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WELL- AND ILL-POSEDNESS ISSUES FOR ENERGY SUPERCRITICAL WAVES

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We investigate the initial value problem for some energy supercritical semilinear wave equations. We establish local existence in suitable spaces with continuous flow. The proof uses the finite speed of propagation and a quantitative study of the associated ODE. It does not require any scaling invariance of the equation. We also obtain some ill-posedness and weak ill-posedness results.

1. Introduction

In this work, we discuss some well-posedness issues of the Cauchy problem associated to the semilinear wave equation

$$\partial_t^2 u - \Delta u + F'(u) = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^d, \tag{1}$$

where $d \geq 2$ and $F : \mathbb{R} \to \mathbb{R}$ is an *even* regular function satisfying

$$F(0) = F'(0) = 0 \quad \text{and} \quad uF'(u) \geq 0. \tag{2}$$

These assumptions on F include the massive case, that is, the Klein–Gordon equation. With hypothesis (2), one can construct a global weak solution with finite energy data using a standard compactness argument; see, for example [Strauss 1989]. However, the construction of (even local) strong solutions requires some control on the growth at infinity and more tools. As regards the growth of the nonlinearity F, we distinguish two cases. For dimensions $d \geq 3$ we shall assume that our Cauchy problem is H^1-supercritical in the sense that

$$\frac{F(u)}{|u|^{2d/(d-2)}} \nearrow +\infty, \quad u \to \infty\,. \tag{3}$$

In two space dimensions and thanks to Sobolev embedding, any Cauchy problem with polynomially growing nonlinearities is locally well-posed regardless of the sign of the nonlinearity and the growth of F at infinity. This is a limit case of (3). Square exponential nonlinearities were investigated first in [Nakamura and Ozawa 1999b], where global existence and scattering for small Cauchy data were proved, then in [Atallah-Baraket 2004], where local existence was obtained under restrictive conditions, and finally in [Ibrahim et al. 2007a], where a new notion of criticality based on the size of the energy

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appears. In this paper, we examine the situation of other growths of exponential nonlinearities (not necessarily square). More precisely, when $d=2$, we assume either

$$\frac{\log(F(u))}{|u|^2} \nearrow +\infty \quad \text{as } u \to \infty, \tag{4}$$

or

$$\text{for some } q \text{ with } 0 < q \leq 2, \quad \frac{\log(F(u))}{|u|^q} = O(1) \quad \text{as } u \to \infty. \tag{5}$$

The model example that we are going to work with when $d=3$ is given by

$$\partial_t^2 u - \Delta u + u^7 = 0. \tag{6}$$

It is a good prototype for all higher dimensions $d \geq 3$ illustrating assumption (3). In two dimensions, we take

$$\partial_t^2 u - \Delta u + u\,(1+u^2)^{((q-2)/2)} \mathrm{e}^{4\pi((1+u^2)^{(q/2)}-1)} = 0, \tag{7}$$

with $q > 0$, illustrating either the cases (4) or (5), depending on whether $q > 2$ or $q \leq 2$.

Define the total energy of u by

$$E(u(t)) \overset{\text{def}}{=} \|\nabla_{t,x} u(t)\|_{L_x^2}^2 + \int_{\mathbb{R}^d} 2F(u(t))\,dx.$$

The energy of data $(\varphi, \psi) \in \dot{H}^1 \times L^2$ is given by

$$E(\varphi, \psi) \overset{\text{def}}{=} \|\nabla\varphi\|_{L_x^2}^2 + \|\psi\|_{L_x^2}^2 + \int_{\mathbb{R}^d} 2F(\varphi)\,dx.$$

When $\psi = 0$, we abbreviate $E(\varphi, 0)$ to simply $E(\varphi)$.

In the sequel, we adopt the following definitions of weak solution and local/global well-posedness of the Cauchy problem associated to (1).

Definition 1.1. Let $\boldsymbol{X} := X_1 \times X_0$ be a Banach space.[1] A weak solution of (1) is a function $u : \mathbb{R} \to X_1$ with $(\partial_t u, \nabla_x u) \in L^\infty(\mathbb{R}, X_0)$ satisfying (1) in the distributional sense and having finite propagation speed. When $\boldsymbol{X} = H^1 \times L^2$ is the energy space, we have in addition $F(u) \in L^\infty(\mathbb{R}, L^1)$ and $E(u(t)) \leq E(u(0))$ for all t.

The existence of such solutions will one of our results.

Definition 1.2.

- The Cauchy problem associated to (1) is *locally well-posed* in $\boldsymbol{X}$, abbreviated as LWP, if for every data $(u_0, u_1) \in \boldsymbol{X}$, there exists a time $T > 0$ and a unique[2] (distributional) solution

$$u : [-T, T] \times \mathbb{R}^d \to \mathbb{R}$$

  to (1) such that $(u, \partial_t u) \in \mathscr{C}([-T, T]; \boldsymbol{X})$, $(u, \partial_t u)(t=0) = (u_0, u_1)$, and such that the solution map $(u_0, u_1) \mapsto (u, \partial_t u)$ is continuous from $\boldsymbol{X}$ to $\mathscr{C}([-T, T]; \boldsymbol{X})$.
- The Cauchy problem is *globally well-posed* (GWP) if the time T can be taken arbitrary.

[1] Typically, $\boldsymbol{X} = B^s_{p,q} \times B^{s-1}_{p,q}$, for some suitable choice of s, p and q.

[2] In some cases the uniqueness holds in more restrictive space.

- The Cauchy problem is *strongly well-posed* (SWP) if the solution map is uniformly continuous.
- The Cauchy problem is *ill-posed* (IP) if the solution map is not continuous.
- The Cauchy problem is *weakly ill-posed* on a set $\boldsymbol{Y} \subset \boldsymbol{X}$ (WIP) if the solution map

$$(u_0, u_1) \in \boldsymbol{Y} \mapsto (u, \partial_t u)$$

is not uniformly continuous from $\boldsymbol{Y}$ to $\mathscr{C}([-T, T]; \boldsymbol{X})$.

We recall a few historic facts about this problem. First, in space dimensions $d \geq 3$, the defocusing semilinear wave equation with power p reads

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0, \tag{8}$$

where $p > 1$. This problem has been widely investigated and there is a large literature dealing with the well-posedness theory of (8) in the scale of the Sobolev spaces H^s. Second, for the global solvability in the energy space $\dot{H}^1 \times L^2$, there are mainly three cases. In the subcritical case

$$p < p^* \stackrel{\text{def}}{=} \frac{d+2}{d-2},$$

Ginibre and Velo [1985] finally settled global well-posedness in the energy space, by using the Strichartz estimate, nonlinear estimates in Besov space, and energy conservation.

The critical case $p = p^*$ is more delicate, due to possibility of energy concentration. Struwe [1988] proved global existence of radially symmetric regular solutions. Then Grillakis [1990; 1992] extended this result to nonradial data. In the energy space, Ginibre, Soffer and Velo [Ginibre et al. 1992] proved global well-posedness in the radial case, where the Morawetz estimate effectively precludes concentration. The case of general data was solved by Shatah and Struwe [1994], and Kapitanski [1994]. See also [Ibrahim and Majdoub 2003] for variable metrics. Note that uniqueness in the energy space is not yet fully solved. We refer to [Planchon 2003] for $d \geq 4$, to [Struwe 1999; Masmoudi and Planchon 2006] for partial results in $d = 3$, and to [Struwe 2006] for the case of classical solutions.

The supercritical case $p > p^*$ is even harder, and the global well-posedness problem for general data remains open, except for the existence of global weak solutions [Strauss 1989], local well-posedness in higher Sobolev spaces (H^s with $s \geq d/2 - 2/p > 1$) as well as global well-posedness with scattering for small data [Lindblad and Sogge 1995; Wang 1998], and some negative results concerning nonuniform continuity of the solution map [Burq et al. 2007; Christ et al. 2003; Lebeau 2001]. See also [Lebeau 2005] for a result concerning a loss of regularity and [Tao 2007] for a result about global regularity for a logarithmically energy-supercritical wave equation in the radial case.

It is worth noticing that the nonlinearities considered in [Burq et al. 2007; Christ et al. 2003; Lebeau 2001; 2005] are homogeneous, and thus at first glance, the proofs cannot be adapted to the case of inhomogeneous nonlinearities. But as suggested in [Alazard and Carles 2009], it might be that homogeneity is used only to guess a suitable ansatz. We also mention the NLS analogues of [Lebeau 2005] (see for example [Alazard and Carles 2009; Carles 2007; Thomann 2008]). Several different techniques are used there, to get some results which seem out of reach with an ODE approach (in [Alazard and Carles 2009], the case $d = 1$ is allowed, and the trick used in [Lebeau 2005] and [Burq et al. 2007] cannot be adapted, apparently). See also [Burq and Tzvetkov 2008] about random data Cauchy theory for supercritical wave equations.

In dimension two, H^1-critical nonlinearities seem to be of exponential type[3], since every power is H^1-subcritical. On the one hand, in a recent work [Ibrahim et al. 2006], the case $F(u) = 1/8\pi\,(e^{4\pi u^2} - 1)$ was investigated and an energy threshold was proposed. Local strong well-posedness was shown under the size restriction $\|\nabla u_0\|_{L^2} < 1$ and the global well-posedness was obtained in both the sub and critical cases (when the energy is below or equal to the energy threshold). Very recently, Struwe [2009] has constructed global smooth solutions with radially symmetric data of arbitrary size. On the other hand, the ill posedness results of [Lebeau 2005; Christ et al. 2003; Burq et al. 2002] show the nonuniform continuity of the solution map (or sometimes its noncontinuity at the zero data). In the two-dimensional exponential case and since small data are in the subcritical regime, we prove only the nonuniform continuity of the solution map. It is worth to note that the results of [Christ et al. 2003] are based on the scaling invariances of the wave and Schrödinger equations with homogeneous nonlinearities. The idea developed there [Christ et al. 2003] is to approximate the solution by its corresponding ODE (at the zero dispersion limit). Since solutions of the ODE are periodic in time, then a decoherence phenomena occurs for small time since the ODE solutions oscillate fast. Note that the original result in this field appears in [Lebeau 2001].

Hence, in this paper our main aim is to investigate the local well and ill posedness regardless of the size of the initial data. Our idea to overcome the absence of scaling invariance is to choose regularized step functions as initial data (i.e., functions constant near zero). The presence of the step immediately guarantees the equality between the PDE and the ODE solutions in a backward light cone, thanks to the finite speed of propagation. The length of the step can be adjusted (in the supercritical regime) so that ill-posedness/weak ill-posedness occurs inside the light cone.

This paper is organized as follows. In Section 2, we state our main results. In Section 3, we recall some basic definitions and auxiliary lemmas. In Section 4, we investigate the energy regularity regime. Section 5 is devoted to the low regularity data.

Finally, we mention that, C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main results

Energy regularity data. First we show that if the general assumptions (2)+(3) or (2)+(4) are satisfied, the nonlinearity is too strong to ensure the local well-posedness in the energy space:

Theorem 2.1. *Assume that $d \geq 3$ and (2)+(3), or $d = 2$ and (2)+(4).*

(1) *There exist a sequence (φ_k) in $\dot{H}^1$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\nabla\varphi_k\|_{L^2_x} \to 0, \quad t_k \to 0, \quad \sup_k E(\varphi_k) < \infty,$$

and such that any weak solution u_k of (6) with initial data $(\varphi_k, 0)$ satisfies

$$\liminf_{k\to+\infty} \|\partial_t u_k(t_k)\|_{L^2_x} \gtrsim 1.$$

In particular the Cauchy problem is ill-posed in $H^1 \times L^2$.

[3] In fact, the critical nonlinearity is of exponential type in any dimension d with respect to $H^{d/2}$ norm.

(2) *If we relax the condition* $\sup_k E(\varphi_k) < \infty$ *by taking* $\lim_{k\to+\infty} \int F(\varphi_k) = +\infty$, *we can even get*

$$\lim_{k\to+\infty} \|\partial_t u_k(t_k)\|_{L^2_x} = \infty\,.$$

Remark 2.2. Lebeau [2001] proved a loss of regularity result for energy supercritical homogeneous wave equation; see also [Christ et al. 2003]. Recently, Tao [2007] has shown the global well-posedness in the radial case of a logarithmic energy supercritical wave equation in $H^{1+\varepsilon} \times H^{\varepsilon}$ for any $\varepsilon > 0$. The above Theorem shows that ε cannot be taken zero.

The above theorem covers model (7) in two space dimensions with $q > 2$. When $q < 2$, recall that the global well-posedness in the energy space can easily be obtained through the sharp Trudinger–Moser inequality combined with the simple observation that for $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|(1+u^2)^{(q-2)/2} e^{4\pi(1+u^2)^{q/2}} - e^{4\pi}\right| \le C_\varepsilon\,(e^{\varepsilon u^2} - 1) \quad \text{for all } u \in \mathbb{R}.$$

In the case $q = 2$, the local well-posedness for the Cauchy problem associated to (7) in the energy space was first established in [Nakamura and Ozawa 1999a; 1999b] for small Cauchy data. Later on, optimal smallness for well-posedness was investigated, first in [Atallah-Baraket 2004] for radially symmetric initial data $(0, u_1)$, and then in [Ibrahim et al. 2006; 2007b] for general data. The following result generalizes the previous results to any data in the energy space regardless of its size.

Theorem 2.3. *Let* $(u_0, u_1) \in H^1 \times L^2$. *There exists a time* $T > 0$ *and a unique solution* u *of* (7) *with* $q = 2$ *in the space* $C_T(H^1) \cap C^1_T(L^2)$ *satisfying* $u(0, x) = u_0(x)$ *and* $\dot u(0, x) = u_1(x)$. *Moreover, the solution map is continuous on* $H^1 \times L^2$.

In [Ibrahim et al. 2007b] it is shown that the local solutions of (7) (with $q = 2$) are global whenever the total energy $E \le 1$, where

$$E(u(t)) \stackrel{\text{def}}{=} \|\nabla_{t,x} u(t)\|^2_{L^2_x} + \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{4\pi u^2} - 1\, dx.$$

Indeed, in that case, the Cauchy problem is strongly well-posed. The following result shows the weak ill-posedness on the set $\{\, E < 1 + \delta \,\}$ for any $\delta > 0$. More precisely

Theorem 2.4. *Let* $\nu > 0$. *There exist a sequence of positive real numbers* (t_k) *tending to zero and two sequences* (u_k) *and* (v_k) *of solutions of the nonlinear Klein–Gordon equation*

$$\Box u + u e^{4\pi u^2} = 0, \tag{9}$$

satisfying

$$\|(u_k - v_k)(t = 0, \cdot)\|^2_{H^1} + \|\partial_t(u_k - v_k)(t = 0, \cdot)\|^2_{L^2} = o(1) \quad \textit{as } k \to +\infty,$$
$$0 < E(u^k, 0) - 1 \le e^3 \nu^2, \quad 0 < E(v^k, 0) - 1 \le \nu^2,$$
$$\liminf_{k\to\infty} \|\partial_t(u_k - v_k)(t_k, \cdot)\|^2_{L^2} \ge \frac{\pi}{4}(e^2 + e^{3-8\pi})\nu^2.$$

Notice that Theorem 2.3 yields the continuity with respect to the initial data and Theorem 2.4 yields that there is no uniform continuity if the energy is larger than 1 (supercritical regime).

Remark 2.5. Struwe [2009] has constructed global smooth solutions for the two-dimensional energy critical wave equation with radially symmetric data. Although the techniques are different, this result might be seen as an analogue of Tao's result [2007] for the three-dimensional energy supercritical wave equation. Our Theorem 2.4 shows just the weak ill-posedness in the supercritical case. This is weaker than the result in higher dimensions where the flow fails to be continuous at zero as shown in [Christ et al. 2003]. The reason behind this is that small data are always subcritical in the exponential case.

***Low regularity data for the model* (7).** Now that the local well/ill-posedness is clarified in the energy space for dimension $d \geq 2$, our next task in this paper is to seek for the "largest possible spaces" in which we have local well-posedness for the Cauchy problem associated to the model (7). Recall that we have the embeddings

$$H^1(\mathbb{R}^2) \hookrightarrow B^1_{2,\infty}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2), \quad s < 1\,. \tag{10}$$

The next theorem show the failure of the well-posedness in spaces slightly bigger than the energy space in the case $q = 2$. This means that the Cauchy problem posed either in $B^1_{2,\infty}$ or H^s with $s < 1$ becomes supercritical. More specifically:

Theorem 2.6. *Assume $q = 2$. Let $\mathcal{W} := \{u \in L^2 : \nabla u \in L^{2,\infty}\}$, where $L^{2,\infty}$ is the classical Lorentz space.*[4]

(1) *There exists a sequence (φ_k) in $\mathcal{W}$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{\mathcal{W}} \to 0 \quad \text{as } t_k \to 0,$$

and such that any weak solution u_k of (7) with initial data $(\varphi_k, 0)$ satisfies

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{L^{2,\infty}} = \infty.$$

(2) *There exists a sequence (φ_k) in $\mathcal{B}^1_{2,\infty}$ and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{\mathcal{B}^1_{2,\infty}} \to 0 \quad \text{as } t_k \to 0,$$

and such that any weak solution u_k of (7) with initial data $(\varphi_k, 0)$ satisfies

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{\mathcal{B}^0_{2,\infty}} = \infty.$$

In particular, the flow fails to be continuous at 0 in the $\mathcal{W} \times L^{2,\infty}$ topology or $\mathcal{B}^1_{2,\infty} \times \mathcal{B}^0_{2,\infty}$ topology.

(3) *Let $s < 1$. There exists a sequence (φ_k) in H^s and a sequence (t_k) in $(0, 1)$ satisfying*

$$\|\varphi_k\|_{H^s} \to 0 \quad \text{as } t_k \to 0,$$

and such that any weak solution u_k of (7) with initial data $(\varphi_k, 0)$ satisfies

$$\lim_{k\to\infty} \|\partial_t u_k(t_k)\|_{H^{s-1}} = \infty.$$

In particular, the flow fails to be continuous at 0 in the $H^s \times H^{s-1}$ topology.

This theorem can be seen as a consequence of the following general result about arbitrary $1 \leq q < \infty$. Indeed, Equation (7) is subcritical at the regularity of the Besov space $\mathcal{B}^1_{2,q'}$ but supercritical at the H^s regularity level with $s < 1$, where, as usual, q' denotes the Lebesgue conjugate exponent of q. More precisely:

[4] It is defined by its norm $\|u\|_{L^{2,\infty}} := \sup_{\sigma>0}(\sigma \operatorname{meas}^{1/2}\{|u(x)| > \sigma\})$.

Theorem 2.7. *Assume that* $1 \le q < \infty$.

(1) *Let* $(u_0, u_1) \in \mathcal{B}^1_{2,q'} \times \mathcal{B}^0_{2,q'}$.[5] *There exists a time* $T > 0$ *and a unique solution* u *of* (7) *with initial data* (u_0, u_1) *in the space* $C_T(\mathcal{B}^1_{2,q'}) \cap C^1_T(\mathcal{B}^0_{2,q'})$.

(2) *Let* $s < 1$. *There exists a sequence* (φ_k) *in* H^s *and a sequence* (t_k) *in* $(0, 1)$ *satisfying*

$$\|\varphi_k\|_{H^s} \to 0 \quad \text{as } t_k \to 0,$$

and such that any weak solution u_k *of* (7) *with initial data* $(\varphi_k, 0)$ *satisfies*

$$\lim_{k\to+\infty} \|\partial_t u_k(t_k)\|_{H^{s-1}} = \infty.$$

In particular, the flow fails to be continuous at 0 *in the* $H^s \times H^{s-1}$ *topology.*

Remark 2.8. The same well-posedness results can be derived for the corresponding two dimensional nonlinear Schrödinger equations.

We end this section with a table summarizing the picture of well/ill-posedness.

Setting	Data regularity		
	H^1	$\mathcal{B}^1_{2,\infty}$	H^s with $s < 1$
$d \ge 3$ and (3)	WIP	IP	IP
$d = 2$ and (4)	IP	IP	IP
$d = 2$ and $q < 2$	GWP & SWP	LWP	IP
$d = q = 2$ and $E > 1$	LWP & WIP	IP	IP
$d = q = 2$ and $E \le 1$	GWP & SWP	IP	IP

3. Background

Besov spaces. For the convenience of the reader, we recall the definition and some properties of Besov spaces.

Definition 3.1. Let χ be a function in $\mathcal{S}(\mathbb{R}^d)$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| > 2$. Define the function $\psi(\xi) = \chi(\xi/2) - \chi(\xi)$. The (homogeneous) frequency localization operators are defined by

$$\widehat{\dot\Delta_j u}(\xi) = \psi(2^{-j}\xi)\hat{u}(\xi) \quad \text{for all } j \in \mathbb{Z}.$$

If $s < d/p$, then u belongs to the homogeneous Besov space $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^d)$ if and only if the partial sum $\sum_{-m}^{m} \dot\Delta_j u$ converges to u as a tempered distribution and the sequence $(2^{sj}\,\|\Delta_j u\|_{L^p})$ belongs to $\ell^q(\mathbb{Z})$.

To define the inhomogeneous Besov spaces, we need an inhomogeneous frequency localization.

Definition 3.2. The inhomogeneous frequency localization operators are defined by

$$\widehat{\Delta_j u}(\xi) = \begin{cases} 0 & \text{if } j \le -2, \\ \chi(\xi)\hat{u}(\xi) & \text{if } j = -1, \\ \psi(2^{-j}\xi)\,\hat{u}(\xi) & \text{if } j \ge 0. \end{cases}$$

[5]As we will see in the proof, when $q' = \infty$ the appropriate space is $\tilde{\mathcal{B}}^1_{2,\infty}$, the closure of smooth compactly supported function in the usual Besov space $\mathcal{B}^1_{2,\infty}$.

For $N \in \mathbb{N}$, set

$$S_N = \sum_{j \le N-1} \triangle_j .$$

We say that u belongs to the inhomogeneous Besov space $\mathcal{B}^s_{p,q}(\mathbb{R}^d)$ if $u \in \mathcal{S}'$ and $\|u\|_{\mathcal{B}^s_{p,q}} < \infty$, where

$$\|u\|_{\mathcal{B}^s_{p,q}} = \begin{cases} \|\triangle_{-1} u\|_{L^p} + \Big(\sum_{j=0}^{\infty} 2^{jqs} \, \|\triangle_j u\|^q_{L^p} \Big)^{1/q} & \text{if } q < \infty, \\ \|\triangle_{-1} u\|_{L^p} + \sup_{j \ge 0} \, 2^{js} \, \|\triangle_j u\|_{L^p} & \text{if } q = \infty . \end{cases}$$

We recall without proof the following properties of the operators $\triangle_j$ and Besov spaces [Runst and Sickel 1996; Triebel 1983; 1992; 1978].

- Bernstein's inequality: For all $1 \le p \le q \le \infty$ we have

$$\|\triangle_j u\|_{L^q(\mathbb{R}^d)} \le C\, 2^{jd(1/p-1/q)} \, \|\triangle_j u\|_{L^p(\mathbb{R}^d)} .$$

- Embeddings:

$$\mathcal{B}^s_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{B}^{s_1}_{p_1,q_1}(\mathbb{R}^d), \tag{11}$$

  whenever

$$s - \frac{d}{p} \ge s_1 - \frac{d}{p_1}, \quad 1 \le p \le p_1 \le \infty, \quad 1 \le q \le q_1 \le \infty, \quad s, s_1 \in \mathbb{R} .$$

- Equivalent norm: For $s > 0$ we have

$$\|u\|_{\mathcal{B}^s_{p,q}} \approx \|u\|_{L^p} + \|\nabla u\|_{\dot{\mathcal{B}}^{s-1}_{p,q}} . \tag{12}$$

Sobolev spaces and Hölder spaces are special cases of Besov spaces: $H^s = \mathcal{B}^s_{2,2}$ and $C^\sigma = \mathcal{B}^\sigma_{\infty,\infty}$, for noninteger $\sigma > 0$.

We shall also use a result about functions that operate by pointwise multiplication in Besov spaces:

Theorem 3.3 [Runst and Sickel 1996, Theorem 4.6.2]. *Let $|s| < d/2$. Any function in $\dot{\mathcal{B}}^{d/2}_{2,\infty} \cap L^\infty(\mathbb{R}^d)$ is a pointwise multiplier in the Besov space $\dot{\mathcal{B}}^s_{2,q}(\mathbb{R}^d)$.*

An important application of this theorem[6] which will be used in the sequel is the fact that the function $f(x) := x/r$ operates on $\dot{\mathcal{B}}^0_{2,\infty}(\mathbb{R}^2)$ via pointwise multiplication. Indeed, according to Theorem 3.3 it suffices to show that f belongs to $\dot{\mathcal{B}}^1_{2,\infty}(\mathbb{R}^2)$. For this, note that $\hat{f}$ is an homogeneous distribution of degree -2, belonging to the C^∞ class outside the origin. We can then define $g \in \mathcal{S}$ by $\hat{g} = \psi \hat{f}$. Hence $\triangle_j f(x) = g(2^j x)$ and $\|\triangle_j f\|_{L^2} = 2^{-j} \|g\|_{L^2}$.

Two-dimensional Strichartz estimate and logarithmic inequality.

Proposition 3.4 [Miao et al. 2004; Nakamura and Ozawa 2001].

$$\|u\|_{L^4((0,T);B^{1/4}_{\infty,2})} \lesssim \|\partial_t^2 u - \Delta u + u\|_{L^1((0,T);L^2)} + \|u(0)\|_{H^1} + \|\partial_t u(0)\|_{L^2}. \tag{13}$$

Using the embedding (11), we can replace $B^{1/4}_{\infty,2}$ with the Hölder space $C^{1/4}$.

[6] We are grateful to Gérard Bourdaud for providing us this reference and a proof of the application.

The following lemma shows that we can estimate the L^∞ norm by a stronger norm but with a weaker growth (namely logarithmic).

Lemma 3.5. *Let $0<\alpha<1$ and $1\le q\le\infty$. There exists a constant C such that*

$$\|u\|_{L^\infty}\le C\|u\|_{\mathscr{B}^1_{2,q'}}\log^{1/q}\left(\mathrm{e}+\frac{\|u\|_{\mathscr{C}^\alpha}}{\|u\|_{\mathscr{B}^1_{2,q'}}}\right). \tag{14}$$

Similar inequalities appeared in [Brézis and Gallouet 1980]; they have been improved (with respect to the best constant) as follows:

Lemma 3.6 [Ibrahim et al. 2007a, Theorem 1.3]. *Let $0<\alpha<1$. For any $\lambda>1/(2\pi\alpha)$ and any $0<\mu\le 1$, a constant $C_\lambda>0$ exists such that, for any function $u\in H^1(\mathbb{R}^2)\cap\mathscr{C}^\alpha(\mathbb{R}^2)$*

$$\|u\|^2_{L^\infty}\le\lambda\|u\|^2_{H_\mu}\log\left(C_\lambda+\frac{8^\alpha\mu^{-\alpha}\|u\|_{\mathscr{C}^\alpha}}{\|u\|_{H_\mu}}\right), \tag{15}$$

where H_μ is defined by the norm $\|u\|^2_{H_\mu}:=\|\nabla u\|^2_{L^2}+\mu^2\|u\|^2_{L^2}$.

Proof of Lemma 3.5. Write $u=\sum_{j=-1}^{N-1}\triangle_j u+\sum_{j=N}^{\infty}\triangle_j u$, with $N\ge 0$ an integer to be chosen later. Using Bernstein's inequality, we get

$$\|u\|_{L^\infty}\le C\sum_{j=-1}^{N-1}2^j\|\triangle_j u\|_{L^2}+\sum_{j=N}^{\infty}2^{-j\alpha}\left(2^{j\alpha}\|\triangle_j u\|_{L^\infty}\right)\le C\left(N^{1/q}\|u\|_{\mathscr{B}^1_{2,q'}}+\frac{2^{-N\alpha}}{1-2^{-\alpha}}\|u\|_{\mathscr{C}^\alpha}\right).$$

Choosing $N\sim\dfrac{1}{\alpha\log 2}\log\left(\mathrm{e}+\dfrac{\|u\|_{\mathscr{C}^\alpha}}{\|u\|_{\mathscr{B}^1_{2,q'}}}\right)$, we obtain (14) as desired. □

Oscillating second order ODE. Here we recall a classical result about ordinary differential equations.

Lemma 3.7 [Arnaudiès and Lelong-Ferrand 1997, Section III.5]. *Let $F:\mathbb{R}\to\mathbb{R}$ be a smooth function. The ODE*

$$\ddot{x}(t)+F'(x(t))=0, \tag{16}$$

with initial conditions $x(0)=x_0>0$ and $\dot{x}(0)=0$, has a nonconstant periodic solution if and only if the function $G:y\mapsto 2(F(x_0)-F(y))$ has two distinct simple zeros α and β with $\alpha\le x_0\le\beta$ and G has no zero in the interval $]\alpha,\beta[$. The period is then given by

$$T=2\int_\alpha^\beta\frac{dy}{\sqrt{G(y)}}=\sqrt{2}\int_\alpha^\beta\frac{dy}{\sqrt{F(x_0)-F(y)}}.$$

In addition, x is decreasing on $[0,T/4]$ and $x(T/4)=0$.

Trudinger–Moser inequalities. It is known that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces L^p for $2\le p<\infty$ but not in L^∞. Moreover, H^1 functions are in the so-called Orlicz space, that is, their exponentials are integrable for every growth less than e^{u^2}. Precisely, we have the following Trudinger–Moser inequality (see [Adachi and Tanaka 2000; Ruf 2005] and references therein).

Proposition 3.8. *Let $\alpha \in (0, 4\pi)$. A constant c_α exists such that*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - 1\right) dx \le c_\alpha \|u\|_{L^2}^2 \tag{17}$$

for all u in $H^1(\mathbb{R}^2)$ such that $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$. Moreover, if $\alpha \ge 4\pi$, then (17) is false.

We point out that $\alpha = 4\pi$ becomes admissible in (17) if we require $\|u\|_{H^1(\mathbb{R}^2)} \le 1$ rather than $\|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$. Precisely, we have

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)} \le 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1\right) dx < \infty$$

and this is false for $\alpha > 4\pi$. See [Ruf 2005] for more details.

The estimates above obviously control any exponential power with smaller growth ($q < 2$). However, no estimate holds if the growth is higher ($q > 2$). Hence, the value $q = 2$ is also another criticality threshold for problems involving such nonlinearities.

Some technical lemmas.

Lemma 3.9. *For any $0 < a < 1$,*

$$\int_a^1 r\mathrm{e}^{4a^2 \log^2 r}\, dr \le 2. \tag{18}$$

Proof. Let $I(a)$ be the integral in (18). The change of variable $s = -2a \log r$ yields

$$I(a) = \frac{1}{2a}\mathrm{e}^{-1/(4a^2)} \int_0^{-2a\log a} \mathrm{e}^{(s-1/2a)^2} ds = \frac{1}{2a}\mathrm{e}^{-1/(4a^2)} \int_{-1/(2a)}^{-2a\log a - 1/(2a)} \mathrm{e}^{y^2} dy.$$

But $-2a\log a - \dfrac{1}{2a} \le \dfrac{1}{2a}$ for $0 < a < 1$; thus

$$I(a) \le 2A\mathrm{e}^{-A^2} \int_0^A \mathrm{e}^{y^2} dy,$$

where $A = \dfrac{1}{2a}$. It remains to prove that for all nonnegative A

$$\int_0^A \mathrm{e}^{y^2}\, dy \le \frac{\mathrm{e}^{A^2}}{A}. \tag{19}$$

Estimate (19) is obvious when $A \le 1$. If $A \ge 1$, we write

$$\int_0^A \mathrm{e}^{y^2} dy = \int_0^1 \mathrm{e}^{y^2} dy + \int_1^A 2y\mathrm{e}^{y^2} \frac{dy}{2y},$$

and an integration by parts gives

$$\int_0^A \mathrm{e}^{y^2} dy \le \frac{\mathrm{e}}{2} + \frac{\mathrm{e}^{A^2}}{2A} + \int_1^A \frac{\mathrm{e}^{y^2}}{2y^2} dy.$$

Using the monotonicity of the function $y \mapsto \mathrm{e}^{y^2}/(2y^2)$, the estimate (19) follows. $\square$

Lemma 3.10. *For any $a \geq 1$ and $k \geq 1$,*

$$\int_{e^{-k/2}}^{1} r e^{(4a^2/k)\log^2 r}\, dr \leq 2e^{(a^2-1)k}. \tag{20}$$

Proof. Let $I(a,k)$ be the integral in (20). The change of variable $u = -\dfrac{2a}{\sqrt{k}}\log r$ yields

$$I(a,k) = \frac{\sqrt{k}}{2a}\, e^{-k/(4a^2)} \int_0^{a\sqrt{k}} e^{(u-\sqrt{k}/(2a))^2}\, du.$$

Changing once more the variable to $v = u - \dfrac{\sqrt{k}}{2a}$ yields

$$I(a,k) = \frac{\sqrt{k}}{2a}\, e^{-k/(4a^2)} \int_{-\sqrt{k}/(2a)}^{(2a^2-1)\sqrt{k}/(2a)} e^{v^2}\, dv.$$

Hence, for any $a \geq 1$ we have

$$I(a,k) \leq \frac{\sqrt{k}}{a}\, e^{-k/(4a^2)} \int_0^{(2a^2-1)\sqrt{k}/(2a)} e^{v^2}\, dv.$$

Now, using the estimate $\displaystyle\int_0^A e^{u^2}\, du \leq \frac{e^{A^2}-1}{A} \leq \frac{e^{A^2}}{A}$, true for all nonnegative A, we obtain (20). □

Lemma 3.11. *For any $\lambda > 0$ and $A > \lambda$,*

$$\int_{A-\lambda^2/A}^{A} \frac{du}{\sqrt{e^{A^2}-e^{u^2}}} \leq \frac{A\, e^{2\lambda^2}}{A^2-\lambda^2}\, e^{-A^2/2}. \tag{21}$$

Proof. Choosing $h(u) = \dfrac{-1}{ue^{u^2}}$ and $g'(u) = \dfrac{ue^{u^2}}{\sqrt{e^{A^2}-e^{u^2}}}$, and integrating by parts, we deduce (21). □

Lemma 3.12. *For any $A > 1$,*

$$\int_0^A \frac{du}{\sqrt{e^{A^2}-e^{u^2}}} \approx A\, e^{-A^2/2}. \tag{22}$$

Proof. Let $I(A)$ be the integrating in (22). In one hand, it is clear that

$$I(A) \geq A\, e^{-A^2/2}.$$

In the other hand, write

$$I(A) = \int_0^{A-1/(4A)} \frac{du}{\sqrt{e^{A^2}-e^{u^2}}} + J\big(A, \tfrac{1}{2}\big). \tag{23}$$

By Lemma 3.11, we get

$$J\big(A, \tfrac{1}{2}\big) \leq \frac{A\, e^{1/2}}{A^2-\frac{1}{4}}\, e^{-A^2/2} \lesssim A\, e^{-A^2/2}.$$

For any $0 \leq u \leq A - \dfrac{1}{4A}$, we have

$$\frac{1}{\sqrt{e^{A^2}-e^{u^2}}} \leq \frac{1}{\sqrt{e^{A^2}-e^{(A-1/4A)^2}}} \lesssim e^{-\frac{A^2}{2}}.$$

Hence, the first integral in (23) can be estimated by $\displaystyle\int_0^{A-1/(4A)} \frac{du}{\sqrt{e^{A^2}-e^{u^2}}} \lesssim A\, e^{-A^2/2}$, and (22) follows. □

4. Energy regularity data

This section is devoted to the well-posedness issues in energy space stated in Section 2. Some of these results were announced in [Ibrahim et al. 2007b]. We begin with Theorem 2.1.

Proof of Theorem 2.1. First, consider the case $d \geq 3$. We prove statement (1) of the theorem.

- Construction of φ_k. For $k \geq 1$ and $\varepsilon > 0$ (depending on k as we will see later) define φ_k by

$$\varphi_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ a(k,\varepsilon)(|x|^{2-d}-1) & \text{if } \varepsilon/k \leq |x| \leq 1, \\ k^{(d-2)/2} & \text{if } |x| \leq \varepsilon/k, \end{cases}$$

where $a(k,\varepsilon) = \dfrac{\varepsilon^{d-2}k^{(d-2)/2}}{k^{d-2}-\varepsilon^{d-2}}$ is chosen such that φ_k is continuous. An easy computation yields

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim \frac{\varepsilon^{d-2}k^{d-2}}{k^{d-2}-\varepsilon^{d-2}} \lesssim \varepsilon^{d-2}.$$

Using assumption (3), we get

$$\begin{aligned}\int_{\mathbb{R}^d} F(\varphi_k(x))dx &\lesssim F(k^{(d-2)/2})\Big(\frac{\varepsilon}{k}\Big)^d + \int_{\varepsilon/k}^1 F\big(a(k,\varepsilon)(r^{2-d}-1)\big)r^{d-1}dr \\ &\lesssim F(k^{(d-2)/2})\Big(\frac{\varepsilon}{k}\Big)^d\bigg(1+\frac{1-(\varepsilon/k)^d}{\big(1-(\varepsilon/k)^{d-2}\big)^{2d/(d-2)}}\bigg).\end{aligned}$$

Since $k\big(F(k^{(d-2)/2})\big)^{-1/d} \to 0$ we will choose

$$\varepsilon = \varepsilon_k \stackrel{\text{def}}{=} k\big(F(k^{(d-2)/2})\big)^{-1/d}.$$

With this choice, we can see that $\|\nabla\varphi_k\|_{L^2} \to 0$ and $\sup_k E(\varphi_k) < \infty$.

- Construction of t_k. Consider the ordinary differential equation associated to (1):

$$\ddot{\Phi} + F'(\Phi) = 0, \qquad (\Phi(0), \dot{\Phi}(0)) = (k^{(d-2)/2}, 0). \tag{24}$$

Using Lemma 3.7 and the assumptions on F, we can see that (24) has a unique global periodic solution Φ_k with period

$$T_k = 2\sqrt{2}\int_0^{k^{(d-2)/2}} \frac{d\Phi}{\sqrt{F(k^{(d-2)/2})-F(\Phi)}} = 2\sqrt{2}\frac{k^{(d-2)/2}}{\sqrt{F(k^{(d-2)/2})}}\int_0^1\bigg(1-\frac{F(vk^{(d-2)/2})}{F(k^{(d-2)/2})}\bigg)^{-1/2}dv.$$

By assumption (3), we get

$$T_k \leq 2\sqrt{2}\frac{k^{(d-2)/2}}{\sqrt{F(k^{(d-2)/2})}}\int_0^1\big(1-v^{2d/(d-2)}\big)^{-1/2}dv \lesssim k^{(d-2)/2}\big(F(k^{(d-2)/2})\big)^{-1/2}.$$

It follows that (for k large enough)

$$T_k \ll \frac{\varepsilon_k}{k}.$$

Now we are in a position to construct the sequence (t_k). Recall that by finite speed of propagation, any weak solution u_k of (1) with data $(\varphi_k, 0)$ satisfies

$$u_k(t,x) = \Phi_k(t) \qquad \text{if } 0 < t < \frac{\varepsilon_k}{k} \text{ and } |x| < \frac{\varepsilon_k}{k} - t.$$

Hence

$$|\partial_t u_k(t,x)| = |\dot{\Phi}_k(t)| = \sqrt{2}\sqrt{F(k^{(d-2)/2}) - F(\Phi_k(t))}.$$

Let us choose $t_k = \dfrac{T_k}{4}$; then $\Phi_k(t_k) = 0$, $t_k \ll \dfrac{\varepsilon_k}{k}$ and, for $|x| < \dfrac{\varepsilon_k}{k} - t_k$,

$$|\partial_t u_k(t_k,x)| = \sqrt{2}\sqrt{F(k^{(d-2)/2}) - F(\Phi_k(t_k))} \gtrsim \sqrt{F(k^{(d-2)/2})}.$$

So

$$\|\partial_t u_k(t_k)\|_{L^2}^2 \gtrsim F(k^{(d-2)/2})\Big(\frac{\varepsilon_k}{k} - t_k\Big)^d = \Big(\frac{\varepsilon_k}{k}\Big)^d F(k^{(d-2)/2})\Big(1 - t_k\frac{k}{\varepsilon_k}\Big)^d,$$

and the conclusion follows.

Now we turn to the proof of the second claim of Theorem 2.1. For clarity, we restrict ourselves to the model example (6). For any real $a > 0$, we denote by Φ_a the unique global solution of

$$\ddot{\Phi}(t) + \Phi^7(t) = 0, \quad (\Phi(0), \dot{\Phi}(0)) = (a, 0)\ . \tag{25}$$

By Lemma 3.7, Φ_a is periodic with period $T(a)$. By a scaling argument, we have $T(a) = a^{-3}T(1)$, and therefore

$$T(a) = C\, a^{-3}, \tag{26}$$

for some absolute positive constant C.

- Construction of t_k. Let (M_k) be a sequence of integers tending to infinity and such that

$$M_k = o(k^{1/6}) \quad \text{as } k \to \infty\,. \tag{27}$$

We denote by (η_k) the unique sequence in $(0,\infty)$ satisfying

$$4M_k = \frac{1}{1-(1-\eta_k)^3}\,. \tag{28}$$

As a consequence of these choices, we obtain the crucial identity

$$M_k\, T(\sqrt{k}) = \big(M_k - \tfrac{1}{4}\big)\, T(\sqrt{k}(1-\eta_k))\,. \tag{29}$$

A good choice for the sequence (t_k) is then

$$t_k = M_k\, T(\sqrt{k})\,. \tag{30}$$

Taking advantage of (26) and (27), we get $t_k \ll k^{-4/3}$.

- Construction of φ_k. The idea is to take a function φ_k oscillating between $\sqrt{k}$ and $\sqrt{k}(1-\eta_k)$ a certain number of times. Choose a sequence (N_k) of even integers tending to infinity and such that

$$N_k \sim C\,k^{1/6}\,M_k^2, \tag{31}$$

and set $\alpha_k := 10\,t_k\,N_k\,k^{4/3} \sim C\,M_k^3$. Divide the radial interval $k^{-4/3} \le r \le (\alpha_k+1)k^{-4/3}$ into N_k subintervals each of them has a length $10\,t_k$ and write

$$[k^{-4/3},\,(\alpha_k+1)k^{-4/3}] = \bigcup_{j=0}^{N_k-1}\,[a_k^{(j)},\,a_k^{(j+1)}],$$

where $a_k^{(j)} = k^{-4/3} + 10\,j\,t_k$. Now consider a φ_k that is continuous and oscillates between $\sqrt{k}$ and $\sqrt{k}(1-\eta_k)$ as follows:

$$\begin{aligned}
&\varphi_k(r) = \sqrt{k} && \text{if } r \le k^{-4/3},\\
&\varphi_k(r) = \sqrt{k}(1-\eta_k) && \text{if } k^{-4/3}+t_k \le r \le k^{-4/3}+9t_k,\\
&\varphi_k(r) = \sqrt{k} && \text{if } k^{-4/3}+11t_k \le r \le k^{-4/3}+19t_k,\\
&\varphi_k(r) = \cdots, &&\\
&\varphi_k(r) = \sqrt{k} && \text{if } k^{-4/3}+(10N_k-9)t_k \le r \le k^{-4/3}+(10N_k-1)t_k,\\
&\varphi_k(r) = \sqrt{k} && \text{if } r \ge k^{-4/3}+10N_kt_k;
\end{aligned}$$

in the remaining intervals, φ_k is affine. An easy computation shows that

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim N_k\Big(\frac{\sqrt{k}\eta_k}{t_k}\Big)^2\,(k^{-4/3})^3\,t_k\,k^{4/3} \lesssim \frac{1}{M_k}\,. \tag{32}$$

Moreover, using the finite speed of propagation and the fact that

$$\Phi_{\sqrt{k}}(t_k) = \sqrt{k}, \quad \Phi_{\sqrt{k}(1-\eta_k)}(t_k) = 0,$$

we conclude that any weak solution u_k to (6) with data $(\varphi_k, 0)$ satisfies

$$\|\partial_t\,u_k(t_k)\|_{L^2}^2 \gtrsim N_k k^4 (k^{-4/3})^4 t_k k^{4/3} \gtrsim M_k^3\,. \tag{33}$$

This finishes the proof for $d \ge 3$. The case $d=2$ can be handled in a similar way. We have just to make a suitable choice of the initial data.

- Construction of φ_k. For $k \ge 1$, we define φ_k by

$$\varphi_k(x) = \begin{cases} 0 & \text{if } |x| \ge 1,\\ \dfrac{-2\sqrt{k}}{\log F(\sqrt{k})}\log|x| & \text{if } \varepsilon_k \mathrm{e}^{-k/2} \le |x| \le 1,\\ \sqrt{k} & \text{if } |x| \le \varepsilon_k \mathrm{e}^{-k/2}, \end{cases}$$

where $\varepsilon_k = \mathrm{e}^{k/2}(F(\sqrt{k}))^{-1/2}$. Remark that, by (4), we have $\varepsilon_k \to 0$. An easy computation using (4) yields

$$\|\nabla\varphi_k\|_{L^2}^2 \lesssim \frac{-1}{\log\varepsilon_k},$$

and

$$\int_{\mathbb{R}^2} F(\varphi_k(x))\,dx \lesssim \varepsilon_k^2\mathrm{e}^{-k}F(\sqrt{k}) + \int_{\varepsilon_k\mathrm{e}^{-k/2}}^1 r\exp\biggl(4\frac{\log^2 r}{(\log F(\sqrt{k}))^2}\biggr)\,dr.$$

The choice of ε_k implies that the first summand on the right side is $\lesssim 1$. For the second summand, we use Lemma 3.9.

- Construction of t_k. As in higher dimensions, we consider the associated ordinary differential equation with data $(\sqrt{k}, 0)$. This equation has a unique global periodic solution with period

$$T_k = 2\sqrt{2}\int_0^{\sqrt{k}} \frac{d\Phi}{\sqrt{F(\sqrt{k}) - F(\Phi)}}.$$

By assumption (4), we get

$$T_k \lesssim \sqrt{k}\frac{1}{A}\int_0^A \frac{du}{\sqrt{\mathrm{e}^{A^2} - \mathrm{e}^{u^2}}},$$

where $A = \sqrt{\log F(\sqrt{k})}$. It follows from Lemma 3.12 that $T_k \ll \varepsilon_k\mathrm{e}^{-k/2}$. Now, arguing exactly in the same manner as in higher dimensions, we finish the proof for $d = 2$. □

Proof of Theorem 2.3. The idea here is to split the initial data into a small part in $H^1 \times L^2$ and a smooth one. First we solve the IVP with smooth initial data to obtain a local and bounded solution v. Then we consider the perturbed equation satisfied by $w := u - v$ and with small initial data. (A similar idea was used in [Gallagher and Planchon 2003; Germain 2008; Kenig et al. 2000; Planchon 2000].) Now we come to the details.

Existence. Given initial data (u_0, u_1) in the energy space $H^1 \times L^2$, we decompose it as

$$(u_0, u_1) = S_n(u_0, u_1) + (I - S_n)(u_0, u_1),$$

where the first term is defined as $(u_0, u_1)_{<n}$ and the second as $(u_0, u_1)_{>n}$, for n a (large) integer to be chosen later. Note that

$$(u_0, u_1)_{>n} \to 0 \quad \text{in } H^1 \times L^2 \quad \text{as } n \to \infty,$$

and that, for every n, $(u_0, u_1)_{<n} \in H^2 \times H^1$. First we consider the IVP with regular data

$$\Box v + v + f(v) = 0, \quad (v(0,x), \partial_t v(0,x)) = (u_0, u_1)_{<n}, \quad f(v) = v(\mathrm{e}^{4\pi v^2} - 1). \tag{34}$$

It is known that (34) is well-posed. More precisely, there exist a time $T_n = T(\|(u_0, u_1)_{<n}\|_{H^2\times H^1}) > 0$ and a unique solution v to (34) in $C_{T_n}(H^2) \cap C^1_{T_n}(H^1)$. Moreover, we can choose T_n such that $\|v\|_{L^\infty_{T_n}(H^2)} \le (\|((u_0)_{<n}, (u_1)_{<n})\|_{H^2\times H^1} + 1)$.

Next we consider the perturbed IVP with small data

$$\Box w + w + f(w+v) - f(v) = 0, \quad (w(0,x), \partial_t w(0,x)) = (u_0, u_1)_{>n}. \tag{35}$$

We shall prove that (35) has a local in time solution in the space $\mathscr{E}_T := C_T(H^1) \cap C_T^1(L^2) \cap L_T^4(C^{1/4})$ for suitable time $T > 0$. This will be achieved by a standard fixed point argument. We denote by w_ℓ the solution of the linear Klein–Gordon equation with data $(u_0, u_1)_{>n}$,

$$\Box w_\ell + w_\ell = 0, \quad (w_\ell(0,x), \partial_t w_\ell(0,x)) = (u_0, u_1)_{>n}.$$

For a positive time $T \le T_n$ and a positive real number δ, we denote by $\mathscr{E}_T(\delta)$ the closed ball in $\mathscr{E}_T$ of radius δ and center at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\Phi : w \in \mathscr{E}_T(\delta) \mapsto \tilde{w}$$

where

$$\Box\tilde{w} + \tilde{w} + f(w + w_\ell + v) - f(v) = 0, \qquad (\tilde{w}(0,x), \partial_t \tilde{w}(0,x)) = (0,0).$$

By energy and Strichartz estimates, we get

$$\|\Phi(w)\|_{\mathscr{E}_T} \lesssim \|f(w + w_\ell + v) - f(v)\|_{L_T^1(L^2)} \lesssim \|w + w_\ell\|_{L_T^\infty(L^2)} \left\| \mathrm{e}^{C\|w+w_\ell+v\|_\infty^2} + \mathrm{e}^{C\|v\|_\infty^2} \right\|_{L_T^1}$$

It is clear that

$$\left\| \mathrm{e}^{C\|v\|_\infty^2} \right\|_{L_T^1} \lesssim T \mathrm{e}^{C(\|(u_0)_{<n}\|_{H^2}+1)^2}.$$

On the other hand, using the logarithmic inequality we infer

$$\mathrm{e}^{C\|w+w_\ell+v\|_\infty^2} \lesssim \mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2} \left(C + \frac{\|w + w_\ell\|_{C^{1/4}}}{\delta + \varepsilon} \right)^{C(\delta+\varepsilon)^2},$$

where $\varepsilon^2 = \|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2$. By the Hölder inequality in time we deduce

$$\left\| \mathrm{e}^{C\|w+w_\ell+v\|_\infty^2} \right\|_{L_T^1} \lesssim \mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2} T^{1-\beta/4} (T^{1/4} + \delta + \varepsilon)^\beta,$$

where $\beta := C(\delta+\varepsilon)^2 < 4$ for δ and ε small enough. Finally, we get

$$\|\Phi(w)\|_{\mathscr{E}_T} \lesssim (\delta+\varepsilon) \mathrm{e}^{C\|(u_0)_{<n}\|_{H^2}^2} \big(T + T^{1-\beta/4} (T^{1/4} + \delta + \varepsilon)^\beta \big).$$

From this inequality it follows immediately that Φ maps $\mathscr{E}_T(\delta)$ into itself if T is small enough. To prove that Φ is a contraction (at least for T small), we consider two elements w_1 and w_2 in $\mathscr{E}_T(\delta)$ and define

$$w = w_1 - w_2, \quad \tilde{w} = \tilde{w_1} - \tilde{w_2}, \quad \bar{w} = (1-\theta)(w_\ell + w_1) + \theta(w_\ell + w_2) + v \quad \text{with } 0 \le \theta \le 1.$$

We can write

$$f(w_\ell + w_1) - f(w_\ell + w_2) = w[(1 + 8\pi\bar{w}^2)\mathrm{e}^{4\pi\bar{w}^2} - 1]$$

for some choice of $0 \le \theta(t,x) \le 1$. By the energy estimate and the Strichartz inequality we have

$$\left\| \Phi(w_1) - \Phi(w_2) \right\|_{\mathscr{E}_T} \lesssim \left\| w \mathrm{e}^{C|\bar{w}|^2} \right\|_{L_T^1(L_x^2)}.$$

By convexity, we obtain

$$\|\Phi(w_1) - \Phi(w_2)\|_{\mathscr{E}_T} \lesssim \|w(\mathrm{e}^{C|w_\ell+w_1|^2} + \mathrm{e}^{C|w_\ell+w_2|^2})\|_{L_T^1(L_x^2)}.$$

So arguing as before, we get

$$\begin{aligned}\|\Phi(w_1)-\Phi(w_2)\|_{\mathscr{E}_T} &\lesssim \|w\|_{L^\infty_T(L^2)}\big(\|\mathrm{e}^{C\|w_\ell+w_1\|^2_\infty}\|_{L^1_T}+\|\mathrm{e}^{C\|w_\ell+w_2\|^2_\infty}\|_{L^1_T}\big),\\ &\lesssim T^{1-\beta/4}(T^{1/4}+\delta+\varepsilon)^\beta\|w_1-w_2\|_T,\end{aligned}$$

for some $\beta<4$. If the parameters $\varepsilon>0$, $\delta>0$ and $T>0$ are suitably chosen, then Φ is a contraction map on $\mathscr{E}_T(\delta)$ and thus a local in time solution is constructed.

Uniqueness. We shall prove the uniqueness in the space

$$\mathscr{F}_\eta := C_T(H^2)\cap C^1_T(H^1)+\{w\in\mathscr{E}_T : \|w\|_T\le\eta\},$$

for any $\eta<1/\sqrt{2}$. Let $u:=v+w$ and $U:=V+W$ be two solutions of (9) in $\mathscr{F}_\eta$ with the same initial data. Since $v,V\in C_t(H^2)$ and H^2 is embedded in L^∞, we can choose a time $T>0$ such that (for some constant C)

$$\|v\|_{L^\infty([0,T],L^\infty)}\le C \quad\text{and}\quad \|V\|_{L^\infty([0,T],L^\infty)}\le C\,. \tag{36}$$

The difference $U-u$ satisfies

$$\Box(U-u)+U-u=f(v+w)-f(V+W),\qquad ((U-u),\partial_t(U-u))(t=0)=(0,0).$$

Using the energy estimate and Strichartz inequality, we get

$$\begin{aligned}\|U-u\|_{\mathscr{E}_T} &\lesssim \big\|f(v+w)-f(V+W)\big\|_{L^1_T(L^2)}\\ &\lesssim \big\|(U-u)\big(U^2(\mathrm{e}^{4\pi U^2}-1)+u^2(\mathrm{e}^{4\pi u^2}-1)\big)\big\|_{L^1_T(L^2)}\\ &\lesssim \|U-u\|_{L^\infty_T(L^{2/\varepsilon})}\big\|U^2(\mathrm{e}^{4\pi U^2}-1)+u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})},\end{aligned}$$

where $\varepsilon>0$ is to be chosen small enough. To conclude the proof of the uniqueness, we have to estimate the term

$$\big\|u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})},$$

for example. Observe that, for any $\beta>0$ and $a>1$,

$$x^2(\mathrm{e}^{4\pi x^2}-1)\le C_\beta(\mathrm{e}^{4\pi(1+\beta)x^2}-1), \tag{37}$$

and

$$(x+y)^2\le\frac{a}{a-1}\,x^2+a\,y^2\,. \tag{38}$$

Hence

$$\big\|u^2(\mathrm{e}^{4\pi u^2}-1)\big\|_{L^1_T(L^{2/(1-\varepsilon)})}\lesssim\int_0^T\bigg(\int_{\mathbb{R}^2}\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}u^2}-1\Big)\,dx\bigg)^{(1-\varepsilon)/2}dt\,.$$

Moreover, using (38), we can write

$$\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}u^2}-1\le\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}\frac{a}{a-1}v^2}-1\Big)+\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}aw^2}-1\Big)+\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}\frac{a}{a-1}v^2}-1\Big)\Big(\mathrm{e}^{8\pi\frac{1+\beta}{1-\varepsilon}aw^2}-1\Big). \tag{39}$$

To estimate the first term on the right-hand side of (39), we use (36). For the second term, observe that

$$\sqrt{2}\,\eta\sqrt{\frac{(1+\beta)a}{1-\varepsilon}}\to\eta\sqrt{2}<1\quad\text{as } a\to1 \text{ and } \varepsilon,\ \beta\to0\,.$$

This enables us to use the Trudinger–Moser inequality. We do the same for the last term. This concludes the proof of the uniqueness in the space $\mathscr{F}_\eta$. Note that we can weaken the hypothesis $\eta < \frac{1}{\sqrt{2}}$ to $\eta < 1$ if we use the sharp logarithmic inequality (15). $\square$

Remark 4.1. In higher dimensions $d \geq 3$, we obtain a similar result in $H^{d/2} \times H^{d/2-1}$ for (1) by using a decomposition in $H^{d/2+1} \times H^{d/2}$ and small in $H^{d/2} \times H^{d/2-1}$.

Proof of Theorem 2.4. For any $k \geq 1$ define f_k by

$$f_k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ -\dfrac{\log|x|}{\sqrt{k\pi}} & \text{if } \mathrm{e}^{-k/2} \leq |x| \leq 1, \\ \sqrt{k/4\pi} & \text{if } |x| \leq \mathrm{e}^{-k/2}. \end{cases}$$

These functions were introduced in [Moser 1971] to show the optimality of the exponent 4π in Trudinger–Moser inequality. An easy computation shows that $\|\nabla f_k\|_{L^2(\mathbb{R}^2)} = 1$ and $\|f_k\|_{L^2(\mathbb{R}^2)} \lesssim 1/\sqrt{k}$. Denote by u_k and v_k any weak solutions of (9) with initial data $((1+\frac{1}{k}) f_k(\frac{\cdot}{\nu}), 0)$ and $(f_k(\frac{\cdot}{\nu}), 0)$, respectively. By construction,

$$\big\|(u_k - v_k)(0)\big\|_{H^1}^2 + \big\|\partial_t(u_k - v_k)0)\big\|_{L^2}^2 = \frac{1}{k^2}\Big\|f_k\Big(\frac{\cdot}{\nu}\Big)\Big\|_{H^1}^2 = o(1) \quad \text{as } k \to \infty.$$

Also, using estimate (20), it is clear that

$$0 < E\Big(\Big(1+\frac{1}{k}\Big) f_k\Big(\frac{\cdot}{\nu}\Big)\Big) - 1 \leq \mathrm{e}^3 \nu^2 \quad \text{and} \quad 0 < E\Big(f_k\Big(\frac{\cdot}{\nu}\Big)\Big) - 1 \leq \nu^2.$$

Now, we shall construct the sequence of time t_k. A good approximation of u_k and v_k is provided by the corresponding ordinary differential equation,

$$\ddot{\Phi} + \Phi \mathrm{e}^{4\pi\Phi^2} = 0. \tag{40}$$

More precisely, let Φ_k and Ψ_k be the solutions of (40) with initial data

$$\Phi_k(0) = \Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}}, \quad \dot{\Phi}_k(0) = 0,$$

and

$$\Psi_k(0) = \sqrt{\frac{k}{4\pi}}, \quad \dot{\Psi}_k(0) = 0,$$

respectively. Note that by finite speed of propagation, we have $\Phi_k = u_k$ and $\Psi_k = v_k$ in the backward light cone $|x| < \nu \mathrm{e}^{-k/2} - t,\ t < \nu \mathrm{e}^{-k/2}$.

On the other hand, recall that the period T_k of Φ_k is given by

$$T_k = 2\int_0^{(1+1/k)\sqrt{k}} \frac{du}{\sqrt{\mathrm{e}^{(1+1/k)^2 k} - \mathrm{e}^{u^2}}};$$

hence, using Lemma 3.12 we can prove that $T_k \approx \sqrt{k}\, \mathrm{e}^{-(1+1/k)^2 k/2}$. Therefore, one need to choose time $t_k << \mathrm{e}^{-(1+1/k)^2 k/2}$ and check that the decoherence of Φ_k and Ψ_k occurs at time t_k. Choose $t_k \in\,]0, T_k/4[$

such that

$$\Phi_k(t_k) = \Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}} - \left(\Big(1+\frac{1}{k}\Big)\sqrt{\frac{k}{4\pi}}\right)^{-1}.$$

It follows that

$$t_k = \int_{\sqrt{k}+1/\sqrt{k}-4\pi\sqrt{k}/(k+1)}^{\sqrt{k}+1/\sqrt{k}} \frac{du}{\sqrt{e^{k(1+1/k)^2} - e^{u^2}}}.$$

Using (21), we obtain $t_k \lesssim (1/\sqrt{k})e^{-k/2}$. In particular, if k is large enough then $t_k \lesssim (\nu/2)e^{-k/2}$. Now we show that this time t_k is sufficient to let instability occurs. Since Ψ_k is decreasing on the interval $[0, (T_k/4)]$, we have

$$e^{4\pi\psi_k(0)^2} - e^{4\pi\psi_k(t_k)^2} = |e^k - e^{4\pi\psi_k(t_k)^2}| \lesssim e^k,$$

Therefore,

$$\big|(\dot\Phi_k(t_k))^2 - (\dot\Psi_k(t_k))^2\big| = \frac{1}{4\pi}\big|(e^{4\pi\Phi_k(0)^2} - e^{4\pi\Phi_k(t_k)^2}) - (e^{4\pi\Psi_k(0)^2} - e^{4\pi\Psi_k(t_k)^2})\big| \gtrsim e^k.$$

Finally, we deduce that

$$\int_{\mathbb{R}^2} \big|\partial_t(u_k - v_k)(t_k)\big|^2\,dx \gtrsim \int_{|x|<(\nu/2)e^{-k/2}} \big|\partial_t(u_k-v_k)(t_k)|^2\,dx \gtrsim \nu^2 e^{-k}|\dot\Phi_k(t_k)) - \dot\Psi_k(t_k)\big|^2$$

and the conclusion follows. □

5. Low regularity data

Proof of Theorem 2.6. (1) For $k \ge 1$ and $\gamma > 1$, let $\varphi_k = \gamma f_k$. An easy computation shows that

$$\|\nabla\varphi_k\|_{L^{2,\infty}} \lesssim \frac{\gamma}{\sqrt{k}}.$$

Next we consider the solution Φ_k of the associated ODE with Cauchy data $(\gamma\sqrt{k/4\pi}, 0)$. The period T_k of Φ_k satisfies

$$T_k \approx \gamma\sqrt{k}\, e^{-(\gamma^2/2)k} \ll e^{-k/2}.$$

Arguing as in the previous section, we construct a sequence (t_k) going to zero such that any weak solution u_k with Cauchy data $(\varphi_k, 0)$ satisfies

$$\|\partial_t u_k(t_k)\|^2_{L^{2,\infty}} \gtrsim e^{(\gamma^2-1)k},$$

and we are done.

(2) Now we will prove the ill-posedness in $\mathscr{B}^1_{2,\infty}$. The main difficulty is the construction of the initial data. For this end, consider a radial smooth function $h \in C_0^\infty(\mathbb{R}^2)$ satisfying $h(r) = 0$ if $r \ge 2$ and $h(r) = 1$ if $r < 1$. For $a > 0$, set $h_a(r) = h(r/a)$. Since $\hat h_a(\xi) = a^2\hat h(a\xi)$, we get

$$|\hat h_a(\xi)| \le \frac{C}{|\xi|^2} \quad \text{uniformly in } a. \tag{41}$$

Now we define the function g_a via

$$g_a(r) = \frac{1 - h_a(r)}{r}.$$

Proposition 5.1. $$|\hat{g}_a(\xi)| \le \frac{C}{|\xi|} \quad \textit{uniformly in } a.$$

Proof. Write
$$\hat{g}_a(\xi) = \frac{C}{|\xi|} - C\Big(\frac{1}{|\xi|} \star \hat{h}_a(\xi)\Big),$$
using the fact that $\widehat{r^{-1}} = C|\xi|^{-1}$. (The convolution here is well defined.) Thus, we have to prove that, for fixed ξ,
$$\left|\int \frac{\hat{h}_a(\eta)}{|\xi-\eta|}\,d\eta\right| \lesssim \frac{1}{|\xi|} \quad \text{uniformly in } a.$$
The idea now is the following: fix ξ such that $|\xi| \sim 2^j$ for some $j \in \mathbb{Z}$ and write
$$\int \frac{\hat{h}_a(\eta)}{|\xi-\eta|}\,d\eta = \int_{|\eta|\le c2^j} \frac{\hat{h}_a(\eta)}{|\xi-\eta|}\,d\eta + \int_{|\eta|\sim 2^j} \frac{\hat{h}_a(\eta)}{|\xi-\eta|}\,d\eta + \int_{|\eta|\ge C2^j} \frac{\hat{h}_a(\eta)}{|\xi-\eta|}\,d\eta. \tag{42}$$
Using (41), we can easily estimate the second and third terms on the right-hand side. To estimate the first term, we use the fact that $\hat{h}_a$ is uniformly in L^1. □

Corollary 5.2. $$\sup_{a>0} \|g_a\|_{\dot{\mathcal{B}}^0_{2,\infty}} < \infty.$$

Proof. Write $\|g_a\|_{\dot{\mathcal{B}}^0_{2,\infty}} \approx \sup_{j\in\mathbb{Z}} \displaystyle\int_{2^{j-1}<|\xi|<2^{j+1}} |\hat{g}_a(\xi)|^2\,d\xi \lesssim \sup_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^{j+1}} \frac{dr}{r} \lesssim 1$, uniformly in a. □

Now we are ready to construct the sequence of initial data (φ_k). Let $\theta \in C_0^\infty(\mathbb{R}^2)$ be a radial function such that $\theta(r) = 1$ if $r \le 1$ and $\theta(r) = 0$ if $r \ge 2$. For $k \ge 1$, set
$$\tilde{g}_k(r) = \frac{1}{\sqrt{k}} g_{e^{-k/2}}(r)\theta(r). \tag{43}$$
It follows from Corollary 5.2 that $\|\tilde{g}_k\|_{\dot{\mathcal{B}}^0_{2,\infty}} \lesssim \dfrac{1}{\sqrt{k}}$. Moreover, one can see easily that
$$\frac{1}{C}\sqrt{k} \le \int_0^2 \tilde{g}_k(r)\,dr\, C\sqrt{k}\,.$$
To finish the construction set
$$\varphi_k(r) = \gamma\sqrt{\frac{k}{4\pi}} - c_k \int_0^r \tilde{g}_k(\tau)\,d\tau,$$
where $\gamma > 1$ and c_k is chosen so that $\varphi_k(2) = 0$. We now summarize some crucial properties of φ_k.

Proposition 5.3. (a) $\varphi_k(r) = \gamma\sqrt{k/4\pi}$ *if* $r \le e^{-k/2}$. (b) $\varphi_k \to 0$ *in* $\mathcal{B}^1_{2,\infty}(\mathbb{R}^2)$.

Proof. Part (a) follows directly from the definition of the function $\tilde{g}_k$. To prove (b), recall that
$$\|\varphi_k\|_{\mathcal{B}^1_{2,\infty}} \approx \|\varphi_k\|_{L^2} + \|\nabla\varphi_k\|_{\dot{\mathcal{B}}^0_{2,\infty}}\,.$$
Since $\|\varphi_k\|_{L^2} \lesssim 1/\sqrt{k}$ we have just to prove that $\|\nabla\varphi_k\|_{\dot{\mathcal{B}}^0_{2,\infty}}$ goes to zero. As $\nabla\varphi_k = (x/r)\tilde{g}_k(r)$, it suffices to apply Theorem 3.3 together with the fact that $x/r \in \dot{\mathcal{B}}^1_{2,\infty} \cap L^\infty$. □

We resume the proof of Theorem 2.6, considering the associated ODE with Cauchy data $(\gamma\sqrt{k/4\pi}, 0)$ and denoting by Φ_k the (global periodic) solution with period

$$T_k \lesssim \int_0^{\gamma\sqrt{k}} \frac{du}{\sqrt{e^{\gamma^2 k} - e^{u^2}}} \lesssim \gamma\sqrt{k}\, e^{-(\gamma^2/2)k} \ll e^{-k/2} \quad (\gamma > 1).$$

Set $t_k = T_k/4$ so that $\Phi_k(t_k) = 0$. Note that by finite speed of propagation any weak solution u_k of (7) with Cauchy data $(\varphi_k, 0)$ satisfies

$$u_k(t, x) = \Phi_k(t) \quad \text{for } 0 < t < e^{-k/2} \text{ and } |x| < e^{-k/2} - t.$$

Hence

$$-\partial_t u_k(t_k, x) \gtrsim e^{(\gamma^2/2)k} \quad \text{for } |x| < e^{-k/2} - t_k. \tag{44}$$

It remains to estimate from below the norm $\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}}$. To get the desired estimate we proceed in the following way. First recall that

$$\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}} = \sup_{\|v\|_{\dot{\mathcal{B}}^0_{2,1}}=1} \int_{\mathbb{R}^2} v(x)\, \partial_t u_k(t_k, x)\, dx.$$

Then we have to make a suitable choice of v. Let v be a smooth compactly supported function such that

$$v(x) = 1 \quad \text{for } |x| \le \tfrac{1}{4}, \qquad v(x) = 0 \quad \text{for } |x| \ge \tfrac{1}{2}.$$

For $k \ge 1$ let $v_k(x) = e^{k/2}\, v(e^{k/2}x)$. We remark that $\|v_k\|_{\dot{\mathcal{B}}^0_{2,\infty}} = \|v\|_{\dot{\mathcal{B}}^0_{2,\infty}}$ is a constant. Using (44), we get

$$\begin{aligned}\|\partial_t u_k(t_k)\|_{\dot{\mathcal{B}}^0_{2,\infty}} &\ge \int -\partial_t u_k(t_k, x)\, v_k(x)\, dx \ge e^{k/2} \int_{|x|\le\frac{1}{4}e^{-k/2}} -\partial_t u_k(t_k, x)\, dx \\ &\gtrsim e^{k/2} (e^{-k/2})^2\, e^{(\gamma^2/2)k} = e^{(\gamma^2-1)/2k}.\end{aligned}$$

This finishes the proof of the part (2) of the theorem, since $\gamma > 1$.

(3) Without loss of generality, we may assume that $0 \le s < 1$. Let $0 < \gamma < \frac{1}{2}(1-s)$ and consider $\varphi_k = k^\gamma f_k$. It is clear that

$$\|\varphi_k\|_{H^s} \lesssim k^\gamma\, k^{-(1-s)/2} \to 0$$

Denote by u_k any weak solution of (9) with initial data $(\varphi_k, 0)$ and Φ_k the solution of the associated ODE with Cauchy data $(k^\gamma \sqrt{k/4\pi}, 0)$. The period T_k of Φ_k satisfies

$$T_k \lesssim k^{\gamma+1/2}\, e^{-(k^{2\gamma+1})/2} \ll e^{-k/2}.$$

Choose $t_k = \dfrac{T_k}{4}$, so that $\Phi_k(t_k) = 0$. By finite speed of propagation, we have

$$u_k(t, x) = \Phi_k(t) \quad \text{for } |x| < e^{-k/2} - t, \;\; 0 < t < e^{-k/2}.$$

Hence $|x| < e^{-k/2} - t_k$,

$$-\partial_t u_k(t_k, x) \;=\; -\dot{\Phi}_k(t_k) = \frac{1}{2\sqrt{\pi}} \sqrt{e^{k^{2\gamma+1}} - e^{4\pi\,\Phi_k^2(t_k)}} = \frac{1}{2\sqrt{\pi}} e^{k^{2\gamma+1}/2}. \tag{45}$$

To conclude the proof we need to estimate from below $\|\partial_t u_k(t_k)\|_{H^{s-1}}$. Write

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} = \sup_{\|v\|_{H^{1-s}}=1} \int_{\mathbb{R}^2} v(x)\,\partial_t u_k(t_k,x)\,dx.$$

Set $v_k(x) = \mathrm{e}^{sk/2}\, v(\mathrm{e}^{k/2}\, x)$, where v is as above. It follows that

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} \geq \int -\partial_t u_k(t_k,x)\, v_k(x)\,dx \geq \mathrm{e}^{sk/2} \int_{|x|\leq \frac{1}{4}\mathrm{e}^{-k/2}} -\partial_t u_k(t_k,x)\,dx \gtrsim \mathrm{e}^{sk/2}\,(\mathrm{e}^{-k/2})^2\,\mathrm{e}^{\frac{1}{2}k^{2\gamma+1}} = \mathrm{e}^{(s/2-1)k+\frac{1}{2}k^{2\gamma+1}},$$

which goes to infinity when $k\to\infty$. □

Proof of Theorem 2.7. (1) Our aim here is to prove the local well-posedness of (7) in $\mathcal{B}^1_{2,q'}\times\mathcal{B}^0_{2,q'}$ for any $1\leq q<\infty$. The strategy is the same as in the proof of Theorem 2.3. We decompose the initial data (u_0,u_1) into a small part[7] in $\mathcal{B}^1_{2,q'}\times\mathcal{B}^0_{2,q'}$ and a regular one:

$$(u_0,u_1) = (u_0,u_1)_{>N} + (u_0,u_1)_{<N}\,.$$

First we solve the IVP with regular data to obtain a local regular solution v, and then we solve the perturbed IVP with small data using a fixed point argument to obtain finally the expected solution u. Let us start by studying the free equation. For a given $(u^0_\ell,u^1_\ell)\in\mathcal{B}^1_{2,q'}\times\mathcal{B}^0_{2,q'}$ we denote by u_ℓ the free solution with data (u^0_ℓ,u^1_ℓ), that is

$$\Box u_\ell + u_\ell = 0,\quad (u_\ell,\partial_t\, u_\ell)(t=0) = (u^0_\ell,u^1_\ell)\,. \tag{46}$$

Using a localization in frequency, an energy estimate and te Strichartz inequality (13), we derive the following result.

Proposition 5.4. *Let $T>0$. For any $1<q'\leq\infty$, there exists $0\leq\varepsilon(q')<\frac{1}{4}$ such that*

$$\|u_\ell\|_{L^\infty_T(\mathcal{B}^1_{2,q'})} + \|u_\ell\|_{L^4_T(\mathscr{C}^{\frac{1}{4}-\varepsilon})} \lesssim \|u^0_\ell\|_{\mathcal{B}^1_{2,q'}} + \|u^1_\ell\|_{\mathcal{B}^0_{2,q'}}\,. \tag{47}$$

(In fact, when $q'\leq 4$, we have a zero loss of derivatives, meaning $\varepsilon(q')=0$, and when $q'>4$, one can choose an arbitrary $0<\varepsilon<\frac{1}{4}$.)

Proof. From the energy and Strichartz estimates applied to $\Delta_j u_\ell$, we have

$$2^j\|\Delta_j u_\ell\|_{L^\infty_T(L^2)} + 2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)} \lesssim 2^j\|\Delta_j u^0_\ell\|_{L^2} + \|\Delta_j u^1_\ell\|_{L^2}\,. \tag{48}$$

Summing this estimate in $\ell^{q'}$ we have $\big\|2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)}\big\|_{\ell^{q'}} \leq \|u^0_\ell\|_{\mathcal{B}^1_{2,q'}} + \|u^1_\ell\|_{\mathcal{B}^0_{2,q'}}$. In the case $q'\leq 4$, the proposition follows from the observation

$$\|u_\ell\|_{L^4(\mathcal{B}^{1/4}_{\infty,q'})} \leq \big\|2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)}\big\|_{\ell^{q'}},$$

together with the Sobolev embedding $\mathcal{B}^{1/4}_{\infty,q'}\to\mathscr{C}^{1/4}$. When $q'>4$, notice that for any $0<\varepsilon<\frac{1}{4}$,

$$\|u_\ell\|_{L^4_T(\mathcal{B}^{1/4-\varepsilon}_{\infty,4})} = \big\|(2^{j/4-j\varepsilon}\|\Delta_j u_\ell\|_{L^\infty})_{\ell^4}\big\|_{L^4_T} = \big\|2^{-j\varepsilon}(2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)})\big\|_{\ell^4}.$$

[7] To do so in the case $q'=\infty$ we have to work with $\tilde{\mathcal{B}}^1_{2,\infty} := \overline{\mathcal{D}}^{\mathcal{B}^1_{2,\infty}}$ and $\tilde{\mathcal{B}}^0_{2,\infty} := \overline{\mathcal{D}}^{\mathcal{B}^0_{2,\infty}}$.

Using (48) and Hölder's inequality in j — writing $\frac{1}{4}=\frac{1}{q'}+\frac{1}{r}$, with $r=\frac{4q'}{q'-4}$ — we get

$$\|u_\ell\|_{L^4_T(\mathscr{B}^{1/4-\varepsilon}_{\infty,4})}\le \|(2^{-j\varepsilon})\|_{\ell^r}\,\|(2^{j/4}\|\Delta_j u_\ell\|_{L^4_T(L^\infty)})\|_{\ell^{q'}}\lesssim \|u^0_\ell\|_{\mathscr{B}^1_{2,q'}}+\|u^1_\ell\|_{\mathscr{B}^0_{2,q'}}\,.$$

Again, Sobolev embedding enables us to finish the proof. □

Define $g_q(u):=u\big((1+u^2)^{(q-2)/2}\mathrm{e}^{4\pi((1+u^2)^{q/2}-1)}-1\big)$, so that (7) reads

$$\Box u+u+g_q(u)=0. \tag{49}$$

An easy computation shows that

$$|g_q(u)-g_q(v)|\le\begin{cases}C|u-v|(\mathrm{e}^{C|u|^q}-1+\mathrm{e}^{C|v|^q}-1) & \text{if } 1\le q\le 2,\\ C|u-v|(u^2+\mathrm{e}^{C|u|^q}-1+v^2+\mathrm{e}^{C|v|^q}-1) & \text{if } 2<q<\infty.\end{cases} \tag{50}$$

According to (50) and the Sobolev embeddings

$$H^1\hookrightarrow\mathscr{B}^1_{2,q'}\quad\text{if } q\le 2,\qquad H^2\hookrightarrow\mathscr{B}^1_{2,q'}\hookrightarrow H^1\quad\text{if } q>2,$$

we will distinguish two cases.

Case $1\le q<2$. We solve $\Box v+v+g_q(v)=0$ with Cauchy data $(u_0,u_1)_{<N}\in H^1\times L^2$ to obtain a global solution $v\in\mathscr{C}(\mathbb{R},H^1)$. Next we have to solve

$$\Box w+w+g_q(v+w)-g_q(v)=0,\quad (w,\partial_t w)(t=0)=(u_0,u_1)_{>N}\,. \tag{51}$$

We seek w in the form

$$w=u_\ell+\boldsymbol{w},$$

where u_ℓ is the free solution with Cauchy data $(u_0,u_1)_{>N}$. Hence $\boldsymbol{w}$ solves

$$\Box\boldsymbol{w}+\boldsymbol{w}+g_q(v+u_\ell+\boldsymbol{w})-g_q(v)=0,\quad (\boldsymbol{w},\partial_t\boldsymbol{w})(t=0)=(0,0)\,. \tag{52}$$

We rely on estimates for the linear part u_ℓ given by Proposition 5.4 in order to choose appropriate functional spaces for which a fixed point argument can be performed. We introduce, for any nonnegative time T and some $0\le\varepsilon<\frac{1}{4}$, the complete metric space

$$\mathscr{E}_T=\mathscr{C}\big([0,T],H^1(\mathbb{R}^2)\big)\cap\mathscr{C}^1\big([0,T],L^2(\mathbb{R}^2)\big)\cap L^4_T\big(\mathscr{C}^{\frac{1}{4}-\varepsilon}(\mathbb{R}^2)\big),$$

endowed with the norm

$$\|u\|_{\mathscr{E}_T}:=\sup_{0\le t\le T}\big[\|u(t,\cdot)\|_{H^1}+\|\partial_t u(t,\cdot)\|_{L^2}\big]+\|u\|_{L^4_T(\mathscr{C}^{\frac{1}{4}-\varepsilon})}.$$

For a positive real number δ, we denote by $\mathscr{E}_T(\delta)$ the ball in $\mathscr{E}_T$ of radius δ and centered at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\boldsymbol{w}\mapsto\Phi(\boldsymbol{w}):=\tilde{\boldsymbol{w}}, \tag{53}$$

where

$$\Box\tilde{\boldsymbol{w}}+\tilde{\boldsymbol{w}}=g_q(v)-g_q(v+u_\ell+\boldsymbol{w}),\quad (\tilde{\boldsymbol{w}},\partial_t\tilde{\boldsymbol{w}})(t=0)=(0,0). \tag{54}$$

To show that, for small T and δ, Φ maps $\mathscr{E}_T(\delta)$ into itself and it is a contraction, we use Proposition 5.4 together with Lemma 3.5 and (50). We skip the details here and refer to [Ibrahim et al. 2006] for similar arguments.

Case $2 < q < \infty$. The method is almost the same as above, except for the choice of the functional spaces. First we solve $\Box v + v + g_q(v) = 0$ with Cauchy data $(u_0, u_1)_{<N} \in H^2 \times H^1$ to obtain a local solution $v \in \mathscr{C}((-T, T), H^2)$. Remember that in this case, the nonlinearity is too strong to solve the Cauchy problem in $H^1 \times L^2$ (see Theorem 2.1). Next we have to solve

$$\Box w + w + g_q(v+w) - g_q(v) = 0, \quad (w, \partial_t w)(t=0) = (u_0, u_1)_{>N}\,. \tag{55}$$

We seek w in the form

$$w = u_\ell + \boldsymbol{w},$$

where u_ℓ is the free solution with Cauchy data $(u_0, u_1)_{>N}$. Hence $\boldsymbol{w}$ solves

$$\Box \boldsymbol{w} + \boldsymbol{w} + g_q(v + u_\ell + \boldsymbol{w}) - g_q(v) = 0, \quad (\boldsymbol{w}, \partial_t \boldsymbol{w})(t=0) = (0,0)\,. \tag{56}$$

We introduce, for any nonnegative time T, the complete metric space

$$\mathscr{E}_T = \mathscr{C}\big([0,T], H^2(\mathbb{R}^2)\big) \cap \mathscr{C}^1\big([0,T], H^1(\mathbb{R}^2)\big) \cap L^4_T\big(\mathscr{C}^{1/4}(\mathbb{R}^2)\big),$$

endowed with the norm

$$\|u\|_{\mathscr{E}_T} := \sup_{0\le t\le T} \big[\|u(t,\cdot)\|_{H^2} + \|\partial_t u(t,\cdot)\|_{H^1}\big] + \|u\|_{L^4_T(\mathscr{C}^{1/4})}.$$

We denote by $\mathscr{E}_T(\delta)$ the ball in $\mathscr{E}_T$ of radius δ and centered at the origin. On the ball $\mathscr{E}_T(\delta)$, we define the map Φ by

$$\boldsymbol{w} \mapsto \Phi(\boldsymbol{w}) := \tilde{\boldsymbol{w}}, \tag{57}$$

where

$$\Box \tilde{\boldsymbol{w}} + \tilde{\boldsymbol{w}} = g_q(v) - g_q(v + u_\ell + \boldsymbol{w}), \quad (\tilde{\boldsymbol{w}}, \partial_t \tilde{\boldsymbol{w}})(t=0) = (0,0). \tag{58}$$

Having in hand Proposition 5.4, Lemma 3.5, and (50), we proceed in a similar way as in the previous case (see also [Ibrahim et al. 2006]) but now we need to be more careful since the source term has to be estimated in $L^1_T(H^1)$ instead of $L^1_T(L^2)$. We refer also to [Colliander et al. 2009] for similar computation in the context of nonlinear Schrödinger equation.

(2) We turn to the second part of the theorem. Without loss of generality, we may assume that $0 \le s < 1$. Also, for the sake of simplicity, we take $q = 1$. Let $\gamma > \frac{1}{2}$ and, for $k \ge 1$, consider the function g_k defined by

$$g_k(x) = \begin{cases} \sqrt{k} & \text{if } |x| \le \mathrm{e}^{-k/2}, \\ -\dfrac{\sqrt{k}}{\log 2} \log|x| + \sqrt{k} - \dfrac{k^{3/2}}{2\log 2} & \text{if } \mathrm{e}^{-k/2} \le |x| \le 2\mathrm{e}^{-k/2}, \\ 0 & \text{if} |x| \ge 2\mathrm{e}^{-k/2}. \end{cases}$$

We remark that

$$\|k^\gamma g_k\|_{H^s} \lesssim k^{\gamma - s + 3/2}\, \mathrm{e}^{-(1-s)k/2} \to 0 \quad \text{as } k \to \infty.$$

Denote by Φ_k the solution of the associated ODE with Cauchy data $(k^{\gamma+\frac{1}{2}}, 0)$. The period T_k of Φ_k satisfies

$$T_k \lesssim k^{\gamma+1/2}\, \mathrm{e}^{-1/2k^{\gamma+1/2}} \ll \mathrm{e}^{-k/2}.$$

Choose $t_k = \dfrac{T_k}{4}$ so that $\Phi_k(t_k) = 0$. By finite speed of propagation, any weak solution u_k of (7) satisfies

$$-\partial_t u_k(t_k, x) \;=\; \dot{\Phi}_k(t_k) = \frac{\mathrm{e}^{-2\pi}}{2\sqrt{\pi}}\sqrt{\mathrm{e}^{\sqrt{k^{2\gamma+1}+1}} - \mathrm{e}^{\sqrt{\Phi_k^2(t_k)+1}}} \gtrsim \mathrm{e}^{\frac{1}{2}k^{\gamma+\frac{1}{2}}} \qquad \text{for } |x| < \mathrm{e}^{-k/2} - t_k.$$

So arguing exactly as before, we get

$$\|\partial_t u_k(t_k)\|_{H^{s-1}} \gtrsim (\mathrm{e}^{-k/2})^2\, \mathrm{e}^{sk/2}\, \mathrm{e}^{\frac{1}{2}k^{\gamma+\frac{1}{2}}} = \mathrm{e}^{(s/2-1)k+\frac{1}{2}k^{\gamma+\frac{1}{2}}}.$$

This concludes the proof once $\gamma > \frac{1}{2}$. □

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