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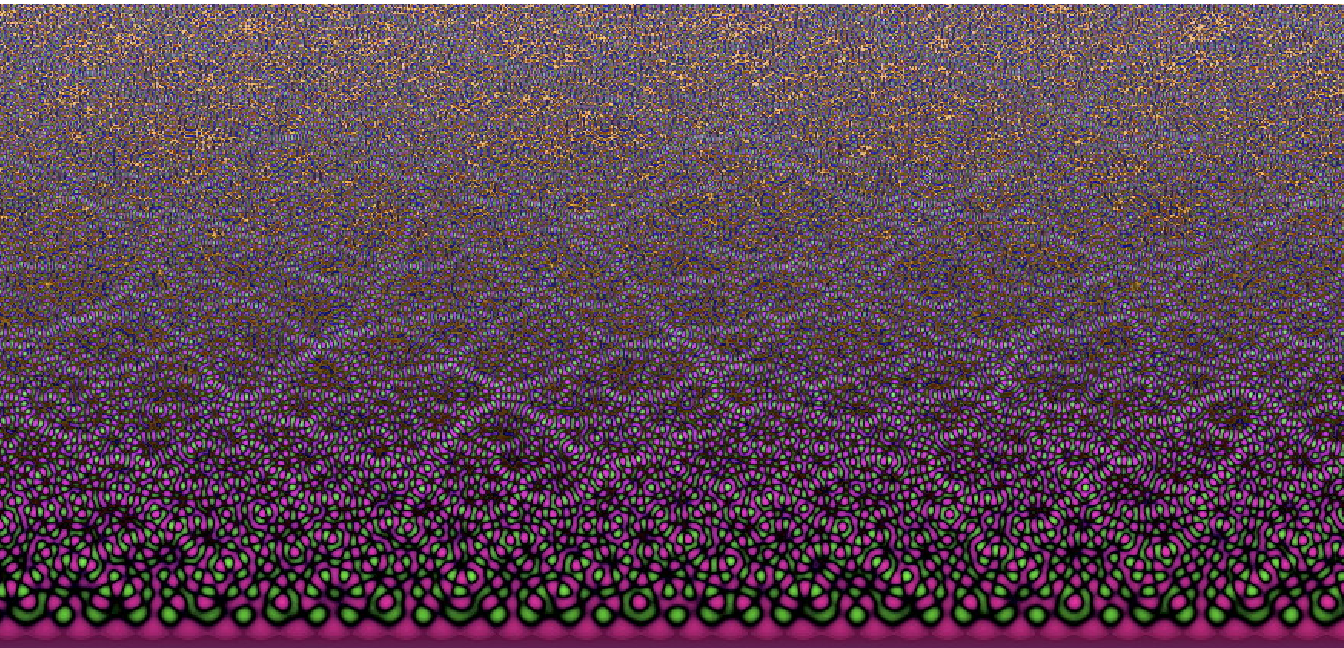

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THE PSEUDOSPECTRUM OF SYSTEMS OF SEMICLASSICAL OPERATORS

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The pseudospectrum (or spectral instability) of non-self-adjoint operators is a topic of current interest in applied mathematics. In fact, for non-self-adjoint operators the resolvent could be very large outside the spectrum, making numerical computation of the complex eigenvalues very hard. This has importance, for example, in quantum mechanics, random matrix theory and fluid dynamics.

The occurrence of false eigenvalues (or pseudospectrum) of non-self-adjoint semiclassical differential operators is due to the existence of quasimodes, that is, approximate local solutions to the eigenvalue problem. For scalar operators, the quasimodes appear generically since the bracket condition on the principal symbol is not satisfied for topological reasons.

In this paper we shall investigate how these results can be generalized to square systems of semiclassical differential operators of principal type. These are the systems whose principal symbol vanishes of first order on its kernel. We show that the resolvent blows up as in the scalar case, except in a nowhere dense set of degenerate values. We also define quasisymmetrizable systems and systems of subelliptic type, for which we prove estimates on the resolvent.

1. Introduction

In this paper we shall study the pseudospectrum or spectral instability of square non-self-adjoint semiclassical systems of principal type. Spectral instability of non-self-adjoint operators is currently a topic of interest in applied mathematics; see [Davies 2002] and [Trefethen and Embree 2005]. It arises from the fact that, for non-self-adjoint operators, the resolvent could be very large in an open set containing the spectrum. For semiclassical differential operators, this is due to the bracket condition and is connected to the problem of solvability. In applications where one needs to compute the spectrum, the spectral instability has the consequence that discretization and round-off errors give false spectral values, so-called pseudospectra; see [Trefethen and Embree 2005] and references there.

We shall consider bounded systems $P(h)$ of semiclassical operators given by (2.2), and we shall generalize the results of the scalar case in [Dencker et al. 2004]. Actually, the study of unbounded operators can in many cases be reduced to the bounded case; see Proposition 2.20 and Remark 2.21. We shall also study semiclassical operators with analytic symbols in the case when the symbols can be extended analytically to a tubular neighborhood of the phase space satisfying (2.3). The operators we study will be of principal type, which means that the principal symbol vanishes of first order on the kernel; see Definition 3.1.

The definition of *semiclassical* pseudospectrum in [Dencker et al. 2004] is essentially the bracket condition, which is suitable for symbols of principal type. By instead using the definition of (injectivity)

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pseudospectrum in [Pravda-Starov 2006a] we obtain a more refined view of the spectral instability; see Definition 2.27. For example, z is in the pseudospectrum of infinite index for $P(h)$ if for any N the resolvent norm blows up faster than any power of the semiclassical parameter:

$$\|(P(h) - z\,\mathrm{Id})^{-1}\| \geq C_N h^{-N} \qquad 0 < h \ll 1 \tag{1.1}$$

In [Dencker et al. 2004] it was proved that (1.1) holds almost everywhere in the *semiclassical* pseudospectrum. We shall generalize this to systems and prove that for systems of principal type, except for a nowhere dense set of degenerate values, the resolvent blows up as in the scalar case; see Theorem 3.10. The complication is that the eigenvalues don't have constant multiplicity in general, only almost everywhere.

At the boundary of the semiclassical pseudospectrum, we obtained in [Dencker et al. 2004] a bound on the norm of the semiclassical resolvent, under the additional condition of having no unbounded (or closed) bicharacteristics. In the systems case, the picture is more complicated and it seems to be difficult to get an estimate on the norm of the resolvent using only information about the eigenvalues, even in the principal type case; see Example 4.1. In fact, the norm is essentially preserved under multiplication with elliptic systems, but the eigenvalues are changed. Also, the multiplicities of the eigenvalues could be changing at all points on the boundary of the eigenvalues; see Example 3.9. We shall instead introduce *quasisymmetrizable* systems, which generalize the normal forms of the scalar symbols at the boundary of the eigenvalues; see Definition 4.5. Quasisymmetrizable systems are of principal type and we obtain estimates on the resolvent as in the scalar case; see Theorem 4.15.

For boundary points of *finite type*, we obtained in [Dencker et al. 2004] subelliptic types of estimates on the semiclassical resolvent. This is the case when one has nonvanishing higher order brackets. For systems the situation is less clear; there seems to be no general results on the subellipticity for systems. In fact, the real and imaginary parts do not commute in general, making the bracket condition meaningless. Even when they do, Example 5.2 shows that the bracket condition is not sufficient for subelliptic types of estimates. Instead we shall introduce invariant conditions on the order of vanishing of the symbol along the bicharacteristics of the eigenvalues. For systems, there could be several (limit) bicharacteristics of the eigenvalues going through a characteristic point; see Example 5.9. Therefore we introduce the *approximation* property in Definition 5.10 which gives that the all (limit) bicharacteristics of the eigenvalues are parallel at the characteristics; see Remark 5.11. The general case presently looks too complicated to handle. We shall generalize the property of being of finite type to systems, introducing systems of *subelliptic type*. These are quasisymmetrizable systems satisfying the approximation property, such that the imaginary part on the kernel vanishes of finite order along the bicharacteristics of the real part of the eigenvalues. This definition is invariant under multiplication with invertible systems and taking adjoints, and for these systems we obtain subelliptic types of estimates on the resolvent; see Theorem 5.20.

As an example, we may look at

$$P(h) = h^2 \Delta\,\mathrm{Id}_N + i K(x)$$

where $\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2$ is the positive Laplacian, and $K(x) \in C^\infty(\mathbb{R}^n)$ is a symmetric $N \times N$ system. If we assume some conditions of ellipticity at infinity for $K(x)$, we may reduce to the case of bounded symbols by Proposition 2.20 and Remark 2.21; see Example 2.22. Then we obtain that $P(h)$ has discrete spectrum in the right half plane $\{z : \operatorname{Re} z \geq 0\}$, and in the first quadrant if $K(x) \geq 0$, by Proposition 2.19.

We obtain from Theorem 3.10 that the L^2 operator norm of the resolvent grows faster than any power of h as $h \to 0$, thus (1.1) holds for almost all values z such that $\operatorname{Re} z > 0$ and $\operatorname{Im} z$ is an eigenvalue of K; see Example 3.12.

For $\operatorname{Re} z = 0$ and almost all eigenvalues $\operatorname{Im} z$ of K, we find from Theorem 5.20 that the norm of the resolvent is bounded by $Ch^{-2/3}$; see Example 5.22. In the case $K(x) \ge 0$ and $K(x)$ is invertible at infinity, we find from Theorem 4.15 that the norm of the resolvent is bounded by Ch^{-1} for $\operatorname{Re} z > 0$ and $\operatorname{Im} z = 0$ by Example 4.17. The results in this paper are formulated for operators acting on the trivial bundle over $\mathbb{R}^n$. But since our results are mainly local, they can be applied to operators on sections of fiber bundles.

2. The definitions

We shall consider $N \times N$ systems of semiclassical pseudo-differential operators, and use the Weyl quantization:

$$P^w(x, hD_x)u = \frac{1}{(2\pi)^n} \iint_{T^*\mathbb{R}^n} P\Big(\frac{x+y}{2}, h\xi\Big) e^{i\langle x-y,\xi\rangle} u(y)\, dy\, d\xi \tag{2.1}$$

for matrix valued $P \in C^\infty(T^*\mathbb{R}^n, \mathscr{L}(\mathbb{C}^N, \mathbb{C}^N))$. We shall also consider the semiclassical operators

$$P(h) \sim \sum_{j=0}^{\infty} h^j P_j^w(x, hD) \tag{2.2}$$

with $P_j \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n, \mathscr{L}(\mathbb{C}^N, \mathbb{C}^N))$. Here C_{b}^∞ is the set of C^∞ functions having all derivatives in L^∞ and $P_0 = \sigma(P(h))$ is the principal symbol of $P(h)$. The operator is said to be elliptic if the principal symbol P_0 is invertible, and of principal type if P_0 vanishes of first order on the kernel; see Definition 3.1. Since the results in the paper only depend on the principal symbol, one could also have used the Kohn–Nirenberg quantization because the different quantizations only differ in the lower order terms. We shall also consider operators with analytic symbols; then we shall assume that $P_j(w)$ are bounded and holomorphic in a tubular neighborhood of $T^*\mathbb{R}^n$ satisfying

$$\|P_j(z,\zeta)\| \le C_0 C^j j^j \qquad |\operatorname{Im}(z,\zeta)| \le 1/C \quad \forall\, j \ge 0 \tag{2.3}$$

which will give exponentially small errors in the calculus, here $\|A\|$ is the norm of the matrix A. But the results hold for more general analytic symbols; see Remarks 3.11 and 4.19. In the following, we shall use the notation $w = (x, \xi) \in T^*\mathbb{R}^n$.

We shall consider the spectrum $\operatorname{Spec} P(h)$ which is the set of values λ such that the resolvent $(P(h) - \lambda \operatorname{Id}_N)^{-1}$ is a bounded operator, here Id_N is the identity in $\mathbb{C}^N$. The spectrum of $P(h)$ is essentially contained in the spectrum of the principal symbol $P(w)$, which is given by

$$|P(w) - \lambda \operatorname{Id}_N| = 0$$

where $|A|$ is the determinant of the matrix A. For example, if $P(w) = \sigma(P(h))$ is bounded and z_1 is not an eigenvalue of $P(w)$ for any $w = (x, \xi)$ (or a limit eigenvalue at infinity) then $P(h) - z_1 \operatorname{Id}_N$ is invertible by Proposition 2.19. When $P(w)$ is an unbounded symbol one needs additional conditions; see for example Proposition 2.20. We shall mostly restrict our study to bounded symbols, but we can reduce to this case if $P(h) - z_1 \operatorname{Id}_N$ is invertible, by considering

$$(P(h) - z_1 \operatorname{Id}_N)^{-1}(P(h) - z_2 \operatorname{Id}_N) \qquad z_2 \neq z_1$$

see Remark 2.21. But unless we have conditions on the eigenvalues at infinity, this does not always give a bounded operator.

Example 2.1. Let

$$P(\xi) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \qquad \xi \in \mathbb{R}.$$

Then 0 is the only eigenvalue of $P(\xi)$ but

$$(P(\xi) - z \operatorname{Id}_N)^{-1} = -1/z \begin{pmatrix} 1 & \xi/z \\ 0 & 1 \end{pmatrix}$$

and $(P^w - z \operatorname{Id}_N)^{-1} P^w = -z^{-1} P^w$ is unbounded for any $z \neq 0$.

Definition 2.2. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system. We denote the closure of the set of eigenvalues of P by

$$\Sigma(P) = \overline{\{\lambda \in \mathbb{C} : \exists\, w \in T^*\mathbb{R}^n,\ |P(w) - \lambda \operatorname{Id}_N| = 0\}}$$

and the eigenvalues at infinity:

$$\Sigma_\infty(P) = \left\{\lambda \in \mathbb{C} : \exists\, w_j \to \infty\ \exists\, u_j \in \mathbb{C}^N \setminus 0;\ |P(w_j)u_j - \lambda u_j|/|u_j| \to 0,\ j \to \infty\right\}$$

which is closed in $\mathbb{C}$.

In fact, that $\Sigma_\infty(P)$ is closed follows by taking a suitable diagonal sequence. Observe that as in the scalar case, we could have $\Sigma_\infty(P) = \Sigma(P)$, for example if $P(w)$ is constant in one direction. It follows from the definition that $\lambda \notin \Sigma_\infty(P)$ if and only if the resolvent is defined and bounded when $|w|$ is large enough:

$$\|(P(w) - \lambda \operatorname{Id}_N)^{-1}\| \le C \qquad |w| \gg 1 \tag{2.4}$$

In fact, if (2.4) does not hold there would exist $w_j \to \infty$ such that $\|(P(w_j) - \lambda \operatorname{Id}_N)^{-1}\| \to \infty$, $j \to \infty$. Thus, there would exist $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $P(w_j)u_j - \lambda u_j \to 0$. On the contrary, if (2.4) holds then $|P(w)u - \lambda u| \ge |u|/C$ for any $u \in \mathbb{C}^N$ and $|w| \gg 1$.

It is clear from the definition that $\Sigma_\infty(P)$ contains all finite limits of eigenvalues of P at infinity. In fact, if $P(w_j)u_j = \lambda_j u_j$, $|u_j| = 1$, $w_j \to \infty$ and $\lambda_j \to \lambda$ then

$$P(w_j)u_j - \lambda u_j = (\lambda_j - \lambda)u_j \to 0.$$

Example 2.1 shows that in general $\Sigma_\infty(P)$ could be a larger set.

Example 2.3. Let $P(\xi)$ be given by Example 2.1; then $\Sigma(P) = \{0\}$ but $\Sigma_\infty(P) = \mathbb{C}$. In fact, for any $\lambda \in \mathbb{C}$ we find

$$|P(\xi)u_\xi - \lambda u_\xi| = \lambda^2 \quad \text{when} \quad u_\xi = {}^t(\xi, \lambda).$$

We have that $|u_\xi| = \sqrt{|\lambda|^2 + \xi^2} \to \infty$ so $|P(\xi)u_\xi - \lambda u_\xi|/|u_\xi| \to 0$ when $|\xi| \to \infty$.

For bounded symbols we get equality according to the following proposition.

Proposition 2.4. *If $P \in C_{\mathrm{b}}^{\infty}(T^*\mathbb{R}^n)$ is an $N \times N$ system then $\Sigma_\infty(P)$ is the set of all limits of the eigenvalues of P at infinity.*

Proof. Since $\Sigma_\infty(P)$ contains all limits of eigenvalues of P at infinity, we only have to prove the opposite inclusion. Let $\lambda \in \Sigma_\infty(P)$ then by the definition there exist $w_j \to \infty$ and $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $|P(w_j)u_j - \lambda u_j| = \varepsilon_j \to 0$. Then we may choose $N \times N$ matrix A_j such that $\|A_j\| = \varepsilon_j$ and $P(w_j)u_j = \lambda u_j + A_j u_j$ thus λ is an eigenvalue of $P(w_j) - A_j$. Now if A and B are $N \times N$ matrices and $d(\mathrm{Eig}(A), \mathrm{Eig}(B))$ is the minimal distance between the sets of eigenvalues of A and B under permutations, then we have that $d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \to 0$ when $\|A - B\| \to 0$. In fact, a theorem of Elsner [1985] gives

$$d(\mathrm{Eig}(A), \mathrm{Eig}(B)) \le N(2\max(\|A\|, \|B\|))^{1-1/N} \|A - B\|^{1/N}.$$

Since the matrices $P(w_j)$ are uniformly bounded we find that they have an eigenvalue μ_j such that $|\mu_j - \lambda| \le C_N \varepsilon_j^{1/N} \to 0$ as $j \to \infty$, thus $\lambda = \lim_{j\to\infty} \mu_j$ is a limit of eigenvalues of $P(w)$ at infinity. □

One problem with studying systems $P(w)$, is that the eigenvalues are not very regular in the parameter w, generally they depend only continuously (and eigenvectors measurably) on w.

Definition 2.5. For an $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ we define

$$\kappa_P(w, \lambda) = \mathrm{Dim}\,\mathrm{Ker}(P(w) - \lambda\,\mathrm{Id}_N)$$
$$K_P(w, \lambda) = \max\left\{k : \partial_\lambda^j p(w, \lambda) = 0 \text{ for } j < k\right\}$$

where $p(w, \lambda) = |P(w) - \lambda\,\mathrm{Id}_N|$ is the characteristic polynomial. We have $\kappa_P \le K_P$ with equality for symmetric systems but in general we need not have equality; see Example 2.7. If

$$\Omega_k(P) = \left\{(w, \lambda) \in T^*\mathbb{R}^n \times \mathbb{C} : K_P(w, \lambda) \ge k\right\} \qquad k \ge 1,$$

then $\varnothing = \Omega_{N+1}(P) \subseteq \Omega_N(P) \subseteq \cdots \subseteq \Omega_1(P)$ and we may define

$$\Xi(P) = \bigcup_{j>1} \partial\Omega_j(P)$$

where $\partial\Omega_j(P)$ is the boundary of $\Omega_j(P)$ in the relative topology of $\Omega_1(P)$.

Clearly, $\Omega_j(P)$ is a closed set for any $j \ge 1$. By definition we find that the multiplicity K_P of the zeros of $|P(w) - \lambda\,\mathrm{Id}_N|$ is locally constant on $\Omega_1(P) \setminus \Xi(P)$. If $P(w)$ is symmetric then $\kappa_P = \mathrm{Dim}\,\mathrm{Ker}(P(w) - \lambda\,\mathrm{Id}_N)$ also is constant on $\Omega_1(P) \setminus \Xi(P)$. This is not true in general; see Example 3.9.

Remark 2.6. We find that $\Xi(P)$ is closed and nowhere dense in $\Omega_1(P)$ since it is the union of boundaries of closed sets. We also find that

$$(w, \lambda) \in \Xi(P) \Leftrightarrow (w, \bar{\lambda}) \in \Xi(P^*)$$

since $|P^* - \bar{\lambda}\,\mathrm{Id}_N| = \overline{|P - \lambda\,\mathrm{Id}_N|}$.

Example 2.7. If

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$, then

$$\Omega_1(P) = \{(w, \lambda) : \lambda = \lambda_j(w),\ \ j = 1,\ 2\}$$
$$\Omega_2(P) = \{(w, \lambda) : \lambda = \lambda_1(w) = \lambda_2(w)\},$$

but $\kappa_P \equiv 1$ on $\Omega_1(P)$.

Example 2.8. Let

$$P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \qquad t \in \mathbb{R}.$$

Then $P(t)$ has the eigenvalues $\pm\sqrt{t}$, and $\kappa_P \equiv 1$ on $\Omega_1(P)$.

Example 2.9. If

$$P = \begin{pmatrix} w_1 + w_2 & w_3 \\ w_3 & w_1 - w_2 \end{pmatrix}$$

then

$$\Omega_1(P) = \left\{(w; \lambda_j) : \lambda_j = w_1 + (-1)^j \sqrt{w_2^2 + w_3^2},\ \ j = 1,\ 2\right\}.$$

We have that $\Omega_2(P) = \{(w_1, 0, 0; w_1) : w_1 \in \mathbb{R}\}$ and $\kappa_P = 2$ on $\Omega_2(P)$.

Definition 2.10. Let π_j be the projections

$$\pi_1(w, \lambda) = w \quad \text{and} \quad \pi_2(w, \lambda) = \lambda.$$

Then we define for $\lambda \in \mathbb{C}$ the closed sets

$$\Sigma_\lambda(P) = \pi_1\big(\Omega_1(P) \cap \pi_2^{-1}(\lambda)\big) = \{w : |P(w) - \lambda \operatorname{Id}_N| = 0\}$$
$$X(P) = \pi_1(\Xi(P)) \subseteq T^*\mathbb{R}^n.$$

Remark 2.11. Observe that $X(P)$ is nowhere dense in $T^*\mathbb{R}^n$ and $P(w)$ has constant characteristics near $w_0 \notin X(P)$. This means that $|P(w) - \lambda \operatorname{Id}_N| = 0$ if and only if $\lambda = \lambda_j(w)$ for $j = 1, \dots k$, where the eigenvalues $\lambda_j(w) \neq \lambda_k(w)$ for $j \neq k$ when $|w - w_0| \ll 1$.

In fact, $\pi_1^{-1}(w)$ is a finite set for any $w \in T^*\mathbb{R}^n$ and since the eigenvalues are continuous functions of the parameters, the relative topology on $\Omega_1(P)$ is generated by $\pi_1^{-1}(\omega) \cap \Omega_1(P)$ for open sets $\omega \subset T^*\mathbb{R}^n$.

Definition 2.12. For an $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ we define the *weakly singular eigenvalue set*

$$\Sigma_{\mathrm{ws}}(P) = \pi_2(\Xi(P)) \subseteq \mathbb{C}$$

and the *strongly singular eigenvalue set*

$$\Sigma_{\mathrm{ss}}(P) = \left\{\lambda : \pi_2^{-1}(\lambda) \cap \Omega_1(P) \subseteq \Xi(P)\right\}.$$

Remark 2.13. It is clear from the definition that $\Sigma_{\mathrm{ss}}(P) \subseteq \Sigma_{\mathrm{ws}}(P)$. We have that $\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$ and $\Sigma_{\mathrm{ss}}(P) \cup \Sigma_\infty(P)$ are closed, and $\Sigma_{\mathrm{ss}}(P)$ is nowhere dense.

In fact, if $\lambda_j \to \lambda \notin \Sigma_\infty(P)$, then $\pi_2^{-1}(\lambda_j) \cap \Omega_1(P)$ is contained in a compact set for $j \gg 1$, which then either intersects $\Xi(P)$ or is contained in $\Xi(P)$. Since $\Xi(P)$ is closed, these properties are preserved in the limit.

Also, if $\lambda \in \overline{\Sigma_{\mathrm{ss}}(P)}$, then there exists $(w_j, \lambda_j) \in \Xi(P)$ such that $\lambda_j \to \lambda$ as $j \to \infty$. Since $\Xi(P)$ is nowhere dense in $\Omega_1(P)$, there exists $(w_{jk}, \lambda_{jk}) \in \Omega_1(P) \setminus \Xi(P)$ converging to (w_j, λ_j) as $k \to \infty$. Then $\Sigma(P) \setminus \Sigma_{\mathrm{ss}}(P) \ni \lambda_{jj} \to \lambda$, so $\Sigma_{\mathrm{ss}}(P)$ is nowhere dense. On the other hand, it is possible that $\Sigma_{\mathrm{ws}}(P) = \Sigma(P)$ by the following example.

Example 2.14. Let $P(w)$ be the system in Example 2.9; then we have

$$\Sigma_{\mathrm{ws}}(P) = \Sigma(P) = \mathbb{R}$$

and $\Sigma_{\mathrm{ss}}(P) = \varnothing$. In fact, the eigenvalues coincide only when $w_2 = w_3 = 0$ and the eigenvalue $\lambda = w_1$ is also attained at some point where $w_2 \neq 0$. If we multiply $P(w)$ with $w_4 + iw_5$, we obtain that $\Sigma_{\mathrm{ws}}(P) = \Sigma(P) = \mathbb{C}$. If we set $\widetilde{P}(w_1, w_2) = P(0, w_1, w_2)$ we find that $\Sigma_{\mathrm{ss}}(\widetilde{P}) = \Sigma_{\mathrm{ws}}(\widetilde{P}) = \{0\}$.

Lemma 2.15. *Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system. If $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$ then there exists a unique C^∞ function $\lambda(w)$ so that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of (w_0, λ_0). If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ then there is $\lambda(w) \in C^\infty$ such that $(w, \lambda) \in \Omega_1(P)$ if and only if $\lambda = \lambda(w)$ in a neighborhood of $\Sigma_{\lambda_0}(P)$.*

We find from Lemma 2.15 that $\Omega_1(P) \setminus \Xi(P)$ is locally given as a C^∞ manifold over $T^*\mathbb{R}^n$, and that the eigenvalues $\lambda_j(w) \in C^\infty$ outside $X(P)$. This is not true if we instead assume that κ_P is constant on $\Omega_1(P)$; see Example 2.8.

Proof. If $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$, then

$$\lambda \to |P(w) - \lambda\,\mathrm{Id}_N|$$

vanishes of exactly order $k > 0$ on $\Omega_1(P)$ in a neighborhood of (w_0, λ_0), so

$$\partial_\lambda^k |P(w_0) - \lambda\,\mathrm{Id}_N| \neq 0 \qquad \text{for } \lambda = \lambda_0.$$

Thus $\lambda = \lambda(w)$ is the unique solution to $\partial_\lambda^{k-1}|P(w) - \lambda\,\mathrm{Id}_N| = 0$ near w_0 which is C^∞ by the Implicit Function Theorem.

If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ then we obtain this in a neighborhood of any $w_0 \in \Sigma_{\lambda_0}(P) \Subset T^*\mathbb{R}^n$. Using a C^∞ partition of unity we find by uniqueness that $\lambda(w) \in C^\infty$ in a neighborhood of $\Sigma_{\lambda_0}(P)$. □

Remark 2.16. Observe that if $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ and $\lambda(w) \in C^\infty$ satisfies $|P(w) - \lambda(w)\,\mathrm{Id}_N| \equiv 0$ near $\Sigma_{\lambda_0}(P)$ and $\lambda|_{\Sigma_{\lambda_0}(P)} = \lambda_0$, then we find by Sard's Theorem that $d\,\mathrm{Re}\,\lambda$ and $d\,\mathrm{Im}\,\lambda$ are linearly independent on the codimension 2 manifold $\Sigma_\mu(P)$ for almost all values μ close to λ_0. Thus for $n = 1$ we find that $\Sigma_\mu(P)$ is a discrete set for almost all values μ close to λ_0.

In fact, since $\lambda_0 \notin \Sigma_\infty(P)$ we find that $\Sigma_\mu(P) \to \Sigma_{\lambda_0}(P)$ when $\mu \to \lambda_0$ so $\Sigma_\mu(P) = \{w : \lambda(w) = \mu\}$ for $|\mu - \lambda_0| \ll 1$.

Definition 2.17. A C^∞ function $\lambda(w)$ is called a *germ of eigenvalues* at w_0 for the $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ if

$$|P(w) - \lambda(w)\,\mathrm{Id}_N| \equiv 0 \qquad \text{in a neighborhood of } w_0.$$

If this holds in a neighborhood of every point in $\omega \Subset T^*\mathbb{R}^n$ then we say that $\lambda(w)$ is a germ of eigenvalues for P on ω.

Remark 2.18. If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ss}}(P) \cup \Sigma_\infty(P))$ then there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$. By Lemma 2.15 there exists a C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$. If $\lambda_0 \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ then there exists a C^∞ germ $\lambda(w)$ of eigenvalues on $\Sigma_{\lambda_0}(P)$.

As in the scalar case we obtain that the spectrum is essentially discrete outside $\Sigma_\infty(P)$.

Proposition 2.19. *Assume that the $N \times N$ system $P(h)$ is given by* (2.2) *with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$. Let Ω be an open connected set, satisfying*

$$\overline{\Omega} \cap \Sigma_\infty(P) = \varnothing \quad \textit{and} \quad \Omega \cap \complement\Sigma(P) \neq \varnothing.$$

Then $(P(h) - z\,\mathrm{Id}_N)^{-1}$, $0 < h \ll 1$, $z \in \Omega$, is a meromorphic family of operators with poles of finite rank. In particular, for h sufficiently small, the spectrum of $P(h)$ is discrete in any such set. When $\Omega \cap \Sigma(P) = \varnothing$ we find that Ω contains no spectrum of $P^w(x, hD)$.

Proof. We shall follow the proof of Proposition 3.3 in [Dencker et al. 2004]. If Ω satisfies the assumptions of the proposition then there exists $C > 0$ such that

$$|(P(w) - z\,\mathrm{Id}_N)^{-1}| \le C \qquad \text{if } z \in \Omega \text{ and } |w| > C\ . \tag{2.5}$$

In fact, otherwise there would exist $w_j \to \infty$ and $z_j \in \Omega$ such that $|(P(w_j) - z_j\,\mathrm{Id}_N)^{-1}| \to \infty$, $j \to \infty$. Thus, there exists $u_j \in \mathbb{C}^N$ such that $|u_j| = 1$ and $P(w_j)u_j - z_j u_j \to 0$. Since $\Sigma(P) \Subset \mathbb{C}$ we obtain after picking a subsequence that $z_j \to z \in \overline{\Omega} \cap \Sigma_\infty(P) = \varnothing$. The assumption that $\Omega \cap \complement\Sigma(p) \neq \varnothing$ implies that for some $z_0 \in \Omega$ we have $(P(w) - z_0\,\mathrm{Id}_N)^{-1} \in C_{\mathrm{b}}^\infty$. Let $\chi \in C_0^\infty(T^*\mathbb{R}^n)$, $0 \le \chi(w) \le 1$ and $\chi(w) = 1$ when $|w| \le C$, where C is given by (2.5). Let

$$R(w, z) = \chi(w)(P(w) - z_0\,\mathrm{Id}_N)^{-1} + (1 - \chi(w))(P(w) - z\,\mathrm{Id}_N)^{-1} \in C_{\mathrm{b}}^\infty$$

for $z \in \Omega$. The symbolic calculus then gives

$$\begin{aligned} R^w(x, hD, z)(P(h) - z\,\mathrm{Id}_N) &= I + hB_1(h, z) + K_1(h, z) \\ (P(h) - z\,\mathrm{Id}_N)R^w(x, hD, z) &= I + hB_2(h, z) + K_2(h, z) \end{aligned}$$

where $K_j(h, z)$ are compact operators on $L^2(\mathbb{R}^n)$ depending holomorphically on z, vanishing for $z = z_0$, and $B_j(h, z)$ are bounded on $L^2(\mathbb{R}^n)$, $j = 1, 2$. By the analytic Fredholm theory we conclude that $(P(h) - z\,\mathrm{Id}_N)^{-1}$ is meromorphic in $z \in \Omega$ for h sufficiently small. When $\Omega \cap \Sigma(P) = \varnothing$ we can choose $R(w, z) = (P(w) - z\,\mathrm{Id}_N)^{-1}$, then $K_j \equiv 0$ for $j = 1, 2$, and $P(h) - z\,\mathrm{Id}_N$ is invertible for small enough h. □

We shall show how the reduction to the case of bounded operator can be done in the systems case, following [Dencker et al. 2004]. Let $m(w)$ be a positive function on $T^*\mathbb{R}^n$ satisfying

$$1 \le m(w) \le C\langle w - w_0\rangle^N m(w_0)\,, \qquad \forall\, w,\ w_0 \in T^*\mathbb{R}^n$$

for some fixed C and N, where $\langle w\rangle = 1+|w|$. Then m is an admissible weight function and we can define the symbol classes $P \in S(m)$ by

$$\|\partial_w^\alpha P(w)\| \le C_\alpha m(w) \qquad \forall\alpha.$$

Following [Dimassi and Sjöstrand 1999] we then define the semiclassical operator $P(h) = P^w(x, hD)$. In the analytic case we require that the symbol estimates hold in a tubular neighborhood of $T^*\mathbb{R}^n$:

$$\|\partial_w^\alpha P(w)\| \le C_\alpha m(\operatorname{Re} w) \qquad \text{for} \quad |\operatorname{Im} w| \le 1/C \qquad \forall\alpha \tag{2.6}$$

One typical example of an admissible weight function is $m(x,\xi) = (\langle\xi\rangle^2 + \langle x\rangle^p)$.

Now we make the ellipticity assumption

$$\|P^{-1}(w)\| \le C_0 m^{-1}(w) \qquad |w| \gg 1 \tag{2.7}$$

and in the analytic case we assume this in a tubular neighborhood of $T^*\mathbb{R}^n$ as in (2.6). By Leibniz' rule we obtain that $P^{-1} \in S(m^{-1})$ at infinity, that is,

$$\|\partial_w^\alpha P^{-1}(w)\| \le C'_\alpha m^{-1}(w) \qquad |w| \gg 1.$$

When $z \notin \Sigma(P) \cup \Sigma_\infty(P)$ we find as before that

$$\|(P(w) - z\operatorname{Id}_N)^{-1}\| \le C \qquad \forall\, w$$

since the resolvent is uniformly bounded at infinity. This implies that $P(w)(P(w) - z\operatorname{Id}_N)^{-1}$ and $(P(w) - z\operatorname{Id}_N)^{-1}P(w)$ are bounded. Again by Leibniz' rule, (2.7) holds with P replaced by $P - z\operatorname{Id}_N$ thus $(P(w)-z\operatorname{Id}_N)^{-1} \in S(m^{-1})$ and we may define the semiclassical operator $((P-z\operatorname{Id}_N)^{-1})^w(x, hD)$. Since $m \ge 1$ we find that $P(w) - z\operatorname{Id}_N \in S(m)$, so by using the calculus we obtain that

$$(P^w - z\operatorname{Id}_N)((P - z\operatorname{Id}_N)^{-1})^w = 1 + hR_1^w$$
$$((P - z\operatorname{Id}_N)^{-1})^w(P^w - z\operatorname{Id}_N) = 1 + hR_2^w$$

where $R_j \in S(1)$, $j = 1, 2$. For small enough h we get invertibility and the following result.

Proposition 2.20. *Assume that $P \in S(m)$ is an $N\times N$ system satisfying* (2.7) *and that $z \notin \Sigma(P)\cup\Sigma_\infty(P)$. Then we find that $P^w - z\operatorname{Id}_N$ is invertible for small enough h.*

This makes it possible to reduce to the case of operators with bounded symbols.

Remark 2.21. If $z_1 \notin \operatorname{Spec}(P)$ we may define the operator

$$Q = (P - z_1\operatorname{Id}_N)^{-1}(P - z_2\operatorname{Id}_N) \qquad z_2 \ne z_1.$$

The resolvents of Q and P are related by

$$(Q - \zeta\operatorname{Id}_N)^{-1} = (1-\zeta)^{-1}(P - z_1\operatorname{Id}_N)\Big(P - \frac{\zeta z_1 - z_2}{\zeta - 1}\operatorname{Id}_N\Big)^{-1} \qquad \zeta \ne 1$$

when $(\zeta z_1 - z_2)/(\zeta - 1) \notin \operatorname{Spec}(P)$.

Example 2.22. Let

$$P(x,\xi)=|\xi|^2\,\mathrm{Id}_N+iK(x)$$

where $0\le K(x)\in C_{\mathrm{b}}^\infty$; then we find that $P\in S(m)$ with $m(x,\xi)=|\xi|^2+1$. If $0\notin\Sigma_\infty(K)$ then $K(x)$ is invertible for $|x|\gg 1$, so $P^{-1}\in S(m^{-1})$ at infinity. Since $\operatorname{Re} z\ge 0$ in $\Sigma(P)$ we find from Proposition 2.20 that $P^w(x,hD)+\mathrm{Id}_N$ is invertible for small enough h and $P^w(x,hD)(P^w(x,hD)+\mathrm{Id}_N)^{-1}$ is bounded in L^2 with principal symbol $P(w)(P(w)+\mathrm{Id}_N)^{-1}\in C_{\mathrm{b}}^\infty$.

In order to measure the singularities of the solutions, we shall introduce the semiclassical wave front sets.

Definition 2.23. For $u\in L^2(\mathbb{R}^n)$ we say that $w_0\notin \mathrm{WF}_h(u)$ if there exists $a\in C_0^\infty(T^*\mathbb{R}^n)$ such that $a(w_0)\ne 0$ and the L^2 norm

$$\|a^w(x,hD)u\|\le C_k h^k \qquad \forall k. \tag{2.8}$$

We call $\mathrm{WF}_h(u)$ the semiclassical wave front set of u.

Observe that this definition is equivalent to Definition (2.5) in [Dencker et al. 2004] which use the FBI transform T given by (4.26): $w_0\notin\mathrm{WF}_h(u)$ if $\|Tu(w)\|=\mathbb{O}(h^\infty)$ when $|w-w_0|\ll 1$. We may also define the *analytic* semiclassical wave front set by the condition that $\|Tu(w)\|=\mathbb{O}(e^{-c/h})$ in a neighborhood of w_0 for some $c>0$; see (2.6) in [Dencker et al. 2004].

Observe that if $u=(u_1,\dots,u_N)\in L^2(\mathbb{R}^n,\mathbb{C}^N)$ we may define $\mathrm{WF}_h(u)=\bigcap_j \mathrm{WF}_h(u_j)$ but this gives no information about which components of u that are singular. Therefore we shall define the corresponding vector valued polarization sets.

Definition 2.24. For $u\in L^2(\mathbb{R}^n,\mathbb{C}^N)$, we say that $(w_0,z_0)\notin\mathrm{WF}_h^{\mathrm{pol}}(u)\subseteq T^*\mathbb{R}^n\times\mathbb{C}^N$ if there exists $A(h)$ given by (2.2) with principal symbol $A(w)$ such that $A(w_0)z_0\ne 0$ and $A(h)u$ satisfies (2.8). We call $\mathrm{WF}_h^{\mathrm{pol}}(u)$ the semiclassical polarization set of u.

We could similarly define the *analytic* semiclassical polarization set by using the FBI transform and analytic pseudodifferential operators.

Remark 2.25. The semiclassical polarization sets are closed, linear in the fiber and has the functorial properties of the C^∞ polarization sets in [Dencker 1982]. In particular, we find that

$$\pi(\mathrm{WF}_h^{\mathrm{pol}}(u)\setminus 0)=\mathrm{WF}_h(u)=\bigcup_j \mathrm{WF}_h(u_j)$$

if π is the projection along the fiber variables: $\pi: T^*\mathbb{R}^n\times\mathbb{C}^N\mapsto T^*\mathbb{R}^n$. We also find that

$$A(\mathrm{WF}_h^{\mathrm{pol}}(u))=\big\{(w,A(w)z):(w,z)\in\mathrm{WF}_h^{\mathrm{pol}}(u)\big\}\subseteq\mathrm{WF}_h^{\mathrm{pol}}(A(h)u)$$

if $A(w)$ is the principal symbol of $A(h)$, which implies that $\mathrm{WF}_h^{\mathrm{pol}}(Au)=A(\mathrm{WF}_h^{\mathrm{pol}}(u))$ when $A(h)$ is elliptic.

This follows from the proofs of Propositions 2.5 and 2.7 in [Dencker 1982].

Example 2.26. Let $u=(u_1,\dots,u_N)\in L^2(T^*\mathbb{R}^n,\mathbb{C}^N)$ where $\mathrm{WF}_h(u_1)=\{w_0\}$ and $\mathrm{WF}_h(u_j)=\varnothing$ for $j>1$. Then

$$\mathrm{WF}_h^{\mathrm{pol}}(u)=\{(w_0,(z,0,\dots)):z\in\mathbb{C}\}$$

since $\|A^w(x, hD)u\| = \mathcal{O}(h^\infty)$ if $A^w u = \sum_{j>1} A^w_j u_j$ and $w_0 \in \mathrm{WF}_h(u)$. By taking a suitable invertible E we obtain

$$\mathrm{WF}_h^{\mathrm{pol}}(Eu) = \{(w_0, zv) : z \in \mathbb{C}\}$$

for any $v \in \mathbb{C}^N$.

We shall use the following definitions from [Pravda-Starov 2006a], here and in the following $\|P(h)\|$ will denote the L^2 operator norm of $P(h)$.

Definition 2.27. Let $P(h)$, $0 < h \le 1$, be a semiclassical family of operators on $L^2(\mathbb{R}^n)$ with domain D. For $\mu > 0$ we define the *pseudospectrum of index* μ as the set

$$\Lambda^{\mathrm{sc}}_\mu(P(h)) = \{z \in \mathbb{C} : \forall C > 0,\ \forall h_0 > 0, \exists\, 0 < h < h_0, \|(P(h) - z\,\mathrm{Id}_N)^{-1}\| \ge Ch^{-\mu}\}$$

and the *injectivity pseudospectrum of index* μ as

$$\lambda^{\mathrm{sc}}_\mu(P(h)) = \{z \in \mathbb{C} : \forall C > 0,\ \forall h_0 > 0, \exists\, 0 < h < h_0,\ \ \exists\, u \in D,\ \|u\| = 1,\ \|(P(h) - z\,\mathrm{Id}_N)u\| \le Ch^{\mu}\}.$$

We define the *pseudospectrum of infinite index* as $\Lambda^{\mathrm{sc}}_\infty(P(h)) = \bigcap_\mu \Lambda^{\mathrm{sc}}_\mu(P(h))$ and correspondingly the *injectivity pseudospectrum of infinite index*.

Here we use the convention that $\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| = \infty$ when λ is in the spectrum $\mathrm{Spec}(P(h))$. Observe that we have the obvious inclusion $\lambda^{\mathrm{sc}}_\mu(P(h)) \subseteq \Lambda^{\mathrm{sc}}_\mu(P(h))$ for all μ. We get equality if, for example, $P(h)$ is Fredholm of index ≥ 0.

3. The interior case

Recall that the scalar symbol $p(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ is of *principal type* if $dp \ne 0$ when $p = 0$. In the following we let $\partial_\nu P(w) = \langle \nu, dP(w)\rangle$ for $P \in C^1(T^*\mathbb{R}^n)$ and $\nu \in T^*\mathbb{R}^n$. We shall use the following definition of systems of principal type, in fact, most of the systems we consider will be of this type. We shall denote $\mathrm{Ker}\, P$ and $\mathrm{Ran}\, P$ the kernel and range of P.

Definition 3.1. The $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is of *principal type* at w_0 if

$$\mathrm{Ker}\, P(w_0) \ni u \mapsto \partial_\nu P(w_0)u \in \mathrm{Coker}\, P(w_0) = \mathbb{C}^N / \mathrm{Ran}\, P(w_0) \tag{3.1}$$

is bijective for some $\nu \in T_{w_0}(T^*\mathbb{R}^n)$. The operator $P(h)$ given by (2.2) is of principal type if the principal symbol $P = \sigma(P(h))$ is of principal type.

Remark 3.2. If $P(w) \in C^\infty$ is of principal type and $A(w)$, $B(w) \in C^\infty$ are invertible then APB is of principal type. We have that $P(w)$ is of principal type if and only if the adjoint P^* is of principal type.

In fact, by Leibniz' rule we have

$$\partial(APB) = (\partial A)PB + A(\partial P)B + AP\partial B \tag{3.2}$$

and $\mathrm{Ran}(APB) = A(\mathrm{Ran}\, P)$ and $\mathrm{Ker}(APB) = B^{-1}(\mathrm{Ker}\, P)$ when A and B are invertible, which gives the invariance under left and right multiplication. Since $\mathrm{Ker}\, P^*(w_0) = \mathrm{Ran}\, P(w_0)^\perp$ we find that P satisfies (3.1) if and only if

$$\mathrm{Ker}\, P(w_0) \times \mathrm{Ker}\, P^*(w_0) \ni (u, v) \mapsto \langle \partial_\nu P(w_0)u, v\rangle$$

is a nondegenerate bilinear form. Since $\langle \partial_\nu P^* v, u\rangle = \overline{\langle \partial_\nu P u, v\rangle}$ we find that P^* is of principal type if and only if P is.

Observe that if P only has one vanishing eigenvalue λ (with multiplicity one) then the condition that P is of principal type reduces to the condition in the scalar case: $d\lambda \neq 0$. In fact, by using the spectral projection one can find invertible systems A and B so that

$$APB = \begin{pmatrix} \lambda & 0 \\ 0 & E \end{pmatrix}$$

with E invertible $(N-1)\times(N-1)$ system, and this system is obviously of principal type.

Example 3.3. Consider the system in Example 2.7

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$. We find that $P(w) - \lambda \operatorname{Id}_2$ is not of principal type when $\lambda = \lambda_1(w) = \lambda_2(w)$ since $\operatorname{Ker}(P(w) - \lambda \operatorname{Id}_2) = \operatorname{Ran}(P(w) - \lambda \operatorname{Id}_2) = \mathbb{C} \times \{0\}$ is preserved by ∂P.

Observe that the property of being of principal type is not stable under C^1 perturbation, not even when $P = P^*$ is symmetric, by the following example.

Example 3.4. The system

$$P(w) = \begin{pmatrix} w_1 - w_2 & w_2 \\ w_2 & -w_1 - w_2 \end{pmatrix} = P^*(w) \qquad w = (w_1, w_2)$$

is of principal type when $w_1 = w_2 = 0$, but *not* of principal type when $w_2 \neq 0$ and $w_1 = 0$. In fact,

$$\partial_{w_1} P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible, and when $w_2 \neq 0$ we have that

$$\operatorname{Ker} P(0, w_2) = \operatorname{Ker} \partial_{w_2} P(0, w_2) = \{z(1, 1) : z \in \mathbb{C}\}$$

which is mapped to $\operatorname{Ran} P(0, w_2) = \{z(1, -1) : z \in \mathbb{C}\}$ by $\partial_{w_1} P$.

We shall obtain a simple characterization of systems of principal type. Recall κ_P, K_P and $\Xi(P)$ given by Definition 2.5.

Proposition 3.5. *Assume $P(w) \in C^\infty$ is an $N \times N$ system and that $(w_0, \lambda_0) \in \Omega_1(P) \setminus \Xi(P)$; then $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at w_0 if and only if $\kappa_P \equiv K_P$ at (w_0, λ_0) and $d\lambda(w_0) \neq 0$ for the C^∞ germ of eigenvalues $\lambda(w)$ for P at w_0 satisfying $\lambda(w_0) = \lambda_0$.*

Thus, in the case $\lambda_0 = 0 \notin \Sigma_{\mathrm{ws}}(P)$ we find that $P(w)$ is of principal type if and only if λ is of principal type and we have no nontrivial Jordan boxes in the normal form. Observe that by the proof of Lemma 2.15 the C^∞ germ $\lambda(w)$ is the unique solution to $\partial_\lambda^k p(w, \lambda) = 0$ for $k = K_P(w, \lambda) - 1$ where $p(w, \lambda) = |P(w) - \lambda \operatorname{Id}_N|$ is the characteristic equation. Thus we find that $d\lambda(w) \neq 0$ if and only if $\partial_w \partial_\lambda^k p(w, \lambda) \neq 0$. For symmetric operators we have $\kappa_P \equiv K_P$ and we only need this condition when $(w_0, \lambda_0) \notin \Xi(P)$.

Example 3.6. The system $P(w)$ in Example 3.4 has eigenvalues $-w_2 \pm \sqrt{w_1^2 + w_2^2}$ which are equal if and only if $w_1 = w_2 = 0$, so $\{0\} = \Sigma_{ws}(P)$. When $w_2 \neq 0$ and $w_1 \approx 0$ the eigenvalue close to zero is $w_1^2/2w_2 + \mathcal{O}(w_1^4)$ which has vanishing differential at $w_1 = 0$. The characteristic equation is $p(w,\lambda) = \lambda^2 + 2\lambda w_2 - w_1^2$, so $d_w p = 0$ when $w_1 = \lambda = 0$.

Proof of Proposition 3.5. Of course, it is no restriction to assume $\lambda_0 = 0$. First we note that $P(w)$ is of principal type at w_0 if and only if

$$\partial_\nu^k |P(w_0)| \neq 0 \qquad k = \kappa_P(w_0, 0) \tag{3.3}$$

for some $\nu \in T(T^*\mathbb{R}^n)$. Observe that $\partial^j |P(w_0)| = 0$ for $j < k$. In fact, by choosing bases for $\operatorname{Ker} P(w_0)$ and $\operatorname{Im} P(w_0)$ respectively, and extending to bases of $\mathbb{C}^N$, we obtain matrices A and B so that

$$AP(w)B = \begin{pmatrix} P_{11}(w) & P_{12}(w) \\ P_{21}(w) & P_{22}(w) \end{pmatrix}$$

where $|P_{22}(w_0)| \neq 0$ and P_{11}, P_{12} and P_{21} all vanish at w_0. By the invariance, P is of principal type if and only if $\partial_\nu P_{11}$ is invertible for some ν, so by expanding the determinant we obtain (3.3).

Since $(w_0, 0) \in \Omega_1(P) \setminus \Xi(P)$ we find from Lemma 2.15 that we may choose a neighborhood ω of $(w_0, 0)$ such that $(w,\lambda) \in \Omega_1(P) \cap \omega$ if and only if $\lambda = \lambda(w) \in C^\infty$. Then

$$|P(w) - \lambda \operatorname{Id}_N| = (\lambda(w) - \lambda)^m e(w,\lambda)$$

near w_0, where $e(w,\lambda) \neq 0$ and $m = K_P(w_0, 0) \geq \kappa_P(w_0, 0)$. Letting $\lambda = 0$ we obtain that $\partial_\nu^j |P(w_0)| = 0$ if $j < m$ and $\partial_\nu^m |P(w_0)| = (\partial_\nu \lambda(w_0))^m e(w_0, 0)$. □

Remark 3.7. Proposition 3.5 shows that for a *symmetric* system the property to be of principal type is stable outside $\Xi(P)$: if the symmetric system $P(w) - \lambda \operatorname{Id}_N$ is of principal type at a point $(w_0, \lambda_0) \notin \Xi(P)$ then it is in a neighborhood. It follows from Sard's Theorem that symmetric systems $P(w) - \lambda \operatorname{Id}_N$ are of principal type almost everywhere on $\Omega_1(P)$.

In fact, for symmetric systems we have $\kappa_P \equiv K_P$ and the differential $d\lambda \neq 0$ almost everywhere on $\Omega_1(P) \setminus \Xi(P)$. For C^∞ germs of eigenvalues we can define the following bracket condition.

Definition 3.8. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system; then we define

$$\Lambda(P) = \overline{\Lambda_-(P) \cup \Lambda_+(P)}$$

where $\Lambda_\pm(P)$ is the set of $\lambda_0 \in \Sigma(P)$ such that there exists $w_0 \in \Sigma_{\lambda_0}(P)$ so that $(w_0, \lambda_0) \notin \Xi(P)$ and

$$\pm \{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}(w_0) > 0 \tag{3.4}$$

for the unique C^∞ germ $\lambda(w)$ of eigenvalues at w_0 for P such that $\lambda(w_0) = \lambda_0$.

Observe that $\Lambda_\pm(P) \cap \Sigma_{ss}(P) = \emptyset$, and it follows from Proposition 3.5 that $P(w) - \lambda_0 \operatorname{Id}_N$ is of principal type at $w_0 \in \Lambda_\pm(P)$ if and only if $\kappa_P = K_P$ at (w_0, λ_0), since $d\lambda(w_0) \neq 0$ when (3.4) holds. Because of the bracket condition (3.4) we find that $\Lambda_\pm(P)$ is contained in the interior of the values $\Sigma(P)$.

Example 3.9. Let

$$P(x,\xi) = \begin{pmatrix} q(x,\xi) & \chi(x) \\ 0 & q(x,\xi) \end{pmatrix} \qquad (x,\xi) \in T^*\mathbb{R}$$

where $q(x,\xi) = \xi + ix^2$ and $0 \le \chi(x) \in C^\infty(\mathbb{R})$ such that $\chi(x) = 0$ when $x \le 0$ and $\chi(x) > 0$ when $x > 0$. Then $\Sigma(P) = \{\operatorname{Im} z \ge 0\}$, $\Lambda_\pm(P) = \{\operatorname{Im} z > 0\}$ and $\Xi(P) = \emptyset$. For $\operatorname{Im}\lambda > 0$ we find $\Sigma_\lambda(P) = \{(\pm\sqrt{\operatorname{Im}\lambda}, \operatorname{Re}\lambda)\}$ and $P - \lambda \operatorname{Id}_2$ is of principal type at $\Sigma_\lambda(P)$ only when $x < 0$.

Theorem 3.10. *Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system; then we have that*

$$\Lambda(P) \setminus \big(\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)\big) \subseteq \overline{\Lambda_-(P)} \tag{3.5}$$

when $n \ge 2$. Assume that $P(h)$ is given by (2.2) *with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$, and that $\lambda_0 \in \Lambda_-(P)$, $0 \ne u_0 \in \operatorname{Ker}(P(w_0) - \lambda_0 \operatorname{Id}_N)$ and $P(w) - \lambda \operatorname{Id}_N$ is of principal type on $\Sigma_\lambda(P)$ near w_0 for $|\lambda - \lambda_0| \ll 1$, for the $w_0 \in \Sigma_{\lambda_0}(P)$ in Definition 3.8. Then there exists $h_0 > 0$ and $u(h) \in L^2(\mathbb{R}^n)$, $0 < h \le h_0$, so that $\|u(h)\| \le 1$,*

$$\|(P(h) - \lambda_0 \operatorname{Id}_N)u(h)\| \le C_N h^N \qquad \forall N \qquad 0 < h \le h_0 \tag{3.6}$$

and $\mathrm{WF}_h^{\mathrm{pol}}(u(h)) = \{(w_0, u_0)\}$. There also exists a dense subset of values $\lambda_0 \in \Lambda(P)$ so that

$$\|(P(h) - \lambda_0 \operatorname{Id}_N)^{-1}\| \ge C_N' h^{-N} \qquad \forall N. \tag{3.7}$$

If all the terms P_j in the expansion (2.2) *are analytic satisfying* (2.3) *then $h^{\pm N}$ may be replaced by $\exp(\mp c/h)$ in* (3.6)–(3.7).

Here we use the convention that $\|(P(h) - \lambda \operatorname{Id}_N)^{-1}\| = \infty$ when λ is in the spectrum $\operatorname{Spec}(P(h))$. Condition (3.6) means that λ_0 is in the *injectivity* pseudospectrum $\lambda_\infty^{\mathrm{sc}}(P)$, and (3.7) means that λ_0 is in the pseudospectrum $\Lambda_\infty^{\mathrm{sc}}(P)$.

Remark 3.11. If $P(h)$ is Fredholm of nonnegative index then (3.6) holds for λ_0 in a dense subset of $\Lambda(P)$. In the analytic case, it follows from the proof that it suffices that $P_j(w)$ is analytic satisfying (2.3) in a fixed complex neighborhood of $w_0 \in \Sigma_\lambda(P)$ for all j.

In fact, if $P(h)$ is Fredholm of nonnegative index and $\lambda_0 \in \operatorname{Spec}(P(h))$ then the dimension of $\operatorname{Ker}(P(h) - \lambda_0 \operatorname{Id}_N)$ is positive and (3.6) holds.

Example 3.12. Let

$$P(x,\xi) = |\xi|^2 \operatorname{Id} + iK(x) \qquad (x,\xi) \in T^*\mathbb{R}^n$$

where $K(x) \in C^\infty(\mathbb{R}^n)$ is symmetric for all x. Then we find that

$$\overline{\Lambda_-(P)} = \Lambda(P) = \Big\{\operatorname{Re} z \ge 0 \wedge \operatorname{Im} z \in \overline{\Sigma(K) \setminus \big(\Sigma_{\mathrm{ss}}(K) \cup \Sigma_\infty(K)\big)}\Big\}.$$

In fact, for any $\operatorname{Im} z \in \Sigma(K) \setminus (\Sigma_{\mathrm{ss}}(K) \bigcup \Sigma_\infty(K))$ there exists a germ of eigenvalues $\lambda(x) \in C^\infty(\omega)$ for $K(x)$ in an open set $\omega \subset \mathbb{R}^n$ so that $\lambda(x_0) = \operatorname{Im} z$ for some $x_0 \in \omega$. By Sard's Theorem, we find that almost all values of $\lambda(x)$ in ω are nonsingular, and if $d\lambda \ne 0$ and $\operatorname{Re} z > 0$ we may choose $\xi_0 \in \mathbb{R}^n$ so that $|\xi_0|^2 = \operatorname{Re} z$ and $\langle \xi_0, \partial_x \lambda\rangle < 0$. Then the C^∞ germ of eigenvalues $|\xi|^2 + i\lambda(x)$ for P satisfies (3.4) at (x_0, ξ_0) with the minus sign. Since $K(x)$ is symmetric, we find that $P(w) - z \operatorname{Id}_N$ is of principal type.

Proof of Theorem 3.10. First we are going to prove (3.5) in the case $n \ge 2$. Let

$$W = \Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$$

which is a closed set by Remark 2.13; then we find that every point in $\Lambda(P) \setminus W$ is a limit point of

$$\big(\Lambda_-(P) \cup \Lambda_+(P)\big) \setminus W = (\Lambda_-(P) \setminus W) \cup (\Lambda_+(P) \setminus W).$$

Thus, we only have to show that $\lambda_0 \in \overline{\Lambda_-(P)}$ if

$$\lambda_0 \in \Lambda_+(P) \setminus W = \Lambda_+(P) \setminus \big(\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)\big). \tag{3.8}$$

By Lemma 2.15 and Remark 2.16 we find from (3.8) that there exists a C^∞ germ of eigenvalues $\lambda(w) \in C^\infty$ so that $\Sigma_\mu(P)$ is equal to the level sets $\{w : \lambda(w) = \mu\}$ for $|\mu - \lambda_0| \ll 1$. By definition we find that the Poisson bracket $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ does not vanish identically on $\Sigma_{\lambda_0}(P)$. Now by Remark 2.16, $d \operatorname{Re}\lambda$ and $d \operatorname{Im}\lambda$ are linearly independent on $\Sigma_\mu(P)$ for almost all μ close to λ_0, and then $\Sigma_\mu(P)$ is a C^∞ manifold of codimension 2. By using Lemma 3.1 of [Dencker et al. 2004] we obtain that $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ changes sign on $\Sigma_\mu(P)$ for almost all values μ near λ_0, so we find that those values also are in $\Lambda_-(P)$. By taking the closure we obtain (3.5).

Next, assume that $\lambda \in \Lambda_-(P)$, it is no restriction to assume $\lambda = 0$. By the assumptions there exists $w_0 \in \Sigma_0(P)$ and $\lambda(w) \in C^\infty$ such that $\lambda(w_0) = 0$, $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\} < 0$ at w_0, $(w_0, 0) \notin \Xi(P)$, and $P(w) - \lambda\,\mathrm{Id}_N$ is of principal type on $\Sigma_\lambda(P)$ near w_0 when $|\lambda| \ll 1$. Then Proposition 3.5 gives that $\kappa_P \equiv K_P$ is constant on $\Omega_1(P)$ near (w_0, λ_0), so

$$\operatorname{Dim}\operatorname{Ker}(P(w) - \lambda(w)\,\mathrm{Id}_N) \equiv K > 0$$

in a neighborhood of w_0. Since the dimension is constant we can construct a base $\{u_1(w), \dots, u_K(w)\} \in C^\infty$ for $\operatorname{Ker}(P(w) - \lambda(w)\,\mathrm{Id}_N)$ in a neighborhood of w_0. By orthonormalizing it and extending to $\mathbb{C}^N$ we obtain orthogonal $E(w) \in C^\infty$ so that

$$E^*(w)P(w)E(w) = \begin{pmatrix} \lambda(w)\,\mathrm{Id}_K & P_{12} \\ 0 & P_{22} \end{pmatrix} = P_0(w). \tag{3.9}$$

If $P(w)$ is analytic in a tubular neighborhood of $T^*\mathbb{R}^n$ then $E(w)$ can be chosen analytic in that neighborhood. Since P_0 is of principal type at w_0 by Remark 3.2 and $\partial P_0(w_0)$ maps $\operatorname{Ker} P_0(w_0)$ into itself, we find that $\operatorname{Ran} P_0(w_0) \cap \operatorname{Ker} P_0(w_0) = \{0\}$ and thus $|P_{22}(w_0)| \neq 0$. In fact, if there exists $u'' \neq 0$ such that $P_{22}(w_0)u'' = 0$, then by applying $P(w_0)$ on $u = (0, u'') \notin \operatorname{Ker} P_0(w_0)$ we obtain

$$0 \neq P_0(w_0)u = (P_{12}(w_0)u'', 0) \in \operatorname{Ker} P_0(w_0) \cap \operatorname{Ran} P_0(w_0)$$

which gives a contradiction. Clearly, the norm of the resolvent $P(h)^{-1}$ only changes with a multiplicative constant under left and right multiplication of $P(h)$ by invertible systems. Now $E^w(x, hD)$ and $(E^*)^w(x, hD)$ are invertible in L^2 for small enough h, and

$$(E^*)^w P(h) E^w = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

where $\sigma(P_{11}) = \lambda\,\mathrm{Id}_N$, $P_{21} = \mathbb{O}(h)$ and $P_{22}(h)$ is invertible for small h. By multiplying from the right by

$$\begin{pmatrix} \mathrm{Id}_K & 0 \\ -P_{22}(h)^{-1}P_{21}(h) & \mathrm{Id}_{N-K} \end{pmatrix}$$

for small h, we obtain that $P_{21}(h) \equiv 0$ and this only changes lower order terms in $P_{11}(h)$. Then by multiplying from the left by

$$\begin{pmatrix} \mathrm{Id}_K & -P_{12}(h)P_{22}(h)^{-1} \\ 0 & \mathrm{Id}_{N-K} \end{pmatrix}$$

we obtain that $P_{12}(h) \equiv 0$ without changing $P_{11}(h)$ or $P_{22}(h)$.

Thus, in order to prove (3.6) we may assume $N = K$ and $P(w) = \lambda(w)\,\mathrm{Id}_K$. By conjugating similarly as in the scalar case (see the proof of Proposition 26.3.1 in Volume IV of [Hörmander 1983–1985]), we can reduce to the case when $P(h) = \lambda^w(x, hD)\,\mathrm{Id}_K$. In fact, let

$$P(h) = \lambda^w(x, hD)\,\mathrm{Id}_K + \sum_{j\ge 1} h^j P_j^w(x, hD) \tag{3.10}$$

$A(h) = \sum_{j\ge 0} h^j A_j^w(x, hD)$ and $B(h) = \sum_{j\ge 0} h^j B_j^w(x, hD)$ with $B_0(w) \equiv A_0(w)$. Then the calculus gives

$$P(h)A(h) - B(h)\lambda^w(x, hD) = \sum_{j\ge 1} h^j E_j^w(x, hD)$$

with

$$E_k = \frac{1}{2i} H_\lambda(A_{k-1} + B_{k-1}) + P_1 A_{k-1} + \lambda(A_k - B_k) + R_k \qquad k \ge 1.$$

Here H_λ is the Hamilton vector field of λ, R_k only depends on A_j and B_j for $j < k-1$ and $R_1 \equiv 0$. Now we can choose A_0 so that $A_0 = \mathrm{Id}_K$ on $V_0 = \{w : \operatorname{Im}\lambda(w) = 0\}$ and $\frac{1}{i} H_\lambda A_0 + P_1 A_0$ vanishes of infinite order on V_0 near w_0. In fact, since $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\} \ne 0$ we find $d\operatorname{Im}\lambda \ne 0$ on V_0, and V_0 is noncharacteristic for $H_{\operatorname{Re}\lambda}$. Thus, the equation determines all derivatives of A_0 on V_0, and we may use the Borel Theorem to obtain a solution. Then, by taking

$$B_1 - A_1 = \Big(\frac{1}{i} H_\lambda A_0 + P_1 A_0\Big)\lambda^{-1} \in C^\infty$$

we obtain $E_0 \equiv 0$. Lower order terms are eliminated similarly, by making

$$\frac{1}{2i} H_\lambda(A_{j-1} + B_{j-1}) + P_1 A_{j-1} + R_j$$

vanish of infinite order on V_0. Observe that only the difference $A_{j-1} - B_{j-1}$ is determined in the previous step. Thus we can reduce to the case $P = \lambda^w(x, hD)\,\mathrm{Id}$ and then the C^∞ result follows from the scalar case (see Theorem 1.2 in [Dencker et al. 2004]) by using Remark 2.25 and Example 2.26.

The analytic case follows as in the proof of Theorem 1.2′ in [Dencker et al. 2004] by applying a holomorphic WKB construction to $P = P_{11}$ on the form

$$u(z, h) \sim e^{i\phi(z)/h} \sum_{j=0}^{\infty} A_j(z) h^j \qquad z = x + iy \in \mathbb{C}^n$$

which will be an approximate solution to $P(h)u(z, h) = 0$. Here $P(h)$ is given by (2.2) with $P_0(w) = \lambda(w)\,\mathrm{Id}$, P_j satisfying (2.3) and $P_j^w(z, hD_z)$ given by the formula (2.1) where the integration may be deformed to a suitable chosen contour instead of $T^*\mathbb{R}^n$ (see [Sjöstrand 1982, Section 4]). The holomorphic phase function $\phi(z)$ satisfying $\lambda(z, d_z\phi) = 0$ is constructed as in [Dencker et al. 2004] so that

$d_z\phi(x_0) = \xi_0$ and $\operatorname{Im}\phi(x) \ge c|x - x_0|^2$, $c > 0$, and $w_0 = (x_0, \xi_0)$. The holomorphic amplitude $A_0(z)$ satisfies the transport equation

$$\sum_j \partial_{\zeta_j}\lambda(z, d_z\phi(z))D_{z_j}A_0(z) + P_1(z, d_z\phi(z))A_0(z) = 0$$

with $A_0(x_0) \ne 0$. The lower order terms in the expansion solve

$$\sum_j \partial_{\zeta_j}\lambda(z, d_z\phi(z))D_{z_j}A_k(z) + P_1(z, d_z\phi(z))A_k(z) = S_k(z)$$

where $S_k(z)$ only depends on A_j and P_{j+1} for $j < k$. As in the scalar case, we find from (2.3) that the solutions satisfy $\|A_k(z)\| \le C_0C^kk^k$ see Theorem 9.3 in [Sjöstrand 1982]. By solving up to $k < c/h$, cutting of near x_0 and restricting to $\mathbb{R}^n$ we obtain that $P(h)u = \mathbb{O}(e^{-c/h})$. The details are left to the reader; see the proof of Theorem 1.2′ in [Dencker et al. 2004].

For the last result, we observe that $\{\operatorname{Re}\bar\lambda, \operatorname{Im}\bar\lambda\} = -\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$, $\lambda \in \Sigma(P) \Leftrightarrow \bar\lambda \in \Sigma(P^*)$, P^* is of principal type if and only if P is, and Remark 2.6 gives $(w, \lambda) \in \Xi(P) \Leftrightarrow (w, \bar\lambda) \in \Xi(P^*)$. Thus, $\lambda \in \Lambda_+(P)$ if and only if $\bar\lambda \in \Lambda_-(P^*)$ and

$$\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| = \|(P^*(h) - \bar\lambda\,\mathrm{Id}_N)^{-1}\|.$$

From the definition, we find that any $\lambda_0 \in \Lambda(P)$ is an accumulation point of $\Lambda_\pm(P)$, so we obtain the result from (3.6). □

Remark 3.13. In order to get the estimate (3.6) it suffices that there exists a semibicharacteristic Γ of $\lambda - \lambda_0$ through w_0 such that $\Gamma \times \{\lambda_0\} \cap \Xi(P) = \emptyset$, $P(w) - \lambda\,\mathrm{Id}_N$ is of principal type near Γ for λ near λ_0 and that condition $(\overline{\Psi})$ is not satisfied on Γ; see [Hörmander 1983–1985, Definition 26.4.6, Volume IV]. This means that there exists $0 \ne q \in C^\infty$ such that Γ is a bicharacteristic of $\operatorname{Re} q(\lambda - \lambda_0)$ through w_0 and $\operatorname{Im} q(\lambda - \lambda_0)$ changes sign from $+$ to $-$ when going in the positive direction on Γ.

In fact, once we have reduced to the normal form (3.10), the construction of approximate local solutions in the proof of [Hörmander 1983–1985, Theorem 26.4.7, Volume IV] can be adapted to this case, since the principal part is scalar. See also Theorem 1.3 in [Pravda-Starov 2006b, Section 3.2] for a similar scalar semiclassical estimate.

When $P(w)$ is not of principal type, the reduction in the proof of Theorem 3.10 may not be possible since P_{22} in (3.9) needs not be invertible by the following example.

Example 3.14. Let

$$P(h) = \begin{pmatrix} \lambda^w(x, hD) & 1 \\ h & \lambda^w(x, hD) \end{pmatrix}$$

where $\lambda \in C^\infty$ satisfies the bracket condition (3.4). The principal symbol is

$$P(w) = \begin{pmatrix} \lambda(w) & 1 \\ 0 & \lambda(w) \end{pmatrix}$$

with eigenvalue $\lambda(w)$ and we have

$$\operatorname{Ker}(P(w) - \lambda(w)\,\mathrm{Id}_2) = \operatorname{Ran}(P(w) - \lambda(w)\,\mathrm{Id}_2) = \{(z, 0) :\ z \in \mathbb{C}\} \qquad \forall\, w.$$

We find that P is *not* of principal type since $dP = d\lambda\,\mathrm{Id}_2$. Observe that $\Xi(P) = \varnothing$ since K_P is constant on $\Omega_1(P)$.

When the dimension is equal to one, we have to add some conditions in order to get the inclusion (3.5).

Lemma 3.15. *Let $P(w) \in C^\infty(T^*\mathbb{R})$ be an $N \times N$ system. Then for every component Ω of $\mathbb{C} \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ which has nonempty intersection with $\complement\Sigma(P)$ we find that*

$$\Omega \subseteq \overline{\Lambda_-(P)}. \tag{3.11}$$

The condition of having nonempty intersection with the complement is necessary even in the scalar case; see the remark and Lemma 3.2' on page 394 in [Dencker et al. 2004].

Proof. If $\mu \notin \Sigma_\infty(P)$ we find that the index

$$i = \operatorname{var}\arg_\gamma |P(w) - \mu\,\mathrm{Id}_N| \tag{3.12}$$

is well-defined and continuous when γ is a positively oriented circle $\{w : |w| = R\}$ for $R \gg 1$. If $\lambda_0 \notin \Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P)$ then we find from Lemma 2.15 that the characteristic polynomial is equal to

$$|P(w) - \mu\,\mathrm{Id}_N| = (\lambda(w) - \mu)^k e(w, \mu)$$

near $w_0 \in \Sigma_{\lambda_0}(P)$, here λ, $e \in C^\infty$, $e \neq 0$ and $k = K_P(w_0)$. By Remark 2.16 we find for almost all μ close to λ_0 that $d\operatorname{Re}\lambda \wedge d\operatorname{Im}\lambda \neq 0$ on $\lambda^{-1}(\mu) = \Sigma_\mu(P)$, which is then a finite set of points on which the Poisson bracket is nonvanishing. If $\mu \notin \Sigma(P)$ we find that the index (3.12) vanishes, since one can then let $R \to 0$. Thus, if a component Ω of $\mathbb{C} \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ has nonempty intersection with $\complement\Sigma(P)$, we obtain that $i = 0$ in Ω. When $\mu_0 \in \Omega \cap \Lambda(P)$ we find from the definition that the Poisson bracket $\{\operatorname{Re}\lambda, \operatorname{Im}\lambda\}$ cannot vanish identically on $\Sigma_\mu(P)$ for all μ close to μ_0. Since the index is equal to the sum of positive multiples of the values of the Poisson brackets at $\Sigma_\mu(P)$, we find that the bracket must be negative at some point $w_0 \in \Sigma_\mu(P)$, for almost all μ near λ_0, which gives (3.11). ☐

4. The quasisymmetrizable case

First we note that if the system $P(w) - z\,\mathrm{Id}_N$ is of principal type near $\Sigma_z(P)$ for z close to $\lambda \in \partial\Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$ and $\Sigma_\lambda(P)$ has no closed semibicharacteristics, then one can generalize Theorem 1.3 in [Dencker et al. 2004] to obtain

$$\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| \le C/h \qquad h \to 0. \tag{4.1}$$

In fact, by using the reduction in the proof of Theorem 3.10 this follows from the scalar case; see Example 4.12. But then the eigenvalues close to λ have constant multiplicity.

Generically, we have that the eigenvalues of the principal symbol P have constant multiplicity almost everywhere since $\Xi(P)$ is nowhere dense. But at the boundary $\partial\Sigma(P)$ this needs not be the case. For example, if

$$P(t, \tau) = \tau\,\mathrm{Id} + iK(t)$$

where $C^\infty \ni K \ge 0$ is unbounded and $0 \in \Sigma_{\mathrm{ss}}(K)$, then $\mathbb{R} = \partial\Sigma(P) \subseteq \Sigma_{\mathrm{ss}}(P)$.

When the multiplicity of the eigenvalues of the principal symbol is not constant the situation is more complicated. The following example shows that then it is not sufficient to have conditions only on the eigenvalues in order to obtain the estimate (4.1), not even in the principal type case.

Example 4.1. Let $a_1(t)$, $a_2(t) \in C^\infty(\mathbb{R})$ be real valued, $a_2(0) = 0$, $a_2'(0) > 0$ and let

$$P^w(t, hD_t) = \begin{pmatrix} hD_t + a_1(t) & a_2(t) - ia_1(t) \\ a_2(t) + ia_1(t) & -hD_t + a_1(t) \end{pmatrix} = P^w(t, hD_t)^*.$$

Then the eigenvalues of $P(t, \tau)$ are

$$\lambda = a_1(t) \pm \sqrt{\tau^2 + a_1^2(t) + a_2^2(t)} \in \mathbb{R}$$

which coincide if and only if $\tau = a_1(t) = a_2(t) = 0$. We have that

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} P \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \begin{pmatrix} hD_t + ia_2(t) & 0 \\ 2a_1(t) & hD_t - ia_2(t) \end{pmatrix} = \widetilde{P}(h).$$

Thus we can construct $u_h(t) = {}^t(0, u_2(t))$ so that $\|u_h\| = 1$ and $\widetilde{P}(h)u_h = \mathcal{O}(h^N)$ for $h \to 0$; see Theorem 1.2 in [Dencker et al. 2004]. When a_2 is analytic we may obtain that $\widetilde{P}(h)u_h = \mathcal{O}(\exp(-c/h))$ by Theorem 1.2′ in [Dencker et al. 2004]. By the invariance, we see that P is of principal type at $t = \tau = 0$ if and only if $a_1(0) = 0$. If $a_1(0) = 0$ then $\Sigma_{\mathrm{ws}}(P) = \{0\}$ and when $a_1 \neq 0$ we have that P^w is a self-adjoint diagonalizable system. In the case $a_1(t) \equiv 0$ and $a_2(t) \equiv t$ the eigenvalues of $P(t, hD_t)$ are $\pm\sqrt{2nh}$, $n \in \mathbb{N}$; see the proof of Proposition 3.6.1 in [Helffer and Sjöstrand 1990].

Of course, the problem is that the eigenvalues are not invariant under multiplication with elliptic systems. To obtain the estimate (4.1) for operators that are *not* of principal type, it is not even sufficient that the eigenvalues are real having constant multiplicity.

Example 4.2. Let $a(t) \in C^\infty(\mathbb{R})$ be real valued, $a(0) = 0$, $a'(0) > 0$ and

$$P^w(t, hD_t) = \begin{pmatrix} hD_t & a(t) \\ -ha(t) & hD_t \end{pmatrix}.$$

Then the principal symbol is

$$P(t, \tau) = \begin{pmatrix} \tau & a(t) \\ 0 & \tau \end{pmatrix}$$

so the only eigenvalue is τ. Thus $\Xi(P) = \varnothing$ but the principal symbol is not diagonalizable, and when $a(t) \neq 0$ the system is not of principal type. We have

$$\begin{pmatrix} h^{1/2} & 0 \\ 0 & -1 \end{pmatrix} P \begin{pmatrix} h^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{h}\begin{pmatrix} \sqrt{h}D_t & a(t) \\ a(t) & -\sqrt{h}D_t \end{pmatrix}$$

thus we obtain that $\|P^w(t, hD_t)^{-1}\| \geq C_N h^{-N}$ for all N, when $h \to 0$ by using Example 4.1 with $a_1 \equiv 0$ and $a_2 \equiv a$. When a is analytic we obtain $\|P(t, hD_t)^{-1}\| \geq \exp(c/\sqrt{h})$.

For nonprincipal type operators, to obtain the estimate (4.1) it is not even sufficient that the principal symbol has real eigenvalues of multiplicity one.

Example 4.3. Let $a(t) \in C^\infty(\mathbb{R})$, $a(0) = 0$, $a'(0) > 0$ and

$$P(h) = \begin{pmatrix} 1 & hD_t \\ h & iha(t) \end{pmatrix}$$

with principal symbol

$$\begin{pmatrix} 1 & \tau \\ 0 & 0 \end{pmatrix}$$

thus the eigenvalues are 0 and 1, so $\Xi(P) = \emptyset$. Since

$$\begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} P(h) \begin{pmatrix} 1 & -hD_t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & hD_t - ia(t) \end{pmatrix}$$

we obtain as in Example 4.1 that $\|P(h)^{-1}\| \geq C_N h^{-N}$ when $h \to 0$ for all N, and for analytic $a(t)$ we obtain $\|P(h)^{-1}\| \geq Ce^{c/h}$, $h \to 0$. Now $\partial_\tau P$ maps $\operatorname{Ker} P(0)$ into $\operatorname{Ran} P(0)$ so the system is not of principal type. Observe that this property is not preserved under the multiplications above, since the systems are not elliptic.

Instead of using properties of the eigenvalues of the principal symbol, we shall use properties that are invariant under multiplication with invertible systems. First we consider the scalar case, recall that a scalar $p \in C^\infty$ is of *principal type* if $dp \neq 0$ when $p = 0$. We have the following normal form for scalar principal type operators near the boundary $\partial\Sigma(P)$. Recall that a *semibicharacteristic* of p is a nontrivial bicharacteristic of $\operatorname{Re} qp$, for $q \neq 0$.

Example 4.4. Assume that $p(x,\xi) \in C^\infty(T^*\mathbb{R}^n)$ is of principal type and $0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$. Then we find from the proof of Lemma 4.1 in [Dencker et al. 2004] that there exists $0 \neq q \in C^\infty$ so that

$$\operatorname{Im} qp \geq 0 \quad \text{and} \quad d\operatorname{Re} qp \neq 0$$

in a neighborhood of $w_0 \in \Sigma_0(p)$. In fact, condition (1.7) in that lemma is not needed to obtain a local preparation. By making a symplectic change of variables and using the Malgrange preparation theorem we then find that

$$p(x,\xi) = e(x,\xi)(\xi_1 + if(x,\xi')) \qquad \xi = (\xi_1, \xi') \tag{4.2}$$

in a neighborhood of $w_0 \in \Sigma_0(p)$, where $e \neq 0$ and $f \geq 0$. If there are no closed semibicharacteristics of p then we obtain this in a neighborhood of $\Sigma_0(p)$ by a partition of unity.

This normal form in the scalar case motivates the following definition.

Definition 4.5. We say that the $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is *quasisymmetrizable* with respect to the real C^∞ vector field V in $\Omega \subseteq T^*\mathbb{R}^n$ if $\exists$ $N \times N$ system $M(w) \in C^\infty(T^*\mathbb{R}^n)$ so that in Ω we have

$$\operatorname{Re}\langle M(VP)u, u\rangle \geq c\|u\|^2 - C\|Pu\|^2 \qquad c > 0 \tag{4.3}$$

$$\operatorname{Im}\langle MPu, u\rangle \geq -C\|Pu\|^2 \tag{4.4}$$

for any $u \in \mathbb{C}^N$, the system M is called a *symmetrizer* for P.

The definition is clearly independent of the choice of coordinates in $T^*\mathbb{R}^n$ and choice of base in $\mathbb{C}^N$. When P is elliptic, we may take $M = iP^*$ as multiplier; then P is quasisymmetrizable with respect to any vector field because $\|Pu\| \cong \|u\|$. Observe that for a *fixed* vector field V the set of multipliers M satisfying (4.3)–(4.4) is a convex cone, a sum of two multipliers is also a multiplier. Thus, given a vector field V it suffices to make a local choice of multiplier M and then use a partition of unity to get a global one.

Taylor has studied *symmetrizable* systems of the type $D_t \operatorname{Id} + iK$, for which there exists $R > 0$ making RK symmetric (see Definition 4.3.2 in [Taylor 1981]). These systems are quasisymmetrizable with respect to ∂_τ with symmetrizer R. We see from Example 4.4 that the scalar symbol p of principal type is quasisymmetrizable in neighborhood of any point at $\partial\Sigma(p) \setminus \Sigma_\infty(p)$.

The invariance properties of quasisymmetrizable systems are partly due to the following simple and probably well-known result on semibounded matrices. In the following, we shall denote $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $i \operatorname{Im} A = \frac{1}{2}(A - A^*)$ the symmetric and antisymmetric parts of the matrix A. Also, if U and V are linear subspaces of $\mathbb{C}^N$, then we let $U + V = \{u + v : u \in U \ \wedge \ v \in V\}$.

Lemma 4.6. *Assume that Q is an $N \times N$ matrix such that* $\operatorname{Im} zQ \ge 0$ *for some* $0 \ne z \in \mathbb{C}$. *Then we find*

$$\operatorname{Ker} Q = \operatorname{Ker} Q^* = \operatorname{Ker}(\operatorname{Re} Q) \cap \operatorname{Ker}(\operatorname{Im} Q) \tag{4.5}$$

and $\operatorname{Ran} Q = \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q)$ *is orthogonal to* $\operatorname{Ker} Q$.

Proof. By multiplying with z we may assume that $\operatorname{Im} Q \ge 0$, clearly the conclusions are invariant under multiplication with complex numbers. If $u \in \operatorname{Ker} Q$, then we have $\langle \operatorname{Im} Qu, u\rangle = \operatorname{Im}\langle Qu, u\rangle = 0$. By using the Cauchy–Schwarz inequality on $\operatorname{Im} Q \ge 0$ we find that $\langle \operatorname{Im} Qu, v\rangle = 0$ for any v. Thus $u \in \operatorname{Ker}(\operatorname{Im} Q)$ so $\operatorname{Ker} Q \subseteq \operatorname{Ker} Q^*$. We get equality and (4.5) by the rank theorem, since $\operatorname{Ker} Q^* = \operatorname{Ran} Q^\perp$.

For the last statement we observe that $\operatorname{Ran} Q \subseteq \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q) = (\operatorname{Ker} Q)^\perp$ by (4.5) where we also get equality by the rank theorem. □

Proposition 4.7. *Assume that $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is a quasisymmetrizable system*; *then we find that P is of principal type. Also*, *the symmetrizer M is invertible if* $\operatorname{Im} MP \ge cP^*P$ *for some* $c > 0$.

Observe that by adding $i\varrho P^*$ to M we may assume that $Q = MP$ satisfies

$$\operatorname{Im} Q \ge (\varrho - C)P^*P \ge P^*P \ge cQ^*Q \qquad c > 0 \tag{4.6}$$

for $\varrho \ge C + 1$, and then the symmetrizer is invertible by Proposition 4.7.

Proof. Assume that (4.3)–(4.4) hold at w_0, $\operatorname{Ker} P(w_0) \ne \{0\}$ but (3.1) is not a bijection. Then there exists $0 \ne u \in \operatorname{Ker} P(w_0)$ and $v \in \mathbb{C}^N$ such that $VP(w_0)u = P(w_0)v$, so (4.3) gives

$$\operatorname{Re}\langle MP(w_0)v, u\rangle = \operatorname{Re}\langle MVP(w_0)u, u\rangle \ge c\|u\|^2 > 0.$$

This means that

$$\operatorname{Ran} MP(w_0) \not\subseteq \operatorname{Ker} P(w_0)^\perp. \tag{4.7}$$

Let $M_\varrho = M + i\varrho P^*$ then we have that

$$\operatorname{Im}\langle M_\varrho Pu, u\rangle \ge (\varrho - C)\|Pu\|^2$$

so for large enough ϱ we have $\operatorname{Im} M_\varrho P \ge 0$. By Lemma 4.6 we find

$$\operatorname{Ran} M_\varrho P \perp \operatorname{Ker} M_\varrho P.$$

Since $\operatorname{Ker} P \subseteq \operatorname{Ker} M_\varrho P$ and $\operatorname{Ran} P^*P \subseteq \operatorname{Ran} P^* \perp \operatorname{Ker} P$ we find that $\operatorname{Ran} MP \perp \operatorname{Ker} P$ for any ϱ. This gives a contradiction to (4.7), thus P is of principal type.

Next, we shall show that M is invertible at w_0 if $\operatorname{Im} MP \ge cP^*P$ at w_0 for some $c > 0$. Then we find as before from Lemma 4.6 that $\operatorname{Ran} MP(w_0) \perp \operatorname{Ker} MP(w_0)$. By choosing a base for $\operatorname{Ker} P(w_0)$ and completing it to a base of $\mathbb{C}^N$ we may assume that

$$P(w_0) = \begin{pmatrix} 0 & P_{12}(w_0) \\ 0 & P_{22}(w_0) \end{pmatrix}$$

where P_{22} is $(N-K) \times (N-K)$ system, $K = \operatorname{Dim} \operatorname{Ker} P(w_0)$. Now, by multiplying P from the left with an orthogonal matrix E we may assume that $P_{12}(w_0) = 0$. In fact, this only amounts to choosing an orthonormal base for $\operatorname{Ran} P(w_0)^\perp$ and completing to an orthonormal base for $\mathbb{C}^N$. Observe that MP is unchanged if we replace M with ME^{-1}, which is invertible if and only if M is. Since $\operatorname{Dim} \operatorname{Ker} P(w_0) = K$ we obtain $|P_{22}(w_0)| \neq 0$. Letting

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

we find

$$MP = \begin{pmatrix} 0 & 0 \\ 0 & M_{22}P_{22} \end{pmatrix} \qquad \text{at } w_0.$$

In fact, $(MP)_{12}(w_0) = M_{12}(w_0)P_{22}(w_0) = 0$ since $\operatorname{Ran} MP(w_0) = \operatorname{Ker} MP(w_0)^\perp$. We obtain that $M_{12}(w_0) = 0$, and by assumption

$$\operatorname{Im} M_{22}P_{22} \ge cP_{22}^*P_{22} \qquad \text{at } w_0,$$

which gives $|M_{22}(w_0)| \neq 0$. Since P_{11}, P_{21} and M_{12} vanish at w_0 we find

$$\operatorname{Re} V(MP)_{11}(w_0) = \operatorname{Re} M_{11}(w_0)VP_{11}(w_0) > c$$

which gives $|M_{11}(w_0)| \neq 0$. Since $M_{12}(w_0) = 0$ and $|M_{22}(w_0)| \neq 0$ we obtain that $M(w_0)$ is invertible. □

Remark 4.8. The $N \times N$ system $P \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetrizable with respect to V if and only if there exists an invertible symmetrizer M such that $Q = MP$ satisfies

$$\operatorname{Re}\langle (VQ)u, u\rangle \ge c\|u\|^2 - C\|Qu\|^2 \qquad c > 0 \tag{4.8}$$

$$\operatorname{Im}\langle Qu, u\rangle \ge 0 \tag{4.9}$$

for any $u \in \mathbb{C}^N$.

In fact, by the Cauchy–Schwarz inequality we find

$$|\langle (VM)Pu, u\rangle| \le \varepsilon\|u\|^2 + C_\varepsilon\|Pu\|^2 \qquad \forall\, \varepsilon > 0 \quad \forall\, u \in \mathbb{C}^N.$$

Since M is invertible, we also have that $\|Pu\| \cong \|Qu\|$.

Definition 4.9. If the $N \times N$ system $Q \in C^\infty(T^*\mathbb{R}^n)$ satisfies (4.8)–(4.9) then Q is *quasisymmetric* with respect to the real C^∞ vector field V.

Proposition 4.10. *Let $P(w) \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ quasisymmetrizable system; then P^* is quasisymmetrizable. If $A(w)$ and $B(w) \in C^\infty(T^*\mathbb{R}^n)$ are invertible $N \times N$ systems then BPA is quasisymmetrizable.*

Proof. Clearly (4.8)–(4.9) are invariant under *left* multiplication of P with invertible systems E, just replace M with ME^{-1}. Since we may write $BPA = B(A^*)^{-1}A^*PA$ it suffices to show that E^*PE is quasisymmetrizable if E is invertible. By Remark 4.8 there exists a symmetrizer M so that $Q = MP$ is quasisymmetric, that is, satisfies (4.8)–(4.9). It then follows from Proposition 4.11 that

$$Q_E = E^*QE = E^*M(E^*)^{-1}E^*PE$$

also satisfies (4.8) and (4.9), so E^*PE is quasisymmetrizable.

Finally, we shall prove that P^* is quasisymmetrizable if P is. Since $Q = MP$ is quasisymmetric, we find from Proposition 4.11 that $Q^* = P^*M^*$ is quasisymmetric. By multiplying with $(M^*)^{-1}$ from the right, we find from the first part of the proof that P^* is quasisymmetrizable. □

Proposition 4.11. *If $Q \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetric, then Q^* is quasisymmetric. If $E \in C^\infty(T^*\mathbb{R}^n)$ is invertible, then E^*QE are quasisymmetric.*

Proof. First we note that (4.8) holds if and only if

$$\operatorname{Re}\langle (VQ)u, u\rangle \ge c\|u\|^2 \qquad \forall\, u \in \operatorname{Ker} Q \tag{4.10}$$

for some $c > 0$. In fact, Q^*Q has a positive lower bound on the orthogonal complement $\operatorname{Ker} Q^\perp$ so that

$$\|u\| \le C\|Qu\| \qquad \text{for } u \in \operatorname{Ker} Q^\perp.$$

Thus, if $u = u' + u''$ with $u' \in \operatorname{Ker} Q$ and $u'' \in \operatorname{Ker} Q^\perp$ we find that $Qu = Qu''$,

$$\operatorname{Re}\langle (VQ)u', u''\rangle \ge -\varepsilon\|u'\|^2 - C_\varepsilon\|u''\|^2 \ge -\varepsilon\|u'\|^2 - C'_\varepsilon\|Qu\|^2 \qquad \forall\, \varepsilon > 0$$

and $\operatorname{Re}\langle (VQ)u'', u''\rangle \ge -C\|u''\|^2 \ge -C'\|Qu\|^2$. By choosing ε small enough we obtain (4.8) by using (4.10) on u'.

Next, we note that $\operatorname{Im} Q^* = -\operatorname{Im} Q$ and $\operatorname{Re} Q^* = \operatorname{Re} Q$, so $-Q^*$ satisfies (4.9) and (4.10) with V replaced by $-V$, and thus it is quasisymmetric. Finally, we shall show that $Q_E = E^*QE$ is quasisymmetric when E is invertible. We obtain from (4.9) that

$$\operatorname{Im}\langle Q_E u, u\rangle = \operatorname{Im}\langle QEu, Eu\rangle \ge 0 \qquad \forall\, u \in \mathbb{C}^N.$$

Next, we shall show that Q_E satisfies (4.10) on $\operatorname{Ker} Q_E = E^{-1}\operatorname{Ker} Q$, which will give (4.8). We find from Leibniz' rule that $VQ_E = (VE^*)QE + E^*(VQ)E + E^*Q(VE)$ where (4.10) gives

$$\operatorname{Re}\langle E^*(VQ)Eu, u\rangle \ge c\|Eu\|^2 \ge c'\|u\|^2 \qquad u \in \operatorname{Ker} Q_E$$

since then $Eu \in \operatorname{Ker} Q$. Similarly we obtain that $\langle (VE^*)QEu, u\rangle = 0$ when $u \in \operatorname{Ker} Q_E$. Now since $\operatorname{Im} Q_E \ge 0$ we find from Lemma 4.6 that

$$\operatorname{Ker} Q_E^* = \operatorname{Ker} Q_E$$

which gives $\langle E^* Q(VE)u, u\rangle = \langle E^{-1}(VE)u, Q_E^* u\rangle = 0$ when $u \in \operatorname{Ker} Q_E = \operatorname{Ker} Q_E^*$. Thus Q_E satisfies (4.10) so it is quasisymmetric. □

Example 4.12. Assume that $P(w) \in C^\infty$ is an $N \times N$ system such that $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cap \Sigma_\infty(P))$ and that $P(w) - \lambda \operatorname{Id}_N$ is of principal type when $|\lambda - z| \ll 1$. By Lemma 2.15 and Proposition 3.5 there exists a C^∞ germ of eigenvalues $\lambda(w) \in C^\infty$ for P so that $\operatorname{Dim} \operatorname{Ker}(P(w) - \lambda(w) \operatorname{Id}_N)$ is constant near $\Sigma_z(P)$. By using the spectral projection as in the proof of Proposition 3.5 and making a base change $B(w) \in C^\infty$ we obtain

$$P(w) = B^{-1}(w) \begin{pmatrix} \lambda(w) \operatorname{Id}_K & 0 \\ 0 & P_{22}(w) \end{pmatrix} B(w) \tag{4.11}$$

in a neighborhood of $\Sigma_z(P)$, here $|P_{22} - \lambda(w) \operatorname{Id}| \neq 0$. We find from Proposition 3.5 that $d\lambda \neq 0$ when $\lambda = z$, so $\lambda - z$ is of principal type. Proposition 4.10 gives that $P - z \operatorname{Id}_N$ is quasisymmetrizable near any $w_0 \in \Sigma_z(P)$ if $z \in \partial\Sigma(\lambda)$. In fact, by Example 4.4 there exists $q(w) \in C^\infty$ so that

$$|d \operatorname{Re} q(\lambda - z)| \neq 0 \tag{4.12}$$

$$\operatorname{Im} q(\lambda - z) \geq 0 \tag{4.13}$$

and we get the normal form (4.2) for λ near $\Sigma_z(P) = \{\lambda(w) = z\}$. One can then take V normal to $\Sigma = \{\operatorname{Re} q(\lambda - z) = 0\}$ at $\Sigma_z(P)$ and use

$$M = B^* \begin{pmatrix} q \operatorname{Id}_K & 0 \\ 0 & M_{22} \end{pmatrix} B$$

with $M_{22}(w) = (P_{22}(w) - z \operatorname{Id})^{-1}$ for example. Then

$$Q = M(P - z \operatorname{Id}_N) = B^* \begin{pmatrix} q(\lambda - z) \operatorname{Id}_K & 0 \\ 0 & \operatorname{Id}_{N-K} \end{pmatrix} B. \tag{4.14}$$

If there are no closed semibicharacteristics of $\lambda - z$ then we also find from Example 4.4 that $P - z \operatorname{Id}_N$ is quasisymmetrizable in a neighborhood of $\Sigma_z(P)$; see the proof of Lemma 4.1 in [Dencker et al. 2004].

Example 4.13. Let

$$P(x, \xi) = |\xi|^2 \operatorname{Id}_N + i K(x)$$

where $0 \leq K(x) \in C^\infty$. When $z > 0$ we find that $P - z \operatorname{Id}_N$ is quasisymmetric in a neighborhood of $\Sigma_z(P)$ with respect to the exterior normal $\langle \xi, \partial_\xi \rangle$ to $\Sigma_z(P) = \big\{ |\xi|^2 = z \big\}$.

For scalar symbols, we find that $0 \in \partial\Sigma(p)$ if and only if p is quasisymmetrizable, see Example 4.4. But in the system case, this needs not be the case according to the following example.

Example 4.14. Let

$$P(w) = \begin{pmatrix} w_2 + i w_3 & w_1 \\ w_1 & w_2 - i w_3 \end{pmatrix} \qquad w = (w_1, w_2, w_3),$$

which is quasisymmetrizable with respect to ∂_{w_1} with symmetrizer

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In fact, $\partial_{w_1} MP = \mathrm{Id}_2$ and

$$MP(w) = \begin{pmatrix} w_1 & w_2 - iw_3 \\ w_2 + iw_3 & w_1 \end{pmatrix} = (MP(w))^*$$

so $\mathrm{Im}\, MP \equiv 0$. Since eigenvalues of $P(w)$ are $w_2 \pm \sqrt{w_1^2 - w_3^2}$ we find that $\Sigma(P) = \mathbb{C}$ so $0 \in \mathring{\Sigma}(P)$ is not a boundary point of the eigenvalues.

For quasisymmetrizable systems we have the following result.

Theorem 4.15. *Let the $N \times N$ system $P(h)$ be given by* (2.2) *with principal symbol $P \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$. Assume that $z \notin \Sigma_\infty(P)$ and there exists a real valued function $T(w) \in C^\infty$ such that $P(w) - z\,\mathrm{Id}_N$ is quasisymmetrizable with respect to the Hamilton vector field $H_T(w)$ in a neighborhood of $\Sigma_z(P)$. Then for any $K > 0$ we have*

$$\left\{\zeta \in \mathbb{C} : |\zeta - z| < Kh\log(1/h)\right\} \cap \mathrm{Spec}(P(h)) = \varnothing \tag{4.15}$$

for $0 < h \ll 1$, *and*

$$|(P(h) - z)^{-1}| \le C/h \quad 0 < h \ll 1. \tag{4.16}$$

If P is analytic in a tubular neighborhood of $T^*\mathbb{R}^n$ then there exists $c_0 > 0$ such that

$$\left\{\zeta \in \mathbb{C} : |\zeta - z| < c_0\right\} \cap \mathrm{Spec}(P(h)) = \varnothing.$$

Condition (4.16) means that $\lambda \notin \Lambda_1^{\mathrm{sc}}(P)$, which is the pseudospectrum of index 1 by Definition 2.27. The reason for the difference between (4.15) and (4.16) is that we make a change of norm in the proof that is not uniform in h. The conditions in Theorem 4.15 give some geometrical information on the bicharacteristic flow of the eigenvalues according to the following result.

Remark 4.16. The conditions in Theorem 4.15 imply that the limit set at $\Sigma_z(P)$ of the nontrivial semibicharacteristics of the eigenvalues close to zero of $Q = M(P - z\,\mathrm{Id}_N)$ is a union of compact curves on which T is strictly monotone, thus they cannot form closed orbits.

In fact, locally $(w, \lambda) \in \Omega_1(P) \setminus \Xi(P)$ if and only if $\lambda = \lambda(w) \in C^\infty$ by Lemma 2.15. Since $P(w) - \lambda\,\mathrm{Id}_N$ is of principal type by Proposition 4.7, we find that $\mathrm{Dim}\,\mathrm{Ker}(P(w) - \lambda(w)\,\mathrm{Id}_N)$ is constant by Proposition 3.5. Thus we obtain the normal form (4.14) as in Example 4.12. This shows that the Hamilton vector field H_λ of an eigenvalue is determined by $\langle dQu, u\rangle$ with $0 \neq u \in \mathrm{Ker}(P - \nu\,\mathrm{Id}_N)$ for ν close to $z = \lambda(w)$ by the invariance property given by (3.2). Now $\langle (H_T\,\mathrm{Re}\,Q)u, u\rangle > 0$ for $0 \neq u \in \mathrm{Ker}(P - z\,\mathrm{Id}_N)$, and $d\langle \mathrm{Im}\,Qu, u\rangle = 0$ for $u \in \mathrm{Ker}\,M(P - z\,\mathrm{Id}_N)$ by (4.9). Thus by picking subsequences we find that the limits of nontrivial semibicharacteristics of eigenvalues λ of Q close to 0 give curves on which T is strictly monotone. Since $z \notin \Sigma_\infty(P)$ these limit bicharacteristics are compact and cannot form closed orbits.

Example 4.17. Consider the system in Example 4.13

$$P(x, \xi) = |\xi|^2\,\mathrm{Id}_N + iK(x)$$

where $0 \le K(x) \in C^\infty$. Then for $z > 0$ we find that $P - z\,\mathrm{Id}_N$ is quasisymmetric in a neighborhood of $\Sigma_z(P)$ with respect to $V = H_T$, for $T(x, \xi) = -\langle \xi, x\rangle$. If $K(x) \in C_{\mathrm{b}}^\infty$ and $0 \notin \Sigma_\infty(K)$ then we obtain from Proposition 2.20, Remark 2.21, Example 2.22 and Theorem 4.15 that

$$\|(P^w(x,hD)-z)^{-1}\| \le C/h \qquad 0<h\ll 1$$

since $z \notin \Sigma_\infty(P)$.

Proof of Theorem 4.15. We shall first consider the C_b^∞ case. We may assume without loss of generality that $z=0$, and we shall follow the proof of Theorem 1.3 in [Dencker et al. 2004]. By the conditions, we find from Definition 4.5, Remark 4.8 and (4.6) that there exists a function $T(w)\in C_0^\infty$ and a multiplier $M(w)\in C_b^\infty(T^*\mathbb{R}^n)$ so that $Q=MP$ satisfies

$$\operatorname{Re} H_T Q \ge c - C\operatorname{Im} Q \tag{4.17}$$

$$\operatorname{Im} Q \ge c\, Q^*Q \tag{4.18}$$

for some $c>0$ and then M is invertible by Proposition 4.7. In fact, outside a neighborhood of $\Sigma_0(P)$ we have $P^*P\ge c_0$; then we may choose $M=iP^*$ so that $Q=iP^*P$ and use a partition of unity to get a global multiplier. Let

$$C_1 h \le \varepsilon \le C_2 h \log\frac{1}{h} \tag{4.19}$$

where $C_1\gg 1$ will be chosen large. Let $T=T^w(x,hD)$

$$Q(h)=M^w(x,hD)P(h)=Q^w(x,hD)+\mathcal{O}(h) \tag{4.20}$$

$$Q_\varepsilon(h)=e^{\varepsilon T/h}Q(h)e^{-\varepsilon T/h}=e^{\frac{\varepsilon}{h}\operatorname{ad}_T}Q(h)\sim\sum_{k=0}^{\infty}\frac{\varepsilon^k}{h^k k!}(\operatorname{ad}_T)^k(Q(h))$$

where $\operatorname{ad}_T Q(h)=[T(h),Q(h)]=\mathcal{O}(h)$. By the assumption on ε and the boundedness of ad_T/h we find that the asymptotic expansion makes sense. Since $\varepsilon^2=\mathcal{O}(h)$ we see that the symbol of $Q_\varepsilon(h)$ is equal to

$$Q_\varepsilon = Q + i\varepsilon\{T,Q\}+\mathcal{O}(h).$$

Since T is a scalar function, we obtain

$$\operatorname{Im} Q_\varepsilon = \operatorname{Im} Q + \varepsilon\operatorname{Re} H_T Q + \mathcal{O}(h). \tag{4.21}$$

Now to simplify notation, we drop the parameter h in the operators $Q(h)$ and $P(h)$, and we shall use the same letters for operators and the corresponding symbols. Using (4.17) and (4.18) in (4.21), we obtain for small enough ε that

$$\operatorname{Im} Q_\varepsilon \ge (1-C\varepsilon)\operatorname{Im} Q + c\varepsilon - Ch \ge c\varepsilon - Ch \tag{4.22}$$

Since the symbol of $\frac{1}{2i}(Q_\varepsilon-(Q_\varepsilon)^*)$ is equal to the expression (4.22) modulo $\mathcal{O}(h)$, the sharp Gårding inequality for systems in Proposition A.5 gives

$$\operatorname{Im}\langle Q_\varepsilon u,u\rangle \ge (c\varepsilon - C_0 h)\|u\|^2 \ge \frac{\varepsilon c}{2}\|u\|^2$$

for $h\ll\varepsilon\ll 1$. By using the Cauchy–Schwarz inequality, we obtain

$$\frac{\varepsilon c}{2}\|u\| \le \|Q_\varepsilon u\|. \tag{4.23}$$

Since $Q = MP$ the calculus gives

$$Q_\varepsilon = M_\varepsilon P_\varepsilon + \mathcal{O}(h) \tag{4.24}$$

where $P_\varepsilon = e^{-\varepsilon T/h} P e^{\varepsilon T/h}$ and $M_\varepsilon = e^{-\varepsilon T/h} M e^{\varepsilon T/h} = M + \mathcal{O}(\varepsilon)$ is bounded and invertible for small enough ε. For $h \ll \varepsilon$ we obtain from (4.23)–(4.24) that

$$\|u\| \le \frac{C}{\varepsilon} \|P_\varepsilon u\| \tag{4.25}$$

so P_ε is injective with closed range. Now $-Q^*$ satisfies the conditions (4.3)–(4.4), with T replaced by $-T$. Thus we also obtain the estimate (4.23) for $Q_\varepsilon^* = P_\varepsilon^* M_\varepsilon^* + \mathcal{O}(h)$. Since M_ε^* is invertible for small enough h we obtain the estimate (4.25) for P_ε^*, thus P_ε is surjective. Because the conjugation by $e^{\varepsilon T/h}$ is uniformly bounded on L^2 when $\varepsilon \le Ch$ we obtain the estimate (4.16) from (4.25).

Now conjugation with $e^{\varepsilon T/h}$ is bounded in L^2 (but not uniformly) also when (4.19) holds. By taking C_2 arbitrarily large in (4.19) we find from the estimate (4.25) for P_ε and P_ε^* that

$$D\left(0, Kh \log \frac{1}{h}\right) \cap \operatorname{Spec}(P) = \varnothing$$

for any $K > 0$ when $h > 0$ is sufficiently small.

The analytic case. We assume as before that $z = 0$ and

$$P(h) \sim \sum_{j\ge 0} h^j P_j^w(x, hD) \qquad P_0 = P$$

where the P_j are bounded and holomorphic in a tubular neighborhood of $T^*\mathbb{R}^n$, satisfy (2.3), and $P_j^w(z, hD_z)$ is defined by the formula (2.1), where we may change the integration to a suitable chosen contour instead of $T^*\mathbb{R}^n$ (see [Sjöstrand 1982, Section 4]). As before, we shall follow the proof of Theorem 1.3 in [Dencker et al. 2004] and use the theory of the weighted spaces $H(\Lambda_{\varrho T})$ developed in [Helffer and Sjöstrand 1990] (see also [Martinez 2002]).

The complexification $T^*\mathbb{C}^n$ of the symplectic manifold $T^*\mathbb{R}^n$ is equipped with a complex symplectic form $\omega_{\mathbb{C}}$ giving two natural real symplectic forms $\operatorname{Im}\omega_{\mathbb{C}}$ and $\operatorname{Re}\omega_{\mathbb{C}}$. We find that $T^*\mathbb{R}^n$ is Lagrangian with respect to the first form and symplectic with respect to the second. In general, a submanifold satisfying these two conditions is called an *IR-manifold*.

Assume that $T \in C_0^\infty(T^*\mathbb{R}^n)$; then we may associate to it a natural family of IR-manifolds:

$$\Lambda_{\varrho T} = \{w + i\varrho H_T(w) : \ w \in T^*\mathbb{R}^n\} \subset T^*\mathbb{C}^n \quad \text{with } \varrho \in \mathbb{R} \text{ and } |\varrho| \text{ small}$$

where as before we identify $T(T^*\mathbb{R}^n)$ with $T^*\mathbb{R}^n$; see [Dencker et al. 2004, page 391]. Since $\operatorname{Im}(\zeta\, dz)$ is closed on $\Lambda_{\varrho T}$,we find that there exists a function G_ϱ on $\Lambda_{\varrho T}$ such that

$$dG_\varrho = -\operatorname{Im}(\zeta\, dz)|_{\Lambda_{\varrho T}}$$

In fact, we can write it down explicitly by parametrizing $\Lambda_{\varrho T}$ by $T^*\mathbb{R}^n$:

$$G_\varrho(z, \zeta) = -\langle \xi, \varrho \nabla_\xi T(x, \xi)\rangle + \varrho T(x, \xi) \quad \text{for} \quad (z, \zeta) = (x, \xi) + i\varrho H_T(x, \xi)$$

The associated spaces $H(\Lambda_{\varrho T})$ are going to be defined by using the FBI transform:

$$T : L^2(\mathbb{R}^n) \to L^2(T^*\mathbb{R}^n)$$

given by

$$Tu(x,\xi) = c_n h^{-3n/4} \int_{\mathbb{R}^n} e^{i(\langle x-y,\xi\rangle + i|x-y|^2)/(2h)} u(y)\, dy. \tag{4.26}$$

The FBI transform may be continued analytically to $\Lambda_{\varrho T}$ so that $T_{\Lambda_{\varrho T}} u \in C^\infty(\Lambda_{\varrho T})$. Since $\Lambda_{\varrho T}$ differs from $T^*\mathbb{R}^n$ on a compact set only, we find that $T_{\Lambda_{\varrho T}} u$ is square integrable on $\Lambda_{\varrho T}$. The FBI transform can of course also be defined on $u \in L^2(\mathbb{R}^n)$ having values in $\mathbb{C}^N$, and the spaces $H(\Lambda_{\varrho T})$ are defined by putting h dependent norms on $L^2(\mathbb{R}^n)$:

$$\|u\|^2_{H(\Lambda_{\varrho T})} = \int_{\Lambda_{\varrho T}} |T_{\Lambda_{\varrho T}} u(z,\zeta)|^2 e^{-2G_\varrho(z,\zeta)/h} (\omega|_{\Lambda_{\varrho T}})^n/n! = \|T_{\Lambda_{\varrho T}} u\|^2_{L^2(\varrho,h)}$$

Suppose that P_1 and P_2 are bounded and holomorphic $N \times N$ systems in a neighbourhood of $T^*\mathbb{R}^n$ in $T^*\mathbb{C}^n$ and that $u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. Then we find for $\varrho > 0$ small enough

$$\begin{aligned}\langle P_1^w(x,hD)u,\, P_2^w(x,hD)v\rangle_{H(\Lambda_{\varrho T})} \\ = \langle (P_1|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} u,\, (P_2|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} v\rangle_{L^2(\varrho,h)} + \mathcal{O}(h)\|u\|_{H(\Lambda_{\varrho T})}\|v\|_{H(\Lambda_{\varrho T})}.\end{aligned}$$

By taking $P_1 = P_2 = P$ and $u = v$ we obtain

$$\|P^w(x,hD)u\|^2_{H(\Lambda_{\varrho T})} = \|(P|_{\Lambda_{\varrho T}}) T_{\Lambda_{\varrho T}} u\|^2_{L^2(\varrho,h)} + \mathcal{O}(h)\|u\|^2_{H(\Lambda_{\varrho T})} \tag{4.27}$$

as in the scalar case; see [Helffer and Sjöstrand 1990] or [Martinez 2002].

By Remark 4.8 we may assume that $MP = Q$ satisfies (4.8)–(4.9), with invertible M. The analyticity of P gives

$$P(w + i\varrho H_T) = P(w) + i\varrho H_T P(w) + \mathcal{O}(\varrho^2) \qquad |\varrho| \ll 1$$

by Taylor's formula; thus

$$\operatorname{Im} M(w)P(w + i\varrho H_T(w)) = \operatorname{Im} Q(w) + \varrho \operatorname{Re} M(w) H_T P(w) + \mathcal{O}(\varrho^2).$$

Since we have $\operatorname{Re} M H_T P > c - C \operatorname{Im} Q$, $c > 0$, by (4.8) and $\operatorname{Im} Q \ge 0$ by (4.9), we obtain for sufficiently small $\varrho > 0$ that

$$\operatorname{Im} M(w)P(w + i\varrho H_T(w)) \ge (1 - C\varrho)\operatorname{Im} Q(w) + c\varrho + \mathcal{O}(\varrho^2) \ge c\varrho/2 \tag{4.28}$$

which gives by the Cauchy–Schwarz inequality that $\|P\restriction_{\Lambda_{\varrho T}} u\| \ge c'\varrho\|u\|$. Thus

$$\|P^{-1}\restriction_{\Lambda_{\varrho T}}\| \le C/\varrho. \tag{4.29}$$

Now recall that $H(\Lambda_{\varrho T})$ is equal to L^2 as a space and that the norms are equivalent for every fixed h (but not uniformly). Thus the spectrum of $P(h)$ does not depend on whether the operator is realized on L^2 or on $H(\Lambda_{\varrho T})$. We conclude from (4.27) and (4.29) that 0 has an h-independent neighbourhood which is disjoint from the spectrum of $P(h)$, when h is small enough. □

Summing up, we have proved the following result.

Proposition 4.18. *Assume that $P(h)$ is an $N \times N$ system on the form given by* (2.2) *with analytic principal symbol $P(w)$, and that there exists a real valued function $T(w) \in C^\infty(T^*\mathbb{R}^n)$ such that $P(w) - z\,\mathrm{Id}_N$ is quasisymmetrizable with respect to H_T in a neighborhood of $\Sigma_z(P)$. Define the IR-manifold*

$$\Lambda_{\varrho T} = \{w + i\varrho H_T(w);\ w \in T^*\mathbb{R}^n\}$$

for $\varrho > 0$ small enough. Then

$$P(h) - z :\ H(\Lambda_{\varrho T}) \longrightarrow H(\Lambda_{\varrho T})$$

has a bounded inverse for h small enough, which gives

$$\mathrm{Spec}(P(h)) \cap D(z, \delta) = \varnothing \qquad 0 < h < h_0$$

for δ small enough.

Remark 4.19. It is clear from the proof of Theorem 4.15 that in the analytic case it suffices that P_j is analytic in a fixed complex neighborhood of $\Sigma_z(P) \Subset T^*\mathbb{R}^n$, $j \geq 0$.

5. The subelliptic case

We shall investigate when we have an estimate of the resolvent which is better than the quasisymmetric estimate, for example the subelliptic type of estimate

$$\|(P(h) - \lambda\,\mathrm{Id}_N)^{-1}\| \leq Ch^{-\mu} \qquad h \to 0$$

with $\mu < 1$, which we obtain in the scalar case under the bracket condition; see Theorem 1.4 in [Dencker et al. 2004].

Example 5.1. Consider the scalar operator $p^w = hD_t + if^w(t, x, hD_x)$ where $0 \leq f(t, x, \xi) \in C_{\mathrm{b}}^\infty$, $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, and $0 \in \partial\Sigma(f)$. Then we obtain from Theorem 1.4 in [Dencker et al. 2004] the estimate

$$h^{k/k+1}\|u\| \leq C\|p^w u\| \qquad h \ll 1 \qquad \forall\, u \in C_0^\infty \tag{5.1}$$

if $0 \notin \Sigma_\infty(f)$ and

$$\sum_{j \leq k} |\partial_t^j f| \neq 0. \tag{5.2}$$

These conditions are also necessary. For example, if $|f(t)| \leq C|t|^k$ then an easy computation gives $\|hD_t u + ifu\|/\|u\| \leq ch^{k/k+1}$ if $u(t) = \phi(th^{-1/k+1})$ with $0 \neq \phi(t) \in C_0^\infty(\mathbb{R})$.

The following example shows that condition (5.2) is not sufficient for systems.

Example 5.2. Let $P = hD_t\,\mathrm{Id}_2 + iF(t)$ where

$$F(t) = \begin{pmatrix} t^2 & t^3 \\ t^3 & t^4 \end{pmatrix}.$$

Then we have

$$F^{(3)}(0) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$$

which gives that

$$\bigcap_{j\le 3} \operatorname{Ker} F^{(j)}(0) = \{0\}\,.$$

But by taking $u(t) = \chi(t)(t, -1)^t$ with $0 \neq \chi(t) \in C_0^\infty(\mathbb{R})$, we obtain $F(t)u(t) \equiv 0$ so we find $\|Pu\|/\|u\| \le ch$. Observe that

$$F(t) = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}.$$

Thus $F(t) = t^2 B^*(t)\Pi(t)B(t)$ where $B(t)$ is invertible and $\Pi(t)$ is a projection of rank one.

Example 5.3. Let $P = hD_t \operatorname{Id}_2 + iF(t)$ where

$$F(t) = \begin{pmatrix} t^2+t^8 & t^3-t^7 \\ t^3-t^7 & t^4+t^6 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t^6 \end{pmatrix} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}.$$

Then we have that

$$P = (1+t^2)^{-1} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} hD_t + i(t^2+t^4) & 0 \\ 0 & hD_t + i(t^6+t^8) \end{pmatrix} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} + \mathcal{O}(h).$$

Thus we find from the scalar case that $h^{6/7}\|u\| \le C\|Pu\|$ for $h \ll 1$; see [Dencker et al. 2004, Theorem 1.4]. Observe that this operator is, element for element, a higher order perturbation of the operator of Example 5.2.

Definition 5.4. Let $0 \le F(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be an $N \times N$ system; then we define

$$\Omega_\delta(F) = \Big\{ t : \min_{\|u\|=1} \langle F(t)u, u\rangle \le \delta \Big\} \qquad 0 < \delta \le 1$$

which is well-defined almost everywhere and contains $\Sigma_0(F) = |F|^{-1}(0)$.

Observe that one can also use this definition in the scalar case, then $\Omega_\delta(f) = f^{-1}([0,\delta])$ for nonnegative functions f.

Remark 5.5. Observe that if $F \ge 0$ and E is invertible then we find that

$$\Omega_\delta(E^*FE) \subseteq \Omega_{C\delta}(F)$$

where $C = \|E^{-1}\|^2$.

Example 5.6. For the scalar symbols $p(x,\xi) = \tau + if(t,x,\xi)$ in Example 5.1 we find from Proposition A.1 that (5.2) is equivalent to

$$|\{t : f(t,x,\xi) \le \delta\}| = |\Omega_\delta(f_{x,\xi})| \le C\delta^{1/k} \qquad 0 < \delta \ll 1 \quad \forall x, \xi,$$

where $f_{x,\xi}(t) = f(t,x,\xi)$.

Example 5.7. For the matrix $F(t)$ in Example 5.3 we find from Remark 5.5 that $|\Omega_\delta(F)| \le C\delta^{1/6}$, and for the matrix in Example 5.2 we find that $|\Omega_\delta(F)| = \infty$.

We also have examples when the semidefinite imaginary part vanishes of infinite order.

Example 5.8. Let $p(x,\xi) = \tau + if(t,x,\xi)$ where $0 \le f(t,x,\xi) \le Ce^{-1/|t|^{\sigma}}$, $\sigma > 0$, then we obtain that

$$|\Omega_\delta(f_{x,\xi})| \le C_0 |\log\delta|^{-1/\sigma} \qquad 0 < \delta \ll 1 \quad \forall\, x, \xi.$$

(We owe this example to Y. Morimoto.)

The following example shows that for subelliptic type of estimates it is not sufficient to have conditions only on the vanishing of the symbol, we also need conditions on the semibicharacteristics of the eigenvalues.

Example 5.9. Let

$$P = hD_t \operatorname{Id}_2 + \alpha h \begin{pmatrix} D_x & 0 \\ 0 & -D_x \end{pmatrix} + i(t-\beta x)^2 \operatorname{Id}_2 \qquad (t,x) \in \mathbb{R}^2$$

with α, $\beta \in \mathbb{R}$. Then we see from the scalar case that P satisfies the estimate (5.1) with $\mu = 2/3$ if and only either $\alpha = 0$ or $\alpha \ne 0$ and $\beta \ne \pm 1/\alpha$.

Definition 5.10. Let $Q(w) \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system and let $w_0 \in \Sigma \subset T^*\mathbb{R}^n$. We say that Q satisfies the *approximation property* on Σ near w_0 if there exists a Q invariant C^∞ subbundle $\mathcal{V}$ of $\mathbb{C}^N$ over $T^*\mathbb{R}^n$ such that $\mathcal{V}(w_0) = \operatorname{Ker} Q^N(w_0)$ and

$$\operatorname{Re}\langle Q(w)v, v\rangle = 0 \qquad v \in \mathcal{V}(w) \qquad w \in \Sigma \tag{5.3}$$

near w_0. That $\mathcal{V}$ is Q invariant means that $Q(w)v \in \mathcal{V}(w)$ for $v \in \mathcal{V}(w)$.

Here $\operatorname{Ker} Q^N(w_0)$ is the space of the generalized eigenvectors corresponding to the zero eigenvalue. The symbol of the system in Example 5.9 satisfies the approximation property on $\Sigma = \{\tau = 0\}$ if and only if $\alpha = 0$.

Let $\widetilde{Q} = Q|_{\mathcal{V}}$ then since $\operatorname{Im} i\widetilde{Q} = \operatorname{Re}\widetilde{Q}$ we obtain from Lemma 4.6 that $\operatorname{Ran}\widetilde{Q} \perp \operatorname{Ker}\widetilde{Q}$ on Σ. Thus $\operatorname{Ker}\widetilde{Q}^N = \operatorname{Ker}\widetilde{Q}$ on Σ, and since $\operatorname{Ker}\widetilde{Q}^N(w_0) = \mathcal{V}(w_0)$ we find that $\operatorname{Ker} Q^N(w_0) = \mathcal{V}(w_0) = \operatorname{Ker} Q(w_0)$. It follows from Example 5.13 that $\operatorname{Ker} Q \subseteq \mathcal{V}$ near w_0.

Remark 5.11. Assume that Q satisfies the approximation property on the C^∞ hypersurface Σ and is quasisymmetric with respect to $V \notin T\Sigma$. Then the limits of the nontrivial semibicharacteristics of the eigenvalues of Q close to zero coincide with the bicharacteristics of Σ.

In fact, the approximation property in Definition 5.10 and Example 5.13 give that $\langle \operatorname{Re} Qu, u\rangle = 0$ for $u \in \operatorname{Ker} Q \subseteq \mathcal{V}$ on Σ. Since $\operatorname{Im} Q \ge 0$ we find that

$$\langle dQu, u\rangle = 0 \qquad \forall\, u \in \operatorname{Ker} Q \qquad \text{on } T\Sigma. \tag{5.4}$$

By Remark 4.16 the limits of the nontrivial semibicharacteristics of the eigenvalues close to zero of Q are curves with tangents determined by $\langle dQu, u\rangle$ for $u \in \operatorname{Ker} Q$. Since $V \operatorname{Re} Q \ne 0$ on $\operatorname{Ker} Q$ we find from (5.4) that the limit curves coincide with the bicharacteristics of Σ, which are the flow-outs of the Hamilton vector field.

Example 5.12. Observe that Definition 5.10 is empty if $\operatorname{Dim}\operatorname{Ker} Q^N(w_0) = 0$. If $\operatorname{Dim}\operatorname{Ker} Q^N(w_0) > 0$, then there exists $\varepsilon > 0$ and a neighborhood ω to w_0 so that

$$\Pi(w) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} (z \operatorname{Id}_N - Q(w))^{-1}\, dz \in C^\infty(\omega) \tag{5.5}$$

is the spectral projection on the (generalized) eigenvectors with eigenvalues having absolute value less than ε. Then $\operatorname{Ran}\Pi$ is a Q invariant bundle over ω so that $\operatorname{Ran}\Pi(w_0) = \operatorname{Ker} Q^N(w_0)$. Condition (5.3) with $\mathscr{V} = \operatorname{Ran}\Pi$ means that $\Pi^* \operatorname{Re} Q\Pi \equiv 0$ in ω. When $\operatorname{Im} Q(w_0) \geq 0$ we find that $\Pi^* Q\Pi(w_0) = 0$; then Q satisfies the approximation property on Σ near w_0 with $\mathscr{V} = \operatorname{Ran}\Pi$ if and only if

$$d(\Pi^*(\operatorname{Re} Q)\Pi)|_{T\Sigma} \equiv 0 \qquad \text{near } w_0.$$

Example 5.13. If Q satisfies the approximation property on Σ, then by choosing an orthonormal base for $\mathscr{V}$ and extending it to an orthonormal base for $\mathbb{C}^N$ we obtain the system on the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} \tag{5.6}$$

where Q_{11} is $K \times K$ system such that $Q_{11}^N(w_0) = 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and $|Q_{22}| \neq 0$. By multiplying from the left with

$$\begin{pmatrix} \operatorname{Id}_K & -Q_{12}Q_{22}^{-1} \\ 0 & \operatorname{Id}_{N-K} \end{pmatrix}$$

we obtain that $Q_{12} \equiv 0$ without changing Q_{11} or Q_{22}.

In fact, the eigenvalues of Q are then eigenvalues of either Q_{11} or Q_{22}. Since $\mathscr{V}(w_0)$ are the (generalized) eigenvectors corresponding to the zero eigenvalue of $Q(w_0)$ we find that all eigenvalues of $Q_{22}(w_0)$ are nonvanishing, thus Q_{22} is invertible near w_0,

Remark 5.14. If Q satisfies the approximation property on Σ near w_0, then it satisfies the approximation property on Σ near w_1, for w_1 sufficiently close to w_0.

In fact, let Q_{11} be the restriction of Q to $\mathscr{V}$ as in Example 5.13, then since $\operatorname{Re} Q_{11} = \operatorname{Im} i Q_{11} = 0$ on Σ we find from Lemma 4.6 that $\operatorname{Ran} Q_{11} \perp \operatorname{Ker} Q_{11}$ and $\operatorname{Ker} Q_{11} = \operatorname{Ker} Q_{11}^N$ on Σ. Since Q_{22} is invertible in (5.6), we find that $\operatorname{Ker} Q \subseteq \mathscr{V}$. Thus, by using the spectral projection (5.5) of Q_{11} near $w_1 \in \Sigma$ for small enough ε we obtain an invariant subbundle $\widetilde{\mathscr{V}} \subseteq \mathscr{V}$ so that $\widetilde{\mathscr{V}}(w_1) = \operatorname{Ker} Q_{11}(w_1) = \operatorname{Ker} Q^N(w_1)$.

If $Q \in C^\infty$ satisfies the approximation property and $Q_E = E^* Q E$ with invertible $E \in C^\infty$, then it follows from the proof of Proposition 5.18 below that there exist invertible $A, B \in C^\infty$ such that AQ_E and $Q^* B$ satisfy the approximation property.

Definition 5.15. Let $P \in C^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system and let $\phi(r)$ be a positive nondecreasing function on $\mathbb{R}_+$. We say that P is of *subelliptic type ϕ* if for any $w_0 \in \Sigma_0(P)$ there exists a neighborhood ω of w_0, a C^∞ hypersurface $\Sigma \ni w_0$, a real C^∞ vector field $V \notin T\Sigma$ and an invertible symmetrizer $M \in C^\infty$ so that $Q = MP$ is quasisymmetric with respect to V in ω and satisfies the approximation property on $\Sigma \cap \omega$. Also, for every bicharacteristic γ of Σ the arc length

$$\left|\gamma \cap \Omega_\delta(\operatorname{Im} Q) \cap \omega\right| \leq C\phi(\delta) \qquad 0 < \delta \ll 1 \tag{5.7}$$

We say that z is of subelliptic type ϕ for $P \in C^\infty$ if $P - z\operatorname{Id}_N$ is of subelliptic type ϕ. If $\phi(\delta) = \delta^\mu$ then we say that the system is of finite type of order $\mu \geq 0$, which generalizes the definition of finite type for scalar operators in [Dencker et al. 2004].

Recall that the bicharacteristics of a hypersurface in T^*X are the flow-outs of the Hamilton vector field of Σ. Of course, if P is elliptic then by choosing $M = iP^{-1}$ we obtain $Q = i\operatorname{Id}_N$, so P is trivially of subelliptic type. If P is of subelliptic type, then it is quasisymmetrizable by definition and thus of principal type.

Remark 5.16. Observe that we may assume that

$$\operatorname{Im}\langle Qu, u\rangle \geq c\|Qu\|^2 \qquad \forall u \in \mathbb{C}^N \tag{5.8}$$

in Definition 5.15.

In fact, by adding $i\varrho P^*$ to M we obtain (5.8) for large enough ϱ by (4.6), and this only increases $\operatorname{Im} Q$.

Since Q is in C^∞ the estimate (5.7) cannot be satisfied for any $\phi(\delta) \ll \delta$ (unless Q is elliptic) and it is trivially satisfied with $\phi \equiv 1$, thus we shall only consider $c\delta \leq \phi(\delta) \ll 1$ (or finite type of order $0 < \mu \leq 1$). Actually, for C^∞ symbols of finite type, the only relevant values in (5.7) are $\mu = 1/k$ for even $k > 0$; see Proposition A.2 in the Appendix.

Actually, the condition that ϕ is nondecreasing is unnecessary, since the left-hand side in (5.7) is nondecreasing (and upper semicontinuous) in δ, we can replace $\phi(\delta)$ by $\inf_{\varepsilon>\delta} \phi(\varepsilon)$ to make it nondecreasing (and upper semicontinuous).

Example 5.17. Assume that Q is quasisymmetric with respect to the real vector field V, satisfying (5.7) and the approximation property on Σ. Then by choosing an orthonormal base as in Example 5.13 we obtain the system on the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} is $K \times K$ system such that $Q_{11}^N(w_0) = 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and $|Q_{22}| \neq 0$. Since Q is quasisymmetric with respect to V we also obtain that $Q_{11}(w_0) = 0$, $\operatorname{Re} V Q_{11} > 0$, $\operatorname{Im} Q_{jj} \geq 0$ for $j = 1$, 2. In fact, then Lemma 4.6 gives that $\operatorname{Im} Q \perp \operatorname{Ker} Q$ so $\operatorname{Ker} Q^N = \operatorname{Ker} Q$. Since Q satisfies (5.7) and $\Omega_\delta(\operatorname{Im} Q_{11}) \subseteq \Omega_\delta(\operatorname{Im} Q)$ we find that Q_{11} satisfies (5.7). By multiplying from the left as in Example 5.13 we obtain that $Q_{12} \equiv 0$ without changing Q_{11} or Q_{22}.

Proposition 5.18. *If the $N \times N$ system $P(w) \in C^\infty(T^*\mathbb{R}^n)$ is of subelliptic type ϕ then P^* is of subelliptic type ϕ. If $A(w)$ and $B(w) \in C^\infty(T^*\mathbb{R}^n)$ are invertible $N \times N$ systems, then APB is of subelliptic type ϕ.*

Proof. Let M be the symmetrizer in Definition 5.15 so that $Q = MP$ is quasisymmetric with respect to V. By choosing a suitable base and changing the symmetrizer as in Example 5.17, we may write

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \tag{5.9}$$

where Q_{11} is $K \times K$ system such that $Q_{11}(w_0) = 0$, $V \operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and that Q_{22} is invertible. We also have $\operatorname{Im} Q \geq 0$ and that Q satisfies (5.7). Let $\mathcal{V}_1 = \{u \in \mathbb{C}^N : u_j = 0 \text{ for } j > K\}$ and $\mathcal{V}_2 = \{u \in \mathbb{C}^N : u_j = 0 \text{ for } j \leq K\}$, these are Q invariant bundles such that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{C}^N$.

First we are going to show that $\widetilde{P} = APB$ is of subelliptic type. By taking $\widetilde{M} = B^{-1}MA^{-1}$ we find that

$$\widetilde{M}\widetilde{P} = \widetilde{Q} = B^{-1}QB$$

and it is clear that $B^{-1}\mathcal{V}_j$ are $\widetilde{Q}$ invariant bundles, $j = 1, 2$. By choosing bases in $B^{-1}\mathcal{V}_j$ for $j = 1, 2$, we obtain a base for $\mathbb{C}^N$ in which $\widetilde{Q}$ has a block form:

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & 0 \\ 0 & \widetilde{Q}_{22} \end{pmatrix}$$

Here $\widetilde{Q}_{jj} : B^{-1}\mathcal{V}_j \mapsto B^{-1}\mathcal{V}_j$, is given by $\widetilde{Q}_{jj} = B_j^{-1} Q_{jj} B_j$ with

$$B_j : B^{-1}\mathcal{V}_j \ni u \mapsto Bu \in \mathcal{V}_j \qquad j = 1,\ 2.$$

By multiplying $\widetilde{Q}$ from the left with

$$\mathcal{B} = \begin{pmatrix} B_1^* B_1 & 0 \\ 0 & B_2^* B_2 \end{pmatrix}$$

we obtain that

$$\overline{Q} = \mathcal{B}\widetilde{Q} = \mathcal{B}\widetilde{M}\widetilde{P} = \begin{pmatrix} B_1^* Q_{11} B_1 & 0 \\ 0 & B_2^* Q_{22} B_2 \end{pmatrix} = \begin{pmatrix} \overline{Q}_{11} & 0 \\ 0 & \overline{Q}_{22} \end{pmatrix}.$$

It is clear that $\operatorname{Im} \overline{Q} \geq 0$, $Q_{11}(w_0) = 0$, $\operatorname{Re} \overline{Q}_{11} = 0$ on Σ, $|\overline{Q}_{22}| \neq 0$ and $V \operatorname{Re} \overline{Q}_{11} > 0$ by Proposition 4.11. Finally, we obtain from Remark 5.5 that

$$\Omega_\delta(\operatorname{Im} \overline{Q}) \subseteq \Omega_{C\delta}(\operatorname{Im} Q)$$

for some $C > 0$, which proves that $\widetilde{P} = APB$ is of subelliptic type. Observe that $\overline{Q} = AQ_B$, where $Q_B = B^* Q B$ and $A = \mathcal{B} B^{-1}(B^*)^{-1}$.

To show that P^* also is of subelliptic type, we may assume as before that $Q = MP$ is on the form (5.9) with $Q_{11}(w_0) = 0$, $V \operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ, Q_{22} is invertible, $\operatorname{Im} Q \geq 0$ and Q satisfies (5.7). Then we find that

$$-P^* M^* = -Q^* = \begin{pmatrix} -Q_{11}^* & 0 \\ 0 & -Q_{22}^* \end{pmatrix}$$

satisfies the same conditions with respect to $-V$, so it is of subelliptic type with multiplier Id_N. By the first part of the proof we obtain that P^* is of subelliptic type. □

Example 5.19. In the scalar case, $p \in C^\infty(T^*\mathbb{R}^n)$ is quasisymmetrizable with respect to $H_t = \partial_\tau$ near w_0 if and only if

$$p(t, x; \tau, \xi) = q(t, x; \tau, \xi)(\tau + i f(t, x, \xi)) \qquad \text{near } w_0 \tag{5.10}$$

with $f \geq 0$ and $q \neq 0$; see Example 4.4. If $0 \notin \Sigma_\infty(p)$ we find by taking q^{-1} as symmetrizer that p in (5.10) is of finite type of order μ if and only if $\mu = 1/k$ for an even k such that

$$\sum_{j \leq k} |\partial_t^k f| > 0$$

by Proposition A.1. In fact, the approximation property is trivial since f is real. Thus we obtain the case in [Dencker et al. 2004, Theorem 1.4]; see Example 5.1.

Theorem 5.20. *Assume that the $N \times N$ system $P(h)$ is given by the expansion* (2.2) *with principal symbol $P \in C_b^\infty(T^*\mathbb{R}^n)$. Assume that $z \in \Sigma(P) \setminus \Sigma_\infty(P)$ is of subelliptic type ϕ for P, where $\phi > 0$ is nondecreasing on $\mathbb{R}_+$. Then there exists $h_0 > 0$ so that*

$$\|(P(h) - z \operatorname{Id}_N)^{-1}\| \le C/\psi(h) \qquad 0 < h \le h_0 \tag{5.11}$$

where $\psi(h) = \delta$ is the inverse to $h = \delta\phi(\delta)$. It follows that there exists $c_0 > 0$ such that

$$\{w : |w - z| \le c_0\psi(h)\} \cap \sigma(P(h)) = \varnothing \qquad 0 < h \le h_0.$$

Theorem 5.20 will be proved in Section 6. Observe that if $\phi(\delta) \to c > 0$ as $\delta \to 0$ then $\psi(h) = \mathcal{O}(h)$ and Theorem 5.20 follows from Theorem 4.15. Thus we shall assume that $\phi(\delta) \to 0$ as $\delta \to 0$, then we find that $h = \delta\phi(\delta) = o(\delta)$ so $\psi(h) \gg h$ when $h \to 0$. In the finite type case: $\phi(\delta) = \delta^\mu$ we find that $\delta\phi(\delta) = \delta^{1+\mu}$ and $\psi(h) = h^{1/\mu+1}$. When $\mu = 1/k$ we find that $1 + \mu = (k+1)/k$ and $\psi(h) = h^{k/k+1}$. Thus Theorem 5.20 generalizes Theorem 1.4 in [Dencker et al. 2004] by Example 5.19. Condition (5.11) with $\psi(h) = h^{1/\mu+1}$ means that $\lambda \notin \Lambda^{\mathrm{sc}}_{1/\mu+1}(P)$, which is the pseudospectrum of index $1/\mu + 1$.

Example 5.21. Assume that $P(w) \in C^\infty$ is $N \times N$ and $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cup \Sigma_\infty(P))$. Then $\Sigma_\mu(P) = \{\lambda(w) = \mu\}$ for μ close to z, where $\lambda \in C^\infty$ is a germ of eigenvalues for P at $\Sigma_z(P)$; see Lemma 2.15. If $z \in \partial\Sigma(\lambda)$ we find from Example 4.12 that $P(w) - z \operatorname{Id}_N$ is quasisymmetrizable near $w_0 \in \Sigma_z(P)$ if $P(w) - \lambda \operatorname{Id}_N$ is of principal type when $|\lambda - z| \ll 1$. Then P is on the form (4.11) and there exists $q(w) \in C^\infty$ so that (4.12)–(4.13) hold near $\Sigma_z(P)$. We can then choose the multiplier M so that Q is on the form (4.14). By taking $\Sigma = \{\operatorname{Re} q(\lambda - z) = 0\}$ we obtain that $P - z \operatorname{Id}_N$ is of subelliptic type ϕ if (5.7) is satisfied for $\operatorname{Im} q(\lambda - z)$. In fact, by the invariance we find that the approximation property is trivially satisfied since $\operatorname{Re} q\lambda \equiv 0$ on Σ.

Example 5.22. Let

$$P(x, \xi) = |\xi|^2 \operatorname{Id}_N + iK(x) \qquad (x, \xi) \in T^*\mathbb{R}^n$$

where $K(x) \in C^\infty(\mathbb{R}^n)$ is symmetric as in Example 3.12. We find that $P - z \operatorname{Id}_N$ is of finite type of order $1/2$ when $z = i\lambda$ for almost all $\lambda \in \Sigma(K) \setminus (\Sigma_{\mathrm{ws}}(K) \cup \Sigma_\infty(K))$ by Example 5.21. In fact, then $z \in \Sigma(P) \setminus (\Sigma_{\mathrm{ws}}(P) \cap \Sigma_\infty(P))$ and the C^∞ germ of eigenvalues for P near $\Sigma_z(P)$ is $\lambda(x, \xi) = |\xi|^2 + i\kappa(x)$, where $\kappa(x)$ is a C^∞ germ of eigenvalues for $K(x)$ near $\Sigma_\lambda(K) = \{\kappa(x) = \lambda\}$. For almost all values λ we have $d\kappa(x) \ne 0$ on $\Sigma_\lambda(K)$. By taking $q = i$ we obtain for such values that (5.7) is satisfied for $\operatorname{Im} i(\lambda(w) - i\lambda) = |\xi|^2$ with $\phi(\delta) = \delta^{1/2}$, since $\operatorname{Re} i(\lambda(w) - i\lambda) = \lambda - \kappa(x) = 0$ on $\Sigma = \Sigma_\lambda(K)$. If $K(x) \in C_b^\infty$ and $0 \notin \Sigma_\infty(K)$ then we may use Theorem 5.20, Proposition 2.20, Remark 2.21 and Example 2.22 to obtain the estimate

$$\|(P^w(x, hD) - z \operatorname{Id}_N)^{-1}\| \le Ch^{-2/3} \qquad 0 < h \ll 1$$

on the resolvent.

Example 5.23. Let

$$P(t, x; \tau, \xi) = \tau M(t, x, \xi) + iF(t, x, \xi) \in C^\infty$$

where $M \ge c_0 > 0$ and $F \ge 0$ satisfies

$$\left|\left\{t : \inf_{|u|=1} \langle F(t, x, \xi)u, u\rangle \le \delta\right\}\right| \le C\phi(\delta) \qquad \forall x, \xi. \tag{5.12}$$

Then P is quasisymmetrizable with respect to ∂_τ with symmetrizer Id_N. When $\tau = 0$ we obtain that $\mathrm{Re}\, P = 0$, so by taking $\mathscr{V} = \mathrm{Ran}\, \Pi$ for the spectral projection Π given by (5.5) for F, we find that P satisfies the approximation property with respect to $\Sigma = \{\tau = 0\}$. Since $\Omega_\delta(\mathrm{Im}\, P) = \Omega_\delta(F)$ we find from (5.12) that P is of subelliptic type ϕ. Observe that if $0 \notin \Sigma_\infty(F)$ we obtain from Proposition A.2 that (5.12) is satisfied for $\phi(\delta) = \delta^\mu$ if and only if $\mu \le 1/k$ for an even $k \ge 0$ so that

$$\sum_{j \le k} |\partial_t^j \langle F(t, x, \xi) u(t), u(t)\rangle| > 0 \qquad \forall\, t, x, \xi$$

for any $0 \ne u(t) \in C^\infty(\mathbb{R})$.

6. Proof of Theorem 5.20

By subtracting $z\,\mathrm{Id}_N$ we may assume $z = 0$. Let $\widetilde{w}_0 \in \Sigma_0(P)$; then by Definition 5.15 and Remark 5.16 there exist a C^∞ hypersurface Σ and a real C^∞ vector field $V \notin T\Sigma$, an invertible symmetrizer $M \in C^\infty$ so that $Q = MP$ satisfies (5.7), the approximation property on Σ, and

$$V\,\mathrm{Re}\, Q \ge c - C\,\mathrm{Im}\, Q \tag{6.1}$$

$$\mathrm{Im}\, Q \ge c\, Q^* Q \tag{6.2}$$

in a neighborhood ω of $\widetilde{w}_0$, here $c > 0$.

Since (6.1) is stable under small perturbations in V we can replace V with H_t for some real $t \in C^\infty$ after shrinking ω. By solving the initial value problem $H_t \tau \equiv -1$, $\tau|_\Sigma = 0$, and completing to a symplectic C^∞ coordinate system (t, τ, x, ξ) we obtain that $\Sigma = \{\tau = 0\}$ in a neighborhood of $\widetilde{w}_0 = (0, 0, w_0)$. We obtain from Definition 5.15 that

$$\mathrm{Re}\langle Qu, u\rangle = 0 \qquad \text{when } \tau = 0 \text{ and } u \in \mathscr{V} \tag{6.3}$$

near $\widetilde{w}_0$. Here $\mathscr{V}$ is a Q invariant C^∞ subbundle of $\mathbb{C}^N$ such that $\mathscr{V}(\widetilde{w}_0) = \mathrm{Ker}\, Q^N(\widetilde{w}_0) = \mathrm{Ker}\, Q(\widetilde{w}_0)$ by Lemma 4.6. By condition (5.7) we have that

$$\left|\Omega_\delta(\mathrm{Im}\, Q_w) \cap \{|t| < c\}\right| \le C\phi(\delta) \qquad 0 < \delta \ll 1 \tag{6.4}$$

when $|w - w_0| < c$, here $Q_w(t) = Q(t, 0, w)$. Since these are all local conditions, we may assume that M and $Q \in C_b^\infty$. We shall obtain Theorem 5.20 from the following estimate.

Proposition 6.1. *Assume that $Q \in C_b^\infty(T^*\mathbb{R}^n)$ is an $N \times N$ system satisfying* (6.1)–(6.4) *in a neighborhood of $\widetilde{w}_0 = (0, 0, w_0)$ with $V = \partial_\tau$ and nondecreasing $\phi(\delta) \to 0$ as $\delta \to 0$. Then there exist $h_0 > 0$ and $R \in C_b^\infty(T^*\mathbb{R}^n)$ so that $\widetilde{w}_0 \notin \mathrm{supp}\, R$ and*

$$\psi(h)\|u\| \le C(\|Q^w(t, x, hD_{t,x})u\| + \|R^w(t, x, hD_{t,x})u\| + h\|u\|) \qquad 0 < h \le h_0 \tag{6.5}$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$. Here $\psi(h) = \delta \gg h$ is the inverse to $h = \delta\phi(\delta)$.

Let ω be a neighborhood of $\widetilde{w}_0$ such that $\mathrm{supp}\, R \cap \omega = \emptyset$, where R is given by Proposition 6.1. Take $\varphi \in C_0^\infty(\omega)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighborhood of $\widetilde{w}_0$. By substituting $\varphi^w(t, x, hD_{t,x})u$ in (6.5) we obtain from the calculus that for any N we have

$$\psi(h)\|\varphi^w(t, x, hD_{t,x})u\| \le C_N(\|Q^w(t, x, hD_{t,x})\varphi^w(t, x, hD_{t,x})u\| + h^N\|u\|) \qquad \forall\, u \in C_0^\infty \tag{6.6}$$

for small enough h since $R\varphi \equiv 0$. Now the commutator

$$\|[Q^w(t,x,hD_{t,x}), \varphi^w(t,x,hD_{t,x})]u\| \le Ch\|u\| \qquad u \in C_0^\infty.$$

Since $Q = MP$ the calculus gives

$$\|Q^w(t,x,hD_{t,x})u\| \le \|M^w(t,x,hD_{t,x})P(h)u\| + Ch\|u\| \le C'(\|P(h)u\| + h\|u\|) \quad u \in C_0^\infty. \tag{6.7}$$

The estimates (6.6)–(6.7) give

$$\psi(h)\|\varphi^w(t,x,hD_{t,x})u\| \le C(\|P(h)u\| + h\|u\|). \tag{6.8}$$

Since $0 \notin \Sigma_\infty(P)$ we obtain by using the Borel Theorem finitely many functions $\phi_j \in C_0^\infty$, $j = 1, \dots, N$, such that $0 \le \phi_j \le 1$, $\sum_j \phi_j = 1$ on $\Sigma_0(P)$ and the estimate (6.8) holds with $\phi = \phi_j$. Let $\phi_0 = 1 - \sum_{j\ge 1} \phi_j$; then since $0 \notin \Sigma_\infty(P)$ we find that $\|P^{-1}\| \le C$ on $\operatorname{supp}\phi_0$. Thus $\phi_0 = \phi_0 P^{-1} P$ and the calculus gives

$$\|\phi_0^w(t,x,hD_{t,x})u\| \le C(\|P(h)u\| + h\|u\|) \qquad u \in C_0^\infty.$$

By summing up, we obtain

$$\psi(h)\|u\| \le C(\|P(h)u\| + h\|u\|) \qquad u \in C_0^\infty. \tag{6.9}$$

Since $h = \delta\phi(\delta) \ll \delta$ we find $\psi(h) = \delta \gg h$ when $h \to 0$. Thus, we find for small enough h that the last term in the right hand side of (6.9) can be cancelled by changing the constant; then $P(h)$ is injective with closed range. Since $P^*(h)$ also is of subelliptic type ϕ by Proposition 5.18 we obtain the estimate (6.9) for $P^*(h)$. Thus $P^*(h)$ is injective making $P(h)$ is surjective, which together with (6.9) gives Theorem 5.20.

Proof of Proposition 6.1. First we shall prepare the symbol Q locally near $\widetilde{w}_0 = (0, 0, w_0)$. Since $\operatorname{Im} Q \ge 0$ we obtain from Lemma 4.6 that $\operatorname{Ran} Q(\widetilde{w}_0) \perp \operatorname{Ker} Q(\widetilde{w}_0)$ which gives $\operatorname{Ker} Q^N(\widetilde{w}_0) = \operatorname{Ker} Q(\widetilde{w}_0)$. Let $\operatorname{Dim}\operatorname{Ker} Q(\widetilde{w}_0) = K$ then by choosing an orthonormal base and multiplying from the left as in Example 5.17, we may assume that

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} is $K \times K$ matrix, $Q_{11}(\widetilde{w}_0) = 0$ and $|Q_{22}(\widetilde{w}_0)| \ne 0$. Also, we find that Q_{11} satisfies the conditions (6.1)–(6.4) with $\mathcal{V} = \mathbb{C}^K$ near $\widetilde{w}_0$.

Now it suffices to prove the estimate with Q replaced by Q_{11}. In fact, by using the ellipticity of Q_{22} at $\widetilde{w}_0$ we find

$$\|u''\| \le C(\|Q_{22}^w u''\| + \|R_1^w u''\| + h\|u''\| \qquad u'' \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^{N-K})$$

where $u = (u', u'')$ and $\widetilde{w}_0 \notin \operatorname{supp} R_1$. Thus, if we have the estimate (6.5) for Q_{11}^w with $R = R_2$, then since $\psi(h)$ is bounded we obtain the estimate for Q^w:

$$\psi(h)\|u\| \le C_0(\|Q_{11}^w u'\| + \|Q_{22}^w u''\| + \|R^w u\| + h\|u\|) \le C_1(\|Q^w u\| + \|R^w u\| + h\|u\|)$$

where $\widetilde{w}_0 \notin \operatorname{supp} R$, $R = (R_1, R_2)$.

Thus, in the following we may assume that $Q = Q_{11}$ is $K \times K$ system satisfying the conditions (6.1)–(6.4) with $\mathcal{V} = \mathbb{C}^K$ near $\widetilde{w}_0$. Since $\partial_\tau \operatorname{Re} Q > 0$ at $\widetilde{w}_0$ by (6.1), we find from the matrix version of the Malgrange Preparation Theorem in [Dencker 1993, Theorem 4.3] that

$$Q(t,\tau,w) = E(t,\tau,w)(\tau \operatorname{Id} + K_0(t,w)) \qquad \text{near } \widetilde{w}_0$$

where $E, K_0 \in C^\infty$, and $\operatorname{Re} E > 0$ at $\widetilde{w}_0$. By taking $M(t,w) = E(t,0,w)$ we find $\operatorname{Re} M > 0$ and

$$Q(t,\tau,w) = E_0(t,\tau,w)(\tau M(t,w) + iK(t,w)) = E_0(t,\tau,w) Q_0(t,\tau,w)$$

where $E_0(t,0,w) \equiv \operatorname{Id}$. Thus we find that Q_0 satisfies (6.2), (6.3) and (6.4) when $\tau = 0$ near $\widetilde{w}_0$. Since $K(0,w_0) = 0$ we obtain that $\operatorname{Im} K \equiv 0$ and $K \ge cK^2 \ge 0$ near $(0,w_0)$. We have $\operatorname{Re} M > 0$ and

$$|\langle \operatorname{Im} M u, u\rangle| \le C \langle K u, u\rangle^{1/2} \|u\| \qquad \text{near } (0,w_0). \tag{6.10}$$

In fact, we have

$$0 \le \operatorname{Im} Q \le K + \tau(\operatorname{Im} M + \operatorname{Re}(E_1 K)) + C\tau^2$$

where $E_1(t,w) = \partial_\tau E(t,0,w)$. Lemma A.7 gives

$$|\langle \operatorname{Im} M u, u\rangle + \operatorname{Re}\langle E_1 K u, u\rangle| \le C\langle K u, u\rangle^{1/2}\|u\|$$

and since $K^2 \le CK$ we obtain

$$|\operatorname{Re}\langle E_1 K u, u\rangle| \le C\|Ku\|\|u\| \le C_0 \langle K u, u\rangle^{1/2}\|u\|$$

which gives (6.10). Now by cutting off when $|\tau| \ge c > 0$ we obtain that

$$Q^w = E_0^w Q_0^w + R_0^w + hR_1^w$$

where $R_j \in C_b^\infty$ and $\widetilde{w}_0 \notin \operatorname{supp} R_0$. Thus, it suffices to prove the estimate (6.5) for Q_0^w. We may now reduce to the case when $\operatorname{Re} M \equiv \operatorname{Id}$. In fact,

$$Q_0^w \cong M_0^w((\operatorname{Id} + iM_1^w)hD_t + iK_1^w)M_0^w \qquad \text{modulo } \mathbb{O}(h)$$

where $M_0 = (\operatorname{Re} M)^{1/2}$ is invertible, $M_1^* = M_1$ and $K_1 = M_0^{-1} K M_0^{-1} \ge 0$. By changing M_1 and K_1 and making $K_1 > 0$ outside a neighborhood of $(0,w_0)$ we may assume that $M_1, K_1 \in C_b^\infty$ and iK_1 satisfies (6.4) for all $c > 0$ and any w, by the invariance given by Remark 5.5. Observe that condition (6.10) also is invariant under the mapping $Q_0 \mapsto E^* Q_0 E$.

We shall use the symbol classes $f \in S(m,g)$ defined by

$$|\partial_{\nu_1} \dots \partial_{\nu_k} f| \le C_k m \prod_{j=1}^{k} g(\nu_j)^{1/2} \qquad \forall\, \nu_1, \dots, \nu_k \quad \forall\, k$$

for constant weight m and metric g, and $\operatorname{Op} S(m,g)$ the corresponding Weyl operators f^w. We shall need the following estimate for the model operator Q_0^w.

Proposition 6.2. *Assume that*

$$Q = M^w(t,x,hD_x)hD_t + iK^w(t,x,hD_x),$$

where $M(t, w)$ *and* $0 \le K(t, w) \in L^\infty(\mathbb{R}, C_b^\infty(T^*\mathbb{R}^n))$ *are* $N \times N$ *system such that* $\operatorname{Re} M \equiv \operatorname{Id}$, $\operatorname{Im} M$ *satisfies* (6.10) *and* iK *satisfies* (6.4) *for all* w *and* $c > 0$ *with non-decreasing* $0 < \phi(\delta) \to 0$ *as* $\delta \to 0$. *Then there exists a real valued* $B(t, w) \in L^\infty(\mathbb{R}, S(1, H|dw|^2/h))$ *such that* $hB(t, w)/\psi(h) \in \operatorname{Lip}(\mathbb{R}, S(1, H|dw|^2/h))$, *and*

$$\psi(h)\|u\|^2 \le \operatorname{Im}\langle Qu, B^w(t, x, hD_x)u\rangle + Ch^2\|D_t u\|^2 \qquad 0 < h \ll 1 \tag{6.11}$$

for any $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$. *Here the bounds on* $B(t, w)$ *are uniform*, $\psi(h) = \delta \gg h$ *is the inverse to* $h = \delta\phi(\delta)$ *so* $0 < H = \sqrt{h/\psi(h)} \ll 1$ *as* $h \to 0$.

Observe that $H^2 = h/\psi(h) = \phi(\psi(h)) \to 0$ and $h/H = \sqrt{\psi(h)h} \ll \psi(h) \to 0$ as $h \to 0$, since $0 < \phi(\delta)$ is non-decreasing.

To prove Proposition 6.1 we shall cut off where $|\tau| \gtrless \varepsilon\sqrt{\psi}/h$. Take $\chi_0(r) \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi_0 \le 1$, $\chi_0(r) = 1$ when $|r| \le 1$ and $|r| \le 2$ in $\operatorname{supp}\chi_0$. Then $1 - \chi_0 = \chi_1$ where $0 \le \chi_1 \le 1$ is supported where $|r| \ge 1$. Let $\phi_{j,\varepsilon}(r) = \chi_j(hr/\varepsilon\sqrt{\psi})$, $j = 0, 1$, for $\varepsilon > 0$; then $\phi_{0,\varepsilon}$ is supported where $|r| \le 2\varepsilon\sqrt{\psi}/h$ and $\phi_{1,\varepsilon}$ is supported where $|r| \ge \varepsilon\sqrt{\psi}/h$. We have that $\phi_{j,\varepsilon}(\tau) \in S(1, h^2 d\tau^2/\psi)$, $j = 0$, 1, and $u = \phi_{0,\varepsilon}(D_t)u + \phi_{1,\varepsilon}(D_t)u$, where we shall estimate each term separately. Observe that we shall use the ordinary quantization and not the semiclassical for these operators.

To estimate the first term, we substitute $\phi_{0,\varepsilon}(D_t)u$ in (6.11). We find that

$$\begin{aligned}\psi(h)\|\phi_{0,\varepsilon}(D_t)u\|^2 \le{}& \operatorname{Im}\langle Qu, \phi_{0,\varepsilon}(D_t)B^w(t, x, hD_x)\phi_{0,\varepsilon}(D_t)u\rangle \\ &+ \operatorname{Im}\langle [Q, \phi_{0,\varepsilon}(D_t)]u, B^w(t, x, hD_x)\phi_{0,\varepsilon}(D_t)u\rangle + 4C\varepsilon^2\psi\|u\|^2 \end{aligned} \tag{6.12}$$

In fact, $h\|D_t\phi_{0,\varepsilon}(D_t)u\| \le 2\varepsilon\sqrt{\psi}\|u\|$ since it is a Fourier multiplier and $|h\tau\phi_{0,\varepsilon}(\tau)| \le 2\varepsilon\sqrt{\psi}$. Next we shall estimate the commutator term. Since $\operatorname{Re} Q = hD_t \operatorname{Id} - h\partial_t \operatorname{Im} M^w/2$ and $\operatorname{Im} Q = h \operatorname{Im} M^w D_t + K^w + h\partial_t \operatorname{Im} M^w/2i$ we find that $[\operatorname{Re} Q, \phi_{0,\varepsilon}(D_t)] \in \operatorname{Op} S(h, \mathscr{G})$ and

$$[Q, \phi_{0,\varepsilon}(D_t)] = i[\operatorname{Im} Q, \phi_{0,\varepsilon}(D_t)] = i[K^w, \phi_{0,\varepsilon}(D_t)] = -h\partial_t K^w\phi_{2,\varepsilon}(D_t)/\varepsilon\sqrt{\psi}$$

is a symmetric operator modulo $\operatorname{Op} S(h, \mathscr{G})$, where $\mathscr{G} = dt^2 + h^2 d\tau^2/\psi + |dx|^2 + h^2|d\xi|^2$ and $\phi_{2,\varepsilon}(\tau) = \chi_0'(h\tau/\varepsilon\sqrt{\psi})$. In fact, we have that $h^2/\psi(h) \le Ch$, $h[\partial_t \operatorname{Im} M^w, \phi_{0,\varepsilon}(D_t)]$ and $[\operatorname{Im} M^w, \phi_{0,\varepsilon}(D_t)]hD_t \in \operatorname{Op} S(h, \mathscr{G})$, since $|\tau| \le \varepsilon\sqrt{\psi}/h$ in $\operatorname{supp}\phi_{0,\varepsilon}(\tau)$. Thus, we find that

$$\begin{aligned} -2i \operatorname{Im}\big(\phi_{0,\varepsilon}(D_t)B^w[Q, \phi_{0,\varepsilon}(D_t)]\big) &= 2ih\varepsilon^{-1}\psi^{-1/2}\operatorname{Im}\big(\phi_{0,\varepsilon}(D_t)B^w\partial_t K^w\phi_{2,\varepsilon}(D_t)\big) \\ &= h\varepsilon^{-1}\psi^{-1/2}\Big(\phi_{0,\varepsilon}(D_t)B^w[\partial_t K^w, \phi_{2,\varepsilon}(D_t)] + \phi_{0,\varepsilon}(D_t)[B^w, \phi_{2,\varepsilon}(D_t)]\partial_t K^w \\ &\quad + \phi_{2,\varepsilon}(D_t)[\phi_{0,\varepsilon}(D_t), B^w]\partial_t K^w + \phi_{2,\varepsilon}(D_t)B^w[\phi_{0,\varepsilon}(D_t), \partial_t K^w]\Big) \end{aligned} \tag{6.13}$$

modulo $\operatorname{Op} S(h, \mathscr{G})$. As before, the calculus gives that $[\phi_{j,\varepsilon}(D_t), \partial_t K^w] \in \operatorname{Op} S(h\psi^{-1/2}, \mathscr{G})$ for any j. Since $t \to hB^w/\psi \in \operatorname{Lip}(\mathbb{R}, \operatorname{Op} S(1, \mathscr{G}))$ uniformly and $\phi_{j,\varepsilon}(\tau) = \chi_j(h\tau/\varepsilon\sqrt{\psi})$ with $\chi_j' \in C_0^\infty(\mathbb{R})$, Lemma A.4 with $\kappa = \varepsilon\sqrt{\psi}/h$ gives that

$$\big\|[\phi_{j,\varepsilon}(D_t), B^w]\big\|_{\mathscr{L}(L^2(\mathbb{R}^{n+1}))} \le C\sqrt{\psi}/\varepsilon$$

uniformly. If we combine the estimates above we can estimate the commutator term:

$$|\operatorname{Im}\langle[Q, \phi_{0,\varepsilon}(D_t)]u, B^w(t,x,hD_x)\phi_{0,\varepsilon}(D_t)u\rangle| \le Ch\|u\|^2 \ll \psi(h)\|u\|^2 \qquad h \ll 1 \tag{6.14}$$

which together with (6.12) will give the estimate for the first term for small enough ε and h.

We also have to estimate $\phi_{1,\varepsilon}(D_t)u$; then we shall use that Q is elliptic when $|\tau| \ne 0$. We have

$$\|\phi_{1,\varepsilon}(D_t)u\|^2 = \langle \chi^w(D_t)u, u\rangle$$

where $\chi(\tau) = \phi^2_{1,\varepsilon}(\tau) \in S(1, h^2 d\tau^2/\psi)$ is real with support where $|\tau| \ge \varepsilon\sqrt{\psi}/h$. Thus, we may write $\chi(D_t) = \varrho(D_t)hD_t$ where $\varrho(\tau) = \chi(\tau)/h\tau \in S(\psi^{-1/2}, h^2 d\tau^2/\psi)$ by Leibniz' rule since $|\tau|^{-1} \le h/\varepsilon\sqrt{\psi}$ in $\operatorname{supp}\varrho$. Now $hD_t \operatorname{Id} = \operatorname{Re} Q + h\partial_t \operatorname{Im} M^w/2$ so we find

$$\langle \chi(D_t)u, u\rangle = \operatorname{Re}\langle \varrho(D_t)Qu, u\rangle + \frac{h}{2}\operatorname{Re}\langle \varrho(D_t)(\partial_t \operatorname{Im} M^w)u, u\rangle + \operatorname{Im}\langle \varrho(D_t)\operatorname{Im} Qu, u\rangle$$

where $|h\operatorname{Re}\langle \varrho(D_t)(\partial_t \operatorname{Im} M^w)u, u\rangle| \le Ch\|u\|^2/\varepsilon\sqrt{\psi}$ and

$$|\operatorname{Re}\langle \varrho(D_t)Qu, u\rangle| \le \|Qu\|\|\varrho(D_t)u\| \le \|Qu\|\|u\|/\varepsilon\sqrt{\psi}$$

since $\varrho(D_t)$ is a Fourier multiplier and $|\varrho(\tau)| \le 1/\varepsilon\sqrt{\psi}$. We have that

$$\operatorname{Im} Q = K^w(t,x,hD_x) + hD_t \operatorname{Im} M^w(t,x,hD_x) - \frac{h}{2i}\partial_t \operatorname{Im} M^w(t,x,hD_x)$$

where $\operatorname{Im} M^w(t,x,hD_x)$ and $K^w(t,x,hD_x) \in \operatorname{Op} S(1,\mathscr{G})$ are symmetric. Since $\varrho = \chi/h\tau \in S(\psi^{-1/2},\mathscr{G})$ is real we find that

$$\begin{aligned}\operatorname{Im}(\varrho(D_t)\operatorname{Im} Q) &= \operatorname{Im}\varrho(D_t)K^w + \operatorname{Im}\chi(D_t)\operatorname{Im} M^w \\ &= \frac{1}{2i}\big([\varrho(D_t), K^w(t,x,hD_x)] + [\chi(D_t), \operatorname{Im} M^w(t,x,hD_x)]\big)\end{aligned}$$

modulo terms in $\operatorname{Op} S(h/\sqrt{\psi},\mathscr{G}) \subseteq \operatorname{Op} S(h/\psi,\mathscr{G})$. Here the calculus gives

$$[\varrho(D_t), K^w(t,x,hD_x)] \in \operatorname{Op} S(h/\psi,\mathscr{G})$$

and similarly we have that

$$[\chi(D_t), \operatorname{Im} M^w(t,x,hD_x)] \in \operatorname{Op} S(h/\sqrt{\psi},\mathscr{G}) \subseteq \operatorname{Op} S(h/\psi,\mathscr{G})$$

which gives that $|\operatorname{Im}\langle \varrho(D_t)\operatorname{Im} Qu, u\rangle| \le Ch\|u\|^2/\psi$. In fact, since the metric $\mathscr{G}$ is constant, it is uniformly σ temperate for all $h > 0$. We obtain that

$$\psi(h)\|\phi_{1,\varepsilon}(D_t)u\|^2 \le C_\varepsilon(\sqrt{\psi}\|Qu\|\|u\| + h\|u\|^2)$$

which together with (6.12) and (6.14) gives the estimate (6.5) for small enough ε and h, since $h/\psi(h) \to 0$ as $h \to 0$. □

Proof of Proposition 6.2. We shall do a second microlocalization in $w = (x,\xi)$. By making a linear symplectic change of coordinates $(x,\xi) \mapsto (h^{1/2}x, h^{-1/2}\xi)$ we see that $Q(t,\tau,x,h\xi)$ is changed into

$$Q(t,\tau,h^{1/2}w) \in S(1, dt^2 + d\tau^2 + h|dw|^2) \qquad \text{when } |\tau| \le c.$$

In these coordinates we find $B(h^{1/2}w) \in S(1, G)$, $G = H|dw|^2$, if $B(w) \in S(1, H|dw|^2/h)$. In the following, we shall use ordinary Weyl quantization in the w variables.

We shall follow an approach similar to the one of [Dencker et al. 2004, Section 5]. To localize the estimate we take $\{\phi_j(w)\}_j$, $\{\psi_j(w)\}_j \in S(1, G)$ with values in ℓ^2, such that $0 \le \phi_j$, $0 \le \psi_j$, $\sum_j \phi_j^2 \equiv 1$ and $\phi_j\psi_j = \phi_j$ for all j. We may also assume that ψ_j is supported in a G neighborhood of w_j. This can be done uniformly in H, by taking $\phi_j(w) = \Phi_j(H^{1/2}w)$ and $\psi_j(w) = \Psi_j(H^{1/2}w)$, with $\{\Phi_j(w)\}_j$ and $\{\Psi_j(w)\}_j \in S(1, |dw|^2)$. Since $\sum \phi_j^2 = 1$ and $G = H|dw|^2$ the calculus gives

$$\sum_j \|\phi_j^w(x, D_x)u\|^2 - CH^2\|u\|^2 \le \|u\|^2 \le \sum_j \|\phi_j^w(x, D_x)u\|^2 + CH^2\|u\|^2$$

for $u \in C_0^\infty(\mathbb{R}^n)$, thus for small enough H we find

$$\sum_j \|\phi_j^w(x, D_x)u\|^2 \le 2\|u\|^2 \le 4\sum_j \|\phi_j^w(x, D_x)u\|^2 \qquad \text{for } u \in C_0^\infty(\mathbb{R}^n). \tag{6.15}$$

Observe that since ϕ_j has values in ℓ^2 we find that $\{\phi_j^w R_j^w\}_j \in \operatorname{Op} S(H^\nu, G)$ also has values in ℓ^2 if $R_j \in S(H^\nu, G)$ uniformly. Such terms will be summable:

$$\sum_j \|r_j^w u\|^2 \le CH^{2\nu}\|u\|^2 \tag{6.16}$$

for $\{r_j\}_j \in S(H^\nu, G)$ with values in ℓ^2; see [Hörmander 1983–1985, Volume III, page 169]. Now we fix j and let

$$Q_j(t, \tau) = Q(t, \tau, h^{1/2}w_j) = M_j(t)\tau + iK_j(t)$$

where $M_j(t) = M(t, h^{1/2}w_j)$ and $K_j(t) = K(t, h^{1/2}w_j) \in L^\infty(\mathbb{R})$. Since $K(t, w) \ge 0$ we find from Lemma A.7 and (6.10) that

$$|\langle \operatorname{Im} M_j(t)u, u\rangle| + |\langle d_w K(t, h^{1/2}w_j)u, u\rangle| \le C\langle K_j(t)u, u\rangle^{1/2}\|u\| \qquad \forall u \in \mathbb{C}^N \qquad \forall t \tag{6.17}$$

and condition (6.4) means that

$$\left|\left\{t : \inf_{|u|=1} \langle K_j(t)u, u\rangle \le \delta\right\}\right| \le C\phi(\delta). \tag{6.18}$$

We shall prove an estimate for the corresponding one-dimensional operator

$$Q_j(t, hD_t) = M_j(t)hD_t + iK_j(t)$$

by using the following result.

Lemma 6.3. *Assume that*

$$Q(t, hD_t) = M(t)hD_t + iK(t)$$

where $M(t)$ and $0 \le K(t)$ are $N \times N$ systems, which are uniformly bounded in $L^\infty(\mathbb{R})$, such that $\operatorname{Re} M \equiv \operatorname{Id}$, $\operatorname{Im} M$ *satisfies* (6.10) *for almost all t and iK satisfies* (6.4) *for any $c > 0$ with non-decreasing $\phi(\delta) \to 0$ as $\delta \to 0$. Then there exists a uniformly bounded real $B(t) \in L^\infty(\mathbb{R})$ so that $hB(t)/\psi(h) \in$* $\operatorname{Lip}(\mathbb{R})$ *uniformly and*

$$\psi(h)\|u\|^2 + \langle Ku, u\rangle \le \operatorname{Im}\langle Qu, Bu\rangle + Ch^2\|D_t u\|^2 \qquad 0 < h \ll 1 \tag{6.19}$$

for any $u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$. *Here* $\psi(h) = \delta \gg h$ *is the inverse to* $h = \delta\phi(\delta)$.

Proof. Let $0 \le \Phi_h(t) \le 1$ be the characteristic function of the set $\Omega_\delta(K)$ with $\delta = \psi(h)$. Since $\delta = \psi(h)$ is the inverse of $h = \delta\phi(\delta)$ we find that $\phi(\psi(h)) = h/\delta = h/\psi(h)$. Thus, we obtain from (6.18) that

$$\int \Phi_h(t)\, dt = |\Omega_\delta(K)| \le Ch/\psi(h)$$

Letting

$$E(t) = \exp\Big(\frac{\psi(h)}{h}\int_0^t \Phi_h(s)\, ds\Big),$$

we find that E and $E^{-1} \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ uniformly and $E' = \psi(h)h^{-1}\Phi_h E$ in $\mathscr{D}'(\mathbb{R})$. We have

$$\begin{aligned} E(t)Q(t, hD_t)E^{-1}(t) &= Q(t, hD_t) + E(t)h[M(t)D_t, E^{-1}(t)]\,\mathrm{Id}_N \\ &= Q(t, hD_t) + i\psi(h)\Phi_h(t)\,\mathrm{Id}_N - \psi(h)\Phi_h(t)\,\mathrm{Im}\, M(t) \end{aligned} \tag{6.20}$$

since $(E^{-1})' = -E'E^{-2}$. In the following, we let

$$F(t) = K(t) + \psi(h)\,\mathrm{Id}_N \ge \psi(h)\,\mathrm{Id}_N\,.$$

By definition we have $\Phi_h(t) < 1 \implies K(t) \ge \psi(h)\,\mathrm{Id}_N$, so

$$K(t) + \psi(h)\Phi_h(t)\,\mathrm{Id}_N \ge \tfrac{1}{2}F(t).$$

Thus by taking the inner product in $L^2(\mathbb{R})$ we find from (6.20) that

$$\mathrm{Im}\langle E(t)Q(t, hD_t)E^{-1}(t)u, u\rangle \ge \tfrac{1}{2}\langle F(t)u, u\rangle + \langle \mathrm{Im}\, M(t)hD_t u, u\rangle - ch\|u\|^2 \qquad u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$$

since $\mathrm{Im}\, Q(t, hD_t) = K(t) + \mathrm{Im}\, M(t)hD_t + \frac{h}{2i}\partial_t\,\mathrm{Im}\, M(t)$. Now we may use (6.10) to estimate for any $\varepsilon > 0$

$$|\langle \mathrm{Im}\, MhD_t u, u\rangle| \le \varepsilon\langle Ku, u\rangle + C_\varepsilon(h^2\|D_t u\|^2 + h\|u\|^2) \qquad \forall u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N). \tag{6.21}$$

In fact, $u = \chi_0(hD_t)u + \chi_1(hD_t)u$ where $\chi_0(r) \in C_0^\infty(\mathbb{R})$ and $|r| \ge 1$ in supp χ_1. We obtain from (6.10) for any $\varepsilon > 0$ that

$$|\langle \mathrm{Im}\, M(t)\chi_0(h\tau)h\tau u, u\rangle| \le C\langle K(t)u, u\rangle^{1/2}|\chi_0(h\tau)h\tau|\|u\| \le \varepsilon\langle K(t)u, u\rangle + C_\varepsilon\|\chi_0(h\tau)h\tau u\|^2$$

so by using Gårdings inequality in Proposition A.5 on

$$\varepsilon K(t) + C_\varepsilon\chi_0^2(hD_t)h^2D_t^2 \pm \mathrm{Im}\, M(t)\chi_0(hD_t)hD_t$$

we obtain

$$|\langle \mathrm{Im}\, M(t)\chi_0(hD_t)hD_t u, u\rangle| \le \varepsilon\langle K(t)u, u\rangle + C_\varepsilon h^2\|D_t u\|^2 + C_0 h\|u\|^2 \quad \forall u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$$

since $\|\chi_0(hD_t)hD_t u\| \le C\|hD_t u\|$. The other term is easier to estimate:

$$|\langle \mathrm{Im}\, M(t)\chi_1(hD_t)hD_t u, u\rangle| \le C\|hD_t u\|\|\chi_1(hD_t)u\| \le C_1 h^2\|D_t u\|^2$$

since $|\chi_1(h\tau)| \le C|h\tau|$. By taking $\varepsilon = 1/6$ in (6.21) we obtain

$$\langle F(t)u, u\rangle \le 3\,\mathrm{Im}\langle E(t)Q(t, hD_t)E^{-1}(t)u, u\rangle + C(h^2\|D_t u\|^2 + h\|u\|^2).$$

Now $hD_tEu = EhD_tu - i\psi(h)\Phi_hEu$ so we find by substituting $E(t)u$ that

$$\psi(h)\|E(t)u\|^2 + \langle KE(t)u, E(t)u\rangle \\ \le 3\,\mathrm{Im}\langle Q(t, hD_t)u, E^2(t)u\rangle + C(h^2\|D_tu\|^2 + h\|u\|^2 + \psi^2(h)\|E(t)u\|^2)$$

for $u \in C_0^\infty(\mathbb{R}, \mathbb{C}^N)$. Since $E \ge c$, $K \ge 0$ and $h \ll \psi(h) \ll 1$ when $h \to 0$ we obtain (6.19) with scalar $B = \varrho E^2$ for $\varrho \gg 1$ and $h \ll 1$. □

To finish the proof of Proposition 6.2, we substitute $\phi_j^w u$ in the estimate (6.19) with $Q = Q_j$ to obtain

$$\psi(h)\|\phi_j^w u\|^2 + \langle K_j\phi_j^w u, \phi_j^w u\rangle \le \mathrm{Im}\langle\phi_j^w Q_j(t, hD_t)u, B_j(t)\phi_j^w u\rangle + Ch^2\|\phi_j^w D_tu\|^2 \tag{6.22}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$, since $\phi_j^w(x, D_x)$ and $Q_j(t, hD_t)$ commute. Next, we shall replace the approximation Q_j by the original operator Q. In a G neighborhood of $\operatorname{supp}\phi_j$ we may use the Taylor expansion in w to write for almost all t

$$Q(t, \tau, h^{1/2}w) - Q_j(t, \tau) = i(K(t, h^{1/2}w) - K_j(t)) + (M(t, h^{1/2}w) - M_j(t))\tau. \tag{6.23}$$

We shall start by estimating the last term in (6.23). Since $M(t, w) \in C_b^\infty$ we have

$$|M(t, h^{1/2}w) - M_j(t)| \le Ch^{1/2}H^{-1/2} \qquad \text{in } \operatorname{supp}\phi_j \tag{6.24}$$

because then $|w - w_j| \le cH^{-1/2}$. Since $M(t, h^{1/2}w) \in S(1, h|dw|^2)$ and $h \ll H$ we find from (6.24) that $M(t, h^{1/2}w) - M_j(t) \in S(h^{1/2}H^{-1/2}, G)$ in $\operatorname{supp}\phi_j$ uniformly in t. By the Cauchy–Schwarz inequality we find

$$|\langle\phi_j^w(M^w - M_j)hD_tu, B_j(t)\phi_j^w u\rangle| \le C(\|\chi_j^w hD_tu\|^2 + hH^{-1}\|\phi_j^w u\|^2) \tag{6.25}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ where $\chi_j^w = h^{-1/2}H^{1/2}\phi_j^w(M^w - M_j) \in \mathrm{Op}\,S(1, G)$ uniformly in t with values in ℓ^2. Thus we find from (6.16) that

$$\sum_j \|\chi_j^w hD_tu\|^2 \le C\|hD_tu\|^2 \qquad u \in C_0^\infty(\mathbb{R}^{n+1})$$

and for the last terms in (6.25) we have

$$hH^{-1}\sum_j \|\phi_j^w u\|^2 \le 2hH^{-1}\|u\|^2 \ll \psi(h)\|u\|^2 \qquad h \to 0 \qquad u \in C_0^\infty(\mathbb{R}^{n+1})$$

by (6.15). For the first term in the right hand side of (6.23) we find from Taylor's formula

$$K(t, h^{1/2}w) - K_j(t) = h^{1/2}\langle S_j(t), W_j(w)\rangle + R_j(t, \tau, w) \qquad \text{in } \operatorname{supp}\phi_j$$

where $S_j(t) = \partial_w K(t, h^{1/2}w_j) \in L^\infty(\mathbb{R})$, $R_j \in S(hH^{-1}, G)$ uniformly for almost all t and $W_j \in S(h^{-1/2}, h|dw|^2)$ such that $\phi_j(w)W_j(w) = \phi_j(w)(w - w_j) = \mathcal{O}(H^{-1/2})$. Here we could take $W_j(w) = \chi(h^{1/2}(w - w_j))(w - w_j)$ for a suitable cut-off function $\chi \in C_0^\infty$. We obtain from the calculus that

$$\phi_j^w K_j(t) = \phi_j^w K^w(t, h^{1/2}x, h^{1/2}D_x) - h^{1/2}\phi_j^w\langle S_j(t), W_j^w\rangle + \widetilde{R}_j^w,$$

where $\{\widetilde{R}_j\}_j \in S(hH^{-1}, G)$ with values in ℓ^2 for almost all t. Thus we may estimate the sum of these error terms by (6.16) to obtain

$$\sum_j |\langle \widetilde{R}_j^w u, B_j(t)\phi_j^w u\rangle| \le ChH^{-1}\|u\|^2 \ll \psi(h)\|u\|^2 \quad \text{as } h \to 0 \quad \text{for } u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N). \tag{6.26}$$

Observe that it follows from (6.17) for any $\kappa > 0$ and almost all t that

$$|\langle S_j(t)u, u\rangle| \le C\langle K_j(t)u, u\rangle^{1/2}\|u\| \le \kappa\langle K_j(t)u, u\rangle + C\|u\|^2/\kappa \qquad \forall\, u \in \mathbb{C}^N.$$

Let $F_j(t) = F(t, h^{1/2}w_j) = K_j(t) + \psi(h)\,\mathrm{Id}_N$; then by taking $\kappa = \varrho H^{1/2}h^{-1/2}$ we find that for any $\varrho > 0$ there exists $h_\varrho > 0$ so that

$$h^{1/2}H^{-1/2}|\langle S_j u, u\rangle| \le \varrho\langle K_j u, u\rangle + ChH^{-1}\|u\|^2/\varrho \le \varrho\langle F_j u, u\rangle \quad \forall\, u \in \mathbb{C}^N \quad 0 < h \le h_\varrho \tag{6.27}$$

since $hH^{-1} \ll \psi(h)$ when $h \ll 1$. Now F_j and S_j only depend on t, so by (6.27) we may use Remark A.6 in the Appendix for fixed t with $A = h^{1/2}H^{-1/2}S_j$, $B = \varrho F_j$, u replaced with $\phi_j^w u$ and v with $B_j H^{1/2}\phi_j^w W_j^w u$. Integration then gives

$$h^{1/2}|\langle B_j\phi_j^w\langle S_j(t), W_j^w\rangle u, \phi_j^w u\rangle| \le \frac{3\varrho}{2}(\langle F_j(t)\phi_j^w u, \phi_j^w u\rangle + \langle F_j(t)\psi_j^w u, \psi_j^w u\rangle) \tag{6.28}$$

for $u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$, $0 < h \le h_\varrho$, where

$$\psi_j^w = B_j H^{1/2}\phi_j^w W_j^w \in \mathrm{Op}\, S(1, G) \qquad \text{with values in } \ell^2.$$

In fact, since $\phi_j \in S(1, G)$ and $W_j \in S(h^{-1/2}, h|dw|^2)$ we find that

$$\phi_j^w W_j^w = (\phi_j W_j)^w \text{ modulo } \mathrm{Op}\, S(H^{1/2}, G).$$

Also, since $|\phi_j W_j| \le CH^{-1/2}$ we find from Leibniz' rule that $\phi_j W_j \in S(H^{-1/2}, G)$. Now $F \ge \psi(h)\,\mathrm{Id}_N \gg hH^{-1}\,\mathrm{Id}_N$ so by using Proposition A.9 in the Appendix and then integrating in t we find that

$$\sum_j \langle F_j(t)\psi_j^w u, \psi_j^w u\rangle \le C\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \qquad u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N).$$

We obtain from (6.15) that

$$\psi(h)\|u\|^2 \le 2\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \qquad u \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N).$$

Thus, for any $\varrho > 0$ we obtain from (6.22) and (6.25)–(6.28) that

$$(1 - C_0\varrho)\sum_j \langle F_j(t)\phi_j^w u, \phi_j^w u\rangle \le \sum_j \mathrm{Im}\langle \phi_j^w Qu, B_j(t)\phi_j^w u\rangle + C_\varrho h^2\|D_t u\|^2 \qquad 0 < h \le h_\varrho.$$

We have that $\sum_j B_j\phi_j^w\phi_j^w \in S(1, G)$ is a scalar symmetric operator uniformly in t. When $\varrho = 1/2C_0$ we obtain the estimate (6.11) with $B^w = 4\sum_j B_j\phi_j^w\phi_j^w$, which finishes the proof of Proposition 6.2. □

Appendix

We shall first study the condition for the one-dimensional model operator

$$hD_t \operatorname{Id}_N + iF(t) \qquad 0 \le F(t) \in C^\infty(\mathbb{R})$$

to be of finite type of order μ:

$$|\Omega_\delta(F)| \le C\delta^\mu \qquad 0 < \delta \ll 1 \tag{A.1}$$

and we shall assume that $0 \notin \Sigma_\infty(P)$. When $F(t) \notin C^\infty(\mathbb{R})$ we may have any $\mu > 0$ in (A.1), for example with $F(t) = |t|^{1/\mu} \operatorname{Id}_N$. But when $F \in C^1_{\mathrm{b}}$ the estimate cannot hold with $\mu > 1$, and since it trivially holds for $\mu = 0$ the only interesting cases are $0 < \mu \le 1$.

When $0 \le F(t)$ is diagonalizable for any t with eigenvalues $\lambda_j(t) \in C^\infty$, $j = 1, \dots, N$, then condition (A.1) is equivalent to

$$\left|\Omega_\delta(\lambda_j)\right| \le C\delta^\mu \qquad \forall j \quad 0 < \delta \ll 1$$

since $\Omega_\delta(F) = \bigcup_j \Omega_\delta(\lambda_j)$. Thus we shall start by studying the scalar case.

Proposition A.1. *Assume that* $0 \le f(t) \in C^\infty(\mathbb{R})$ *such that* $f(t) \ge c > 0$ *when* $|t| \gg 1$, *that is,* $0 \notin \Sigma_\infty(f)$. *We find that f satisfies* (A.1) *with* $\mu > 0$ *if and only if* $\mu \le 1/k$ *for an even* $k \ge 0$ *so that*

$$\sum_{j \le k} |\partial_t^j f(t)| > 0 \qquad \forall t. \tag{A.2}$$

Simple examples as $f(t) = e^{-t^2}$ show that the condition that $0 \notin \Sigma_\infty(f)$ is necessary for the conclusion of Proposition A.1.

Proof. Assume that (A.2) does not hold with $k \le 1/\mu$; then there exists t_0 such that $f^{(j)}(t_0) = 0$ for all integer $j \le 1/\mu$. Then Taylor's formula gives that $f(t) \le c|t - t_0|^k$ and $|\Omega_\delta(f)| \ge c\delta^{1/k}$ where $k = [1/\mu] + 1 > 1/\mu$, which contradicts condition (A.1).

Assume now that condition (A.2) holds for some k, then $f^{-1}(0)$ consists of finitely many points. In fact, else there would exist t_0 where f vanishes of infinite order since $f(t) \ne 0$ when $|t| \gg 1$. Also note that $\bigcap_{\delta > 0} \Omega_\delta(f) = f^{-1}(0)$, in fact f must have a positive infimum outside any neighborhood of $f^{-1}(0)$. Thus, in order to estimate $|\Omega_\delta(f)|$ for $\delta \ll 1$ we only have to consider a small neighborhood ω of $t_0 \in f^{-1}(0)$. Assume that

$$f(t_0) = f'(t_0) = \dots = f^{(j-1)}(t_0) = 0 \text{ and } f^{(j)}(t_0) \ne 0$$

for some $j \le k$. Since $f \ge 0$ we find that j must be even and $f^{(j)}(t_0) > 0$. Taylor's formula gives as before $f(t) \ge c|t - t_0|^j$ for $|t - t_0| \ll 1$ and thus we find that

$$|\Omega_\delta(f) \cap \omega| \le C\delta^{1/j} \le C\delta^{1/k} \qquad 0 < \delta \ll 1$$

if ω is a small neighborhood of t_0. Since $f^{-1}(0)$ consists of finitely many points we find that (A.1) is satisfied with $\mu = 1/k$ for an even k. □

So if $0 \le F \in C^\infty(\mathbb{R})$ is C^∞ diagonalizable system and $0 \notin \Sigma_\infty(P)$, condition (A.1) is equivalent to

$$\sum_{j \le k} |\partial_t^j \langle F(t)u(t), u(t)\rangle| / \|u(t)\|^2 > 0 \qquad \forall t$$

for any $0 \neq u(t) \in C^\infty(\mathbb{R})$, since this holds for diagonal matrices and is invariant. This is true also in the general case by the following proposition.

Proposition A.2. *Assume that* $0 \le F(t) \in C^\infty(\mathbb{R})$ *is an* $N \times N$ *system such that* $0 \notin \Sigma_\infty(F)$. *We find that* F *satisfies* (A.1) *with* $\mu > 0$ *if and only if* $\mu \le 1/k$ *for an even* $k \ge 0$ *so that*

$$\sum_{j \le k} |\partial_t^j \langle F(t)u(t), u(t)\rangle| / \|u(t)\|^2 > 0 \qquad \forall\, t \tag{A.3}$$

for any $0 \neq u(t) \in C^\infty(\mathbb{R})$.

Observe that since $0 \notin \Sigma_\infty(F)$ it suffices to check condition (A.3) on a compact interval.

Proof. First we assume that (A.1) holds with $\mu > 0$, let $u(t) \in C^\infty(\mathbb{R}, \mathbb{C}^N)$ such that $|u(t)| \equiv 1$, and $f(t) = \langle F(t)u(t), u(t)\rangle \in C^\infty(\mathbb{R})$. Then we have $\Omega_\delta(f) \subset \Omega_\delta(F)$ so (A.1) gives

$$|\Omega_\delta(f)| \le |\Omega_\delta(F)| \le C\delta^\mu \qquad 0 < \delta \ll 1.$$

The first part of the proof of Proposition A.1 then gives (A.3) for some $k \le 1/\mu$.

For the proof of the sufficiency of (A.3) we need the following simple lemma.

Lemma A.3. *Assume that* $F(t) = F^*(t) \in C^k(\mathbb{R})$ *is an* $N \times N$ *system with eigenvalues* $\lambda_j(t) \in \mathbb{R}$, $j = 1, \dots, N$. *Then, for any* $t_0 \in \mathbb{R}$, *there exist analytic* $v_j(t) \in \mathbb{C}^N$, $j = 1, \dots, N$, *so that* $\{v_j(t_0)\}$ *is a base for* $\mathbb{C}^N$ *and*

$$\left|\lambda_j(t) - \langle F(t)v_j(t), v_j(t)\rangle\right| \le C|t - t_0|^k \qquad \text{for } |t - t_0| \le 1$$

after a renumbering of the eigenvalues.

By a well-known theorem of Rellich, the eigenvalues $\lambda(t) \in C^1(\mathbb{R})$ for symmetric $F(t) \in C^1(\mathbb{R})$; see [Kato 1966, Theorem II.6.8].

Proof. It is no restriction to assume $t_0 = 0$. By Taylor's formula

$$F(t) = F_k(t) + R_k(t)$$

where F_k and R_k are symmetric, $F_k(t)$ is a polynomial of degree $k - 1$ and $R_k(t) = \mathcal{O}(|t|^k)$. Since $F_k(t)$ is symmetric and holomorphic, it has a base of normalized holomorphic eigenvectors $v_j(t)$ with real holomorphic eigenvalues $\widetilde{\lambda}_j(t)$ by [Kato 1966, Theorem II.6.1]. Thus $\widetilde{\lambda}_j(t) = \langle F_k(t)v_j(t), v_j(t)\rangle$ and by the minimax principle we may renumber the eigenvalues so that

$$|\lambda_j(t) - \widetilde{\lambda}_j(t)| \le \|R_k(t)\| \le C|t|^k \qquad \forall\, j.$$

The result then follows since

$$|\langle (F(t) - F_k(t))v_j(t), v_j(t)\rangle| = |\langle R_k(t)v_j(t), v_j(t)\rangle| \le C|t|^k \qquad \forall\, j. \qquad \square$$

Assume now that Equation (A.3) holds for some k. As in the scalar case, we have that k is even and $\bigcap_{\delta > 0} \Omega_\delta(F) = \Sigma_0(F) = |F|^{-1}(0)$. Thus, for small δ we only have to consider a small neighborhood of $t_0 \in \Sigma_0(F)$. Then by using Lemma A.3 we have after renumbering that for each eigenvalue $\lambda(t)$ of $F(t)$ there exists $v(t) \in C^\infty$ so that $|v(t)| \ge c > 0$ and

$$|\lambda(t) - \langle F(t)v(t), v(t)\rangle| \le C|t - t_0|^{k+1} \qquad \text{when } |t - t_0| \le c. \tag{A.4}$$

Now if $\Sigma_0(F) \ni t_j \to t_0$ is an accumulation point, then after choosing a subsequence we obtain that for some eigenvalue λ_k we have $\lambda_k(t_j) = 0$ for all j. Then λ_k vanishes of infinite order at t_0, contradicting (A.3) by (A.4). Thus, we find that $\Sigma_0(F)$ is a finite collection of points. By using (A.3) with $u(t) = v(t)$ we find as in the second part of the proof of Proposition A.1 that

$$\langle F(t)v(t), v(t)\rangle \geq c|t - t_0|^j \qquad |t - t_0| \ll 1$$

for some even $j \leq k$, which by (A.4) gives that

$$\lambda(t) \geq c|t - t_0|^j - C|t - t_0|^{k+1} \geq c'|t - t_0|^j \qquad |t - t_0| \ll 1.$$

Thus $|\Omega_\delta(\lambda) \cap \omega| \leq c\delta^{1/j}$ if ω for $\delta \ll 1$ if ω is a small neighborhood of $t_0 \in \Sigma_0(F)$. Since $\Omega_\delta(F) = \bigcup_j \Omega_\delta(\lambda_j)$, where $\{\lambda_j(t)\}_j$ are the eigenvalues of $F(t)$, we find by adding up that $|\Omega_\delta(F)| \leq C\delta^{1/k}$. Thus the largest μ satisfying (A.1) must be $\geq 1/k$. □

Let $A(t) \in \mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$ be the $L^2(\mathbb{R}^n)$ bounded operators which are Lipschitz continuous in the parameter $t \in \mathbb{R}$. This means that

$$\frac{A(s) - A(t)}{s - t} = B(s, t) \in \mathscr{L}(L^2(\mathbb{R}^n)) \qquad \text{uniformly in } s \text{ and } t. \tag{A.5}$$

One example is $A(t) = a^w(t, x, D_x)$ where $a(t, x, \xi) \in \mathrm{Lip}(\mathbb{R}, S(1, G))$ for a σ temperate metric G which is constant in t such that $G/G^\sigma \leq 1$.

Lemma A.4. *Assume that $A(t) \in \mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$ and $\phi(\tau) \in C^\infty(\mathbb{R})$ such that $\phi'(\tau) \in C_0^\infty(\mathbb{R})$. Then for $\kappa > 0$ we can estimate the commutator*

$$\big\|\big[\phi(D_t/\kappa), A(t)\big]\big\|_{\mathscr{L}(L^2(\mathbb{R}^{n+1}))} \leq C\kappa^{-1},$$

where the constant only depends on ϕ and the bound on $A(t)$ in $\mathrm{Lip}(\mathbb{R}, \mathscr{L}(L^2(\mathbb{R}^n)))$.

Proof. In the following, we shall denote by $A(t, x, y)$ the distribution kernel of $A(t)$. Then we find from (A.5) that

$$A(s, x, y) - A(t, x, y) = (s - t)B(s, t, x, y), \tag{A.6}$$

where $B(s, t, x, y)$ is the kernel for $B(s, t)$ for $s, t \in \mathbb{R}$. Then

$$\begin{aligned}\big\langle\big[\phi(D_t/\kappa), A(t)\big]u, v\big\rangle \\ = (2\pi)^{-1} \int e^{i(t-s)\tau}\phi(\tau/\kappa)(A(s, x, y) - A(t, x, y))u(s, x)\overline{v(t, y)}\, d\tau\, ds\, dt\, dx\, dy\end{aligned} \tag{A.7}$$

for $u, v \in C_0^\infty(\mathbb{R}^{n+1})$, and by using (A.6) we obtain that the commutator has kernel

$$\frac{1}{2\pi}\int e^{i(t-s)\tau}\phi(\tau/\kappa)(s-t)B(s,t,x,y)\,d\tau = \frac{1}{\kappa}\int e^{i(t-s)\tau}\rho(\tau/\kappa)B(s,t,x,y)\,d\tau = \widehat{\rho}(\kappa(s-t))B(s,t,x,y)$$

in $\mathscr{D}(\mathbb{R}^{2n+2})$, where $\rho \in C_0^\infty(\mathbb{R})$. Thus, we may estimate (A.7) by using Cauchy–Schwarz:

$$\int |\widehat{\varrho}(\kappa s)\langle B(s + t, t)u(s + t), v(t)\rangle_{L^2(\mathbb{R}^n)}|\, dt\, ds \leq C\kappa^{-1}\|u\|\|v\|$$

where the norms are in $\mathscr{L}(L^2(\mathbb{R}^{n+1}))$. □

We shall need some results about the lower bounds of systems, and we shall use the following version of the Gårding inequality for systems. A convenient way for proving the inequality is to use the Wick quantization of $a \in L^\infty(T^*\mathbb{R}^n)$ given by

$$a^{\mathrm{Wick}}(x, D_x)u(x) = \int_{T^*\mathbb{R}^n} a(y,\eta)\Sigma^w_{y,\eta}(x, D_x)u(x)\,dy\,d\eta \qquad u \in \mathscr{S}(\mathbb{R}^n)$$

using the rank one orthogonal projections $\Sigma^w_{y,\eta}(x, D_x)$ in $L^2(\mathbb{R}^n)$ with Weyl symbol

$$\Sigma_{y,\eta}(x,\xi) = \pi^{-n}\exp\left(-|x-y|^2 - |\xi-\eta|^2\right)$$

(see [Dencker 1999, Appendix B] or [Lerner 1997, Section 4]). We find that $a^{\mathrm{Wick}}\colon \mathscr{S}(\mathbb{R}^n) \mapsto \mathscr{S}'(\mathbb{R}^n)$ is symmetric on $\mathscr{S}(\mathbb{R}^n)$ if a is real-valued,

$$a \ge 0 \implies \left(a^{\mathrm{Wick}}(x, D_x)u, u\right) \ge 0 \qquad u \in \mathscr{S}(\mathbb{R}^n), \tag{A.8}$$
$$\|a^{\mathrm{Wick}}(x, D_x)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \le \|a\|_{L^\infty(T^*\mathbb{R}^n)},$$

which is the main advantage with the Wick quantization. If $a \in S(1, h|dw|^2)$ we find that

$$a^{\mathrm{Wick}} = a^w + r^w \tag{A.9}$$

where $r \in S(h, h|dw|^2)$. For a reference; see [Lerner 1997, Proposition 4.2].

Proposition A.5. *Let $0 \le A \in C_{\mathrm{b}}^\infty(T^*\mathbb{R}^n)$ be an $N \times N$ system, then we find that*

$$\langle A^w(x, hD)u, u\rangle \ge -Ch\|u\|^2 \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

This result is well known (see for example Theorem 18.6.14 in Volume III of [Hörmander 1983–1985]) but we shall give a short and direct proof.

Proof. By making a L^2 preserving linear symplectic change of coordinates: $(x,\xi) \mapsto (h^{1/2}x, h^{-1/2}\xi)$ we may assume that $0 \le A \in S(1, h|dw|^2)$. Then we find from (A.9) that $A^w = A^{\mathrm{Wick}} + R^w$ where $R \in S(h, h|dw|^2)$. Since $A \ge 0$ we obtain from (A.8) that

$$\langle A^w u, u\rangle \ge \langle R^w u, u\rangle \ge -Ch\|u\|^2 \qquad \forall\, u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N). \qquad \square$$

Remark A.6. Assume that A and B are $N \times N$ matrices such that $\pm A \le B$. Then we find

$$|\langle Au, v\rangle| \le \tfrac{3}{2}\left(\langle Bu, u\rangle + \langle Bv, v\rangle\right) \qquad \forall\, u,\ v \in \mathbb{C}^n.$$

In fact, since $B \pm A \ge 0$ we find by the Cauchy–Schwarz inequality that

$$2\,|\langle (B \pm A)u, v\rangle| \le \langle (B \pm A)u, u\rangle + \langle (B \pm A)v, v\rangle \qquad \forall\, u,\ v \in \mathbb{C}^n$$

and $2\,|\langle Bu, v\rangle| \le \langle Bu, u\rangle + \langle Bv, v\rangle$. The estimate can then be expanded to give the inequality, since

$$|\langle Au, u\rangle| \le \langle Bu, u\rangle \qquad \forall\, u \in \mathbb{C}^n$$

by the assumption.

Lemma A.7. *Assume that $0 \le F(t) \in C^2(\mathbb{R})$ is an $N \times N$ system such that $F'' \in L^\infty(\mathbb{R})$. Then we have*

$$|\langle F'(0)u, u\rangle|^2 \le C\|F''\|_{L^\infty}\langle F(0)u, u\rangle\|u\|^2 \qquad \forall\, u \in \mathbb{C}^N.$$

Proof. Take $u \in \mathbb{C}^N$ with $|u| = 1$ and let $0 \le f(t) = \langle F(t)u, u\rangle \in C^2(\mathbb{R})$. Then $|f''| \le \|F''\|_{L^\infty}$ so Lemma 7.7.2 in Volume I of [Hörmander 1983–1985] gives

$$|f'(0)|^2 = |\langle F'(0)u, u\rangle|^2 \le C\|F''\|_{L^\infty} f(0) = C\|F''\|_{L^\infty}\langle F(0)u, u\rangle. \qquad \square$$

Lemma A.8. *Assume that $F \ge 0$ is an $N \times N$ matrix and that A is a L^2 bounded scalar operator. Then*

$$0 \le \langle FAu, Au\rangle \le \|A\|^2\langle Fu, u\rangle$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$.

Proof. Since $F \ge 0$ we can choose an orthonormal base for $\mathbb{C}^N$ such that $\langle Fu, u\rangle = \sum_{j=1}^N f_j |u_j|^2$ for $u = (u_1, u_2, \dots) \in \mathbb{C}^N$, where $f_j \ge 0$ are the eigenvalues of F. In this base we find

$$0 \le \langle FAu, Au\rangle = \sum_j f_j \|Au_j\|^2 \le \|A\|^2 \sum_j f_j \|u_j\|^2 = \|A\|^2\langle Fu, u\rangle$$

for $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$. $\square$

Proposition A.9. *Assume that $h/H \le F \in S(1, g)$ is an $N \times N$ system, $\{\phi_j\}$ and $\{\psi_j\} \in S(1, G)$ with values in ℓ^2 such that $\sum_j |\phi_j|^2 \ge c > 0$ and ψ_j is supported in a fixed G neighborhood of $w_j \in \operatorname{supp}\phi_j$ for all j. Here $g = h|dw|^2$ and $G = H|dw|^2$ are constant metrics, $0 < h \le H \le 1$. If $F_j = F(w_j)$ we find for $H \ll 1$ that*

$$\sum_j \langle F_j \psi_j^w(x, D_x)u, \psi_j^w(x, D_x)u\rangle \le C \sum_j \langle F_j \phi_j^w(x, D_x)u, \phi_j^w(x, D_x)u\rangle \tag{A.10}$$

for any $u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$.

Proof. Since $\chi = \sum_j |\phi_j|^2 \ge c > 0$ we find that $\chi^{-1} \in S(1, G)$. The calculus gives

$$(\chi^{-1})^w \sum_j \overline{\phi}_j^w \phi_j^w = 1 + r^w$$

where $r \in S(H, G)$ uniformly in H. Thus, the mapping $u \mapsto (\chi^{-1})^w \sum_j \overline{\phi}_j^w \phi_j^w u$ is a homeomorphism on $L^2(\mathbb{R}^n)$ for small enough H. Now the constant metric $G = H|dw|^2$ is trivially *strongly σ temperate* according to Definition 7.1 in [Bony and Chemin 1994], so Theorem 7.6 in the same reference gives $B \in S(1, G)$ such that

$$B^w(\chi^{-1})^w \sum_j \overline{\phi}_j^w \phi_j^w = \sum_j B_j^w \phi_j^w = 1$$

where $B_j^w = B^w(\chi^{-1})^w \overline{\phi}_j^w \in \operatorname{Op} S(1, G)$ uniformly, which gives $1 = \sum_j \overline{\phi}_j^w \overline{B}_j^w$ since $(B_j^w)^* = \overline{B}_j^w$. Now we shall put

$$\widetilde{\mathcal{F}}^w(x, D_x) = \sum_j \overline{\psi}_j^w(x, D_x) F_j \psi_j^w(x, D_x).$$

Then

$$\widetilde{\mathcal{F}}^w = \sum_{jk} \overline{\phi}_j^w \overline{B}_j^w \widetilde{\mathcal{F}}^w B_k^w \phi_k^w = \sum_{jkl} \overline{\phi}_j^w \overline{B}_j^w \overline{\psi}_l^w F_l \psi_l^w B_k^w \phi_k^w. \tag{A.11}$$

Let $C^w_{jkl} = \bar{B}^w_j \overline{\Psi}^w_l \psi^w_l B^w_k$; then we find from (A.11) that

$$\langle \widetilde{\mathscr{F}}^w u, u\rangle = \sum_{jkl} \langle F_l C^w_{jkl} \phi^w_k u, \phi^w_j u\rangle.$$

Let d_{jk} be the $H^{-1}|dw|^2$ distance between the G neighborhoods in which ψ_j and ψ_k are supported. The usual calculus estimates (see [Hörmander 1983–1985, Volume III, page 168] or [Bony and Chemin 1994, Theorem 2.6]) gives that the L^2 operator norm of C^w_{jkl} can be estimated by

$$\|C^w_{jkl}\| \le C_N(1 + d_{jl} + d_{lk})^{-N}$$

for any N. We find by Taylor's formula, Lemma A.7 and the Cauchy–Schwarz inequality that

$$|\langle (F_j - F_k)u, u\rangle| \le C_1|w_j - w_k|\langle F_k u, u\rangle^{1/2} h^{1/2}\|u\| + C_2 h|w_j - w_k|^2\|u\|^2 \le C\langle F_k u, u\rangle(1 + d_{jk})^2$$

since $|w_j - w_k| \le C(d_{jk} + H^{-1/2})$ and $h \le hH^{-1} \le F_k$. Since $F_l \ge 0$ we obtain that

$$2|\langle F_l u, v\rangle| \le \langle F_l u, u\rangle^{1/2}\langle F_l v, v\rangle^{1/2} \le C\langle F_j u, u\rangle^{1/2}\langle F_k v, v\rangle^{1/2}(1 + d_{jl})(1 + d_{lk})$$

and Lemma A.8 gives

$$\langle F_k C^w_{jkl}\phi^w_k u, F_k C^w_{jkl}\phi^w_k u\rangle \le \|C_{jkl}\|^2\langle F_k \phi^w_k u, \phi^w_k u\rangle.$$

Thus we find that

$$\begin{aligned}\sum_{jkl}\langle F_l C^w_{jkl}\phi^w_k u, \phi^w_j u\rangle &\le C_N \sum_{jkl}(1 + d_{jl} + d_{lk})^{2-N}\langle F_k \phi^w_k u, \phi^w_k u\rangle^{1/2}\langle F_j \phi^w_j u, \phi^w_j u\rangle^{1/2} \\ &\le C_N \sum_{jkl}(1 + d_{jl})^{1-N/2}(1 + d_{lk})^{1-N/2}\big(\langle F_j\phi^w_j u, \phi^w_j u\rangle + \langle F_k \phi^w_k u, \phi^w_k u\rangle\big)\end{aligned}$$

Since $\sum_j (1 + d_{jk})^{-N} \le C$ for all k for N large enough by [Hörmander 1983–1985, Volume III, page 168]), we obtain the estimate (A.10) and the result. □

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