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ON THE GLOBAL WELL-POSEDNESS OF THE ONE-DIMENSIONAL SCHRÖDINGER MAP FLOW

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We establish the global well-posedness of the initial value problem for the Schrödinger map flow for maps from the real line into Kähler manifolds and for maps from the circle into Riemann surfaces. This partially resolves a conjecture of W.-Y. Ding.

1. Introduction

In this article we study the Schrödinger map flow from a one-dimensional domain into a complete Kähler manifold. First, we show that when the domain is the real line the flow exists for all time. Second, we show that when the domain is the circle and the target is a Riemann surface the flow also exists for all time. The main contribution of this article is to bring Bourgain's work on the periodic cubic nonlinear Schrödinger equation (NLS) to bear on the geometric situation at hand.

Let (M, g) be a complete Riemannian manifold of dimension m, and let (N, ω, J, h) be a complete symplectic manifold of dimension $2n$ with a compatible almost complex structure J, that is, such that $\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,,\,\cdot\,)$ and such that $h(\,\cdot\,,\,\cdot\,) = \omega(\,\cdot\,, J\,\cdot\,)$ defines a complete Riemannian metric on N. Associated to this data is the space of all smooth maps from M to N, the Fréchet manifold $X := C^\infty(M, N)$, endowed with a symplectic structure,

$$\Omega(V, W)\big|_u = \int_M u^\star\omega(V, W)\, dV_{M,g} \quad \text{for all } V, W \in T_uX = \Gamma(M, u^\star TN),$$

where the tangent space to X at a map $u : M \to N$ is the space of smooth sections of $u^\star TN \to M$ and where $dV_{M,g}$ denotes the volume form on M induced by g. The form Ω is nondegenerate, i.e., it endows X with an injective map $TX \to T^\star X$.

Define the energy function on X by

$$E(u) = \tfrac{1}{2}\int_M |du|^2_{g^\sharp\otimes u^\star h}\, dV_{M,g},$$

where we denote by $g^\sharp$ the metric induced by g on $T^\star M$ and where we view du as a section of $T^\star M \otimes u^\star TN \to M$ and equip this bundle with the metric $g^\sharp \otimes u^\star h$.

The almost-complex structure on N induces one on X and a corresponding compatible Riemannian metric defined by

$$G(V, W)\big|_u = \int_M u^\star h(V, W)\, dV_{M,g} \quad \text{for all } V, W \in T_uX = \Gamma(M, u^\star TN).$$

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In the infinite-dimensional setting not every function will necessarily have a gradient. However, if we let $\{u_t\}_{t\in[-1,1]}$ be a smooth family of maps with $u_0 = u$ and denote by $W = (\partial u_t/\partial t)|_0$ a variation, then

$$dE(W)|_{u_0} = \frac{\partial E(u_t)}{\partial t}\bigg|_0 = \int_M g^\sharp \otimes u^\star h(du, dW)\, dV_{M,g} = -\int_M u^\star h(\mathrm{tr}_{g^\sharp} \nabla du, W)\, dV_{M,g}, \tag{1}$$

and hence the gradient of E exists and is given by

$$\nabla^G E\big|_u = -\tau(u), \tag{2}$$

where $\tau(u) := \mathrm{tr}_{g^\sharp} \nabla du$ is called the tension field of u and ∇ is the connection on $T^\star M \otimes u^\star TN \to M$ induced from the Levi-Civita connection on (M, g) and the pulled-back Levi-Civita connection from (N, h). The corresponding gradient flow

$$\frac{\partial u}{\partial t} = \tau(u), \quad u|_{\{0\}\times M} = u_0, \tag{3}$$

is the classical harmonic map flow introduced by Eells and Sampson [1964], which has been extensively studied.

Now the symplectic gradient of E also exists and is given by

$$\nabla^\Omega E\big|_u = -J\tau(u).$$

The corresponding Hamiltonian flow

$$\frac{\partial u}{\partial t} = -J\tau(u), \quad u|_{\{0\}\times M} = u_0, \tag{4}$$

on (X, Ω), introduced in [Ding and Wang 1998; Terng and Uhlenbeck 2006], is called the Schrödinger map flow.

While the energy decreases along (3), for (4) the flow is contained in an energy level set, since for maps of finite energy we have by (1)

$$\frac{dE(u(t))}{dt} = -\int_M u^\star h\Big(\tau(u), \frac{\partial u}{\partial t}\Big)\, dV_{M,g} = 0. \tag{5}$$

For (3) one typically expects to converge to a harmonic representative of the homotopy class of u_0 under some geometric assumptions (for example, negatively curved target [Eells and Sampson 1964]) while (4) seems to be describing some rather very different behavior. Analytically this may be described by the transition from the parabolic (3) to the borderline case (4) whose symbol has purely imaginary eigenvalues. Note also that for the Schrödinger flow there is no preferred time direction.

One problem common to both flows is the question of existence and uniqueness. Indeed since the flows are defined on infinite-dimensional spaces one cannot expect global existence[1] or well-posedness[2] in general. Restricting to the Kähler case, there is a similarity between the two flows as far as local existence is concerned: Results of Ding, Wang and McGahagan show that at least locally (4) can be approximated by equations of either parabolic (in the sense of Petrovskiĭ: see, for example, [Eidelman and Zhitarashu 1998]) or hyperbolic character. As a consequence the following result holds for maps of finite energy.

[1] For the Schrödinger flow the question of global existence is equivalent to the existence of a "symmetry" of (X, Ω), that is, a one-parameter subgroup of Hamiltonian diffeomorphisms of (X, Ω) for the energy function E integrating $\nabla^\Omega E$.

[2] Until recently no results were known for general symplectic targets; see [Chihara 2008] for recent work on local well-posedness in this setting.

Theorem 1.1 [Ding and Wang 2001; McGahagan 2007]. *Let (M^m, g) be a complete Riemannian manifold and let (N, J, h) be a complete Kähler manifold with bounded geometry. For integers $k > m/2+1$ the flow equation (4) with $u_0 \in W^{k,2}(M, N)$ admits a unique solution $u \in C^0\big([0, T], W^{k,2}(M, N)\big)$ where $T < T_0$ and T_0 depends on $\|\nabla u_0\|_{W^{[m/2]+1,2}}$ and the geometry of N alone. Moreover, there exist positive constants C_1, C_2 depending only on these quantities such that*

$$\|\nabla u(t)\|_{W^{[m/2]+1,2}} \leq C_1/(T_0 - t)^{C_2} \quad \text{for all } t \in [0, T_0).$$

In particular, if $u_0 \in W^{k,2}(M, N)$ for all $k \geq 2$ then $u \in C^\infty\big([0, T] \times M, N\big)$.

Here by bounded geometry we mean uniform bounds on the injectivity radius and the curvature tensor and its derivatives. This is automatically true for compact targets.

The main difficulty lies, therefore, in understanding the global behavior.

Previous results of a global nature are mostly concerned with the one-dimensional domain case and are all restricted to the case of a special target Kähler manifold. We recall the following nonexhaustive list of works. The flow on $(S^1, \text{can}) \to (S^2, \text{can})$, where "can" denotes the canonical metric, corresponding to the classical model for an isotropic ferromagnet was studied from the mathematical point of view by Sulem et al. [1986], who obtained local well-posedness for the initial value problem as well as partial global results. Zhou et al. [1991] studied the global well-posedness problem using a parabolic approximation which was later put to use in [Ding and Wang 1998; Pang et al. 2001] to prove global existence and uniqueness of smooth solutions of maps from (S^1, can) into a constant sectional curvature Kähler target (that is, a Riemann surface equipped with a constant curvature metric or a flat complex torus) as well as to Hermitian locally symmetric spaces [Pang et al. 2002] using a conservation law. The latter also treats the inhomogeneous flow which can be essentially viewed as the Schrödinger flow with domain S^1 equipped with a different metric. Terng and Uhlenbeck [2006] studied in detail the flow from the Euclidean line into Grassmannians. Chang et al. [2000] proved existence and uniqueness of global smooth solutions for maps of the Euclidean line into a compact Riemann surface. In addition, they treated maps of the Euclidean plane into a compact Riemann surface under the assumption of small initial energy and certain symmetries. Finally, see [Bejenaru et al. 2007; 2008] for recent work on global well-posedness in the case of maps from Euclidean space into (S^2, can) under a certain smallness assumption.

Note that in all of these results one restricts the target to a rather small class of Kähler manifolds.

We recall the following conjecture:

Conjecture 1.2 [Ding 2002]. *The Schrödinger map flow is globally well-posed for maps from one-dimensional domains into compact Kähler manifolds.*

The main results of this article are a partial answer to this conjecture. Namely, we establish the global well-posedness of the one-dimensional Schrödinger flow into general Kähler manifolds when the domain is the real line, and into Riemann surfaces when the domain is the circle.

Theorem 1.3. *Let $(M, g) = (\mathbb{R}, dx \otimes dx)$, let (N, J, h) be a complete Kähler manifold with bounded geometry, and let $k \geq 2$ be an integer. The flow equation (4) with $u_0 \in W^{k,2}(\mathbb{R}, N)$ admits a unique solution $u \in C^0(\mathbb{R}, W^{k,2}(\mathbb{R}, N))$. In particular u is smooth if u_0 is in $W^{k,2}(\mathbb{R}, N)$ for all $k \geq 2$.*

Theorem 1.4. *Let* $(M, g) = (S^1, dx \otimes dx)$, *let* (N, J, h) *be a complete Riemann surface with bounded geometry, and let* $k \geq 2$ *be an integer. The flow equation* (4) *with* $u_0 \in W^{k,2}(S^1, N)$ *admits a unique solution* $u \in C^0(\mathbb{R}, W^{k,2}(S^1, N))$. *In particular u is smooth if* u_0 *is in* $W^{k,2}(S^1, N)$ *for all* $k \geq 2$.

Remark 1.5. From the physical point of view, the Schrödinger map flow may also be introduced as a generalization of the Heisenberg model for a ferromagnetic spin system. The classical model for this physical system precisely corresponds to maps from the standard circle into $N = S^2$ with the standard metric and complex structure [Landau and Lifschitz 1935] (for some background see, for example, [Ding 2002; Ding and Wang 1998; McGahagan 2004; Sulem et al. 1986]). Perhaps the most physically natural generalization of the classical model would be to vary the metric on the target S^2, however it seems that even for small perturbations of the round metric on S^2 global well-posedness was not known before. Theorem 1.4 establishes the global well-posedness of the Cauchy problem describing this physical model when the metric on S^2 is arbitrary.

The global well-posedness thus established in these cases, several natural questions arise related to more precise information regarding the long-time behavior of the Schrödinger map flow. For example, for the case of maps from the real line it would be interesting to determine whether certain scattering occurs in some cases. In addition, we pose the following conjecture regarding the length of the image along the flow.

Conjecture 1.6. *In the setting of Theorem 1.3 one has* $\lim_{t\to\infty} \|u(t)\|_{W^{1,1}(\mathbb{R},N)} = \infty$. *In addition, for every* $\epsilon > 0$, *there exists a time* t_0 *and a geodesic ball* $B \subset N$ *of radius* ϵ *such that the image of* $u(t_0)$ *is contained in* B.

Is not hard to show this conjecture holds for the case $N = \mathbb{C}^n$, equipped with the Euclidean metric, with an estimate $\|u(t)\|_{W^{1,1}(\mathbb{R},N)} \geq Ct^{1/2}$.

Outline of proofs and organization of the paper. According to Theorem 1.1 we have existence of a time-local solution. The strategy of the proof is this: First, using the Kähler condition, we translate the flow equation into a system of nonlinear Schrödinger (NLS) equations. Then, for this system of equations we obtain an a priori estimate in a weaker norm than that in Theorem 1.1, namely in an appropriate Strichartz norm for $M = \mathbb{R}$ and in L^4 for $M = S^1$. These estimates are crucial since they only depend on the initial energy (which is a conserved quantity) and that in a manner that can be readily converted into a global a priori estimate in the same space. Taking derivatives of the flow equation and after additional work we then obtain global a priori estimates in stronger norms and these in turn may be converted back to imply global well-posedness for our original Cauchy problem in $W^{k,2}$ for all $k \geq 2$.

While the proofs of both Theorem 1.3 and Theorem 1.4 follow the same general scheme, nevertheless there are substantial differences between the two, as we now explain.

We start in Section 2 with the case of the real line, which is simpler due to simple connectivity and dispersiveness. Here we follow Chang, Shatah and Uhlenbeck [2000] and write the flow equation in terms of a parallel frame, with the added observation that the Kähler condition allows one to readily generalize their computations from the Riemann surface case to higher dimensions. The flow equation then reduces to a system of NLS equations. The same Strichartz-type calculations as in their study of the Riemann surface case then apply.

We then treat in Section 3 the case of maps from $M = S^1$ into a Riemann surface, which is considerably more difficult and is the main contribution of this article. There are two main difficulties. First, using a parallel frame introduces holonomy, so the resulting NLS equation lives on $\mathbb{R}$, the universal cover of S^1, instead of on S^1 itself. To overcome this we use a certain space-time transformation in order to obtain an NLS equation on S^1 in terms of the holonomy representation of N. In addition we need to estimate the variation of the holonomy along the flow. Second, since S^1 is compact the equations are no longer dispersive. To overcome that we adapt Bourgain's results on the cubic NLS to our setting in order to prove a time-local a priori estimate in L^4 that depends in such a way on the initial data that it may be used to obtain a global a priori estimate in the same space. Finally, we take derivatives of the NLS equation and after some more work obtain higher derivative a priori estimates.

In Section 4 we discuss some of the difficulties that arise when trying to apply our approach to treat maps from the circle into higher-dimensional Kähler manifolds. It is conceivable that some of these ideas might be related to showing finite time blow-up for higher-dimensional domains.

2. Maps from the real line into a Kähler manifold

In the case of maps from the Euclidean real line into a complete Kähler manifold (N, J, h) of complex dimension n, the Schrödinger equation (4) becomes

$$J\nabla_t u - \nabla_x \nabla_x u = 0, \qquad u(0) = u_0, \tag{6}$$

and we define the energy as

$$E(u_0) = \int_{\mathbb{R}} |du_0|^2_{\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x}\otimes u^\star h} dx < \infty \tag{7}$$

(departing from the convention in the Introduction by a factor of 2). Here we have used the abbreviated notation $\nabla_t = \nabla_{u_\star \partial/\partial t}$, $\nabla_x = \nabla_{u_\star \partial/\partial x}$, $\nabla_t u = u_\star \partial/\partial t = \partial u/\partial t$, $\nabla_x u = u_\star \partial/\partial x = \partial u/\partial x$, and we denote the derivatives of a function f by $f_{,x}$ and $f_{,t}$. The key idea in this section, going back to [Chang et al. 2000], is to rewrite (6) in an appropriate frame along the image, in such a way that (6) reduces to a system of nonlinear Schrödinger (NLS) equations. In fact our proof closely follows their approach for the Riemann surface case observing that it readily generalizes to Kähler targets of arbitrary dimension.

Assume that $u : I \times \mathbb{R} \to N$ is a solution of (6), where I is a neighborhood of 0 in $\mathbb{R}$ (given, for example, by Theorem 1.1). Choose an orthonormal frame $\{e_1, \ldots, e_{2n}\}$ for $u^\star TN$ with respect to h. We further reduce the structure group to $U(n) \subseteq O(2n)$ by assuming $e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$. We identify $U(n)$ with its image in $O(2n)$ under the map $\imath : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$ given by

$$\imath(A + \sqrt{-1}\, B) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Note that if $v = x + \sqrt{-1}\, y \in \mathbb{C}^n$ and $\imath(v) = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\imath(Av) = \imath(A)\imath(v).$$

We will use this identification frequently, sometimes omitting the reference to the map $\imath$. In the following we let Latin indices take values in $\{1, \ldots, 2n\}$ and Greek indices in $\{1, \ldots, n\}$. For both alphabets we use the notation

$$\bar{\cdot} = \cdot + n - 1 \ (\mathrm{mod}\ 2n)\ + 1.$$

Therefore barred Greek indices take values in $\{n+1,\dots,2n\}$. As mentioned $e_{\bar{j}} = Je_j$ so $e_{\bar{\alpha}} = Je_\alpha$ and $e_{\overline{\alpha+n}} = -e_\alpha$. We abbreviate the spaces $L^p(\mathbb{R}, dx)$ and $L^p(\mathbb{R}, dt)$ as $L^p(\mathbb{R}_x)$ and $L^p(\mathbb{R}_t)$, and so on. Finally, given a map $u:(M,g)\to(N,h)$, by $u\in W^{k,p}(M,N)$ we will mean that $\sum_{j=0}^{k-1}\big\||\nabla^j du|\big\|_{L^p}<\infty$. For example, in this notation we have $E(u)=\|u\|^2_{W^{1,2}(M,N)}$.

Now we may view the flow equation (6) in this frame. Write, for each $(t,x)\in I\times\mathbb{R}$,

$$\nabla_x u = \sum_{j=1}^{2n} h(\nabla_x u, e_j)e_j =: a^j e_j, \quad \nabla_t u = \sum_{j=1}^{2n} h(\nabla_t u, e_j)e_j =: b^j e_j, \tag{8}$$

where we use the Einstein summation convention, namely the appearance of an index both as a subscript and a superscript indicates summation. Then (6) can be rewritten as

$$b^j e_{\bar{j}} - a^j_{,x} e_j - a^j \nabla_x e_j = 0. \tag{9}$$

The conservation of energy (see (5)) is expressed as

$$E(u(t)) = E(u_0) = \int_{\mathbb{R}} \sum_{l=1}^{2n} (a_l)^2 dx = \|a(t)\|^2_{L^2(\mathbb{R}_x)} \quad \text{for all } t\in\mathbb{R}. \tag{10}$$

Note that

$$0 = u_\star\Big[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big] = [\nabla_t u, \nabla_x u]. \tag{11}$$

Since $\nabla J = 0$, differentiating (6) in space yields $J\nabla_x\nabla_t u - \nabla_x\nabla_x\nabla_x u = 0$, which becomes, using (11) and (8),

$$J(a^j_{,t} e_j + a^j \nabla_t e_j) - a^j_{,xx} e_j - 2a^j_{,x}\nabla_x e_j - a^j \nabla_x\nabla_x e_j = 0. \tag{12}$$

We impose the gauge-fixing condition

$$\nabla_x e_j = 0, \quad j = 1,\dots,2n. \tag{13}$$

The resulting frame along the image is still unitary, since the complex structure commutes with parallel transport. Equation (9) becomes

$$b^j = a^{\bar{j}}_{,x}. \tag{14}$$

Note that $u^\star TN \to \mathbb{R}$ is trivial and that (13) amounts to fixing a trivializing parallel frame. With this choice, the flow on $u^\star TN$ is given by

$$a^j_{,t} e_{\bar{j}} - a^j_{,xx} e_j = -a^j \nabla_t e_{\bar{j}}. \tag{15}$$

Along the image, using (13) and (14), and letting R denote the curvature tensor of (N,h), we have

$$\begin{aligned}
\nabla_x\nabla_t e_{\bar{j}} &= R(\nabla_x u, \nabla_t u) e_{\bar{j}} = a^k b^l R_{kl\bar{j}}{}^q e_q \\
&= \big(a^\alpha b^\beta R_{\alpha\beta\bar{j}}{}^q + a^{\bar\alpha} b^\beta R_{\bar\alpha\beta\bar{j}}{}^q + a^\alpha b^{\bar\beta} R_{\alpha\bar\beta\bar{j}}{}^q + a^{\bar\alpha} b^{\bar\beta} R_{\bar\alpha\bar\beta\bar{j}}{}^q\big) e_q \\
&= \big(a^\alpha a^{\bar\beta}_{,x} R_{\alpha\beta\bar{j}}{}^q + a^{\bar\alpha} a^{\bar\beta}_{,x} R_{\bar\alpha\beta\bar{j}}{}^q - a^\alpha a^{\beta}_{,x} R_{\alpha\bar\beta\bar{j}}{}^q - a^{\bar\alpha} a^{\beta}_{,x} R_{\bar\alpha\bar\beta\bar{j}}{}^q\big) e_q \\
&= \sum_{\alpha,\beta}\big[(a^\alpha a^{\bar\beta})_{,x} R_{\alpha\beta\bar{j}}{}^q + \tfrac{1}{2}[(a^{\bar\alpha} a^{\bar\beta})_{,x} + (a^\alpha a^\beta)_{,x}] R_{\bar\alpha\beta\bar{j}}{}^q\big] e_q,
\end{aligned} \tag{16}$$

where we have used the Kähler condition once more:

$$R_{\alpha\beta\bar{j}}{}^{q} = R_{\bar{\alpha}\bar{\beta}\bar{j}}{}^{q}, \quad R_{\bar{\alpha}\beta\bar{j}}{}^{q} = -R_{\alpha\bar{\beta}\bar{j}}{}^{q}.$$

Equation (7) implies $\lim_{x\to\pm\infty} a^i(t,x) = 0$. Therefore, since $\nabla_x h(\nabla_t e_{\bar{j}}, e_q) = h(\nabla_x \nabla_t e_{\bar{j}}, e_q)$, we have

$$\begin{aligned}\nabla_t e_{\bar{j}}(t,x) = \sum_{q=1}^{2n} h(\nabla_t e_{\bar{j}}, e_q)(t,-\infty) e_q(t,x) \\ + \Bigg(\sum_{\alpha,\beta} \big[a^\alpha a^{\bar{\beta}} R_{\alpha\beta\bar{j}}{}^{q} + \tfrac{1}{2}[a^{\bar{\alpha}} a^{\bar{\beta}} + a^\alpha a^\beta] R_{\bar{\alpha}\beta\bar{j}}{}^{q}\big](t,x) \\ - \int_{(-\infty,x]} \sum_{\alpha,\beta} \big[a^\alpha a^{\bar{\beta}} R_{\alpha\beta\bar{j}}{}^{q}{}_{,x} + \tfrac{1}{2}(a^{\bar{\alpha}} a^{\bar{\beta}} + a^\alpha a^\beta) R_{\bar{\alpha}\beta\bar{j}}{}^{q}{}_{,x}\big](t,y)\,dy \Bigg) e_q(t,x). \quad (17)\end{aligned}$$

Defining

$$\begin{aligned} A^q_{\bar{j}}(t,-\infty) &:= h(\nabla_t e_{\bar{j}}, e_q)(t,-\infty), \\ P^q_{\bar{j}}(t,x) &:= \sum_{\alpha,\beta} \big[a^\alpha a^{\bar{\beta}} R_{\alpha\beta\bar{j}}{}^{q} + \tfrac{1}{2}[a^{\bar{\alpha}} a^{\bar{\beta}} + a^\alpha a^\beta] R_{\bar{\alpha}\beta\bar{j}}{}^{q}\big](t,x), \\ Q^q_{\bar{j}}(t,x) &:= -\int_{(-\infty,x]} \sum_{\alpha,\beta} \big[a^\alpha a^{\bar{\beta}} R_{\alpha\beta\bar{j}}{}^{q}{}_{,x} + \tfrac{1}{2}(a^{\bar{\alpha}} a^{\bar{\beta}} + a^\alpha a^\beta) R_{\bar{\alpha}\beta\bar{j}}{}^{q}{}_{,x}\big](t,y)\,dy, \end{aligned}$$

we thus have

$$\nabla_t e_{\bar{j}}(t,x) = A^q_{\bar{j}}(t,-\infty) e_q(t,x) + \big[P^q_{\bar{j}}(t,x) + Q^q_{\bar{j}}(t,x)\big] e_q(t,x). \quad (18)$$

We now estimate these terms. Using (9) we have

$$\nabla_t e_{\bar{j}} = b^k \Gamma^p_{k\bar{j}} e_p = a^{\bar{k}}_{,x} \Gamma^p_{k\bar{j}} e_p. \quad (19)$$

Hence, $h(\nabla_t e_{\bar{j}}, e_q) = a^{\bar{k}}_{,x} \Gamma^q_{k\bar{j}}$ and so we may assume that $A^q_{\bar{j}}$ vanishes at $(t,-\infty)$. To justify this, note that this is indeed the case for the local solution of our equation given by Theorem 1.1; even though this assumption makes use of the finiteness of the $W^{2,2}$ norm of that local solution, the important point is that eventually our estimates will not depend on the $W^{2,2}$ norm of u (equivalently on the $W^{1,2}$ norm of a), and so the proof of the a priori estimate for the system of NLS equations (23) below (for a) goes through, with this assumption. Next, using (8), note that $R_{klp}{}^{q}{}_{,x} = a^s R_{klp}{}^{q}{}_{,s}$. Therefore,

$$|P^q_j(t,x)| < C|\mathrm{a}(t,x)|^2, \qquad |Q^q_j(t,x)| < C\int_{\mathbb{R}} |\mathrm{a}(t,y)|^3 dy, \quad (20)$$

where $C > 0$ depends only on the geometry and where we use the notation $|\mathrm{a}| := \big(\sum_{j=1}^{2n} (a^j)^2\big)^{1/2}$.

To summarize the discussion, we have shown that (15) transforms to the following system of NLS equations

$$-a^{\bar{\gamma}}_{,t} - a^{\gamma}_{,xx} = -a^j P^{\gamma}_j - a^j Q^{\gamma}_j, \qquad \gamma = 1,\dots,n, \quad (21)$$

$$a^{\gamma}_{,t} - a^{\bar{\gamma}}_{,xx} = -a^j P^{\bar{\gamma}}_j - a^j Q^{\bar{\gamma}}_j, \qquad \gamma = 1,\dots,n, \quad (22)$$

or, letting $J_0 = \imath(\sqrt{-1}\,I)$,

$$J_0 \mathrm{a}_{,t} = \mathrm{a}_{,xx} - \mathrm{P}\cdot\mathrm{a} - \mathrm{Q}\cdot\mathrm{a}, \quad (23)$$

where $\mathrm{a} = (a^1, \dots, a^{2n})^T$, $\mathrm{P} = (P^k_j)$, and $\mathrm{Q} = (Q^k_j)$. Equivalently, using the aforementioned identification $\imath$ of $\mathrm{GL}(n, \mathbb{C})$ with a subset of $\mathrm{GL}(2n, \mathbb{R})$,

$$\sqrt{-1}\Phi_{,t} = \Phi_{,xx} - \mathrm{S} \cdot \Phi - \mathrm{T} \cdot \Phi, \tag{24}$$

where

$$\Phi := \imath^{-1}(\mathrm{a}) = (a^1 + \sqrt{-1}\, a^{\bar{1}}, \dots, a^n + \sqrt{-1}\, a^{\bar{n}})^T, \quad \mathrm{S} := (S^\beta_\alpha) = \imath^{-1}(\mathrm{P}), \quad \mathrm{T} := (T^\beta_\alpha) = \imath^{-1}(\mathrm{Q}), \tag{25}$$

and from (20) we have

$$|S^\beta_\alpha| < C|\Phi|^2, \qquad |T^\beta_\alpha| < C \int_{\mathbb{R}} |\Phi|^3 dy. \tag{26}$$

Here we have set $|\Phi| := \big(\sum_{j=1}^n |\Phi^j|^2\big)^{1/2}$.

Remark 2.1. In the case of a variable complete smooth metric on the domain $(M, g) = (\mathbb{R}, \alpha^{-1} dx \otimes dx)$ with $\alpha > 0$ the flow equation (4) becomes

$$b^k e_{\bar{k}} = \alpha a^k_{,x} e_k + \tfrac{1}{2}\alpha_{,x} a^k e_k, \tag{27}$$

which can then be transformed, as before, to

$$J_0 \mathrm{a}_{,t} = \alpha \mathrm{a}_{,xx} + \tfrac{3}{2}\alpha_{,x}\mathrm{a}_{,x} + \tfrac{1}{2}\alpha_{,xx}\mathrm{a} - \mathrm{P} \cdot \mathrm{a} - \mathrm{Q} \cdot \mathrm{a}. \tag{28}$$

Equivalently, again using the map $\imath : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$,

$$\sqrt{-1}\Phi_{,t} = \alpha\Phi_{,xx} + \tfrac{3}{2}\alpha_{,x}\Phi_{,x} + \tfrac{1}{2}\alpha_{,xx}\Phi - \mathrm{S} \cdot \Phi - \mathrm{T} \cdot \Phi. \tag{29}$$

The only obstacle to treating this equation using the methods below is the first derivative term on the right-hand side (cf. Remark 3.1).

Therefore we have reduced the original flow equation for the map to a system of NLS equations for the frame coefficients of the gradient of the map. Thus we have reduced ourselves to the same situation as in [Chang et al. 2000] (the only difference in (24) from the case where the target is a Riemann surface is that the equation for each Φ^j depends also on the other Φ^k, $k = 1, \dots, n$; however this dependence is only in the nonlinear terms and not in the terms involving derivatives) and their work now implies the following theorem which is the main result of this section.

Theorem 2.2. *Let $(M, g) = (\mathbb{R}, dx \otimes dx)$ and let (N, J, h) be a complete Kähler manifold with bounded geometry. Then for integers $k \geq 2$ the flow equation (4) with $u_0 \in W^{k,2}(\mathbb{R}, N)$ admits a unique solution $u \in C^0(\mathbb{R}, W^{k,2}(\mathbb{R}, N))$.*

For the benefit of the reader that may not be familiar with standard Strichartz estimates techniques we include here the detailed proof of the Chang–Shatah–Uhlenbeck $L^4(\mathbb{R}_{t,loc}, L^\infty(\mathbb{R}_x))$ estimate and how it implies global well-posedness in $W^{k,2}(\mathbb{R}, N)$. No originality is claimed here. This also serves to provide some perspective on the differences between this case and the case of the circle, treated in the following sections. In addition, it serves to explain the three basic steps in obtaining global well-posedness in $W^{k,2}$ that are also (at least schematically) needed in the case of the circle.

Proof. First, we have by Theorem 1.1 local well-posedness of the original Schrödinger map flow (6) in $W^{k,2}(\mathbb{R}, N)$ for $k \geq 2$. The key to obtaining global well-posedness in these spaces will be a local a priori estimate in a space that is morally larger than $C^0(\mathbb{R}, W^{2,2}(\mathbb{R}, N))$. Equivalently, we will prove an estimate for the frame coefficients a in a norm "weaker" than $C^0(\mathbb{R}, W^{1,2}(\mathbb{R}, \mathbb{R}^{2n})) \equiv C^0(\mathbb{R}, W^{1,2}(\mathbb{R}))$.

More precisely, the proof of global well-posedness is divided into three steps:

First, given an initial data u_0 in $W^{2,2}(\mathbb{R}, N)$ (equivalently, $\Phi(0) \in W^{1,2}(\mathbb{R})$) we will prove an a priori estimate on the $L^4(\mathbb{R}_{t,loc}, L^\infty(\mathbb{R}))$ norm of the frame coefficients Φ depending only on the initial energy $\|\Phi\|_{L^2(\mathbb{R})}$ and the geometry. In particular it will imply that $\|\Phi\|_{L^4([0,T],L^\infty(\mathbb{R}))}$ is finite for all $T > 0$. This is the most fundamental step.

Second, taking a derivative of the system of NLS equations (24) for Φ, using the estimate from the first step, and applying similar calculations we prove that $\|\Phi\|_{L^4([0,T],W^{1,\infty}(\mathbb{R}))}$ and $\|\Phi\|_{C^0([0,T],W^{1,2}(\mathbb{R}))}$ are finite for all $T > 0$.

Third, we let $k \geq 3$ and assume our initial data u_0 lies in $W^{k,2}(\mathbb{R}, N)$. Taking further derivatives of the equations (24) and working inductively, one proves that $\|\Phi\|_{C^0([0,T],W^{k-1,2}(\mathbb{R}))}$ is finite for all $T > 0$. This step, sometimes called *propagation of regularity*, is considered as routine once the first two steps have been carried out.

The key feature of the analysis involved here is that while one is interested only in proving that the $W^{1,2}(\mathbb{R})$ norm of $\Phi(t)$ stays finite, one is forced to use the auxiliary space $L^4(\mathbb{R}_t, W^{1,\infty}(\mathbb{R}_x))$.

In fact, although we will not carry this out here, in the first step one may prove local (and hence global) a priori estimates for (24) in other Strichartz spaces (these are by definition the spaces $L^q(\mathbb{R}_t, L^r(\mathbb{R}_x))$ specified by Lemma 2.3 below) as well, for example, $L^6(\mathbb{R} \times \mathbb{R})$.

Having thus outlined the different steps of the proof, we now turn to the proof itself.

Step 1. Suppose a function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies the NLS equation

$$\sqrt{-1}\, c_{,t} = c_{,xx} + F \quad \text{for all } t \in [0, T], \qquad c(0) = f, \tag{30}$$

for some function $F : [0, T] \times \mathbb{R} \to \mathbb{C}$ that may depend on c nonlinearly (but not on its derivatives). One then has the integral expression (Duhamel formula)

$$c(t, x) = \int_{\mathbb{R}} f(y) \frac{e^{-\sqrt{-1}|x-y|^2/4t}}{\sqrt{2\pi t}} dy - \sqrt{-1} \int_0^t \int_{\mathbb{R}} F(s, y) \frac{e^{-\sqrt{-1}|x-y|^2/4(t-s)}}{\sqrt{2\pi(t-s)}} dy \wedge ds. \tag{31}$$

Denote the Schrödinger operator by

$$S(t)f := e^{-\sqrt{-1}\, t\partial^2/\partial x^2}; \tag{32}$$

more explicitly, we have for $M = \mathbb{R}$,

$$S(t)f = \int_{\mathbb{R}} f(y) \frac{e^{-\sqrt{-1}|x-y|^2/4t}}{\sqrt{2\pi t}} dy. \tag{33}$$

We now recall the Strichartz estimates (on $\mathbb{R}$). For appropriate q, r we denote by $L^q(\mathbb{R}, L^r(\mathbb{R}))$ the Banach space equipped with the norm

$$\|f\|_{L^q(\mathbb{R},L^r(\mathbb{R}))} := \big\| \|f\|_{L^r(\mathbb{R}_x)} \big\|_{L^q(\mathbb{R}_t)}. \tag{34}$$

Lemma 2.3 [Cazenave 2003, page 33]. *Let q, r satisfy $\frac{2}{q}+\frac{1}{r}=\frac{1}{2}$ with $r\in[2,\infty]$ and let $f\in L^2(\mathbb{R}_x)$. Then the function $t\mapsto S(t)f$ belongs to $L^q(\mathbb{R}_t, L^r(\mathbb{R}_x))\cap C^0(\mathbb{R}_t, L^2(\mathbb{R}_x))$ and there is a constant independent of (q,r) and of $f\in L^2(\mathbb{R})$ such that*

$$\|S(\cdot)f\|_{L^q(\mathbb{R}_t,L^r(\mathbb{R}_x))}\le C\|f\|_{L^2(\mathbb{R})}. \tag{35}$$

In our situation we know that $\|\Phi\|_{L^2}$ is constant in time (recall (10)). Assume also that Φ lies in $L^4([0,T], L^\infty(\mathbb{R}_x))$. We will now show that the $L^4([0,T], L^\infty(\mathbb{R}_x))$ norm of Φ is controlled by its L^2 norm and the geometry. This will imply local and eventually global estimates in $L^4(\mathbb{R}_t, L^\infty(\mathbb{R}_x))$.

Let $F=-\mathrm{S}\cdot\Phi-\mathrm{T}\cdot\Phi$. In what follows we restrict t to the interval $[t_1,t_2]$. Then

$$\Phi^j(t)=S(t-t_1)\Phi^j(t_1)-\sqrt{-1}\int_{t_1}^t S(t-s)F(s,\cdot)\,ds. \tag{36}$$

The first term of (36) is in $L^4([t_1,t_2], L^\infty(\mathbb{R}))$ by the Strichartz estimate (35). We will now show that the second term is also in this space.

First, we consider the term $\mathrm{S}\cdot\Phi\le C\|\Phi\|^3$. We need to estimate

$$\left\|\int_{t_1}^t S(t-s)(\mathrm{S}\cdot\Phi)(s,\cdot)ds\right\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}. \tag{37}$$

From (33) we have the dispersive estimate $|S(t)f|\le Ct^{-1/2}\|f\|_{L^1(\mathbb{R})}$. Hence,

$$\begin{aligned}\left\|\int_{t_1}^t S(t-s)(\mathrm{S}\cdot\Phi)(s,\cdot)ds\right\|_{L^\infty(\mathbb{R})} &\le C\int_{t_1}^t (t-s)^{-1/2}\big\||\Phi(s,\cdot)|^3\big\|_{L^1(\mathbb{R})}ds\\ &\le C\cdot E(u_0)\int_{t_1}^t (t-s)^{-1/2}\|\Phi(s,\cdot)\|_{L^\infty(\mathbb{R})}ds\\ &\le C'\left(\int_{t_1}^t ((t-s)^{-1/2})^{4/3}ds\right)^{3/4}\left(\int_{t_1}^t \|\Phi(s,\cdot)\|^4_{L^\infty(\mathbb{R})}ds\right)^{1/4}\\ &= C''|t-t_1|^{1/4}\|\Phi\|_{L^4([t_1,t],L^\infty(\mathbb{R}))},\end{aligned} \tag{38}$$

and it follows that

$$\left\|\int_{t_1}^t S(t-s)(\mathrm{S}\cdot\Phi)(s,\cdot)ds\right\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}\le C|t_2-t_1|^{1/2}\|\Phi\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}. \tag{39}$$

Next, we consider the term $\mathrm{T}\cdot\Phi\le C\|\Phi\|\int_{\mathbb{R}}\|\Phi\|^3dx$. By applying a Strichartz estimate under the integral sign and using energy conservation we obtain

$$\begin{aligned}&\left\|\int_{t_1}^t S(t-s)(\mathrm{T}\cdot\Phi)(s,\cdot)\,ds\right\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}\\ &\qquad\le C'\int_{t_1}^t \Big\|\Phi\,\big\||\Phi|^3\big\|_{L^1(\mathbb{R})}\Big\|_{L^2(\mathbb{R})}ds\le C'\int_{t_1}^t \Big\|\Phi\,\|\Phi\|_{L^\infty(\mathbb{R})}\|\Phi\|^2_{L^2(\mathbb{R})}\Big\|_{L^2(\mathbb{R})}ds\\ &\qquad\le C''\int_{t_1}^{t_2}\|\Phi\|_{L^\infty(\mathbb{R})}ds\le C''|t_2-t_1|^{3/4}\|\Phi\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}.\end{aligned} \tag{40}$$

Combining (39) and (40) we thus obtain, by choosing $|t_2 - t_1|$ small enough (depending only on the initial energy and the geometry of (N, h)), an estimate on $\|\Phi\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}))}$, depending only on the geometry of (N, h) and the initial energy. This then implies that for all $T > 0$ we have the a priori estimate

$$\|\Phi\|_{L^4([0,T],L^\infty(\mathbb{R}_x))} < \infty.$$

Step 2. To prove global well-posedness of the flow equation (6) in $W^{2,2}(\mathbb{R}, N)$, as outlined earlier, one differentiates (24) and follows similar computations as above. The main difference from Step 1 is that now the $L^2(\mathbb{R}_x)$ norm of $\Phi_{,x}$ is no longer preserved and one needs to work in the intersection of the spaces $C^0([0, T], W^{1,2}(\mathbb{R}_x))$ and $L^4([0, T], W^{1,\infty}(\mathbb{R}_x))$. Nevertheless the nonlinearity becomes milder after differentiation and can be readily controlled using the estimate from Step 1. We carry out the details for the sake of completeness.

The differentiated equation takes the form $\sqrt{-1}(\Phi_{,x})_{,t} = (\Phi_{,x})_{,xx} + G$, where

$$|G| < C\big(|\Phi|^4 + |\Phi_{,x}|\big(|\Phi|^2 + \|\Phi\|^3_{L^3(\mathbb{R}_x)}\big)\big).$$

Using the Duhamel formula (31) we have

$$\|\Phi_{,x}\|_{C^0([t_1,t_2],L^2(\mathbb{R}_x))} < C\|\Phi_{,x}(t_1)\|_{L^2(\mathbb{R}_x)} + \|G\|_{L^1([t_1,t_2],L^2(\mathbb{R}_x))}, \tag{41}$$

and using the Duhamel formula together with a Strichartz estimate we have

$$\|\Phi_{,x}\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))} < C\|\Phi_{,x}(t_1)\|_{L^2(\mathbb{R}_x)} + \|G\|_{L^1([t_1,t_2],L^2(\mathbb{R}_x))}. \tag{42}$$

Now, we have rather large freedom in estimating G. For example,

$$\||\Phi|^4\|_{L^1([t_1,t_2],L^2(\mathbb{R}_x))} = \|\Phi\|^4_{L^4([t_1,t_2],L^8(\mathbb{R}_x))} \leq \|\Phi\|^4_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))}, \tag{43}$$

with the latter uniformly bounded from Step 1, while

$$\begin{aligned}\big\||\Phi_{,x}||\Phi|^2\big\|_{L^1([t_1,t_2],L^2(\mathbb{R}_x))} &\leq E(u_0)^{1/2}\int_{t_1}^{t_2}\|\Phi_{,x}\|_{L^\infty(\mathbb{R})}\|\Phi\|_{L^\infty(\mathbb{R})}ds \\ &\leq C\|\Phi\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))}\|\Phi_{,x}\|_{L^{4/3}([t_1,t_2],L^\infty(\mathbb{R}_x))} \\ &\leq C\|\Phi\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))}\|\Phi_{,x}\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))}|t_2-t_1|^{1/2}, \end{aligned} \tag{44}$$

and the same calculation applies also to the term $|\Phi_{,x}| \cdot \|\Phi\|^3_{L^3(\mathbb{R})}$. Plugging (43) and (44) back into (42) and choosing $|t_2 - t_1|$ small enough (depending only on the energy and the geometry (here we are using the uniform local estimate found in Step 1)) we obtain an estimate on $\|\Phi_{,x}\|_{L^4([t_1,t_2],L^\infty(\mathbb{R}_x))}$ in terms of $\|\Phi_{,x}(t_1)\|_{L^2(\mathbb{R}_x)}$, the energy, and the geometry. Using this back in (41) we then obtain an estimate on $\|\Phi_{,x}\|_{C^0([t_1,t_2],L^2(\mathbb{R}_x))}$ in terms of $\|\Phi_{,x}(t_1)\|_{L^2(\mathbb{R}_x)}$. This then implies that $\|\Phi_{,x}(t)\|_{L^2(\mathbb{R}_x)}$ increases at most exponentially in t, in particular remains finite for all $t > 0$, as desired. More precisely, for all $T > 0$,

$$\|\Phi_{,x}\|_{C^0([0,T],L^2(\mathbb{R}_x))} < C'e^{CT}, \tag{45}$$

for some uniform constants $C, C' > 0$.

Step 3. The higher derivatives estimates follow similar computations; this step is commonly called *propagation of regularity*. Essentially, each time the system of NLS equations are differentiated we obtain a new system of NLS equations where the nonlinearity is milder than in the previous stage. Hence, for initial data $u_0 \in W^{k,2}(\mathbb{R}, N)$, inductively using the estimates from the previous $k-1$ systems of equations yields an a priori estimate on $\|\Phi\|_{C^0([0,T],W^{k-1,2}(\mathbb{R}))}$ for each $k \geq 3$, again by using the auxiliary spaces $L^4([0,T], W^{k-1,\infty}(\mathbb{R}))$. This concludes the proof of Theorem 2.2. □

Remark 2.4. It is essential to use Theorem 1.1, since even after we reduce the Schrödinger map flow to a system of NLS equations and after proving that a unique global solution for (24) exists it is not completely obvious how to go from such a solution for $\nabla_x u$ to an actual map u into N.

Remark 2.5. The proof of Theorem 2.2 can also be used to obtain uniqueness of solutions for initial data in $W^{1,2}(\mathbb{R}, N)$ intersected with the appropriate Strichartz space. In particular this shows that the uniqueness result of Theorem 1.1 is not optimal.

3. Maps from the circle into a Riemann surface

In this section we consider the Schrödinger map flow with the domain being the round circle. Compared with the previous section, the discussion here is more delicate due to the fact that the domain is no longer simply-connected (introduces holonomy) nor noncompact (lack of dispersion).

Let $u : I \times S^1 \to N$ where $I \subset \mathbb{R}$ is a neighborhood of 0. The bundle $u^\star TN \to I \times S^1$ is no longer trivial and so fixing a frame satisfying (13) does not yield a trivialization. To describe the solution of (13) we work instead with $\mathbb{Z}$-invariant objects over $\mathbb{R}$. We therefore make the identifications

$$\mathrm{Maps}(S^1, N) \cong \mathrm{Maps}(\mathbb{R}, N)^{\mathbb{Z}}, \qquad \Gamma(I \times S^1, u^\star TN) \cong \Gamma(I \times \mathbb{R}, u^\star TN)^{\mathbb{Z}}, \tag{46}$$

the superscript denoting $\mathbb{Z}$-invariant objects, and take the freedom to use either one of these identifications interchangeably. Similar identifications will be made for all the other tensor bundles encountered over $I \times S^1$ (for example, $u^\star(T^\star N \otimes T^\star N \otimes T^\star N \otimes TN)$).

Recall that parallel transport is defined as a map $P : u \mapsto \mathrm{Aut}(u(0)^\star TN, u(1)^\star TN)$ for all $u \in C^\infty([0,1], N)$, which on any Kähler manifold restricts to an operator $P : C^\infty((S^1, \mathrm{pt}), N) \to \imath(U(n))$ on base-pointed loops. Formally, a solution of (13) is given by $e(t,x) = P(u(t)|_{[0,x]})e(t,0)$. This can be described somewhat more explicitly as follows.

Let U denote a contractible open set in N and let $e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$ denote a local orthonormal frame. Assume $u : I \times \mathbb{R} \to N$ is a solution of (4), a collection of loops in N which we will initially assume to be contained in U (and so, in effect, these loops are all contractible in N). Along the image of our flow we denote by $\alpha^1, \ldots, \alpha^{2n}$ the dual 1-forms to $e_1, \ldots, e_{2n}$. The Levi-Civita connection along our flow restricted to this patch is represented by a section $A_U = \Gamma^k_{ij}\alpha^i$ of $T^\star N|_U \otimes u(n)$ which pulls back to a connection form $u^\star A_U = \Gamma^k_{ij} a^i dx =: B_U dx$ for the pulled-back bundle. A section $e = E^j e_j$ of the pulled-back bundle (as in (46)) is then (locally) parallel when

$$0 = \nabla e = \frac{\partial E^j}{\partial x} e_j \otimes dx + B_U \cdot e \otimes dx = (E^j_{,x} + B_U{}^j_k E^k) e_j \otimes dx. \tag{47}$$

The solution of this first-order matrix equation simplifies considerably in the case $n = 1$. The matrices B_U then lie in the trivial Lie algebra $so(2) \cong u(1)$ and so their exponentials commute. One may therefore integrate (47) to obtain

$$e(t,1) = \exp\Big(-\int_0^1 B_U dx\Big) e(t,0). \tag{48}$$

If D_u is the disc bounded by u and contained in U, and K denotes the Gaussian curvature of N, then Stokes' Theorem gives

$$e(t,1)=\exp\Big(-\int_{D_u} dA_U\Big)e(t,0)=\exp\Big(-\int_{D_u} K\,dV_{N,h}\Big)e(t,0) \tag{49}$$

(possibly up to a factor of 2π, depending on conventions) from which it becomes evident that one may relax the assumption above (for the moment still restricting to contractible loops) and work globally (one might have two choices for D_u then). Also, we see that the holonomy factor is independent of the starting point on the loop. In fact this last fact is seen to be true for noncontractible loops as well. We have therefore a well-defined holonomy map

$$P : C^\infty((S^1,\mathrm{pt}),N)\to \mathrm{SO}(2)=\imath(U(1)).$$

Next, for general u, since $u(0,S^1)$ and $u(t,S^1)$ are homotopic for any $t\in I$ we may define the surface $D_u=u([0,t]\times S^1)$ and as chains on N $\partial D_u=u(t,S^1)-u(0,S^1)$. Let K denote the Gaussian curvature of (N,h). Then we have once again by Stokes' Theorem

$$e(t,1)=P(u)e(t,0)=\exp\Big(-\int_{D_u} K\,dV_{N,h}\Big)P(u_0)\,e(t,0),$$

or for any $x\in\mathbb{R}$ and $l\in\mathbb{N}$

$$e(t,x+l)=P(u)^l e(t,x)=\exp\Big(-l\int_{D_u} K\,dV_{N,h}\Big)P(u_0)^l e(t,0). \tag{50}$$

Therefore a solution of (13) produces a parallel section of $\Gamma(\mathbb{R},u^\star TN)$ rather than of $\Gamma(\mathbb{R},u^\star TN)^{\mathbb{Z}}$. In expressing our $\mathbb{Z}$-invariant tensors in terms of the frame $\{e_j\}_{j=1}^{2n}$ we therefore use coefficients satisfying a relation appropriately proportional to (50). For example if $v\in\Gamma(\mathbb{R},u^\star TN)^{\mathbb{Z}}$ then we may write $v=v^j e_j$ with $v^j(x+l)=P(u)^{-l}v^j(x)$ (while on the other hand sections of endomorphism tensor bundles require no adjustment when $n=1$).

The main difficulty though is that the lifted frame coefficients that live on $\mathbb{R}$ have infinite energy ($L^2(\mathbb{R})$ norm), and so our goal is to still extract an equation for objects that live on S^1, eventually.

Going through the computations of Section 2 it follows that (15) still holds. We then obtain

$$\begin{aligned}\nabla_t e_{\bar{\jmath}}(t,x)&=\sum_{q=1}^{2n} h(\nabla_t e_{\bar{\jmath}},e_q)(t,x_0)e_q(t,x)\\&+\Bigg(\sum_{\alpha,\beta}\Big(\big[a^\alpha a^{\bar\beta}R_{\alpha\beta\bar{\jmath}}{}^q+\tfrac12[a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta]R_{\bar\alpha\beta\bar{\jmath}}{}^q\big](t,x)-\big[a^\alpha a^{\bar\beta}R_{\alpha\beta\bar{\jmath}}{}^q+\tfrac12(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{\jmath}}{}^q\big](t,x_0)\Big)\\&\qquad-\int_{[x_0,x]}\sum_{\alpha,\beta}\big[a^\alpha a^{\bar\beta}R_{\alpha\beta\bar{\jmath}}{}^q{}_{,x}+\tfrac12(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{\jmath}}{}^q{}_{,x}\big](t,y)\,dy\Bigg)e_q(t,x)\end{aligned} \tag{51}$$

The terms depending on the fixed point x_0 are in a sense worse than those that depend on the variable point x since the former must be evaluated in the $L^\infty(\mathbb{R}_x)$ norm. To overcome this apparent obstacle we average over S^1 (namely, x_0 in the range $(x-1,x)$) to obtain

$$\nabla_t e_{\bar{j}}(t,x) = \sum_{q=1}^{2n}\Big(\int_{S^1} h(\nabla_t e_{\bar{j}}, e_q)(t,x_0)\,dx_0\Big)\, e_q(t,x)$$
$$+\Big[\sum_{\alpha,\beta}\Big(\big[a^\alpha a^{\bar\beta} R_{\alpha\beta\bar{j}}{}^q + \tfrac{1}{2}(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{j}}{}^q\big](t,x) - \int_{S^1}\big[a^\alpha a^{\bar\beta} R_{\alpha\beta\bar{j}}{}^q + \tfrac{1}{2}(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{j}}{}^q\big](t,x_0)\,dx_0\Big)$$
$$-\int_{[x_0,x]}\sum_{\alpha,\beta}\big[a^\alpha a^{\bar\beta} a^s R_{\alpha\beta\bar{j}}{}^q{}_{,s} + \tfrac{1}{2}a^s(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{j}}{}^q{}_{,s}\big](t,y)\,dy\Big] e_q(t,x), \quad (52)$$

which, upon setting

$$O^q_{\bar{j}}(t,x) := \int_{S^1} h(\nabla_t e_{\bar{j}}, e_q)(t,x_0)\,dx_0,$$
$$P^q_{\bar{j}}(t,x) := \sum_{\alpha,\beta}\big[a^\alpha a^{\bar\beta} R_{\alpha\beta\bar{j}}{}^q + \tfrac{1}{2}(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{j}}{}^q\big](t,x),$$
$$Q^q_{\bar{j}}(t,x) := -\int_{[x_0,x]}\sum_{\alpha,\beta}\big[a^\alpha a^{\bar\beta} a^s R_{\alpha\beta\bar{j}}{}^q{}_{,s} + \tfrac{1}{2}a^s(a^{\bar\alpha}a^{\bar\beta}+a^\alpha a^\beta)R_{\bar\alpha\beta\bar{j}}{}^q{}_{,s}\big](t,y)\,dy,$$

becomes

$$\nabla_t e_{\bar{j}}(t,x) = \big(O^q_{\bar{j}} + P^q_{\bar{j}} - \int_{S^1} P^q_{\bar{j}}(t,x_0)\,dx_0 + Q^q_{\bar{j}}\big)e_q. \quad (53)$$

Switching to complex notation, as in (24), we have

$$\sqrt{-1}\Phi_{,t} = \Phi_{,xx} - \mathrm{U}\cdot\Phi - \mathrm{S}\cdot\Phi + \mathrm{W}\cdot\Phi - \mathrm{T}\cdot\Phi, \quad (54)$$
$$\Phi^\alpha(t,x+l) = P(u(t)|_{[0,1]})^{-l}\Phi^\alpha(t,x), \quad (55)$$

where we have set

$$\mathrm{U} := \iota^{-1}(O^q_{\bar{j}}), \quad \mathrm{S} := \iota^{-1}(P^q_{\bar{j}}), \quad \mathrm{T} := \iota^{-1}(Q^q_{\bar{j}}), \quad \mathrm{W} := \iota^{-1}\Big(\int_{S^1} P^q_{\bar{j}}(t,x_0)\,dx_0\Big).$$

To estimate U we note that according to (19) we have $\nabla_t e_{\bar{j}} = a^{\bar{k}}_{,x}\Gamma^p_{k\bar{j}} e_p$, hence

$$h(\nabla_t e_{\bar{j}}, e_q) = a^{\bar{k}}_{,x}\Gamma^q_{k\bar{j}}. \quad (56)$$

Note that in (56) the left-hand side, hence also the right-hand side, are bona fide functions on S^1 (even though each term separately in the product on the right-hand side is not). Therefore, using (8),

$$\int_{S^1} h(\nabla_t e_{\bar{j}}, e_q)(t,x_0)\,dx_0 = \int_{S^1} a^{\bar{k}}_{,x}\Gamma^q_{k\bar{j}}dx_0 = -\int_{S^1} a^{\bar{k}}\Gamma^q_{k\bar{j},x}dx_0 = -\int_{S^1} a^{\bar{k}}a^p\Gamma^q_{k\bar{j},p}dx_0.$$

Hence we have the estimates

$$\|\mathrm{U}\| < C\int_{S^1}\|\Phi\|^2 dx, \quad \|\mathrm{W}\| < C\int_{S^1}\|\Phi\|^2 dx, \quad \|\mathrm{S}\| < C\|\Phi\|^2, \quad \|\mathrm{T}\| < C\int_{S^1}\|\Phi\|^3 dx, \quad (57)$$

where $C > 0$ depends only on the geometry of (N,h). We stress that (54) and (55) are equations on $I\times\mathbb{R}$.

Next, we try to derive equations that will be defined on $I \times S^1$. Define a real-valued function θ by

$$P(u(t)|_{[0,1]}) =: e^{\sqrt{-1}\,\theta(t)} \in U(1), \tag{58}$$

and set

$$a := \Phi^1 = a^1 + \sqrt{-1}\,a^{\bar{1}}.$$

Note that the holonomy factor (58) is independent of x as noted after (49). Also note that we cannot restrict θ to $[-\pi, \pi)$ in order not to violate continuity of θ.

As remarked in the paragraph after (50), the functions $\mathrm{U} = U_1^1$, $\mathrm{S} = S_1^1$, $\mathrm{W} = W_1^1$, $\mathrm{T} = T_1^1$ are $\mathbb{Z}$-invariant. Also

$$\varphi(t, x) := e^{\sqrt{-1}\,\theta x} a(t, x) \tag{59}$$

is $\mathbb{Z}$-invariant. Moreover, so are all of its x-derivatives. To wit,

$$\varphi(t, x)_{,x} = \sqrt{-1}\,\theta\varphi(t, x) + e^{\sqrt{-1}\,\theta x} a(t, x)_{,x} = \varphi(t, x+1)_{,x} \tag{60}$$

since $(e^{\sqrt{-1}\,\theta} a(t, x+1))_{,x} = a(t, x)_{,x}$, and the claim now follows by induction. It follows that the estimates we will obtain for φ will imply the same estimates for a.

Equation (54) becomes, after the change of variable (59),

$$\sqrt{-1}\,\varphi_{,t} = \varphi_{,xx} - 2\sqrt{-1}\,\theta\varphi_{,x} - \big(\theta^2 + x\theta_{,t} + Q_1^1 + S_1^1 - W_1^1 + T_1^1\big)\varphi. \tag{61}$$

Let $\beta : I \times \mathbb{R} \to I \times \mathbb{R}$ be given by

$$\beta(t, x) = \left(t,\ x - 2\int_{[0,t]} \theta\, ds\right).$$

Let

$$\tilde{x} := x + 2\int_{[0,t]} \theta\, ds.$$

Writing $(t, x) = \beta\big(t, x + 2\int_{[0,t]} \theta\, ds\big) = \beta(t, \tilde{x})$, (54) becomes

$$\begin{aligned}\sqrt{-1}(\varphi\circ\beta)_{,t}(t, \tilde{x}) &= (\varphi\circ\beta)_{,\tilde{x}\tilde{x}}(t, \tilde{x}) \\ &\quad - \big(\theta^2(t) + (\tilde{x} - 2\textstyle\int_{[0,t]}\theta\, ds)\theta_{,t}(t) + (Q_1^1\circ\beta + S_1^1\circ\beta - W_1^1\circ\beta + T_1^1\circ\beta)(t, \tilde{x})\big)(\varphi\circ\beta)(t, \tilde{x}).\end{aligned} \tag{62}$$

This equation is on $I \times S^1$.

Remark 3.1. Note that here it was crucial that θ does not depend on x in order to have $\partial\tilde{x}/\partial x = 1$. This is also the difference from the situation in (29).

The main result of this section is the following a priori estimate:

Theorem 3.2. *Let $(M, g) = (S^1, dx\otimes dx)$ and let (N, J, h) be a complete Riemann surface with bounded geometry. Given $u_0 \in W^{2,2}(S^1, N)$, the solution $\varphi(t, x)$ of the system of NLS equations (62) satisfies for all $T > 0$ the a priori estimate*

$$\|\varphi\|_{L^4([0,T], L^4(S^1, \mathbb{R}^{2n}))} < \infty.$$

This will be shown to imply:

Corollary 3.3. *Let $(M, g) = (S^1, dx \otimes dx)$ and let (N, J, h) be a complete Riemann surface with bounded geometry. Then for integers $k \geq 2$ the flow equation (4) with $u_0 \in W^{k,2}(S^1, N)$ admits a unique solution $u \in C^0(\mathbb{R}, W^{k,2}(S^1, N))$.*

Proof of Theorem 3.2. We will use (62) to obtain a priori estimates on

$$\tilde{\varphi}(t,\tilde{x}) := \varphi \circ \beta(t,\tilde{x}).$$

The estimates on $\tilde{\varphi}$ and on φ are equivalent since the two functions only differ by a time-dependent translation in the space direction. We will localize in time: Indeed it is enough to prove local (in time) a priori estimates for solutions of (62) in $C^0(\mathbb{R}_{t,loc}, L^2(S^1)) \cap L^4(\mathbb{R}_{t,loc} \times S^1)$ depending in a good manner only on $\|\tilde{\varphi}\|_{L^2(S^1)} = E(u_0)^{1/2}$ and a bounded constant depending on time, since that will rule out finite-time blow-up.

We now recall some work of Bourgain that will be of central importance later (see also [Ginibre 1996] for an exposition). We start with some Fourier restriction estimates:

Lemma 3.4 [Bourgain 1993, page 112]. *Let φ be a periodic solution of the linear Schrödinger equation on S^1. Then*

$$\|\varphi\|_{L^4(S^1\times S^1)} \le \sqrt{2}\|\varphi(0)\|_{L^2(S^1)},$$

and dually

$$\|\varphi\|_{L^2(S^1\times S^1)} \le \sqrt{2}\|\varphi\|_{L^{4/3}(S^1\times S^1)}.$$

More generally, Bourgain proved the following fundamental result that allows for the same estimate — now with appropriate weights — even for an arbitrary function whose Fourier modes are not necessarily restricted to the parabola $\{(p, p^2) : p \in \mathbb{Z}\}$. We state the result although we will only directly use a consequence of it.

Lemma 3.5 [Bourgain 1993, Proposition 2.33]. *Let $f(x,t) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{\sqrt{-1}(mx+nt)}$ be a function on $S^1 \times S^1$. Then*

$$\Big(\sum_{m,n\in\mathbb{Z}} \big(|n-m^2|+1\big)^{-3/4} |a_{m,n}|^2\Big)^{1/2} \le C\|f\|_{L^{4/3}(S^1\times S^1)}.$$

In addition, if $|\lambda_{m,n}| \le (1+|n-m^2|)^{-3/4}$, then

$$\Big\|\sum_{m,n\in\mathbb{Z}} \lambda_{m,n} a_{m,n} e^{\sqrt{-1}(mx+nt)}\Big\|_{L^4(S^1\times S^1)} \le C\|f\|_{L^{4/3}(S^1\times S^1)}.$$

In both estimates $C > 0$ is some universal constant.

Using this estimate, Bourgain obtains the following L^4 estimate for the nonlinear contribution in Duhamel's formula. This estimate will play a central role below. Let $f(x) = \sum_{m\in\mathbb{Z}} a_m e^{\sqrt{-1}\,mx} \in L^2(S^1)$. On $M = S^1$ the Schrödinger operator (see (32)) takes the form

$$(S(t)f)(x) = \sum_{m\in\mathbb{Z}} a_m e^{\sqrt{-1}(mx+m^2 t)}. \tag{63}$$

Lemma 3.6 [Bourgain 1993, §4]. *Let $F \in L^{4/3}(S^1 \times S^1)$. For any $0 < \delta < 1/8$ and $0 < B < \frac{1}{100\delta}$ there holds*

$$\Big\|\int_0^{2\delta} S(t-\tau)F(\tau,x)\,d\tau\Big\|_{L^4(S^1\times S^1)} \le C(B^{-1/4} + \delta B)\|F\|_{L^{4/3}(S^1\times S^1)},$$

where $C > 0$ is some universal constant.

The constant B can be thought of as a Fourier mode cut-off parameter, measuring distance of a lattice point in $\mathbb{Z}^2$ from the parabola $\{(m, m^2) : m \in \mathbb{Z}\}$. The constant δ is the time cut-off parameter.

Equation (62) is equivalent to the integral equation

$$\tilde{\varphi}(t, \tilde{x}) = S(t)\tilde{\varphi}(0, \tilde{x}) - \sqrt{-1}\int_0^t S(t-\tau)F(\tau, \tilde{x})\,d\tau, \tag{64}$$

with

$$F(\tau, \tilde{x}) = -\big(\theta^2(\tau) + (\tilde{x} - 2\textstyle\int_{[0,\tau]}\theta ds)\theta_{,t}(\tau) + (Q_1^1\circ\beta + S_1^1\circ\beta - W_1^1\circ\beta + T_1^1\circ\beta)(\tau, \tilde{x})\big)\tilde{\varphi}(\tau, \tilde{x}). \tag{65}$$

There is a subtlety here: The time derivative of $\tilde{\varphi}$ (or of φ) is not necessarily $\mathbb{Z}$-invariant (in $\tilde{x}$). However, (62) holds on $I \times S^1$ and it is equivalent to the integral equation (64).

We would like to obtain an a priori L^4 estimate on $\tilde{\varphi}$. We localize in time, namely multiply (64) by a smooth cut-off function in time $\psi(t)$ satisfying $\psi = 1$ on $[-\delta, \delta]$ and $\psi = 0$ for $|t| \geq 2\delta$. Here δ is a positive number smaller than $1/8$ to be specified later. We may thus regard $\psi\tilde{\varphi}$ as a function on $S^1 \times S^1$ with period 1 in both the t and $\tilde{x}$ variables and Bourgain's estimates apply.

First, the linear term satisfies

$$\|\psi S(t)\tilde{\varphi}(0, \tilde{x})\|_{L^4(S^1\times S^1)} \leq \sqrt{2}\|\tilde{\varphi}(0, \cdot)\|_{L^2(S^1)} = \sqrt{2E(u_0)},$$

according to Lemma 3.4.

Next, we estimate the integral term. The terms involving Q and W are simpler since |Q| and |W| are uniformly bounded according to (57) and conservation of energy.

We now turn to the other terms. First, using Lemma 3.4 under the integral sign, and assuming $\|\theta\|_{L^\infty} \leq C$, we have

$$\left\|\psi\int_0^t S(t-\tau)\big(\theta^2(\tau)\tilde{\varphi}(\tau, \tilde{x})\big)\,d\tau\right\|_{L^4(S^1\times S^1)} \leq \int_0^{2\delta}\|\theta^2(\tau)\tilde{\varphi}(\tau, \cdot)\|_{L^2(S^1)}d\tau \leq 2C^2\delta\|\varphi(0)\|_{L^2(S^1)}. \tag{66}$$

To show that this assumption holds, use the representation of the holonomy given by (48): $|\theta(t)| \leq \int_0^1 |\Gamma_{ij}^k||a^i|dx \leq C'E(u_0)^{1/2}$, where we have used the assumption of bounded geometry — indeed it implies that Christoffel symbols are uniformly bounded [Eichhorn 1991].

Second, $|\tilde{x}| \leq 1$ and so $|\tilde{x} - 2\int_{[0,\tau]}\theta\,ds| \leq 1 + 2\cdot 1\cdot C$. Let $\{\alpha_1, \alpha_{\bar{1}}\}$ be an orthonormal coframe dual to $\{e_1, e_{\bar{1}}\}$. To compute the time derivative of θ, recall that by (50) we have

$$\theta(t) = \int_{D_u} K\,dV_{N,h} = \int_{D_u} K\alpha_1\wedge\alpha_{\bar{1}} = \int_{I\times S^1} K\circ u(t, x)[a^1b^{\bar{1}} - a^{\bar{1}}b^1]\,dx\wedge dt,$$

since $u^\star\alpha_1 = a^1dx + b^1dt$, $u^\star\alpha_{\bar{1}} = a^{\bar{1}}dx + b^{\bar{1}}dt$. Combining this with the equality $b^k = a^{\bar{k}}_{,x}$ given by (9), we have

$$\theta_{,t} = \int_{S^1} K\circ u(t, x)(a^1b^{\bar{1}} - a^{\bar{1}}b^1)\,dx = -\tfrac{1}{2}\int_{S^1} K\circ u(t, x)((a^1)^2 + (a^{\bar{1}})^2)_{,x}dx.$$

Integrating by parts this becomes

$$\theta_{,t} = \tfrac{1}{2}\int_{S^1}(K\circ u(t, x))_{,x}((a^1)^2 + (a^{\bar{1}})^2)\,dx = \tfrac{1}{2}\int_{S^1} K_{,s}\circ u(t, x)a^s((a^1)^2 + (a^{\bar{1}})^2)\,dx.$$

By bounded geometry we therefore have

$$\|\theta_{,t}\|_{L^\infty} \le C\|a\|^3_{L^3(S^1)}. \tag{67}$$

Therefore the term $\big(\tilde{x} - 2\int_{[0,\tau]}\theta\, ds\big)\theta_{,t}(\tau)\tilde{\varphi}(\tau,\tilde{x})$ behaves in the same way as the term $T_1^1 \circ \beta\tilde{\varphi}(\tau,\tilde{x})$ in (65), and so it's enough to treat the latter. We will do that shortly.

Third, $|S_1^1 \circ \beta \cdot \tilde{\varphi}| < C|\tilde{\varphi}|^3$, and therefore this term may be estimated in $L^4(S^1 \times S^1)$ just like in Bourgain's estimates for a cubic nonlinearity. More precisely, by Lemma 3.6 we have

$$\left\| \psi \int_0^t S(t-\tau)\big(|\tilde{\varphi}|^3(\tau,\tilde{x})\big)\, d\tau \right\|_{L^4(S^1\times S^1)} \le C(\delta B + B^{-1/4})\|\psi\tilde{\varphi}\|^3_{L^4(S^1\times S^1)}, \tag{68}$$

with $B > 0$ as in the lemma, δ is as before the time cut-off parameter, and $C > 0$ is a uniform constant.

Fourth, using Lemma 3.4 and energy conservation we have

$$\begin{aligned} \left\| \psi \int_0^t S(t-\tau)\Big(\tilde{\varphi}\int_{S^1}|\tilde{\varphi}|^3 d\tilde{x}(\tau,\tilde{x})\Big)\, d\tau \right\|_{L^4(S^1\times S^1)} &\le C\int_0^{2\delta} \left\| \tilde{\varphi}(\tau,\,\cdot\,)\int_{S^1}|\tilde{\varphi}(\tau,\,\cdot\,)|^3 d\tilde{x} \right\|_{L^2(S^1)} d\tau \\ &\le C'\int_0^{2\delta}\int_{S^1}|\tilde{\varphi}(\tau,\,\cdot\,)|^3 d\tilde{x}\, d\tau \\ &\le C''\delta^{1/4}\|\varphi\|^3_{L^4(S^1\times S^1)}. \end{aligned} \tag{69}$$

Combining Equations (66)–(69) we have

$$\|\psi\tilde{\varphi}\|_{L^4([0,2\delta]\times S^1)} \le C\big((1+\delta)\|\varphi\|_{L^2(S^1)} + (\delta B + B^{-1/4} + \delta^{1/4})\|\psi\tilde{\varphi}\|^3_{L^4([0,2\delta]\times S^1)}\big). \tag{70}$$

In fact, due to energy conservation, the time interval may be taken to be $[t_1, t_1+2\delta]$ for any $t_1 \in \mathbb{R}$. Now, by choosing B large enough and then choosing δ small enough, in such a manner that δB is also small enough (all of these choices depend only on the initial energy and the geometry) we therefore may argue similarly to Bourgain to obtain a uniform estimate on $\|\tilde{\varphi}\|_{L^4([t_1,t_1+2\delta]\times S^1)} = \|\varphi\|_{L^4([t_1,t_1+2\delta]\times S^1)} = \|a\|_{L^4([t_1,t_1+2\delta]\times S^1)}$:

$$\|\varphi\|_{L^4([t_1,t_1+2\delta]\times S^1)} < C, \tag{71}$$

where $C > 0$ is a uniform constant depending only on the initial energy and the geometry. We therefore obtain the global a priori estimate

$$\|a\|_{L^4([0,T]\times S^1)} < \infty,$$

for all $T > 0$. This concludes the proof of Theorem 3.2. □

Remark 3.7. A difference between our situation and that of Bourgain [1993, page 139] is that while Bourgain actually proves the existence (and uniqueness) of a local solution of the periodic cubic NLS in $L^4(\mathbb{R}_{t,loc} \times S^1)$ using energy conservation, we only need to prove an a priori estimate in this norm for the unique local solution given by Theorem 1.1.

Conclusion of the proof of Corollary 3.3. To obtain global well-posedness for our original flow equation in $W^{k,2}$ we take $k-1$ derivatives of (62). In fact, we obtain certain terms that are worse than those that arise when one differentiates the cubic NLS. For example, for $k = 2$ we obtain several extra terms the worst of which are of order $|\tilde{\varphi}|^4$ and $|\tilde{\varphi}_{,x}| \cdot \|\tilde{\varphi}\|^3_{L^3(S^1)}$. Such terms may be handled nevertheless. We carry out the computations in detail in the case $k = 2$, omitting the details in the case $k \geq 3$ as they are similar (see the remarks in Step 3 of Section 2).

Set $w := \tilde{\varphi}_{,\tilde{x}}$. Taking a derivative of (62) we obtain

$$\sqrt{-1}\, w_{,t} = w_{,\tilde{x}\tilde{x}} + G, \tag{72}$$

where

$$|G| < C\big(|\tilde{\varphi}|^4 + |\tilde{\varphi}|^2 + |w|\big(1 + |\tilde{\varphi}|^2 + \|\tilde{\varphi}\|^3_{L^3(S^1)}\big)\big). \tag{73}$$

As before we would like to obtain an $L^4(\mathbb{R}_{t,loc} \times S^1)$ estimate, this time for w. We use Lemma 3.4 in order to handle the term $|\tilde{\varphi}|^4$ (more precisely, the corresponding term in the Duhamel formula); the corresponding contribution is bounded by

$$C \int_{t_1}^{t_1+2\delta} \|\varphi(\tau,\,\cdot\,)^4\|_{L^2(S^1)} d\tau.$$

Using the Gagliardo–Nirenberg inequality [Aubin 1998, page 93], we have

$$\|\varphi\|^4_{L^8(S^1)} \leq C\big(1 + \|\varphi_{,x}\|^{1/2}_{L^4(S^1)}\big)\big(1 + \|\varphi\|^{7/2}_{L^4(S^1)}\big), \tag{74}$$

Note that the Gagliardo–Nirenberg inequality as cited requires $\int_{S^1} \varphi\, dx = 0$; nevertheless we know that $\|\varphi\|_{L^1(S^1)}$ is uniformly bounded in time due to the Cauchy–Schwarz inequality and conservation of energy, and so (74) holds in our case. It is enough to treat the worst term on the right-hand side of (74), namely the term $\|\varphi_{,x}\|^{1/2}_{L^4(S^1)}\|\varphi\|^{7/2}_{L^4(S^1)}$. The Hölder inequality and (71) give

$$\int_{t_1}^{t_1+2\delta} \|\varphi_{,x}\|^{1/2}_{L^4(S^1)}\|\varphi\|^{7/2}_{L^4(S^1)} ds \leq \|\varphi_{,x}\|^{1/2}_{L^4([t_1,t_1+2\delta]\times S^1)}\|\varphi\|^{7/2}_{L^4([t_1,t_1+2\delta]\times S^1)} \leq C\|\varphi_{,x}\|^{1/2}_{L^4([t_1,t_1+2\delta]\times S^1)}. \tag{75}$$

Since this is sublinear in the norm we are estimating it will be possible to use this inequality to obtain the a priori estimate we are after. Next, of course the nonlinear term $|\tilde{\varphi}|^2$ in (73) is even easier to handle:

$$\|\varphi^2\|_{L^1([t_1,t_1+2\delta],L^2(S^1))}. \leq C\sqrt{\delta}\|\varphi\|_{L^4([t_1,t_1+2\delta],L^4(S^1))} \leq C'\sqrt{\delta}. \tag{76}$$

Let us now treat the other nonlinearities. To handle the contribution of the term $|\tilde{\varphi}^2\tilde{\varphi}_{,x}|$ to the Duhamel formula, we apply Lemma 3.6 and the Hölder inequality

$$\begin{aligned}\left\|\int_{t_1}^{t} S(t-\tau)(\varphi^2\varphi_{,x})(\tau,\,\cdot\,)\, d\tau\right\|_{L^4([t_1,t_1+2\delta]\times S^1)} &\leq C(\delta B + B^{-1/4})\|\varphi^2\varphi_{,x}\|_{L^{4/3}([t_1,t_1+2\delta]\times S^1)}\\ &\leq C(\delta B + B^{-1/4})\|\varphi_{,x}\|_{L^4([t_1,t_1+2\delta]\times S^1)}\|\varphi\|^2_{L^4([t_1,t_1+2\delta]\times S^1)};\end{aligned} \tag{77}$$

this term can be controlled by a small uniform constant times $\|\varphi_{,x}\|_{L^4([t_1,t_1+2\delta]\times S^1)}$, by choosing δ, B appropriately.

Next, using Lemma 3.4, the contribution of the term $|\tilde{\varphi}_{,x}| \cdot \|\tilde{\varphi}\|^3_{L^3(S^1)}$ to the Duhamel formula can be bounded as follows:

$$\left\|\int_{t_1}^{t} S(t-\tau)(|\tilde{\varphi}_{,x}| \cdot \|\tilde{\varphi}\|^3_{L^3(S^1)})(\tau,\,\cdot\,)\, d\tau\right\|_{L^4([t_1,t_1+2\delta]\times S^1)} \leq C\int_{t_1}^{t_1+2\delta} \big\||\varphi_{,x}| \cdot |\varphi\|^3_{L^3(S^1)}\big\|_{L^2(S^1)} dt, \tag{78}$$

and using the interpolation inequality $\|\varphi\|^3_{L^3(S^1)} \le \|\varphi\|^2_{L^4(S^1)}\|\varphi\|_{L^2(S^1)} \le C\|\varphi\|^2_{L^4(S^1)}$, this may be estimated as follows:

$$\begin{aligned}(78) &\le C' \int_{t_1}^{t_1+2\delta} \|\varphi_{,x}\|_{L^2(S^1)}\|\varphi\|^2_{L^4(S^1)} dt \le C'' \int_{t_1}^{t_1+2\delta} \|\varphi_{,x}\|_{L^4(S^1)}\|\varphi\|^2_{L^4(S^1)} dt \\ &\le C'' ||\varphi_{,x}||_{L^4([t_1,t_1+2\delta]\times S^1)} \Big(\int_{t_1}^{t_1+2\delta} \|\varphi\|^{8/3}_{L^4(S^1)} dt \Big)^{3/4} \\ &\le C''(2\delta)^{1/4}\|\varphi_{,x}\|_{L^4([t_1,t_1+2\delta]\times S^1)}\|\varphi\|^2_{L^4([t_1,t_1+2\delta]\times S^1)} \le C'''\delta^{1/4}\|\varphi_{,x}\|_{L^4([t_1,t_1+2\delta]\times S^1)}. \qquad (79)\end{aligned}$$

To summarize, combining (75), (76), (77), and (79), we may thus find $\delta, C > 0$, depending only on the initial energy and the geometry, for which (71) still holds and for which we also have the a priori estimate

$$\|\varphi_{,x}\|_{L^4([t_1,t_1+2\delta]\times S^1)} \le C(1 + \|\varphi_{,x}(t_1)\|_{L^2(S^1)}). \qquad (80)$$

The main point here is that δ does not depend on $\|\varphi_{,x}(t_1)\|_{L^2(S^1)}$.

We now need to "close" the argument by estimating the $L^\infty([t_1, t_1+2\delta], L^2(S^1))$ norm of $\varphi_{,x}$, making use of the auxiliary estimate (80). For each $t \in [t_1, t_1 + 2\delta]$ we have (see (72))

$$\begin{aligned}\|\varphi_{,x}\|_{L^\infty([t_1,t_1+2\delta],L^2(S^1))} &\le C\|\varphi_{,x}(t_1)\|_{L^2(S^1)} + \big\| \|G\|_{L^1([t_1,t],L^2(S^1))} \big\|_{L^\infty([t_1,t_1+2\delta])} \\ &= C\|\varphi_{,x}(t_1)\|_{L^2(S^1)} + \|G\|_{L^1([t_1,t_1+2\delta],L^2(S^1))}. \qquad (81)\end{aligned}$$

With the exception of the term $|\varphi|^2|\varphi_{,x}|$, we have already estimated all of the nonlinearities occurring in G in $L^1([t_1, t_1 + 2\delta], L^2(S^1))$ in the process of proving (80) (see (75), (76), and (79)). Let us now estimate that term in this norm. Using the Gagliardo–Nirenberg inequality (together with the remark following (74)), energy conservation and the Hölder inequality we have

$$\begin{aligned}\int_{t_1}^{t_1+2\delta} \|\varphi^2\varphi_{,x}\|_{L^2(S^1)} d\tau &\le \int_{t_1}^{t_1+2\delta} \|\varphi\|_{L^\infty(S^1)}\|\varphi\|_{L^4(S^1)}\|\varphi_{,x}\|_{L^4(S^1)} d\tau \\ &\le C \int_{t_1}^{t_1+2\delta} \big(1 + \|\varphi\|^{1/2}_{L^2(S^1)}\big)\big(1 + \|\varphi_{,x}\|^{1/2}_{L^2(S^1)}\big)\|\varphi\|_{L^4(S^1)}\|\varphi_{,x}\|_{L^4(S^1)} d\tau. \qquad (82)\end{aligned}$$

As earlier, it suffices to estimate the worst term on the right-hand side. Namely, we estimate

$$\begin{aligned}\int_{t_1}^{t_1+2\delta} \|\varphi\|^{1/2}_{L^2(S^1)}\|\varphi_{,x}\|^{1/2}_{L^2(S^1)}\|\varphi\|_{L^4(S^1)}\|\varphi_{,x}\|_{L^4(S^1)} d\tau &\le C' \int_{t_1}^{t_1+2\delta} \|\varphi\|_{L^4(S^1)}\|\varphi_{,x}\|^{3/2}_{L^4(S^1)} d\tau \\ &\le C''\delta^{3/8}\|\varphi\|_{L^4([t_1,t_1+2\delta]\times S^1)}\|\varphi_{,x}\|^{3/2}_{L^4([t_1,t_1+2\delta]\times S^1)}. \qquad (83)\end{aligned}$$

It follows that

$$\|\varphi_{,x}\|_{L^\infty([t_1,t_1+2\delta],L^2(S^1))} \le C\big(1 + \|\varphi_{,x}(t_1)\|_{L^2(S^1)} + \|\varphi_{,x}(t_1)\|^{3/2}_{L^2(S^1)}\big). \qquad (84)$$

Therefore there exists uniform constants $C, C', C'' > 0$ such that for all $T > 0$,

$$\|\varphi_{,x}\|_{L^\infty([0,T],L^2(S^1))} < C'' e^{C' e^{CT}} < \infty. \qquad (85)$$

This concludes the proof of Corollary 3.3. □

4. Maps of the circle into a Kähler manifold

In this section we explain the difficulties encountered when one tries to apply the same methods to treat the case of maps from the circle to Kähler manifolds of arbitrary dimension $n \geq 1$.

For general n, one gets an expression for a solution of (13) given by the chronological exponential

$$\begin{aligned} e(t, x+1) &= A^{-1}(t, x)e(t, x) \\ &:= \lim_{n\to\infty} \exp\big(-\tfrac{1}{n} B_U(t, x+1)\big) \exp\big(-\tfrac{1}{n} B_U(t, x + \tfrac{n-1}{n})\big) \cdots \exp\big(-\tfrac{1}{n} B_U(t, x + \tfrac{1}{n})\big) e(t, 0); \quad (86) \end{aligned}$$

see for example [Dubrovin et al. 1985]. Applying ∇_x to (86) and using the fact that $\nabla_x e = 0$, we obtain $A_{,x} = 0$, that is, $A(t, x)$ does not depend on x. From now on we simply write $A(t)$.

Now $\Phi(t, x+1) = A(t)\Phi(t, x)$. Since $A(t)$ is unitary (and hence normal) it is unitarily diagonalizable and we set

$$A(t) = U(t)^\star D(t) U(t), \qquad \text{with } U(t) \in U(n) \text{ and } D(t) = \operatorname{diag}(e^{\sqrt{-1}\,\theta_1}, \dots, e^{\sqrt{-1}\,\theta_n}).$$

The vector-valued function $\tilde{\Phi}$ defined by

$$\tilde{\Phi}(t, x) := U(t)^\star D(t)^{-x} U(t)\Phi(t, x) = A(t)^{-x}\Phi(t, x).$$

is periodic in x. Moreover, by a computation similar to (60), so are all of its x-derivatives. We have

$$\begin{aligned} \Phi_{,t} &= (A(t)^x)_{,t}\tilde{\Phi} + A(t)^x \tilde{\Phi}_{,t}, \\ \Phi_{,x} &= U(t)^\star D(t)^x \operatorname{diag}(\sqrt{-1}\,\theta_i) U(t)\tilde{\Phi} + U(t)^\star D(t)^x U(t)\tilde{\Phi}_{,x}, \\ \Phi_{,xx} &= U(t)^\star D(t)^x \operatorname{diag}(-\theta_i^2) U(t)\tilde{\Phi} + 2U(t)^\star D(t)^x \operatorname{diag}(\sqrt{-1}\,\theta_i) U(t)\tilde{\Phi}_{,x} + U(t)^\star D(t)^x U(t)\tilde{\Phi}_{,xx}. \end{aligned}$$

It follows that

$$\begin{aligned} \sqrt{-1}\Phi_{,t} - \Phi_{,xx} = A(t)^x\big[&\sqrt{-1}\tilde{\Phi}_{,t} - \tilde{\Phi}_{,xx} + (A(t)^x)_{,t}\tilde{\Phi} \\ &- U(t)^\star \operatorname{diag}(-\theta_i^2) U(t)\tilde{\Phi} - 2U(t)^\star \operatorname{diag}(\sqrt{-1}\,\theta_i) U(t)\tilde{\Phi}_{,x}\big]. \quad (87) \end{aligned}$$

Therefore equations (54)–(55) may be rewritten as

$$\begin{aligned} \sqrt{-1}\,\tilde{\Phi}_{,t} = \tilde{\Phi}_{,xx} - (A(t)^x)_{,t}\tilde{\Phi} &+ U(t)^\star \operatorname{diag}(-\theta_i^2) U(t)\tilde{\Phi} \\ &+ 2U(t)^\star \operatorname{diag}(\sqrt{-1}\,\theta_i) U(t)\tilde{\Phi}_{,x} - A(t)^{-x}\big(\mathrm{Q}\cdot\Phi + \mathrm{S}\cdot\Phi - \mathrm{W}\cdot\Phi + \mathrm{T}\cdot\Phi\big). \quad (88) \end{aligned}$$

Note that the last term is expressed in terms of Φ instead of $\tilde{\Phi}$. However as far as the estimates are concerned this is not important since it involves no derivatives and the two vectors differ by a unitary transformation. Two problems now arise. First, one needs to obtain an estimate on the variation of the holonomy matrix $A(t)$ along the flow. Such an estimate was available in the one-dimensional setting due to the Gauss-Bonnet theorem. Second, the matrix multiplying the first derivative term is not diagonal and so it is not clear how to eliminate this term.

Although this requires some work, and we will not attempt to provide the details here, the first difficulty may likely be overcome using the theory developed by Chacon and Fomenko [1991] for a noncommutative version of the Stokes' Theorem for product integrals (see also the classical references [Nijenhuis

1953; Schlesinger 1928]). To approach the second difficulty one may consider $\hat{\Phi} := U(t)^{\star} D(t)^{-x} \Phi$ instead of $\tilde{\Phi}$. Then the matrix multiplying the first derivative of $\hat{\Phi}$ is diagonal. Therefore, we may apply the space-time transformation as in the Riemann surface case, however for each equation in the system separately. However, this introduces a new obstacle. Indeed, then one needs to control the time derivative of $D(t)$ as well as of $U(t)$. The main difficulty comes from the latter. In general, the unitary diagonalizing matrix does not vary smoothly (or even continuously) even when a family of matrices does [Kato 1966, page 111]. Instead one may try to diagonalize $A(t)$ smoothly. However, to the best of our knowledge, even given such a diagonalization, the problem is that even though the diagonalizing matrix is then smooth one has essentially no control over its derivatives (that is, estimates on these derivatives in terms of derivatives of $A(t)$). We hope to come back to this problem in the future. In some sense the two transformations (to $\tilde{\Phi}$ and to $\hat{\Phi}$) are dual to each other, and one may ask whether for higher-dimensional domains the two troublesome terms, namely the first derivative term and the derivative of the holonomy, may be a source for finite-time blow-up.

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References

[Aubin 1998] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer, Berlin, 1998. MR 99i:58001 Zbl 0896.53003

[Bejenaru et al. 2007] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, "Global existence and uniqueness of Schrödinger maps in dimensions $d \geq 4$", *Adv. Math.* **215**:1 (2007), 263–291. MR 2008i:35124 Zbl 1152.35049

[Bejenaru et al. 2008] I. Bejenaru, A. D. Ionescu, C. E. Kenig, and D. Tataru, "Global Schrödinger maps", preprint, 2008. arXiv 0807.0265

[Bourgain 1993] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations", *Geom. Funct. Anal.* **3**:2 (1993), 107–156. MR 95d:35160a Zbl 0787.35097

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, Courant Institute, New York, 2003. MR 2004j:35266 Zbl 1055.35003

[Chacon and Fomenko 1991] R. V. Chacon and A. T. Fomenko, "Stokes' formula for Lie algebra valued connection and curvature forms", *Adv. Math.* **88**:2 (1991), 258–300. MR 92m:53116 Zbl 0741.53021

[Chang et al. 2000] N.-H. Chang, J. Shatah, and K. Uhlenbeck, "Schrödinger maps", *Comm. Pure Appl. Math.* **53**:5 (2000), 590–602. MR 2001a:35161 Zbl 1028.35134

[Chihara 2008] H. Chihara, "Schrödinger flow into almost Hermitian manifolds", preprint, 2008. arXiv 0807.3395

[Ding 2002] W.-Y. Ding, "On the Schrödinger flows", pp. 283–291 in *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), vol. 2, Higher Education Press, 2002.

[Ding and Wang 1998] W. Ding and Y. Wang, "Schrödinger flow of maps into symplectic manifolds", *Sci. China Ser. A* **41**:7 (1998), 746–755. MR 99m:58062 Zbl 0918.53017

[Ding and Wang 2001] W. Ding and Y. Wang, "Local Schrödinger flow into Kähler manifolds", *Sci. China Ser. A* **44**:11 (2001), 1446–1464. MR 2002k:53129 Zbl 1019.53032

[Dubrovin et al. 1985] B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern geometry—methods and applications*, vol. II, Graduate Texts in Math. **104**, Springer, New York, 1985. MR 86m:53001 Zbl 0565.57001

[Eells and Sampson 1964] J. Eells, Jr. and J. H. Sampson, "Harmonic mappings of Riemannian manifolds", *Amer. J. Math.* **86** (1964), 109–160. MR 29 #1603 Zbl 0122.40102

[Eichhorn 1991] J. Eichhorn, "The boundedness of connection coefficients and their derivatives", *Math. Nachr.* **152** (1991), 145–158. MR 92k:53069 Zbl 0736.53031

[Eidelman and Zhitarashu 1998] S. D. Eidelman and N. V. Zhitarashu, *Parabolic boundary value problems*, Operator Theory: Advances and Applications **101**, Birkhäuser, Basel, 1998. MR 99h:35080 Zbl 0893.35001

[Ginibre 1996] J. Ginibre, "Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain)", pp. 163–187 (Exp. 796) in *Séminaire Bourbaki, 1994/95*, Astérisque **237**, Soc. Mat. de France, Paris, 1996. MR 98e:35154 Zbl 0870.35096

[Kato 1966] T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften **132**, Springer, New York, 1966. MR 34 #3324

[Landau and Lifschitz 1935] L. D. Landau and E. M. Lifschitz, "On the theory of the dispersion of magnetic permeability in ferromagnetic bodies", *Phys. Z. Sow.* **8** (1935), 153–169. Reprinted as pp. 101–114 in *Collected papers of L. D. Landau*, Gordon and Breach, New York (1967), and as pp. 14–22 in a special issue of *Ukr. J. Physics* **53** (2008). Zbl 0565.57001

[McGahagan 2004] H. McGahagan, *Some existence and uniqueness results for Schroedinger maps and Landau–Lifshitz–Maxwell equations*, Ph.D. thesis, New York University, 2004.

[McGahagan 2007] H. McGahagan, "An approximation scheme for Schrödinger maps", *Comm. Partial Differential Equations* **32**:1-3 (2007), 375–400. MR 2009e:35265 Zbl 1122.35138

[Nijenhuis 1953] A. Nijenhuis, "On the holonomy groups of linear connections. IA. General properties of affine connections", *Nederl. Akad. Wetensch. Proc. Ser. A.* **56** = *Indagationes Math.* **15** (1953), 233–240. MR 16,171c Zbl 0051.13203

[Pang et al. 2001] P. Y. H. Pang, H.-Y. Wang, and Y.-D. Wang, "Schrödinger flow for maps into Kähler manifolds", *Asian J. Math.* **5**:3 (2001), 509–533. MR 2003h:53092 Zbl 1021.53039

[Pang et al. 2002] P. Y. H. Pang, H.-Y. Wang, and Y.-D. Wang, "Schrödinger flow on Hermitian locally symmetric spaces", *Comm. Anal. Geom.* **10**:4 (2002), 653–681. MR 2003i:53102 Zbl 1032.53057

[Schlesinger 1928] L. Schlesinger, "Parallelverschiebung und Krümmungstensor", *Math. Ann.* **99** (1928), 413–434. MR 1512459 JFM 54.0766.04

[Sulem et al. 1986] P.-L. Sulem, C. Sulem, and C. Bardos, "On the continuous limit for a system of classical spins", *Comm. Math. Phys.* **107**:3 (1986), 431–454. MR 87k:82024 Zbl 0614.35087

[Terng and Uhlenbeck 2006] C.-L. Terng and K. Uhlenbeck, "Schrödinger flows on Grassmannians", pp. 235–256 in *Integrable systems, geometry, and topology*, edited by C.-L. Terng, AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, RI, 2006. MR 2007a:37079 Zbl 1110.37056

[Zhou et al. 1991] Y. L. Zhou, B. L. Guo, and S. B. Tan, "Existence and uniqueness of smooth solution for system of ferromagnetic chain", *Sci. China Ser. A* **34**:3 (1991), 257–266. MR 92i:82030 Zbl 0752.35074

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