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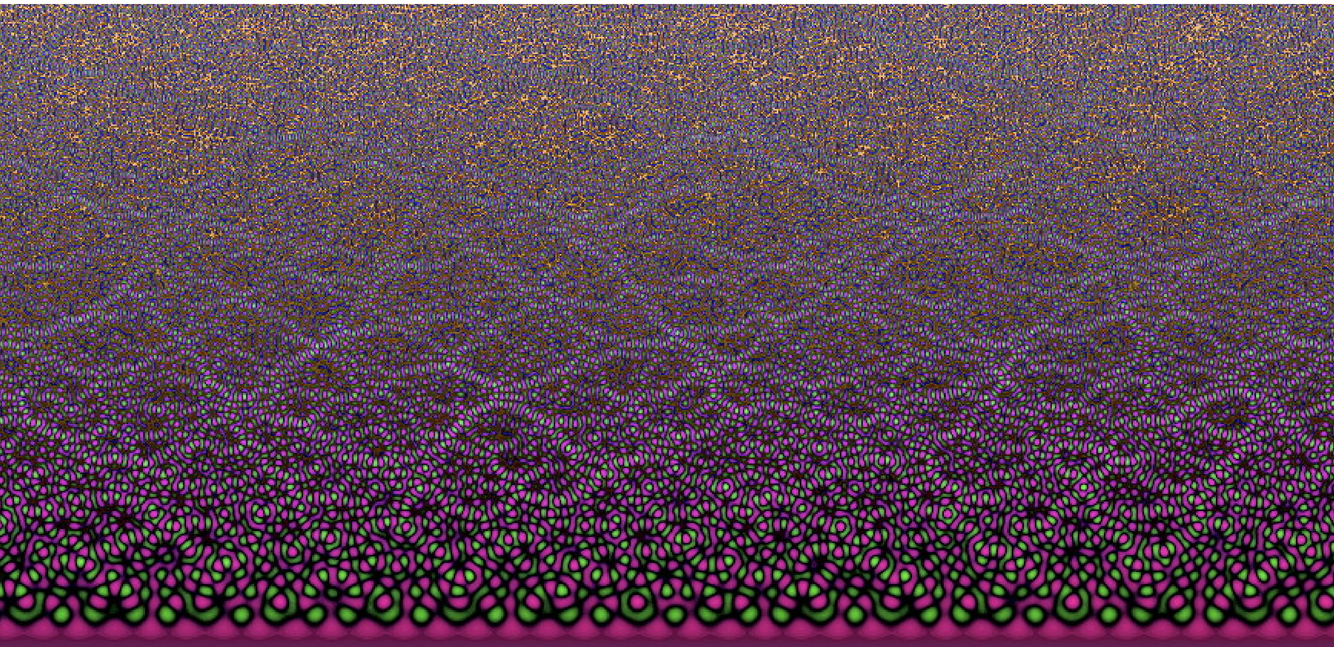

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# DYNAMICS OF NONLINEAR SCHRÖDINGER/GROSS–PITAEVSKII EQUATIONS: MASS TRANSFER IN SYSTEMS WITH SOLITONS AND DEGENERATE NEUTRAL MODES

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Nonlinear Schrödinger/Gross–Pitaevskii equations play a central role in the understanding of nonlinear optical and macroscopic quantum systems. The large time dynamics of such systems is governed by interactions of the nonlinear ground state manifold, discrete neutral modes ("excited states") and dispersive radiation. Systems with symmetry, in spatial dimensions larger than one, typically have degenerate neutral modes. Thus, we study the large time dynamics of systems with degenerate neutral modes. This requires a new normal form (nonlinear matrix Fermi Golden Rule) governing the system's large time asymptotic relaxation to the ground state (soliton) manifold.



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## 1. Introduction

Nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equations are a class of dispersive Hamiltonian partial differential equations (PDEs) of the form:

$$i\partial_t\psi(x,t) = -\Delta\psi(x,t) + \big(V(x) - f(|\psi(x,t)|^2)\big)\psi(x,t). \tag{1-1}$$

Here, $\psi = \psi(x,t)$ is a scalar complex-valued function of position, $x \in \mathbb{R}^d$ and time, $t \in \mathbb{R}$. The function $V : \mathbb{R}^d \to \mathbb{R}$ denotes a linear *potential* and $f : \mathbb{R}_+ \to \mathbb{R}$, a nonlinear potential. For example, $V$ can be taken to be a smooth, nonpositive potential well, with rapid decay as $|x| \to \infty$ and $f(|\psi|^2) = -g|\psi|^2$, $g$ constant. For $g > 0$, the nonlinearity is called *repulsive* or *defocusing*. For $g < 0$ it is called *attractive* or *focusing*. In this paper, we focus on spatial dimensions $d \geq 3$. Precise hypotheses on $V$ and $f$ are given below. We are interested in the initial value problem (IVP) for (1-1) with finite energy data $\psi(x,0)$ and solutions $\psi(x,t)$, which are sufficiently regular and decaying to zero as $|x| \to \infty$. A precise well-posedness result is cited below; see Theorem 3.1.

NLS/GP equations play a central role in the understanding of nonlinear optical [Moloney and Newell 2004; Boyd 2008; Sulem and Sulem 1999] and macroscopic quantum systems [Erdős and Yau 2001]. A striking and important feature of NLS/GP is that it can have localized standing waves or *nonlinear bound state* solutions, some of which are stable and play a central role in the general dynamics. In particular, for a wide variety of potentials and nonlinearities there exists an interval $\mathcal{I} \subset \mathbb{R}$ such that for any $\lambda \in \mathcal{I}$, (1-1) has *nonlinear ground state* solutions. These are solutions of the form

$$\psi(x,t) = e^{i\lambda t}\phi^\lambda(x),$$

where

$$-\Delta\phi^\lambda + \big(V - f(|\phi^\lambda|^2)\big)\phi^\lambda = -\lambda\phi^\lambda \tag{1-2}$$

with $\phi^\lambda \in H^1$ and $\phi^\lambda > 0$.

The gauge (phase-translation) invariance of (1-1),

$$\psi \mapsto e^{i\gamma}\psi, \quad \gamma \in [0, 2\pi),$$

generates a *nonlinear ground state* or *"soliton"* [1] *manifold*:

$$\mathcal{M}_{\mathcal{I}} := \{e^{i\gamma}\phi^\lambda,\ \lambda \in \mathcal{I},\ \gamma \in [0,2\pi)\}. \tag{1-3}$$

If $V$ is identically zero, then NLS/GP admits a larger group of symmetries and the definition of soliton manifold (which exists in the focusing case, $g < 0$) is naturally extended to incorporate these additional symmetries; see, for example, [Weinstein 1986; Grillakis et al. 1987].

***Orbital stability.*** The soliton manifold $\mathcal{M}_{\mathcal{I}}$ is said to be *orbitally stable* if any initial condition $\psi_0$, which is close to $\mathcal{M}_{\mathcal{I}}$ in $H^1$, gives rise to a solution $\psi(t)$, which is $H^1$ close for $t \neq 0$. There is an extensive literature on the *orbital stability* of the soliton manifold. For the case $V \equiv 0$, orbital stability (stability modulo spatial and phase translations) of global energy minimizers was proved in [Cazenave and Lions 1982] by compactness arguments. In [Weinstein 1985; 1986], it is shown that positive solutions,

[1] The term *soliton* sometimes refers, more specifically, to particle-like solutions of completely integrable PDEs.

which are index one critical points (Hessian with one strictly negative eigenvalue) and satisfy the slope condition[2]:

$$\frac{d}{d\lambda}\int_{\mathbb{R}^d} |\phi^\lambda(x)|^2\,dx > 0, \tag{1-4}$$

are $H^1$ orbitally stable. For the focusing case

$$V \equiv 0, \quad f(|\psi|^2) = -g|\psi|^2, \quad g < 0,$$

(1-4) is equivalent to $\sigma < 2/d$. Orbital stability of solitary waves of NLS/GP for a class of potentials $V$ was studied by Rose and Weinstein [1988] and, for a semiclassical setting, by Oh [1988]. A general formulation of a stability/instability theory is presented in [Grillakis et al. 1987].

***Asymptotic stability.*** We say the soliton manifold $\mathcal{M}_\mathcal{G}$ is *asymptotically stable* if $\psi_0$ close to $\mathcal{M}_\mathcal{G}$ in a suitable norm implies that $\psi(t)$ remains close to and *converges to* $\mathcal{M}_\mathcal{G}$ (in a possibly different norm), as $t$ tends to infinity.

Are solitary waves *asymptotically stable*? This is a local variant of the problem of *asymptotic resolution* [Tao 2008], that is, whether general initial conditions resolve into stable nonlinear bound states of the system plus dispersive radiation. A great deal of progress has been made on this problem in recent years. The study of asymptotic stability of solitary waves was initiated in [Soffer and Weinstein 1990; 1992]; see also [Buslaev and Perel′man 1992; Pillet and Wayne 1997; Gustafson et al. 2004; Weder 2000]. In the translation invariant case, asymptotic stability was then investigated by [Buslaev and Perel′man 1995]. Asymptotic stability analysis requires two new analytical features: one dynamical systems and the other harmonic analysis / spectral theory.

First, since we do not know in advance which nonlinear ground state in $\mathcal{M}_\mathcal{G}$ emerges in the large time limit, a decomposition with flexibility allowing for the asymptotic soliton to dynamically emerge is required[3]. To this end, the solution is decomposed in terms of a motion along the soliton manifold and components symplectic orthogonal or biorthogonal to it. Dynamics along the soliton manifold, $\mathcal{M}_\mathcal{G}$, are governed by *modulation equations*; see, for example, [Weinstein 1985; Fröhlich et al. 2004; Holmer and Zworski 2007].

Secondly, in order to prove convergence to the soliton manifold $\mathcal{M}_\mathcal{G}$, we need to show that the deviation of the solution from $\mathcal{M}_\mathcal{G}$ decays with advancing time. This requires time-decay estimates ($L^p$, weighted $L^2(\mathbb{R}^d)$ or space-time norms) for the linearized (about the soliton) propagator on the subspace symplectic orthogonal or biorthogonal to the discrete spectral subspace. The discrete subspace is the union of a *zero frequency mode subspace* spanned by infinitesimal generators of the NLS/GP symmetries (translation, gauge) acting on $\phi^\lambda$, and often a subspace of *neutral modes* (sometimes called internal modes) with nonzero frequencies.

Since a typical perturbation of the ground state solitary wave in $\mathcal{M}_\mathcal{G}$ excites all discrete spectral components, one must understand the mechanisms, due to which these do not interfere with the asymptotic convergence of $\psi(x,t)$ to $\mathcal{M}_\mathcal{G}$. In brief: Concerning the zero modes, the choice of modulation equations

[2] $\mu = -\lambda$ is the typical definition of soliton frequency. Therefore the slope condition (1-4) often appears as a rate of change with respect to $\mu$ being negative.

[3] The case of integrable systems, such as one-dimensional NLS $V = 0$, $f(|\psi|^2)\psi = |\psi|^2\psi$ is an important class for which it *is* possible to determine the emerging coherent structures from the *scattering transform* of the initial data.

"quotients out" the zero modes; perturbations exciting these induce motion *along* the soliton manifold. And concerning the nonzero frequency neutral modes, these are shown to damp to zero, as $t \to \infty$, due to the resonant nonlinear coupling of discrete to radiation modes. Related to this is a further dynamical systems aspect of the analysis. The neutral mode amplitudes are governed by nonlinear oscillator equations, coupled to a dispersive wave field. Near-identity changes of variables are used to put the system in an appropriate normal form, wherein the mechanism of energy transfer from the neutral modes to the evolving soliton and propagating radiation is made explicit. Energy transfer shows up as an explicit (nonlinear) damping term in the normal form; see the discussion below. The positive damping coefficient (matrix, in the present work) is a nonlinear variant of Fermi Golden Rule [Cohen-Tannoudji et al. 1992]. See [Buslaev and Perel′man 1995] regarding the dynamics near solitary waves of the translation invariant NLS equations and [Soffer and Weinstein 1999] for "breathers" of a class of nonlinear wave equations. In [Soffer and Weinstein 2004] this mechanism was proved to be responsible for *ground state selection* in NLS/GP equations; see also [Weinstein 2006]. Experimental verification of the prediction in [Soffer and Weinstein 2004; 2005] is reported in [Mandelik et al. 2005]. Related work on resonant radiation damping appears in [Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Cuccagna et al. 2006; Cuccagna and Mizumachi 2008]. The role of the Fermi Golden Rule in the nonpersistence of coherent structures for nonlinear wave equations was first demonstrated, via Floquet analysis, in [Sigal 1993]. There is a close relation to the perturbation theory of embedded eigenvalues for linear problems [Reed and Simon 1979; Soffer and Weinstein 1998; Cuccagna et al. 2005].

The above works on nonlinear resonance required that the neutral modes frequencies (a) lie sufficiently close to the essential spectrum and (b) are of geometric multiplicity one. For example, for the cubic nonlinearity, $f(|\psi|^2) = -g|\psi|^2$, *close* means that coupling to radiation modes occurs at order $|g|^2$. The situation where simple neutral modes are with a large spectral gap has been studied in [Gang and Sigal 2006; 2007; Gang 2007; Cuccagna and Mizumachi 2008; Cuccagna 2008]. Here, coupling of the discrete to continuous modes occurs at some high order in $g$. Thus, the normal form expansion gives a damping term at some even order $|g|^{2k}$ with $k \geq 2$.

***Results of this paper — systems with degenerate neutral modes.*** An important situation, not covered by previous results, is the dynamics in the presence of degenerate neutral modes. This case arises naturally in systems of spatial dimensions $d \geq 2$ with symmetry. For example, if the potential is spherically symmetric, $V = V(|x|)$, then the first and higher excited states are degenerate, with the degree of degeneracy related to the order of the associated spherical harmonics. Another interesting class of examples is a class of multiwell potentials; see Appendix A.

In this paper we prove the asymptotic stability of the ground state / soliton manifold, $\mathcal{M}_{\mathcal{G}}$, of NLS/GP when the linearized spectrum has degenerate neutral modes. We show that the solution has three *interacting* parts:

(i) a modulating soliton, parametrized by the motion along $\mathcal{M}_{\mathcal{G}}$,

(ii) oscillatory, spatially localized, neutral modes, which decay with time and

(iii) a dispersive part, which decays in a local energy norm.

The neutral modes and dispersive waves decay via transferring their mass to the soliton manifold or to spatial infinity. Additionally, degenerate neutral modes are coupled and exchange mass among themselves in addition to with the soliton and radiation. These degenerate modes cannot be viewed as very weakly coupled "oscillators" [Tsai 2003]. We require instead a new normal form expansion. This is related to ideas developed in [Kirr and Weinstein 2001], where a parametrically forced linear Hamiltonian PDE was considered, and a normal form, uniform in discrete eigenvalue spacing, was required.

We outline the perspective we take and give a rough form of the main theorem, Theorem 7.1. Consider NLS/GP, where $-\Delta + V$ has a ground state $\xi_0(x) > 0$, whose energy $e_0 < 0$, and a *degenerate* excited state, whose energy $e_1$ with $e_0 < e_1 < 0$ is assumed sufficiently close to zero. Typical solutions of the linear Schrödinger equation, evolving from localized initial data $\psi_0$,

$$\psi(t) = \exp(-i(-\Delta + V)\,t)\psi_0$$

will be a time-quasiperiodic superposition of spatially localized ground state and time-periodic excited states, plus a part which disperses to zero, that is, tends to zero as $t$ advances in $L^2_{\mathrm{loc}}$. This picture emerges from the spectral decomposition of $-\Delta + V$ in $L^2$, with respect to which the bound state projections of the solution evolve as independent *oscillators* and the continuous spectral part of the solution has a character, qualitatively like a solution to the free Schrödinger equation.

For NLS/GP, for example $-g|\psi|^2\psi$ with $g \neq 0$, the dynamics of discrete and continuous modes are coupled. We consider an appropriate open set of initial conditions near the soliton manifold. *In contrast to the linear Schrödinger equation*, we show that the solution converges to a nonlinear ground state. To see this, we view NLS/GP as a infinite-dimensional Hamiltonian system comprising two subsystems: (i) a finite-dimensional system governing dynamics on the soliton manifold $\mathcal{M}_{\mathcal{I}}$, parametrized by $(\lambda(t), \gamma(t))$, the zero modes amplitudes $(a_1, a_2)$ and the neutral mode amplitudes $z = (z_1, z_2, \ldots, z_n)^T$; (ii) an infinite-dimensional dispersive Schrödinger wave equation. A very detailed analysis of this coupled system (the bulk of this paper) yields the following (rough) form for the asymptotic behavior of small amplitude solutions of NLS/GP:

**Main Theorem.** *Consider the initial value problem for NLS/GP. Assume arbitrary localized initial data, which are sufficiently near a small amplitude nonlinear bound state $\phi^{\lambda_0}$. Then the solution of NLS/GP evolves as a modulated soliton plus decaying error in the following form*:

$$\psi(t) = \exp\Bigl(i\int_0^t \lambda(s)\,ds\Bigr)\cdot\exp\bigl(i\bigl(\gamma(t)+a_2(z(t),\bar{z}(t))\bigr)\bigr)\cdot\bigl(\phi^{\lambda(t)+a_1(z(t),\bar{z}(t))} + \mathbb{O}(|z(t)|) + R(t)\bigr),$$

*where $\lambda(t) \to \lambda_\infty$, $\mathbb{O}(|z(t)|)$ represents a localized nonspreading decaying part satisfying*

$$|z(t)| \le C\langle t\rangle^{-1/2},$$

*$a_j = a_j(z, \bar{z}) = \mathbb{O}(|z|^2)$ and $R(t)$ represents a spreading dispersively decaying part and tends to zero as $t \to \infty$ in $L^2_{\mathrm{loc}}$, more precisely $\|\langle x\rangle^{-\nu}R(t)\|_2 \to 0$ with $\nu > 0$.*

For the precise statement, see Theorem 7.1.

A key part of the proof of this theorem is to show that $|z(t)|$ tends to zero and that $\lambda(t)$ has a limiting value $\lambda_\infty \in \mathcal{I}$ as $t$ tends to infinity. We prove the latter by showing $\partial_t\lambda(t) \in L^1(\mathbb{R}^+)$. We have two comments on the approach of this article to these issues:

*New normal form.* We show that there exist a *nonnegative symmetric matrix* $\Gamma(z,\bar{z}) = \mathbb{O}(|z|^2)$ and a skew symmetric matrix $\Lambda(z,\bar{z}) = \mathbb{O}(|z|^2)$ (see (7-3) below) such that

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathbb{O}\big((1+t)^{-\frac{3}{2}-\delta}\big), \tag{1-5}$$

with $\delta > 0$. The matrix $\Gamma$ is defined in terms of the spectral decomposition of the $L(\lambda) = JH(\lambda)$, the generator of the linearized flow about the nonlinear bound state $\phi^\lambda$; see Section 5. Our analysis requires that $\Gamma = \Gamma(z,\bar{z};\lambda)$ is positive-definite for an open $\lambda$-interval. A variant of this hypothesis appears in [Soffer and Weinstein 2004; Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Gang and Sigal 2006; 2007; Cuccagna et al. 2006; Cuccagna and Mizumachi 2008]. It is expected to hold, in some sense, generically. In Section 6 we state a hypothesis under which positive-definiteness holds for a class of potentials of multiwell type, constructed in Appendix A. This hypothesis, denoted (FGR) (see also Theorem 6.1), is a nonlinear variant of the Fermi Golden Rule [Cohen-Tannoudji et al. 1992; Reed and Simon 1979; Soffer and Weinstein 1998]. We note that for finite-dimensional Hamiltonian systems a damping term is absent; it would violate phase-volume conservation. This term arises due to nonlinearity induced by the coupling between discrete and continuous (radiational) spectral modes, a phenomenon associated with continuous spectra, arising in PDEs on spatially infinite domains; see [Soffer and Weinstein 1999; Weinstein 2006]. We show that (1-5) and (FGR) imply the bound $|z(t)| = \mathbb{O}(t^{-1/2})$. For the case of multiple simple bound states with well-separated frequencies, a system of type (1-5) holds with $\Gamma$, a *diagonal* matrix [Tsai 2003]. Equation (1-5) can be viewed as a new normal form, a special case of one valid uniformly in neutral mode eigenfrequency-separation.

*Choice of basis for the neutral mode subspace.* We prove that $\lambda(t)$ approaches some $\lambda_\infty$ as $t \to \infty$, by proving that $\partial_t\lambda(t)$ is integrable. If there are $n$ simple well-separated neutral modes, one initially finds

$$\partial_t\lambda(t) = \sum_{m=1}^{n} a_m|z_m|^2 + \mathbb{O}(t^{-3/2}).$$

Since we expect $|z_m| = \mathbb{O}(t^{-1/2})$ we can not conclude integrability of $\partial_t\lambda(t)$. However, it can be shown that, after near identity change of variables $z \mapsto z + \mathbb{O}(|z|^2)$, we can take $a_m = 0$; see the normal form expansion in [Gang and Sigal 2006; 2007; Soffer and Weinstein 2004]. In the degenerate (similarly, not well-separated) case, $\lambda(t)$ satisfies:

$$\partial_t\lambda(t) = \sum_{m,k} a_{m,k} z_m \bar{z}_k + \mathbb{O}(t^{-3/2}).$$

In the present paper we show very generally that, by appropriate choice of neutral subspace basis, we can take $a_{m,k} = 0$.

Finally, we expect that our techniques can be extended to more complicated situations, for example, where coupling of neutral to continuum modes occurs at higher order in the nonlinearity.

***Outline of the paper.*** The paper is organized as follows. Section 2 displays notation which is often used. Section 3 is a brief section outlining structural properties of NLS/GP and gives a statement of a basic well-posedness result. Section 4 introduces solitary waves (solitons) in the regime of weak nonlinearity. Section 5 has a detailed discussion of the spectral properties of $L(\lambda) = JH(\lambda)$, the generator of the linearized dynamics about the soliton: zero energy subspace, degenerate neutral subspace and continuous spectral subspace. Projections associated with theses subspaces are defined and decay estimates of the linearized evolution on the continuous spectral subspace are recalled. In Section 6 the Fermi Golden Rule matrix $\Gamma$ is introduced explicitly in Theorem 6.1. The detailed calculations, proving the symmetry and nonnegativity, are given in Appendix B. Note that the main theorem requires positive-definiteness of $\Gamma$. Proposition 6.2 is a result reducing the required positive-definiteness to a condition involving the spectral properties of $-\Delta + V$. Section 7 contains a statement of the main theorem, Theorem 7.1. In Section 8 we give a more precise formulation of Theorem 7.1. This formulation makes explicit the dynamical (modulation) equations for the solitary wave parameters, the neutral mode amplitudes and the dispersive part. These are proved via normal form methods in Sections 9 and 10. In Section 11 we prove the reformulated Theorem 7.1 in the setting of Theorem 8.1. Appendix contains some important calculations used in the body of the paper. Of particular interest is Appendix A, where a class of multiwell three-dimensional potentials is constructed, to which we apply Theorem 7.1.

## 2. Notation

(1) $\alpha_+ = \max\{\alpha, 0\}$ and $[\tau] = \max_{\tilde{\tau}\in Z}\{\tilde{\tau} \le \tau\}$.

(2) $\Re z$ denotes the real part of $z$ and $\Im z$ the imaginary part of $z$.

(3) Multiindices:

$$w = (w_1, \dots, w_N), \quad \bar{w} = (\bar{w}_1, \dots, \bar{w}_N) \in \mathbb{C}^N,$$
$$a = (a_1, \dots, a_N) \in \mathbb{N}^N, \quad z^a = z_1^{a_1} \cdots z_N^{a_N}, \quad |a| = |a_1| + \cdots + |a_N|,$$

where $z$ denotes the vector of *neutral mode* amplitudes, $\xi$ denotes the vector whose $j$-th entry $\xi_j$ is the $j$-th neutral vector-mode of $JL(\lambda)$.

(4) $Q_{m,n}$ denotes an expression of the form

$$Q_{m,n} = \sum_{\substack{|a|=m\\|b|=n}} q_{a,b} z^a \bar{z}^b = \sum_{\substack{|a|=m\\|b|=n}} q_{a,b} \prod_{k=1}^{N} z_k^{a_k} \bar{z}_k^{b_k}.$$

(5)
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad L = JH = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}.$$

(6) $\sigma_{\rm ess}(L) = \sigma_c(L)$ is the essential (continuous) spectrum and $\sigma_{\rm disc}(L)$ the discrete spectrum of $L$.

(7) Riesz projections:

$$P_c(L) = I - P_{\rm disc}(L),$$

where $P_{\rm disc}(L)$ projects to the discrete spectral of $L$ and $P_c(L)$ to the continuous spectral of $L$.

(8)
$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx.$$

(9)
$$\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p\,dx, \qquad 1 \le p \le \infty.$$

(10)
$$\|f\|_{H^{s,\nu}}^2 = \int_{\mathbb{R}^d} \left|\langle x\rangle^\nu (I-\Delta)^{s/2} f(x)\right|^2 dx.$$

## 3. Hamiltonian structure

NLS/GP can be expressed as a Hamiltonian system

$$i\partial_t\psi = \frac{\delta\mathscr{E}[\psi,\bar\psi]}{\delta\bar\psi}$$

where the Hamiltonian energy $\mathscr{E}[\cdot]$ is defined by

$$\mathscr{E}[\psi]=\mathscr{E}[\psi,\bar\psi]=\int\left(\tfrac{1}{2}\nabla\psi\cdot\nabla\bar\psi+\tfrac{1}{2}V(x)\psi\bar\psi-F(\psi\bar\psi)\right)dx$$

with

$$F(u)=\tfrac{1}{2}\int_0^u f(\xi)\,d\xi.$$

Equation (1-1) is a Hamiltonian system on Sobolev space $H^1(\mathbb{R}^d,\mathbb{C})$ viewed as a real space

$$H^1(\mathbb{R}^d,\mathbb{R})\oplus H^1(\mathbb{R}^d,\mathbb{R}),$$

that is,

$$H^1(\mathbb{R}^d,\mathbb{C})\ni f\leftrightarrow(\Re f,\Im f)\in H^1(\mathbb{R}^d,\mathbb{R})\oplus H^1(\mathbb{R}^d,\mathbb{R}),$$

with the symplectic form

$$\omega(\psi,\phi)=\Im\int_{\mathbb{R}^d}\psi\bar\phi\,dx.$$

Equation (1-1) is invariant under time-translation and gauge-translation (phase-translation):

$$t\mapsto t+t_0,\qquad \phi\mapsto\phi e^{i\gamma}$$

with $\gamma\in\mathbb{R}$, yielding, by Noether's Theorem, the conservation laws of energy

$$\mathscr{E}[\psi(t)]=\mathscr{E}[\psi(0)]$$

and of particle number (optical power)

$$\mathcal{N}[\psi(t)]=\mathcal{N}[\psi(0)]$$

where

$$\mathcal{N}[\psi]=\int|\psi|^2\,dx.$$

### Assumptions on the potential $V$ and nonlinearity $f$

**(fA)** $f(\tau)$ is a smooth function satisfying $f(\tau)=\mathbb{O}(\tau)$ for $|x|$ is small. Thus, the nonlinearity in NLS is cubic at small amplitudes, that is, $f(|\psi|^2)\psi\sim g|\psi|^2\psi$.

**(VA)** $V$ is smooth and decays exponentially as $|x|$ tends to $\infty$.

To ensure the global well-posedness of the initial value problem for (1-1) we impose:

**(fB) Subcritical nonlinearity for large amplitudes**

$$|f(\xi)| \le c(1+|\xi|^{\beta})$$

for some $\beta \in [0, 2/d)$ and

$$|f'(\xi)| \le c(1+|\xi|^{\alpha-1})$$

for some $\alpha \in [0, 2/(d-2)_+)$ where $s_+ = \max\{s, 0\}$.

The following well-posedness theorem can be found in [Cazenave 2003; Sulem and Sulem 1999].

**Theorem 3.1.** *Assume that the nonlinearity $f$ satisfies the condition (fB), and the potential $V$ satisfies (VA). Then (1-1) is globally well-posed in $H^1$, that is, the Cauchy problem for (1-1) with the initial data $\psi(0) \in H^1$ has a unique solution $\psi(t)$ in the space $H^1$, which depends continuously on $\psi(0)$. Moreover, the solution $\psi(t)$ satisfies conservation of energy and conservation of particle number.*

## 4. Bifurcation and Lyapunov stability of solitons in the weakly nonlinear regime

In this section we discuss the existence of solitons in the weakly nonlinear regime. The following arguments are similar to those in [Rose and Weinstein 1988; Tsai and Yau 2002c] except that the excited states are degenerate. We assume that the linear operator $-\Delta + V$ has the following properties:

**(Eig$_V$)** The linear operator $-\Delta+V$ has two eigenvalues $e_0 < e_1 < 0$ with $2e_1 > e_0$. $e_0$ is the lowest eigenvalue with ground state $\phi_{\text{lin}} > 0$. The eigenvalue $e_1$ is degenerate with multiplicity $N$ and eigenfunctions $\xi_1^{\text{lin}}, \xi_2^{\text{lin}}, \ldots, \xi_N^{\text{lin}}$.

**Remark.** In Appendix A we construct a class of double-well examples $V$ for $d = 3$ and with multiplicity $N = 2$.

The following result shows that nonlinear bound state solutions $(\phi^{\lambda}, \lambda)$ of NLS/GP (1-2) bifurcate from the zero state and the linear ground state energy $(0, \lambda = -e_0)$.

**Proposition 4.1.** *Suppose $-\Delta+V$ satisfies the conditions in (Eig$_V$). Then there exists a constant $\delta_0 > 0$ and a nonempty interval $\mathcal{I}_{\delta_0} \subset [-e_0-\delta_0, -e_0+\delta_0]$ such that for any $\lambda \in \mathcal{I}_{\delta_0}$ (1-1) has solutions of the form*

$$\psi(x,t) = e^{i\lambda t}\phi^{\lambda} \in L^2$$

*with*

$$\phi^{\lambda} = \delta(\lambda)\cdot\big(\phi_{\text{lin}} + \mathcal{O}(\delta(\lambda))\big), \qquad \delta(\lambda) = \mathcal{O}\Big(\big|\lambda-|e_0|\big|^{1/2}\Big)$$

*for $\big|\lambda-|e_0|\big|$ small. Moreover, for some $c > 0$ independent of $\lambda$,*

$$|\phi^{\lambda}(x)| \le ce^{-c|x|}, \qquad |\partial_{\lambda}\phi^{\lambda}(x)| \le ce^{-c|x|},$$

*and similarly for the spatial derivatives of $\phi^{\lambda}$ and $\partial_{\lambda}\phi^{\lambda}$.*

**Remark.** Suppose $f(|\psi|^2)\psi = -g|\psi|^2 + o(|\psi|^2)$. Then for $g > 0$ (repulsive case) we have

$$\mathcal{I}_{\delta_0} = (-e_0, -e_0 + \delta_0)$$

and for $g < 0$ (attractive case) we have

$$\mathcal{I}_{\delta_0} = (-e_0 - \delta_0, -e_0).$$

Finally, we conclude this section by noting that for $\delta' \le \delta_0$ sufficiently small that soliton manifold $\mathcal{M}_{\delta'}$ (see (1-3)) is $H^1$ orbitally stable; see the discussions in the introduction and [Weinstein 1986; Rose and Weinstein 1988; Grillakis et al. 1987].

## 5. $L(\lambda) = JH(\lambda)$, the linearized operator about $\phi^\lambda$

We now turn to a discussion of the operator obtained by linearization around the soliton and the existence of neutral modes with nonzero frequencies. Rewrite (1-1) as

$$\frac{\partial \psi}{\partial t} = G(\psi),$$

where the nonlinear map $G(\psi)$ is defined by

$$G(\psi) = -i(-\Delta + \lambda + V)\psi + i f(|\psi|^2)\psi.$$

Then the linearization of (1-1) can be written as

$$\frac{\partial \chi}{\partial t} = dG(\phi^\lambda)\chi,$$

where $dG(\phi^\lambda)$ is the Fréchet derivative of $G(\psi)$ at $\phi^\lambda$. It is computed to be

$$dG(\phi^\lambda)\chi = -i(-\Delta + \lambda + V)\chi + i f[(\phi^\lambda)^2]\chi + i f'[(\phi^\lambda)^2](\phi^\lambda)^2(\chi + \bar{\chi}).$$

This operator is real linear but not complex linear. To convert it to a complex linear operator we pass from complex functions to real vector-functions

$$\chi \longleftrightarrow \vec{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

where $\chi_1 = \Re\chi$ and $\chi_2 = \Im\chi$. Then $dG(\phi^\lambda)\chi \longleftrightarrow L(\lambda)\vec{\chi}$ where the operator $L(\lambda)$ is given by

$$L(\lambda) = JH(\lambda) \tag{5-1}$$

where $J$ is a skew-symmetric matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $H(\lambda)$ is a selfadjoint matrix

$$H(\lambda) := \begin{pmatrix} L_+(\lambda) & 0 \\ 0 & L_-(\lambda) \end{pmatrix}$$

with

$$L_-(\lambda) := -\Delta + \lambda + V - f[(\phi^\lambda)^2]$$

and

$$L_+(\lambda) := -\Delta + \lambda + V - f[(\phi^\lambda)^2] - 2f'[(\phi^\lambda)^2](\phi^\lambda)^2.$$

We extend the operator $L(\lambda)$ to the complex space $H^2(\mathbb{R}^d, \mathbb{C}) \oplus H^2(\mathbb{R}^d, \mathbb{C})$.

**5A.** ***The spectrum of $L(\lambda)$.*** The operator $L(\lambda)$ has neutral modes.

**Proposition 5.1.** *Let $L(\lambda)$, or more explicitly, $L(\lambda(\delta), \delta)$ denote the linearized operator about the bifurcating state $\phi^\lambda$, $\lambda = \lambda(\delta)$. Note that $\lambda(0) = -e_0$. Corresponding to the degenerate energy value $e_1$ of $-\Delta + V$, the matrix operator*

$$L(\lambda = -e_0, \delta = 0)$$

*has degenerate eigenvalues $\pm iE(-e_0) = \pm i(e_1 - e_0)$, each with multiplicity $N$. For $\delta > 0$ and small, these bifurcate to (possibly degenerate) eigenvalues $\pm iE_1(\lambda), \dots, \pm iE_N(\lambda)$ with eigenfunctions*

$$\begin{pmatrix} \xi_1 \\ \pm i\eta_1 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \pm i\eta_2 \end{pmatrix}, \dots, \begin{pmatrix} \xi_N \\ \pm i\eta_N \end{pmatrix}$$

*with*

$$\langle \xi_m, \eta_n \rangle = \delta_{m,n}$$

*and*

$$0 \neq \lim_{\lambda \to e_0} \xi_j = \lim_{\lambda \to e_0} \eta_j \in \operatorname{span}\{\xi_j^{\text{lin}}, j = 1, 2, \dots, N\} \subset H^k \text{ for any } k > 0.$$

*Moreover, for $\delta$ sufficiently small $2E_j(\lambda) > \lambda$ for $j = 1, 2, \dots, N$ (nonlinear coupling of discrete to continuous spectrum at second order).*

For the case of a radial potential $V = V(|x|)$, the neutral modes have the following structure:

**Proposition 5.2.** *If the potential is radial $V = V(|x|)$, then $\phi^\lambda$, hence $\partial_\lambda \phi^\lambda$, is spherically symmetric. If the degenerate linear excited states $\xi_n^{\text{lin}}$ are of the form $\xi_j^{\text{lin}} = \frac{x_j}{|x|}\xi^{\text{lin}}(|x|)$ for some function $\xi^{\text{lin}}$, then $E_j = E_1$ for any $j = 1, 2, \dots, N = d$ and we can choose $\xi_j$ and $\eta_j$ such that $\xi_j = \frac{x_j}{|x|}\xi(|x|)$ and $\eta_j = \frac{x_j}{|x|}\eta(|x|)$ for some real functions $\xi$ and $\eta$.*

**Remark.** For $d = 3$, the hypothesis on the linear excited states says that they are proportional to $\xi^{\text{lin}}(|x|)Y_1^m(\theta, \phi)$ for $m = -1, 0, 1$, where $Y_1^m$ are the spherical harmonics of degree one.

*Sketch of proof.* If $V$ is spherically symmetric then by the uniqueness of the ground state and the fact $-\Delta + V$ is invariant under unitary transformations we have $\phi^\lambda$, hence $\partial_\lambda \phi^\lambda$ is spherically symmetric.

We now outline a proof of the existence of $\xi_j$ and $\eta_j$ with desired structure. Define a linear space

$$\mathcal{Y}^k = \left\{ J \in H^k, J(x) = \frac{x_1}{|x|} g(|x|) \right\}.$$

By definition $L(\lambda) : \mathcal{Y}^2 \to \mathcal{Y}^0$. Note that, restricted to $\mathcal{Y}^2$, $\frac{x_1}{|x|}\xi^{\text{lin}}(|x|)$ is an eigenfunction of $-\Delta + V$ of multiplicity one. Applying the bifurcation theory to $\mathcal{Y}^2$, we prove there exists an eigenfunction $(\xi_1, i\eta_1)^T \in \mathcal{Y}^2$ with eigenvalue $E_1$. The other eigenfunctions with the same eigenvalue are obtained by noting that this computation can be carried out for any $x_j$ with $j = 1, \dots, d$. $\square$

Based on the above discussion, we assume:

**(SA) Structure of the discrete spectrum of $L(\lambda) = JH(\lambda)$.**

(1) $\sigma_d(L(\lambda))$ consists of an eigenvalue at 0 and complex conjugate eigenvalues at $\pm iE(\lambda)$.

(2) The discrete subspace, corresponding to the eigenvalue 0, is spanned by the associated eigenfunctions
$$\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}, \qquad \begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}.$$

(3) The discrete subspace, corresponding to the eigenvalue $iE(\lambda)$ with $E(\lambda) > 0$, is $N$-dimensional and is spanned by the (complex) eigenfunctions $v_1, v_2, \dots, v_N$.

(4) Thus, $\overline{v_1}, \overline{v_2}, \dots, \overline{v_N}$ are the eigenfunctions which span the discrete subspace corresponding to the eigenvalue $-iE(\lambda)$.

(5) Moreover we observe that $Jv_n$ are eigenfunctions of the adjoint operator $L(\lambda)^*$ with eigenvalue $-iE(\lambda)$:
$$L(\lambda)^* Jv_n = -JL(\lambda)v_n = -iE(\lambda)Jv_n.$$

Concerning the continuous spectrum of $L(\lambda)$, we apply Weyl's Theorem to the stability of the essential spectrum for localized perturbations of $J(-\Delta)$ [Hislop and Sigal 1996; Reed and Simon 1979] to obtain
$$\sigma_{\text{ess}}(L(\lambda)) = (-i\infty, -i\lambda] \cup [i\lambda, i\infty)$$
if the potential $V$ in (1-1) decays sufficiently rapidly as $|x|$ tends to infinity.

The end points of the essential spectrum are called threshold energies.

**Definition 5.3.** Let $d \geq 3$. A function $h$ is called a *threshold resonance function* of $L(\lambda)$ at $\mu = \pm i\lambda$, the endpoints of the essential spectrum, if $h \notin L^2$, $|h(x)| \leq c\langle x\rangle^{-(d-2)_+}$ and $h$ is $C^2$ and solves the equation
$$(L(\lambda) - \mu)h = 0.$$

In this paper we make the following spectral assumption on the thresholds $\pm i\lambda$:

**(Thresh$_\lambda$)** There exists $\delta'$ with $0 < \delta' \leq \delta_0$ (see Proposition 4.1) such that for $\lambda \in \mathcal{I}_{\delta'}$, $L(\lambda)$ has no threshold resonances at $\pm i\lambda$.

In the weak amplitude limit, property (Thresh$_\lambda$) can be referred to the question of whether the scalar operator $-\Delta + V(x)$ has a threshold (zero energy) resonance. In [Jensen and Kato 1979] it was shown that $-\Delta + V$ has a zero energy resonance or eigenvector if and only if the operator
$$I + (-\Delta + i0)^{-1}V : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$
is not invertible. Moreover, this operator is generically invertible. That is, if we replace $V$ by $qV$ where $q$ is a real number, then we have noninvertibility for only a discrete set of $q$ values [Rauch 1978; Jensen and Kato 1979].

The reduction from the properties of $L(\lambda)$ to those of $-\Delta + V$ is seen as follows. Let
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$
so $U^*U = I$. Then,
$$\sigma_3 \mathcal{H}(\lambda) = -iU^* L(\lambda) U. \tag{5-2}$$
It follows that $\pm i\lambda$ are threshold resonances of $L(\lambda)$ if and only if $\pm\lambda$ are threshold resonances of $\sigma_3\mathcal{H}(\lambda)$.

We next observe that $\sigma_3\mathcal{H}$ is a small perturbation of $\sigma_3(-\Delta+V+\lambda)$. Indeed, a computation of $\sigma_3\mathcal{H}(\lambda)$ yields

$$\sigma_3\mathcal{H} = \sigma_3(\mathcal{H}_0+V)+V_{\text{small}}$$

where

$$\sigma_3\mathcal{H}_0 := (-\Delta+\lambda)\sigma_3, \qquad |V_{\text{small}}| \le e^{-c|x|}o(1)$$

for some $c>0$, where $o(1)\to 0$ as $\big|\lambda-|e_0|\big|\to 0$.

Therefore, the generic validity of (Thresh$_\lambda$) follows from the generic absence of zero energy threshold resonances for $-\Delta+V$ by the following result proved for $d=3$ using the results in [Cuccagna et al. 2005]. The proof for general dimensions is similar.

**Proposition 5.4.** *Let $d=3$. If the operator*

$$I+(-\Delta+i0)^{-1}V : \langle x\rangle^2L^2 \to \langle x\rangle^2L^2$$

*is invertible, then* (Thresh$_\lambda$) *holds when $\big|\lambda-|e_0|\big|$ is sufficiently small.*

*Proof.* We begin by proving that the operator

$$I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}(\sigma_3V+V_{\text{small}}) : \langle x\rangle^2L^2 \to \langle x\rangle^2L^2$$

is invertible. Observe that $-2\lambda\approx 2e_0$ is not an eigenvalue of the operator $-\Delta+V$ so $I+(-\Delta+2\lambda)^{-1}V$ is invertible. This, together with the hypothesis, implies that $I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}\sigma_3V$ is invertible with a uniformly bounded inverse. On the other hand the norm of the operator $(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}V_{\text{small}}$ is small when $|e_0+\lambda|$ is small. Hence

$$I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}(\sigma_3V+V_{\text{small}}) = (I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}\sigma_3V)\big(1+(1+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}\sigma_3V)^{-1}V_{\text{small}}\big)$$

is invertible when $\big|\lambda-|e_0|\big|$ is small. Moreover in [Cuccagna et al. 2005] it is proved that the operator $L(\lambda)$ has no threshold resonance functions if the operator

$$I+(\sigma_3\mathcal{H}_0\pm\lambda+i0)^{-1}(\sigma_3V+V_{\text{small}}) : \langle x\rangle^2L^2 \to \langle x\rangle^2L^2$$

is invertible. This completes the proof. $\square$

***Choice of basis for degenerate subspaces.*** In our analysis, it is important that we choose an appropriate basis of the degenerate eigenspaces corresponding to $\pm iE(\lambda)$. We present this choice of basis and its construction here.

**Proposition 5.5.** *There exist real functions $\xi_n$, $\eta_n$ for $n=1,2,\dots,N$ such that*

$$\operatorname{span}\left\{\begin{pmatrix}\xi_n\\ i\eta_n\end{pmatrix}\right\} = \operatorname{span}\{v_1,v_2,\dots,v_N\}$$

*and for any $m,n\in\{1,2,\dots,N\}$,*

$$\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_m\eta_n-\xi_n\eta_m)\,dx = 0 \tag{5-3}$$

*and*

$$\langle\phi^\lambda,\xi_n\rangle = \langle\partial_\lambda\phi^\lambda,\eta_n\rangle = 0, \qquad \langle\xi_n,\eta_m\rangle = \delta_{m,n}. \tag{5-4}$$

The proof is given in Appendix D.

**Remark.** If $\phi^\lambda$ is spherically symmetric, then

$$\xi_n = \frac{x_n}{|x|}\xi(|x|), \qquad \eta_n = \frac{x_n}{|x|}\eta(|x|)$$

for $n \in \{1, 2, \dots, N = d\}$; see Proposition 5.2). Therefore (5-3) trivially holds because $\xi_m\eta_n - \xi_n\eta_m = 0$.

We conclude this section with the explicit form of the projection $P_{\text{disc}}$, whose proof for dimension one can be found in [Gang and Sigal 2005]. The proof for general dimensions is similar, and hence omitted. Recall that $\langle \xi_m, \eta_n\rangle = \delta_{m,n}$.

**Proposition 5.6.** *For the nonselfadjoint operator $L(\lambda)$, the (Riesz) projection onto the discrete spectrum subspace of $L(\lambda)$, $P_{\text{disc}} = P_{\text{disc}}(L(\lambda)) = P^\lambda_{\text{disc}}$, is given by*

$$P_{\text{disc}} = \frac{2}{\partial_\lambda\|\phi^\lambda\|^2}\left(\left|\begin{matrix}0\\ \phi^\lambda\end{matrix}\right\rangle\left\langle\begin{matrix}0\\ \partial_\lambda\phi^\lambda\end{matrix}\right| + \left|\begin{matrix}\partial_\lambda\phi^\lambda\\ 0\end{matrix}\right\rangle\left\langle\begin{matrix}\phi^\lambda\\ 0\end{matrix}\right|\right) - \frac{i}{2}\sum_{n=1}^{N}\left(\left|\begin{matrix}\xi_n\\ i\eta_n\end{matrix}\right\rangle\left\langle\begin{matrix}-i\eta_n\\ \xi_n\end{matrix}\right| - \left|\begin{matrix}\xi_n\\ -i\eta_n\end{matrix}\right\rangle\left\langle\begin{matrix}i\eta_n\\ \xi_n\end{matrix}\right|\right).$$

We define the projection onto the continuous spectral subspace of $L(\lambda)$ by

$$P_c = P_c(L(\lambda)) = P^\lambda_c \equiv I - P_{\text{disc}}. \tag{5-5}$$

**5B.** ***Estimates of the propagator.*** We will need estimates of the evolution operator $U(t) := e^{tL(\lambda_1)}$ for $\lambda_1 \in \mathcal{I}$. Recall that $L(\lambda_1)$ has two branches of essential spectrum: $[i\lambda_1, i\infty)$ and $(-i\infty, -i\lambda_1]$. We denote by $P_+ = P_+^{\lambda_1}$ and $P_- = P_-^{\lambda_1}$ the spectral projections associated with these two branches of the essential spectrum. Hence, $P_c^{\lambda_1} = P_+ + P_-$.

**Theorem 5.7.** *Let $d \ge 3$ and define $k := [\frac{d}{2}] + 1$ and $\nu := \frac{5+d}{2}$. Assume that $2E(\lambda_1) > \lambda$ so that $\pm 2iE(\lambda_1) \in \sigma_{\text{ess}}(L(\lambda_1))$. Then, for any time $t \ge 0$ and $\lambda_1 \in \mathcal{I}$ there exists a constant $c$ such that*

$$\big\|\langle x\rangle^{-\nu}(-\Delta+1)^{k/2}U(t)\big(L(\lambda_1) \pm 2iE(\lambda_1) - 0\big)^{-n}P_\pm h\big\|_2 \le c(1+t)^{-d/2}\|\langle x\rangle^{\nu}(-\Delta+1)^{k/2}h\|_2 \tag{5-6}$$

*with $n = 0, 1, 2$. For any time $t \in (-\infty, \infty)$ and $\lambda_1 \in \mathcal{I}$ there exists a constant $C_{\mathcal{I}}$ such that*

$$\|\langle x\rangle^{-\nu}(-\Delta+1)^{k/2}U(t)P_\pm h\|_2 \le C_{\mathcal{I}}(1+|t|)^{-d/2}\|\langle x\rangle^{\nu}(-\Delta+1)^{k/2}h\|_2, \tag{5-7}$$

$$\|U(t)P_\pm h\|_\infty \le C_{\mathcal{I}}|t|^{-d/2}\|h\|_1, \tag{5-8}$$

$$\|U(t)P_\pm h\|_\infty \le C_{\mathcal{I}}(1+|t|)^{-d/2}(\|h\|_{H^k} + \|h\|_1), \tag{5-9}$$

$$\|U(t)P_\pm h\|_3 \le C_{\mathcal{I}}(1+|t|)^{-d/6}(\|h\|_{H^k} + \|h\|_1), \tag{5-10}$$

$$\|\langle x\rangle^{-\nu}U(t)P_\pm h\|_2 \le C_{\mathcal{I}}(1+|t|)^{-d/2}(\|h\|_1 + \|h\|_2). \tag{5-11}$$

We refer the proof of the estimates to [Soffer and Weinstein 1999; Gang and Sigal 2007; Tsai and Yau 2002a; Goldberg and Schlag 2004]. For the constant $C_{\mathcal{I}}$ can be taken uniformly for $\lambda_1 \in \mathcal{I}$, see [Cuccagna 2001; 2003].

## 6. Matrix Fermi Golden Rule

As highlighted in the introduction, the decay of neutral mode components, associated with the linearized NLS/GP equation, is necessary for asymptotic stability of the soliton manifold $\mathcal{M}_{\mathcal{G}}$. We shall prove that, after near-identity transformations, the system governing these neutral mode amplitudes is (1-5):

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathbb{O}((1+t)^{-3/2-\delta}), \qquad \delta > 0,$$

where $\pm iE(\lambda)$ are complex conjugate $N$-fold degenerate neutral eigenfrequencies of $L(\lambda) = JH(\lambda)$, $\Gamma$ is symmetric and $\Lambda$ is skew symmetric. It follows that

$$\partial_t |z(t)|^2 = -2z^*\Gamma(z,\bar{z})z + \cdots . \tag{6-1}$$

Our strategy to show that $|z(t)|$ tends to zero is based on proving that $\Gamma$ is positive-definite and that the corrections to (6-1) decay sufficiently rapidly as $t$ tends to infinity. If $L(\lambda)$ has a complex conjugate pair of simple neutral eigenvalues, then $\Gamma$ reduces to a nonnegative scalar. If $L(\lambda)$ has multiple, well-separated pairs of neutral modes, then $\Gamma$ reduces to a diagonal matrix [Soffer and Weinstein 1999; 2004; Tsai and Yau 2002a; 2002b; 2002c; Tsai 2003; Buslaev and Sulem 2003]. The present case of problem of degenerate neutral modes is more involved due to coupling among the various discrete modes and with the continuous spectrum. Our computation yields a nondiagonal FGR matrix, $\Gamma$. In this section, we display the expression for $\Gamma$ and state a result on its general properties. The detailed derivation of the expression for $\Gamma$ is carried out in Section 10.

***The FGR matrix* $\boldsymbol{\Gamma(z,\bar{z})}$.** To construct $\Gamma$ we must first introduce some notation.

Define vector functions $G_k$ for $k = 1, 2, \ldots, N$ as

$$G_k(z,x) := \begin{pmatrix} B(k) \\ D(k) \end{pmatrix} \tag{6-2}$$

with the functions $B(k)$ and $D(k)$ defined as

$$\begin{aligned} B(k) &:= -if'[(\phi^\lambda)^2]\phi^\lambda((z\cdot\xi)\eta_k + (z\cdot\eta)\xi_k), \\ D(k) &:= -f'[(\phi^\lambda)^2]\phi^\lambda(3(z\cdot\xi)\xi_k - (z\cdot\eta)\eta_k) - 2f''[(\phi^\lambda)^2](\phi^\lambda)^3(z\cdot\xi)\xi_k, \end{aligned}$$

where

$$z\cdot\xi := \sum_{n=1}^N z_n\xi_n, \qquad z\cdot\eta := \sum_{n=1}^N z_n\eta_n.$$

In terms of the column 2-vector $G_k$, we define a $N\times N$ matrix $Z(z,\bar{z})$ as

$$Z(z,\bar{z}) = \big(Z^{(k,l)}(z,\bar{z})\big), \tag{6-3}$$

for $1 \le k, l \le N$, where

$$Z^{(k,l)}(z,\bar{z}) \equiv -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c G_l, iJG_k \big\rangle.$$

Since $P_c(L)^*J = JP_c(L)$, a consequence of $L = JH$ and $H^* = H$ (see (5-1) and Proposition E.1), we have the more symmetric expression for $Z^{(k,l)}$:

$$Z^{(k,l)} = -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c G_l, iJP_c G_k \big\rangle.$$

Finally, we define $\Gamma(z,\bar z)$ as

$$\Gamma(z,\bar z) := \tfrac{1}{2}(Z(z,\bar z)+Z^*(z,\bar z)).$$

Thus,

$$\big(\Gamma(z,\bar z)\big)_{kl} = -\Re\big\langle (L(\lambda)+2iE(\lambda)-0)^{-1}P_cG_l,\, iJP_cG_k\big\rangle.$$

Concerning the properties of $\Gamma$, we have the following general result:

**Theorem 6.1** (Matrix nonlinear Fermi Golden Rule). (1) $\Gamma(z,\bar z)=\Gamma(z,\bar z;\lambda)$ *is a nonnegative symmetric $N\times N$ matrix.*

(2) *Define*

$$K(\lambda,\vec G) := \min_{s,z\neq 0}\frac{s^*\Gamma(z,\bar z)s}{|s|^2|z|^2},$$

*where $\vec G=(G_1,\dots,G_N)$ defined in (6-2). Then, $K(\lambda,\vec G)$ depends continuously on $\lambda$ and $\vec G$ (in the space $\langle x\rangle^3L^\infty$).*

We shall require the following *Fermi Golden Rule resonance condition*:

**(FGR)** There exists $\delta'$ with $0<\delta'\le\delta$ (see Proposition 4.1) and a constant $C>0$ such that for any $s=(s_1,\dots,s_N)^T$ and $z=(z_1,\dots,z_N)^T\in\mathbb{C}^N$, we have

$$s^*\Gamma(z,\bar z;\lambda)s\ge C|s|^2|z|^2,$$

where $\lambda\in\mathcal{I}_{\delta'}$.

**Remark.** In the weakly nonlinear regime (see Section 5A) $E(\lambda)\sim e_1-e_0$, $\lambda\sim -e_0$ and therefore the condition for resonance with the continuous spectrum at second order is

$$2E(\lambda)-\lambda\sim 2(e_1-e_0)+e_0=2e_1-e_0>0.$$

Our next result is a reduction of the condition (FGR) for the class of multiwell potentials discussed in Appendix A to an explicit condition on the operator $V$.

**Proposition 6.2.** *Let $V$ denote the multiwell potential satisfying condition ($\mathrm{Eig}_V$) and constructed according to Appendix A. Thus, $-\Delta+V$ has two negative eigenvalues $e_0<e_1<0$ with $2e_1-e_0>0$. The excited state eigenvalue $e_2$ is degenerate of multiplicity $N=2$ with spanning eigenfunctions $\{\xi_1^{\mathrm{lin}},\xi_2^{\mathrm{lin}}\}$.*

*Let $f(|\psi|^2)=-g|\psi|^2$. Assume the nonnegative matrix*

$$\Big(\Im\big\langle(-\Delta+V-(2e_1-e_0)-i0)^{-1}P_c\phi_{\mathrm{lin}}\xi_m^{\mathrm{lin}}\xi_n^{\mathrm{lin}},\,\phi_{\mathrm{lin}}\xi_m^{\mathrm{lin}}\xi_n^{\mathrm{lin}}\big\rangle\Big)_{1\le m,n\le 2} \tag{6-4}$$

*is positive-definite. Then there exists $\delta'>0$ such that, for $\phi^\lambda$ denotes the soliton of Proposition 4.1, if $\big|\lambda-|e_0|\big|<\delta'$ then $K(\lambda,\vec G)>0$. And the Fermi Golden Rule condition holds by taking*

$$C=\inf_{\lambda\in\overline{\mathcal{I}_{\delta'}}}K(\lambda,\vec G(\lambda))>0$$

*in (FGR). Here $\mathcal{I}_{\delta'}$ denotes a sufficiently small subinterval of the range of $\lambda$-values for which the soliton exists*; *see Proposition 4.1.*

**Remark.** Positive-definiteness of the matrix in (6-4) is equivalent to

$$\Im\big\langle(-\Delta+V-(2e_1-e_0)-i0)^{-1}P_c\phi_{\text{lin}}(z_1\xi_1^{\text{lin}}+z_2\xi_2^{\text{lin}})^2,\,\phi_{\text{lin}}(z_1\xi_1^{\text{lin}}+z_2\xi_2^{\text{lin}})^2\big\rangle\geq C|z|^2,$$

for all $z_1, z_2\in\mathbb{C}$.

*Proof of Proposition 6.2.* In what follows we sketch the proof, which is very similar to the case $N=1$ (see [Soffer and Weinstein 1999; Tsai and Yau 2002c]).

Recall the transformation of $L(\lambda)$ in (5-2):

$$\begin{aligned}
&(L(\lambda)+2iE(\lambda)-0)^{-1}\\
&\quad=\big(iU\sigma_3\mathcal{H}U^*+2iE(\lambda)-0\big)^{-1}=-iU(\sigma_3\mathcal{H}+2E(\lambda)+i0)^{-1}U^*\\
&\quad=-iU\big(\sigma_3(\mathcal{H}_0+V)+2E(\lambda)+i0\big)^{-1}U^*\\
&\qquad\qquad+iU\big(\sigma_3\mathcal{H}+2E(\lambda)+i0\big)^{-1}V_{\text{small}}\big(\sigma_3(\mathcal{H}_0+V)+2E(\lambda)+i0\big)^{-1}U^*
\end{aligned}$$

and

$$\big(\sigma_3(\mathcal{H}_0+V)+2E(\lambda)+i0\big)^{-1}=\begin{pmatrix}\big(-\Delta+V-(-\lambda-2E(\lambda))\big)^{-1} & 0\\ 0 & -\big(-\Delta-(2E(\lambda)-\lambda)-i0\big)^{-1}\end{pmatrix}. \tag{6-5}$$

Furthermore, by Propositions 4.1 and 5.1 we have, in the space $H^2$, that

$$\frac{1}{\|\phi^\lambda\|_{H^2}}\phi^\lambda\to\frac{1}{\|\phi_{\text{lin}}\|_{H^2}}\phi_{\text{lin}},\qquad\Big(\frac{1}{\|\xi_n\|_{H^2}}\xi_n,\,\frac{1}{\|\eta_n\|_{H^2}}\eta_n\Big)\to\frac{1}{\|\xi_n^{\text{lin}}\|_{H^2}}(\xi_n^{\text{lin}},\xi_n^{\text{lin}})$$

for some $\xi_n^{\text{lin}}$ as $\big|\lambda-|e_0|\big|\to 0$. If the nonlinearity $f(\tau)=\tau^\sigma$ with $\sigma\geq 1$, we have

$$U^*P_c\sum_l z_lG_l=C\,\|\phi_{\text{lin}}\|_{H^2}^{2\sigma-1}\begin{pmatrix} * \\ P_c\phi_{\text{lin}}^{2\sigma-1}(z_1\xi_1^{\text{lin}}+z_2\xi_2^{\text{lin}})^2\end{pmatrix}(1+o(1))$$

for some constant $\tilde{c}\in\mathbb{C}$.

In considering (6-5), note that $-\lambda-2E(\lambda)\sim e_0-2(e_1-e_0)<0$ and $2E(\lambda)-\lambda\sim 2e_1-e_0<0$. Thus

$$\Im\big\langle(-\Delta+V+\lambda+2E(\lambda))^{-1}F,\,F\big\rangle=0$$

for any $F$. Furthermore, $\|e^{-\tau|x|}V_{\text{small}}\|_{L^\infty}$ is small for some $\tau>0$, we have

$$K(\lambda,\vec{G})=|\tilde{c}|^2\|\phi_{\text{lin}}\|_{H^2}^{4\sigma-2}K_0\big(1+o(1)\big)$$

with

$$\begin{aligned}
K_0&:=(1+o(1))\\
&\quad\times\Im\big\langle(-\Delta+V+e_0-2e_1-i0)^{-1}P_c(\phi_{\text{lin}})^{2\sigma-1}(z_1\xi_1^{\text{lin}}+z_2\xi_2^{\text{lin}})^2,\,(\phi_{\text{lin}})^{2\sigma-1}(z_1\xi_1^{\text{lin}}+z_2\xi_2^{\text{lin}})^2\big\rangle.
\end{aligned}$$

The proof is complete. In Appendix C we have a simpler formula for (FGR) when the potential $V$ is spherical symmetric. □

The proof of Theorem 6.1 is deferred to Appendix B.

## 7. Main theorem

In this section we state precisely the main theorem of this paper. Recall the notations $\xi = (\xi_1, \dots, \xi_N)$ and $\eta = (\eta_1, \dots, \eta_N)$ for components of the neutrally stable modes of frequencies $\pm i E(\lambda)$ of the linearized operator. Recall the definition of the interval $\mathcal{I}$ in (1-4).

**Theorem 7.1.** *Assume* (fA), (fB) *on the nonlinearity* $f(|\psi|^2)$ (page 274), (VA) *on the potential* $V(x)$ (p. 274), (SA) *on the structure of the discrete spectral subspace of the linearization about* $\phi^\lambda$ (page 278), (Thresh$_\lambda$) *on the absence of threshold resonances* (page 278), *and* (FGR), *the nonlinear Fermi Golden Rule resonance condition* (page 282). *Fix* $\nu > 0$ *sufficiently large and let* $k = [\frac{d}{2}]+1$ *where* $d \geq 3$ *denotes the spatial dimension. Then there exist constants* $c, \epsilon_0 > 0$ *such that, if for some* $\lambda_0 \in \mathcal{I}$

$$\inf_{\gamma \in \mathbb{R}} \left\| \psi_0 - e^{i\gamma}(\phi^{\lambda_0} + (\Re z^{(0)}) \cdot \xi + i(\Im z^{(0)}) \cdot \eta) \right\|_{H^{k,\nu}} \leq c|z^{(0)}| \leq \epsilon_0, \tag{7-1}$$

*then there exist smooth functions*

$$\lambda(t) : \mathbb{R}^+ \to \mathcal{I}, \quad \gamma(t) : \mathbb{R}^+ \to \mathbb{R}, \quad z(t) : \mathbb{R}^+ \to \mathbb{C}^N, \quad R(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$$

*such that the solution of NLS evolves in the form*:

$$\psi(x,t) = e^{i\int_0^t \lambda(s)\,ds} e^{i\gamma(t)} \big(\phi^\lambda + a_1(z,\bar{z})\,\partial_\lambda \phi^\lambda + i a_2(z,\bar{z})\phi^\lambda + \Re\tilde{z}\cdot\xi + i\Im\tilde{z}\cdot\eta + R\big) \tag{7-2}$$

*where* $\lim_{t\to\infty} \lambda(t) = \lambda_\infty$ *for some* $\lambda_\infty \in \mathcal{I}$, $a_1(z,\bar{z})$, $a_2(z,\bar{z}) : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{R}$ *and* $\tilde{z} - z : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ *are polynomials of* $z$ *and* $\bar{z}$, *beginning with terms of order* $|z|^2$. *Moreover*,

(A) $|z(t)| \leq c(1+t)^{-1/2}$ *and* $z$ *satisfies the initial value problem*

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathcal{O}\big((1+t)^{-19/5}\big) \tag{7-3}$$

*where* $\Gamma(z,\bar{z})$ *is a symmetric, positive-definite matrix defined in* (6-3) *and* $\Lambda(z,\bar{z})$ *is a skew symmetric matrix.*

(B) $\vec{R}(t) = (\Re R(t), \Im R(t))^T$ *lies in the essential spectral part of* $L(\lambda(t))$. *Equivalently*, $R(\cdot, t)$ *satisfies the symplectic orthogonality conditions*:

$$\begin{aligned} &\omega\langle R, i\phi^\lambda\rangle = \omega\langle R, \partial_\lambda \phi^\lambda\rangle = 0, \\ &\omega\langle R, i\eta_n\rangle = \omega\langle R, \xi_n\rangle = 0, \quad \textit{for } n = 1, 2, \dots, N, \end{aligned}$$

*where* $\omega\langle X, Y\rangle = \Im \int X\overline{Y}\,dx$ *and satisfies the decay estimate*:

$$\|(1+x^2)^{-\nu}\vec{R}(t)\|_2 \leq c(1+t)^{-1}.$$

**Remark.** We conclude this section by stating that all the hypotheses except (FGR) in our main result apply to the multiwell example of Appendix A; see Proposition 6.2 for a reduction of (FGR) is an explicit condition on the spectral condition on $-\Delta + V$. We expect (FGR) to hold generically in an appropriate sense.

## 8. Reformulation of the main theorem

In proving Theorem 7.1 we establish more detailed characterization of the perturbation about $\mathcal{M}_{\mathcal{J}}$.

First, we introduce the following simplying

**Notation.** We always use $z$ to stand for a complex $N$-dimensional vector $z = (z_1, z_2, \dots, z_N)$ and an upper case letter or a Greek letter with two subindices, for example, $Q_{m,n}$ to stand for

$$Q_{m,n}(\lambda) = \sum_{\substack{a,b \in \mathbb{N}^N \\ |a|=m, |b|=n}} q_{a,b}(\lambda) \prod_{k=1}^{N} z_k^{a_k} \bar{z}_k^{b_k},$$

where $|a| := \sum_{k=1}^{N} a_k$. We refer to this kind term as $(m, n)$ term.

**Theorem 8.1.** *The following more precise decomposition of the solution in Theorem 7.1 holds: The perturbation $\vec{R}$ in (7-2) can be decomposed as*

$$\vec{R} = \sum_{m+n=2} R_{m,n}(\lambda) + \tilde{R} \tag{8-1}$$

*where $R_{m,n}$ are functions of the form*

$$R_{m,n} = (L(\lambda) + iE(\lambda)(m-n) - 0)^{-1} \phi_{m,n},$$

*$\phi_{m,n}$ are polynomials of $z$ and $\bar{z}$ with coefficients being smooth, exponentially decaying functions. The function $\tilde{R}$ satisfies*

$$\partial_t \tilde{R} = L(\lambda)\tilde{R} + M_2(z, \bar{z})\tilde{R} + P_c N_2(\vec{R}, z) + P_c S_2(z, \bar{z}). \tag{8-2}$$

*In this formula, $S_2(z, \bar{z}) = \mathbb{O}(|z|^3)$ is a polynomial in $z$ and $\bar{z}$ with $\lambda$-dependent coefficients, and each coefficient can be written as the sum of functions of either of the following two forms:*

$$(L(\lambda) + 2iE(\lambda) - 0)^{-k} P_c \phi_{+k}(\lambda), \qquad (L(\lambda) - 2iE(\lambda) - 0)^{-k} P_c \phi_{-k}(\lambda), \tag{8-3}$$

*where $k = 0, 1, 2$ and the functions $\phi_{\pm k}(\lambda)$ are smooth and decay exponentially fast at $\infty$. $M_2(z, \bar{z})$ is an operator defined by*

$$M_2(z, \bar{z}) := \dot{\gamma} P_c J + \dot{\lambda} P_{c\lambda} + X, \tag{8-4}$$

*where $X$ is a $2\times 2$ matrix satisfying the bound $|X| \le c|z|e^{-\epsilon_0|x|}$. $N_2(\vec{R}, z)$ can be separated into localized term and nonlocal term*

$$N_2 = \mathrm{Loc} + \mathrm{NonLoc} \tag{8-5}$$

*where* Loc *consists of terms spatially localized (exponentially) function of $x \in \mathbb{R}^d$ as a factor and satisfies the estimate*

$$\|\langle x\rangle^{\nu}(-\Delta+1)\mathrm{Loc}\|_2 + \|\mathrm{Loc}\|_1 + \|\mathrm{Loc}\|_{4/3} \le c\big(|z(t)|^3 + |z(t)|\,\|\langle x\rangle^{-\nu}(-\Delta+1)\vec{R}\|_2\big) \tag{8-6}$$

*and* NonLoc *is given by*

$$\mathrm{NonLoc} := f(R_1^2 + R_2^2) J \vec{R} \tag{8-7}$$

*and consists of purely nonlinear terms in* $\vec{R}$ *with no spatially localized factors.* (Here $\nu$ is the same as in Theorem 7.1.)

*Denote by* $\mathrm{Remainder}(t)$ *any quantity which satisfies the estimate*

$$|\mathrm{Remainder}(t)| \lesssim |z(t)|^4 + \|\langle x\rangle^{-\nu}(-\Delta+1)\vec{R}(t)\|_2^2 + \|\vec{R}(t)\|_\infty^2 + |z(t)|\,\|\langle x\rangle^{-\nu}\tilde{R}(t)\|_2. \tag{8-8}$$

*The functions* $\lambda$, $\gamma$, $z$ *have the following properties*:

$$\dot{\lambda} = \mathrm{Remainder}(t), \tag{8-9}$$

$$\dot{\gamma} = \Upsilon_{1,1} + \mathrm{Remainder}(t), \tag{8-10}$$

$$\partial_t z = -iE(\lambda)z - \Gamma(z,\bar{z})z + \Lambda(z,\bar{z})z + \mathrm{Remainder}(t) \tag{8-11}$$

*where*

$$\Upsilon_{1,1} := \frac{\left\langle \phi^\lambda\left(\frac{3}{2}f'[(\phi^\lambda)^2] + f''[(\phi^\lambda)^2](\phi^\lambda)^2\right)|z\cdot\xi|^2 + \frac{1}{2}f'[(\phi^\lambda)^2]|z\cdot\eta|^2,\ \partial_\lambda\phi^\lambda\right\rangle}{\langle \phi^\lambda, \partial_\lambda\phi^\lambda\rangle}, \tag{8-12}$$

$\Gamma(z,\bar{z})$ *is the* $N\times N$ *positive-definite matrix used in* (FGR) *and* $\Lambda(z,\bar{z})$ *is skew symmetric.*

## 9. Modulation equations for $z(t)$, $\lambda(t)$, $\gamma(t)$ and the dispersive part, $R(\cdot,t)$

In this section we derive evolution equations for $z$, $\lambda$, $\gamma$ and $R$.

We decompose the solution as

$$\begin{aligned}\psi(x,t) &= e^{i\int_0^t \lambda(s)\,ds}e^{i\gamma(t)}\Bigl(\phi^\lambda + a_1\phi^\lambda_\lambda + ia_2\phi^\lambda + \sum_{n=1}^N(\alpha_n+p_n)\xi_n + i\sum_{n=1}^N(\beta_n+q_n)\eta_n + R\Bigr)\\ &= e^{i\int_0^t \lambda(s)\,ds}e^{i\gamma(t)}(\phi^\lambda + a_1\phi^\lambda_\lambda + ia_2\phi^\lambda + (\alpha+p)\cdot\xi + (\beta+q)\cdot\eta + R).\end{aligned} \tag{9-1}$$

Here and going forward let

$$\alpha = (\alpha_1,\dots,\alpha_N)^T,\ \beta = (\beta_1,\dots,\beta_N)^T,\ \xi = (\xi_1,\dots,\xi_N)^T,\ \eta = (\eta_1,\dots,\eta_N)^T.$$

Let $z = \alpha + i\beta$ then

$$\alpha = \tfrac{1}{2}(z+\bar{z}), \qquad \beta = \tfrac{1}{2i}(z-\bar{z}).$$

We seek polynomials in $z$ and $\bar{z}$, which are of degree two or higher:

$$a_j = a_j(z,\bar{z}) = \mathbb{O}(|z|^2), \quad p_n = p_n(z,\bar{z}) = \mathbb{O}(|z|^2), \quad q_n = q_n(z,\bar{z}) = \mathbb{O}(|z|^2)$$

where $j = 1,2$ and $n = 1,\dots,N$. Substituting Ansatz (9-1) into NLS (1-1), we have the system of equations

$$\begin{aligned}\partial_t\vec{R} = L(\lambda)\vec{R} + \dot{\gamma}J\vec{R} - J\vec{N}(\vec{R},z) - \begin{pmatrix}\partial_\lambda\phi^\lambda(\dot{\lambda}+\partial_t a_1)\\ \phi^\lambda(\dot{\gamma}+\partial_t a_2 - a_1)\end{pmatrix} + \begin{pmatrix}\xi\cdot\bigl(E(\lambda)(\beta+q) - \partial_t(\alpha+p)\bigr)\\ -\eta\cdot\bigl(E(\lambda)(\alpha+p)+\partial_t(\beta+q)\bigr)\end{pmatrix}\\ + \dot{\gamma}\begin{pmatrix}(\beta+q)\cdot\eta\\ -(\alpha+p)\cdot\xi\end{pmatrix} - \dot{\lambda}\begin{pmatrix}a_1\partial_\lambda^2\phi^\lambda + (\alpha+p)\cdot\partial_\lambda\xi\\ a_2\partial_\lambda\phi^\lambda + (\beta+q)\cdot\partial_\lambda\eta\end{pmatrix},\end{aligned} \tag{9-2}$$

where

$$\vec{R} \equiv \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad \vec{N} = \begin{pmatrix} \Re N(\vec{R}, z) \\ \Im N(\vec{R}, z) \end{pmatrix}, \quad J\vec{N}(\vec{R}, z) = \begin{pmatrix} \Im N(\vec{R}, z) \\ -\Re N(\vec{R}, z) \end{pmatrix} \tag{9-3}$$

with $R_1 \equiv \Re R$, $R_2 \equiv \Im R$ and

$$\Im N(\vec{R}, z) := f[|\phi^\lambda + I_1 + i I_2|^2] I_2 - f[(\phi^\lambda)^2] I_2,$$
$$\Re N(\vec{R}, z) := (f[|\phi^\lambda + I_1 + i I_2|^2] - f[(\phi^\lambda)^2])(\phi^\lambda + I_1) - 2f'[(\phi^\lambda)^2](\phi^\lambda)^2 I_1,$$

in which

$$I_1 = A_1 + A_2 + R_1, \quad I_2 = B_1 + B_2 + R_2$$
$$A_1 = \alpha \cdot \xi, \quad A_2 = a_1 \partial_\lambda \phi^\lambda + p \cdot \xi,$$
$$B_1 = \beta \cdot \eta, \quad B_2 = a_2 \phi^\lambda + q \cdot \eta.$$

From (9-2) and the orthogonality conditions (5-4) we obtain equations for $\dot{\lambda}$, $\dot{\gamma}$ and $z_n = \alpha_n + i\beta_n$ with $n = 1, \ldots, N$:

$$\partial_t(\alpha_n + p_n) - E(\lambda)(\beta_n + q_n) + \langle \Im N(\vec{R}, z), \eta_n \rangle = F_{1n}, \tag{9-4}$$

$$\partial_t(\beta_n + q_n) + E(\lambda)(\alpha_n + p_n) - \langle \Re N(\vec{R}, z), \xi_n \rangle = F_{2n}, \tag{9-5}$$

$$\dot{\gamma} + \partial_t a_2 - a_1 - \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle \Re N(\vec{R}, z), \partial_\lambda \phi^\lambda \rangle = F_3, \tag{9-6}$$

$$\dot{\lambda} + \partial_t a_1 + \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda \rangle} \langle \Im N(\vec{R}, z), \phi^\lambda \rangle = F_4, \tag{9-7}$$

where the scalar functions $F_{1n}$, $F_{2n}$, $F_3$ and $F_4$ are defined as

$$F_{1n} = \dot{\gamma}\langle (\beta + q) \cdot \eta, \eta_n \rangle - \dot{\lambda} a_1 \langle \partial_\lambda^2 \phi^\lambda, \eta_n \rangle - \dot{\lambda}\langle (\alpha + p) \cdot \partial_\lambda \xi, \eta_n \rangle - \dot{\gamma}\langle R_2, \eta_n \rangle + \dot{\lambda}\langle R_1, \partial_\lambda \eta_n \rangle,$$

$$F_{2n} = -\dot{\gamma}\langle (\alpha + p) \cdot \xi, \xi_n \rangle - \dot{\lambda} a_2 \langle \phi_\lambda^\lambda, \xi_n \rangle - \dot{\lambda}\langle (\beta + q) \cdot \partial_\lambda \eta, \xi_n \rangle + \dot{\gamma}\langle R_1, \xi_n \rangle + \dot{\lambda}\langle R_2, \partial_\lambda \xi_n \rangle,$$

$$F_3 = \frac{1}{\langle \phi^\lambda, \phi_\lambda^\lambda \rangle} \big( \dot{\lambda}\langle R_2, \phi_{\lambda\lambda}^\lambda \rangle - \dot{\gamma}\langle R_1, \phi_\lambda^\lambda \rangle - \langle \dot{\gamma}(\alpha + p) \cdot \xi + \dot{\lambda} a_2 \phi_\lambda^\lambda + \dot{\lambda}(\beta + q) \cdot \partial_\lambda \eta, \phi_\lambda^\lambda \rangle \big),$$

$$F_4 = \frac{1}{\langle \phi^\lambda, \phi_\lambda^\lambda \rangle} \big( \dot{\lambda}\langle R_1, \phi_\lambda^\lambda \rangle + \dot{\gamma}\langle R_2, \phi^\lambda \rangle + \langle \dot{\gamma}(\beta + q) \cdot \eta - \dot{\lambda} a_1 \partial_\lambda^2 \phi^\lambda - \dot{\lambda}(\alpha + p) \cdot \partial_\lambda \xi, \phi^\lambda \rangle \big).$$

**Remarks.** (a) Recall the estimate of Remainder in (8-8). By (9-4)–(9-7) we have

$$\dot{\lambda}, \dot{\gamma}, \partial_t z_n + i E(\lambda) z_n = \mathbb{O}(|z|^2) + \text{Remainder}. \tag{9-8}$$

(b) The functions $a_j(z, \bar{z})$, $p_n(z, \bar{z})$ and $q_n(z, \bar{z})$ for $j = 1, 2$ and $n = 1, \ldots, N$ will be chosen to eliminate "nonresonant" terms $z^m \bar{z}^n$ with $2 \le |m| + |n| \le 3$.

Finally, we derive an equation for

$$\vec{R} = P_c^{\lambda(t)} \vec{R} = P_c \vec{R},$$

the continuous spectral part of the solution, relative to the operator $L(\lambda(t))$. Applying $P_c = P_c^{\lambda(t)}$ to (9-2) and using the commutator identity:

$$P_c \partial_t \vec{R} = \partial_t \vec{R} - \dot{\lambda} \partial_\lambda P_c \vec{R},$$

we obtain

$$\partial_t \vec{R} = L(\lambda(t))\vec{R} - P_c^{\lambda(t)} J \vec{N}(\vec{R}, z) + L_{(\dot{\lambda},\dot{\gamma})}\vec{R} + \mathcal{G}. \tag{9-9}$$

The operator $L_{(\dot{\lambda},\dot{\gamma})}$ and the vector function $\mathcal{G}$ are defined as

$$L_{(\dot{\lambda},\dot{\gamma})} = \dot{\lambda}(\partial_\lambda P_c^{\lambda(t)}) + \dot{\gamma} P_c^{\lambda(t)} J, \tag{9-10}$$

$$\mathcal{G} = P_c^{\lambda(t)} \begin{pmatrix} \dot{\gamma}(\beta+q)\cdot\eta - \dot{\lambda}a_1\partial_\lambda^2\phi^\lambda - \dot{\lambda}(\alpha+p)\cdot\partial_\lambda\xi \\ -\dot{\gamma}(\alpha+p)\cdot\xi - \dot{\lambda}a_2\phi_\lambda^\lambda - \dot{\lambda}(\beta+q)\cdot\partial_\lambda\eta \end{pmatrix}. \tag{9-11}$$

We now summarize the preceding calculation in

**Proposition 9.1** (Reformulation of NLS). *Using the Ansatz (9-1)*

$$\psi(x,t) = e^{i\int_0^t \lambda(s)\,ds} e^{i\gamma(t)}\big(\phi^\lambda + a_1\phi_\lambda^\lambda + ia_2\phi^\lambda + (\alpha+p)\cdot\xi + (\beta+q)\cdot\eta + R\big),$$

*NLS can be equivalently expressed as a coupled system of equations (9-4)–(9-7) for modulating solitary wave parameters $\lambda(t)$ and $\gamma(t)$, neutral mode amplitudes $z_n(t) = \alpha_n(t) + i\beta_n(t)$ for $n = 1, \dots, N$, together with Equation (9-9) governing "dispersive part" $\vec{R}(t)$ which evolves in the continuous spectral subspace of $L(\lambda(t))$, that is, $P_c^{\lambda(t)}\vec{R}(t) = \vec{R}(t)$; see (5-5). Moreover, the functions $a_j = a_j(z,\bar{z})$ for $j = 1, 2$, $\big(p(z,\bar{z}), q(z,\bar{z})\big) = (p_n, q_n)_{n=1,\dots,N}$ are $\mathbb{O}(|z|^2)$ polynomials chosen (in what follows) to eliminate "nonresonant" terms of the form $z^a\bar{z}^b$ with $2 \le |a| + |b| \le 3$.*

***Extracting the $\mathbb{O}(|z|^2)$ part of $\vec{R}(t)$; proof of (8-2).*** For fixed $z(t) \in \mathbb{C}^N$, the equation for $\vec{R}(t)$ is forced by terms of order $|z(t)|^2$; linear terms are removed due to the equations satisfied by $z(t) = \alpha(t) + i\beta(t)$. In our analysis, we need to explicitly extract the quadratic part in $z$, $\bar{z}$ of $\vec{R}(t)$.

Thus, we consider the quadratic terms generated by the nonlinearity:

$$\sum_{m+n=2} J\vec{N}_{m,n} = J\vec{N}_{2,0} + J\vec{N}_{1,1} + J\vec{N}_{0,2} = \begin{pmatrix} 2f'[(\phi^\lambda)^2]\phi^\lambda A_1 B_1 \\ -\big(3f'[(\phi^\lambda)^2]\phi^\lambda + 2f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)A_1^2 - f'[(\phi^\lambda)^2]\phi^\lambda B_1^2 \end{pmatrix}, \tag{9-12}$$

where $A_1 = \alpha\cdot\xi$, $B_1 = \beta\cdot\eta$.

**Theorem 9.2.** *Define*

$$\vec{R}_{m,n} := (L(\lambda) + iE(\lambda)(m-n) - 0)^{-1} P_c J\vec{N}_{m,n} \tag{9-13}$$

*and decompose $\vec{R}(t)$ as*

$$\vec{R} = \sum_{m+n=2} \vec{R}_{m,n} + \tilde{R}. \tag{9-14}$$

*The function $\tilde{R}(x,t)$ satisfies (8-2).*

*Proof.* $\tilde{R}$, defined in Equation (9-14), satisfies the equation:

$$\partial_t\tilde{R} = L(\lambda)\tilde{R} + L_{(\dot{\lambda},\dot{\gamma})}\tilde{R} + \sum_{m+n=2} L_{(\dot{\lambda},\dot{\gamma})}R_{m,n} + \mathcal{G} - \sum_{m+n=2}(\partial_t\vec{R}_{m,n} + iE(\lambda)(m-n)\vec{R}_{m,n}) - P_c J\vec{N}_{>2}$$

where, recall the definitions of $\vec{R}_{m,n}$ in (9-13), the definitions of the operator $L_{\dot{\lambda},\dot{\gamma}}$ and the term $\mathcal{G}$ in (9-9), and we define

$$J\vec{N}_{>2} := J\vec{N}(\vec{R}, z) - \sum_{m+n=2} J\vec{N}_{m,n}.$$

Next we further decompose $J\vec{N}_{>2}$ and find $M_2$, $S_2$ and $N_2$ in (8-2). We consider the functions $JN_{m,n}$ with $m+n=3$, the third order terms of $J\vec{N}_{>2}$:

$$\sum_{m+n=3} J\vec{N}_{m,n} = X\left(\sum_{m+n=2} \vec{R}_{m,n} + \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}\right) + \begin{pmatrix} G_1(A_1^2, B_1^2)B_1 \\ -G_2(A_1^2, B_1^2)A_1 \end{pmatrix} \tag{9-15}$$

where, recall the definitions of $A_1$, $B_1$, $A_2$ and $B_2$ from (9-3),

$$G_1(A_1^2, B_1^2) := f'[(\phi^\lambda)^2](A_1^2+B_1^2) + 2f''[(\phi^\lambda)^2](\phi^\lambda)^2 A_1^2,$$
$$G_2(A_1^2, B_1^2) := \big(f'[(\phi^\lambda)^2] + 2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(A_1^2+B_1^2) + \big(2f''[(\phi^\lambda)^2](\phi^\lambda)^2 + \tfrac{4}{3}f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)A_1^2$$

and $X$ is a $2\times 2$ matrix of order $|z|$ defined as

$$X = X_{0,1} + X_{1,0} = \begin{pmatrix} 2f'[(\phi^\lambda)^2]\phi^\lambda B_1 & 2f'[(\phi^\lambda)^2]\phi^\lambda A_1 \\ -\big(6f'[(\phi^\lambda)^2]\phi^\lambda + 4f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)A_1 & -2f'[(\phi^\lambda)^2]\phi^\lambda B_1 \end{pmatrix}. \tag{9-16}$$

We define the linear operator $M_2(z, \bar{z})$ as

$$M_2(z, \bar{z}) := X + L_{(\dot\lambda, \dot\gamma)}$$

which satisfies (8-4).

The function $S_2$ in the statement of Theorem 7.1 is defined as

$$S_2(z, \bar{z}) := \sum_{m+n=2} L_{(\dot\lambda,\dot\gamma)} R_{m,n} + \mathscr{G} - \sum_{m+n=2} (\partial_t R_{m,n} + iE(\lambda)(m-n)R_{m,n}) + \sum_{m+n=3} JN_{m,n}.$$

By (9-8) and

$$\begin{aligned}[\partial_t, (L(\lambda) \pm iE(\lambda) - 0)^{-1}] &\equiv \partial_t (L(\lambda) \pm iE(\lambda) - 0)^{-1} - (L(\lambda) \pm iE(\lambda) - 0)^{-1}\partial_t \\ &= \dot\lambda (L(\lambda) \pm iE(\lambda) - 0)^{-1} \partial_\lambda (L(\lambda) \pm iE(\lambda))(L(\lambda) \pm iE(\lambda) - 0)^{-1}\end{aligned}$$

we have that $S_2(z, \bar{z})$ satisfies the estimate in the first part of Theorem 8.1. For the details we refer to [Gang and Sigal 2007].

Lastly, we define the nonlinear term

$$\vec{N}_2(\vec{R}, z) := -\Big(J\vec{N}(\vec{R}, z) - \sum_{m+n=2,3} JN_{m,n}\Big). \tag{9-17}$$

Using the smoothness of the nonlinearity $f[\cdot]$ and removing $\mathbb{O}(|z|^2)$ and $\mathbb{O}(|z|^3)$ terms, we have that $N_2(\vec{R}, z) = \mathrm{Loc} + \mathrm{NonLoc}$ (see (8-5)) satisfying (8-6) and (8-7). The computation is straightforward but tedious and is therefore omitted.

Collecting the various definitions and estimates above we have (8-2). □

***$z(t)$ dependence of equations for $\lambda(t)$ and $\gamma(t)$.*** In this subsection we present the proofs of (8-9) and (8-10), crucial to controlling the large time behavior.

Here's the idea. Central to our claim about the large time dynamics of NLS is that the solution settles into an asymptotic solitary wave $\phi^{\lambda_\infty}$ where $\lambda(t) \to \lambda_\infty$. We show this by establishing the integrability and uniform smallness of $\dot\lambda$. Since we expect the neutral mode amplitudes $z(t)$ to decay with a rate $t^{-1/2}$, we require that there be no $\mathbb{O}(|z(t)|^2)$ terms in the (9-7): $\dot\lambda(t) + \partial_t a_1(z, \bar{z}) = \ldots$. The strategy is to choose the quadratic part of the polynomial $a_1(z,\bar{z})$ so as to eliminate all quadratic nonresonant terms. The latter are terms whose $z$-behavior is like $(z_k)^2$ or $(\bar{z}_k)^2$ and are oscillatory with frequencies $\sim \pm 2iE(\lambda)$. But what about the terms of the form $z_k\bar{z}_m$, which are resonant (nonoscillatory)? This is where we use the choice of basis for the degenerate subspace; see Appendix D. A consequence of this choice is that there are *no* resonant quadratic terms appearing in the equation for $\lambda$! The calculation is carried out below; see Lemma 9.4.

In what follows we use the notations $N^{\Im}_{m,n}$ and $N^{\Re}_{m,n}$ to denote functions satisfying

$$\begin{pmatrix} N^{\Im}_{m,n} \\ -N^{\Re}_{m,n}\end{pmatrix} = JN_{m,n}.$$

We define the polynomials $a_1$, $a_2$ and $p_k$, $q_k$ for $k=1,2,\dots,N$ in (9-1) (see also (7-2)) as

$$a_j(z,\bar{z}) := \sum_{\substack{m+n=2,3\\ m\neq n}} A^{(j)}_{m,n}(\lambda), \quad p_k(z,\bar{z}) := \sum_{m+n=2,3} P^{(k)}_{m,n}(\lambda), \quad q_k(z,\bar{z}) := \sum_{m+n=2,3} Q^{(k)}_{m,n}(\lambda), \tag{9-18}$$

with $j=1,2$, $k=1,2,\dots,N$, and the explicit forms

$$\begin{aligned} 2iE(\lambda)A^{(1)}_{2,0} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Im}_{2,0},\phi^\lambda\rangle, \qquad 3iE(\lambda)A^{(1)}_{3,0} := \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Im}_{3,0},\phi^\lambda\rangle,\\ iE(\lambda)A^{(1)}_{2,1} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\big(\langle N^{\Im}_{2,1},\phi^\lambda\rangle - \tfrac{i}{2}\Upsilon_{1,1},\langle z\cdot\eta,\phi^\lambda\rangle\big)\end{aligned} \tag{9-19}$$

where $\Upsilon_{1,1}$ is given in (8-12); similarly

$$\begin{aligned} -2iE(\lambda)A^{(2)}_{2,0}+A^{(1)}_{2,0} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Re}_{2,0},\partial_\lambda\phi^\lambda\rangle, \qquad -3iE(\lambda)A^{(2)}_{3,0}+A^{(1)}_{3,0} := \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\langle N^{\Re}_{3,0},\partial_\lambda\phi^\lambda\rangle,\\ -iE(\lambda)A^{(2)}_{2,1}+A^{(1)}_{2,1} &:= \frac{1}{\langle\phi^\lambda,\partial_\lambda\phi^\lambda\rangle}\big(\langle N^{\Re}_{2,1},\partial_\lambda\phi^\lambda\rangle - \tfrac{1}{2}\Upsilon_{1,1}\langle z\cdot\xi,\partial_\lambda\phi^\lambda\rangle\big)\end{aligned} \tag{9-20}$$

and

$$\begin{aligned} -2iE(\lambda)P^{(n)}_{2,0}-E(\lambda)Q^{(n)}_{2,0} &:= -\langle N^{\Im}_{2,0},\eta_n\rangle, \qquad -2iE(\lambda)Q^{(n)}_{2,0}+E(\lambda)P^{(n)}_{2,0} := \langle N^{\Re}_{2,0},\xi_n\rangle,\\ -3iE(\lambda)P^{(n)}_{3,0}-E(\lambda)Q^{(n)}_{3,0} &:= -\langle N^{\Im}_{3,0},\eta_n\rangle, \qquad -3iE(\lambda)Q^{(n)}_{3,0}+E(\lambda)P^{(n)}_{3,0} := \langle N^{\Re}_{3,0},\xi_n\rangle,\\ 2iE(\lambda)P^{(n)}_{1,2}-2E(\lambda)Q^{(n)}_{1,2} &:= -\langle N^{\Im}_{1,2},\eta_n\rangle + i\langle N^{\Re}_{1,2},\xi_n\rangle + i\Upsilon_{1,1}\sum_{k=1}^{N}\bar{z}_k(\langle\eta_k,\eta_n\rangle-\langle\xi_k,\xi_n\rangle),\\ E(\lambda)Q^{(n)}_{1,1} &:= \langle N^{\Im}_{1,1},\eta_n\rangle, \qquad E(\lambda)P^{(n)}_{1,1} := \langle N^{\Re}_{1,1},\xi_n\rangle\end{aligned} \tag{9-21}$$

$$A^{(j)}_{a,b} := \overline{A^{(j)}_{a,b}}, \quad P^{(n)}_{a,b} := \overline{P^{(n)}_{a,b}}, \quad Q^{(n)}_{a,b} := \overline{Q^{(n)}_{a,b}}$$

for $j=1,2$, $a+b=2,3$, $a\neq b$.

The following is the main result.

**Proposition 9.3.** *Define the polynomials $a_1(z,\bar z)$, $a_2(z,\bar z)$, $p_n(z,\bar z)$, $q_n(z,\bar z)$ as above. Then,* (8-9)–(8-10) *hold and*

$$\begin{gathered}\partial_t\lambda = \text{Remainder}(t), \qquad \partial_t\gamma = \Upsilon_{1,1} + \text{Remainder}(t),\\ \partial_t z_n + iE(\lambda)z_n = -\left\langle JN_{2,1}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\right\rangle + \tfrac{1}{2}\Upsilon_{1,1}\sum_{m=1}^{N} z_m\left\langle\begin{pmatrix}-i\eta_m\\ \xi_m\end{pmatrix}, \begin{pmatrix}\eta_n\\ i\xi_n\end{pmatrix}\right\rangle + \text{Remainder}(t),\end{gathered} \tag{9-22}$$

*where $\Upsilon_{1,1}$ is defined in* (8-12), *and moreover,*

$$|\text{Remainder}(t)| \lesssim |z(t)|^4 + \|\langle x\rangle^{-\nu}(-\Delta+1)\vec R(t)\|_2^2 + \|\vec R(t)\|_\infty^2 + |z(t)|\cdot\|\langle x\rangle^{-\nu}\tilde R(t)\|_2.$$

Before proving the proposition we state the following key observation.

**Lemma 9.4.** $$\langle N_{1,1}^{\Im}, \phi^\lambda\rangle = 0.$$

*Proof.* Recall that

$$A_1 = \alpha\cdot\xi = \tfrac{1}{2}(z\cdot\xi + \bar z\cdot\xi), \qquad B_1 = \beta\cdot\eta = \tfrac{1}{2i}(z\cdot\eta - \bar z\cdot\eta).$$

The explicit form of

$$JN_{2,0} + JN_{1,1} + JN_{0,2}$$

in (9-12) implies that

$$\begin{aligned}N_{2,0}^{\Im} + N_{1,1}^{\Im} + N_{0,2}^{\Im} &= 2f'[(\phi^\lambda)^2]\phi^\lambda A_1 B_1\\ &= \tfrac{1}{2i}f'[(\phi^\lambda)^2]\phi^\lambda\left(\sum_{n=1}^N z_n\xi_n + \sum_{n=1}^N \bar z_n\xi_n\right)\left(\sum_{m=1}^N z_m\eta_m - \sum_{m=1}^N \bar z_m\eta_m\right).\end{aligned}$$

By taking the relevant terms we have

$$\begin{aligned}N_{1,1}^{\Im} &= \frac{1}{2i}f'[(\phi^\lambda)^2]\phi^\lambda\left(\sum_{n=1}^N \bar z_n\xi_n\sum_{m=1}^N z_m\eta_m - \sum_{n=1}^N z_n\xi_n\sum_{m=1}^N \bar z_m\eta_m\right)\\ &= \frac{1}{2i}\sum_{n=1}^N\sum_{m=1}^N \bar z_n z_m f'[(\phi^\lambda)^2]\phi^\lambda(\xi_n\eta_m - \xi_m\eta_n),\end{aligned}$$

which, together with (5-3), yields

$$\langle N_{1,1}^{\Im}, \phi^\lambda\rangle = \frac{1}{2i}\sum_{n=1}^N\sum_{m=1}^N \bar z_n z_m\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_n\eta_m - \xi_m\eta_n) = 0. \qquad \square$$

*Proof of Proposition 9.3.* Recall the estimate of any term denoted Remainder in (8-8). We put (9-6) and (9-7) in the matrix form

$$(\mathrm{Id} + M(z,\vec R,p,q))\begin{pmatrix}\dot\lambda\\ \dot\gamma - \Upsilon_{1,1}\end{pmatrix} = \Omega + \text{Remainder}, \tag{9-23}$$

where the matrix $\Omega$ is defined as

$$\Omega := \begin{pmatrix} -\frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda\rangle}\big(\langle \Im N, \phi^\lambda\rangle + \frac{i}{2}\Upsilon_{1,1}\langle (z-\bar z)\cdot \eta, \phi^\lambda\rangle\big) - \partial_t a_1 \\ \frac{1}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda\rangle}\big(\langle \Re N, \partial_\lambda\phi^\lambda\rangle - \frac{1}{2}\Upsilon_{1,1}\langle (z+\bar z)\cdot \xi, \partial_\lambda\phi^\lambda\rangle\big) - \Upsilon_{1,1} - \partial_t a_2 + a_1 \end{pmatrix}, \tag{9-24}$$

the term Remainder is produced by

$$\frac{\Upsilon_{1,1}}{\langle \phi^\lambda, \partial_\lambda \phi^\lambda\rangle}\begin{pmatrix} -\langle R_1, \partial_\lambda\phi^\lambda\rangle + p\langle \xi, \partial_\lambda \phi^\lambda\rangle \\ \langle R_2, \phi^\lambda\rangle + q\langle \eta, \phi^\lambda\rangle \end{pmatrix},$$

Id is the $2\times 2$ identity matrix, $M(z, \vec R, p, q)$ is a vector depending on $z$, $\vec R$, $p$ and $q$ and satisfies the estimate

$$\|M(z, \vec R, p, q)\| = \mathbb{O}(|z|) + \text{Remainder}. \tag{9-25}$$

Now by the definitions of $a_1$ and $a_2$ in (9-18), we remove the lower order terms in $z, \bar z$ from

$$\langle \Im N, \phi^\lambda\rangle - \tfrac{i}{2}\Upsilon_{1,1}\langle (z-\bar z)\cdot\eta, \phi^\lambda\rangle \quad \text{and} \quad \langle \Re N, \partial_\lambda \phi^\lambda\rangle + \tfrac{1}{2}\Upsilon_{1,1}\langle (z+\bar z)\cdot \xi, \partial_\lambda\phi^\lambda\rangle$$

to get

$$\Omega = D_1 + D_2 \tag{9-26}$$

with

$$D_1 := \frac{1}{\langle \phi^\lambda, \partial_\lambda\phi^\lambda\rangle}\begin{pmatrix} -\big\langle \Im N - \sum_{m+n=2,3} N^{\Im}_{m,n}, \phi^\lambda\big\rangle \\ \big\langle \Re N - \sum_{m+n=2,3} N^{\Re}_{m,n}, \partial_\lambda\phi^\lambda\big\rangle \end{pmatrix}$$

by Lemma 9.4 and

$$D_2 := -\sum_{m+n=2,3}\begin{pmatrix} \partial_t A^{(1)}_{m,n} + iE(\lambda)(m-n)A^{(1)}_{m,n} \\ \partial_t A^{(2)}_{m,n} + iE(\lambda)(m-n)A^{(2)}_{m,n} \end{pmatrix}.$$

We claim that

$$D_1, D_2 = \text{Remainder}. \tag{9-27}$$

If the claim holds then estimates (8-9) and (8-10) follow from (9-26) and the estimates (9-23), (9-25). Next we prove the claim (9-27) together with (9-22).

Since we removed all the second and third order terms of $J\vec N$ we obtain $D_1 = \text{Remainder}$. Recall the estimate of Remainder in (8-8). To estimate $D_2$ we have to start with studying the equation for $z$. By the fact that

$$\partial_t z_n + iE(\lambda)z_n = \mathbb{O}(|z|^2) + \text{Remainder}$$

in (9-8) we obtain $D_2 = \mathbb{O}(|z|^3) + \text{Remainder}$. Hence,

$$\dot\lambda = \mathbb{O}(|z|^3) + \text{Remainder}, \qquad \dot\gamma - \Upsilon_{1,1} = \mathbb{O}(|z|^3) + \text{Remainder}, \tag{9-28}$$

which, together with the expansion of $J\vec N$ in (9-17), yields

$$\partial_t(\alpha_n + p_n) - E(\lambda)(\beta_n + q_n) + \sum_{k+l=2,3}\langle N^{\Im}_{k,l}, \eta_n\rangle = -\tfrac{i}{2}\Upsilon_{1,1}\langle (z-\bar z)\cdot\eta, \eta_n\rangle + \text{Remainder},$$

$$\partial_t(\beta_n + q_n) + E(\lambda)(\alpha_n + p_n) - \sum_{k+l=2,3}\langle N^{\Re}_{k,l}, \xi_n\rangle = -\tfrac{1}{2}\Upsilon_{1,1}\langle (z+\bar z)\cdot\xi, \xi_n\rangle + \text{Remainder},$$

where the real function $\Upsilon_{1,1}$ is defined in (8-10). Choose $p_n$ and $q_n$ as in (9-21) to remove the lower order terms as in the equations for $\dot{\lambda}$ and $\dot{\gamma}$, which, together with the definition $z_n = \alpha_n + i\beta_n$, enables us to obtain

$$\partial_t z_n + iE(\lambda)z_n = -\left\langle JN_{2,1} + \tfrac{1}{2}\Upsilon_{1,1}\begin{pmatrix} iz\cdot\eta \\ z\cdot\xi \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle + D_3(n) + \text{Remainder}$$

with $D_3(n)$ defined as

$$D_3(n) := -\sum_{k+l=2,3}\big(\partial_t P_{k,l}^{(n)} + i(k-l)E(\lambda)P_{k,l}^{(n)}\big) - i\sum_{k+l=2,3}\big(\partial_t Q_{k,l}^{(n)} + i(k-l)E(\lambda)Q_{k,l}^{(n)}\big).$$

We claim that this together with the equations for $\dot{\lambda}$ in (9-23) implies that

$$|D_2|,\ |D_3(n)| = \text{Remainder}. \tag{9-29}$$

Indeed, by (9-8), we have

$$\partial_t z_n + iE(\lambda)z_n = \mathbb{O}(|z|^2) + \text{Remainder},$$

which, together with the equation for $\dot{\lambda}$ in (9-28), implies $D_3 = \mathbb{O}(|z|^3) + \text{Remainder}$. In turn we have an improved equation for $z_n$ as

$$\partial_t z_n = -iE(\lambda)z_n + \mathbb{O}(|z|^3) + \text{Remainder}.$$

Using this and repeating the analysis we find there is no $\mathbb{O}(|z|^3)$ term in $D_2$ and $D_3$. Hence (9-29) holds which leads to (9-22) and (9-27). □

## 10. Proof of the normal form equation (8-11)

Recall the definitions of the functions $B(n)$ and $D(n)$ after (6-2). Then the function $JN_{2,0}$ in (9-12) admits the form

$$JN_{2,0} = \sum_{n=1}^{N} z_n \begin{pmatrix} B(n) \\ D(n) \end{pmatrix}. \tag{10-1}$$

The following is the result establishing the desired normal form of the differential equation for the neutral mode amplitudes $z(t)$.

**Theorem 10.1.** *With polynomials $a_1, a_2, p_n$ and $q_n$ for $n = 1, 2, \ldots, N$ defined in (9-18)–(9-21), (8-11) holds.*

*Proof.* Recall the definitions of $JN_{m,n}$ with $m+n=3$ in (9-15). The first two terms on the right-hand side of (9-22) admit the expansion

$$\sum_{k=1}^{5} K_k(n)$$

where

$$\begin{aligned}
K_1(n) &:= -\Big\langle X_{0,1}R_{2,0}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle = -\Big\langle R_{2,0}, X_{0,1}^*\begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle,\\
K_2(n) &:= -\Big\langle X_{1,0}\begin{pmatrix}\sum_{k=1}^N P_{1,1}^{(k)}\xi_k\\ \sum_{k=1}^N Q_{1,1}^{(k)}\eta_k\end{pmatrix} + X_{0,1}\begin{pmatrix}\sum_{k=1}^N P_{2,0}^{(k)}\xi_k + A_{2,0}^{(1)}\partial_\lambda\phi^\lambda\\ \sum_{k=1}^N Q_{2,0}^{(k)}\eta_k + A_{2,0}^{(2)}\phi^\lambda\end{pmatrix}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle,\\
K_3(n) &:= -\tfrac{1}{8}\Big\langle \big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)((z\cdot\xi)^2-(z\cdot\eta)^2)\begin{pmatrix}i\bar z\cdot\eta\\ -\bar z\cdot\xi\end{pmatrix}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle\\
&\quad+\tfrac{1}{4}\Big\langle \big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(|z\cdot\xi|^2+|z\cdot\eta|^2)\begin{pmatrix}iz\cdot\eta\\ z\cdot\xi\end{pmatrix}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle\\
&\quad-\tfrac{3}{4}i\Big\langle f''[(\phi^\lambda)^2](\phi^\lambda)^2(z\cdot\eta)^2\begin{pmatrix}\bar z\cdot\eta\\ 0\end{pmatrix}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle\\
&\quad-\Big\langle \big(\tfrac{3}{4}f''[(\phi^\lambda)^2](\phi^\lambda)^2+\tfrac{1}{2}f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)(z\cdot\xi)^2\begin{pmatrix}0\\ -\bar z\cdot\xi\end{pmatrix}, \begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle,\\
K_4(n) &:= \tfrac{1}{2}\Upsilon_{1,1}\Big\langle \begin{pmatrix}-iz\cdot\eta\\ z\cdot\xi\end{pmatrix}, \begin{pmatrix}\eta_n\\ i\xi_n\end{pmatrix}\Big\rangle,\\
K_5(n) &:= -\Big\langle R_{1,1}, X_{1,0}^*\begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix}\Big\rangle
\end{aligned}$$

with $X$ defined in (9-16) divided into two terms $X = X_{1,0}+X_{0,1}$:

$$\begin{aligned}
X_{1,0} &:= \begin{pmatrix} -if'[(\phi^\lambda)^2]\phi^\lambda z\cdot\eta & f'[(\phi^\lambda)^2]\phi^\lambda z\cdot\xi\\ -\big(3f'[(\phi^\lambda)^2]\phi^\lambda+2f''[(\phi^\lambda)^2](\phi^\lambda)^3\big)z\cdot\xi & if'[(\phi^\lambda)^2]\phi^\lambda z\cdot\eta\end{pmatrix},\\
X_{0,1} &= \overline{X_{1,0}}
\end{aligned}$$

and the real function $\Upsilon_{1,1}$ given in (8-10).

Next we study $K_j(n)$ for $j = 1,2,3,4,5$. We start with the important term, $K_1(n)$. Recall the definition of $G_n$ in (6-2). By direct computation we obtain

$$X_{0,1}^*\begin{pmatrix}\eta_n\\ -i\xi_n\end{pmatrix} = -iJ\begin{pmatrix}B(n)\\ D(n)\end{pmatrix} = -iJG_n \tag{10-2}$$

which, together with (9-13) and (10-1), implies that

$$K_1(n) = \sum_{k=1}^N z_k\big\langle (L(\lambda)+2iE(\lambda)-0)^{-1}P_cG_k, iJG_n\big\rangle.$$

Define

$$Z(k,n) := -\big\langle (L(\lambda)+2iE(\lambda)-0)^{-1}P_cG_n, iJG_k\big\rangle$$

and a $N\times N$ matrix

$$\Gamma(z,\bar z) := [A(k,l)] \tag{10-3}$$

with $A(k,l) := \frac{1}{2}(Z(k,l)+\overline{Z(l,k)})$ for $1\le k,l\le d$.

For the sum of $K_2(n)$ through $K_5(n)$ we claim that it can decomposed into the matrix form

$$\sum_{j=2}^{5} K_j(n) = (S(n,1), S(n,2), \ldots, S(n,d))z \tag{10-4}$$

with $S(k,l) + \overline{S(l,k)} = 0$. Define a $N \times N$ skew symmetric matrix

$$\Lambda(z,\bar{z}) := [\Lambda(j,k)]$$

with $\Lambda(j,k) := S_{k,l} + \frac{1}{2}(Z(k,l) - \overline{Z(l,k)})$. This together with (9-22) and (10-3) yields the equation for $z$ in (8-11)

What is left is to prove (10-4). To avoid the tedious but simple computations, we only analyze $K_2(n)$ and $K_3(n)$.

(A) Consider the part of $K_2(n)$ given by

$$\Psi_{2,1}(n) := -\left\langle X_{0,1}\begin{pmatrix} A^{(1)}_{2,0}\partial_\lambda\phi^\lambda \\ A^{(2)}_{2,0}\phi^\lambda \end{pmatrix}, \begin{pmatrix} \eta_n \\ -i\xi_n \end{pmatrix}\right\rangle.$$

The analysis of the other terms is similar. By (10-2) we rewrite $\Psi_{2,1}(n)$ as

$$\Psi_{2,1}(n) = \left\langle\begin{pmatrix} A^{(1)}_{2,0}\partial_\lambda\phi^\lambda \\ A^{(2)}_{2,0}\phi^\lambda \end{pmatrix}, 4i\begin{pmatrix} D(n) \\ -B(n) \end{pmatrix}\right\rangle = -4iA^{(1)}_{2,0}\langle\partial_\lambda\phi^\lambda, D(n)\rangle + 4iA^{(2)}_{2,0}\langle\phi^\lambda, B(n)\rangle.$$

Equation (9-19) relates $\langle N^{\Re}_{2,0}, \phi^\lambda_\lambda\rangle$ and $\langle N^{\Im}_{2,0}, \phi^\lambda\rangle$ to $A^{(1)}_{2,0}$ and $A^{(2)}_{2,0}$, which, together with the expression of $JN_{2,0}$ in (10-1), yields

$$\Psi_{2,1}(n) = \sum_{k=1}^{N} \Psi(n,k)z_k \tag{10-5}$$

with

$$\Psi(n,k) := \frac{2}{E(\lambda)\langle\phi^\lambda, \partial_\lambda\phi^\lambda\rangle}\big(\langle B(k), \phi^\lambda\rangle\langle\partial_\lambda\phi^\lambda, D(n)\rangle - \langle D(k), D_\lambda\phi^\lambda\rangle\langle\phi^\lambda, B(n)\rangle\big) \\ + \frac{i}{E^2(\lambda)\langle\phi^\lambda, \partial_\lambda\phi^\lambda\rangle}\langle B(k), \phi^\lambda\rangle\langle\phi^\lambda, B(n)\rangle.$$

By straightforward computation we have

$$\Psi(n,k) + \overline{\Psi(k,n)} = 0. \tag{10-6}$$

By (10-5) and (10-6) we complete the proof for $\Psi_{2,1}(n)$.

(B) To simplify the notation we introduce

$$\rho := \tfrac{1}{2}(z\cdot\xi) = \tfrac{1}{2}\sum_{n=1}^{N} z_n\xi_n, \quad \omega := \tfrac{1}{2}(z\cdot\eta) = \tfrac{1}{2}\sum_{n=1}^{N} z_n\eta_n.$$

This implies that

$$\rho^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n\xi_n\rho,\ \ \omega^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n\eta_n\omega,\ \ \rho\bar\rho+\omega\bar\omega=\tfrac{1}{2}\sum_{n=1}^{N} z_n(\xi_n\bar\rho+\eta_n\bar\omega),\ \ \rho^2-\omega^2=\tfrac{1}{2}\sum_{n=1}^{N} z_n(\rho\xi_n-\omega\eta_n).$$

By the definition of $K_3(n)$ it is not hard to get

$$K_3(n)=\sum_{k=1}^{N} z_k\Phi(n,k) \tag{10-7}$$

where

$$\begin{aligned}\Phi(n,k):=\tfrac{i}{2}\big\langle\big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(\rho\xi_k-\omega\eta_k),(\rho\xi_n-\omega\eta_n)\big\rangle\\ +i\big\langle\big(f'[(\phi^\lambda)^2]+2f''[(\phi^\lambda)^2](\phi^\lambda)^2\big)(\bar\omega\eta_k+\bar\rho\xi_k),(\bar\omega\eta_n+\bar\rho\xi_n)\big\rangle\\ +i\big\langle\big(3f''[(\phi^\lambda)^2](\phi^\lambda)^2+2f'''[(\phi^\lambda)^2](\phi^\lambda)^4\big)\rho\xi_k,\rho\xi_n\big\rangle\\ -i\langle 3f''[(\phi^\lambda)^2](\phi^\lambda)^2\omega\eta_k,\omega\eta_n\rangle.\end{aligned}$$

Immediately we have

$$\Phi(n,k)+\overline{\Phi(k,n)}=0.$$

This together with (10-7) completes the proof for $K_3(n)$. □

## 11. Proof of Theorem 7.1

For simplicity, we present the proof of Theorem 7.1 for the case $d=3$; the proof can be easily modified to cover $d\ge 3$. The main difference is that, in controlling $\|\vec R(t)\|_{L^\infty(\mathbb{R}^d)}$ by $\|\vec R(t)\|_{H^k(\mathbb{R}^d)}$ for $d=3$ we take $k=2$ while in general we need $k=[d/2]+1$; see Section 5B.

***Estimation strategy.*** This subsection discusses our strategy for studying the large time behavior of solutions.

We begin by introducing a family of space-time norms for measuring the decay of $z(t)$ and $\vec R(t)$ for $0\le t\le T$ with arbitrary $T$. We then prove that this family of norms satisfies a set of coupled inequalities, from which we can infer the desired large time asymptotic behavior.

We claim that

$$T_0:=\big|z^{(0)}\big|^{-1} \tag{11-1}$$

where $z^{(0)}$ is defined in Theorem 7.1.

*Family of Norms.*

$$\begin{aligned}Z(T):=\max_{t\le T}(T_0+t)^{1/2}|z(t)|,\quad &\mathcal{R}_1(T):=\max_{t\le T}(T_0+t)\|\langle x\rangle^{-\nu}\vec R(t)\|_{H^2},\\ \mathcal{R}_2(T):=\max_{t\le T}(T_0+t)\|\vec R(t)\|_\infty,\quad &\mathcal{R}_3(T):=\max_{t\le T}(T_0+t)^{7/5}\|\langle x\rangle^{-\nu}\tilde R(t)\|_2,\\ \mathcal{R}_4(T):=\max_{t\le T}\|\vec R(t)\|_{H^2},\quad &\mathcal{R}_5(T):=\max_{t\le T}\frac{(T_0+t)^{1/2}}{\log(T_0+t)}\|\vec R(t)\|_3\end{aligned} \tag{11-2}$$

where the constant $\nu$ is defined in Theorem 7.1.

*Remark on choice of norms.* It is clear that a combination of $H^2$, spatially weighted $H^2$ and $L^\infty$ norms of $\vec{R}(t)$, as well as a bound on $|z(t)|$, are plausible choices of norms to control the large time behavior. This accounts for the definitions of $Z(T)$, $\mathcal{R}_1(T)$, $\mathcal{R}_2(T)$ and $\mathcal{R}_4(T)$. Our list of norms also includes estimation of the time decay of $\|\vec{R}(t)\|_3$, that is, $\mathcal{R}_5$, and the local $L^2$ norm of an auxiliary function $\tilde{R}(t)$, that is, $\mathcal{R}_3$. Why these two additional norms? As will be seen, $\zeta(t) = |z(t)|$ satisfies an equation of the form

$$\dot{\zeta} \sim -\kappa^2 \zeta^3 + c(t),$$

where $c(t)$ consists of various coupling terms (products) involving neutral mode amplitudes $z(t)$, the ground state $\phi^{\lambda(t)}$ and dispersive terms $\vec{R}(t)$. First, neglecting $c(t)$, we observe that $\zeta(t) \sim t^{-1/2}$. To treat $c(t)$ as a small perturbation for large $t$, it is necessary that it decays more rapidly than the term $\zeta^3(t) \sim t^{-3/2}$. Without any further decomposition of $\vec{R}(t)$, we find among the coupling terms one is of order $|z(t)| \cdot \|\langle x\rangle^{-\nu}\ \vec{R}(t)\|_2$. The expected decay rate of each factor implies this term is of order $t^{-3/2}$ for large $t$, which is of the *same* order as $\zeta^3(t)$. The resolution is to expand $\vec{R}(t)$ as a leading order part consisting of terms $R_{m,n} = z^m \bar{z}^n$ with $m + n = 2$ plus a more rapidly decaying correction $\tilde{R}(t)$ with $\|\langle x\rangle^{-\nu} \tilde{R}(t)\|_2 = \mathbb{O}(t^{-1-\delta})$ $(\delta > 0)$; see (8-1). This modification yields an equation with an improved correction term of order $|z(t)| \cdot \|\langle x\rangle^{-\nu} \tilde{R}(t)\|_2 = t^{-\frac{3}{2}-\delta}$ $(\delta > 0)$, which can be treated as a small perturbation in the large time dynamics.

*Remark on the estimation strategy.* See also [Buslaev and Sulem 2003; Soffer and Weinstein 2004]. Estimation of the norms $\mathcal{R}_j(T)$ proceeds as follows. We first express $\vec{R}$, the solution to Equation (9-2), in terms of the Duhamel integral equation, relative to the linear operator, $L(\lambda_1)$. Here, $\lambda_1 = \lambda(T)$, $T > 0$ is fixed and arbitrary. Namely,

$$\begin{aligned} \partial_t \vec{R} = L(\lambda)\vec{R} + \cdots &\implies \partial_t P_c^{\lambda_1}\vec{R} = L(\lambda_1) P_c^{\lambda_1}\vec{R} + P_c^{\lambda_1}\big(L(\lambda(t)) - L(\lambda_1)\big)\vec{R} + \cdots \\ &\implies P_c^{\lambda_1}\vec{R}(t) = e^{L(\lambda_1)t}\vec{R}(0) + \int_0^t e^{L(\lambda_1)(t-s)}(\cdots)\, ds. \end{aligned}$$

We can therefore apply the time-decay estimates of Theorem 5.7 to obtain bounds on local decay and $L^\infty$ norms of $P_c^{\lambda_1}\vec{R}(t)$. However, we need bounds on $\vec{R}(t) = P_c^{\lambda(t)}\vec{R}(t)$. Since

$$\vec{R}(t) = P_c^{\lambda_1}\vec{R}(t) + P_{\text{disc}}^{\lambda_1}\vec{R}(t),$$

it suffices to bound $P_{\text{disc}}^{\lambda_1}\vec{R}(t)$. This is done as follows.

$$\begin{aligned} P_{\text{disc}}^{\lambda_1} R &= (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})R(t) + P_{\text{disc}}^{\lambda(t)}R(t) = (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})R(t) \qquad (\text{because } P_{\text{disc}}^{\lambda(t)}R(t) = 0) \\ &= (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})P_{\text{disc}}^{\lambda_1}R(t) + (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})P_c^{\lambda_1}R(t), \end{aligned}$$

which implies

$$\big(I - (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})\big)P_{\text{disc}}^{\lambda_1}R = (P_{\text{disc}}^{\lambda_1} - P_{\text{disc}}^{\lambda(t)})P_c^{\lambda_1}R(t).$$

Therefore,

$$P_{\text{disc}}^{\lambda_1}R(t) = (I - \delta(\lambda, \lambda_1))^{-1}\delta(\lambda, \lambda_1)P_c^{\lambda_1}R(t)$$

and we estimate $R(t)$ in either a local energy $H^2(\mathbb{R}^d; \langle x\rangle^{-\sigma}\, dx)$ or $L^\infty(\mathbb{R}^d)$ via

$$\|R(t)\|_X \le \|P_c^{\lambda_1}R(t)\|_X + \|P_{\text{disc}}^{\lambda_1}R(t)\|_X \le \|P_c^{\lambda_1}R(t)\|_X + \|P_c^{\lambda_1}R(t)\|_X.$$

Here $\delta(\lambda,\lambda_1)=P^{\lambda_1}_{\mathrm{disc}}-P^{\lambda(t)}_{\mathrm{disc}}$ is of finite rank and of small norm proportional to $\int_t^T|\dot\lambda(s)|\,ds$.

We now derive the integral equation for $P_c^{\lambda_1}\vec R$, which is the basis for our time-decay estimates. If we write

$$L(\lambda(t))=L(\lambda_1)+L(\lambda(t))-L(\lambda_1),$$

then (9-9) for $P_c^{\lambda_1}\vec R$, which takes the form

$$\partial_t P_c^{\lambda_1}\vec R=L(\lambda_1)P_c^{\lambda_1}\vec R+(\lambda-\lambda_1+\dot\gamma)P_c^{\lambda_1}J\vec R+\cdots.$$

Recall that $L(\lambda)$ has two branches of essential spectrum $[i\lambda,i\infty)$ and $(-i\infty,-i\lambda]$. We use $P_+$ and $P_-$ to denote the projection operators onto these two branches of the essential spectrum of $L(\lambda_1)$.

**Lemma 11.1.** *For any function $h$ and any large constant $\nu>0$, we have*

$$\big\|\langle x\rangle^{\nu}(-\Delta+1)(P_c^{\lambda_1}J-i(P_+-P_-))h\big\|_2\le c\|\langle x\rangle^{-\nu}(-\Delta+1)h\|_2.$$

For $d=1$ the proof of this lemma can be found in [Buslaev and Sulem 2003]; the proof for $d\ge 3$ is similar, hence omitted here.

Equation (9-9) can be rewritten as

$$\partial_t P_c^{\lambda_1}\vec R=L(\lambda_1)P_c^{\lambda_1}\vec R+i(\dot\gamma+\lambda-\lambda_1)(P_+-P_-)\vec R+P_c^{\lambda_1}O_1\vec R+P_c^{\lambda_1}P_c^{\lambda(t)}\mathcal{G}-P_c^{\lambda_1}P_c^{\lambda(t)}JN(\vec R,z),\quad (11\text{-}3)$$

where $O_1$ is the operator defined by

$$O_1:=\dot\lambda P_{c\lambda}+L(\lambda)-L(\lambda_1)+\dot\gamma P_c^{\lambda}J-i(\dot\gamma+\lambda-\lambda_1)(P_+-P_-).\quad (11\text{-}4)$$

By (11-3) and the observation that the operators $P_+$, $P_-$ and $L(\lambda_1)$ commute with each other, we have

$$P_c^{\lambda_1}\vec R=e^{tL(\lambda_1)+a(t,0)(P_+-P_-)}P_c^{\lambda_1}\vec R(0)+\int_0^t e^{(t-s)L(\lambda_1)+a(t,s)(P_+-P_-)}P_c^{\lambda_1}\big(O_1\vec R+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec R,z)\big)\,ds \quad (11\text{-}5)$$

with $a(t,s)=i\int_s^t(\dot\gamma(\tau)+\lambda(\tau)-\lambda_1)\,d\tau$. We observe that $P_+P_-=P_-P_+=0$ and for any $t_1\le t_2$ the operator

$$e^{a(t_2,t_1)(P_+-P_-)}=e^{a(t_2,t_1)}P_++e^{-a(t_2,t_1)}P_-:\ H^2\to H^2$$

is uniformly bounded.

The following result, whose proof is given in the Appendix F, will be used repeatedly in our estimates:

**Proposition 11.2.** *Let $T_0\ge 2$. There exists a constant $c>0$ such that*

$$\int_0^t\frac{1}{(1+t-s)^{3/2}}\frac{1}{(T_0+s)^{\sigma}}\,ds\le\frac{c}{(T_0+t)^{\sigma}},\qquad \sigma\in[0,\tfrac32],\quad (11\text{-}6)$$

$$\int_0^t(t-s)^{-1/2}(T_0+s)^{-1}\,ds\le c(T_0+t)^{-1/2}\log(T_0+t).\quad (11\text{-}7)$$

Similar versions can be found in many literature, for example [Soffer and Weinstein 1999; Buslaev and Sulem 2003].

***Estimate for*** $\mathcal{R}_1(T) := \max_{t\le T}(T_0+t)\|\langle x\rangle^{-\nu}\vec{R}(t)\|_{H^k}$.

**Proposition 11.3.** $\mathcal{R}_1 \le c\big(T_0\|\langle x\rangle^{\nu}\vec{R}(0)\|_{H^2} + \mathcal{R}_4^2\mathcal{R}_2 + Z^2 + T_0^{-1/2}(Z^3 + Z\mathcal{R}_1 + \mathcal{R}_4\mathcal{R}_2^2)\big)$.

With a view toward proving the time decay estimate of Proposition 11.3, we now first give appropriate norm-estimates of the latter terms in (11-3).

First from the norm definitions (11-2) and Lemma 11.1, we estimate the $O_1\vec{R}$ and $\mathcal{G}$ terms

$$\begin{aligned}\|\langle x\rangle^{\nu}(-\Delta+1)O_1\vec{R}\|_2 &\le c(T_0+t)^{-3/2}Z\mathcal{R}_1,\\ \|\langle x\rangle^{\nu}(-\Delta+1)\mathcal{G}\|_2 &\le c(T_0+t)^{-3/2}Z^3.\end{aligned} \tag{11-8}$$

Next, we estimate the nonlinear term $JN$:

**Lemma 11.4.**

$$\|(-\Delta+1)JN(\vec{R},z)\|_1+\|(-\Delta+1)JN(\vec{R},z)\|_2 \le c(T_0+t)^{-1}(\mathcal{R}_4^2\mathcal{R}_2+Z^2)+c(T_0+t)^{-3/2}(Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2).$$

*Proof.* Recall the definition

$$N_2(\vec{R},z) := -J\vec{N}(\vec{R},z) + \sum_{m+n=2,3} JN_{m,n}$$

in (9-17) and the decomposition $N_2$ as the sum of Loc and NonLoc in (8-5). By the fact $JN_{m,n}$ for $m+n=2,3$ are localized functions we have the estimate

$$\begin{aligned}&\|(-\Delta+1)(JN(\vec{R},z)-\mathrm{NonLoc})\|_1+\|(-\Delta+1)(JN(\vec{R},z)-\mathrm{NonLoc})\|_2\\ &\qquad\le c|z|(|z|+\|\langle x\rangle^{-\nu}\vec{R}\|_2) \le c((T_0+t)^{-1}Z^2+(T_0+t)^{-3/2}Z\mathcal{R}_1).\end{aligned}$$

More challenging is the term $NonLoc$ defined in (8-7), which is purely nonlinear, having no spatially localized factors. We use the estimate

$$\begin{aligned}&\|(-\Delta+1)\mathrm{NonLoc}\|_1+\|(-\Delta+1)\mathrm{NonLoc}\|_2\\ &\qquad\le c(\|\vec{R}\|_{H^2}^2\|\vec{R}\|_\infty+\|\vec{R}\|_{H^2}\|\vec{R}\|_\infty^2) \le c(T_0+t)^{-1}\mathcal{R}_4^2\mathcal{R}_2+c(T_0+t)^{-3/2}\mathcal{R}_4\mathcal{R}_2^2\end{aligned}$$

by the fact $f(x^2)x$ is of the order $x^3$ around $x=0$ for $d=3$. □

*Proof of Proposition 11.3.* By (11-5) and estimates (5-7), (5-11) for $d=3$ we have

$$\begin{aligned}&\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}(t)\|_2\\ &\quad\le \|\langle x\rangle^{-\nu}(-\Delta+1)e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_2\\ &\qquad+\left\|\int_0^t\langle x\rangle^{-\nu}(-\Delta+1)e^{(t-s)L(\lambda_1)}P_c^{\lambda_1}(O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec{R},z))\,ds\right\|_2\\ &\quad\le c(1+t)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)\vec{R}(0)\|_2+\int_0^t(1+t-s)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)(O_1\vec{R}+P_c^{\lambda}\mathcal{G})\,ds\|_2\\ &\qquad+\int_0^t(1+t-s)^{-3/2}\big(\|(-\Delta+1)P_c^{\lambda}JN(\vec{R}(s),z)\|_1+\|(-\Delta+1)P_c^{\lambda}JN(\vec{R}(s),z)\|_2\big)\,ds.\end{aligned}$$

Therefore, by the estimates (11-8) and Lemma 11.4 we have

$$\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}\|_2 \le c(1+t)^{-3/2}\|\langle x\rangle^{\nu}(-\Delta+1)\vec{R}(0)\|_2 + \big(\mathcal{R}_4^2\mathcal{R}_2+Z^2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2)\big)\int_0^t(1+t-s)^{-3/2}(T_0+s)^{-1}\,ds.$$

Using the time convolution estimate (11-6) we obtain

$$\|\langle x\rangle^{-\nu}(-\Delta+1)P_c^{\lambda_1}\vec{R}\|_2 \le c(T_0+t)^{-1}\big(T_0\|\langle x\rangle^{\nu}\vec{R}(0)\|_2+\mathcal{R}_4^2\mathcal{R}_2+Z^2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_4\mathcal{R}_2^2)\big).$$

This implies Proposition 11.3. □

***Estimate for $\mathcal{R}_2(T) := \max_{t\le T}(T_0+t)\|\vec{R}(t)\|_\infty$.***

**Proposition 11.5.** $\mathcal{R}_2 \le c\big(T_0(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})+Z^2+\mathcal{R}_4^2\mathcal{R}_2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2)\big)$.

To prove this we use the following result whose proof is very similar to that of Lemma 11.4.

**Lemma 11.6.**

$$\|P_c^{\lambda}JN(\vec{R},z)\|_1+\|P_c^{\lambda}JN(\vec{R},z)\|_{H^2} \le c(T_0+t)^{-1}(Z^2+\mathcal{R}_4^2\mathcal{R}_2)+c(T_0+t)^{-3/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2).$$

*Proof of Proposition 11.5.* By estimate (5-9) for $d=3$ and (11-3) we have that

$$\begin{aligned}\|P_c^{\lambda_1}\vec{R}(t)\|_\infty &\le \|e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_\infty\\ &\le c(1+t)^{-3/2}(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})\\ &\quad + c\int_0^t(1+t-s)^{-3/2}(\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_1+\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{H^2})\,ds\\ &\quad + c\int_0^t(1+t-s)^{-3/2}(\|P_c^{\lambda}JN(\vec{R},z)\|_1+\|P_c^{\lambda}JN(\vec{R},z)\|_{H^2})\,ds.\end{aligned}$$

By the properties of $O_1$ (see (11-4)) and $\mathcal{G}$ (see (9-9)) we have

$$\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_1+\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{H^2} \le c(T_0+t)^{-3/2}(Z\mathcal{R}_1+Z^3).$$

This, together with Lemma 11.6, yields

$$\|P_c^{\lambda_1}\vec{R}(t)\|_\infty \le c(T_0+t)^{-1}\big(T_0\|\vec{R}(0)\|_1+T_0\|\vec{R}(0)\|_{H^2}+Z^2+\mathcal{R}_4^2\mathcal{R}_2+T_0^{-1/2}(Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2)\big).$$ □

***Estimate for $\mathcal{R}_5(T) := \max_{t\le T}\dfrac{(T_0+t)^{1/2}}{\log(T_0+t)}\|\vec{R}(t)\|_3$@.***

**Proposition 11.7.** $\mathcal{R}_5 \le c\big(T_0(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})+Z^2+T_0^{-1/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)\big)$.

**Lemma 11.8.** $\|JN(\vec{R},z)\|_{3/2} \le c(T_0+t)^{-1}Z^2+c(T_0+t)^{-3/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)$.

*Proof.* As in the proof of Lemma 11.4 we decompose $JN$ into the localized term

$$JN(\vec{R},z)-\text{NonLoc}$$

and nonlocalized term NonLoc. The estimate of the first term is similar to that of Lemma 11.4 and hence omitted. The nonlocal term NonLoc defined in (8-7) admits the estimate

$$\|\mathrm{NonLoc}\|_{3/2} \le c\Big(\int |\vec{R}|^{5/4}\Big)^{2/3} \le c\|\vec{R}\|_3^2\|\vec{R}\|_\infty.$$

By using the definitions of estimating functions on all the terms above we have the lemma. □

*Proof of Proposition 11.7.* By estimate (5-10) for $d=3$ and Lemma 11.4 we have that

$$\begin{aligned}\|P_c^{\lambda_1}\vec{R}(t)\|_3 &\le \|e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_3 + \int_0^t \|e^{(t-s)L(\lambda_1)}P_c^{\lambda_1}(O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}-P_c^{\lambda}JN(\vec{R},z))\|_3\,ds\\ &\le c(1+t)^{-1/2}(\|\vec{R}(0)\|_1+\|\vec{R}(0)\|_{H^2})\\ &\qquad + c\int_0^t (t-s)^{-1/2}\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{3/2}\,ds + \int_0^t (t-s)^{-1/2}\|P_c^{\lambda}JN(\vec{R},z)\|_{3/2}\,ds.\end{aligned}$$

By the properties of $O_1$ (see (11-4)) and $\mathcal{G}$ (see (9-9)) we have

$$\|O_1(s)\vec{R}+P_c^{\lambda}\mathcal{G}\|_{3/2} \le c(T_0+t)^{-3/2}(Z\mathcal{R}_1+Z^3).$$

This, together with Lemma 11.6 and (11-7), implies

$$\begin{aligned}&\|P_c^{\lambda_1}\vec{R}(t)\|_3\\ &\le c(T_0+t)^{-1/2}\log(T_0+t)\big(T_0^{1/2}\|\vec{R}(0)\|_1+T_0\|\vec{R}(0)\|_{H^2}+Z^2+T_0^{-1/2}(\mathcal{R}_5^2\mathcal{R}_2+Z^3+Z\mathcal{R}_1+\mathcal{R}_1^2+\mathcal{R}_2^2)\big).\end{aligned}$$

This estimate and the definition of $\mathcal{R}_5$ yield the proposition. □

***Estimate for*** $\boldsymbol{\mathcal{R}_3(T) := \max_{t\le T}(T_0+t)^{7/5}\|\langle x\rangle^{-\nu}\tilde{R}(t)\|_2}$.

**Proposition 11.9.** *Let the constant $\nu$ the same as that in (5-6) and (5-7) with $d=3$. Then*

$$\mathcal{R}_3 \le c\big(T_0^{3/2}(\|\langle x\rangle^{\nu}\vec{R}(0)\|_2+|z(0)|^2)\big)+cT_0^{-1/20}(Z^3+Z\mathcal{R}_3+Z\mathcal{R}_1+\mathcal{R}_5^3+R_2^2\mathcal{R}_4).$$

As usual we estimate the nonlinear term $N_2(\vec{R},z)$.

**Lemma 11.10.** $\displaystyle\int_0^t \|\langle x\rangle^{-\nu}e^{(t-s)L(\lambda)}P_c^{\lambda_1}P_c^{\lambda}N_2(\vec{R},z)\|_2\,ds \le c(T_0+t)^{-7/5}T_0^{-1/20}(Z^3+Z\mathcal{R}_1+\mathcal{R}_5^3+\mathcal{R}_2^2\mathcal{R}_4).$

*Proof.* We start with the function $N_2$. Recall that $N_2 = \mathrm{Loc}+\mathrm{NonLoc}$ in (8-5) and the estimate of Loc after that. The nonlocal term NonLoc defined in (8-7) admits the estimate

$$\|\mathrm{NonLoc}\|_1+\|\mathrm{NonLoc}\|_2 \le c(\|\vec{R}\|_3^3+\|\vec{R}\|_6^3) \le c(\|\vec{R}\|_3^3+\|\vec{R}\|_\infty^2\|\vec{R}\|_2).$$

By the definition of estimating function we have

$$\begin{aligned}\|\mathrm{NonLoc}\|_1+\|\mathrm{NonLoc}\|_2 &\le c(T_0+t)^{-3/2}(\log(T_0+t))^{3/2}\mathcal{R}_5^3+(T_0+t)^{-2}\mathcal{R}_2^2\mathcal{R}_4\\ &\le c(T_0+t)^{-7/5}T_0^{-1/20}(\mathcal{R}_5^3+\mathcal{R}_2^2\mathcal{R}_4).\end{aligned}$$

Finally, by the propagator estimates (5-9) and (5-11), we have

$$\begin{aligned}\int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda)} P_c^{\lambda_1} P_c^{\lambda} N_2(\vec{R},z)\|_2\,ds \\ \leq c\int_0^t (1+t-s)^{-3/2}\big(\|\mathrm{NonLoc}(\vec{R},z)\|_1+\|\mathrm{NonLoc}(\vec{R},z)\|_2+\|\langle x\rangle^{\nu}\mathrm{Loc}\|_2\big)\,ds.\end{aligned}$$

This together with the estimates of Loc and NonLoc above yields the lemma. □

*Proof of Proposition 11.9.* By the same techniques as in deriving (11-3) we have

$$\partial_t P_c^{\lambda_1}\tilde{R} = L(\lambda_1)P_c^{\lambda_1}\tilde{R} + i(\dot{\gamma}+\lambda-\lambda_1)(P_+-P_-)\tilde{R} + P(z,\bar{z})\tilde{R} + P_c^{\lambda_1}S_2(z,\bar{z}) + P_c^{\lambda_1}P_c^{\lambda}N_2(\vec{R},z), \tag{11-9}$$

where the operator $P(z,\bar{z})$ is defined as

$$P(z,\bar{z}) := P_c^{\lambda_1}M_2(z,\bar{z}) - i(\dot{\gamma}+\lambda-\lambda_1)(P_+-P_-) + P_c^{\lambda_1}(L(\lambda)-L(\lambda_1))$$

and the terms $P_c^{\lambda}N_2(\vec{R},z)$, $S_2(z,\bar{z})$, $M_2(z,\bar{z})$ are defined in Theorem 8.1.

Rewrite (11-9) in the integral form by the Duhamel principle to obtain

$$\begin{aligned}\|\langle x\rangle^{-\nu}P_c^{\lambda_1}\tilde{R}(t)\|_2 \leq \|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}\tilde{R}(0)\|_2 \\ +\int_0^t \big\|\langle x\rangle^{-\nu}e^{(t-s)L(\lambda_1)}\big(P(z,\bar{z})\tilde{R}+P_c^{\lambda_1}S_2(z,\bar{z})+P_c^{\lambda_1}P_c^{\lambda}N_2(\vec{R},z)\big)\big\|_2\,ds.\end{aligned} \tag{11-10}$$

For the left-hand side we claim that

$$\|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}\tilde{R}(0)\|_2 \leq c(1+t)^{-3/2}(\|\langle x\rangle^{\nu}\vec{R}(0)\|_2+|z(0)|^2). \tag{11-11}$$

Indeed recall that

$$\tilde{R} = \vec{R} - \sum_{m+n=2} R_{m,n}$$

with $R_{m,n}$ defined in (9-13). Therefore, with the time-dependent of $\tilde{R}$, $\lambda$ and $z$, we have

$$\|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}\tilde{R}(0)\|_2 \leq \|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_2 + \sum_{m+n=2}\|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}R_{m,n}(0)\|_2.$$

By (5-6) and the fact that $R_{m,n}$ is the summation of terms of order $|z|^2$ we have

$$\|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}R_{m,n}(0)\|_2 \leq c|z(0)|^2(1+t)^{-3/2}.$$

This, together with the estimate

$$\|\langle x\rangle^{-\nu}e^{tL(\lambda_1)}P_c^{\lambda_1}\vec{R}(0)\|_2 \leq c(1+t)^{-3/2}\|\langle x\rangle^{\nu}\vec{R}(0)\|_2,$$

implies (11-11).

Use (5-6) on the right-hand side of (11-10) to obtain

$$\int_0^t \big\|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)}\big(P(z,\bar z)\tilde R + P_c^{\lambda_1} S_2(z,\bar z) + P_c^{\lambda_1} P_c^{\lambda} N_2(\vec R, z)\big)\big\|_2\, ds$$
$$\le \int_0^t (1+t-s)^{-3/2}\big(\|\langle x\rangle^{\nu} P(z,\bar z)\tilde R\|_2 + \|N_2(\vec R,z)\|_1 + \|N_2(\vec R,z)\|_2\big)\, ds$$
$$+ \int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} S_2(z,\bar z)\|_2\, ds.$$

We estimate these terms in detail:

(A) By the definition of $S_2(z,\bar z)$ in (8-3) and estimate (5-6) with $d=3$ we have that

$$\left|\int_0^t \|\langle x\rangle^{-\nu} e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} S_2(z,\bar z)\|_2\, ds\right| \le cZ^3 \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-3/2}\, ds \le cZ^3 (T_0+t)^{-3/2}.$$

(B) By the definition of $P(z,\bar z)$ and the estimate of $M_2(z,\bar z)$ in (8-4),

$$\|\langle x\rangle^{\nu} P(z(s),\bar z(s))\tilde R(s)\|_2 \le c|z|\cdot\|\langle x\rangle^{-\nu}\tilde R(s)\|_2 \le c(T_0+s)^{-19/20} Z\mathcal{R}_3.$$

Hence by (11-6),

$$\int_0^t (1+t-s)^{-3/2}\|\langle x\rangle^{\nu} P(z(s),\bar z(s))\tilde R\|_2\, ds$$
$$\le cT_0^{-1/20} Z\mathcal{R}_3 \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-7/5}\, ds \le cT_0^{-1/20} Z\mathcal{R}_3 (T_0+t)^{-7/5}.$$

These, together with Lemma 11.10, implies

$$\|\langle x\rangle^{-\nu} P_c^{\lambda_1}\tilde R\|_2$$
$$\le c(1+t)^{-3/2}(\|\langle x\rangle^{\nu}\vec R(0)\|_2 + |z(0)|^2) + c(T_0+t)^{-7/5} T_0^{-1/20}(Z\mathcal{R}_1 + Z\mathcal{R}_3 + Z^3 + \mathcal{R}_5^3 + \mathcal{R}_2^2\mathcal{R}_4)$$
$$\le c(T_0+t)^{-7/5}\big(T_0^{7/5}\|\langle x\rangle^{\nu}\vec R(0)\|_2 + T_0^{7/5}|z(0)|^2 + T_0^{-1/20}(Z\mathcal{R}_1 + Z\mathcal{R}_3 + Z^3 + \mathcal{R}_5^3 + \mathcal{R}_2^2\mathcal{R}_4)\big),$$

which implies the proposition. □

**Proposition 11.11.** $\mathcal{R}_4^2 \le \|\vec R(0)\|_{H^2}^2 + cT_0^{-1}(\mathcal{R}_1^2 + Z^2\mathcal{R}_1 + Z^2\mathcal{R}_1^2 + \mathcal{R}_4^2\mathcal{R}_2^2).$

Before the proof we estimate the nonlinear terms.

**Lemma 11.12.** $\big|\big\langle(-\Delta+1)P_c^{\lambda} JN(\vec R,z), (-\Delta+1)\vec R\big\rangle\big| \le c(T_0+t)^{-2}(Z^2\mathcal{R}_1 + \mathcal{R}_4^2\mathcal{R}_2^2).$

*Proof.* As in Lemma 11.4 we decompose $J\vec N$ into the localized term Loc and the nonlocalized NonLoc defined in (8-7). The Localized part satisfies the estimate

$$|\langle(-\Delta+1)\mathrm{Loc}, (-\Delta+1)\vec R\rangle| \le c(|z|^2 + \|\langle x\rangle^{-\nu}\vec R\|_2^2).$$

By the definition of NonLoc in (8-7) we obtain

$$|\langle(-\Delta+1)\mathrm{NonLoc}, (-\Delta+1)\vec R\rangle| \le c\|(-\Delta+1)\vec R\|_2^2\|\vec R\|_\infty^2.$$

This together with the definitions of estimating functions implies the lemma. □

*Proof of Proposition 11.11.* By (9-9) we have

$$\partial_t \langle(-\Delta+1)\vec{R},(-\Delta+1)\vec{R}\rangle = \Big\langle(-\Delta+1)\frac{d}{dt}\vec{R},(-\Delta+1)\vec{R}\Big\rangle + \Big\langle(-\Delta+1)\vec{R},(-\Delta+1)\frac{d}{dt}\vec{R}\Big\rangle = \sum_{n=1}^{4} K_n,$$

with

$$\begin{aligned}
K_1 &:= \langle(-\Delta+1)(L(\lambda)+\dot{\gamma}J)\vec{R},(-\Delta+1)\vec{R}\rangle + \langle(-\Delta+1)\vec{R},(-\Delta+1)(L(\lambda)+\dot{\gamma}J)\vec{R}\rangle,\\
K_2 &:= \dot{\lambda}\langle(-\Delta+1)P_{c\lambda}\vec{R},(-\Delta+1)\vec{R}\rangle + \dot{\lambda}\langle(-\Delta+1)\vec{R},(-\Delta+1)P_{c\lambda}\vec{R}\rangle,\\
K_3 &:= -\langle(-\Delta+1)P_c^{\lambda}JN(\vec{R},z),(-\Delta+1)\vec{R}\rangle - \langle(-\Delta+1)\vec{R},(-\Delta+1)P_c^{\lambda}JN(\vec{R},z)\rangle,\\
K_4 &:= \langle(-\Delta+1)P_c^{\lambda}\mathcal{G},(-\Delta+1)\vec{R}\rangle + \langle(-\Delta+1)\vec{R},(-\Delta+1)P_c^{\lambda}\mathcal{G}\rangle.
\end{aligned}$$

Recall the definition of the operator $L(\lambda)$ in (5-1). By the observation $J^* = -J$ and the fact that $JL(\lambda)$ is selfadjoint we cancel all the nonlocal terms in $K_1$:

$$|K_1| \le c\|\langle x\rangle^{-\nu}\vec{R}\|_{H^2}^2 \le c(T_0+t)^{-2}\mathcal{R}_1^2.$$

By observing that $|\dot{\lambda}| = \mathcal{O}(|z|^2)$ and $P_{c\lambda}\vec{R}$ is localized we have that

$$|K_2| \le c|z(t)|^2\|\langle x\rangle^{-\nu}\vec{R}(t)\|_{H^2}^2 \le c(T_0+t)^{-2}Z^2(t)\mathcal{R}_1^2(t).$$

By the lemma we just prove, we have

$$|K_3| \le c(T_0+t)^{-2}(Z^2\mathcal{R}_1^2 + \mathcal{R}_4^2\mathcal{R}_2^2).$$

By the property of $P_c^{\lambda}\mathcal{G}$ in (9-9) we have

$$|K_4| \le c|z|^2\|\langle x\rangle^{-\nu}\vec{R}\|_{H^2} \le c(T_0+t)^{-2}Z^2\mathcal{R}_1.$$

Collecting all the estimates above, we obtain

$$\Big|\frac{d}{dt}\langle(-\Delta+1)\vec{R},(-\Delta+1)\vec{R}\rangle\Big| \le c(T_0+t)^{-2}(\mathcal{R}_1^2 + Z^2\mathcal{R}_1 + Z^2\mathcal{R}_1^2 + \mathcal{R}_4^2\mathcal{R}_2^2).$$

After integrating the equation above from 0 to $t$ we have proposition. □

***Estimate for $Z(T) = \max_{t\le T}(T_0+t)^{1/2}|z(t)|$.*** Recall that by (FGR)

$$z^*(Z(z,\bar{z}) + Z^*(z,\bar{z}))z \ge C|z|^4.$$

**Proposition 11.13.** *There exists an order one constant $m>0$ such that if $m < T_0 < |z(0)|^{-2}$ then*

$$Z(T) \le 1 + \frac{K}{T_0^{2/5}}Z(T)(Z(T) + \mathcal{R}_1^2(T) + \mathcal{R}_2^2(T) + Z(T)\mathcal{R}_3(T)).$$

*Proof.* By (8-11) we have

$$\frac{d}{dt}|z|^2 = -z^*(Z(z,\bar{z}) + Z^*(z,\bar{z}))z + \Re(\bar{z}\mathrm{Remainder}(t))$$

which can be transformed into a Riccati inequality:

$$\partial_t|z(t)|^2 \le -C|z(t)|^4 + 2|z(t)||\mathrm{Remainder}(t)|.$$

By (8-8),

$$|z(t)|\,|\mathrm{Remainder}(t)| \le \frac{c}{(T_0+t)^{2+\delta}} Z(T)(Z(T)+\mathcal{R}_1^2(T)+\mathcal{R}_2^2(T)+Z(T)\mathcal{R}_3(T)),$$

where $\delta = 2/5$.

**Lemma 11.14.** *Suppose that $z(t)$ is any function satisfying the equation*

$$\partial_t |z(t)|^2 \le -|z(t)|^4 + g(t), \qquad z(0) = z_0, \tag{11-12}$$

*where $g(t)$ is a function satisfying the estimate*

$$|g(t)| \le c_\#(T_0+t)^{-2-\delta} \tag{11-13}$$

*with the constants $c_\#, \delta > 0$. Then there exists $K > 0$ independent of $T_0$ and $c_\#$ such that if $c_\# T_0^{-\delta}$ is sufficiently small then the function $z(t)$ in (11-12) admits the bound*

$$|z(t)| \le \frac{1+Kc_\# T_0^{-\delta}}{(\kappa+t)^{1/2}} \tag{11-14}$$

*where $\kappa = \min\{T_0, |z_0|^{-2}\}$.*

The proof of this lemma is in Appendix G.

We now chose

$$m < T_0 < |z(0)|^{-2}$$

where $m$ is an order one positive constant. Then,

$$Z(T) \le 1 + \frac{K}{T_0^{2/5}} Z(T)(Z(T)+\mathcal{R}_1^2(T)+\mathcal{R}_2^2(T)+Z(T)\mathcal{R}_3(T)). \qquad \square$$

***Closing the estimates.***

*Proof of Theorem 7.1.* We seek to obtain $T$-independent bounds on $\mathcal{R}_j(T)$ and $Z(T)$ defined in (11-2). This will be achieved by choosing the parameter $T_0$ in the norm definitions sufficiently large and the data $R(0)$ sufficiently small with $T_0$ and $R(0)$ related in a manner to be specified.

Define

$$M(T) := \sum_{n\neq 4} \mathcal{R}_n(T), \qquad S := T_0^{3/2}(\|\vec{R}(0)\|_{H^2} + \|\langle x\rangle^\nu \vec{R}(0)\|_2)$$

where $T_0$ is defined in (11-1). By the conditions in (7-1) we have that $\mathcal{R}_4(0)$ is small and $M(0)$ and $Z(0)$ are bounded.

Recall the estimates of $\mathcal{R}_n$ for $n = 1, 2, 3, 4, 5$ and $Z$ in Propositions 11.3, 11.5, 11.9, 11.11, 11.7 and 11.13. By plugging the estimate of $Z$ and $\mathcal{R}_4$ in Propositions 11.13, 11.11 into Propositions 11.3, 11.5 and 11.7, we obtain

$$\begin{aligned} M(T) &\le c(S+1) + (R_4(T) + T_0^{-1/20})P(M(T), Z(T)),\\ Z(T) &\le 1 + T_0^{-1/20} P(M(T), Z(T)),\\ \mathcal{R}_4^2(T) &\le \|\vec{R}(0)\|_{H^2}^2 + T_0^{-1} P(M(T), Z(T)) \end{aligned} \tag{11-15}$$

where $P(x, y) > 0$ is a polynomial in $x$ and $y$. Using an implicit-function-theorem type argument (see below) we have that if $S$ and $M(0)$ are bounded then

$$M(T) + Z(T) \le \mu(S) \quad \text{and} \quad \mathcal{R}_4 \ll 1 \tag{11-16}$$

where $\mu$ is a bounded function for $S$ bounded. By the definitions of $\mathcal{R}_j(T)$ and $Z(T)$ there exists some constant $c$ such that

$$\|\langle x\rangle^{-\nu}\vec{R}(t)\|_2, \|\vec{R}(t)\|_\infty \le c(T_0+t)^{-1}, \qquad |z(t)| \le c(T_0+t)^{-1/2}, \tag{11-17}$$

which is statement (B) in Theorem 7.1.

By the bound of Remainder in (8-8) and the estimates (11-17) we have

$$|\text{Remainder}| \le c(T_0+t)^{-19/5}$$

which, together with (8-11), implies statement (A).

The convergence of $\lambda$ comes from (8-9) and the fact that Remainder is integrable at $\infty$. □

In the following we prove (11-15) implies (11-16) by using implicit function theorem. For the other methods we refer to [Soffer and Weinstein 1999; 2004; Tsai and Yau 2002b; 2002c; Buslaev and Sulem 2003; Tsai 2003; Cuccagna and Mizumachi 2008]. First we transform the inequalities by taking square root of the third equation of (11-15) and plugging it into the first one, then

$$\begin{aligned} M(T) &\le c(S+1) + (\|\vec{R}(0)\|_{H^2} + T_0^{-1/20})P(M(T), Z(T)),\\ Z(T) &\le 1 + T_0^{-1/20}P(M(T), Z(T)),\\ \mathcal{R}_4(T) &\le \|\vec{R}(0)\|_{H^2} + T_0^{-1/20}P(M(T), Z(T)). \end{aligned}$$

In what follows we use this equation instead of (11-15). Define a vector function $F_{\epsilon,\delta}(\tilde{M}, \tilde{Z})$ as

$$F_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) := (F^{(1)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}), F^{(2)}_{\epsilon}(\tilde{M}, \tilde{Z}), F^{(3)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}))$$

with

$$F^{(1)}_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) := c(S+1) + (\delta+\epsilon)P(\tilde{M}, \tilde{Z}),\ \ F^{(2)}_{\epsilon}(\tilde{M}, \tilde{Z}) := 1 + \epsilon P(\tilde{M}, \tilde{Z}),\ \ F^{(3)}_{\epsilon,\delta} := \delta + \epsilon P(\tilde{M}, \tilde{Z}).$$

Immediately we can see that

$$M_0 = c(1+S), \quad Z_0 = 1, \quad R_0 = 0$$

is a solution to the equation

$$(M_0, Z_0, R_0) = F_{0,0}(M_0, Z_0).$$

Define a closed set

$$\Sigma := [0, 2c(S+1)] \times [0, 2] \times [0, 1].$$

**Lemma 11.15.** *There exists $\delta_0 \ge 0$ such that if $\epsilon, \delta \in [0, \delta_0]$ then*

$$(\tilde{M}, \tilde{Z}, \tilde{R}) = F_{\epsilon,\delta}(\tilde{M}, \tilde{Z}) \tag{11-18}$$

*has a unique solution in $\Sigma$. Moreover, for any continuous functions $M, Z, \mathcal{R} : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying*

$$\big(M(0), Z(0), \mathcal{R}(0)\big) \le (\tilde{M}, \tilde{Z}, \tilde{R}) \quad \text{and} \quad \big(M(t), Z(t), R(t)\big) \le F_{\epsilon,\delta}\big(M(t), Z(t)\big),$$

*we have*

$$\big(M(t), Z(t), \mathfrak{R}(t)\big) \le (\tilde{M}, \tilde{Z}, \tilde{R}). \tag{11-19}$$

*for any time $t$.*

*Proof.* The proof of existence and uniqueness of the solution is not difficult by observing

$$\big\|\big(\partial_M F_{\epsilon,\delta}(M, Z), \partial_Z F_{\epsilon,\delta}(M, Z), \partial_{\mathfrak{R}} F_{\epsilon,\delta}(M, Z)\big)\big\| \le c(\delta+\epsilon)$$

if $(M, Z, \mathfrak{R}) \in \Sigma$. Hence by implicit function theorem we have that if $c(\epsilon+\delta) \le 1/2$ there exists a unique solution to (11-18).

We next prove (11-19) by contradiction. Suppose that (11-19) fails at time $t$. Since $(M(t), Z(t), \mathfrak{R}(t))$ is continuous there exists a time $t_1 \le t$ such that $(M(t_1), Z(t_1), \mathfrak{R}(t_1)) \in \Sigma$ and (11-19) does not hold. Without loss of generality we assume $t = t_1$. Then by subtracting the inequality for $(M(t), Z(t), \mathfrak{R}(t))$ by (11-18) we get

$$M(t) - \tilde{M} \le (\delta+\epsilon)\big(K_1(M(t) - \tilde{M}) + K_2(Z(t) - \tilde{Z})\big)$$

and

$$Z(t) - \tilde{Z}, \mathfrak{R}(t) - \tilde{R} \le \epsilon\big(K_3(M(t) - \tilde{M}) + K_4(Z(t) - \tilde{Z})\big)$$

for some $K_n$ with $n = 1, 2, 3, 4$ depending on $(M(t), Z(t), \mathfrak{R}(t))$ and $(\tilde{M}, \tilde{Z}, \tilde{R})$. By the fact that

$$(\tilde{M}, \tilde{Z}, \tilde{R}), (M(t), Z(t), \mathfrak{R}(t)) \in \Sigma$$

and $P(x, y)$ is a polynomial with positive coefficient, we have that $K_n$ are *positive* and bounded. By these inequalities and the fact $0 \le \epsilon, \delta \ll 1$ we derive (11-19). This contradicts our assumption. Thus (11-19) holds for any time $t \ge 0$. □

## 12. Summary and discussion

We have extended the asymptotic stability / scattering theory of solitary waves of the nonlinear Schrödinger/Gross–Pitaevskii (NLS/GP) equation to the important case where the linearized dynamics about the Lyapunov stable bound state has degenerate neutral modes. This is the prevalent case in situation where the equation is invariant under a nontrivial symmetry. We construct a class of multiwell potentials to which the theory applies. The current theory, as all previous work on soliton scattering in systems with nontrivial neutral modes, requires a Fermi Golden Rule (FGR) nondegeneracy hypothesis. The analytical verification of this hypothesis for either specific or generic NLS/GP systems is an open question. Numerical experiments for the time-dependent NLS/GP equations, in which decay rates of neutral modes are measured, are consistent with the generic validity of the (FGR) nondegeneracy hypothesis.

We conclude by mentioning an interesting direction for further exploration:

***Semiclassical limits and higher order nonlinear Fermi Golden Rule.*** A problem of great interest is NLS/GP on $\mathbb{R}^d$ in the semiclassical limit:

$$i\partial_t\psi = -\Delta\psi + V(hx)\psi - f(|\psi|^2)\psi, \qquad \psi(x,0) = \psi_0(x)$$

where $0<h\ll1$. The nonlinearity is taken to be focusing (attractive) but subcritical. Using the Lyapunov–Schmidt method it has been shown in [Floer and Weinstein 1986; Oh 1988; Ambrosetti et al. 1996] that for $h$ sufficiently small a soliton concentrated at a nondegenerate critical point of $V$ can be constructed.

The soliton, constructed in this manner, is soliton of the translation invariant nonlinear Schrödinger equation, scaled to be highly concentrated about the critical point of $V$. Therefore, the linearized operator $JH^h(\lambda)$ is expected to have spectrum, quite closely related to the linearization about the translation invariant NLS soliton. If the soliton is concentrated near a minimum of $V$, then it is Lyapunov stable [Oh 1988], and therefore the spectrum of $JH^h(\lambda)$ is a subset of the imaginary axis. As we have seen for NLS/GP, there is a two-dimensional generalized eigenspace corresponding to an eigenvalue zero. $h$ being small implies that the $2 \times d$ *zero modes* associated with the translation symmetry

$$\psi(x,t) \mapsto \psi(x + x_0, t)$$

and Galilean symmetry

$$\psi(x,t) \mapsto e^{iv\cdot(x-vt)}\psi(x - 2vt, t)$$

perturb to $d$ complex conjugate pairs of eigenvalues. Although we expect semiclassical, highly localized solitons to be asymptotically stable and for the degenerate neutral modes to damp by resonant radiation damping, as elucidated in this article, we note that for $h$ very small, the complex conjugate neutral modes of $JH^h(\lambda)$ are very close to zero and the condition $2E(\lambda) - \lambda > 0$, which is necessary (although not sufficient) for the Fermi Golden Rule resonance condition (FGR) to hold, fails. It remains an open question to derive the normal form when resonance of discrete modes with the continuum occurs at some arbitrary order in the coupling parameter $g$ (recall $f(|\psi|^2)\psi = -g|\psi|^2\psi$ and see also the discussion in Section 1). For results in this direction, see [Gang 2007; Cuccagna and Mizumachi 2008].

## Appendix A. A class of multiwell potentials for which $-\Delta + V$ satisfies condition (Eig$_V$) and $L(\lambda)$ satisfies (SA) and (Thresh$_\lambda$)

In this section we find an example $-\Delta + V$ in a subspace of $L^2(\mathbb{R}^3)$ satisfying condition (**Eig$_{\mathbf{V}}$**), motivated by the study of double well potentials. Define

$$\mathcal{A} := \{f : \mathbb{R}^3 \to \mathbb{C} \mid f(-x) = f(x) \text{ for any } x\}.$$

Observe that $\mathcal{A}$ is a self-closed subspace, that is, if $f_1, f_2 \in \mathcal{A}$ then $f_1 + f_2, f_1 f_2, \Delta f_1 \in \mathcal{A}$. Hence we can study (1-1) in the space $\mathcal{A} \cap L^2(\mathbb{R}^3)$ and obtain all the results. The following is the main result

**Proposition A.1.** *There exists a potential $V$ such that the linear operator $-\Delta + V$ acting on the subspace $\mathcal{A} \cap L^2(\mathbb{R}^3)$ has two eigenvalues $e_0 < e_1 < 0$ with $2e_1 > e_0$. $e_0$, the lowest eigenvalue, is simple, and eigenvalue $e_1$ is degenerate with multiplicity 2. Moreover the operator*

$$1 + (-\Delta + i0)^{-1} V : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

*is invertible.*

If the nonlinearity $f(x) = x$ and if $|\lambda - |e_0||$ is sufficiently small and $\phi^\lambda$ is the ground state satisfying

$$-\Delta\phi^\lambda + V\phi^\lambda + \lambda\phi^\lambda - (\phi^\lambda)^3 = 0$$

then we have the following results for the linearized operator $L(\lambda)$ defined in (5-1).

**Proposition A.2.** *The operator $L(\lambda)$ satisfies the spectral conditions (SA) and (Thresh$_\lambda$).*

Proposition A.1 is implied by Proposition A.5 below. Proposition A.2 will be proved at the end of this section.

As proved in [Albeverio et al. 2005, Theorem 1.1.4 on page 116] the operator $-\Delta - q\delta(x)$ has only one eigenfunction, that is, the ground state, for any $q > 0$. By this observation we have:

**Lemma A.3.** *For any $q > 0$, there exists a constant $\lambda \in (0, \infty)$ such that the operators*

$$-\Delta - q\lambda^{-\frac{3}{2}} e^{-\frac{|x|^2}{\lambda}}, \qquad -\Delta - \frac{1}{3} q\lambda^{-\frac{3}{2}} e^{-\frac{|x|^2}{\lambda}}$$

*each have only one eigenfunction in $\mathcal{A}$.*

To facilitate later discussions we define

$$W := q\lambda^{-3/2} e^{-|x|^2/\lambda}.$$

We start with constructing a family of operators. Define

$$M_1 := (m, 0, 0), \quad M_2 := (0, m, 0), \quad M_3 := (0, 0, m),$$

and

$$W_{M_k}(x) := \tfrac{1}{2}(W(x + M_k) + W(x - M_k)) \quad \text{for } k = 1, 2, 3.$$

**Lemma A.4.** *If $m$ is sufficiently large then in the subspace $\mathcal{A} \cap L^2(\mathbb{R}^3)$ each of the operators $-\Delta - W_{M_k}$ and $-\Delta - \frac{1}{3} W_{M_k}$, for $k = 1, 2, 3$, has only one eigenfunction.*

*Proof.* We only prove the result for $-\Delta - W_{M_1}$. The proof of the other cases is similar, hence omitted.

First we have that if $m$ is sufficiently large then

$$\big\langle (-\Delta - W_{M_1})\big(\phi(\cdot + M_1) + \phi(\cdot - M_1)\big),\ \phi(\cdot + M_1) + \phi(\cdot - M_1)\big\rangle < 0.$$

This principle [Reed and Simon 1979] implies that the operator $-\Delta - W_{M_1}$ has at least one eigenstate.

Second the min-max principle implies that any function $f \perp \phi(\cdot + M_1), \phi(\cdot - M_1)$ satisfies

$$\langle (-\Delta - W_{M_1}) f, f\rangle = \tfrac{1}{2}\big(\langle (-\Delta - W(\cdot + M)) f, f\rangle + \langle (-\Delta - W(\cdot - M)) f, f\rangle\big) \geq 0.$$

This, together with the facts

$$\phi(\cdot + M_1) - \phi(\cdot - M_1) \perp L^2(\mathbb{R}^3) \cap \mathcal{A} \ \text{ and } \operatorname{span}\{\phi(\cdot - M_1), \phi(\cdot + M_1)\} = \operatorname{span}\{\phi(\cdot - M_1) \pm \phi(\cdot + M_1)\},$$

yields that

$$\langle (-\Delta - W_{M_1}) f, f\rangle \geq 0$$

for any $f \in \mathcal{A} \cap L^2(\mathbb{R}^3)$ and $f \perp \phi(\cdot + M_1) + \phi(\cdot - M_1)$.

Collecting what was proved we have that the operator $-\Delta - W_{M_1}$ has only one eigenfunction, its ground state. □

To prove the main result we define

$$V_m := \tfrac{1}{3}(W_{M_1} + W_{M_2} + W_{M_3}).$$

**Proposition A.5.** *There exists at least one $m \in [0, \infty)$ such that $-\Delta - V_m$ has all the properties in Proposition A.1.*

*Proof.* We need the following facts:

(A) *For any $m \in [0, \infty)$ the operator $-\Delta - V_m$ has at most three eigenfunctions in $\mathcal{A} \cap L^2$.* Recall in Lemma A.4 we proved that if $f \perp \phi(\cdot + M_k) + \phi(\cdot - M_k)$ for $k = 1, 2, 3$ and $f \in \mathcal{A} \cap L^2$ then $\langle(-\Delta - W_{M_k}) f, f\rangle \geq 0$. Consequently if $f \perp \phi(\cdot + M_k) + \phi(\cdot - M_k)$ for $k = 1, 2, 3$ then

$$\langle(-\Delta - V_m) f, f\rangle = \tfrac{1}{3}\big(\langle(-\Delta - W_{M_1}) f, f\rangle + \langle(-\Delta - W_{M_2}) f, f\rangle + \langle(-\Delta - W_{M_3}) f, f\rangle\big) \geq 0.$$

The min-max principle [Reed and Simon 1979] implies that there are at most three eigenfunctions.

(B) *If $m$ is sufficiently large then in the space $L^2 \cap \mathcal{A}$ the operator $-\Delta - V_m$ has three eigenfunctions and two eigenvalues: one ground state and two degenerate eigenstates.* The fact that $-\Delta + V_m$ has three eigenfunctions follows from the min-max principle. The proof is similar to the case of double-well potentials [Harrell 1980] and is omitted. We need to prove that these eigenstates are degenerate. Indeed, as $m \to \infty$ the three eigenfunctions converge to a linear combination of the functions:

$$\phi(\cdot + M_k) + \phi(\cdot - M_k) \quad \text{for } k = 1, 2, 3.$$

In particular, the ground state converges to

$$\sum_{k=1}^{3} \phi(\cdot + M_k) + \phi(\cdot - M_k).$$

Moreover, the ground state is simple and orthogonal to the excited eigenstates. The excited eigenstates are not invariant under a permutation: $(x_1, x_2, x_3) \mapsto (x_{n(1)}, x_{n(2)}, x_{n(3)})$. Since $V_m$ is invariant under permutation, a second, linearly independent, eigenstate may be obtained from a particular choice via permutation.

(C) *When $m = 0$, $-\Delta - V_m$ has only one eigenfunction, the ground state.* This is clear since $V_m = W$ when $m = 0$.

(D) *For any $m \geq 0$, $-\Delta - V_m$ has at least one eigenfunction with eigenvalue less than some $-c_0 < 0$.* Let $\phi_2$ be the normalized ground state of $-\Delta - \frac{1}{3} W_{M_2}$ with eigenvalue $-c_0 < 0$. Then we have

$$\langle(-\Delta - V_m)\phi_2, \phi_2\rangle < -c_0$$

by the facts $\phi_2 > 0$ and $W > 0$. By the min-max principle $-\Delta - V_m$ has a ground state with eigenvalue $< -c_0$.

The definition of $W$ implies that $(-\Delta + k)^{-1} W(\cdot + z)$ is analytic in $z$ if $k \in \mathbb{C}\backslash\mathbb{R}^+$. By [Reed and Simon 1979] we have that the eigenvalues are analytic functions of $z$ in a suitable subset of $\mathbb{C}$. Since the eigenvalues of the excited states are degenerate for sufficiently large $m$ (see (B)), they are degenerate for any $m$ before the excited states disappear into the essential spectrum. Hence there exists at least one $m$ such that $-\Delta - V_m$ has one eigenvalue, $e_0$, less than $-c_0$ (defined in (D)) and two degenerate excited states with eigenvalue $e_1$, sufficiently close to the essential spectrum (see (A), (C)); $e_1 - e_0 > -e_1$ or $2e_1 - e_0 > 0$.

In the final step we find $m$ and $q$ such that the operator

$$1 + (-\Delta + i0)^{-1} V_m : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$$

is invertible. Recall that $V_m = qV_2(m)$, with $V_2(m)$ independent of $q$ by its definition. For a fixed $q_0$ we proved that there exists at least one $m = m_0$ such that the eigenvalues of $-\Delta - q_0 V_2(m_0)$ have the desired properties. We now consider the family of operators

$$X(q) := 1 - q(-\Delta + i0)^{-1} V_2(m_0)$$

which is analytic in $q$. Moreover, the operator $q(-\Delta + i0)^{-1} V_2(m_0) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$ is compact. By [Reed and Simon 1979] the operators $X(q) : \langle x\rangle^2 L^2 \to \langle x\rangle^2 L^2$ are either invertible everywhere (that is, no threshold resonance) except for some discrete points or not invertible anywhere. The first case holds because the operator is invertible when $q = 0$.

Now we consider $-\Delta - qV_2(m_0)$ with $q \in [q_0 - \epsilon, q_0 + \epsilon]$. Choose $\epsilon$ sufficiently small such that for every $q$ the operator $-\Delta - qV_2(m_0)$ has at least three eigenvectors. On the other hand by what we proved above it has at most three eigenvectors and two of them must be degenerate. Since $1 - q(-\Delta + i0)^{-1} V_2(m_0)$ is not invertible only at discrete points we obtain the desired result. □

*Proof of Proposition A.2.* The fact $L(\lambda)$ has no resonances at $\pm i\lambda$ follows from the invertibility of $I + (-\Delta + i0)^{-1} V$ and $|\,\lambda - |e_0|\,|$ being small.

Next we prove the neutral modes are degenerate. Recall that the potential we constructed is of the form $V = V_{m_0}$ for some $m_0$. For each $m > 0$ there are $\lambda = \lambda_m$ and $\phi^\lambda = \phi^{\lambda,m}$ satisfying

$$-\Delta\phi^{\lambda,m} + \lambda_m \phi^{\lambda,m} + V_m \phi^{\lambda,m} - (\phi^{\lambda,m})^3 = 0$$

with $\lambda_m$ and $\phi_m$ analytic in $m$ in some proper neighborhood of positive real axis.

Recall that when $m$ is sufficiently large the neutral modes of $-\Delta + V_m$ can be generated by permuting one of them. Hence the neutral modes of $L(\lambda) = L(\lambda, m)$ are degenerate when $m$ is large. Moreover the eigenvalues of $L(\lambda, m)$ are analytic in $m$, thus the neutral modes must be degenerate. □

## Appendix B. The Fermi Golden Rule

The proof of Theorem 6.1, given at the end of this section, requires the following:

**Proposition B.1.** *Given smooth functions $\mathcal{F}, \mathcal{G} : \mathbb{R}^d \to \mathbb{C}^2$, there exists $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^T$ and $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)^T$ (see the definitions below) such that*

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathcal{F}, iJP_c\mathcal{G}\big\rangle = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle \tag{B-1}$$

*Proof of Proposition B.1.* The entries of $\Gamma$ are expressions of the form

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathcal{F},\ iJP_c\mathcal{G}\big\rangle \tag{B-2}$$

which we now proceed to simplify. Recall $L(\lambda)$ is of the form

$$L(\lambda) = (-\Delta + \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix}$$

where $V_1$ and $V_2$ are real-valued and exponentially decaying as $|x|$ tends to infinity. Introduce the unitary matrix

$$U = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Note that

$$L(\lambda) = iU\sigma_3\mathcal{H}(\lambda)U^*, \qquad \mathcal{H}^* = \mathcal{H}$$

where

$$\mathcal{H} := \mathcal{H}_0 + \tilde{V},\ \mathcal{H}_0 := (-\Delta + \lambda)\,\mathrm{Id},\ \tilde{V} := \begin{pmatrix} V_1 - V_2 & -i(V_1+V_2) \\ i(V_1+V_2) & V_1 - V_2 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now use the unitary transformation $U$ to obtain an expression in terms of the operator $\sigma_3\mathcal{H}$:

$$\begin{aligned} -\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c\mathcal{F},\ iJP_c\mathcal{G}\big\rangle &= -\big\langle \big(iU(\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)U^*\big)^{-1} P_c\mathcal{F},\ iJP_c\mathcal{G}\big\rangle \\ &= \big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^*P_c\mathcal{F}, U^*JP_c\mathcal{G}\big\rangle \\ &= \big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^*P_c\mathcal{F}, (U^*JU)U^*P_c\mathcal{G}\big\rangle \\ &= -i\big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^*P_c\mathcal{F}, \sigma_3U^*P_c\mathcal{G}\big\rangle \end{aligned} \tag{B-3}$$

where we have used that $U^*JU = i\sigma_3$.

Next we introduce $P_c(\sigma_3\mathcal{H})$, the projection onto the continuous spectral of $\sigma_3\mathcal{H}$ and wave operators $W : L^2 \to P_c(\sigma_3\mathcal{H})L^2$ and $Z : P_c(\sigma_3\mathcal{H})L^2 \to L^2$ (see [Cuccagna et al. 2005]), which satisfy

$$P_c(\sigma_3\mathcal{H})^*\sigma_3 = \sigma_3P_c(\sigma_3\mathcal{H}),\ \ W^*\sigma_3 = \sigma_3Z,\ \ Z^*\sigma_3 = \sigma_3W,\ \ Z\sigma_3\mathcal{H} = \sigma_3\mathcal{H}_0Z. \tag{B-4}$$

Now we use the wave operators $W$ and $Z$ to transform the previous expression into one in terms of the "free operator" $\sigma_3(-\Delta+\lambda)$. First note that $U^*P_c\mathcal{F}$ lies in the range of $P_c(\sigma_3\mathcal{H})$ and therefore there exists $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^T$ such that $W\tilde{\mathcal{F}} = U^*P_c\mathcal{F}$. Similarly, there exists $\tilde{\mathcal{G}} = (\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)^T$ such that $W\tilde{\mathcal{G}} = U^*P_c\mathcal{G}$. Substituting into the final expression in (B-3) and using of the properties (B-4) we have

$$\begin{aligned} &i\big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^*P_c\mathcal{F}, \sigma_3U^*P_c\mathcal{G}\big\rangle \\ &\qquad = i\big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} W\tilde{\mathcal{F}}, \sigma_3W\tilde{\mathcal{G}}\big\rangle = i\big\langle (\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} W\tilde{\mathcal{F}}, Z^*\sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = i\big\langle Z(\sigma_3\mathcal{H}(\lambda) + 2E(\lambda) + i0)^{-1} W\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle = i\big\langle (\sigma_3(-\Delta+\lambda) + 2E(\lambda) + i0)^{-1} ZW\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = i\big\langle (\sigma_3(-\Delta+\lambda) + 2E(\lambda) + i0)^{-1}\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle. \end{aligned}$$

Referring back to (B-2) we recall that we are interested in the real part of this expression:

$$\begin{aligned} &-\Re\, i\big\langle (\sigma_3(-\Delta+\lambda) + 2E(\lambda) + i0)^{-1}\tilde{\mathcal{F}},\ \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = \Im\,\big\langle (\sigma_3(-\Delta+\lambda) + 2E(\lambda) + i0)^{-1}\tilde{\mathcal{F}}, \sigma_3\tilde{\mathcal{G}}\big\rangle \\ &\qquad = \Im\,\left\langle \begin{pmatrix} (-\Delta+\lambda+2E(\lambda)+i0)^{-1} & 0 \\ 0 & -(-\Delta+\lambda-2E(\lambda)-i0)^{-1} \end{pmatrix}\tilde{\mathcal{F}},\ \sigma_3\tilde{\mathcal{G}}\right\rangle \\ &\qquad = \Im\,\left\langle \begin{pmatrix} (-\Delta+\lambda+2E(\lambda)+i0)^{-1} & 0 \\ 0 & (-\Delta+\lambda-2E(\lambda)-i0)^{-1}) \end{pmatrix}\tilde{\mathcal{F}},\ \tilde{\mathcal{G}}\right\rangle \\ &\qquad = \Im\,\big\langle (-\Delta - (2E(\lambda)-\lambda) - i0)^{-1}\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle \\ &\qquad = \pi\big\langle \delta(-\Delta - (2E(\lambda)-\lambda))\tilde{\mathcal{F}}_2, \tilde{\mathcal{G}}_2\big\rangle. \end{aligned}$$

We uses that

$$0 < 2E(\lambda) - \lambda \in \sigma_c(-\Delta), \qquad -2E(\lambda) - \lambda \notin \sigma_c(-\Delta)$$

and the distributional (Plemelj) identity:

$$\Im(x - i0)^{-1} = \lim_{\varepsilon \downarrow 0} \Im(x - i\varepsilon)^{-1} = \pi\delta(x)$$

to get the last equality.

Summarizing, we have shown

$$-\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c \mathscr{F},\, iJP_c\mathscr{G}\big\rangle = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathscr{F}}_2,\, \tilde{\mathscr{G}}_2\big\rangle. \qquad \square$$

*Proof of Theorem 6.1.* We use Proposition B.1 with $\mathscr{F} = G_k$ and $\mathscr{G} = G_l$, $\tilde{\mathscr{F}} = \tilde{G}_k$ and $\tilde{\mathscr{G}} = \tilde{G}_l$. By (B-1) we have

$$\Gamma_{k,l} = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{G}_{l,2},\, \tilde{G}_{k,2}\big\rangle.$$

To see that $\Gamma_{k,l}$ is nonnegative, observe that for any $s \in \mathbb{C}^N$ we have

$$s^*\Gamma s = \sum_{k,l=1}^{N} \Gamma_{k,l} s_k \bar{s}_l = \pi\big\langle \delta(-\Delta - (2E(\lambda) - \lambda))\tilde{\mathscr{G}},\, \tilde{\mathscr{G}}\big\rangle \geq 0$$

where $\tilde{\mathscr{G}} = \sum_{k=1}^{N} s_k \tilde{G}_{k,2}$.

For the second statement we only sketch the proof. Recall the transformation of $L(\lambda)$ in (5-2). Then for any $2 \times 1$ vector functions $\vec{F}$ and $\vec{G}$ we have

$$\begin{aligned}\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c\vec{F},\, P_c\vec{G}\big\rangle &= -i\big\langle (\sigma_3\mathscr{H}(\lambda) + 2E(\lambda) + i0)^{-1} U^* P_c\vec{F},\, U^* P_c\vec{G}\big\rangle \\ &= -i\big\langle K(\lambda)\big(\sigma_3(\mathscr{H}_0 + V) + 2E(\lambda) + i0\big)^{-1} U^* P_c\vec{F},\, U^* P_c\vec{G}\big\rangle\end{aligned}$$

where $K(\lambda)$ is the operator defined as $(1 + K_{\text{small}})^{-1}$ with

$$K_{\text{small}} := (\sigma_3(\mathscr{H}_0 + V) + 2E(\lambda) + i0)^{-1} V_{\text{small}}.$$

The operator $\big(\sigma_3(\mathscr{H}_0 + V) + 2E(\lambda) + i0\big)^{-1}$ is well defined since

$$-\lambda - 2E(\lambda) = e_0 - 2(e_1 - e_0)$$

is not an eigenvalue of $-\Delta + V$ and the operator $-\Delta + V$ has no embedded eigenvalues in the essential spectrum.

Since the operator $K_{\text{small}} : \langle x\rangle^2 L^\infty \to \langle x\rangle^2 L^\infty$ has a small norm and is continuous in $\lambda$ we have

$$(1 + K_{\text{small}})^{-1} = \sum_{n=0}^{\infty} (-K_{\text{small}})^n$$

is continuous in $\lambda$. This, together with the fact

$$(\sigma_3(\mathscr{H}_0 + V) + 2E(\lambda) + i0)^{-1} U^* P_c\vec{F} \in \langle x\rangle^2 L^\infty$$

is continuous in $\lambda$, implies that $\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c\vec{F},\, P_c\vec{G}\big\rangle$ is continuous in $\lambda$. $\square$

## Appendix C. Fermi Golden Rule for symmetric potentials

In this section we derive the simpler form of the FGR matrix and condition for positivity in the case where the potential $V(x)$ is a function of $|x|$. In fact, it is proved in Proposition 5.2 that if the potential $V$, hence $\phi^\lambda$, is spherically symmetric then the functions $\xi_n$, $\eta_n$ satisfy

$$\xi_n = \frac{x_n}{|x|}\xi(|x|), \qquad \eta_n = \frac{x_n}{|x|}\eta(|x|)$$

for some functions $\xi(|x|)$ and $\eta(|x|)$. By the assumptions on $V$, $\phi^\lambda$, $\xi_k$ and $\eta_k$ with $k = 1, 2, \ldots, N = d$ we have

$$G_k(z, x) = x_k(z \cdot x)G(|x|)$$

for some radial vector function $G(|x|)$.

Before stating the results we define two constants

$$\begin{aligned}\Re Z_0^{(1,1)} &= -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1^2 G(|x|), iJx_1^2 G(|x|)\big\rangle,\\ \Re Z_0^{(2,2)} &= -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1 x_2 G(|x|), iJx_1x_2 G(|x|)\big\rangle.\end{aligned}$$

**Proposition C.1.** *(i) Suppose that $V$, $\xi_n$, $\eta_n$ satisfy the conditions above. Then the assumption (FGR) holds provided that*

$$\Re Z_0^{(1,1)} > 0, \qquad \Re Z_0^{(2,2)} > 0. \tag{C-1}$$

*(ii) From Proposition B.1, it follows that*

$$\Re Z_0^{(1,1)} \ge 0, \qquad \Re Z_0^{(2,2)} \ge 0.$$

*And, generically, the strict positivity in (C-1) holds.*

*Proof.* For any vectors $s, \beta, z \in \mathbb{C}^N$, we define

$$\mathfrak{Q}(s, \beta; z) := -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c (z\cdot x)(s \cdot x) G(|x|), iJ(z\cdot x)(\beta\cdot x)G(|x|)\big\rangle.$$

Note that

$$\mathfrak{Q}(s, s; z) = \tfrac{1}{2}s^*(Z(z, \bar z) + Z^*(z, \bar z))s = \Re s^* Z(z, \bar z)s.$$

Therefore, verifying (FGR) is equivalent to checking that there is a constant $C > 0$ for which

$$\mathfrak{Q}(s, s; z) \ge C|s|^2|z|^2$$

with $s, z \in \mathbb{C}^d$.

To simplify $\mathfrak{Q}(s, s; z)$, first note that since operator $L(\lambda)$ and $G(|x|)$ are invariant under $x \mapsto T^*x$, where $T$ is a unitary transformations, the value of $\mathfrak{Q}(s, \beta; z)$ is unchanged when $x$ is replaced by $T^*x$. Therefore,

$$\mathfrak{Q}(s, \beta; z) = \mathfrak{Q}(Ts, T\beta; Tz). \tag{C-2}$$

Now choose $T$ to be a unitary matrix such that

$$Tz = |z|e_1 = |z|(1, 0, \ldots, 0)^T.$$

With this choice of $T$, we have by (C-2) with $\beta = s$,

$$\mathfrak{Q}(s, s; z) = -\Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c |z| x_1 (Ts \cdot x) G(|x|), i|z| x_1 (Ts \cdot x) JG(|x|) \big\rangle. \tag{C-3}$$

The following argument will show that

$$\mathfrak{Q}(s, s; z) \geq C|Ts|^2 |z|^2 = C|s|^2|z|^2,$$

the latter holding since $T$ is unitary. Therefore, without any loss of generality, consider (C-3) with $T$ set equal to the identity. Explicitly writing out the inner products and using bilinearity and symmetry, we have

$$\begin{aligned}
\mathfrak{Q}(s, s; z) &= -|z|^2 \Re\Big( \sum_{p,q=1}^{d} \big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1 x_p G(|x|), i x_1 x_q JG(|x|) \big\rangle s_p \overline{s_q} \Big) \\
&= -|z|^2 \Re\Big( \sum_{p=1}^{d} \big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1 x_p G(|x|), i x_1 x_p JG(|x|) \big\rangle |s_p|^2 \Big) \\
&= -|z|^2 |s_1|^2 \Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1^2 G(|x|), i x_1^2 JG(|x|) \big\rangle \\
&\quad - |z|^2 \sum_{q=2}^{d} |s_q|^2 \Re\big\langle (L(\lambda) + 2iE(\lambda) - 0)^{-1} P_c x_1 x_2 G(|x|), i x_1 x_2 JG(|x|) \big\rangle \\
&= |z|^2 \Big( |s_1|^2 \, \Re Z_0^{(1,1)} + \sum_{q=2}^{d} |s_q|^2 \, \Re Z_0^{(2,2)} \Big) \\
&\geq |s|^2 |z|^2 \min\{\Re Z_0^{(1,1)}, \Re Z_0^{(2,2)}\} \equiv C|s|^2|z|^2 > 0.
\end{aligned}$$

□

## Appendix D. Choice of basis for the degenerate subspace

In the proof of Proposition 5.5 we need the following lemma.

**Lemma D.1.** *If* $u = \begin{pmatrix} u_1 \\ iu_2 \end{pmatrix} \neq 0$ *is an eigenfunction of* $L(\lambda)$ *with eigenvalue* $iE(\lambda)$, $E(\lambda) > 0$ *then*

$$\langle u_1, u_2 \rangle > 0. \tag{D-1}$$

*Proof.* The fact $L(\lambda)u = iE(\lambda)u$ yields

$$L_-(\lambda) u_2 = E(\lambda) u_1, \qquad L_+(\lambda) u_1 = E(\lambda) u_2. \tag{D-2}$$

Therefore,

$$\langle u_1, u_2 \rangle = \frac{1}{E(\lambda)} \langle L_-(\lambda) u_2, u_2 \rangle.$$

Equation (D-1) follows from the two claims that $L_-(\lambda)$ is a positive-definite selfadjoint operator on the space $\{v \mid v \perp \phi^\lambda\}$ and $u_2 \notin \operatorname{span}\{\phi^\lambda\}$. The first fact is well known; see for example [Weinstein 1986]. We prove the second by contradiction. Suppose that $u_2 = c\phi^\lambda$ for some constant $c$ then we have $L_-(\lambda) u_2 = 0$, which, together with (D-2) and the fact $E(\lambda) \neq 0$, implies $u_1 = u_2 = 0$, that is, $u = 0$. This contradicts to the fact $u \neq 0$. Thus $u_2 \notin \operatorname{span}\{\phi^\lambda\}$. □

*Proof of Proposition 5.5.* We start by constructing $N$ independent vectors $u_n \in \mathrm{span}\{v_1,\ v_2, \dots, v_N\}$ for $n = 1, 2, \dots, N$ such that the vector

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} u_n$$

is real. Suppose that

$$v_n = \begin{pmatrix} v_1^{(n)} \\ v_2^{(n)} \end{pmatrix}.$$

Then the definition of $L(\lambda)$ in (5-1) implies

$$\begin{pmatrix} \Re v_1^{(n)} \\ i\Im v_2^{(n)} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \Im v_1^{(n)} \\ -i\Re v_2^{(n)} \end{pmatrix}$$

are also eigenfunctions of $L(\lambda)$ with eigenvalues $iE(\lambda)$. This, together with the fact

$$\left\{ \begin{pmatrix} \Re v_1^{(n)} \\ i\Im v_2^{(n)} \end{pmatrix}, \begin{pmatrix} \Im v_1^{(n)} \\ -i\Re v_2^{(n)} \end{pmatrix},\ n = 1, 2, \dots, N \right\} = \{v_n,\ n = 1, 2, \dots, N\},$$

enables us to choose $N$ independent eigenfunctions $u_n$ with $n = 1, 2, \dots, N$ for $iE(\lambda)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} u_n$$

are real vectors.

Using (D-1) and a standard Gram–Schmidt procedure in linear algebra, one can find $N$ pairs of real functions $(\xi_n, \eta_n)$ for $n = 1, 2, \dots, N$ such that

$$\mathrm{span}\left\{ \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix},\ n = 1, 2, \dots, N \right\} = \mathrm{span}\{v_n, n = 1, 2, \dots, N\} \quad \text{and} \quad \langle \xi_n, \eta_m \rangle = \delta_{n,m}.$$

We now turn to the verification of (5-3). The observations

$$L_-(\lambda) - L_+(\lambda) = 2f'[(\phi^\lambda)^2](\phi^\lambda)^2$$

and

$$L_-(\lambda)\eta_n = E(\lambda)\xi_n, \qquad L_+(\lambda)\xi_n = E(\lambda)\eta_n$$

for $n = 1, 2, \dots, N$ yield

$$\int f'[(\phi^\lambda)^2](\phi^\lambda)^2(\xi_m\eta_n - \xi_n\eta_m)\,dx$$
$$= \tfrac{1}{2}\big(\langle \xi_m, L_-(\lambda)\eta_n\rangle - \langle L_+(\lambda)\xi_m, \eta_n\rangle - \langle \xi_n, L_-(\lambda)\eta_m\rangle + \langle L_+(\lambda)\xi_n, \eta_m\rangle\big) = 0.$$

Finally (5-4) is seen as follows:

$$\langle \phi^\lambda, \xi_n\rangle = \frac{1}{E(\lambda)}\langle \phi^\lambda, L_-(\lambda)\eta_n\rangle = \frac{1}{E(\lambda)}\langle L_-(\lambda)\phi^\lambda, \eta_n\rangle = 0,$$
$$\langle \partial_\lambda\phi^\lambda, \eta_n\rangle = \frac{1}{E(\lambda)}\langle \partial_\lambda\phi^\lambda, L_+(\lambda)\xi_n\rangle = -\frac{1}{E(\lambda)}\langle \phi^\lambda, \xi_n\rangle = 0 \qquad \square$$

## Appendix E. The identity $P_c(JH)^* J = J P_c(JH)$

**Proposition E.1.** *$L = JH$ and $H = H^*$ imply*

$$P_c(L)^* J = J P_c(L).$$

*Proof.* Represent $P_c(L)$ as a Riesz projection

$$P_c(L) = \frac{1}{2\pi i} \oint (zI - JH)^{-1}\, dz$$

where the contour of the integration is counterclockwise. Moreover, the essential spectrum of $L$ is

$$(-i\infty, -i\lambda] \cup [i\lambda, i\infty).$$

The spectrum associated with the upper branch $[i\lambda, i\infty)$ is given by

$$P^+(JH) = \tfrac{1}{2\pi}(A - B),$$

where

$$A = \int_\lambda^\infty (i\tau + 0 - JH)^{-1}\, d\tau, \qquad B = \int_\lambda^\infty (i\tau - 0 - JH)^{-1}\, d\tau.$$

We claim that

$$A^* J = -JB, \qquad B^* J = -JA. \tag{E-1}$$

This implies

$$(P^+(JH))^* J = J P^+(JH), \qquad (P_c(JH))^* J = J P_c(JH).$$

To complete the proof of the proposition, we now prove (E-1). By direct computation using $J^* = -J$ we have

$$A^* = \int_\lambda^\infty (-i\tau + 0 + HJ)^{-1}\, d\tau.$$

Therefore,

$$A^* J = \int_\lambda^\infty (J(i\tau)J - J0J - JJHJ)^{-1}\, d\tau\; J = \int_\lambda^\infty (-J)(i\tau - 0 - JH)^{-1}(-J)\, d\tau\; J = -JB,$$

thus proving the first identity in (E-1). The second can be proved similarly. □

## Appendix F. Time convolution lemmas

*Proof of Proposition 11.2.* In what follows we only prove the case $\sigma = 1$ of (11-6); the other cases and (11-7) are similar.

$$I(t) := \int_0^t \frac{1}{(1+t-s)^{3/2}} \frac{1}{T_0 + s}\, ds \le \frac{1}{(1+\frac{t}{2})^{3/2}} \int_0^{t/2} \frac{1}{T_0+s}\, ds + \frac{1}{T_0 + \frac{t}{2}} \int_{t/2}^t \frac{1}{(1+t-s)^{3/2}}\, ds$$
$$\le \frac{\log(1 + \frac{t}{2T_0})}{(1+\frac{t}{2})^{3/2}} + \frac{2}{T_0 + \frac{t}{2}}.$$

On the other hand, we also have

$$\int_0^t \frac{1}{(1+t-s)^{3/2}} \frac{1}{T_0+s}\, ds \le \frac{2}{T_0}.$$

Thus,

$$I(t) \le c_1 \min\left\{\frac{1}{T_0}, \frac{1}{1+t}\right\}.$$

We now claim that for some constant $c > 0$,

$$I(t) \le \frac{c}{T_0+t}.$$

It sufficies to find a constant $c$ independent of $T_0$ and $t$ such that

$$m(t) := (T_0+t) \min\left\{\frac{1}{T_0}, \frac{1}{1+t}\right\} \le c.$$

If $t$ is such that the above minimum is $T_0^{-1}$ then $T_0^{-1} \le (1+t)^{-1}$, that is, $t \le T_0 - 1$. Therefore,

$$m(t) \le \frac{2T_0-1}{T_0} \le \frac{3}{2}.$$

If $t$ is such that the above minimum is $(1+t)^{-1}$ then $t \ge T_0 - 1$. Therefore,

$$m(t) \le \frac{2T_0-1}{T_0}$$

since $m(t)$ is decreasing with $t$. Since $T \ge 2$, $m(t) \le 3/2$. This completes the proof. □

## Appendix G. Bounds on solutions to a weakly perturbed ODE

*Proof of Lemma 11.14.* Let $\beta$ denote the solution to the differential equation

$$\partial_t |\beta_\rho|^2 = -|\beta_\rho|^4 + g, \qquad |\beta_\rho|^2(0) = |z(0)|^2 - \rho$$

for $\rho > 0$. Since

$$\partial_t(|z(t)|^2 - |\beta_\rho(t)|^2) \le -|z(t)|^4 + |\beta_\rho(t)|^4 = -(|z(t)|^2 + |\beta_\rho(t)|^2)\,(|z(t)|^2 - |\beta_\rho(t)|^2)$$

with the initial condition

$$|z(0)|^2 - |\beta_\rho(0)|^2 = \rho > 0.$$

Thus $|z(t)|^2 \le |\beta_\rho(t)|^2$ for all $t \ge 0$. Letting $\rho$ tend to zero, we have

$$|z(t)|^2 \le |\beta(t)|^2$$

so it suffices to prove the bound:

$$|\beta(t)| \le (1 + K c_\# T_0^{-\delta})\,(\kappa + t)^{-1/2},$$

where $\kappa = \min\{T_0, |w_0|^{-2}\}$ and $\beta(t)$ solves the initial value problem

$$\partial_t |\beta|^2 = -|\beta|^4 + g, \qquad |\beta(0)|^2 = |w_0|^2. \tag{G-1}$$

The proof of (11-14) for $\beta$ is divided into two cases:

*Case* $|w_0| \geq T_0^{-1/2}$. By local existence of the solutions for the initial value problem (G-1), we have that for some $t_1 > 0$

$$\frac{1}{2(T_0 + t)^{1/2}} \leq |\beta(t)| \tag{G-2}$$

with $t \in [0, t_1]$. Then using the assumed bound on $g(t)$ in (11-13) we have

$$|g(t)| \leq \frac{c_\#}{(T_0 + t)^{2+\delta}} = \frac{c_\#}{(T_0 + t)^2} \frac{1}{(T_0 + t)^\delta} \leq 2^4 c_\# |\beta(t)|^4 \cdot \frac{1}{T_0^\delta} = c_{1\#} T_0^{-\delta} |\beta(t)|^4$$

where $c_{1\#} := 2^4 c_\#$. It follows from (G-1) that

$$\partial_t |\beta(t)|^2 \leq -(1 - c_{1\#} T_0^{-\delta}) |\beta(t)|^4$$

or

$$\partial_t |\beta(t)|^{-2} \geq 1 - c_{1\#} T_0^{-\delta}.$$

Integration over the interval $[0, t]$ for $t \leq t_1$ yields

$$|\beta(t)| \leq \frac{1 + c_{2\#} T_0^{-\delta}}{(|w_0|^{-2} + t)^{1/2}} \tag{G-3}$$

where $c_{2\#} \sim c_{1\#} \sim c_\#$ and we use that $c_\# T_0^{-\delta}$ is sufficiently small. Now set $\kappa = \min\{|w_0|^{-2}, T_0\}$ and we have

$$|z(t)| \leq |\beta(t)| \leq \frac{1 + c_{2\#} T_0^{-\delta}}{(\kappa + t)^{1/2}}$$

for $0 \leq t \leq t_1$. Now let $[0, \Xi)$ denote the maximal subset of $\mathbb{R}_+$, on which the upper bound in (G-3) holds. If $\Xi < \infty$ then by continuity and the assumption that $|w_0| \geq T_0^{-1/2}$ we have

$$|\beta(\Xi)| = \frac{1 + c_{2\#} T_0^{-\delta}}{(|w_0|^{-2} + \Xi)^{1/2}} \geq \frac{1}{(|w_0|^{-2} + \Xi)^{1/2}} \geq \frac{3}{4} \frac{1}{(T_0 + \Xi)^{1/2}},$$

implying (see (G-2)) that the above argument can be applied beyond $t = \Xi$, contradicting its maximality.

*Case* $|w_0| < T_0^{-1/2}$. Denote by $\beta_1(t)$ the solution to (11-12) with the initial condition $\beta_1(0) = T_0^{-1/2}$. As shown in the previous case

$$|\beta_1(t)| \leq (1 + K\ c_\# T_0^{-\delta})(T_0 + t)^{-1/2}.$$

Observing that

$$\partial_t (|\beta|^2 - |\beta_1|^2) = -(|\beta|^2 + |\beta_1|^2)(|\beta|^2 - |\beta_1|^2), \qquad |\beta(0)|^2 - |\beta_1(0)|^2 < 0,$$

we have $|\beta(t)|^2 \leq |\beta_1(t)|^2$ for any time $t$. This, together with the estimate of $\beta_1$, completes the proof of the second case. □

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