\xi) = \frac{1}{t}\int_0^t G(s,x,\xi)\,ds. \tag{3.6.14}$$

Note that $H(t,x,\xi)$ is a polynomial in ξ, homogeneous of degree 2, with coefficients smooth in (t,x), and

$$H(t,x,\xi) \geq C|\xi|^2, \tag{3.6.15}$$

for some $C>0$. For $j \geq 1$, a_j solves

$$\frac{\partial a_j}{\partial t} = -\nu G(t,x,\xi)a_j + \Omega_j(\nu,t,x,\xi), \quad a_j(\nu,0,x,\xi)=0, \tag{3.6.16}$$

where

$$\Omega_j(\nu,t,x,\xi) = \nu \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)a_{j-\ell}(\nu,t,x,\xi), \tag{3.6.17}$$

with the convention (operative for $j=1,\ \ell=2$) that $a_{-1}\equiv 0$. We hence have

$$a_j(\nu,t,x,\xi) = e^{-\nu t H(t,x,\xi)} \int_0^t e^{\nu s H(s,x,\xi)}\Omega_j(\nu,s,x,\xi)\,ds. \tag{3.6.18}$$

Another way to display these terms in the amplitude is to set

$$a_j(\nu,t,x,\xi) = A_j(\nu,t,x,\xi)e^{-\nu t H(t,x,\xi)}. \tag{3.6.19}$$

Also set

$$\Omega_j(\nu,t,x,\xi) = \Gamma_j(\nu,t,x,\xi)e^{-\nu t H(t,x,\xi)}, \tag{3.6.20}$$

so (3.6.17) becomes

$$\Gamma_j(\nu,t,x,\xi) = \nu e^{\nu t H(t,x,\xi)} \sum_{\ell=1}^{2} B_{2-\ell}(t,x,\xi,D_x)\big(A_{j-\ell}e^{-\nu t H(t,x,\xi)}\big), \tag{3.6.21}$$

and (3.6.18) becomes

$$A_j(\nu,t,x,\xi) = \int_0^t \Gamma_j(\nu,s,x,\xi)\,ds. \tag{3.6.22}$$

We next take an explicit look at the case $j=1$. In that case, (3.6.17) gives

$$\Omega_1 = \nu B_1(t,x,\xi,D_x)e^{-\nu t H(t,x,\xi)} = -\nu^2 t e^{-\nu t H(t,x,\xi)} B_1(t,x,\xi,D_x)H(t,x,\xi), \tag{3.6.23}$$

and recall that B_1 is a differential operator of order 1, whose coefficients are polynomials of degree 1 in ξ. A formula equivalent to (3.6.23) is

$$\Gamma_1 = -\nu^2 t B_1(t,x,\xi,D_x)H(t,x,\xi) = -\nu^2 t \sum_{|\alpha|\le 3} C_1^\alpha(t,x)\xi^\alpha, \tag{3.6.24}$$

with $C_1^\alpha(t,x)$ smooth. Then, by (3.6.22),

$$A_1(\nu,t,x,\xi) = -\nu^2 \sum_{|\alpha|\le 3}\Big(\int_0^t sC_1^\alpha(s,x)\,ds\Big)\xi^\alpha = -(\nu t)^2 \sum_{|\alpha|\le 3} D_1^\alpha(t,x)\xi^\alpha, \tag{3.6.25}$$

with $D_1^\alpha(t,x)$ smooth, and we have

$$a_1(\nu,t,x,\xi) = -(\nu t)^2 \sum_{|\alpha|\le 3} D_1^\alpha(t,x)\xi^\alpha\, e^{-\nu t H(t,x,\xi)}. \tag{3.6.26}$$

Let us now recall the definition of a symbol class, important in the theory of pseudodifferential operators. Given $m \in \mathbb{R}$, we say

$$p(x,\xi) \in S^m_{1,0} \Longleftrightarrow |D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}, \tag{3.6.27}$$

and we say a family $\{p(\nu,t,x,\xi) : t \in [0,T],\ \nu \in (0,1]\}$ is bounded in $S^m_{1,0}$ provided such estimates hold with $C_{\alpha\beta}$ independent of ν and t. In follows from (3.6.14) that

$$\{a_0(\nu,t,x,\xi) : t \in [0,T], \nu \in (0,1]\} \ \text{ is bounded in } \ S^0_{1,0}, \tag{3.6.28}$$

or as we say for short, $a_0(\nu,t,x,\xi)$ is bounded in $S^0_{1,0}$. Similarly, from (3.6.26) we have

$$a_1(\nu,t,x,\xi) \ \text{ bounded in } S^{-1}_{1,0} \ \text{ and } \ O((\nu t)^{1/2}) \text{ in } S^0_{1,0}, \tag{3.6.29}$$

the latter property meaning that $(\nu t)^{-1/2}a_1(\nu,t,x,\xi)$ is bounded in $S^0_{1,0}$.

To extend (3.6.28)–(3.6.29) to a_j for larger j, it is convenient to have another presentation. Set

$$\zeta = (\nu t)^{1/2}\xi, \quad \omega = \nu t\xi. \tag{3.6.30}$$

Now (3.6.14) and (3.6.26) give

$$\begin{aligned} a_0(\nu,t,x,\xi) &= e^{-H(t,x,\zeta)}, \\ a_1(\nu,t,x,\xi) &= \nu t \mathscr{A}_1(\nu t,t,x,\xi,\zeta)e^{-H(t,x,\zeta)}, \end{aligned} \tag{3.6.31}$$

where $\mathscr{A}_1(\tau,t,x,\xi,\zeta)$ is a polynomial in τ of degree 1, in ξ of degree 1 and in ζ of degree 2, with coefficients smooth in (t,x). It will be useful to have the following:

Definition. The space $\mathscr{P}_k$ is characterized by

$$\begin{aligned} F(\nu t,t,x,\xi,\zeta,\omega) \in \mathscr{P}_k \Longleftrightarrow F \ &\text{ is a polynomial in } \ \nu t,\ \zeta,\ \omega,\ \text{ and } \ \xi,\ \text{ even in } \zeta, \\ &\text{of degree } \ \le k \text{ in } \ \xi,\ \text{withcoefficients smooth in } (t,x). \end{aligned} \tag{3.6.32}$$

Without loss of generality, we can assume the degree in ω is ≤ 1.

Then a_1 satisfies (3.6.31) with

$$\mathcal{A}_1(\nu t, t, x, \xi, \zeta) \in \mathcal{P}_1. \tag{3.6.33}$$

(Actually $\mathcal{A}_1$ is independent of ω, but other amplitudes will have ω dependence.)

Theorem 3.6.1. *For each* $k = 0, 1, 2, \dots$, *we have*

$$\begin{aligned} a_{2k}(\nu, t, x, \xi) &= (\nu t)^k \mathcal{A}_{2k} e^{-H(t,x,\zeta)}, \quad \mathcal{A}_{2k} \in \mathcal{P}_0, \\ a_{2k+1}(\nu, t, x, \xi) &= (\nu t)^{k+1} \mathcal{A}_{2k+1} e^{-H(t,x,\zeta)}, \quad \mathcal{A}_{2k+1} \in \mathcal{P}_1. \end{aligned} \tag{3.6.34}$$

Proof. The results in (3.6.31) give (3.6.34) for $k = 0$. We proceed by induction on k. To set this up, let us assume

$$a_j = (\nu t)^{\alpha_j} \mathcal{A}_j e^{-H(t,x,\zeta)}, \quad \mathcal{A}_j \in \mathcal{P}_{\beta_j}, \tag{3.6.35}$$

for $j \le 2k+1$, with indices α_j and β_j consistent with (3.6.34). Then (3.6.17) gives

$$\Omega_{j+1} = \Omega^1_{j+1} + \Omega^0_{j+1} \tag{3.6.36}$$

with

$$\begin{aligned} \Omega^1_{j+1} &= \nu(\nu t)^{\alpha_j} B_1(t, x, \xi, D_x)\big(\mathcal{A}_j e^{-H(t,x,\zeta)}\big) \\ &= \nu(\nu t)^{\alpha_j} \mathcal{B}^1_{j+1} e^{-H(t,x,\zeta)}, \quad \mathcal{B}^1_{j+1} \in \mathcal{P}_{\beta_j+1}, \end{aligned} \tag{3.6.37}$$

so $\Gamma^1_{j+1} = \nu(\nu t)^{\alpha_j} \mathcal{B}^1_{j+1}$ and

$$A^1_{j+1}(\nu, t, x, \xi) = \int_0^t \Gamma^1_{j+1}(\nu, s, x, \xi)\, ds \in (\nu t)^{\alpha_j+1} \cdot \mathcal{P}_{\beta_j+1}, \tag{3.6.38}$$

and furthermore

$$\begin{aligned} \Omega^0_{j+1} &= \nu(\nu t)^{\alpha_{j-1}} B_0(t, x, D_x)\big(\mathcal{A}_{j-1} e^{-H(t,x,\zeta)}\big) \\ &= \nu(\nu t)^{\alpha_{j-1}} \mathcal{B}^0_{j+1} e^{-H(t,x,\zeta)}, \quad \mathcal{B}^0_{j+1} \in \mathcal{P}_{\beta_{j-1}}, \end{aligned} \tag{3.6.39}$$

so $\Gamma^0_{j+1} = \nu(\nu t)^{\alpha_{j-1}} \mathcal{B}^0_{j+1}$ and

$$A^0_{j+1}(\nu, t, x, \xi) = \int_0^t \Gamma^0_{j+1}(\nu, s, x, \xi)\, ds \in (\nu t)^{\alpha_{j-1}+1} \cdot \mathcal{P}_{\beta_{j-1}}. \tag{3.6.40}$$

We are now ready to verify the induction step in the proof of Theorem 3.6.1. Suppose (3.6.34) holds for a given $k \in \mathbb{Z}^+$, i.e.,

$$A_{2k} \in (\nu t)^k \cdot \mathcal{P}_0, \quad A_{2k+1} \in (\nu t)^{k+1} \cdot \mathcal{P}_1. \tag{3.6.41}$$

(If $k \ge 1$, assume also the counterpart of (3.6.41) with k replaced by $k-1$.) Then, using the fact that (3.6.35) implies (3.6.38) and (3.6.40), we obtain

$$A_{2k+2} = A^1_{2k+2} + A^0_{2k+2} \in (\nu t)^{k+2} \cdot \mathcal{P}_2 + (\nu t)^{k+1} \cdot \mathcal{P}_0 \subset (\nu t)^{k+1} \cdot \mathcal{P}_0, \tag{3.6.42}$$

(upon noting that $(\nu t) \cdot \mathcal{P}_2 \subset \mathcal{P}_0$), and furthermore

$$A_{2k+3} = A^1_{2k+3} + A^0_{2k+3} \in (\nu t)^{k+2} \cdot \mathcal{P}_1. \tag{3.6.43}$$

This completes the proof. □

We can use Theorem 3.6.1 to extend (3.6.28)–(3.6.29), as follows.

Corollary 3.6.2. *In the setting of Theorem 3.6.1, we have*

$$a_{2k}(\nu,t,x,\xi)=O((\nu t)^k) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-2k}_{1,0}, \tag{3.6.44}$$

and

$$a_{2k+1}(\nu,t,x,\xi)=O((\nu t)^{k+1}) \text{ in } S^1_{1,0}, \text{ bounded in } S^{-2k-1}_{1,0}, \tag{3.6.45}$$

hence, for $j\ge 0$,

$$a_j(\nu,t,x,\xi)=O((\nu t)^{j/2}) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-j}_{1,0}. \tag{3.6.46}$$

Proof. The result (3.6.34) directly gives (3.6.44)–(3.6.45), and (3.6.46) follows from this plus the observation that the condition

$$p(\nu,t,x,\xi)=(\nu t)\mathcal{A}e^{-H(t,x,\zeta)}, \quad \mathcal{A}\in\mathcal{P}_1 \tag{3.6.47}$$

implies $p(\nu,t,x,\xi)=O((\nu t)^{1/2})$ in $S^0_{1,0}$. □

Returning to (3.6.9)–(3.6.10), let us fix $N\in\mathbb{N}$ and set

$$a(\nu,t,x,\xi)=\sum_{j=0}^{N} a_j(\nu,t,x,\xi). \tag{3.6.48}$$

We use this to define $\mathfrak{S}^t_\nu f$ in (3.6.9). Then we have

$$(\partial_t-\nu L(t))\mathfrak{S}^t_\nu f(x)=(2\pi)^{-n/2}\int R_N(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.49}$$

with

$$\begin{aligned}R_N&(\nu,t,x,\xi)\\&=\nu B_1(t,x,\xi,D_x)a_N(\nu,t,x,\xi)+\nu B_0(t,x,D_x)\big[a_{N-1}(\nu,t,x,\xi)+a_N(\nu,t,x,\xi)\big].\end{aligned} \tag{3.6.50}$$

Arguments used in the proof of (3.6.34) and (3.6.45) give

$$\begin{aligned}\nu B_1(t,x,\xi,D_x)a_N(\nu,t,x,\xi)&=O(\nu(\nu t)^{N/2}) \text{ in } S^1_{1,0},\\&\quad O(\nu(\nu t)^{(N-1)/2}) \text{ in } S^0_{1,0},\\&\quad O(\nu) \text{ in } S^{-(N-1)}_{1,0},\end{aligned} \tag{3.6.51}$$

and

$$\begin{aligned}\nu B_0(t,x,D_x)[a_{N-1}+a_N]&=O(\nu(\nu t)^{(N-1)/2}) \text{ in } S^0_{1,0},\\&\quad O(\nu) \text{ in } S^{-(N-1)}_{1,0}.\end{aligned} \tag{3.6.52}$$

In conclusion:

Proposition 3.6.3. *If $N\in\mathbb{N}$ is given, a is defined as in (3.6.48), and $\mathfrak{S}^t_\nu$ as in (3.6.9), then*

$$u^\nu(t)=\mathfrak{S}^t_\nu f \tag{3.6.53}$$

solves

$$\frac{\partial u^\nu}{\partial t}=\nu L(t)u+g^\nu, \quad u^\nu(0)=f, \tag{3.6.54}$$

with

$$g^\nu(t,x)=(2\pi)^{-n/2}\int R_N(\nu,t,x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{3.6.55}$$

where

$$R_N(\nu,t,x,\xi) = O(\nu(\nu t)^{(N-1)/2}) \ \ in \ \ S^0_{1,0},$$
$$O(\nu) \ \ in \ \ S^{-(N-1)}_{1,0}. \tag{3.6.56}$$

Using standard pseudodifferential operator estimates, we obtain:

Corollary 3.6.4. *In the setting of Proposition 3.6.3, if $p \in (1,\infty)$, $s \in \mathbb{R}$, then, for $t \in [0,T]$, $\nu \in (0,1]$,*

$$\|g^\nu(t)\|_{H^{s,p}(M)} \le C_T \nu^{(N+1)/2} \|f\|_{H^{s,p}(M)}, \tag{3.6.57}$$

and

$$\|g^\nu(t)\|_{H^{s+N-1,p}(M)} \le C_T \nu \|f\|_{H^{s,p}(M)}, \tag{3.6.58}$$

with C_T independent of ν.

We can compare the approximate solution $\mathfrak{S}^t_\nu f$ with the exact solution $S^t_\nu f$ to (3.6.2) by applying the Duhamel formula to (3.6.54), which gives

$$\mathfrak{S}^t_\nu f = S^t_\nu f + \int_0^t S^{s,t}_\nu g^\nu(s)\,ds, \tag{3.6.59}$$

where, for $0 \le s \le t$, $S^{s,t}_\nu$ is the solution operator to (3.6.2) defined by

$$v(t) = S^{s,t}_\nu v(s), \quad \text{equivalently,} \ \ S^{s,t}_\nu = e^{tX} e^{(t-s)(\nu\Delta - X)} e^{-sX}. \tag{3.6.60}$$

A straightforward analogue of (3.6.4) is

$$\|S^{s,t}_\nu f\|_{H^{\sigma,p}} \le C e^{K(t-s)} \|f\|_{H^{\sigma,p}}, \tag{3.6.61}$$

valid for $p \in [2,\infty)$, $\sigma \in [0,\infty)$, with $C = C_{\sigma,p}$ and $K = K_{\sigma,p}$ independent of $\nu \in (0,1]$. This gives:

Corollary 3.6.5. *In the setting of Proposition 3.6.3, if $p \in [2,\infty)$, $\sigma \ge 0$, then for $t \in [0,T]$, $\nu \in (0,1]$,*

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma,p}(M)} \le C_T \nu^{(N+1)/2} \|f\|_{H^{\sigma,p}(M)}, \tag{3.6.62}$$

and

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma+N-1,p}(M)} \le C_T \nu \|f\|_{H^{\sigma,p}(M)}, \tag{3.6.63}$$

with C_T independent of ν.

Remark. Applying Corollary 3.6.5 with N replaced by $N+2$ and taking into account the fact that this just adds $a_{N+1} + a_{N+2}$ to the amplitude in the formula for $\mathfrak{S}^t_\nu$, we obtain a complement to (3.6.62)–(3.6.63), namely

$$\|\mathfrak{S}^t_\nu f - S^t_\nu f\|_{H^{\sigma+N+1,p}(M)} \le C_T \|f\|_{H^{\sigma,p}(M)}. \tag{3.6.64}$$

The family of operators $S^{s,t}_\nu$ is as important as the family S^t_ν, and it is also of interest to have a parametrix for this family. This is obtained by a slight modification of the previous construction. Parallel to (3.6.9)–(3.6.10), this parametrix has the form

$$\mathfrak{S}^{s,t}_\nu f(x) = (2\pi)^{-n/2} \int a(\nu,s,t,x,\xi) e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi, \tag{3.6.65}$$

with

$$a(\nu, s, t, x, \xi) \sim \sum_{j\ge 0} a_j(\nu, s, t, x, \xi), \tag{3.6.66}$$

given by equations similar to (3.6.12), except that the initial condition is

$$a(\nu, s, s, x, \xi) = 1. \tag{3.6.67}$$

Thus, in place of (3.6.14) we have

$$\begin{aligned} a_0(\nu, s, t, x, \xi) &= e^{-\nu(t-s)H(s,t,x,\xi)}, \\ H(s, t, x, \xi) &= \frac{1}{t-s}\int_s^t G(\sigma, x, \xi)\, d\sigma, \end{aligned} \tag{3.6.68}$$

and in place of (3.6.31) we have

$$a_1(\nu, s, t, x, \xi) = \nu(t-s)\mathcal{A}_1(\nu(t-s), s, t, x, \xi, \zeta)e^{-H(s,t,x,\zeta)}, \tag{3.6.69}$$

this time with

$$\zeta = (\nu(t-s))^{1/2}\xi, \quad \omega = \nu(t-s)\xi, \quad \mathcal{A}_1 \in \mathcal{P}_1, \tag{3.6.70}$$

where now $\mathcal{P}_k$ is defined to consist of functions $F(\nu(t-s), s, t, x, \xi, \zeta, \omega)$, polynomials in $\nu(t-s)$, ζ, ω, and ξ, even in ζ, of degree $\le k$ in ξ and of degree ≤ 1 in ω, with coefficients smooth in (s, t, x), the obvious variant of (3.6.32). (As in (3.6.31), $\mathcal{A}_1$ does not actually depend on ω.) More generally, parallel to (3.6.34), we have

$$\begin{aligned} a_{2k}(\nu, s, t, x, \xi) &= (\nu(t-s))^k \mathcal{A}_{2k}\, e^{-H(s,t,x,\zeta)}, \quad && \mathcal{A}_{2k} \in \mathcal{P}_0, \\ a_{2k+1}(\nu, s, t, x, \xi) &= (\nu(t-s))^{k+1} \mathcal{A}_{2k+1}\, e^{-H(s,t,x,\zeta)}, \quad && \mathcal{A}_{2k+1} \in \mathcal{P}_1, \end{aligned} \tag{3.6.71}$$

except now with $\zeta = (\nu(t-s))^{1/2}\xi$ (as in (3.6.70)), with $\mathcal{A}_j = \mathcal{A}_j(\nu(t-s), s, t, x, \xi, \zeta, \omega)$, and with $\mathcal{P}_k$ as redefined above. In place of (3.6.46), we have

$$a_j(\nu, s, t, x, \xi) = O((\nu(t-s))^{j/2}) \text{ in } S^0_{1,0}, \text{ bounded in } S^{-j}_{1,0}. \tag{3.6.72}$$

The estimates recorded in Corollary 3.6.5 readily extend, to yield:

Proposition 3.6.6. *Given $N \in \mathbb{N}$, take*

$$a(\nu, s, t, x, \xi) = \sum_{j=0}^{N} a_j(\nu, s, t, x, \xi), \tag{3.6.73}$$

and define $\mathfrak{S}^{s,t}_\nu f$ by (3.6.65). Then for $p \in [2, \infty)$, $\sigma \ge 0$, $0 \le s \le t \le T$, and $\nu \in (0, 1]$, we have

$$\begin{aligned} \|\mathfrak{S}^{s,t}_\nu f - S^{s,t}_\nu f\|_{H^{\sigma,p}(M)} &\le C_T \nu^{(N+1)/2} \|f\|_{H^{\sigma,p}(M)}, \\ \|\mathfrak{S}^{s,t}_\nu f - S^{s,t}_\nu f\|_{H^{\sigma+N+1,p}(M)} &\le C_T \|f\|_{H^{\sigma,p}(M)}, \end{aligned} \tag{3.6.74}$$

with C_T independent of ν.

The formula (3.6.65) represents the parametrix $\mathfrak{S}_\nu^{s,t}$ in Fourier integral form. We next obtain a more explicit representation of its integral kernel. We examine the individual terms

$$\mathfrak{S}_{\nu,j}^{s,t} f(x) = (2\pi)^{-n/2} \int a_j(\nu, s, t, x, \xi) e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi = \int K_j(\nu, s, t, x, x-y) f(y)\, dy, \tag{3.6.75}$$

where

$$K_j(\nu, s, t, x, z) = (2\pi)^{-n} \int a_j(\nu, s, t, x, \xi) e^{iz\cdot\xi}\, d\xi, \quad z = x - y. \tag{3.6.76}$$

In case $j = 0$, let us rewrite a_0 as

$$a_0(\nu, s, t, x, \xi) = e^{-\nu(t-s)\mathcal{H}(s,t,x)\xi\cdot\xi}, \tag{3.6.77}$$

where $\mathcal{H}(s, t, x)$ is a positive-definite $n \times n$ matrix. We have a standard Gaussian integral:

$$\begin{aligned} K_0(\nu, s, t, x, z) &= (2\pi)^{-n} \int e^{-\nu(t-s)\mathcal{H}(s,t,x)\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi \\ &= \big(4\pi\nu(t-s)\big)^{-n/2} \det \mathcal{G}(s, t, x)^{1/2} e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.78}$$

where

$$\mathcal{G}(s, t, x) = \mathcal{H}(s, t, x)^{-1}. \tag{3.6.79}$$

Note from (3.6.8) that

$$\mathcal{H}^{ij}(s, t, x) = \frac{1}{t-s} \int_s^t g^{ij}(\sigma, x)\, d\sigma, \tag{3.6.80}$$

where $(g^{ij}) = (g_{ij})^{-1}$, so in particular $\mathcal{H}^{ij}(s, s, x) = g^{ij}(s, x)$ and

$$\mathcal{G}_{ij}(s, s, x) = g_{ij}(s, x). \tag{3.6.81}$$

To compute K_j more generally, we use (3.6.71), which we restate as follows:

$$\begin{aligned} a_{2k}(\nu, s, t, x, \xi) = (\nu(t-s))^k \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} & F_{\alpha\beta}(\nu(t-s), s, t, x) \\ & \times \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi}, \end{aligned} \tag{3.6.82}$$

and

$$\begin{aligned} a_{2k+1}(\nu, s, t, x, \xi) = (\nu(t-s))^{k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1, \ell}} & F_{\alpha\beta\ell}(\nu(t-s), s, t, x) \\ & \times \xi_\ell \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi} \\ + (\nu(t-s))^{k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} & F^0_{\alpha\beta}(\nu(t-s), s, t, x) \\ & \times \big((\nu(t-s))^{1/2}\xi\big)^\alpha \big(\nu(t-s)\xi\big)^\beta e^{-\nu(t-s)\mathcal{H}\xi\cdot\xi}. \end{aligned} \tag{3.6.83}$$

Here $\mathcal{H} = \mathcal{H}(s, t, x)$ is as in (3.6.77), and $F_{\alpha\beta}$, $F_{\alpha\beta\ell}$, and $F^0_{\alpha\beta}$ are smooth functions of their arguments. All the sums are finite. To compute the integrals in (3.6.76), we use the following result:

$$(2\pi)^{-n} \int \xi^\alpha e^{-\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi = \big(\det(4\pi\mathcal{H})\big)^{-1/2} D_z^\alpha e^{-\mathcal{G}z\cdot z/4} = p_\alpha(\mathcal{H}, z) e^{-\mathcal{G}z\cdot z/4}, \tag{3.6.84}$$

where the last identity defines $p_\alpha(\mathcal{H}, z)$, which is a polynomial of degree $|\alpha|$ whose coefficients depend smoothly on $\mathcal{H}$, and $\mathcal{G} = \mathcal{H}^{-1}$. We note that

$$p_\alpha(\mathcal{H}, -z) = (-1)^{|\alpha|} p_\alpha(\mathcal{H}, z). \tag{3.6.85}$$

Taking

$$\mu = \nu(t-s), \tag{3.6.86}$$

we go from (3.6.82)–(3.6.83) to formulas for $K_j(\nu, s, t, x, z)$ via the identities

$$(2\pi)^{-n} \int (\mu^{1/2}\xi)^\alpha (\mu\xi)^\beta e^{-\mu\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi = \mu^{(-n+|\beta|)/2} p_{\alpha+\beta}(\mathcal{H}, \mu^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\mu}, \tag{3.6.87}$$

and

$$(2\pi)^{-n} \int \xi_\ell (\mu^{1/2}\xi)^\alpha (\mu\xi)^\beta e^{-\mu\mathcal{H}\xi\cdot\xi} e^{iz\cdot\xi}\, d\xi = \mu^{(-n+|\beta|-1)/2} p_{\alpha+\beta+\varepsilon_\ell}(\mathcal{H}, \mu^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\mu}. \tag{3.6.88}$$

We obtain

$$K_{2k}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} (\nu(t-s))^{|\beta|/2} F_{\alpha\beta}(\nu(t-s), s, t, x) \times p_{\alpha+\beta}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)}, \tag{3.6.89}$$

hence

$$K_{2k}(\nu, s, t, x, z) = (\nu(t-s))^{-n/2+k} \sum_{b=0}^{1} (\nu(t-s))^{b/2} \Phi_{2k,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \tag{3.6.90}$$

where $\Phi_{2k,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z = Z$, with coefficients smooth in $\nu(t-s)$, s, t, x, satisfying

$$\Phi_{2k,b}(\nu(t-s), s, t, x, -Z) = (-1)^b \Phi_{2k,b}(\nu(t-s), s, t, x, Z). \tag{3.6.91}$$

Similarly,

$$\begin{aligned} K_{2k+1}(\nu, s, t, x, z) = {} & (\nu(t-s))^{-n/2+k+1/2} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1,\ell}} (\nu(t-s))^{|\beta|/2} F_{\alpha\beta\ell}(\nu(t-s), s, t, x) \times p_{\alpha+\beta+\varepsilon_\ell}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)} \\ & + (\nu(t-s))^{-n/2+k+1} \sum_{\substack{\alpha \text{ even} \\ |\beta|\le 1}} (\nu(t-s))^{|\beta|/2} F^0_{\alpha\beta}(\nu(t-s), s, t, x) \times p_{\alpha+\beta}(\mathcal{H}, (\nu(t-s))^{-1/2}z) e^{-\mathcal{G}z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.92}$$

hence

$$\begin{aligned} K_{2k+1}(\nu, s, t, x, z) = {} & (\nu(t-s))^{-n/2+k+1/2} \sum_{b=0}^{1} (\nu(t-s))^{b/2} \Phi_{2k+1,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)} \\ & + (\nu(t-s))^{-n/2+k+1} \sum_{b=0}^{1} (\nu(t-s))^{b/2} \Phi^0_{2k+1,b}(\nu(t-s), s, t, x, (\nu(t-s))^{-1/2}z) \times e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}, \end{aligned} \tag{3.6.93}$$

where $\Phi_{2k+1,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z = Z$, with coefficients smooth in $\nu(t-s), s, t, x$, satisfying

$$\Phi_{2k+1,b}(\nu(t-s), s, t, x, -Z) = (-1)^{b+1}\Phi_{2k+1,b}(\nu(t-s), s, t, x, Z), \tag{3.6.94}$$

and $\Phi^0_{2k+1,b}$ is a polynomial in $(\nu(t-s))^{-1/2}z$ with coefficients smooth in $\nu(t-s), s, t, x$, satisfying

$$\Phi^0_{2k+1,b}(\nu(t-s), s, t, x, -Z) = (-1)^{b}\Phi^0_{2k+1,b}(\nu(t-s), s, t, x, Z). \tag{3.6.95}$$

While the formulas (3.6.89)–(3.6.90) and (3.6.92)–(3.6.93) for the functions $K_j(\nu, s, t, x, z)$ are rather lengthy, they are not difficult to comprehend. The basic result to be gleaned from these calculations is that for $j \geq 1$, $K_j(\nu, s, t, x, z)$ is *smaller* and *smoother* than the dominant term $K_0(\nu, s, t, x, z)$, given by the comparatively simple formula (3.6.78).

3.7. *Boundary layer analysis of $e^{t(\nu\Delta - X)}$*. In this section we examine the fine behavior near $\partial\mathbb{O}$ as $\nu \searrow 0$ of $e^{t(\nu\Delta-X)}f$, with emphasis on the cases $f \in C(\overline{\mathbb{O}})$ and $f \in C^\infty(\overline{\mathbb{O}})$. As in Section 3.2, we work with solutions to

$$\frac{\partial v^\nu}{\partial t} = \nu L(t) v^\nu, \quad v^\nu\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \quad v^\nu(0) = f, \tag{3.7.1}$$

where

$$L(t) = e^{tX}\Delta e^{-tX} \tag{3.7.2}$$

is a smooth family of strongly elliptic operators, as in (3.2.3) and (3.6.6). From this, the behavior of

$$e^{t(\nu\Delta-X)}f = e^{-tX}v^\nu(t) \tag{3.7.3}$$

is easily deduced.

We assume $\mathbb{O}$ is a smoothly bounded open subset of a compact Riemannian manifold M without boundary. To begin the analysis of (3.7.1), we extend f to $\tilde{f}$ on M, having the same degree of smoothness as f, e.g.,

$$f \in C(\overline{\mathbb{O}}) \Rightarrow \tilde{f} \in C(M), \quad f \in C^\infty(\overline{\mathbb{O}}) \Rightarrow \tilde{f} \in C^\infty(M), \quad \text{etc.} \tag{3.7.4}$$

We also extend X to be a smooth vector field on M (we need not assume $\operatorname{div} X = 0$ on M), and define V^ν on $\mathbb{R}^+ \times M$ by

$$\frac{\partial V^\nu}{\partial t} = \nu L(t) V^\nu \text{ on } \mathbb{R}^+ \times M, \quad V^\nu(0, x) = \tilde{f}(x). \tag{3.7.5}$$

Here $L(t)$ is given by (3.7.2). The solution to (3.7.5) has the form

$$V^\nu(t, x) = \int_M \tilde{f}(y) H(\nu, 0, t, x, y)\, dV(y), \tag{3.7.6}$$

where dV is the Riemannian volume element on M. More generally, for $0 \leq s < t$,

$$V^\nu(t, x) = \int_M V^\nu(s, y) H(\nu, s, t, x, y)\, dV_s(y), \tag{3.7.7}$$

where dV_s is the pull-back of dV via the flow generated by X, or equivalently the Riemannian volume element for g_s, the metric tensor g of $\mathbb{O}$ pulled back by this flow. In local coordinates, we have

$$H(\nu, s, t, x, y) = g(s, y)^{-1/2} K(\nu, s, t, x, x - y), \tag{3.7.8}$$

where $K(\nu, s, t, x, x-y)$ has the form

$$K(\nu, s, t, x, z) = \sum_{j=0}^{N} K_j(\nu, s, t, x, z) + R_N(\nu, s, t, x, z), \tag{3.7.9}$$

with R_N the kernel of an operator satisfying the results given in Proposition 3.6.6, i.e., negligible for N large. As seen in (3.6.78),

$$K_0(\nu, s, t, x, z) = (4\pi\nu(t-s))^{-n/2} \det \mathscr{G}(s, t, x)^{1/2} e^{-\mathscr{G}(s,t,x)z\cdot z/4\nu(t-s)}, \tag{3.7.10}$$

and for $j \geq 1$, $K_j(\nu, s, t, x, z)$ are given by (3.6.90) and (3.6.93), as integral kernels that are smaller and smoother then $K_0(\nu, s, t, x, z)$. As before, $n = \dim M = \dim \mathbb{O}$.

Having V^ν, we can write the solution to (3.7.1) as

$$v^\nu(t, x) = V^\nu(t, x) - u^\nu(t, x), \quad t \geq 0,\ x \in \mathbb{O}, \tag{3.7.11}$$

where $u^\nu(t, x)$ is defined by

$$\begin{aligned} \frac{\partial u^\nu}{\partial t} &= \nu L(t) u^\nu \quad \text{on } \mathbb{R} \times \mathbb{O}, \\ u^\nu &= g^\nu \quad \text{on } \mathbb{R} \times \partial\mathbb{O}, \\ u^\nu &= 0 \quad \text{on } (-\infty, 0) \times \mathbb{O}, \end{aligned} \tag{3.7.12}$$

where

$$g^\nu(t, x) = \chi_{\mathbb{R}^+}(t)\, V^\nu(t, x), \quad x \in \partial\mathbb{O}. \tag{3.7.13}$$

We now describe how to use the method of layer potentials to solve (3.7.12).

We start with the case $\nu = 1$ and then explain the modifications that work for $\nu \in (0, 1]$. With H as in (3.7.7)–(3.7.8), we set

$$\mathscr{D}_1 h(t, x) = \int_0^t \int_{\partial\mathbb{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(1, s, t, x, y)\, dS_s(y)\, ds, \quad t \geq 0,\ x \in \mathbb{O}. \tag{3.7.14}$$

Here dS_s is the area element on $\partial\mathbb{O}$ induced by the metric tensor g_s, described as below (3.7.7), and $\partial/\partial n_{s,y}$ is the outward unit normal to $\partial\mathbb{O}$ at $y \in \partial\mathbb{O}$, determined by this metric tensor. The boundary trace relation for $\mathscr{D}_1$ is

$$\mathscr{D}_1 h\big|_{\mathbb{R}\times\partial\mathbb{O}} = \left(\tfrac{1}{2}I + N_1\right)h, \tag{3.7.15}$$

assuming $h(s, y) = 0$ for $s < 0$, where

$$N_1 h(t, x) = \int_0^t \int_{\partial\mathbb{O}} h(s, y) \frac{\partial H}{\partial n_{s,y}}(1, s, t, x, y)\, dS_s(y)\, ds, \quad t \geq 0,\ x \in \partial\mathbb{O}. \tag{3.7.16}$$

The integral formula on the right sides of (3.7.14) and (3.7.16) have an identical appearance, but in the former case $x \in \mathbb{O}$ and in the latter case $x \in \partial\mathbb{O}$. It follows that we can solve (3.7.12), in the case $\nu = 1$, as

$$u^1 = \mathscr{D}_1 h^1, \tag{3.7.17}$$

provided h^1 solves

$$\left(\tfrac{1}{2}I + N_1\right)h^1 = g^1. \tag{3.7.18}$$

For general $\nu > 0$, we have essentially the same situation, except that $\nu L(t)$ is the Laplace operator for the metric tensor $\nu^{-1} g_t$. One has the analogue of (3.7.16), with this scaled metric tensor. This rescaling requires that $\partial/\partial n_{s,y}$ be replaced by $\nu^{1/2}\partial/\partial n_{s,y}$ and that dS_s be replaced by $\nu^{-(n-1)/2}\, dS_s$. Also dV is replaced by $\nu^{-n/2}\, dV$, so we need to replace $H(1,s,t,x,y)$ by $\nu^{n/2} H(\nu,s,t,x,y)$. Since

$$\nu^{1/2}\nu^{-(n-1)/2}\nu^{n/2} = \nu, \tag{3.7.19}$$

we obtain

$$\mathcal{D}_\nu h(t,x) = \nu \int_0^t \int_{\partial\mathcal{O}} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,y)\, dS_s(y)\, ds. \tag{3.7.20}$$

The boundary trace result (3.7.15) becomes

$$\mathcal{D}_\nu h\big|_{\mathbb{R}\times\partial\mathcal{O}} = \big(\tfrac{1}{2} I + \nu N_\nu\big) h, \tag{3.7.21}$$

for supp $h \subset \mathbb{R}^+ \times \partial\mathcal{O}$, where

$$N_\nu h(t,x) = \int_0^t \int_{\partial\mathcal{O}} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,y)\, dS_s(y)\, ds. \tag{3.7.22}$$

Hence the solution to (3.7.12) has the form

$$u^\nu(t,x) = \mathcal{D}_\nu h^\nu(t,x), \tag{3.7.23}$$

provided H^ν solves

$$\big(\tfrac{1}{2} I + \nu N_\nu\big) h^\nu = g^\nu, \tag{3.7.24}$$

with $g^\nu(t,x)$ given by (3.7.13).

We now tackle the problem of inverting $((1/2)I + \nu N_\nu)$ in (3.7.24). The results (3.7.8)–(3.7.10) on H and related estimates on K_j established in Section 3.6 imply

$$\Big\| \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,\cdot)\Big\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1/2}, \quad x \in \partial\mathcal{O}, \tag{3.7.25}$$

and

$$\Big\| \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,\cdot)\Big\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1}, \quad x \in \mathcal{O}, \tag{3.7.26}$$

uniformly for $0 \le s < t \le T_0$. For the present analysis, the focus is on (3.7.25). It implies for $I = [0, T_0]$

$$\|\nu N_\nu h\|_{L^\infty(I\times\partial\mathcal{O})} \le C(T_0)\, \nu^{1/2}. \tag{3.7.27}$$

Hence, given $T_0 \in (0,\infty)$, as long as ν is so small that $C(T_0)\nu^{1/2} \le 1/2$, if $g^\nu \in L^\infty(I \times \partial\mathcal{O})$, Equation (3.7.24) is solved by

$$h^\nu = 2(I + 2\nu N_\nu)^{-1} g^\nu = 2(I - 2\nu N_\nu + 4\nu^2 N_\nu^2 - \cdots) g^\nu. \tag{3.7.28}$$

Note that

$$\|h^\nu - 2g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2} \|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})}. \tag{3.7.29}$$

We are motivated to estimate $\mathcal{D}_\nu(h^\nu - 2g^\nu)$. The estimate (3.7.26) is not adequate for this; instead we argue as follows. Denote the solution to (3.7.12) by

$$u^\nu = \mathrm{PI}_\nu g^\nu. \tag{3.7.30}$$

The content of (3.7.21) and (3.7.28) is that

$$\mathrm{PI}_\nu g^\nu = \mathcal{D}_\nu h^\nu, \quad \big(\tfrac{1}{2} I + \nu N_\nu\big) h^\nu = g^\nu. \tag{3.7.31}$$

Hence

$$\mathcal{D}_\nu (h^\nu - 2g^\nu) = \mathrm{PI}_\nu \big(\tfrac{1}{2} I + \nu N_\nu\big)(h^\nu - 2g^\nu). \tag{3.7.32}$$

Now the maximum principle gives

$$\|\mathrm{PI}_\nu h\|_{L^\infty(I\times\mathcal{O})} \le \|h\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.33}$$

so we have the general estimate

$$\|\mathcal{D}_\nu h\|_{L^\infty(I\times\mathcal{O})} \le C\|h\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.34}$$

with C independent of $\nu \in (0, 1]$, and in particular

$$\|\mathcal{D}_\nu (h^\nu - 2g^\nu)\|_{L^\infty(I\times\mathcal{O})} \le C\|h^\nu - 2g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2}\|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})}, \tag{3.7.35}$$

the last inequality by (3.7.29).

Proposition 3.7.1. *The solution u^ν to (3.7.12) has the property*

$$\|u^\nu - 2\mathcal{D}_\nu g^\nu\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|g^\nu\|_{L^\infty(I\times\partial\mathcal{O})} \le C'(I)\nu^{1/2}\|\tilde{f}\|_{L^\infty(M)}. \tag{3.7.36}$$

Proof. The first inequality in (3.7.36) follows from (3.7.35) and the fact that $u^\nu = \mathcal{D}_\nu h^\nu$. The second follows from the identification of g^ν in (3.7.13) and the maximum principle, applied to (3.7.5). □

Recalling (3.7.11), we have:

Corollary 3.7.2. *The solution v^ν to (3.7.1) has the property*

$$\|v^\nu - (V^\nu - 2\mathcal{D}_\nu g^\nu)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{L^\infty(M)}. \tag{3.7.37}$$

We can obtain simpler approximations to u^ν and v^ν if we assume more regularity on f. Using (3.5.9), we have, for $q \in [2, \infty)$, $\sigma > 0$,

$$\|V^\nu(t,\cdot)\|_{H^{\sigma,q}(M)} \le C\|\tilde{f}\|_{H^{\sigma,q}(M)}, \quad 0 \le t \le T_0, \tag{3.7.38}$$

with C independent of $\nu \in (0, 1]$. Taking $\sigma = 2 + \varepsilon$ and q sufficiently large, we obtain

$$\|V^\nu(t,\cdot)\|_{C^2(M)} \le C\|\tilde{f}\|_{H^{2+\varepsilon,q}(M)}, \quad 0 \le t \le T_0, \tag{3.7.39}$$

for each $\varepsilon > 0$, $q > n/\varepsilon$, with C independent of ν. Hence the solution V^ν to (3.7.5) satisfies

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le C\nu\|\tilde{f}\|_{H^{2+\varepsilon,q}(M)}, \quad 0 \le t \le T_0. \tag{3.7.40}$$

Interpolation with

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le 2\|\tilde{f}\|_{L^\infty(M)} \le C\|\tilde{f}\|_{H^{\varepsilon,q}(M)} \tag{3.7.41}$$

gives

$$\|V^\nu(t) - \tilde{f}\|_{L^\infty(M)} \le C\nu^{1/2}\|\tilde{f}\|_{H^{1+\varepsilon,q}(M)} \le C'\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.42}$$

the last inequality holding provided $\delta > \varepsilon$. We hence have the following.

Proposition 3.7.3. *In the setting of Proposition 3.7.1 and Corollary 3.7.2, we have, for each* $\delta > 0$,

$$\|u^\nu - 2\mathcal{D}_\nu f^b\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)} \tag{3.7.43}$$

and

$$\|v^\nu - (f - 2\mathcal{D}_\nu f^b)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.44}$$

where

$$f^b = \chi_{\mathbb{R}^+}(t)\, f\big|_{\partial\mathcal{O}}. \tag{3.7.45}$$

Proof. From (3.7.42) we have

$$\|g^\nu - f^b\|_{L^\infty(I\times\partial\mathcal{O})} \le C\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}, \tag{3.7.46}$$

and then the estimate (3.7.34) applied to $h = g^\nu - f^b$ gives (3.7.43) from (3.7.36). The proof of (3.7.44) is similar. □

For a further simplification, we compare $\mathcal{D}_\nu$ with $\mathcal{D}_\nu^0$, defined by

$$\mathcal{D}_\nu^0 h(t,x) = \nu\int_0^t\int_{\partial\mathcal{O}} h(s,y)\frac{\partial H_0}{\partial n_{s,y}}(\nu,s,t,x,y)\,dS_s(y)\,ds, \tag{3.7.47}$$

where, parallel to (3.7.8), we set

$$H_0(\nu,s,t,x,y) = g(s,y)^{-1/2}K_0(\nu,s,t,x,x-y), \tag{3.7.48}$$

with K_0 given by (3.7.10). By (3.7.9) we have

$$K - K_0 = \sum_{j=1}^N K_j + R_N. \tag{3.7.49}$$

Parallel to (3.7.26) we have

$$\Big\|\frac{\partial K_1}{\partial n_{s,y}}(\nu,s,t,x,\cdot)\Big\|_{L^1(\partial\mathcal{O})} \le C(\nu(t-s))^{-1/2}, \quad x\in\mathcal{O}, \tag{3.7.50}$$

with better estimates on $\partial K_j/\partial n_{s,y}$ for $j \ge 2$ and on $\partial R_N/\partial n_{s,y}$. This leads to:

Proposition 3.7.4. *With* $\mathcal{D}_\nu^0$ *defined by (3.7.47)–(3.7.48), we have*

$$\|\mathcal{D}_\nu h - \mathcal{D}_\nu^0 h\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|h\|_{L^\infty(I\times\partial\mathcal{O})}. \tag{3.7.51}$$

Hence, in the setting of Proposition 3.7.3, we have, for each $\delta > 0$,

$$\|u^\nu - 2\mathcal{D}_\nu^0 f^b\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)} \tag{3.7.52}$$

and

$$\|v^\nu - (f - 2\mathcal{D}_\nu^0 f^b)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|\tilde{f}\|_{C^{1,\delta}(M)}. \tag{3.7.53}$$

4. Analysis of solutions to $u_t = \nu\Delta u - X_\nu u$

In this chapter, we extend some of the results of Chapter 3 from the setting of solutions to $u_t = \nu\Delta u - Xu$ to the more subtle setting of solutions to $u_t = \nu\Delta u - X_\nu u$, directly relevant to the equation for w^ν in (1.0.8). As in that chapter, we assume $\overline{\mathcal{O}}$ is a compact Riemannian manifold with boundary $\partial\mathcal{O}$, and with Laplace Beltrami operator Δ. We take X_ν, for $\nu \in (0, 1]$, to be a family of (time dependent) vector fields on $\mathcal{O}$ having certain properties that we will specify below, and take $u = u^\nu$ to solve

$$\frac{\partial u}{\partial t} = \nu\Delta u - X_\nu u, \quad u\big|_{\mathbb{R}^+\times\partial\mathcal{O}} = 0, \quad u(0) = f. \tag{4.0.1}$$

In Section 4.1 we estimate $u^\nu(t)$ in the spaces $\mathcal{V}^k(\mathcal{O})$, introduced in Section 3.3, given $f \in \mathcal{V}^k(\mathcal{O})$, extending the scope of the uniform boundedness results of Section 3.3. In Section 4.2 we establish convergence of $u^\nu(t)$ to $e^{-tX}f$ in $\mathcal{V}^k(\mathcal{O})$, for such f, when $\nu \searrow 0$ and $X_\nu \to X$ in an appropriate sense, specified there. We also obtain L^p-norm convergence results, for $p \in [1, \infty)$.

4.1. *Conormal type estimates.* We will find it useful to extend the class of function spaces $\mathcal{V}^k(\mathcal{O})$. Given $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $p \in [1, \infty]$, we define

$$\mathcal{V}^{k,p}(\mathcal{O}) = \{u \in L^p(\mathcal{O}) : Y_1 \cdots Y_j\, u \in L^p(\mathcal{O}),\ \forall j \le k, Y_\ell \in \mathfrak{X}^1\}, \tag{4.1.1}$$

with

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathcal{O}} : Y \parallel \partial\mathcal{O}\}. \tag{4.1.2}$$

Recall that the case $p = 2$ is defined in (3.3.1). As in (3.3.3), there exists a finite set

$$\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1 \tag{4.1.3}$$

with the property that each element of $\mathfrak{X}^1$ is a linear combination, with coefficients in $C^\infty(\overline{\mathcal{O}})$ of these vector fields Y_j. We recall and generalize some further useful notation from Section 3.3. With Y_j as in (4.1.3), let $J = (j_1, \dots, j_k)$ and set

$$Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k. \tag{4.1.4}$$

Also set

$$\mathfrak{X}^k = \operatorname{Span}\{Z_1 \cdots Z_j : j \le k,\ Z_\ell \in \mathfrak{X}^1\}. \tag{4.1.5}$$

We have

$$\mathfrak{X}^k = \text{Span over } C^\infty(\overline{\mathcal{O}}) \text{ of } \{Y^J : |J| \le k\}, \tag{4.1.6}$$

and

$$\begin{aligned}\mathcal{V}^{k,p}(\mathcal{O}) &= \{u \in L^p(\mathcal{O}) : Y^J u \in L^p(\mathcal{O}),\ \forall\, |J| \le k\} \\ &= \{u \in L^p(\mathcal{O}) : Lu \in L^p(\mathcal{O}) : \forall\, L \in \mathfrak{X}^k\}.\end{aligned} \tag{4.1.7}$$

Let us also set

$$\mathcal{V}^{\infty,p}(\mathcal{O}) = \bigcap_k \mathcal{V}^{k,p}(\mathcal{O}). \tag{4.1.8}$$

We now discuss conditions on X_ν. We require

$$X_\nu \in \widehat{\mathfrak{X}}^1, \tag{4.1.9}$$

a space of t-dependent vector fields on $\overline{\mathcal{O}}$, depending on the parameter $\nu \in (0, 1]$, which we proceed to define. We want to include the example arising in (2.2.4)–(2.2.5), i.e.,

$$X_\nu = v^\nu(t, z)\frac{\partial}{\partial x}, \quad v^\nu(t, z) = e^{\nu t A} V(z). \tag{4.1.10}$$

In this case we have $\mathcal{O} = \mathbb{T} \times I$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $I = [0, 1]$, and A is given by (2.1.5).

Lemma 4.1.1. *Given $T_0 \in (0, \infty)$, we have*

$$v^\nu(t, \cdot) \in \mathcal{V}^{\infty,\infty}(\mathcal{O}), \tag{4.1.11}$$

with bounds independent of $t \in [0, T_0]$, $\nu \in (0, 1]$.

Proof. Straightforward from the construction of $e^{\nu t A} V(z)$ via the method of images. There is no x-dependence, so the result is actually $v^\nu(t, \cdot) \in \mathcal{V}^{\infty,\infty}(I)$, with uniform bounds. In this setting, we mention that $\mathfrak{X}^1(I)$ consists of smooth vector fields on I that vanish at the endpoints. □

To define $\widehat{\mathfrak{X}}^1$ in general, we first specify that, on any compact $\overline{\Omega} \Subset \mathcal{O}$, an element $X_\nu(t)$ has uniform bounds in $C^k(\overline{\Omega})$ for all k. To complete the definition, we take a collar neighborhood U of $\partial\mathcal{O}$, diffeomorphic to $\partial\mathcal{O} \times I$, take coordinates $(x, z) \in \partial\mathcal{O} \times I$, and write

$$X_\nu = v^\nu(t, x, z)\frac{\partial}{\partial x} + w^\nu(t, x, z)\beta(x, z)\frac{\partial}{\partial z}. \tag{4.1.12}$$

Here $v^\nu \partial/\partial x$ is shorthand for $\sum_j v_j^\nu \partial/\partial x_j$. We require (with bounds uniform in $t \in [0, T_0]$, $\nu \in (0, 1]$),

$$v^\nu, w^\nu \in \mathcal{V}^{\infty,\infty}(\mathcal{O}), \quad \beta \in C^\infty(\overline{\mathcal{O}}), \quad \beta\big|_{\partial\mathcal{O}} = 0. \tag{4.1.13}$$

These conditions define $\widehat{\mathfrak{X}}^1$.

Lemma 4.1.2. *We have*

$$X_\nu \in \widehat{\mathfrak{X}}^1, \ Y \in \mathfrak{X}^1 \Longrightarrow [X_\nu, Y] \in \widehat{\mathfrak{X}}^1. \tag{4.1.14}$$

Proof. The bounds on $[X_\nu, Y]$ on any $\overline{\Omega} \Subset \mathcal{O}$ are clear. Near $\partial\mathcal{O}$, we represent X_ν as in (4.1.12) and set

$$Y = a(x, z)\frac{\partial}{\partial x} + b(x, z)\frac{\partial}{\partial z}, \quad a, b \in C^\infty(\overline{\mathcal{O}}), \quad b\big|_{\partial\mathcal{O}} = 0. \tag{4.1.15}$$

Then

$$[X_\nu, Y] = \xi^\nu(t, x, z)\frac{\partial}{\partial x} + \eta^\nu(t, x, z)\frac{\partial}{\partial z}, \tag{4.1.16}$$

with

$$\begin{aligned} \xi^\nu &= v^\nu(\partial_x a) + w^\nu\beta(\partial_z a) - a(\partial_x v^\nu) - b(\partial_z v^\nu), \\ \eta^\nu &= v^\nu(\partial_x b) + w^\nu\beta(\partial_z b) - a\partial_x(w^\nu\beta) - b\partial_z(w^\nu\beta). \end{aligned} \tag{4.1.17}$$

Comparison with the defining conditions in (4.1.12)–(4.1.13) gives $[X_\nu, Y] \in \widehat{\mathfrak{X}}^1$. □

Next we define

$$\widehat{\mathfrak{X}}^k = \text{Span}\,\{X_\nu Y^J : X_\nu \in \widehat{\mathfrak{X}}^1, \ Y^J \in \mathfrak{X}^{k-1}\}. \tag{4.1.18}$$

Lemma 4.1.3. *We have*

$$P_\nu \in \widehat{\mathfrak{X}}^k,\ Y \in \mathfrak{X}^1 \implies Y P_\nu \in \widehat{\mathfrak{X}}^{k+1}; \tag{4.1.19}$$

hence

$$P_\nu \in \widehat{\mathfrak{X}}^k,\ Y^I \in \mathfrak{X}^\ell \implies Y^I P_\nu \in \widehat{\mathfrak{X}}^{k+\ell}. \tag{4.1.20}$$

Proof. To prove (4.1.19), note that for $X_\nu \in \widehat{\mathfrak{X}}^1,\ Y^J \in \mathfrak{X}^{k-1}$,

$$Y X_\nu Y^J = X_\nu Y Y^J + [Y, X_\nu] Y^J, \tag{4.1.21}$$

and apply Lemma 4.1.2 to the second term on the right side of (4.1.21). The result (4.1.20) follows directly from (4.1.19). □

Lemma 4.1.3 will prove useful in connection with the following. With Y_j as in (4.1.3), let us set

$$\|u\|_{\mathcal{V}^{k,p}} = \sum_{|J|\le k} \|Y^J u\|_{L^p}. \tag{4.1.22}$$

From the representation (4.1.12), we have

$$X_\nu \in \widehat{\mathfrak{X}}^1 \implies X_\nu = \sum a^j_{\nu,t} Y_j, \quad a^j_{\nu,t} \in L^\infty(\mathcal{O}), \tag{4.1.23}$$

with bounds independent of $\nu \in (0, 1],\ t \in [0, T_0]$, hence, given $X_\nu \in \widehat{\mathfrak{X}}^1$,

$$\|X_\nu u\|_{L^p} \le C\|u\|_{\mathcal{V}^{1,p}}, \tag{4.1.24}$$

and, by (4.1.20),

$$\|X_\nu u\|_{\mathcal{V}^{k,p}} \le C\|u\|_{\mathcal{V}^{k+1,p}}. \tag{4.1.25}$$

We also set

$$P_k^2(u) = \sum_{|J|=k} \|Y^J u\|^2_{L^2}, \tag{4.1.26}$$

so

$$\|u\|^2_{\mathcal{V}^{k,2}} \approx \sum_{j\le k} P_j^2(u). \tag{4.1.27}$$

We also denote $\mathcal{V}^{k,2}$ by $\mathcal{V}^k$.

We now estimate the rate of change of $P_k^2(u(t))$ for $u(t)$ satisfying (4.0.1). We assume

$$X_\nu \in \widehat{\mathfrak{X}}^1, \quad \operatorname{div} X_\nu = 0. \tag{4.1.28}$$

We also assume u is sufficiently smooth on $(0, \infty) \times \overline{\mathcal{O}}$ for the calculations made below to work. We will comment on how to verify this assumption later in this section.

We start with the case $k = 0$:

$$\frac{d}{dt}\|u\|^2_{L^2} = 2(u_t, u)_{L^2} = 2\nu(\Delta u, u)_{L^2} - 2(X_\nu u, u)_{L^2} = -2\nu\|\nabla u\|^2_{L^2}, \tag{4.1.29}$$

Moving on to $k = 1$, we have

$$\begin{aligned}\frac{d}{dt}\|Y_j u\|^2_{L^2} &= 2(Y_j u_t, Y_j u)_{L^2} = 2\nu(Y_j \Delta u, Y_j u)_{L^2} - 2(Y_j X_\nu u, Y_j u)_{L^2} \\ &= 2\nu(\Delta Y_j u, Y_j u)_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2(X_\nu Y_j u, Y_j u)_{L^2} - 2([Y_j, X_\nu]u, Y_j u)L^2 \\ &= -2\nu\|\nabla Y_j u\|^2_{L^2} + 2\nu([Y_j, \Delta]u, Y_j u)_{L^2} - 2([Y_j, X_\nu]u, Y_j u)_{L^2}.\end{aligned} \tag{4.1.30}$$

Of the three terms in the last line, the first has a clear significance. For the third, we have $[Y_j, X_\nu] \in \widehat{\mathfrak{X}}^1$, by Lemma 4.1.2, and hence, by (4.1.23),

$$2([Y_j, X_\nu]u, Y_j u)_{L^2} \le C P_1^2(u). \tag{4.1.31}$$

It remains to estimate the second term. For this, write

$$[Y, \Delta] = \sum_\ell A_\ell B_\ell, \tag{4.1.32}$$

with A_ℓ, B_ℓ smooth vector fields on $\overline{\mathcal{O}}$. We have

$$2\nu([Y_j, \Delta]u, Y_j u)_{L^2} = 2\nu \sum_\ell (B_\ell u, A_\ell^* Y_j u)_{L^2} \le \nu\|\nabla Y_j u\|_{L^2}^2 + \nu\|Y_j u\|_{L^2}^2 + K_1 \nu\|\nabla u\|_{L^2}^2. \tag{4.1.33}$$

Plugging (4.1.31) and (4.1.33) into (4.1.30) and summing over j gives

$$\frac{d}{dt} P_1^2(u) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u) + M K_1 \nu \|\nabla u\|^2. \tag{4.1.34}$$

The term $M K_1 \nu \|\nabla u\|_{L^2}^2$ is tamed by bringing in (4.1.29), to obtain

$$\frac{d}{dt}\Big(P_1^2(u) + M K_1 P_0^2(u)\Big) \le -\nu \sum_j \|\nabla Y_j u\|_{L^2}^2 + (MC + \nu) P_1^2(u). \tag{4.1.35}$$

Proceeding to general k, we take $|J| = k$ and look at

$$\begin{aligned}
\frac{d}{dt}\|Y^J u\|_{L^2}^2 &= 2(Y^J u_t, Y^J u)_{L^2} = 2\nu(Y^J \Delta u, Y^J u)_{L^2} - 2(Y^J X_\nu u, Y^J u)_{L^2} \\
&= 2\nu(\Delta Y^J u, Y^J u)_{L^2} + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2(X_\nu Y^J u, Y^J u)_{L^2} - 2([Y^J, X_\nu]u, Y^J u)_{L^2} \\
&= -2\nu\|\nabla Y^J u\|_{L^2}^2 + 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} - 2([Y^J, X_\nu]u, Y^J u)_{L^2}.
\end{aligned} \tag{4.1.36}$$

As with (4.1.30), of the three terms in the last line of (4.1.36), the first has a clear significance. For the third, we have, by Lemmas 4.1.2–4.1.3,

$$[X_\nu, Y^J] = [X_\nu, Y_{j_1}] Y_{j_2} \cdots Y_{j_k} + \cdots + Y_{j_1} \cdots Y_{j_{k-1}} [X_\nu, Y_{j_k}] \in \widehat{\mathfrak{X}}^k, \tag{4.1.37}$$

and hence, by (4.1.25),

$$([Y^J, X_\nu]u, Y^J u)_{L^2} \le C_k \|u\|_{\mathcal{V}^k}^2. \tag{4.1.38}$$

It remains to estimate the second term in the last line of (4.1.36). For this, write

$$[\Delta, Y^J] = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}} [\Delta, Y_{j_\ell}] Y_{j_{\ell+1}} \cdots Y_{j_k} = \sum_{\ell=1}^k Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k}, \tag{4.1.39}$$

where $L_{j_\ell} = [\Delta, Y_{j_\ell}]$ is a second order differential operator that annihilates constants.. We say a product of k factors

$$Y_{j_1} \cdots Y_{j_{\ell-1}} L_{j_\ell} Y_{j_{\ell+1}} \cdots Y_{j_k} \tag{4.1.40}$$

is of type (k, ℓ), meaning it is a product of k factors, all being vector fields in $\mathfrak{X}^1$ except one, in position ℓ, which is a second order differential operator that annihilates constants. If $\ell \ge 2$, we can write (4.1.40) as

$$Y_{j_1} \cdots Y_{j_\ell-2} L_{j_\ell} \cdots Y_{j_k} + Y_{j_1} \cdots Y_{j_{\ell-2}} [Y_{j_{\ell-1}}, L_{j_\ell}] \cdots Y_{j_k}, \tag{4.1.41}$$

a sum of terms of type $(k, \ell - 1)$ and of type $(k - 1, \ell - 1)$. Repeating this process, we convert (4.1.40) into a sum of terms of type $(j, 1)$, for $j \le k$. Hence we have

$$([Y^J, \Delta]u, Y^J u)_{L^2} = \sum_{|I| \le k-1} (L_I Y^I u, Y^J u)_{L^2}, \tag{4.1.42}$$

where the L_I are differential operators of order 2, annihilating constants; hence

$$L_I = \sum_j A_{Ij} B_{Ij}, \tag{4.1.43}$$

where A_{Ij} are first order differential operators and B_{Ij} are vector fields. We then have

$$\begin{aligned} 2\nu([Y^J, \Delta]u, Y^J u)_{L^2} &= 2\nu \sum_{|I| \le k-1} \sum_j (B_{Ij} Y^I u, A_{Ij}^* Y^J u)_{L^2} \\ &\le \tilde{C}\nu \sum_{|I| \le k-1} \|\nabla Y^I u\|_{L^2} \cdot \big(\|\nabla Y^J u\|_{L^2} + \|Y^J u\|_{L^2}\big) \\ &\le \nu \|\nabla Y^J u\|_{L^2}^2 + \nu \|Y^J u\|_{L^2}^2 + C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|_{L^2}^2. \end{aligned} \tag{4.1.44}$$

Inserting (4.1.38) and (4.1.44) into (4.1.36), we get

$$\frac{d}{dt} \|Y^J u\|_{L^2}^2 \le -\nu \|\nabla Y^J u\|_{L^2}^2 + (C_k + \nu)\|u\|_{\mathcal{V}^k}^2 + C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|_{L^2}^2; \tag{4.1.45}$$

hence, for $\nu \in (0, 1]$, and with $C_k + 1$ re-notated as C_k,

$$\frac{d}{dt} P_k^2(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|_{L^2}^2 + M C_k \|u\|_{\mathcal{V}^k}^2 + M C_k \nu \sum_{|I| \le k-1} \|\nabla Y^I u\|_{L^2}^2. \tag{4.1.46}$$

It follows that there exist $A_{kj} \in (0, \infty)$ and $B_k \in (0, \infty)$ such that if we set

$$\widetilde{N}_k^2(u) = P_k^2(u) + \sum_{j=0}^{k-1} A_{kj} P_j^2(u), \tag{4.1.47}$$

then

$$\frac{d}{dt} \widetilde{N}_k^2(u) \le -\nu \sum_{|J|=k} \|\nabla Y^J u\|_{L^2}^2 + 2B_k \widetilde{N}_k^2(u), \tag{4.1.48}$$

when $u = u(t)$ is given by (4.0.1). In particular, redefining $\|u\|_{\mathcal{V}^k}^2$ as

$$\|u\|_{\mathcal{V}^k}^2 = \widetilde{N}_k^2(u), \tag{4.1.49}$$

we obtain

$$\|u(t)\|_{\mathcal{V}^k} \le e^{(t-s)B_k} \|u(s)\|_{\mathcal{V}^k}, \tag{4.1.50}$$

for $0 < s < t < \infty$.

The estimates (4.1.48)–(4.1.50) have been established under the assumption that $u(t) = u^\nu(t)$ is sufficiently smooth on $\overline{\mathcal{O}}$ for $t > 0$. For example, if we add the assumption

$$X_\nu \in C^\infty((0, \infty) \times \overline{\mathcal{O}}) \tag{4.1.51}$$

for each $\nu \in (0, 1]$, we have such estimates, since well known parabolic regularity results give $u \in C^\infty((0, \infty) \times \overline{\mathcal{O}})$. (We emphasize that we do *not* assume $X_\nu \in C([0, \infty) \times \overline{\mathcal{O}})$.) Let us record this result.

Proposition 4.1.4. *Let $u = u^\nu$ solve (4.0.1). Assume X_ν satisfies (4.1.9) and (4.1.51). Then the estimate (4.1.50) holds, for $0 < s < t < \infty$, with B_k and the $\mathcal{V}^k$-norm independent of $\nu \in (0, 1]$.*

Next we want to pass to the limit $s = 0$ in (4.1.50), obtaining

$$\|u(t)\|_{\mathcal{V}^k} \le e^{tB_k} \|f\|_{\mathcal{V}^k}. \tag{4.1.52}$$

It is clear that we can do this in the context of Proposition 4.1.4 if we also know that

$$u \in C([0, \infty), \mathcal{V}^k(\mathcal{O})). \tag{4.1.53}$$

In turn, since the hypotheses of Proposition 4.1.4 already imply the result $u \in C^\infty((0, \infty) \times \overline{\mathcal{O}})$, it remains to establish that

$$f \in \mathcal{V}^k(\mathcal{O}) \implies u \in C([0, T_\nu], \mathcal{V}^k(\mathcal{O})), \tag{4.1.54}$$

for some $T_\nu > 0$ (possibly depending on ν). We turn to this task.

We set

$$\mathfrak{X} = C([0, T_\nu], \mathcal{V}^k(\mathcal{O})), \tag{4.1.55}$$

and seek $u \in \mathfrak{X}$ as a unique solution to

$$u(t) = e^{t\nu\Delta} f - \int_0^t e^{(t-s)\nu\Delta} X_\nu(s) u(s)\, ds, \tag{4.1.56}$$

i.e., as a fixed point of $\Phi : \mathfrak{X} \to \mathfrak{X}$, defined by

$$\Phi u(t) = e^{t\nu\Delta} f - \int_0^t e^{(t-s)\nu\Delta} X_\nu(s) u(s)\, ds. \tag{4.1.57}$$

This will work if we are able to show $\Phi : \mathfrak{X} \to \mathfrak{X}$ is a contraction map for $T_\nu > 0$ sufficiently small. We have

$$\Phi u(t) - \Phi v(t) = -\int_0^t e^{(t-s)\nu\Delta} X_\nu(x)(u(s) - v(s))\, ds. \tag{4.1.58}$$

Note that, by (4.1.25),

$$\|X_\nu(s)(u(s) - v(s))\|_{\mathcal{V}^{k-1}} \le C\|u(s) - v(s)\|_{\mathcal{V}^k}. \tag{4.1.59}$$

Meanwhile, it follows from (3.4.19) that

$$\|e^{(t-s)\nu\Delta} g\|_{\mathcal{V}^k} \le \frac{C}{\nu^{1/2}(t-s)^{1/2}} \|g\|_{\mathcal{V}^{k-1}}. \tag{4.1.60}$$

Hence

$$\|\Phi u(t) - \Phi v(t)\|_{\mathcal{V}^k} \le C \frac{t^{1/2}}{\nu^{1/2}} \sup_{0 \le s \le t} \|u(s) - v(s)\|_{\mathcal{V}^k}. \tag{4.1.61}$$

A similar estimate works on (4.1.57), and we deduce that Φ is a contraction map on $\mathfrak{X}$ provided $T_\nu \le \nu/2C^2$.

We summarize what has been accomplished.

Proposition 4.1.5. *In the setting of Proposition 4.1.4, given $f \in \mathcal{V}^k(\mathcal{O})$, there is a unique solution $u = u^\nu$ to (4.0.1), satisfying*

$$u \in C([0,\infty), \mathcal{V}^k(\mathcal{O})) \cap C^\infty((0,\infty) \times \overline{\mathcal{O}}), \tag{4.1.62}$$

and we have

$$\|u(t)\|_{\mathcal{V}^k} \le e^{tB_k} \|f\|_{\mathcal{V}^k}. \tag{4.1.63}$$

4.2. *Vanishing ν limits.* As in Section 4.1, we assume $u = u^\nu$ solves

$$\frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu - X_\nu u^\nu, \quad u^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0, \quad u(0) = f, \tag{4.2.1}$$

with $f \in \mathcal{V}^k(\mathcal{O})$. We assume, as in (4.1.28), that

$$X_\nu \in \widehat{\mathfrak{X}}^1, \quad \operatorname{div} X_\nu = 0, \tag{4.2.2}$$

and as in (4.1.51) that

$$X_\nu \in C^\infty((0,\infty) \times \overline{\mathcal{O}}). \tag{4.2.3}$$

We also assume

$$X \in \mathfrak{X}^1, \quad \operatorname{div} X = 0. \tag{4.2.4}$$

Here is our first convergence result.

Proposition 4.2.1. *Under these hypotheses, we have, as $\nu \searrow 0$,*

$$u^\nu(t) \to e^{-tX} f, \quad \textit{weak}^* \textit{ in } \mathcal{V}^k(\mathcal{O}), \tag{4.2.5}$$

provided X_ν also satisfies the following: we can write

$$X_\nu = \sum_j a_j^\nu(t,x) Y_j, \quad X = \sum_j a_j(x) Y_j, \tag{4.2.6}$$

where, as in (4.1.3), the set $\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1$ spans $\mathfrak{X}^1$ over $C^\infty(\overline{\mathcal{O}})$, and we have $\|a_j^\nu(t,\cdot)\|_{L^\infty(\mathcal{O})}$, $\|a_j\|_{L^\infty(\mathcal{O})} \le K$, and

$$\lim_{\nu \searrow 0} [a_j^\nu(t,x) - a_j(x)] = 0, \quad \textit{uniformly on compact subsets of } \mathcal{O}. \tag{4.2.7}$$

Remark. Looking at (4.1.10), we see that (4.2.6)–(4.2.7) hold when X_ν is the family arising in the plane-parallel chanel flow problem.

Proof. Rewrite (4.2.1) as

$$\frac{\partial u^\nu}{\partial t} = (\nu\Delta - X) u^\nu + (X - X_\nu) u^\nu, \tag{4.2.8}$$

so

$$u^\nu(t) = e^{t(\nu\Delta - X)} f + \int_0^t e^{(t-s)(\nu\Delta - X)} (X - X_\nu(s)) u^\nu(s)\, ds. \tag{4.2.9}$$

We have

$$(X - X_\nu(s)) u^\nu(s) = \sum_j [a_j(x) - a_j^\nu(s,x)] Y_j u^\nu(s), \tag{4.2.10}$$

and $u^\nu(s)$ is bounded in $\mathcal{V}^k(\mathbb{O})$. As long as $k \geq 1$, $Y_j u^\nu(s)$ is bounded in $L^2(\mathbb{O})$, and the hypotheses on a_j^ν give

$$\|(X - X_\nu(s))u^\nu(s)\|_{L^p(\mathbb{O})} \to 0, \quad \text{as } \nu \searrow 0, \ \forall\, p < 2, \tag{4.2.11}$$

with uniform bounds in $L^2(\mathbb{O})$. Now $e^{t(\nu\Delta - X)}$ is a contraction semigroup on each space $L^p(\mathbb{O})$, so from (4.2.9) we obtain

$$\lim_{\nu \searrow 0} \|u^\nu(t) - e^{t(\nu\Delta - X)} f\|_{L^p} = 0, \quad \forall\, p < 2. \tag{4.2.12}$$

This result together with the uniform bounds on $u^\nu(t)$ and on $e^{t(\nu\Delta - X)}$ in $\mathcal{V}^k(\mathbb{O})$, and in concert with the result that

$$e^{t(\nu\Delta - X)} f \to e^{-tX} f, \quad \text{weak}^* \text{ in } \mathcal{V}^k(\mathbb{O}), \tag{4.2.13}$$

given in Proposition 3.3.4, yield the asserted convergence (4.2.5), for $k \geq 1$. The case $k = 0$ then follows since $\mathcal{V}^1(\mathbb{O})$ is dense in $\mathcal{V}^0(\mathbb{O}) = L^2(\mathbb{O})$. □

We will improve weak* convergence in (4.2.5) to norm convergence. Here is a first step.

Proposition 4.2.2. *In the setting of Proposition 4.2.1,*

$$f \in L^2(\mathbb{O}) \Longrightarrow u^\nu(t) \to e^{-tX} f, \ \text{ in } L^2\text{-norm}, \ \text{ as } \nu \searrow 0. \tag{4.2.14}$$

Proof. We already have weak* convergence in $L^2(\mathbb{O})$. Also, results of Section 4.1, involving (4.1.29), imply

$$\|u^\nu(t)\|_{L^2(\mathbb{O})} \leq \|f\|_{L^2(\mathbb{O})}, \quad \forall\, \nu, t > 0. \tag{4.2.15}$$

Since for $X \in \mathfrak{X}^1$ such that $\operatorname{div} X = 0$ we have $\|e^{-tX} f\|_{L^2} = \|f\|_{L^2}$, the conclusion in (4.2.14) follows from the weak* convergence. □

An alternative proof of a generalization of Proposition 4.2.2 will be provided in Proposition 4.2.3 below. We begin with the elementary inequality

$$\|u^\nu(t)\|_{L^p} \leq \|f\|_{L^p}, \quad 1 \leq p \leq \infty, \tag{4.2.16}$$

for solutions to (4.2.1) with $f \in L^p(\mathbb{O})$. If also $f \in \mathcal{V}^k(\mathbb{O})$ with $k > n/2$, the result that $u^\nu(t) \to e^{-tX} f$ weak* in $\mathcal{V}^k(\mathbb{O})$, proven in Proposition 4.2.1, implies

$$u^\nu(t) \to e^{-tX} f \ \text{ locally uniformly on } \mathbb{O}. \tag{4.2.17}$$

In particular,

$$f \in C^\infty(\overline{\mathbb{O}}) \Longrightarrow u^\nu(t) \to e^{-tX} f, \ \text{ boundedly and locally uniformly.} \tag{4.2.18}$$

Combining (4.2.16) and (4.2.18) and using standard approximation arguments, we have:

Proposition 4.2.3. *In the setting of Proposition 4.2.1,*

$$f \in C(\overline{\mathbb{O}}) \Longrightarrow u^\nu(t) \to e^{-tX} f, \ \textit{boundedly and locally uniformly on } \mathbb{O}, \tag{4.2.19}$$

and, for $1 \leq p < \infty$,

$$f \in L^p(\mathbb{O}) \Longrightarrow u^\nu(t) \to e^{-tX} f \ \text{ in } L^p\text{-norm}. \tag{4.2.20}$$

We now sharpen Proposition 4.2.1.

Proposition 4.2.4. *In the setting of Proposition 4.2.1, (4.2.5) can be sharpened to*

$$u^\nu(t) \to e^{-tX} f, \quad \text{in } \mathcal{V}^k\text{-norm.} \tag{4.2.21}$$

Proof. In view of uniform bounds on $\|u^\nu(t)\|_{\mathcal{V}^k}$ in (4.1.63), it suffices to establish (4.2.21) for f in a dense subspace of $\mathcal{V}^k(\mathcal{O})$, so take $f \in C_0^\infty(\mathcal{O})$. As in the proof of Proposition 3.3.4, we use the complex interpolation identity

$$\mathcal{V}^k(\mathcal{O}) = [L^2(\mathcal{O}), \mathcal{V}^{2k}(\mathcal{O})]_{1/2}, \tag{4.2.22}$$

established in Proposition A.1.1 of the Appendix, which yields, for $f \in \mathcal{V}^{2k}(\mathcal{O})$,

$$\|u^\nu(t) - e^{-tX} f\|_{\mathcal{V}^k} \le \|u^\nu(t) - e^{-tX} f\|_{L^2}^{1/2} \, \|u^\nu(t) - e^{-tX} f\|_{\mathcal{V}^{2k}}^{1/2}. \tag{4.2.23}$$

The first factor on the right side tends to zero as $\nu \searrow 0$, by Proposition 4.2.2 (or Proposition 4.2.3), and the last factor is uniformly bounded as $\nu \searrow 0$ by (4.1.63) (with k replaced by $2k$). This completes the proof. □

Let us tie these results more closely to estimates obtained in Section 2.2. In such a case we had additional structure to exploit. Namely, X and X_ν were given in (2.2.5) as $V(z)\partial_x$ and $v^\nu(t,z)\partial_x$, respectively, where $v^\nu(t,z) = e^{\nu t \partial_z^2} V(z)$ (see also (4.1.10)). To generalize a bit to our present context, we assume in addition to (4.2.2)–(4.2.4) that

$$X = vZ, \quad X_\nu = v^\nu Z, \quad Z \in \mathfrak{X}^1, \quad Z \text{ commutes with } \Delta \text{ and with } X \text{ and } X_\nu. \tag{4.2.24}$$

The last two conditions are equivalent to

$$Zv = Zv^\nu = 0. \tag{4.2.25}$$

In such a case, (4.2.9) becomes

$$u^\nu(t) = e^{t(\nu\Delta - X)} f + \int_0^t e^{(t-s)(\nu\Delta - X)} \big((v - v^\nu) Z u^\nu(s)\big)\, ds. \tag{4.2.26}$$

The commutation properties yield

$$w^\nu(t) = Zu^\nu(t) \implies \big(\partial_t w^\nu = (\nu\Delta - X_\nu) w^\nu, \quad w^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0, \quad w^\nu(0) = Zf\big). \tag{4.2.27}$$

Then the maximum principle gives

$$\|Zu^\nu(s)\|_{L^\infty} \le \|Zf\|_{L^\infty}. \tag{4.2.28}$$

Let us assume $Zf \in L^\infty(\mathcal{O})$ and set $\|Zf\|_{L^\infty} = K$. Since $e^{(t-s)(\nu\Delta - X)}$ is positivity preserving, we have from (4.2.26) that

$$\big|u^\nu(t,x) - e^{t(\nu\Delta - X)} f(x)\big| \le K \int_0^t e^{(t-s)(\nu\Delta - X)} |v - v^\nu(s)|\, ds. \tag{4.2.29}$$

Now (4.2.24)–(4.2.25) imply $Ze^{(t-s)(\nu\Delta - X)}|v - v^\nu(s)| = 0$, and hence

$$e^{(t-s)(\nu\Delta - X)} |v - v^\nu(s)| = e^{(t-s)\nu\Delta} |v - v^\nu(s)|, \tag{4.2.30}$$

so we have

$$\big|u^\nu(t,x) - e^{t(\nu\Delta - X)} f(x)\big| \le K \int_0^t e^{(t-s)\nu\Delta} |v - v^\nu(s)|\, ds, \tag{4.2.31}$$

which can be compared to (2.2.10)–(2.2.11). To be sure, results of Chapter 3 apply to the right side of (4.2.29), as we have seen in the analysis of (4.2.9), but the analysis of the right side of (4.2.31) is more elementary.

5. Further conclusions on plane parallel flows

This chapter contains further results pertaining to plane parallel flows in a channel. In Section 5.1 we generalize the analysis of the vanishing viscosity limit for plane parallel flows to include flows sheared by a moving boundary, translated at varying speed parallel to the x-axis. In Section 5.2 we consider more general boundary motions, parallel to the x-y-plane. We continue to assume (1.0.1)–(1.0.4) and we take the forcing $F = 0$.

5.1. ***Moving boundary, parallel to x-axis.*** We begin with the case in which both channel walls move with the same velocity $\alpha(t)$, that is, we take the vector field B in (1.0.2) of the form:

$$B(t, p) = (\alpha(t), 0, 0), \quad p \in \partial\mathbb{O}. \tag{5.1.1}$$

Recall $\mathbb{O} = \mathbb{R}/\mathbb{Z} \times [0, 1]$. Since α is spatially constant, this is consistent with the assumption of periodicity in x. Later we extend the analysis to independent motion of the walls, in (5.1.47), and then extend it further in (5.2.1).

The goal is again to study the limit of vanishing viscosity and the corresponding boundary layer, assuming a rough boundary velocity α. The case of circularly symmetric flows in a rotating circle or annulus was studied in [Lopes Filho et al. 2007]. We follow the notation used there.

It is convenient to assume α is defined on the whole $\mathbb{R}$ but supported in $[0, \infty)$. If $\mathfrak{X}$ is a space of distributions on $\mathbb{R}$, we indicate with $\mathfrak{X}_b$ the space of elements of $\mathfrak{X}$ supported on $[0, \infty)$. We then take $\alpha \in \mathrm{BV}_b(\mathbb{R})$ or even $\alpha \in L^p_b(\mathbb{R})$. Since $C^\infty_b(\mathbb{R})$ is dense in these spaces ($p < \infty$), we can first pick $\alpha \in C^\infty_b$ and then use limiting arguments.

In order to highlight the effect of the moving boundary, we again take smooth initial data compatible with (1.0.4) and independent of ν, that is,

$$u^\nu(0, x, y, z) = (V(z), W(x, z), 0), \tag{5.1.2}$$

with $V \in C^\infty([0, 1])$ and $W \in C^\infty(\overline{\mathbb{O}})$. Here u^ν satisfies the system (1.0.8) with $f = g = 0$, which we repeat here for convenience:

$$\frac{\partial v^\nu}{\partial t} = \nu \frac{\partial^2 v^\nu}{\partial z^2}, \tag{5.1.3}$$

$$\frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} = \nu\Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2}\Big). \tag{5.1.4}$$

At the same time, since the inviscid flow does not see the moving boundary due to slip boundary conditions (see below), we do not impose compatibility of the initial data with the motion of the boundary, (i. e., in this context, we do not assume that $V(z) = \alpha(0)$ for $z = 0, 1$). Consequently, the viscous flow has an initial layer at $t = 0$.

As we will demonstrate, the vanishing viscosity limit in this context takes the form $u^\nu \to u^0$, where

$$u^0(t, x, y, z) = (v^0(t, z), w^0(t, x, z), 0), \tag{5.1.5}$$

is the solution of the Euler equations (1.0.15) again with $f = g = 0$, that is,

$$\frac{\partial v^0}{\partial t} = 0, \qquad \frac{\partial w^0}{\partial t} + v^0 \frac{\partial w^0}{\partial x} = 0. \tag{5.1.6}$$

Initial data are as in (5.1.2), so that

$$u^0(0, x, y, z) = (V(z), W(x, z), 0), \tag{5.1.7}$$

and the boundary conditions (1.0.12) are automatically satisfied in this case. In particular, the Euler flow is independent of the moving boundary and there is a boundary layer in the limit $\nu \to 0$.

As in [Lopes Filho et al. 2008; 2007], we pass to a frame moving with the boundary. Equivalently, we set

$$\bar{v}^\nu(t, z) = v^\nu(t, z) - \alpha(t), \qquad \bar{u}^\nu = (\bar{v}^\nu, w^\nu, 0). \tag{5.1.8}$$

We still assume $\alpha \in C_b^\infty(\mathbb{R})$, in particular $\alpha(0) = 0$. Then $\bar{u}^\nu$ must solve the following problem in $\mathbb{O}$:

$$\frac{\partial \bar{v}^\nu}{\partial t} = \nu \frac{\partial^2 \bar{v}^\nu}{\partial z^2} - \alpha'(t), \tag{5.1.9}$$

$$\frac{\partial w^\nu}{\partial t} + V \frac{\partial w^\nu}{\partial x} + (\bar{v}^\nu + \alpha(t) - V) \frac{\partial w^\nu}{\partial x} = \nu \Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2} \Big), \tag{5.1.10}$$

$$\bar{u}^\nu(t, x, z) = 0 \ \text{ on } \ \partial\mathbb{O}, \tag{5.1.11}$$

$$\bar{u}^\nu(0, x, z) = (V(z), W(x, z), 0). \tag{5.1.12}$$

By Duhamel's principle, the system above is equivalent to:

$$\bar{v}^\nu = e^{\nu t\, A} V(z) - \int_0^t [e^{\nu(t-s)\, A}\ 1]\, d\alpha(s), \tag{5.1.13}$$

$$w^\nu = e^{t(\nu\,\Delta - X)} W + \int_0^t e^{(t-s)(\nu\,\Delta - X)} [(V - \bar{v}^\nu - \alpha(s))\, \partial_x w^\nu]\, ds. \tag{5.1.14}$$

The solution to the Euler system is given by

$$v^0(t, z) = V(z), \qquad\qquad t > 0,\ z \in [0, 1], \tag{5.1.15}$$

$$\begin{aligned} w^0(t, x, z) &= e^{-t\, X}\, W_0(x, z) \\ &= W(x - t\, V(z), z), \qquad t > 0,\ x \in \mathbb{R}/\mathbb{Z},\ z \in [0, 1], \end{aligned} \tag{5.1.16}$$

as long as V and W are smooth enough.

We separate the contribution of the boundary conditions by writing (5.1.13) as

$$v^\nu(t) = \bar{v}^\nu(t) + \alpha(t) = e^{\nu t\, A} V(z) + \mathcal{S}^\nu \alpha(t), \quad \text{where } \mathcal{S}^\nu \alpha(t) := \int_0^t [(I - e^{\nu\,(t-s)\, A})\ 1]\, d\alpha(s), \tag{5.1.17}$$

with the integral defined as a Bochner integral. As long as $\nu > 0$, we have

$$\mathcal{S}^\nu : C_b^\infty(\mathbb{R}) \to C_b^1(\mathbb{R}, C^\infty([0, 1])),$$

and in particular the boundary conditions are satisfied pointwise, since $e^{\nu s\, A}\ 1\, \alpha(s)$ is continuous in s and vanishes at $z = 0, 1$ for $s \geq 0$. The trace at the boundary takes value in two copies of $C_b^1(\mathbb{R})$.

To treat less regular α, we observe that for α smooth (5.1.9) is equivalent to (5.1.3), so that $S^\nu\alpha$ is a classical solution of (5.1.3) with $V \equiv 0$. Therefore, the maximum principle for the heat equation gives

$$\mathcal{S}^\nu : C_b(\mathbb{R}) \to C_b\big(\mathbb{R}, C([0,1])\big) \subset L^2_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big). \tag{5.1.18}$$

Next, we observe that if $\beta \in C_b^\infty(\mathbb{R})$ then

$$\alpha = \beta' \implies \mathcal{S}^\nu\alpha = \partial_t \mathcal{S}^\nu\beta.$$

so that

$$\mathcal{S}^\nu\partial_t = \partial_t\mathcal{S}^\nu : C_b^\infty(\mathbb{R}) \to C_b\big(\mathbb{R}, C^\infty([0,1])\big).$$

From (5.1.18) it follows that

$$\mathcal{S}^\nu\partial_t = \partial_t\mathcal{S}^\nu : C_b(\mathbb{R}) \to H^{-1}_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big). \tag{5.1.19}$$

But each $\alpha \in L_b^{p'}(\mathbb{R})$, $p' \geq 1$, has the form $\alpha = \beta'$ with $\beta \in C_b(\mathbb{R})$, namely $\beta(t) = \int_{-\infty}^t \alpha(s)\,ds$. Hence

$$\mathcal{S}^\nu : L_b^{p'}(\mathbb{R}) \to H^{-1}_{b,\mathrm{loc}}\big(\mathbb{R}, C([0,1])\big), \tag{5.1.20}$$

for each $p' \geq 1$. Consequently we have the continuous linear map

$$\mathrm{Tr}\circ\mathcal{S}^\nu : L_b^{p'}(\mathbb{R}) \to \big(H^{-1}_{b,\mathrm{loc}}(\mathbb{R}) \oplus H^{-1}_{b,\mathrm{loc}}(\mathbb{R})\big), \tag{5.1.21}$$

By density, then, the boundary condition $v^\nu(t)|_{\partial\mathbb{O}} = \alpha$ in $H^{-1}_{b,\mathrm{loc}}(\mathbb{R})$ holds for any $\alpha \in L_b^{p'}(\mathbb{R})$, $p' \geq 1$ and also $\alpha \in \mathrm{BV}_b(\mathbb{R}) \subset L_b^1(\mathbb{R})$. The vanishing viscosity limit cannot hold in these spaces, which have good trace properties; in fact, we seek convergence as $\nu \to 0$ in $H^\sigma(\mathbb{O})$, $0 \leq \sigma < 1/2$, locally uniformly in t. Note that $L^2(\mathbb{O})$ is the energy space for solutions to the Euler system, but convergence in L^2-norm is relatively weak compared to the convergence results we are in a position to establish.

We first consider $\alpha \in \mathrm{BV}_b(\mathbb{R})$. Let X be a Banach space of functions on $[0,1]$ such that $1 \in X$ and $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup on X. For example, $X = L^p([0,1])$, $1 \leq p < \infty$. More generally, we could take $X = H^{s,p}([0,1])$, with $p \in (1,\infty)$ and $s \in [0, 1/p)$. Recall that $\mathcal{S}^\nu\alpha$ is given explicitly in (5.1.17) for α smooth. By an approximation argument using mollifiers with support in $(0, 1/k)$, we can extend the validity of that expression to more singular α's (for details, we refer to [Lopes Filho et al. 2007], Proposition 2.1). We observe that the integral in (5.1.17) can be taken over $[0,t)$ or $[0,t]$, since the integrand vanishes at $s = t$.

Lemma 5.1.1. *If X is a space such as described in the previous paragraph, we have*

$$\mathcal{S}^\nu : \mathrm{BV}_b(\mathbb{R}) \to C_b(\mathbb{R}, X),$$

given by

$$\mathcal{S}^\nu\alpha(t) = \int_{I(t)} \big[(I - e^{\nu(t-s)A})1\big]\,d\alpha(s), \qquad I(t) = [0,t], \tag{5.1.22}$$

where the integral is a Lebesgue-Stieltjes-Bochner integral.

Formula (5.1.22) also implies the estimate

$$\|\mathcal{S}^\nu\alpha(t)\|_X \leq \|\alpha\|_{\mathrm{BV}([0,t])} \sup_{s\in[0,t]} \|e^{\nu s A} f_1 - f_1\|_X, \tag{5.1.23}$$

and, if $v^\nu(0) = V \in X$,

$$\|v^\nu(t) - V\|_X \le \|e^{\nu t A} V - V\|_X + \|\mathcal{S}^\nu \alpha(t)\|_X \to 0, \tag{5.1.24}$$

as $\nu \to 0$, which shows the zero-viscosity limit holds in the X-norm, for the v component of the velocity, in view of (5.1.15).

We next consider some rougher α, namely $\alpha \in L^{p'}$, for a certain range of p'. To begin, take $\alpha \in C_b^\infty(\mathbb{R})$, in particular $\alpha(0) = 0$, and integrate by parts in formula (5.1.22):

$$\int_0^t [e^{\nu(t-s) A} \cdot 1]\, d\alpha(s) = \alpha(t) - e^{\nu t A} \alpha(0) - \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \nu\big(A e^{\nu(t-s) A}\, 1\big)\, \alpha(s)\, ds,$$

using that $e^{\nu(t-s) A}\, 1 \in \mathcal{D}(A)$, whenever $s < t$. The limit $\epsilon \to 0$ exists at least in $L^2([0, 1])$ and we write

$$\lim_{\epsilon \to 0} \int_0^{t-\epsilon} \nu\big(A e^{\nu(t-s) A}\, 1\big)\, \alpha(s)\, ds = \int_0^t \nu(A\, e^{\nu(t-s) A}\, 1)\, \alpha(s)\, ds.$$

Equation (5.1.13) then becomes

$$\bar{v}^\nu = e^{\nu t A} V(z) - \alpha(t) + \int_0^t (\nu\, A\, e^{\nu(t-s) A}\, 1)\, \alpha(s)\, ds, \tag{5.1.25}$$

and

$$v^\nu = e^{\nu t A} V(z) + \int_0^t (\nu\, A\, e^{\nu(t-s) A}\, 1)\, \alpha(s)\, ds. \tag{5.1.26}$$

Consequently, to establish convergence of the v component of the velocity to the corresponding Euler solution in the limit $\nu \to 0$ it is enough to prove the last integral vanishes in the limit.

We observe that $e^{\nu t A} 1$ and $\nu A\, e^{\nu t A} 1$ can be explicitly computed using Fourier series. However, it is preferable to use Green's function methods as we are interested in the limit $\nu t \to 0$. To this end, we bring in the Sobolev spaces $H^\sigma([0, 1])$ with $0 \le \sigma < 1/2$. We recall the well-known interpolation estimate

$$[L^2(M), H_0^1(M)]_\sigma = \begin{cases} H_0^\sigma(M) & \text{if } \frac{1}{2} < \sigma \le 1, \\ H^\sigma(M) & \text{if } 0 \le \sigma < \frac{1}{2}, \end{cases} \tag{5.1.27}$$

where $M = [0, 1]$ or $M = \mathbb{O}$ here, which gives

$$\mathcal{D}((-A)^{\sigma/2}) = H^\sigma([0, 1]) \quad \text{for } \sigma \in \big[0, \tfrac{1}{2}\big). \tag{5.1.28}$$

Hence, we first have uniformly in $t \in [0, T]$ for any $0 < T < \infty$,

$$e^{\nu t A}\, V \to V \text{ strongly in } H^\sigma([0, 1]), \quad \text{as } \nu \to 0. \tag{5.1.29}$$

We next observe as in [Lopes Filho et al. 2007, Equations 3.8–3.11] that

$$\begin{aligned} \|\nu A e^{\nu s A}\, 1\|_{H^\sigma([0,1])} &\le C \|\nu(-A)^{1+\sigma/2} e^{\nu s A}\, 1\|_{L^2([0,1])} \\ &= C \|\nu(-A)^{1-(\tau-\sigma)/2} e^{\nu s A} (-A)^{\tau/2}\, 1\|_{L^2([0,1])} \\ &= C \nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1} \|(-\nu s A)^{1-(\tau-\sigma)/2} e^{\nu s A} (-A)^{\tau/2}\, 1\|_{L^2([0,1])} \\ &\le C \nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1} \|1\|_{H^\tau([0,1])}. \end{aligned}$$

for $0 \le \sigma < \tau < 1/2$, so that by Hölder's inequality we have, with p' the conjugate exponent to p,

$$\int_0^t \|\nu A e^{\nu(t-s)A}\, 1\, \alpha(s)\|_{H^\sigma(D)}\, ds \le \|\alpha\|_{L^{p'}([0,t])} \Big(\int_0^t \|\nu A e^{\nu s A}\, 1\|^p_{H^\sigma(D)}\, ds \Big)^{1/p}$$
$$\le C_{p\sigma\tau}\, \nu^{(\tau-\sigma)/2}\, t^{(\tau-\sigma)/2-1+1/p} \|\alpha\|_{L^{p'}([0,t])} \|1\|_{H^\tau(D)}, \qquad (5.1.30)$$

provided $1 \le p < 2/(2-(\tau-\sigma))$. For example, it is enough to have $p' > 4$. The same estimate holds for $\alpha \in L^{p'}_b(\mathbb{R})$ using a smooth approximation by convolutions.

Combining the estimates in (5.1.29) and (5.1.30), we obtain convergence of the v component of the velocity in the limit $\nu \to 0$ in the Sobolev space $H^\sigma([0,1])$. We record this result in a proposition.

Proposition 5.1.2. *Let $0 \le \sigma < \tau < 1/2$ and assume $\alpha \in L^{p'}_b(\mathbb{R})$ with $p' = \dfrac{p}{p-1}$ and $1 \le p < \dfrac{2}{2-(\tau-\sigma)}$. Then $\mathcal{S}^\nu \alpha(t) = \int_0^t (\nu\, A\, e^{\nu(t-s)\, A}\, 1)\, \alpha(s)\, ds$ defines a map*

$$\mathcal{S}^\nu : L^{p'}_b(\mathbb{R}) \to C_b(\mathbb{R}, H^\sigma([0,1])),$$

satisfying estimate (5.1.30). Furthermore, uniformly in $t \in [0,T]$ for any $0 < T < \infty$,

$$v^\nu \to v^0 \text{ strongly in } H^\sigma([0,1]), \quad \text{as } \nu \to 0. \qquad (5.1.31)$$

Having settled the analysis of the first Equation (5.1.3), we now turn to Equation (5.1.4) in its mild fomulation (5.1.14), which we solve as a fixed-point problem, but first we record some useful *a priori* estimates.

We denote again $\partial_x^k w^\nu$ by w_k^ν, $k \in \mathbb{Z}_+$. Since α depends only on t and $\bar{v}^\nu$ depend only on t, z, the same arguments as in (2.1.9) – (2.1.12) gives that w_k^ν also solves (5.1.4). Integrating by parts in that equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} + \int_0^1 \int_0^{2\pi} \Big[(\bar{v}^\nu(t,z)+\alpha(t))\frac{\partial}{\partial x}\frac{|w^\nu(t,x,z)|^2}{2}\Big]\, dx\, dz$$
$$= \frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} + \int_0^1 \int_0^{2\pi} (\bar{v}^\nu(t,z)+\alpha(t))\Big[\frac{|w^\nu(t,x,z)|^2}{2}\Big]_0^{2\pi} dz$$
$$= \frac{1}{2}\frac{d}{dt}\|w^\nu\|^2_{L^2(\mathbb{O})} = -\nu\|\nabla w^\nu\|^2_{L^2(\mathbb{O})} \le 0,$$

using periodicity in x. Therefore

$$\|w_k^\nu(t)\|_{L^2(\mathbb{O})} \le \|\partial_x^k W\|_{L^2(\mathbb{O})}. \qquad (5.1.32)$$

On the other hand the maximum principle gives

$$\|w_k^\nu(t)\|_{L^\infty(\mathbb{O})} \le \|\partial_x^k W\|_{L^\infty(\mathbb{O})}. \qquad (5.1.33)$$

These estimates continue to hold for $\alpha \in \mathrm{BV}$ or $L^{p'}$ $(1 \le p' < +\infty)$ by approximation with smooth functions.

We write (5.1.14) as $w^\nu(t) = e^{t(\nu\Delta - X)}W(t) + \mathfrak{F}^\nu(w^\nu)(t)$, where

$$\mathfrak{F}^\nu(t, V, \alpha, v)(w) = \mathfrak{F}^\nu(w)(t) = \int_0^t e^{(t-s)(\nu\Delta - X)}[(V - \bar{v}^\nu - \alpha(s))\, \partial_x w(s)]\, ds. \qquad (5.1.34)$$

To establish the existence of a unique solution to (5.1.14), it is enough to prove that $\mathfrak{F}^\nu$ is a contraction in $L^\infty([0,T], L^2(\mathbb{O}))$, T small enough, since then continuation of the solution follows from the uniform estimate (5.1.32).

We observe first that Proposition 5.1.2 and the Sobolev embedding implies that

$$V - \bar{v} - \alpha \in L^{p'}([0,T], L^q([0,1]))$$

for any $1 \le q < \infty$. Furthermore, at fixed viscosity, given that V is smooth and bounded on $[0,1]$ with all its derivatives, a scaling argument gives

$$\|e^{t(\nu\Delta - X)} f\|_{H^1(\mathbb{O})} \le C_{\nu,V}\, t^{-(1/r-1/2)-1/2}\, \|f\|_{L^r(\mathbb{O})}, \tag{5.1.35}$$

if $1 \le r \le 2$, $0 < t \le 1$. We apply this estimate below with $1/r = 1/q + 1/2$, q large, so that $r > 1$.

Let $|||w||| = \|w\|_{L^\infty([0,T],L^2(\mathbb{O}))}$. Then, from (5.1.35),

$$\begin{aligned}
|||\mathfrak{F}(w) - \mathfrak{F}(w')||| &\le C_{\nu,V} \int_0^T (t-s)^{-1/r} \|(V - \bar{v}^\nu(s) - \alpha(s))\,(w(s) - w'(s)\|_{L^r(\mathbb{O})}\, ds \\
&\le C_{\nu,V} \int_0^T (t-s)^{-1/r} \|V - \bar{v}^\nu(s) - \alpha(s)\|_{L^q([0,1])} \|w(s) - w'(s)\|_{L^2(\mathbb{O})}\, ds \\
&\le C_{\nu,V} T^{1/p-1/r} \|V - \bar{v} - \alpha\|_{L^{p'}([0,T],L^q([0,1]))}\, |||w - w'|||,
\end{aligned} \tag{5.1.36}$$

using that $V - \bar{v}^\nu - \alpha$ commutes with ∂_x. This estimate holds provided $p < r$, where p is the conjugate exponent to p' and $1/r = 1/q + 1/2$. If $p' > 4$, we can find such an $r > p > 4/3$ by choosing $q > 4$ in (5.1.35). The estimate above gives that $\mathfrak{F}$ is a strict contraction on $L^\infty([0,T], L^2(\mathbb{O}))$ if T is sufficiently small. We therefore have existence and uniqueness of solutions to (5.1.4) in $L^\infty([0,T], L^2(\mathbb{O}))$, and hence in $L^\infty([0,\infty), L^2(\mathbb{O}))$ thanks to (5.1.32). Furthermore, since w_k^ν satisfies the same equation for all $k \in \mathbb{Z}_+$, w_k^ν is the unique solution to (5.1.14) in $L^\infty([0,\infty), L^2(\mathbb{O}))$ and we conclude that $w^\nu \in L^\infty([0,\infty), \mathcal{V}^k(\mathbb{O}))$ for all $k \in \mathbb{Z}_+$. Also w^ν is smooth in x, z for $t > 0$, and satifies the boundary condition $w^\nu \equiv 0$ on $\partial\mathbb{O}$ pointwise.

We now turn to the analysis of the vanishing viscosity limit $w^\nu \to w^0$ as $\nu \to 0$. For this analysis, we rely on the results in Section 3.1 on the behavior of the semigroup $e^{t(\nu\Delta - X)}$ as $\nu \to 0$. In view of (5.1.16), we can write

$$(w^\nu - w^0)(t,x,z) = [e^{t(\nu\Delta - X)} - e^{-tX}]W(x,z) + R^\nu(t,x,z),$$

where

$$R^\nu(t,x,z) = \int_0^t e^{(t-s)(\nu\Delta - X)} [(V(z) - \bar{v}^\nu(s,z) - \alpha(s))\, \partial_x w^\nu(s,x,z)]\, ds.$$

We estimate the easier term $R^\nu(t,x,z)$ first. This can be done exactly as in (2.2.11), using (5.1.33) and the positivity of the kernel of $e^{t(\nu\Delta - X)}$:

$$\begin{aligned}
|R^\nu(t,x,z)| &\le C\, \|\partial_x W\|_{L^\infty(\mathbb{O})} \int_0^t e^{(t-s)(\nu\Delta - X)}\, |V(z) - \bar{v}^\nu(s,z) - \alpha(s)|\, ds \\
&= C\, \|\partial_x W\|_{L^\infty(\mathbb{O})} \int_0^t e^{(t-s)\nu\Delta} |V(z) - v^\nu(s,z)|\, ds,
\end{aligned} \tag{5.1.37}$$

where the equality follows since $V - v^\nu$ is independent of x. Next, since $V - v^\nu \to 0$ strongly in $L^q([0,1])$, $1 \le q < \infty$, uniformly in $t \in [0,T]$ from (5.1.30), and $e^{\nu(t-s)\Delta}$ is uniformly bounded in t and ν on $L^q(\mathbb{O})$, we conclude

$$R^\nu(x,z,t) \to 0 \text{ strongly in } L^q(\mathbb{O}) \text{ uniformly in } t \in [0,T], \text{ as } \nu \to 0. \tag{5.1.38}$$

In fact, when $q = 2$ and $V = 0$, the estimate (5.1.30) gives also an upper bound for the rate of convergence:

$$\sup_{0\le t\le T} \|R^\nu(\cdot,t)\|_{L^2(\mathbb{O})} \le C_V\, \nu^{\tau/2}\, T^{\tau/2+2-1/p} \|1\|_{H^\tau([0,1])} \|\alpha\|_{L^{p'}([0,T]}, \tag{5.1.39}$$

with again $p = p'/(p'-1)$, $0 < \tau < 1/2$. In the case $p = \infty$, we get a rate consistent with estimate (2.1.21) for $\alpha = 0$. We now turn to the more delicate term $[e^{t(\nu\Delta - X)} - e^{-tX}]W(x,z)$ for which we directly use Proposition 4.3 to conclude:

$$[e^{t(\nu\Delta - X)} - e^{-tX}]W \to 0 \text{ strongly in } L^q(\mathbb{O}) \text{ uniformly in } t \in [0,T], \tag{5.1.40}$$

as $\nu \to 0$. Putting together (5.1.40) and (5.1.38) we obtain convergence in $L^q(\mathbb{O})$ of the w component of the velocity in the vanishing viscosity limit, and hence of the Navier–Stokes solution to the Euler solution.

Proposition 5.1.3. *Let $\alpha \in L_b^{p'}(\mathbb{R})$, $p' > 4$. Let $u^\nu = (v^\nu, w^\nu)$ be the solution of the Navier–Stokes system (5.1.3)–(5.1.4) with initial condition (5.1.2) and boundary conditions (5.1.1). Let u^0 be the solution of the Euler system (5.1.6) with the same initial condition, given by formulas (5.1.15)-(5.1.16). Then, as $\nu \to 0$,*

$$u^\nu(t) \to u^0(t) \text{ strongly in } L^q(\mathbb{O}), \qquad \forall\, q \in [1,\infty),$$

locally uniformly in $t \in [0,\infty)$.

Exploiting the analysis of Section 3.2 yields convergence in higher norms in the interior. Recall that v^ν is given by formula (5.1.25), and w^ν by formula (5.1.14) respectively. Below, v^0 and w^0 are the components of the Euler solution, given respectively by (5.1.15) and (5.1.16). Let the set Ω_j be defined as in Proposition 3.2.1, i. e., $\Omega_1 \Subset \Omega_0 \Subset \mathbb{O}$. Projecting along the z-direction we then have two maximal intervals $I_1 \subset\subset I_0 \Subset [0,1]$.

Lemma 5.1.4. *Let $k \in \mathbb{N}$ and fix $0 < T < \infty$. Then v^ν defined in (5.1.25) belongs to $C^\infty([0,T], H^k(I_1))$ and*

$$v^\nu \to V = v^0 \;\text{ in } L^\infty([0,T], H^k(I_1)), \quad \text{as } \nu \to 0. \tag{5.1.41}$$

Proof. The limit $e^{tA} f \to f$ as $t \to 0$ in $H^k(I_1) \cap L^2([0,1])$ follows easily from the explicit formula for the Green's function. Since $V \in C^\infty(\bar{\mathbb{O}})$, we immediately have $e^{\nu t A} V \to V$ as $\nu \to 0$ in $H^k(I_1)$, $\forall k \in \mathbb{N}$. We also have $e^{\nu t A} 1 \to 1$ in $L^\infty([0,T], H^k(I_1))$ as $\nu \to 0$, so that

$$\lim_{\nu\to 0} \mathcal{S}^\nu(\alpha) = 0, \qquad \text{in } L^\infty([0,T], H^k(I_1)),$$

since $\mathcal{S}^\nu \alpha(t) = \displaystyle\int_0^t (\nu A\, e^{\nu(t-s)A}\, 1)\, \alpha(s)\, ds.$ □

From the Lemma, proceeding as in the proof of Proposition 3.2.3, we obtain

$$v^\nu \to V = v^0 \quad \text{as } \nu \to 0, \text{ uniformly on } I_1 \text{ for } t \in [0,T]. \tag{5.1.42}$$

The method of images yields more precise estimates. In fact, from (2.1.20), when $\alpha \in \mathrm{BV}_b(\mathbb{R})$

$$\left|\mathcal{S}^\nu \alpha(z,t)\right| = \left|\int_0^t [1 - e^{\nu(t-s)A}\, 1]\, d\alpha(s)\right| \le C^T\, \|\alpha\|_{TV([0,t])} \sup_{0\le s\le t} \varphi\big((\nu s)^{-1/2}\delta(z)\big), \tag{5.1.43}$$

for $t \in [0,T]$, where $\delta(z) = \operatorname{dist}(z,\{0,1\})$ and $\varphi(\zeta)$ is rapidly decreasing as $\zeta \to \infty$. Similarly, if $\alpha \in L^p_b(\mathbb{R})$, $1 \le p \le \infty$,

$$\left|\mathcal{S}^\nu \alpha(z,t)\right| = \left|\int_0^t (\nu\, \Delta\, e^{\nu(t-s)A}\, 1)\alpha(s)\, ds\right| \le C^T\, \|\alpha\|_{L^1(\mathbb{R})} \sup_{0\le s\le t} \psi\big((s\nu)^{-1/2}\delta(z)\big), \tag{5.1.44}$$

where $\psi(\zeta)$ vanishes at 0 and is rapidly decreasing as $\zeta \to \infty$.

Next, we address convergence of w^ν.

Lemma 5.1.5. *Fix $0 < T < \infty$. Then w^ν defined in (5.1.14) belongs to $C^\infty([0,T], C(\Omega_1))$ and*

$$w^\nu \to w^0 \quad \text{as } \nu \to 0, \text{ uniformly on } \Omega_1 \text{ for } t \in [0,T]. \tag{5.1.45}$$

Proof. We first observe that, since $e^{\nu t\,\Delta}$ is uniformly bounded in $L^\infty(\mathbb{O})$ (though not strongly continuous), estimate (5.1.37) together with (5.1.42) implies

$$R^\nu(t,x,z) \to 0 \quad \text{as } \nu \to 0, \text{ uniformly on } \Omega_1 \text{ for } t \in [0,T]. \tag{5.1.46}$$

Therefore, it is enough to show that $[e^{t(\nu\Delta - X)} - e^{-tX}]W(x,z) \to 0$ uniformly as $\nu \to 0$. In fact, it is equivalent to show

$$e^{tX}\, e^{t(\nu\Delta - X)} W(x,z) \to W(x,z),$$

given that $e^{tX}$is an isometry. This result then follows from Proposition 3.2.3 (via (3.2.1)). ☐

We combine the two lemmas in a proposition (see also Proposition 4.2.3).

Proposition 5.1.6. *In the setting of Proposition 5.1.3 , let $\Omega_1 \Subset \Omega_0 \Subset \mathbb{O}$. Then, as $\nu \to 0$,*

$$u^\nu(t,x,z) \to u^0(t,x,z) \quad \text{uniformly in } (x,z) \in \Omega_1, \quad t \in [0,T].$$

If α is sufficiently regular, then it follows from (2.1.20) and (5.1.25) that $X_\nu = v^\nu(t,z)\partial_x \in \widehat{\mathfrak{X}}_1$ and hence the results in § 3.7 can be applied to w_ν to obtain a more detailed analysis in the boundary layer.

We now generalize the setting to allow for the two channel walls to move with different velocities, that is, we replace the boundary condition (5.1.1) with:

$$(v^\nu(t,j), w^\nu(t,x,j), 0) = (\alpha_j(t), 0, 0), \;\; x \in \mathbb{R}/\mathbb{Z}, \;\; t > 0, \;\; j \in \{0,1\}. \tag{5.1.47}$$

It is straightforward to extend the results derived above to this case. We begin by replacing (5.1.8) with

$$\bar{v}^\nu(t,z) = v^\nu(t,z) - \Phi(t,z), \qquad \bar{u}^\nu = (\bar{v}^\nu, w^\nu, 0), \tag{5.1.48}$$

where Φ is given by

$$\Phi(t,z) = [\alpha_1(t) - \alpha_0(t)]z + \alpha_0(t). \tag{5.1.49}$$

Note that Φ solves

$$\begin{aligned} \partial_z^2 \Phi(t,\cdot) &= 0 && \text{on } [0,1], \\ \Phi(t,0) &= \alpha_0(t), && t>0, \\ \Phi(t,1) &= \alpha_1(t), && t>0. \end{aligned}$$

Formula (5.1.17) is then replaced by

$$\begin{aligned} v^\nu(t) &= e^{\nu t A} V + \mathcal{S}^\nu(\alpha_0,\alpha_1)(t), \\ \mathcal{S}^\nu(\alpha_0,\alpha_1)(t,z) &= \int_{[0,t]} [(I - e^{\nu(t-s)A})\partial_s \Phi(s,z)]\, ds \\ &= \int_{[0,t]} [(I - e^{\nu(t-s)A})(1-z)]\, d\alpha_0(s) + \int_{[0,t]} [(I - e^{\nu(t-s)A})z]\, d\alpha_1(s). \end{aligned} \tag{5.1.50}$$

Integrating by parts we can obtain the analog of (5.1.25). Estimates analogous to those done above on $\mathcal{S}^\nu \alpha(t)$ are readily verified.

5.2. ***Moving boundary, parallel to the x-y-plane.*** In this section, we take a look at the following more general motion of $\partial\mathcal{O}$, namely

$$B(t,x,z) = (\alpha_j(t), \beta_j(t), 0), \quad z = j \in \{0,1\}. \tag{5.2.1}$$

Most of the techniques have been developed in Section 5.1, so we will be brief. First note that allowing β_j to be nonzero has no effect on the component $v^\nu(t,z)$, and (5.1.50) continues to hold.

Let us analyze the effect on $w^\nu(t,x,z)$. Take $\beta_j \in C_b^\infty(\mathbb{R})$ to start (though later we can extend to $\beta_j \in \mathrm{BV}_b(\mathbb{R})$). Set

$$\Psi(t,z) = [\beta_1(t) - \beta_0(t)]z + \beta_0(t). \tag{5.2.2}$$

We see that

$$\overline{w}^\nu(t,x,z) = w^\nu(t,x,z) - \Psi(t,z) \tag{5.2.3}$$

vanishes on $\partial\mathcal{O}$ and satisfies

$$\partial_t \overline{w}^\nu + v^\nu \partial_x \overline{w}^\nu = \nu \Delta \overline{w}^\nu - \partial_t \Psi, \quad \overline{w}^\nu(0,x,z) = W(x,z). \tag{5.2.4}$$

Hence, with $X = V\partial_x$,

$$\begin{aligned} \overline{w}^\nu(t,x,z) = e^{t(\nu\Delta - X)} W(x,z) &+ \int_0^t e^{(t-s)(\nu\Delta - X)} (V - v^\nu(s,z)) \partial_x \overline{w}^\nu(s,x,z)\, ds \\ &- \int_0^t e^{(t-s)(\nu\Delta - X)} \partial_s \Psi(s,z)\, ds, \end{aligned} \tag{5.2.5}$$

so, making use of the fact that $\Psi(s,z)$ is independent of x, we obtain

$$\begin{aligned} w^\nu(t,x,z) = e^{t(\nu\Delta - X)} W(x,z) &+ \int_0^t e^{(t-s)(\nu\Delta - X)} (V - v^\nu(s,z)) \partial_s w^\nu(s,x,z)\, ds \\ &+ \int_0^t (I - e^{(t-s)\nu\Delta}) \partial_s \Psi(s,z)\, ds. \end{aligned} \tag{5.2.6}$$

One can write the last integral as

$$\int_0^t (I - e^{(t-s)\nu\Delta})(1-z)\,d\beta_0(s) + \int_0^t (I - e^{(t-s)\nu\Delta})z\,d\beta_1(s). \tag{5.2.7}$$

Previously developed techniques apply to (5.2.6)–(5.2.7).

Finally, we draw further conclusions when (5.2.1) is specialized to

$$B(t,x,z) = (0, \beta_j(t), 0), \quad z = j \in \{0,1\}. \tag{5.2.8}$$

In such a case, $v^\nu(t,z)$ is as in Chapters 3–4. Consequently, (5.2.4) is

$$\partial_t \overline{w}^\nu = (\nu\Delta - X_\nu)\overline{w}^\nu - \partial_t \Psi, \tag{5.2.9}$$

with initial data $w^\nu(0,x,z) = W(x,z)$, boundary data 0 on $\partial\mathbb{O}$, and with X_ν exactly as in Section 2.2. Hence the results of Chapter 4 apply. We have

$$\overline{w}^\nu(t,x,z) = \Sigma_\nu^{0,t} W(x,z) - \int_0^t \Sigma_\nu^{s,t}\,\partial_s\Psi(s,z)\,ds, \tag{5.2.10}$$

where $\Sigma_\nu^{s,t}$ is the solution operator to

$$\partial_t u = (\nu\Delta - X_\nu)u, \quad u\big|_{\mathbb{R}^+\times\partial\mathbb{O}} = 0, \tag{5.2.11}$$

i.e., $u(t) = \Sigma_\nu^{s,t} u(s)$ for $0 \le s < t$. Hence

$$w^\nu(t,x,z) = \Sigma_\nu^{0,t} W(x,z) + \int_0^t (I - \Sigma_\nu^{s,t})\partial_s\Psi(s,z)\,ds, \tag{5.2.12}$$

and we can write the last integral as

$$\int_0^t (I - \Sigma_\nu^{s,t})(1-z)\,d\beta_0(s) + \int_0^t (I - \Sigma_\nu^{s,t})z\,d\beta_1(s). \tag{5.2.13}$$

Results of Chapter 4 then give convergence

$$w^\nu(t) \to w^0(t) \tag{5.2.14}$$

in various function spaces, including $\mathcal{V}^k(\mathbb{O})$.

Obtaining such convergence in the context of (5.2.1) would require some extra hypotheses on $\alpha_j(t)$, which we will not pursue here.

Appendix A. $\mathcal{V}^k(\mathbb{O})$ and b-Sobolev spaces

We take $\overline{\mathbb{O}}$ to be a compact Riemannian manifold with smooth boundary. Recall from (3.3.1) the definition

$$\mathcal{V}^k(\mathbb{O}) = \{u \in L^2(\mathbb{O}) : Y_1 \cdots Y_j u \in L^2(\mathbb{O}),\ \forall\, j \le k,\ Y_\ell \in \mathfrak{X}^1\}, \tag{A.0.1}$$

for $k \in \{0,1,2,\dots\}$, where

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{\mathbb{O}} : Y \parallel \partial\mathbb{O}\}. \tag{A.0.2}$$

These spaces are special cases of weighted b-Sobolev spaces, introduced and studied in [Melrose 1993] (see also [Melrose 1996]). Here we discuss this matter and draw some conclusions that are useful in Sections 3.3 and 4.2.

The manifold $\mathbb{O}$ carries a complete Riemannian metric, called a "b-metric," which on a collar neighborhood of $\partial\mathbb{O}$, identified with $[0, 1) \times \partial\mathbb{O}$ (with $\{0\} \times \partial\mathbb{O}$ identified with $\partial\mathbb{O} \subset \overline{\mathbb{O}}$) has the form

$$g = \left(\frac{dy}{y}\right)^2 + h, \tag{A.0.3}$$

where h is a smooth metric tensor on $\partial\mathbb{O}$ and y the parameter on $[0, 1)$. We use the symbol $\widetilde{\mathbb{O}}$ to denote $\mathbb{O}$ as a Riemannian manifold with such a Riemannian metric. The b-Sobolev spaces $H^k_b(\mathbb{O})$ are defined by

$$H^k_b(\mathbb{O}) = \{u \in L^2_b(\mathbb{O}) : Y_1 \cdots Y_j u \in L^2_b(\mathbb{O}),\ \forall\, j \le k, Y_\ell \in \mathfrak{X}^1\}, \tag{A.0.4}$$

where $\mathfrak{X}^1$ is as in (A.0.2) and

$$L^2_b(\mathbb{O}) = L^2(\widetilde{\mathbb{O}}). \tag{A.0.5}$$

Different choices of b-metrics on $\mathbb{O}$ give the same spaces, with equivalent norms. To define weighted b-Sobolev spaces, take a defining function ρ for $\partial\mathbb{O}$, i.e., $\rho \in C^\infty(\overline{\mathbb{O}})$, $\rho > 0$ on $\mathbb{O}$, $\rho = 0$ on $\partial\mathbb{O}$, $\nabla\rho(x) \neq 0$, $\forall\, x \in \partial\mathbb{O}$. Thus, for $s \in \mathbb{R}$, set

$$\rho^s H^k_b(\mathbb{O}) = \{\rho^s u : u \in H^k_b(\mathbb{O})\}. \tag{A.0.6}$$

An inductive argument shows that

$$\rho^s H^k_b(\mathbb{O}) = \{u \in \rho^s L^2_b(\mathbb{O}) : Y_1 \cdots Y_j u \in \rho^s L^2_b(\mathbb{O}),\ \forall\, j \le k, Y_\ell \in \mathfrak{X}^1\}. \tag{A.0.7}$$

We also have

$$L^2(\mathbb{O}) = \rho^{-1/2} L^2_b(\mathbb{O}). \tag{A.0.8}$$

Hence

$$\mathcal{V}^k(\mathbb{O}) = \rho^{-1/2} H^k_b(\mathbb{O}). \tag{A.0.9}$$

Remark. The use of "b" as a subscript in names of function spaces is different in this appendix than it was in Chapter 5. We trust this warning will forestall confusion.

A.1. ***Interpolation identities.*** This identity (A.0.9) is of use in establishing the following result, which is valuable in §Section 3.3 and 4.2.

Proposition A.1.1. *If* $0 < k < \ell$ *and* $k = \ell\theta$, *then*

$$[L^2(\mathbb{O}), \mathcal{V}^\ell(\mathbb{O})]_\theta = \mathcal{V}^k(\mathbb{O}), \tag{A.1.1}$$

where the left side is the complex interpolation space.

In light of (A.0.9), this follows straight away from:

Proposition A.1.2. *If* $0 < k < \ell$ *and* $k = \ell\theta$, *then*

$$[L^2_b(\mathbb{O}), H^\ell_b(\mathbb{O})]_\theta = H^k_b(\mathbb{O}). \tag{A.1.2}$$

In turn, Proposition A.1.2 can be proven by identifying $H_b^k(\mathbb{O})$ with a regular Sobolev space of functions on the complete Riemannian manifold $\widetilde{\mathbb{O}}$. (Thanks to R. Mazzeo for pointing this out.) In detail, we set

$$H^k(\widetilde{\mathbb{O}}) = \{u \in L^2(\widetilde{\mathbb{O}}) : \nabla^j u \in L^2(\widetilde{\mathbb{O}}),\ \forall\, j \le k\}, \tag{A.1.3}$$

where a priori $\nabla^j u$ is a distributional section of $\otimes^j T^*\widetilde{\mathbb{O}}$, whose fiber $\otimes^j T_x^*\widetilde{\mathbb{O}}$ inherits an inner product from that of $T_x\widetilde{\mathbb{O}}$ given by the complete Riemannian metric tensor on $\widetilde{\mathbb{O}}$ described above. Since the Riemannian manifold $\widetilde{\mathbb{O}}$ considered here, arising from $\overline{\mathbb{O}}$ via a b-metric, has special structure as a Riemannian manifold with bounded geometry, we can give a convenient alternative characterization of $H^k(\widetilde{\mathbb{O}})$, as follows. There exist $K \in \mathbb{N}$ and smooth maps from the closed unit ball $\overline{B}_1 \subset \mathbb{R}^n$ into $\widetilde{\mathbb{O}}$ ($n = \dim \widetilde{\mathbb{O}}$)

$$\varphi_\nu : \overline{B}_1 \to \widetilde{\mathbb{O}}, \tag{A.1.4}$$

with the following properties:

$$\begin{aligned}
&\varphi_\nu \ \text{ is a diffeomorphism of } \ \overline{B}_1 \ \text{ onto its image;}\\
&\{\varphi_\nu^* g\} \ \text{ is a } C^\infty \text{ bounded family of metric tensors on } \overline{B}_1;\\
&\{\varphi_\nu(B_{1/2})\} \text{ covers } \widetilde{\mathbb{O}};\\
&\text{each } p \in \widetilde{\mathbb{O}} \text{ is contained in at most } K \text{ sets } \varphi_\nu(\overline{B}_1).
\end{aligned} \tag{A.1.5}$$

Given a function $u \in L^1_{\text{loc}}(\widetilde{\mathbb{O}})$, set

$$u_\nu = \varphi_\nu^* u \in L^1(B_1). \tag{A.1.6}$$

Then

$$H^k(\widetilde{\mathbb{O}}) = \Big\{u \in L^2(\widetilde{\mathbb{O}}) : \sum_\nu \sum_{|\alpha| \le k} \|D^\alpha u_\nu\|^2_{L^2(B_1)} < \infty\Big\}. \tag{A.1.7}$$

Note also that

$$u \in H^k(\widetilde{\mathbb{O}}) \Leftrightarrow \sum_\nu \sum_{|\alpha| \le k} \|D^\alpha u_\nu\|^2_{L^2(B_{1/2})} < \infty. \tag{A.1.8}$$

An examination of the behavior of elements of $\mathfrak{X}^1$ when pushed forward to B_1 via φ_ν establishes:

Proposition A.1.3. *For $k \in \mathbb{Z}^+$,*

$$H_b^k(\mathbb{O}) = H^k(\widetilde{\mathbb{O}}). \tag{A.1.9}$$

Hence (A.1.2) follows from the result that

$$[L^2(\widetilde{\mathbb{O}}), H^\ell(\widetilde{\mathbb{O}})]_\theta = H^k(\widetilde{\mathbb{O}}). \tag{A.1.10}$$

To establish this, it is convenient to bring in the Laplace-Beltrami operator of $\widetilde{\mathbb{O}}$, which we denote L. This is defined as an unbounded operator on $L^2(\widetilde{\mathbb{O}})$ via the Friedrichs construction:

$$u \in \mathfrak{D}(L) \ \text{ and } \ Lu = f \iff u \in H^1(\widetilde{\mathbb{O}}) \text{ and } (\nabla u, \nabla g)_{L^2(\widetilde{\mathbb{O}})} = -(f, g)_{L^2(\widetilde{\mathbb{O}})},\ \forall\, g \in H^1(\widetilde{\mathbb{O}}). \tag{A.1.11}$$

The fact that $\widetilde{\mathbb{O}}$ is complete implies L is a negative self adjoint operator and $C_0^\infty(\widetilde{\mathbb{O}})$ is dense in the domain of all powers of L, defined inductively by

$$u \in \mathfrak{D}(L^{k+1}) \implies u \in \mathfrak{D}(L) \text{ and } Lu \in \mathfrak{D}(L^k). \tag{A.1.12}$$

Compare [Chernoff 1973]. More generally, for each $s \in [0, \infty)$, $(-L)^s$ is defined via the spectral theorem as a positive self adjoint operator, and one has the classical interpolation identity

$$[L^2(\widetilde{\mathcal{O}}), \mathcal{D}((-L)^s)]_\theta = \mathcal{D}((-L)^{s\theta}). \tag{A.1.13}$$

Hence the identity (A.1.10) is a consequence of:

Proposition A.1.4. *For $k \in \mathbb{N}$,*

$$H^k(\widetilde{\mathcal{O}}) = \mathcal{D}((-L)^{k/2}). \tag{A.1.14}$$

Proof. That

$$\mathcal{D}((-L)^{1/2}) = H^1(\widetilde{\mathcal{O}}) \tag{A.1.15}$$

is a fundamental property of the Friedrichs construction. Next, from (A.1.11) we have

$$\mathcal{D}(L) = \{u \in H^1(\widetilde{\mathcal{O}}) : Lu \in L^2(\widetilde{\mathcal{O}})\}, \tag{A.1.16}$$

where Lu is a priori a distribution on $\widetilde{\mathcal{O}}$. Clearly $H^2(\widetilde{\mathcal{O}}) \subset \mathcal{D}(L)$. We can use the interior elliptic estimates

$$\sum_{|\alpha|\le 2} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C\Bigl(\|u\|^2_{L^2(B_1)} + \|L_\nu u\|^2_{L^2(B_1)}\Bigr), \tag{A.1.17}$$

with L_ν the image of L on B_1 via φ_ν. The estimate (A.1.17) holds with C independent of ν. We use this together with the equivalence of (A.1.7) and (A.1.8), to obtain the reverse inclusion, hence

$$\mathcal{D}(L) = H^2(\widetilde{\mathcal{O}}). \tag{A.1.18}$$

To continue, we note that (A.1.17) extends to

$$\sum_{|\alpha|\le 2k} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C_k\Bigl(\|u\|^2_{L^2(B_1)} + \|L^k_\nu u\|^2_{L^2(B_1)}\Bigr), \tag{A.1.19}$$

again with C_k independent of ν, and this together with (A.1.7)–(A.1.8) gives

$$\{u \in H^1(\widetilde{\mathcal{O}}) : L^k u \in L^2(\widetilde{\mathcal{O}})\} \subset H^{2k}(\widetilde{\mathcal{O}}). \tag{A.1.20}$$

By comparison, the definition (A.1.12) says

$$\mathcal{D}(L^k) = \{u \in H^1(\widetilde{\mathcal{O}}) : Lu \in \mathcal{D}(L^{k-1})\}. \tag{A.1.21}$$

The right side of (A.1.21) is contained in the left side of (A.1.20). On the other hand, if we know that $\mathcal{D}(L^{k-1}) = H^{2k-2}(\widetilde{\mathcal{O}})$, it readily follows that $H^{2k}(\widetilde{\mathcal{O}}) \subset \mathcal{D}(L^k)$. Hence it follows inductively that

$$\mathcal{D}(L^k) = H^{2k}(\widetilde{\mathcal{O}}). \tag{A.1.22}$$

To complete the proof of (A.1.14), we use

$$\mathcal{D}((-L)^{k+1/2}) = \{u \in \mathcal{D}(L^k) : L^k u \in \mathcal{D}((-L)^{1/2})\} = \{u \in H^{2k}(\widetilde{\mathcal{O}}) : L^k u \in H^1(\widetilde{\mathcal{O}})\}, \tag{A.1.23}$$

and the interior regularity estimate

$$\sum_{|\alpha|\le 2k+1} \|D^\alpha u\|^2_{L^2(B_{1/2})} \le C_k\Bigl(\|u\|^2_{L^2(B_1)} + \|L^k_\nu u\|^2_{H^1(B_1)}\Bigr). \tag{A.1.24}$$

This proves Proposition A.1.4, and hence Propositions A.1.1–A.1.3. □

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