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HEAT-FLOW MONOTONICITY OF STRICHARTZ NORMS

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Our main result is that for $d=1,2$ the classical Strichartz norm $\|e^{is\Delta}f\|_{L^{2+4/d}_{s,x}(\mathbb{R}\times\mathbb{R}^d)}$ associated to the free Schrödinger equation is nondecreasing as the initial datum f evolves under a certain quadratic heat flow.

1. Introduction

For $d\in\mathbb{N}$ let the Fourier transform $\widehat{f}:\mathbb{R}^d\to\mathbb{C}$ of a Lebesgue integrable function f on $\mathbb{R}^d$ be given by

$$\widehat{f}(\xi)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{-ix\cdot\xi}f(x)\,dx.$$

For each $s\in\mathbb{R}$ the Fourier multiplier operator $e^{is\Delta}$ is defined via the Fourier transform by

$$\widehat{e^{is\Delta}f}(\xi)=e^{-is|\xi|^2}\widehat{f}(\xi)$$

for all f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and $\xi\in\mathbb{R}^d$. Thus for each $f\in\mathcal{S}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,

$$e^{is\Delta}f(x)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{i(x\cdot\xi-s|\xi|^2)}\widehat{f}(\xi)\,d\xi.$$

By an application of the Fourier transform in x it is easily seen that $e^{is\Delta}f(x)$ solves the Schrödinger equation

$$i\partial_s u=-\Delta u, \tag{1-1}$$

with initial datum $u(0,x)=f(x)$. It is well known that the solution operator $e^{is\Delta}$ extends to a bounded operator from $L^2(\mathbb{R}^d)$ to $L^p_sL^q_x(\mathbb{R}\times\mathbb{R}^d)$ if and only if (d,p,q) is Schrödinger-admissible; that is, there exists a finite constant $C_{p,q}$ such that

$$\|e^{is\Delta}f\|_{L^p_sL^q_x(\mathbb{R}\times\mathbb{R}^d)}\le C_{p,q}\|f\|_{L^2(\mathbb{R}^d)} \tag{1-2}$$

if and only if

$$p,q\ge 2,\quad (d,p,q)\ne(2,2,\infty),\ \text{and}\quad \frac{2}{p}+\frac{d}{q}=\frac{d}{2}. \tag{1-3}$$

For $p=q=2+4/d$, this classical inequality is due to Strichartz [1977], who followed arguments of Stein and Tomas (see [Tomas 1975]). For $p\ne q$ the reader is referred to [Keel and Tao 1998] for historical references and a full treatment of (1-2) for suboptimal constants $C_{p,q}$.

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Foschi [2007] and independently Hundertmark and Zharnitsky [2006] showed that in the cases where one can "multiply out" the Strichartz norm

$$\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}, \tag{1-4}$$

that is, when q is an even integer dividing p, the sharp constants $C_{p,q}$ in the inequalities above are obtained by testing on isotropic centered Gaussians. (These authors considered $p = q$ only.) The main purpose of this paper is to highlight a startling monotonicity property of such Strichartz norms as the function f evolves under a certain quadratic heat flow.

Theorem 1.1. *Let $f \in L^2(\mathbb{R}^d)$. If (d, p, q) is Schrödinger-admissible and q is an even integer which divides p, the quantity*

$$Q_{p,q}(t) := \big\|e^{is\Delta}(e^{t\Delta}|f|^2)^{1/2}\big\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \tag{1-5}$$

is nondecreasing for all $t > 0$; that is, $Q_{p,q}$ is nondecreasing in the cases $(1, 6, 6)$, $(1, 8, 4)$, and $(2, 4, 4)$.

The heat operator $e^{t\Delta}$ is of course defined to be the Fourier multiplier operator with multiplier $e^{-t|\xi|^2}$, and so

$$e^{t\Delta}|f|^2 = H_t * |f|^2,$$

where the heat kernel $H_t : \mathbb{R}^d \to \mathbb{R}$ is given by

$$H_t(x) = \frac{1}{(4\pi t)^{d/2}}\, e^{-|x|^2/4t}. \tag{1-6}$$

By making an appropriate rescaling one may rephrase the above result in terms of "sliding" Gaussians in the following way. For $f \in L^2(\mathbb{R}^d)$ let $u : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be given by $u(t, x) = H_t * |f|^2(x)$ and $\widetilde{u} : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$\widetilde{u}(t, x) = t^{-d} u(t^{-2}, t^{-1}x) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-tv|^2/4} |f(v)|^2 dv.$$

We interpret $\widetilde{u}$ as a superposition of translates of a fixed Gaussian which simultaneously slide to the origin as t tends to zero. By a simple change of variables it follows that

$$Q_{p,q}(t^{-2}) = \big\|e^{is\Delta}(\widetilde{u}(t, \cdot)^{1/2})\big\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{1-7}$$

The reader familiar with the standard wave-packet analysis in the context of Fourier extension estimates may find it more enlightening to interpret Theorem 1.1 via this rescaling.

The claimed monotonicity of $Q_{p,q}$ yields the sharp constant $C_{p,q}$ in (1-2) as a simple corollary. To see this, suppose that the function f is bounded and has compact support. Then, by rudimentary calculations,

$$\lim_{t\to 0} Q_{p,q}(t) = \big\|e^{is\Delta}|f|\big\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)},$$

which, by virtue of the fact that q is an even integer which divides p, is greater than or equal to $\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}$. Furthermore, because of (1-7) it follows that

$$\lim_{t\to\infty} Q_{p,q}(t) = \big\|e^{is\Delta}(H_1^{1/2})\big\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)},$$

where H_1 is the heat kernel at time $t = 1$. Therefore Theorem 1.1 gives the sharp constant $C_{p,q}$ in (1-2) for the triples $(1, 6, 6)$, $(1, 8, 4)$, and $(2, 4, 4)$, and shows that Gaussians are maximisers. In particular, if

$$C_{p,q} := \sup\{\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} : \|f\|_{L^2(\mathbb{R}^d)} = 1\}$$

then $C_{6,6} = 12^{-1/12}$, $C_{8,4} = 2^{-1/4}$, and $C_{4,4} = 2^{-1/2}$. As we have already noted, $C_{6,6}$ and $C_{4,4}$ were found recently by Foschi [2007] and independently by Hundertmark and Zharnitsky [2006]. In the $(1, 8, 4)$ case, we shall see in the proof of Theorem 1.1 below that the monotonicity (and hence sharp constant) follows easily from the $(2, 4, 4)$ case.

Heat-flow methods have already proved effective in treating certain d-linear analogues of the Strichartz estimate (1-2) [Bennett et al. 2006]. Also intimately related (as we shall see) are the articles [Carlen et al. 2004; Bennett et al. 2008a] in the setting of the multilinear Brascamp–Lieb inequalities.

The proof of Theorem 1.1 is contained in Section 2. We discuss some further results in Section 3. In particular we show that the Strichartz norm is nondecreasing under a certain quadratic Mehler flow and observe that one may relax the quadratic nature of the heat flow in Theorem 1.1 by inserting a mitigating factor which is a power of t. We also consider extensions of Theorem 1.1 to higher dimensions.

2. Proof of Theorem 1.1

The idea behind the proof of Theorem 1.1 is simply to express the Strichartz norm

$$\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}$$

in terms of quantities which are already known to be monotone under the heat flow that we consider. As we shall see, this essentially amounts to bringing together the Strichartz-norm representation formulae of Hundertmark and Zharnitsky [2006] and the following heat-flow monotonicity property inherent in the Cauchy–Schwarz inequality.

Lemma 2.1. *For $n \in \mathbb{N}$ and nonnegative integrable functions f_1 and f_2 on $\mathbb{R}^n$, the quantity*

$$\Lambda(t) := \int_{\mathbb{R}^n} (e^{t\Delta} f_1)^{1/2} (e^{t\Delta} f_2)^{1/2}$$

is nondecreasing for all $t > 0$.

Proof. Let $0 < t_1 < t_2$. If H_t denotes the heat kernel on $\mathbb{R}^n$ given by (1-6) then,

$$\begin{aligned}
\Lambda(t_1) &= \int_{\mathbb{R}^n} (H_{t_1} * f_1)^{1/2} (H_{t_1} * f_2)^{1/2} = \int_{\mathbb{R}^n} H_{t_2-t_1} * \left((H_{t_1} * f_1)^{1/2}\,(H_{t_1} * f_2)^{1/2}\right) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(H_{t_2-t_1}(x-y) H_{t_1} * f_1(y)\big)^{1/2} \big(H_{t_2-t_1}(x-y) H_{t_1} * f_2(y)\big)^{1/2} dy\,dx \\
&\le \int_{\mathbb{R}^n} \big(H_{t_2-t_1} * (H_{t_1} * f_1)\big)^{1/2} \big(H_{t_2-t_1} * (H_{t_1} * f_2)\big)^{1/2} \\
&= \Lambda(t_2),
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality on $L^2(\mathbb{R}^n)$ and the semigroup property of the heat kernel. □

This proof of Lemma 2.1 originates in [Ball 1989] and was developed further in [Bennett et al. 2008a]. An alternative method of proof, used in [Carlen et al. 2004] and [Bennett et al. 2008a], is based on the divergence theorem and produces the explicit formula

$$\Lambda'(t) = \tfrac{1}{4}\int_{\mathbb{R}^n} \big|\nabla(\log e^{t\Delta} f_1) - \nabla(\log e^{t\Delta} f_2)\big|^2 (e^{t\Delta} f_1)^{1/2}(e^{t\Delta} f_2)^{1/2} \tag{2-1}$$

for each $t > 0$, provided f_1 and f_2 are sufficiently well behaved (for instance, bounded with compact support). We remark in passing that the Cauchy–Schwarz inequality on $L^2(\mathbb{R}^n)$ follows from Lemma 2.1 by comparing the limiting values of $\Lambda(t)$ for t at zero and infinity.

The next lemma is an observation of Hundertmark and Zharnitsky [2006], who showed that multiplied out expressions for the Strichartz norm in the $(1, 6, 6)$ and $(2, 4, 4)$ cases have a particularly simple geometric interpretation.

Lemma 2.2. (1) *For nonnegative* $f \in L^2(\mathbb{R})$,

$$\|e^{is\Delta} f\|^6_{L^6_s L^6_x(\mathbb{R}\times\mathbb{R})} = \frac{1}{2\sqrt{3}} \int_{\mathbb{R}^3} (f\otimes f\otimes f)(X) P_1(f\otimes f\otimes f)(X)\, dX,$$

where $P_1 : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ *is the projection operator onto the subspace of functions on* $\mathbb{R}^3$ *invariant under the isometries that fix the direction* $(1, 1, 1)$.

(2) *For nonnegative* $f \in L^2(\mathbb{R}^2)$,

$$\|e^{is\Delta} f\|^4_{L^4_s L^4_x(\mathbb{R}\times\mathbb{R}^2)} = \frac{1}{4} \int_{\mathbb{R}^4} (f\otimes f)(X) P_2(f\otimes f)(X)\, dX,$$

where $P_2 : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$ *is the projection operator onto the subspace of functions on* $\mathbb{R}^4$ *invariant under the isometries that fix the directions* $(1, 0, 1, 0)$ *and* $(0, 1, 0, 1)$.

Proof of Theorem 1.1. We begin with the case where (p, q, d) is equal to $(1, 6, 6)$. For functions $G \in L^2(\mathbb{R}^3)$ we may write

$$P_1 G(X) = \int_O G(\rho X)\, d\mathcal{H}(\rho), \tag{2-2}$$

where O is the group of isometries on $\mathbb{R}^3$ that coincide with the identity on the span of $(1, 1, 1)$ and $d\mathcal{H}$ denotes the right-invariant Haar probability measure on O.

If, for $f \in L^2(\mathbb{R})$, we let $F := f\otimes f\otimes f$ then it is easy to see that

$$e^{t\Delta}|f|^2 \otimes e^{t\Delta}|f|^2 \otimes e^{t\Delta}|f|^2 = e^{t\Delta}|F|^2, \tag{2-3}$$

because, in general, the heat operator $e^{t\Delta}$ commutes with tensor products. It is also easy to check that for each isometry ρ on $\mathbb{R}^3$,

$$\big(e^{t\Delta}|f|^2 \otimes e^{t\Delta}|f|^2 \otimes e^{t\Delta}|f|^2\big)(\rho\,\cdot) = e^{t\Delta}|F_\rho|^2, \tag{2-4}$$

where $F_\rho := F(\rho\,\cdot)$. In (2-3) and (2-4) the Laplacian Δ acts in the number of variables dictated by context. Therefore, by Lemma 2.2(1),

$$Q_{6,6}(t)^6 = \frac{1}{2\sqrt{3}} \int_O \int_{\mathbb{R}^3} (e^{t\Delta}|F|^2)^{1/2}(X)(e^{t\Delta}|F_\rho|^2)^{1/2}(X)\, dX\, d\mathcal{H}(\rho)$$

and, by Lemma 2.1 and the nonnegativity of the measure $d\mathcal{H}$, it follows that $Q_{6,6}(t)$ is nondecreasing for each $t > 0$.

For the $(2, 4, 4)$ case, we use a representation of the form (2-2) for the projection operator P_2 where the averaging group O is replaced by the group of isometries on $\mathbb{R}^4$ which coincide with the identity on the span of $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. Of course, the analogous statements to (2-3) and (2-4) involving two-fold tensor products hold. Hence the nondecreasingness of $Q_{4,4}$ follows from Lemma 2.2(2) and Lemma 2.1.

Finally, for the $(1, 8, 4)$ case we observe that

$$\left\| e^{is\Delta}(e^{t\Delta}|f|^2)^{1/2} \right\|^2_{L^8_s L^4_x(\mathbb{R}\times\mathbb{R})} = \left\| e^{is\Delta}(e^{t\Delta}(|f|^2 \otimes |f|^2))^{1/2} \right\|_{L^4_s L^4_x(\mathbb{R}\times\mathbb{R}^2)} \tag{2-5}$$

because both solution operators $e^{is\Delta}$ and $e^{t\Delta}$ commute with tensor products. Therefore, the claimed monotonicity in the $(1, 8, 4)$ case follows from the corresponding claim in the $(2, 4, 4)$ case. This completes the proof of Theorem 1.1. □

It is transparent from the proof of Theorem 1.1 and (2-1) how one may obtain an explicit formula for $Q'_{p,q}(t)$ provided q is an even integer which divides p and f is sufficiently well behaved (say, bounded with compact support). For example, using the notation used in the proof of Theorem 1.1,

$$\frac{d}{dt}(Q_{6,6}(t)^6) = \frac{1}{8\sqrt{3}} \int_O \int_{\mathbb{R}^3} \left| V(t, X) - \rho^t V(t, \rho X) \right|^2 (e^{t\Delta}|F|^2)^{1/2} (e^{t\Delta}|F_\rho|^2)^{1/2} dX \, d\mathcal{H}(\rho),$$

where $V(t, \cdot)$ denotes the time-dependent vector field on $\mathbb{R}^3$ given by

$$V(t, X) = \nabla\big(\log e^{t\Delta}|F|^2\big)(X)$$

and ρ^t denotes the transpose of ρ.

Lemma 2.2, combined with a further argument from [Hundertmark and Zharnitsky 2006] (where explicit details can be found), shows that Gaussians are the only extremisers of the Strichartz inequality in the cases $(d, p, q) = (1, 6, 6), (2, 4, 4)$. The same conclusion for the case $(d, p, q) = (1, 8, 4)$ follows quickly from that for the case $(d, p, q) = (2, 4, 4)$ by (2-5).

3. Further results

Mehler flow. The operator $L := \Delta - \langle x, \nabla \rangle$ generates the Mehler semigroup e^{tL} (sometimes called the Ornstein–Uhlenbeck semigroup) given by

$$e^{tL} f(x) = \int_{\mathbb{R}^d} f\big(e^{-t}x + \sqrt{1 - e^{-2t}}\, y\big)\, d\gamma_d(y)$$

for suitable functions f on $\mathbb{R}^d$, where $d\gamma_d$ is the Gaussian probability measure on $\mathbb{R}^d$ given by

$$d\gamma_d(y) = \frac{1}{(2\pi)^{d/2}} e^{-|y|^2/2} dy.$$

Naturally, $u(t, \cdot) := e^{tL} f$ satisfies the evolution equation

$$\partial_t u = Lu$$

with initial datum $u(0,x) = f(x)$. It will be convenient to restrict our attention to functions f which are bounded and compactly supported.

The purpose of this remark is to highlight that when (d, p, q) is one of $(1, 6, 6)$, $(1, 8, 4)$, or $(2, 4, 4)$ the Strichartz norm also exhibits a certain monotonicity subject to the input evolving according to a quadratic Mehler flow.

Theorem 3.1. *Suppose f is a bounded and compactly supported function on $\mathbb{R}^d$. If (d, p, q) is Schrödinger admissible and q is an even integer dividing p, then the quantity*

$$Q(t) := \left\| e^{is\Delta}(e^{-|\cdot|^2/2} e^{tL}|f|^2)^{1/2} \right\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)}$$

is nondecreasing for all $t > 0$.

As a consequence of Theorem 3.1, we may again recover sharp forms of the Strichartz estimates in (1-2) for such exponents by considering the limiting values of $Q(t)$ as t approaches zero and infinity. In particular, since

$$e^{tL}|f|^2(x) = \int_{\mathbb{R}^d} |f|^2\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big)\,d\gamma_d(y),$$

it follows that, for each $x \in \mathbb{R}^d$, $e^{tL}|f|^2(x)$ tends to $\int_{\mathbb{R}^d} |f|^2 d\gamma_d$ as t tends to infinity. Thus, the monotonicity of Q implies that

$$\left\| e^{is\Delta}(e^{-|\cdot|^2/4}|f|) \right\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \leq \left\| e^{is\Delta}(e^{-|\cdot|^2/4}) \right\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |f|^2 d\gamma_d \right)^{1/2}$$

for each bounded and compactly supported function f on $\mathbb{R}^d$. Thus,

$$\|e^{is\Delta} g\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \leq \left\| e^{is\Delta}\big(\tfrac{1}{(2\pi)^{d/2}} e^{-|\cdot|^2/2}\big)^{1/2} \right\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}$$

for each $g \in L^2(\mathbb{R}^d)$.

The first key ingredient in the proof of Theorem 3.1 is to observe that an analogue of Lemma 2.1 holds for Mehler flow.

Lemma 3.2. *Let $n \in \mathbb{N}$ and let f_1 and f_2 be nonnegative, bounded and compactly supported functions on $\mathbb{R}^n$. Then the quantity*

$$\Lambda(t) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \big(e^{-|\cdot|^2/2} e^{tL} f_1\big)^{1/2} \big(e^{-|\cdot|^2/2} e^{tL} f_2\big)^{1/2}$$

is nondecreasing for all $t > 0$.

Proof. Notice that

$$e^{\log \frac{1}{\sqrt{1-2T}} L} f_j\Big(\frac{x}{\sqrt{1-2T}}\Big) = e^{T\Delta} f_j(x) = H_T * f_j(x)$$

for each $0 < T < 1/2$. Thus, for $0 < T_1 < T_2 < 1/2$ we have

$$\begin{aligned}\Lambda\Big(\log\frac{1}{\sqrt{1-2T_1}}\Big) &= \int_{\mathbb{R}^n}(f_1 * H_{T_1})^{1/2}(f_2 * H_{T_1})^{1/2} H_{1/2-T_1}\\ &= \int_{\mathbb{R}^n}(f_1 * H_{T_1})^{1/2}(f_2 * H_{T_1})^{1/2}(H_{T_2-T_1} * H_{1/2-T_2})\\ &= \int_{\mathbb{R}^n}\big[H_{T_2-T_1} * ((f_1 * H_{T_1})^{1/2}(f_2 * H_{T_1})^{1/2})\big] H_{1/2-T_2}\end{aligned}$$

using the semigroup property and evenness of the heat kernel. As in the proof of Lemma 2.1 it follows from the Cauchy–Schwarz inequality and another application of the semigroup property of the heat kernel that

$$H_{T_2-T_1} * \big((f_1 * H_{T_1})^{1/2}(f_2 * H_{T_1})^{1/2}\big) \le (f_1 * H_{T_2})^{1/2}(f_2 * H_{T_2})^{1/2},$$

and thus

$$\Lambda\Big(\log\frac{1}{\sqrt{1-2T_1}}\Big) \le \Lambda\Big(\log\frac{1}{\sqrt{1-2T_2}}\Big).$$

Hence, $\Lambda(t_1) \le \Lambda(t_2)$ for $0 < t_1 < t_2$. □

As with Lemma 2.1, it is possible to prove Lemma 3.2 in a way that produces an explicit formula for $\Lambda'(t)$ for each $t > 0$, from which the monotonicity of Λ is manifest. To see this, let $\mathfrak{u}_j : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be given by

$$\mathfrak{u}_j(t,x) = e^{-|x|^2/2} e^{tL} f_j(x) = e^{-|x|^2/2}\int_{\mathbb{R}^n} f_j\big(e^{-t}x + \sqrt{1-e^{-2t}}\,y\big)\,d\gamma_n(y) \tag{3-1}$$

for $j = 1, 2$. It is straightforward to check that

$$\partial_t \mathfrak{u}_j = \Delta\mathfrak{u}_j + \langle x, \nabla\mathfrak{u}_j\rangle + n\mathfrak{u}_j$$

and furthermore

$$\partial_t(\log\mathfrak{u}_j) = \operatorname{div}(v_j) + |v_j|^2 + \langle x, v_j\rangle + n,$$

where $v_j := \nabla(\log\mathfrak{u}_j)$. Therefore,

$$\Lambda'(t) = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\big(\langle x, \tfrac12 v_1 + \tfrac12 v_2\rangle + n\big)(t,x)\,\mathfrak{u}_1(t,x)^{1/2}\mathfrak{u}_2(t,x)^{1/2}dx$$

and

$$\mathrm{II} := \frac{1}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}\big(\operatorname{div}(v_1) + \operatorname{div}(v_2) + |v_1|^2 + |v_2|^2\big)(t,x)\,\mathfrak{u}_1(t,x)^{1/2}\mathfrak{u}_2(t,x)^{1/2}dx.$$

Since

$$\mathrm{I} = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\operatorname{div}(\mathfrak{u}_1(t,x)^{1/2}\mathfrak{u}_2(t,x)^{1/2}x)\,dx,$$

it follows from the divergence theorem that I vanishes. Using the fact that each f_j is bounded with compact support it follows from the explicit formula for $\mathfrak{u}_j$ in (3-1) that $v_j(t,x)$ grows at most polynomially in x for each fixed $t>0$, so $\int_{\mathbb{R}^n} \operatorname{div}(\mathrm{u}_1^{1/2}\mathrm{u}_2^{1/2}v_j)$ vanishes by the divergence theorem. Therefore, for each $t>0$,

$$\Lambda'(t)=\frac{1}{4(2\pi)^{n/2}}\int_{\mathbb{R}^n}\bigl|v_1(t,x)-v_2(t,x)\bigr|^2\mathrm{u}_1(t,x)^{1/2}\mathrm{u}_2(t,x)^{1/2}dx,$$

which is manifestly nonnegative.

The above argument which proves Lemma 3.2 based on the divergence theorem is very much in the spirit of the heat-flow monotonicity results in [Carlen et al. 2004] and [Bennett et al. 2008a] and naturally extends to the setting of the geometric Brascamp–Lieb inequality. In particular, for $j=1,\dots,m$ suppose that $p_j\geq 1$ and $B_j:\mathbb{R}^n\to\mathbb{R}^{n_j}$ is a linear mapping such that $B_j^*B_j$ is a projection and $\sum_{j=1}^m \frac{1}{p_j}B_j^*B_j=I_{\mathbb{R}^n}$. Then the quantity

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\prod_{j=1}^m\bigl(e^{-|B_jx|^2/2}(e^{tL}f_j)(B_jx)\bigr)^{1/p_j}dx=\int_{\mathbb{R}^n}\prod_{j=1}^m(e^{tL}f_j)(B_jx)^{1/p_j}d\gamma_n(x)$$

is nondecreasing for each $t>0$ provided each f_j is a nonnegative, bounded and compactly supported function on $\mathbb{R}^{n_j}$. This is due to Barthe and Cordero-Erausquin [2004] in the case where each B_j has rank one. A modification of the argument gives the general rank case (see [Carlen and Lieb 2008] for closely related results).

By following the same argument employed in our proof of Theorem 1.1, to conclude the proof of Theorem 3.1 it suffices to note that Mehler flow appropriately respects tensor products and isometries. In particular we need that if F is the m-fold tensor product of f then

$$\bigotimes_{j=1}^m e^{-|\cdot|^2/2}e^{tL}|f|^2=e^{-|\cdot|^2/2}e^{tL}|F|^2 \tag{3-2}$$

and, for each isometry ρ on $(\mathbb{R}^d)^m$,

$$\bigotimes_{j=1}^m e^{-|\cdot|^2/2}e^{tL}|f|^2(\rho\,\cdot)=e^{-|\cdot|^2/2}e^{tL}|F_\rho|^2, \tag{3-3}$$

where $F_\rho:=F(\rho\,\cdot)$. Here, the operators $|\cdot|$ and L are acting on the number of variables dictated by context. The verification of (3-2) and (3-3) is an easy exercise.

Mitigating powers of t. It is possible to relax the quadratic nature of the heat flow in the quantity $Q_{p,q}$ in Theorem 1.1 by inserting as a mitigating factor a well-chosen power of t.

Theorem 3.3. *Suppose that (p,q,d) is Schrödinger-admissible and q is an even integer which divides p. If f is a nonnegative integrable function on $\mathbb{R}^d$ and $\alpha\in[1/2,1]$, the quantity*

$$t^{d(\alpha-1/2)/2}\|e^{is\Delta}(e^{t\Delta}f)^\alpha\|_{L^p_sL^q_x(\mathbb{R}\times\mathbb{R}^d)}$$

is nondecreasing for each $t>0$.

By [Bennett et al. 2008a], Lemma 2.1 generalises to the statement that

$$t^{n(\alpha-1/2)}\int_{\mathbb{R}^n}(e^{t\Delta}f_1)^\alpha(e^{t\Delta}f_2)^\alpha \tag{3-4}$$

is nondecreasing for all $t > 0$ provided $n \in \mathbb{N}$, $\alpha \in [1/2, 1]$ and f_1, f_2 are nonnegative integrable functions on $\mathbb{R}^n$. Thus Theorem 3.3 follows by the same argument in our proof of Theorem 1.1.

Higher dimensions. Theorem 1.1 raises obvious questions about higher-dimensional analogues and consequently the potential of our approach to prove the sharp form of (1-2) in all dimensions (at least for nonnegative initial data f). Shao [2009] has shown that for nonendpoint Schrödinger-admissible triples (p, q, d),

$$\sup\big\{\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R}\times\mathbb{R}^d)} : \|f\|_{L^2(\mathbb{R}^d)} = 1\big\}$$

is at least attained, although he does not determine the explicit form of an extremiser. There is some anecdotal evidence in [Bennett et al. 2008b] to suggest that Theorem 1.1 may not extend to all Schrödinger-admissible triples (d, p, q). Nevertheless, we end this section with a discussion of some results in this direction which we believe to be of some interest.

We shall consider the case $p = q = 2+4/d$ and it will be convenient to denote this number by $p(d)$. Since $p(d)$ is not an even integer for $d \geq 3$, one possible approach to the question of monotonicity of $Q_{p(d),p(d)}$, given by (1-5), is to attempt to embed the Strichartz norm

$$|||f|||_{p(d)} := \|e^{is\Delta} f\|_{L^{2+4/d}_{s,x}(\mathbb{R}\times\mathbb{R}^d)}$$

in a one-parameter family of norms $|||\cdot|||_p$ which are appropriately monotone under a quadratic flow for $p \in 2\mathbb{N}$, and for which the resulting monotonicity formula may be extrapolated, in a sign-preserving way, to $p = p(d)$. Such an approach has proved effective in the context of the general Brascamp–Lieb inequalities, and was central to the approach to the multilinear Kakeya and Strichartz inequalities in [Bennett et al. 2006].

Our analysis for $d = 1, 2$ suggests (albeit rather indirectly) a natural candidate for such a family of norms. For each $d \in \mathbb{N}$ and $p > p(d)$, we define a norm $|||\cdot|||_p$ on $\mathcal{S}(\mathbb{R}^d)$ by

$$|||f|||_p^p = \frac{(p(d)/\pi)^{d/2}}{(2\pi)^{d+2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}}\bigg|\int_{\mathbb{R}^d} e^{-|z-\sqrt{\zeta}\xi|^2} e^{i(x\cdot\xi - s|\xi|^2)}\widehat{f}(\xi)\,d\xi\bigg|^p \frac{\zeta^{\nu-1}}{\Gamma(\nu)} ds\,d\zeta\,dz\,dx,$$

where $\nu = d(p-p(d))/4$.

Theorem 3.4. *As p tends to $p(d)$, the norm $|||f|||_p$ converges to the Strichartz norm $\|e^{is\Delta} f\|_{L^{p(d)}_{s,x}}$ for each f belonging to the Schwartz class on $\mathbb{R}^d$. Additionally, if $\alpha \in [1/2, 1]$ and f is a nonnegative integrable function on $\mathbb{R}^d$ then*

$$\widetilde{Q}_{\alpha,p}(t) := t^{d(\alpha-1/2)/2} |||(e^{t\Delta} f)^\alpha|||_p$$

is nondecreasing for all $t > 0$ whenever p is an even integer.

Remarks. (1) This modified Strichartz norm $|||f|||_p$ is related in spirit to the norm

$$\|I_\beta\, e^{is\Delta} f\|_{L^p_{s,x}(\mathbb{R}\times\mathbb{R}^d)},$$

where I_β denotes the fractional integral of order $\beta = d(p-p(d))/2p$. Although it is true that for all $p \geq p(d)$,

$$\|I_\beta\, e^{is\Delta} f\|_{L^p_{s,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$$

for some finite constant C, the desired heat-flow monotonicity for $p \in 2\mathbb{N}$ is far from apparent for these norms.

(2) Both the Strichartz norm and the modified Strichartz norms $|||\cdot|||_p$ are invariant under the Fourier transform; that is

$$\|e^{is\Delta}\widehat{f}\|_{L^{p(d)}_{s,x}(\mathbb{R}\times\mathbb{R}^d)} = \|e^{is\Delta}f\|_{L^{p(d)}_{s,x}(\mathbb{R}\times\mathbb{R}^d)} \tag{3-5}$$

for all $d \in \mathbb{N}$ and

$$|||\widehat{f}|||_p = |||f|||_p \tag{3-6}$$

for all $p > p(d)$ and $d \in \mathbb{N}$. This observation follows by direct computation and simple changes of variables; for the Strichartz norm it was noted for $d = 1, 2$ in [Hundertmark and Zharnitsky 2006]. We note that in the proof of Theorem 3.4 below we use the invariance in (3-6) for even integers p which (as we will see) follows from Parseval's theorem.

(3) For every integer $m \geq 2$ and in all dimensions $d \geq 1$, a corollary to the case $\alpha = 1/2$ of Theorem 3.4 is the sharp inequality

$$|||f|||_{2m} \leq C_{d,m}\|f\|_{L^2(\mathbb{R}^d)},$$

where the constant $C_{d,m}$ is given by

$$C_{d,m}^{2m} = \frac{\pi^\nu}{2^{\nu+1}m^d\Gamma(\nu+1)}\Big(\frac{p(d)}{2}\Big)^{d/2}. \tag{3-7}$$

Here $\nu = d(2m - p(d))/4$ as before.

(4) It is known that for nonnegative integrable functions f on $\mathbb{R}^d$ the quantity

$$\|\widehat{(e^{t\Delta}f)^{1/p}}\|_{L^{p'}(\mathbb{R}^d)}$$

is nondecreasing for each $t > 0$ provided the conjugate exponent p' is an even integer; this follows from [Bennett et al. 2008a] and [Bennett and Bez 2009]. However, tying in with our earlier comment on the extension of Theorem 1.1 to all Schrödinger-admissible exponents, in [Bennett et al. 2008b] we show that whenever $p' > 2$ is not an even integer there exists a nonnegative integrable function f such that $Q(t)$ is *strictly decreasing* for all sufficiently small $t > 0$.

Proof of Theorem 3.4. To see the claimed limiting behaviour of $|||f|||_p$ as p tends to $p(d)$ observe that

$$\lim_{\nu\to 0}\frac{1}{\Gamma(\nu)}\int_0^\infty \phi(\nu,\zeta)\zeta^{\nu-1}\,d\zeta = \phi(0,0) \tag{3-8}$$

for any ϕ on $[0,\infty)\times[0,\infty)$ satisfying certain mild regularity conditions. For example, (3-8) holds if ϕ is continuous at the origin and there exist constants $C, \varepsilon > 0$ such that, locally uniformly in ν, one has $|\phi(\nu,\zeta) - \phi(\nu,0)| \leq C|\zeta|^\varepsilon$ for all ζ in a neighbourhood of zero and $|\phi(\nu,\zeta)| \leq C|\zeta|^{-\varepsilon}$ for all ζ bounded away from a neighbourhood of zero. One can check that standard estimates (for example, Strichartz estimates of the form (1-2) for compactly supported functions) imply that for f belonging to the Schwartz class on $\mathbb{R}^d$,

$$\phi(\nu,\zeta) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\bigg|\int_{\mathbb{R}^d} e^{-|z-\sqrt{\zeta}\xi|^2}e^{i(x\cdot\xi - s|\xi|^2)}\widehat{f}(\xi)\,d\xi\bigg|^p\,ds\,dx\,dz$$

satisfies such conditions.

We now turn to the monotonicity claim, beginning with some notation. Suppose that $p = 2m$ for some positive integer m. For a nonnegative $f \in \mathcal{S}(\mathbb{R}^d)$ let $F : \mathbb{R}^{md} \to \mathbb{R}$ be given by $F(X) = \otimes_{j=1}^m f(X)$, where $X = (\xi_1, \dots, \xi_m) \in (\mathbb{R}^d)^m \cong \mathbb{R}^{md}$. Next we define the subspace W of $\mathbb{R}^{md}$ to be the linear span of $\mathbf{1}_1, \dots, \mathbf{1}_d$, where for each $1 \le j \le d$, $\mathbf{1}_j := (e_j, \dots, e_j)/\sqrt{m}$ and e_j denotes the jth standard basis vector of $\mathbb{R}^d$. For a vector $X \in \mathbb{R}^{md}$ we denote by X_W and $X_{W^\perp}$ the orthogonal projections of X onto W and $W^\perp$ respectively. Now,

$$|||f|||_{2m}^{2m} = \tfrac{1}{2^{d+1}\pi} \left(\tfrac{p(d)}{m\pi}\right)^{d/2} \int \delta(X_W - Y_W)\delta(|X|^2 - |Y|^2) K(X, Y) F(X) F(Y) \, dX dY,$$

where we integrate over $\mathbb{R}^{md} \times \mathbb{R}^{md}$ and

$$\begin{aligned} K(X, Y) &= \int_0^\infty \frac{\zeta^{\nu-1}}{\Gamma(\nu)} e^{-\zeta(|X|^2+|Y|^2)} \int_{\mathbb{R}^d} e^{\sqrt{m\zeta} z\cdot(X_W+Y_W)} e^{-m|z|^2/2} dz \, d\zeta \\ &= \left(\tfrac{2\pi}{m}\right)^{d/2} \int_0^\infty \frac{\zeta^{\nu-1}}{\Gamma(\nu)} e^{-\zeta(|X|^2+|Y|^2)} e^{\zeta|X_W+Y_W|^2/2} d\zeta \end{aligned}$$

for $(X, Y) \in \mathbb{R}^{md} \times \mathbb{R}^{md}$. Thus, on the support of the delta distributions ($X_W = Y_W$ and $|X|^2 = |Y|^2$) we have

$$K(X, Y) = \left(\tfrac{2\pi}{m}\right)^{d/2} \int_0^\infty \frac{\zeta^{\nu-1}}{\Gamma(\nu)} e^{-2\zeta(|X|^2-|X_W|^2)} d\zeta = \tfrac{1}{2^\nu} \left(\tfrac{2\pi}{m}\right)^{d/2} \frac{1}{(|X|^2 - |X_W|^2)^\nu} = \tfrac{1}{2^\nu} \left(\tfrac{2\pi}{m}\right)^{d/2} \frac{1}{|X_{W^\perp}|^{2\nu}}.$$

Therefore

$$|||f|||_{2m}^{2m} = \tfrac{\pi^\nu}{2^{\nu+1} m^d \Gamma(\nu+1)} \left(\tfrac{p(d)}{2}\right)^{d/2} \int_{\mathbb{R}^{md}} F(X) P F(X) \, dX, \tag{3-9}$$

where P is given by

$$PF(X) = \tfrac{\Gamma(\nu+1)}{\pi^{\nu+1}} \frac{1}{|X_{W^\perp}|^{2\nu}} \int_{\mathbb{R}^{md}} \delta(X_W - Y_W)\delta(|X|^2 - |Y|^2) F(Y) \, dY.$$

Using polar coordinates in $W^\perp$ in the above integral and recalling that $\nu = d(2m - p(d))/4$ identifies P as the orthogonal projection onto functions on $\mathbb{R}^{md}$ which are invariant under the action of O, the group of isometries on $\mathbb{R}^{md}$ which coincide with the identity on W; that is,

$$PF(X) = \int_O F(\rho X) \, d\mathcal{H}(\rho),$$

where $d\mathcal{H}$ denotes the right-invariant Haar probability measure on O.

Finally, applying the representation of $|||f|||_{2m}^{2m}$ in (3-9) to the quantity $\widetilde{Q}_{\alpha,2m}$, and appealing to the nondecreasingness of the quantity in (3-4), we conclude that $\widetilde{Q}_{\alpha,2m}(t)$ is nondecreasing for all $t > 0$ and all $\alpha \in [1/2, 1]$. This completes the proof of Theorem 3.4. □

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