ELECTRONIC SUPPLEMENT
TO
THE CORONA THEOREM FOR THE DRURY–ARVESON HARDY SPACE AND OTHER HOLOMORPHIC BESOV–SOBOLEV SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

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Here we collect proofs of formulas and modifications of arguments already in the literature that would otherwise interrupt the main flow of the paper. All references not starting in ES are to results and equations in the main paper:

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Charpentier's solution kernels. Here we prove Theorem 8. In the computation of the Cauchy kernel $\mathscr{C}_n(w,z)$, we need to compute the full exterior derivative of the section $s(w,z)$. By definition one has,

$$s_i(w,z) = \overline{w_i}(1 - w\overline{z}) - \overline{z_i}(1 - |w|^2),$$
$$ds_i(w,z) \equiv (\partial_w + \overline{\partial}_w + \partial_z + \overline{\partial}_z)s_i(w,z)$$

Straightforward computations show that

$$\partial_w s_i(w,z) = \sum_{j=1}^n \left(\overline{z}_i \overline{w}_j - \overline{w}_i \overline{z}_j\right) dw_j \tag{ES-1}$$
$$\overline{\partial}_w s_i(w,z) = (1 - w\overline{z})\, d\overline{w}_i + \sum_{j=1}^n w_j \overline{z}_i d\overline{w}_j$$
$$\overline{\partial}_z s_i(w,z) = -\sum_{j=1}^n \overline{w}_i w_j d\overline{z}_j - \left(1 - |w|^2\right) d\overline{z}_i$$
$$\partial_z s_i(w,z) = 0,$$

as well as

$$\overline{\partial}_w s_k = (1 - w\overline{z})d\overline{w}_k + \overline{z}_k \overline{\partial}_w |w|^2$$
$$\overline{\partial}_z s_k = -(1 - |w|^2)d\overline{z}_k - \overline{w}_k \overline{\partial}_z (w\overline{z}).$$

We also have the following representations of s_k, again following by simple computation. Recall from the Notation on page 509 that $\{1, 2, \ldots, n\} = \{i_\nu\} \cup J_\nu \cup L_\nu$ where J_ν and L_ν are increasing multi-indices of lengths

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$n-q-1$ and q. We will use the following with $k=i_\nu$.

$$\begin{aligned}
s_k &= (\overline{w}_k-\overline{z}_k)+\sum_{l\neq k} w_l(\overline{w}_l\overline{z}_k-\overline{w}_k\overline{z}_l)\\
&= (\overline{w}_k-\overline{z}_k)+\sum_{j\in J_\nu} w_j(\overline{w}_j\overline{z}_k-\overline{w}_k\overline{z}_j)+\sum_{l\in L_\nu} w_l(\overline{w}_l\overline{z}_k-\overline{w}_k\overline{z}_l)\\
&= (\overline{w}_k-\overline{z}_k)+\overline{z}_k\sum_{j\in J_\nu}|w_j|^2-\overline{w}_k\sum_{j\in J_\nu} w_j\overline{z}_j+\overline{z}_k\sum_{l\in L_\nu}|w_l|^2-\overline{w}_k\sum_{l\in L_\nu} w_l\overline{z}_l.
\end{aligned}$$

Remark. Since $A\wedge A=0$ for any form, we have in particular that $\overline{\partial_w}\,|w|^2\wedge\overline{\partial_w}\,|w|^2=0$ and $\overline{\partial_z}\,(w\overline{z})\wedge\overline{\partial_z}\,(w\overline{z})=0$.

Using this remark we next compute $\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j$. We identify J_ν as $j_1<j_2<\cdots<j_{n-q-1}$ and define a map $\imath(j_r)=r$, namely $\imath$ says where j_r occurs in the multi-index. We will frequently abuse notation and simply write $\imath(j)$. Because $\overline{\partial}_w|w|^2\wedge\overline{\partial}_w|w|^2=0$ it is easy to conclude that we can not have any term in $\overline{\partial}_w|w|^2$ of degree greater than one when expanding the wedge product of the $\overline{\partial_w s_j}$.

$$\begin{aligned}
\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j &= \bigwedge_{j\in J_\nu}\left\{(1-w\overline{z})d\overline{w}_j+\overline{z}_j\overline{\partial}_w|w|^2\right\}\\
&= (1-w\overline{z})^{n-q-1}\bigwedge_{j\in J_\nu} d\overline{w}_j+(1-w\overline{z})^{n-q-2}\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\overline{\partial}_w|w|^2\wedge\bigwedge_{j'\in J_\nu\setminus\{j\}} d\overline{w}_{j'}\\
&= (1-w\overline{z})^{n-q-2}\\
&\quad\left(\left(1-w\overline{z}+\sum_{j\in J_\nu} w_j\overline{z}_j\right)\bigwedge_{j\in J_\nu} d\overline{w}_j+\sum_{j\in J_\nu}(-1)^{\imath(j)-1}\overline{z}_j\sum_{k\in L_\nu\cup\{i_\nu\}} w_k d\overline{w}_k\bigwedge_{j'\in J_\nu\setminus\{j\}} d\overline{w}_{j'}\right).
\end{aligned}$$

The last line follows by direct computation using

$$\overline{\partial}_w\,|w|^2=\sum_{j\in J_\nu} w_j d\overline{w}_j+\sum_{k\in L_\nu\cup\{i_\nu\}} w_k d\overline{w}_k.$$

A similar computation yields that

$$\begin{aligned}
&\bigwedge_{l\in L_\nu}\overline{\partial}_z s_l\\
&= (-1)^q\bigwedge_{l\in L_\nu}\left\{(1-|w|^2)d\overline{z}_l+\overline{w}_l\overline{\partial}_z(w\overline{z})\right\}\\
&= (-1)^q\left((1-|w|^2)^q\bigwedge_{l\in L_\nu} d\overline{z}_l+(1-|w|^2)^{q-1}\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\overline{\partial}_z(w\overline{z})\wedge\bigwedge_{l'\in L_\nu\setminus\{l\}} d\overline{z}_{l'}\right)\\
&= (-1)^q(1-|w|^2)^{q-1}\\
&\quad\left(\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{l\in L_\nu} d\overline{z}_l+\sum_{l\in L_\nu}(-1)^{\imath(l)-1}\overline{w}_l\sum_{k\in J_\nu\cup\{i_\nu\}} w_k d\overline{z}_k\bigwedge_{l'\in L_\nu\setminus\{l\}} d\overline{z}_{l'}\right).
\end{aligned}$$

An important remark at this point is that the multi-index J_ν or L_ν can only appear in the first term of the last line above. The terms after the plus sign have multi-indices that are related to J_ν and L_ν, but differ by one element. This fact will play a role later.

Combining things, we see that

$$\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j \bigwedge_{l\in L_\nu}\overline{\partial}_z s_j = (-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}\left(I_\nu + II_\nu + III_\nu + IV_\nu\right),$$

where

$$I_\nu = \left(1-w\overline{z}+\sum_{j\in J_\nu} w_j\overline{z}_j\right)\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{j\in J_\nu} d\overline{w}_j \bigwedge_{l\in L_\nu} d\overline{z}_l,$$

$$II_\nu = \left(1-w\overline{z}+\sum_{j\in J_\nu} w_j\overline{z}_j\right)\bigwedge_{j\in J_\nu} d\overline{w}_j\left(\sum_{l\in L_\nu}(-1)^{\iota(l)-1}\overline{w}_l \sum_{k\in J_\nu\cup\{i_\nu\}} w_k d\overline{z}_k \bigwedge_{l'\in L_\nu\setminus\{l\}} d\overline{z}_{l'}\right),$$

$$III_\nu = \left(\sum_{j\in J_\nu}(-1)^{\iota(j)-1}\overline{z}_j \sum_{k\in L_\nu\cup\{i_\nu\}} w_k d\overline{w}_k \bigwedge_{j'\in J_\nu\setminus\{j\}} d\overline{w}_{j'}\right)\left(1-|w|^2+\sum_{l\in L_\nu}|w_l|^2\right)\bigwedge_{l\in L_\nu} d\overline{z}_l,$$

$$\begin{aligned} IV_\nu = &\left(\sum_{j\in J_\nu}(-1)^{\iota(j)-1}\overline{z}_j \sum_{k\in L_\nu\cup\{i_\nu\}} w_k d\overline{w}_k \bigwedge_{j'\in J_\nu\setminus\{j\}} d\overline{w}_{j'}\right)\\ &\times\left(\sum_{l\in L_\nu}(-1)^{\iota(l)-1}\overline{w}_l \sum_{k\in J_\nu\cup\{i_\nu\}} w_k d\overline{z}_k \bigwedge_{l'\in L_\nu\setminus\{l\}} d\overline{z}_{l'}\right).\end{aligned}$$

We next introduce a little more notation to aid in the computation of the kernel $\mathscr{C}_n^{0,q}(w,z)$. For $1\le k\le n$ we let $P_n^q(k)=\{\nu\in P_n^q : \nu(1)=i_\nu=k\}$. This divides the set P_n^q into n classes with $\frac{(n-1)!}{(n-q-1)!q!}$ elements. At this point, with the notation introduced in the Notation on page 509 and computations performed above, we have reduced the calculation of $\mathscr{C}_n^{0,q}(w,z)$ to

$$\begin{aligned}\mathscr{C}_n^{0,q}(w,z) &= \frac{1}{\triangle(w,z)^n}\sum_{\nu\in P_n^q}\epsilon_\nu s_{i_\nu}\bigwedge_{j\in J_\nu}\overline{\partial}_w s_j \bigwedge_{l\in L_\nu}\overline{\partial}_z s_l \wedge\omega(w)\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_k\sum_{\nu\in P_n^q(k)}\epsilon_\nu\left(I_\nu+II_\nu+III_\nu+IV_\nu\right)\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_k\left(I(k)+II(k)+III(k)+IV(k)\right)\\ &= \frac{(-1)^q(1-w\overline{z})^{n-q-2}(1-|w|^2)^{q-1}}{\triangle(w,z)^n}\sum_{k=1}^n s_k C(k).\end{aligned}$$

Here we have defined $C(k)\equiv I(k)+II(k)+III(k)+IV(k)$, and

$$I(k)\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu I_\nu \quad II(k)\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu II_\nu$$

$$III(k)\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu III_\nu \quad IV(k)\equiv\sum_{\nu\in P_n^q(k)}\epsilon_\nu IV_\nu.$$

For a fixed $\tau \in P_n^q$ we will compute the coefficient of $\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$. We will ignore the functional coefficient in front of the sum since it only needs to be taken into consideration at the final stage. We will show that for this fixed τ the sum on k of s_k times $I(k)$, $II(k)$, $III(k)$ and $IV(k)$ can be replaced by $\epsilon_\tau(1-w\overline{z})(1-|w|^2)(\overline{w}_{i_\tau}-\overline{z}_{i_\tau})\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$. There will also be other terms that appear in this expression that arise from multi-indices J and I that are not disjoint. Using the computations below it can be seen that these terms actually vanish and hence provide no contribution for $\mathscr{C}_n^{0,q}(w,z)$. Since τ is an arbitrary element of P_n^q this will then complete the computation of the kernel.

Note that when $k=i_\tau$ then we have the following contributions. It is easy to see that $II(i_\tau)=III(i_\tau)=0$. It is also easy to see that

$$\begin{aligned}
I(i_\tau) &= \epsilon_\tau\left(1-w\overline{z}+\sum_{j\in J_\tau} w_j\overline{z}_j\right)\left(1-|w|^2+\sum_{l\in L_\tau}|w_l|^2\right)\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l\\
&= \epsilon_\tau(1-w\overline{z})(1-|w|^2)\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l\\
&\quad+\left((1-w\overline{z})\sum_{l\in L_\tau}|w_l|^2+(1-|w|^2)\sum_{j\in J_\tau} w_j\overline{z}_j+\sum_{l\in L_\tau}|w_l|^2\sum_{j\in J_\tau} w_j\overline{z}_j\right)\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l.
\end{aligned}$$

We also receive a contribution from term $IV(i_\tau)$ is this case. This happens by interchanging an index in the multi-index J_τ with one in L_τ. Namely, we consider the permutations $\nu:\{1,\dots,n\}\to\{i_\tau,(J_\tau\setminus\{j\})\cup\{l\},(L_\tau\setminus\{l\})\cup\{j\}\}$. This permutation contributes the term $\overline{z}_l w_l\overline{w}_j w_j$. After summing over all these possible permutations, we arrive at the simplified formula,

$$IV(i_\tau)=-\epsilon_\tau\left(\sum_{j\in J_\tau}|w_j|^2\right)\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right)\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l.$$

Collecting all these terms, when $k=i_\tau$ we have that the coefficient of $\epsilon_\tau\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ is:

$$\begin{aligned}
C(i_\tau) &= (1-w\overline{z})(1-|w|^2)+(1-w\overline{z}+\sum_{j\in J_\tau} w_j\overline{z}_j)\sum_{l\in L_\tau}|w_l|^2\\
&\quad+(1-|w|^2+\sum_{l\in L_\tau}|w_l|^2)\sum_{j\in J_\tau} w_j\overline{z}_j-\sum_{l\in L_\tau}|w_l|^2\sum_{j\in J_\tau} w_j\overline{z}_j-\sum_{j\in J_\tau}|w_j|^2\sum_{l\in L_\tau} w_l\overline{z}_l.
\end{aligned}$$

Next note that when $k\neq i_\tau$ one can still have terms which contribute to the coefficient of $\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$. To see this we further split the conditions on k into the situations where $k\in J_\tau$ and $k\in L_\tau$. First, observe in this situation that if $k\neq i_\tau$ then term $I(k)$ can never contribute. So all contributions must come from terms $II(k)$, $III(k)$, and $IV(k)$. In these terms it is possible to obtain the term $\bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ by replacing some index in ν. Namely, it is possible to have ν and τ differ by one index from each other, or one by replacing an index with i_τ.

Next, observe that when $k\in L_\tau$ there exists a unique $\nu\in P_n^q(k)$ such that $J_\nu=J_\tau$. Namely, we have that $\nu:\{1,\dots,n\}\to\{k,J_\tau,(L_\tau\setminus\{k\})\cup i_\tau\}$. Here, we used that $i_\nu=k$. Terms of this type will contribute to term $II(k)$ but will give no contribution to term $III(k)$. However, they will give a contribution to term $IV(k)$.

Similarly, when $k\in J_\tau$ there will exist a unique $\mu\in P_n^q(k)$ with $L_\mu=L_\tau$. This happens with $\mu:\{1,\dots,n\}\to\{k,(J_\tau\setminus\{k\})\cup i_\tau,L_\tau\}$. Here we used that $i_\mu=k$. Again, we get a contribution to term $III(k)$ and $IV(k)$ and they give no contribution to the term $II(k)$.

Using these observations when $k \in L_\tau$ we arrive at the following for $I(k)$, $II(k)$, $III(k)$, and $IV(k)$:

$$\begin{aligned}
I(k) &= 0\\
II(k) &= -\epsilon_\tau \left(1 - w\overline{z} + \sum_{j\in J_\tau} w_j\overline{z}_j\right)\overline{w}_{i_\tau} w_k \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l\\
III(k) &= 0\\
IV(k) &= \epsilon_\tau \overline{z}_{i_\tau} w_k \left(\sum_{j\in J_\tau} |w_j|^2\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l .
\end{aligned}$$

Similarly, when $k \in J_\tau$ we arrive at the following for $I(k)$, $II(k)$, $III(k)$, and $IV(k)$:

$$\begin{aligned}
I(k) &= 0\\
II(k) &= 0\\
III(k) &= -\epsilon_\tau \left(1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2\right)\overline{z}_{i_\tau} w_k \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l\\
IV(k) &= \epsilon_\tau \overline{w}_{i_\tau} w_k \left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l .
\end{aligned}$$

Collecting these terms, we see the following for the coefficient of $\epsilon_\tau \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$:

$$C(k) = -w_k \left(\overline{z}_{i_\tau}\left(1 - |w|^2 + \textstyle\sum_{l\in L_\tau} |w_l|^2\right) - \overline{w}_{i_\tau}\left(\textstyle\sum_{l\in L_\tau} w_l\overline{z}_l\right)\right) \quad \forall k \in J_\tau,$$

$$C(k) = -w_k \left(\overline{w}_{i_\tau}\left(1 - w\overline{z} + \textstyle\sum_{j\in J_\tau} w_j\overline{z}_j\right) - \overline{z}_{i_\tau}\left(\textstyle\sum_{j\in J_\tau} |w_j|^2\right)\right) \quad \forall k \in L_\tau.$$

This then implies that the *total* coefficient of $\epsilon_\tau \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l$ is given by

$$s_{i_\tau} C(i_\tau) + \sum_{k\in J_\tau} s_k C(k) + \sum_{k\in L_\tau} s_k C(k).$$

At this point the remainder of the proof of the Theorem 8 reduces to tedious algebra. The term $s_{i_\tau} C(i_\tau)$ will contribute the term $(1 - w\overline{z})(1 - |w|^2)(\overline{w}_{i_\tau} - \overline{z}_{i_\tau})$ and a remainder term. The remainder term will cancel with the terms $\sum_{k\neq i_\tau} s_k C(k)$.

We first compute the term $s_k C(k)$ for $k \in J_\tau$. Note that in this case, we have that

$$\begin{aligned}
C(k) &= w_k \left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1 - |w|^2 + \sum_{l\in L_\tau} |w_l|^2\right)\right)\\
&= w_k \left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right) - \overline{z}_{i_\tau}\left(1 - \sum_{l\in J_\tau} |w_l|^2\right)\right) + w_k\overline{z}_{i_\tau}|w_{i_\tau}|^2 .
\end{aligned}$$

Multiplying this by s_k we see that

$$\begin{aligned} s_k C(k) &= (1-w\overline{z})\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right)-\overline{z}_{i_\tau}\left(1-\sum_{l\in J_\tau}|w_l|^2\right)\right)|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right)-\overline{z}_{i_\tau}\left(1-\sum_{l\in J_\tau}|w_l|^2\right)\right)w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2|w_k|^2-(1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2 w_k\overline{z}_k. \end{aligned}$$

Upon summing in $k\in J_\tau$ we find that

$$\begin{aligned} \sum_{k\in J_\tau} s_k C(k) &= (1-w\overline{z})\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right)-\overline{z}_{i_\tau}\left(1-\sum_{j\in J_\tau}|w_j|^2\right)\right)\sum_{k\in J_\tau}|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{w}_{i_\tau}\left(\sum_{l\in L_\tau} w_l\overline{z}_l\right)-\overline{z}_{i_\tau}\left(1-\sum_{j\in J_\tau}|w_j|^2\right)\right)\sum_{k\in J_\tau} w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in J_\tau}|w_k|^2-(1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in J_\tau} w_k\overline{z}_k. \end{aligned}$$

Performing similar computations for $k\in L_\tau$ we find,

$$\begin{aligned} \sum_{k\in L_\tau} s_k C(k) &= (1-w\overline{z})\left(\overline{z}_{i_\tau}\left(\sum_{k\in J_\tau}|w_j|^2\right)-\overline{w}_{i_\tau}\left(1-\sum_{l\in L_\tau} w_l\overline{z}_l\right)\right)\sum_{k\in L_\tau}|w_k|^2 \\ &\quad -(1-|w|^2)\left(\overline{z}_{i_\tau}\left(\sum_{k\in J_\tau}|w_j|^2\right)-\overline{w}_{i_\tau}\left(1-\sum_{l\in L_\tau} w_l\overline{z}_l\right)\right)\sum_{k\in L_\tau} w_k\overline{z}_k \\ &\quad +(1-w\overline{z})\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in L_\tau}|w_k|^2-(1-|w|^2)\overline{z}_{i_\tau}|w_{i_\tau}|^2\sum_{k\in L_\tau} w_k\overline{z}_k. \end{aligned}$$

Putting this all together we find that

$$\begin{aligned} &\sum_{k\neq i_\tau} s_k C(k) \\ &= \overline{w}_{i_\tau}(1-w\overline{z})\left(\left(\sum_{k\in L_\tau} w_l\overline{z}_l\right)\left(\sum_{k\in J_\tau}|w_k|^2\right)-\left(1-\sum_{k\in L_\tau} w_k\overline{z}_k-w_{i_\tau}\overline{z}_{i_\tau}\right)\left(\sum_{k\in L_\tau}|w_k|^2\right)\right) \\ &\quad +\overline{z}_{i_\tau}(1-|w|^2)\left(\left(1-\sum_{k\in J_\tau}|w_k|^2-|w_{i_\tau}|^2\right)\left(\sum_{k\in J_\tau} w_k\overline{z}_k\right)-\left(\sum_{k\in J_\tau}|w_j|^2\right)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right)\right) \\ &\quad -\overline{z}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in J_\tau}|w_j|^2\right)+\overline{w}_{i_\tau}(1-w\overline{z})(1-|w|^2)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right). \end{aligned}$$

We next compute the term $s_{i_\tau} C(i_\tau)$. Using the properties of s_k we have that $s_{i_\tau} C(i_\tau)$ is

$$
\begin{aligned}
&(\overline{w}_{i_\tau} - \overline{z}_{i_\tau})\,(1 - w\overline{z})(1 - |w|^2) \\
&+\overline{z}_{i_\tau}\,(1 - w\overline{z})(1 - |w|^2)\left(\sum_{k\in J_\tau} |w_k|^2\right) - \overline{w}_{i_\tau}\,(1 - w\overline{z})(1 - |w|^2)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right) \\
&+\overline{w}_{i_\tau}\,(1 - w\overline{z})\left\{(1 - w\overline{z})\left(\sum_{k\in L_\tau} |w_k|^2\right) + \left(\sum_{k\in L_\tau} |w_k|^2\right)\left(\sum_{k\in J_\tau} w_k\overline{z}_k\right)\right. \\
&\qquad\qquad\left. - \left(\sum_{k\in J_\tau} |w_k|^2\right)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right)\right\} \\
&+\overline{z}_{i_\tau}\,(1 - |w|^2)\left\{-(1 - |w|^2)\left(\sum_{k\in J_\tau} w_k\overline{z}_k\right) - \left(\sum_{k\in L_\tau} |w_k|^2\right)\left(\sum_{k\in J_\tau} w_k\overline{z}_k\right)\right. \\
&\qquad\qquad\left. + \left(\sum_{k\in J_\tau} |w_k|^2\right)\left(\sum_{k\in L_\tau} w_k\overline{z}_k\right)\right\}.
\end{aligned}
$$

From this point on it is simple to see that the remainder of the term $s_{i_\tau} C(i_\tau)$ cancels with $\sum_{k\neq i_\tau} s_k C(k)$. One simply adds and subtracts a common term in parts of $\sum_{k\neq i_\tau} s_k C(k)$. The only term that remains is $(\overline{w}_{i_\tau} - \overline{z}_{i\tau})(1 - w\overline{z})(1 - |w|^2)$. Thus, we see that the term corresponding to τ in the sum $\mathcal{C}_n^{0,q}(w,z)$ is

$$
\epsilon_\tau \frac{(-1)^q (1 - w\overline{z})^{n-q-2}(1 - |w|^2)^{q-1}}{\triangle(w,z)^n}(1 - w\overline{z})(1 - |w|^2)(\overline{w}_{i_\tau} - \overline{z}_{i_\tau}) \bigwedge_{j\in J_\tau} d\overline{w}_j \bigwedge_{l\in L_\tau} d\overline{z}_l \wedge \omega(w).
$$

Since τ was arbitrary we conclude that $\mathcal{C}_n^{0,q}(w,z)$ equals

$$
\frac{(1 - w\overline{z})^{n-q-1}\left(1 - |w|^2\right)^q}{\triangle\,(w,z)^n}
$$

times

$$
\sum_{\nu\in P_n^q} \epsilon_\nu (\overline{w}_{i_\nu} - \overline{z}_{i_\nu}) \bigwedge_{j\in J_\nu} d\overline{w}_j \bigwedge_{l\in L_\nu} d\overline{z}_l \wedge \omega(w),
$$

which completes the proof of Theorem 8.

Explicit formulas for kernels in $n = 2$ and 3 dimensions . Using the above computations and simplifying algebra we obtain the formulas

$$
\begin{aligned}
&\mathcal{C}_2^{0,0}(w,z) \\
&= \frac{(1 - w\overline{z})}{\triangle(w,z)^2}\left[(\overline{z}_2 - \overline{w}_2)d\overline{w}_1 \wedge dw_1 \wedge dw_2 - (\overline{z}_1 - \overline{w}_1)d\overline{w}_2 \wedge dw_1 \wedge dw_2\right],
\end{aligned}
\tag{ES-2}
$$

and

$$
\begin{aligned}
&\mathcal{C}_2^{0,1}(w,z) \\
&= \frac{(1 - |w|^2)}{\triangle(w,z)^2}\left[(\overline{w}_2 - \overline{z}_2)d\overline{z}_1 \wedge dw_1 \wedge dw_2 - (\overline{w}_1 - \overline{z}_1)d\overline{z}_2 \wedge dw_1 \wedge dw_2\right],
\end{aligned}
\tag{ES-3}
$$

and

$$\begin{aligned}
&\mathscr{C}_3^{0,q}(w,z) \qquad \text{(ES-4)}\\
&= \sum_{\sigma\in\mathscr{S}_3} sgn(\sigma)\, \frac{(1-w\overline{z})^{2-q}\left(1-|w|^2\right)^q\left(\overline{z_{\sigma(1)}}-\overline{w_{\sigma(1)}}\right)}{\triangle(w,z)^3} d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}}\wedge\omega_3(w),
\end{aligned}$$

where $\mathscr{S}_3$ denotes the group of permutations on $\{1,2,3\}$ and q determines the number of $d\overline{z_i}$ in the form $d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}}$:

$$d\overline{\zeta_{\sigma(2)}}\wedge d\overline{\zeta_{\sigma(3)}} = \begin{cases} d\overline{w_{\sigma(2)}}\wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q=0\\ d\overline{z_{\sigma(2)}}\wedge d\overline{w_{\sigma(3)}} & \text{if} \quad q=1\\ d\overline{z_{\sigma(2)}}\wedge d\overline{z_{\sigma(3)}} & \text{if} \quad q=2\end{cases}.$$

Integrating in higher dimensions. Here we give the proof of (2-19). Let

$$B\equiv\frac{\left(1-|z|^2\right)}{|1-w\overline{z}|^2} \text{ and } R\equiv\sqrt{1-|w|^2},$$

so that

$$BR^2=\frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}=1-|\varphi_w(z)|^2.$$

Then with the change of variable $\rho=Br^2$ we obtain

$$\begin{aligned}
&(1-w\overline{z})^{s-q-1}\int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{(1-|w|^2-|w'|^2)^q}{\triangle((w,w'),(z,0))^s}dV(w')\\
&=\frac{(1-w\overline{z})^{s-q-1}}{|1-w\overline{z}|^{2s}}\int_{\sqrt{1-|w|^2}\mathbb{B}_k}\frac{\left(1-|w|^2-|w'|^2\right)^q}{\left(1-\frac{(1-|z|^2)}{|1-w\overline{z}|^2}\left(1-|w|^2-|w'|^2\right)\right)^s}dV\left(w'\right)\\
&=\frac{(1-w\overline{z})^{s-q-1}}{|1-w\overline{z}|^{2s}}\int_0^R\frac{\left(R^2-r^2\right)^q}{\left(1-BR^2+Br^2\right)^s}r^{2k-1}dr\\
&=\frac{(1-w\overline{z})^{s-q-1}}{2B^{k+q}|1-w\overline{z}|^{2s}}\int_0^{BR^2}\frac{\left(BR^2-\rho\right)^q}{\left(1-BR^2+\rho\right)^s}\rho^{k-1}d\rho,
\end{aligned}$$

which with

$$\Psi_{n,k}^{0,q}(t)=\frac{(1-t)^n}{t^k}\int_0^t\frac{(t-\rho)^q}{(1-t+\rho)^{n+k}}\rho^{k-1}d\rho,$$

we rewrite as

$$\begin{aligned}
&\frac{(1-w\overline{z})^{s-q-1}}{2B^{k+q}\,|1-w\overline{z}|^{2s}}\frac{(BR^2)^k}{|\varphi_w(z)|^{2n}}\Psi^{0,q}_{n,k}\left(BR^2\right)\\
&=\frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^k}{2\left(1-|z|^2\right)^q|1-w\overline{z}|^{2s}}\frac{|1-w\overline{z}|^{2q}}{|\varphi_w(z)|^{2n}}\Psi^{0,q}_{n,k}\left(BR^2\right)\\
&=\frac{(1-w\overline{z})^{s-q-1}\left(1-|w|^2\right)^k}{2\left(1-|z|^2\right)^q}\frac{|1-w\overline{z}|^{2q-2k}}{\triangle(w,z)^n}\Psi^{0,q}_{n,k}\left(BR^2\right)\\
&=\frac{1}{2}\Phi^q_n(w,z)\left(\frac{1-|w|^2}{1-\overline{w}z}\right)^{k-q}\left(\frac{1-w\overline{z}}{1-|z|^2}\right)^q\Psi^{0,q}_{n,k}\left(BR^2\right).
\end{aligned}$$

since $\Phi^q_n(w,z)=\frac{(1-w\overline{z})^{n-1-q}\left(1-|w|^2\right)^q}{\triangle(w,z)^n}$.

At this point we claim that

$$\Psi^{0,q}_{n,k}(t)=\frac{(1-t)^n}{t^k}\int_0^t\frac{(t-r)^q}{(1-t+r)^{n+k}}r^{k-1}dr \tag{ES-5}$$

is a polynomial in

$$t=BR^2=1-|\varphi_w(z)|^2$$

of degree $n-1$ that vanishes to order q at $t=0$, so that

$$\Psi^{0,q}_{n,k}(t)=\sum_{j=q}^{n-1}c_{j,n,s}\left(\frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-w\overline{z}|^2}\right)^j,$$

With this claim established, the proof of (2-19) is complete.

To see that $\Psi^{0,q}_{n,k}$ vanishes of order q at $t=0$ is easy since for t small (ES-5) yields

$$\left|\Psi^{0,q}_{n,k}(t)\right|\le Ct^{-k}\int_0^t\frac{t^q}{C}r^{k-1}dr\le Ct^q.$$

To see that$\Psi^{0,q}_{n,k}$ is a polynomial of degree $n-1$ we prove two recursion formulas valid for $0\le t<1$ (we let $t\to 1$ at the end of the argument):

$$\begin{aligned}
\Psi^{0,q}_{n,k}(t)-\Psi^{0,q+1}_{n,k}(t)&=(1-t)\,\Psi^{0,q}_{n-1,k}(t), \\
\Psi^{0,0}_{n,k}(t)&=\frac{1}{k}(1-t)^n+\frac{n+k}{k}t\Psi^{0,0}_{n,k+1}(t).
\end{aligned} \tag{ES-6}$$

The first formula follows from

$$(t-r)^q-(t-r)^{q+1}=(t-r)^q(1-t+r),$$

while the second is an integration by parts:

$$\begin{aligned}\int_0^t \frac{r^{k-1}}{(1-t+r)^{n+k}}dr &= \frac{1}{k}\frac{r^k}{(1-t+r)^{n+k}}\Big|_0^t \\ &\quad +\frac{n+k}{k}\int_0^t \frac{r^k}{(1-t+r)^{n+k+1}}dr \\ &= \frac{1}{k}t^k+\frac{n+k}{k}\int_0^t \frac{r^k}{(1-t+r)^{n+k+1}}dr.\end{aligned}$$

If we multiply this equality through by $\frac{(1-t)^n}{t^k}$ we obtain the second formula in (ES-6).

The recursion formulas in (ES-6) reduce matters to proving that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$. Indeed, once we know that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$, then the second formula in (ES-6) and induction on k shows that $\Psi_{n,k}^{0,0}$ is as well. Then the first formula and induction on q then shows that $\Psi_{n,k}^{0,q}$ is also. To see that $\Psi_{n,1}^{0,0}$ is a polynomial of degree $n-1$ we compute

$$\begin{aligned}\Psi_{n,1}^{0,0}(t) &= \frac{(1-t)^n}{t}\int_0^t \frac{1}{(1-t+r)^{n+1}}dr \\ &= \frac{(1-t)^n}{t}\left\{-\frac{1}{n(1-t+r)^n}\right\}\Big|_0^t \\ &= \frac{1-(1-t)^n}{nt},\end{aligned}$$

which is a polynomial of degree $n-1$. This finishes the proof of the claim, and hence that of (2-19) as well.

Integration by parts formulas in the ball. We begin by proving the generalized analogue of the integration by parts formula of [Ortega and Fàbrega 2000] as given in Lemma 14. For this we will use the following identities.

Lemma ES.1. *For $\ell \in \mathbb{Z}$, we have*

$$\begin{aligned}\overline{\mathfrak{X}}\left\{\triangle(w,z)^\ell\right\} &= \ell\,\triangle(w,z)^\ell, \\ \overline{\mathfrak{X}}\left\{(1-w\overline{z})^\ell\right\} &= 0, \\ \overline{\mathfrak{X}}\left\{\left(1-|w|^2\right)^\ell\right\} &= \ell\left(1-|w|^2\right)^\ell-\ell\left(1-|w|^2\right)^{\ell-1}(1-\overline{z}w).\end{aligned} \tag{ES-7}$$

Proof of Lemma ES.1. The computation

$$\begin{aligned}\frac{\partial\triangle}{\partial\overline{w_j}} &= \frac{\partial}{\partial\overline{w_j}}\left\{|1-w\overline{z}|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)\right\} \\ &= (w\overline{z}-1)\,z_j+\left(1-|z|^2\right)w_j,\end{aligned}$$

shows that $\overline{\mathfrak{X}}\triangle = \triangle$:

$$
\begin{aligned}
\overline{\mathfrak{X}}\,\triangle(w,z) &= \left(\sum_{j=1}^{n}\left(\overline{w_j}-\overline{z_j}\right)\frac{\partial}{\partial\overline{w_j}}\right)\left\{\left|1-w\overline{z}\right|^2-\left(1-|w|^2\right)\left(1-|z|^2\right)\right\}\\
&=\sum_{j=1}^{n}\left(\overline{w_j}-\overline{z_j}\right)\left\{\left(w\overline{z}-1\right)z_j+\left(1-|z|^2\right)w_j\right\}\\
&=\left(\overline{w}z-|z|^2\right)\left(w\overline{z}-1\right)+\left(1-|z|^2\right)\left(|w|^2-\overline{z}w\right)\\
&=-\overline{w\overline{z}}+|z|^2+|\overline{w}z|^2-|z|^2\,w\overline{z}+|w|^2-w\overline{z}-|z|^2\,|w|^2+|z|^2\,w\overline{z}\\
&=-2\operatorname{Re}w\overline{z}+|z|^2+|w\overline{z}|^2+|w|^2-|z|^2\,|w|^2\\
&=|w-z|^2+|w\overline{z}|^2-|z|^2\,|w|^2=\triangle(w,z)
\end{aligned}
$$

by the second line in (2-1) above. Iteration then gives the first line in (ES-7). The second line is trivial since $1-w\overline{z}$ is holomorphic in w. The third line follows by iterating

$$
\overline{\mathfrak{X}}\left(1-|w|^2\right)=\overline{z}w-|w|^2=\left(1-|w|^2\right)-\left(1-\overline{z}w\right). \qquad \square
$$

Proof of Lemma 14. We use the general formula (2-7) for the solution kernels $\mathcal{C}_n^{0,q}$ to prove by induction on m. For $m=0$ we obtain

$$
\mathcal{C}_n^{0,q}\eta(z)=c_0\int_{\mathbb{B}_n}\Phi_n^q(w,z)\left\{\sum_{|J|=q}\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\overline{z}^J\right\}dV(w)\equiv c_0\Phi_n^q\left(\overline{\mathfrak{D}^0}\eta\right)(z), \tag{ES-8}
$$

from (3-5) and the following calculation using (2-6):

$$
\begin{aligned}
&\mathcal{C}_n^{0,q}\eta(z)\\
&\equiv\int_{\mathbb{B}_n}\mathcal{C}_n^{0,q}(w,z)\wedge\eta(w)\\
&=\int_{\mathbb{B}_n}\sum_{|J|=q}\Phi_n^q(w,z)\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\overline{z_k}-\overline{\eta_k}\right)d\overline{z}^J\wedge d\overline{w}^{(J\cup\{k\})^c}\wedge\omega_n(w)\wedge\left(\sum_{|I|=q+1}\eta_I\,d\overline{w_I}\right)\\
&=\left\{\int_{\mathbb{B}_n}\Phi_n^q(w,z)\left[\sum_{|J|=q}\sum_{k\notin J}(-1)^{\mu(k,J)}\left(\overline{z_k}-\overline{w_k}\right)\eta_{J\cup\{k\}}d\overline{z}^J\right]dV(w)\right\}.
\end{aligned}
$$

Now we consider the case $m=1$. First we note that for each J with $|J|=q$,

$$
\overline{\mathfrak{X}\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)-\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)=\overline{\mathfrak{D}^1}\left(\eta\lrcorner d\overline{w}^J\right). \tag{ES-9}
$$

Indeed, we compute

$$\begin{aligned}\overline{\mathfrak{X}}\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)&=\left(\sum_{j=1}^n\left(\overline{w_j}-\overline{z_j}\right)\frac{\partial}{\partial\overline{w_j}}\right)\left(\sum_{k\notin J}\left(\overline{w_k}-\overline{z_k}\right)\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\eta_I\right)\\&=\sum_{j=1}^n\sum_{k\notin J}\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\left(\overline{w_j}-\overline{z_j}\right)\left(\overline{w_k}-\overline{z_k}\right)\frac{\partial}{\partial\overline{w_j}}\eta_I\\&\quad+\sum_{k\notin J}\left(\overline{w_k}-\overline{z_k}\right)\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\eta_I,\end{aligned}$$

so that

$$\begin{aligned}&\overline{\mathfrak{X}}\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)-\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\\&=\sum_{j=1}^n\sum_{k\notin J}\sum_{I\setminus J=\{k\}}(-1)^{\mu(k,J)}\left(\overline{w_j}-\overline{z_j}\right)\left(\overline{w_k}-\overline{z_k}\right)\frac{\partial}{\partial\overline{w_j}}\eta_I=\overline{\mathfrak{D}^1}\left(\eta\lrcorner d\overline{w}^J\right).\end{aligned}$$

For $|J|=q$ and $0\le\ell\le q$ define

$$\mathcal{I}_J^\ell\equiv\sum_{j=1}^n\int_{\mathbb{B}_n}\frac{\partial}{\partial\overline{w_j}}\left\{\frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(w,z)^n}\left(\overline{w_j}-\overline{z_j}\right)\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\right\}\omega\left(\overline{w}\right)\wedge\omega(w).$$

By (3) and (4) of Proposition 16.4.4 in [Rudin 1980] we have

$$\sum_{j=1}^n(-1)^{j-1}\left(\overline{w_j}-\overline{z_j}\right)\bigwedge_{k\neq j}d\overline{w_k}\wedge\omega(w)\,|_{\partial\mathbb{B}_n}=c\left(1-\overline{z}w\right)d\sigma(w),$$

and Stokes' theorem then yields

$$\mathcal{I}_J^\ell=c\int_{\partial\mathbb{B}_n}\frac{(1-w\overline{z})^{n-\ell}\left(1-|w|^2\right)^\ell}{\triangle(w,z)^n}\overline{\mathfrak{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\sigma(w)=0,$$

since $\ell \geq 1$ and $1-|w|^2$ vanishes on $\partial\mathbb{B}_n$. Moreover, from Lemma ES.1 we obtain

$$\begin{aligned}
\mathcal{I}_J^\ell &= n\int_{\mathbb{B}_n} \frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w) \\
&\quad + \int_{\mathbb{B}_n} \overline{\mathcal{X}}\left\{\frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)\right\}dV(w) \\
&= \int_{\mathbb{B}_n} \frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(z,w)^n}\overline{\mathcal{X}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w) \\
&\quad +\ell\int_{\mathbb{B}_n} \frac{(1-w\overline{z})^{n-1-\ell}\left(1-|w|^2\right)^\ell}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w) \\
&\quad -\ell\int_{\mathbb{B}_n} \frac{(1-w\overline{z})^{n-\ell}\left(1-|w|^2\right)^{\ell-1}}{\triangle(z,w)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w).
\end{aligned}$$

Combining this with (ES-9) and (ES-8) yields

$$\begin{aligned}
\Phi_n^\ell\left(\overline{\mathcal{D}^0}\eta\right)(z) &= \sum_J \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&= \sum_J \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\overline{\mathcal{X}}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&\quad -\sum_J \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&= -\sum_J \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\overline{\mathcal{D}^1}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&\quad -\ell\sum_J \int_{\mathbb{B}_n} \Phi_n^\ell(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&\quad +\ell\sum_J \int_{\mathbb{B}_n} \Phi_n^{\ell-1}(w,z)\,\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)dV(w)\,d\overline{z}^J \\
&= -\Phi_n^\ell\left(\overline{\mathcal{D}^1}\eta\right)(z) - \ell\Phi_n^\ell\left(\overline{\mathcal{D}^0}\eta\right)(z) + \ell\Phi_n^{\ell-1}\left(\overline{\mathcal{D}^0}\eta\right)(z).
\end{aligned}$$

Thus we have

$$\Phi_n^\ell\left(\overline{\mathcal{D}^0}\eta\right)(z) = -\frac{1}{\ell+1}\Phi_n^\ell\left(\overline{\mathcal{D}^1}\eta\right)(z) + \frac{\ell}{\ell+1}\Phi_n^{\ell-1}\left(\overline{\mathcal{D}^0}\eta\right)(z). \tag{ES-10}$$

From (ES-8) and then iterating (ES-10) we obtain

$$
\begin{aligned}
&\mathscr{C}_n^{(0,q)}\eta(z) \qquad\qquad \text{(ES-11)}\\
&=\Phi_n^q\left(\overline{\mathcal{D}^0}\eta\right)(z)=-\frac{1}{q+1}\Phi_n^q\left(\overline{\mathcal{D}^1}\eta\right)(z)+\frac{q}{q+1}\Phi_n^{q-1}\left(\overline{\mathcal{D}^0}\eta\right)(z)\\
&=-\frac{1}{q+1}\Phi_n^q\left(\overline{\mathcal{D}^1}\eta\right)(z)+\frac{q}{q+1}\left\{-\frac{1}{q}\Phi_n^{q-1}\left(\overline{\mathcal{D}^1}\eta\right)(z)+\frac{q-1}{q}\Phi_n^{q-2}\left(\overline{\mathcal{D}^0}\eta\right)(z)\right\}\\
&=-\frac{1}{q+1}\sum_{\ell=1}^{q}\Phi_n^\ell\left(\overline{\mathcal{D}^1}\eta\right)(z)+boundary\ term.
\end{aligned}
$$

Thus we have obtained the second sum in with $c_\ell=-\frac{1}{q+1}$ for $1\le\ell\le q$ in the case $m=1$.

We have included $boundary\ term$ in (ES-11) since when we use Stokes' theorem on $\Phi_n^0\left(\overline{\mathcal{D}^0}\eta\right)$ the boundary integral no longer vanishes. In fact when $\ell=0$ the boundary term in Stokes' theorem is

$$
\begin{aligned}
\mathcal{I}_J^0&=c\int_{\partial\mathbb{B}_n}\frac{(1-\zeta\overline{z})^n}{\triangle(\zeta,z)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\sigma(\zeta)\\
&=c\int_{\partial\mathbb{B}_n}\frac{1}{\left(1-\overline{\zeta}z\right)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\sigma(\zeta),
\end{aligned}
$$

since from (2-1) we have

$$
\frac{(1-w\overline{z})^n}{\triangle(z,w)^n}=\frac{(1-w\overline{z})^n}{|1-w\overline{z}|^{2n}|\varphi_z(w)|^{2n}}=\frac{1}{(1-\overline{w}z)^n},\qquad w\in\partial\mathbb{B}_n.
$$

Thus the boundary term in (ES-11) is

$$
c\sum_J\int_{\partial\mathbb{B}_n}\frac{1}{\left(1-\overline{\zeta}z\right)^n}\overline{\mathcal{D}^0}\left(\eta\lrcorner d\overline{w}^J\right)d\sigma(\zeta)\,d\overline{z}^J=c\mathcal{S}_n\left(\overline{\mathcal{D}^0}\eta\right)(z).
$$

This completes the proof of in the case $m=1$. Now we proceed by induction on m to complete the proof of Lemma 14. □

Finally here is the simple proof of the integration by parts formula for the radial derivative in Lemma 15.

Proof of Lemma 15. Since $\left(1-|w|^2\right)^{b+1}$ vanishes on the boundary for $b>-1$, and since

$$
R\left(1-|w|^2\right)^{b+1}=\sum_{j=1}^n w_j\frac{\partial}{\partial w_j}\left(1-|w|^2\right)^{b+1}=-(b+1)\left(1-|w|^2\right)^b|w|^2,
$$

the divergence theorem yields

$$\begin{aligned}
0 &= \int_{\partial\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, w\cdot \mathbf{n} d\sigma(w) \\
&= \int_{\mathbb{B}_n} \sum_{j=1}^{n} \frac{\partial}{\partial w_j} \left\{ w_j \left(1-|w|^2\right)^{b+1} \Psi(w) \right\} dV(w) \\
&= n \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, dV(w) \\
&\quad + (b+1) \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b} \left(-|w|^2\right) \Psi(w)\, dV(w) \\
&\quad + \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} R\Psi(w)\, dV(w),
\end{aligned}$$

which after rearranging becomes

$$\begin{aligned}
&(n+b+1) \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} \Psi(w)\, dV(w) \\
&\quad + \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b+1} R\Psi(w)\, dV(w). \\
&= (b+1) \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{b} \Psi(w)\, dV(w).
\end{aligned}$$

□

We now recall the invertible "radial" operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [Zhu 2005] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. This definition is easily extended to $f \in H\left(\mathbb{B}_n; \ell^2\right)$. If the inverse of $R^{\gamma,t}$ is denoted $R_{\gamma,t}$, then Proposition 1.14 of [Zhu 2005] yields

$$\begin{aligned}
R^{\gamma,t}\left(\frac{1}{(1-\overline{w}z)^{n+1+\gamma}}\right) &= \frac{1}{(1-\overline{w}z)^{n+1+\gamma+t}}, \\
R_{\gamma,t}\left(\frac{1}{(1-\overline{w}z)^{n+1+\gamma+t}}\right) &= \frac{1}{(1-\overline{w}z)^{n+1+\gamma}},
\end{aligned} \tag{ES-12}$$

for all $w \in \mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t.

Equivalent seminorms on Besov–Sobolev spaces. It is a routine matter to take known scalar-valued proofs of the results in this section and replace the scalars with vectors in ℓ^2 to obtain proofs for the ℓ^2-valued versions. We begin illustrating this process by proving the equivalence of certain norms:

Proposition ES.2. *Suppose that $\sigma \geq 0$, $0 < p < \infty$, $n+\gamma$ is not a negative integer, and $f \in H\left(\mathbb{B}_n;\ell^2\right)$. Then the following four conditions are equivalent:*

$$\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for \textbf{some} } m > \frac{n}{p}-\sigma, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for \textbf{all} } m > \frac{n}{p}-\sigma, m \in \mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for \textbf{some} } m > \frac{n}{p}-\sigma, m+n+\gamma \notin -\mathbb{N},$$

$$\left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) \in L^p\left(d\lambda_n;\ell^2\right) \text{ for \textbf{all} } m > \frac{n}{p}-\sigma, m+n+\gamma \notin -\mathbb{N}.$$

Moreover, with $\psi(z) = 1-|z|^2$, we have for $1 < p < \infty$,

$$\begin{aligned} & C^{-1}\left\|\psi^{m_1+\sigma} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n;\ell^2)} \\ & \leq \sum_{k=0}^{m_2-1}\left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m_2+\sigma} \nabla^{m_2} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \\ & \leq C\left\|\psi^{m_1+\sigma} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n;\ell^2)} \end{aligned}$$

for all $m_1, m_2 > \frac{n}{p}-\sigma$, $m_1+n+\gamma \notin -\mathbb{N}$, $m_2 \in \mathbb{N}$, and where the constant C depends only on σ, m_1, m_2, n, γ and p.

Proof. First we note the equivalence of the following two conditions (the case $\sigma = 0$ is Theorem 6.1 of [Zhu 2005]):

(1) The functions

$$\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z), \qquad |k| = N$$

are in $L^p\left(d\lambda_n;\ell^2\right)$ for some $N > \frac{n}{p}-\sigma$,

(2) The functions

$$\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z), \qquad |k| = N$$

are in $L^p\left(d\lambda_n;\ell^2\right)$ for every $N > \frac{n}{p}-\sigma$.

Indeed, $L^p\left(d\lambda_n;\ell^2\right) = L^p\left(\nu_{-n-1};\ell^2\right)$ and $\left(1-|z|^2\right)^{|k|+\sigma} \frac{\partial^{|k|}}{\partial z^k} f(z) \in L^p\left(\nu_{-n-1};\ell^2\right)$ if and only if

$$\frac{\partial^{|k|}}{\partial z^k} f(z) \in L^p\left(\nu_{p(|k|+\sigma)-n-1};\ell^2\right).$$

Provided $p(|k|+\sigma)-n-1 > -1$, Theorem 2.17 of [Zhu 2005] shows that $\left(1-|z|^2\right)^{\ell} \frac{\partial^{|\ell|}}{\partial z^{\ell}}\left(\frac{\partial^{|k|}}{\partial z^k} f\right)(z) \in L^p\left(\nu_{p(|k|+\sigma)-n-1};\ell^2\right)$, which shows that (2) follows from (1).

From the equivalence of (1) and (2) we obtain the equivalence of the first two conditions in Proposition ES.2. The equivalence with the next two conditions follows from the corresponding generalization to $\sigma > 0$ of Theorem 6.4 in [Zhu 2005], which in turn is achieved by arguing as in the previous paragraph. □

Next we prove a lemma whose case scalar $\sigma = 0$ is Lemma 6.4 in [Arcozzi et al. 2006]. Our prove is an adaptation of the one in that reference.

Lemma ES.3. *Let $1 < p < \infty$, $\sigma \geq 0$ and $m > 2\left(\frac{n}{p} - \sigma\right)$. Denote by $B_\beta(c, C)$ the ball center c radius C in the Bergman metric β. Then for $f \in H\left(\mathbb{B}_n; \ell^2\right)$,*

$$\begin{aligned} &\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \qquad \text{(ES-13)} \\ &\equiv \left(\sum_{\alpha \in \mathcal{T}_n} \int_{B_\beta(c_\alpha, C_2)} \left|\left(1-|z|^2\right)^\sigma D^m_{c_\alpha} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| \\ &\approx \left(\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} + \sum_{j=0}^{m-1} \left|\nabla^j f(0)\right| = \|f\|_{B^\sigma_{p,m}(\mathbb{B}_n;\ell^2)}. \end{aligned}$$

Proof of Lemma ES.3. We have

$$|D_a f(z)| = \left|f'(z)\left\{\left(1-|a|^2\right) P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\}\right| \geq \left|\left(1-|a|^2\right) f'(z)\right|, \qquad \text{(ES-14)}$$

and iterating with f replaced by (the components of) $D_a f$ in (ES-14), we obtain

$$\left|D^2_a f(z)\right| \geq \left|\left(1-|a|^2\right)(D_a f)'(z)\right|.$$

Applying (ES-14) once more with f replaced by (the components of) f', we get

$$\left|\left(1-|a|^2\right)(D_a f)'(z)\right| = \left|\left(1-|a|^2\right) D_a\left(f'\right)(z)\right| \geq \left|\left(1-|a|^2\right)^2 f''(z)\right|,$$

which when combined with the previous inequality yields

$$\left|D^2_a f(z)\right| \geq \left|\left(1-|a|^2\right)^2 f''(z)\right|.$$

Continuing by induction we have

$$\left|D^m_a f(z)\right| \geq \left|\left(1-|a|^2\right)^m f^{(m)}(z)\right|, \quad m \geq 1. \qquad \text{(ES-15)}$$

Proposition ES.2 and (ES-15) now show that

$$
\begin{aligned}
&\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} R^{0,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \\
&\leq C\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_\beta(c_\alpha,C_2)}\left|\left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_\beta(c_\alpha,C_2)}\left|\left(1-|c_\alpha|^2\right)^{m+\sigma} f^{(m)}(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&\leq C\left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_\beta(c_\alpha,C_2)}\left|\left(1-|z|^2\right)^{\sigma} D^m_{c_\alpha} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right| \\
&= C\,\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)}+\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right|.
\end{aligned}
$$

For the opposite inequality, just as in [Arcozzi et al. 2006], we employ some of the ideas in the proofs of Theorem 6.11 and Lemma 3.3 in [Zhu 2005], where the case $\sigma=0$ and $m=1>\frac{2n}{p}$ is proved. Suppose $f\in H(\mathbb{B}_n)$ and that the right side of (ES-13) is finite. By Proposition ES.2 and the proof of Theorem 6.7 of [Zhu 2005] we have

$$
f(z)=c\int_{\mathbb{B}_n}\frac{g(w)}{(1-\overline{w}z)^{n+1+\sigma}}dV(w),\qquad z\in\mathbb{B}_n, \tag{ES-16}
$$

for some $g\in L^p(\lambda_n)$ where

$$
\|g\|_{L^p(\lambda_n)}\approx\sum_{j=0}^{m-1}\left|\nabla^j f(0)\right|+\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}. \tag{ES-17}
$$

Indeed, Proposition ES.2 shows that

$$
\begin{aligned}
f\ \in\ B^\sigma_p(\mathbb{B}_n) &\Leftrightarrow \left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\in L^p(\lambda_n) \\
&\Leftrightarrow R^{\sigma,m} f(z)\in L^p\left(\nu_{p(m+\sigma)-n-1}\right)\cap H(\mathbb{B}_n),
\end{aligned}
$$

where as in [Zhu 2005] we write $d\nu_\alpha(z)=\left(1-|z|^2\right)^\alpha dV(z)$. Now Lemma 24 above (see also Proposition 2.11 in [Zhu 2005]) shows that

$$
T_{0,\beta,0}L^p\left(\nu_\gamma\right)=L^p\left(\nu_\gamma\right)\cap H(\mathbb{B}_n)
$$

if and only if $p(\beta+1)>\gamma+1$. Choosing $\beta=m+\sigma$ and $\gamma=p(m+\sigma)-n-1$ we see that $p(\beta+1)>\gamma+1$ and so $f\in B^\sigma_p(\mathbb{B}_n)$ if and only if

$$
R^{\sigma,m} f(z)=c\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{m+\sigma} h(w)}{(1-\overline{w}z)^{n+1+m+\sigma}}dV(w)
$$

for some $h \in L^p\left(\nu_{p(m+\sigma)-n-1}\right)$. If we set $g(w) = \left(1-|w|^2\right)^{m+\sigma} h(w)$ we obtain

$$R^{\sigma,m} f(z) = c \int_{\mathbb{B}_n} \frac{g(w)}{(1-\overline{w}z)^{n+1+m+\sigma}} dV(w) \tag{ES-18}$$

with $g \in L^p(\lambda_n)$. Now apply the inverse operator $R_{\sigma,m} = (R^{\sigma,m})^{-1}$ to both sides of (ES-18) and use (ES-12),

$$R_{\sigma,m}\left(\frac{1}{(1-\overline{w}z)^{n+1+m+\sigma}}\right) = \frac{1}{(1-\overline{w}z)^{n+1+\sigma}},$$

to obtain (ES-16) and (ES-17).

Fix $\alpha \in \mathcal{T}_n$ and let $a = c_\alpha \in \mathbb{B}_n$. We claim that

$$\left|D_a^m f(z)\right| \le C_m \left(1-|a|^2\right)^{\frac{m}{2}} \int_{\mathbb{B}_n} \frac{|g(w)|}{|1-\overline{w}z|^{n+1+\frac{m}{2}+\sigma}} dV(w), \quad m \ge 1, z \in B_\beta(a,C). \tag{ES-19}$$

To see this we compute $D_a^m f(z)$ for $z \in B_\beta(a,C)$, beginning with the case $m=1$. Since

$$\begin{aligned} D_a(\overline{w}z) &= (\overline{w}z)' \varphi_a'(0) = -\overline{w}^t \left\{\left(1-|a|^2\right) P_a + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a\right\} \\ &= -\overline{\left\{\left(1-|a|^2\right) P_a w + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a w\right\}}^t, \end{aligned}$$

we have

$$\begin{aligned} & D_a f(z) \\ &= c_n \int_{\mathbb{B}_n} D_a (1-\overline{w}z)^{-(n+1+\sigma)} g(w)\, dV(w) \\ &= c_n \int_{\mathbb{B}_n} (1-\overline{w}z)^{-(n+2+\sigma)} D_a(\overline{w}z)\, g(w)\, dV(w) \\ &= c_n \int_{\mathbb{B}_n} (1-\overline{w}z)^{-(n+2+\sigma)} \overline{\left\{\left(1-|a|^2\right) P_a w + \left(1-|a|^2\right)^{\frac{1}{2}} Q_a w\right\}}^t g(w)\, dV(w). \end{aligned} \tag{ES-20}$$

Taking absolute values inside, we obtain

$$|D_a f(z)| \le C \left(1-|a|^2\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{\left(1-|a|^2\right)^{\frac{1}{2}} |P_a w| + |Q_a w|}{|1-\overline{w}z|^{n+2+\sigma}} |g(w)|\, dV(w). \tag{ES-21}$$

From the following elementary inequalities

$$\begin{aligned} |Q_a w|^2 &= |Q_a(w-a)|^2 \le |w-a|^2, \\ &= |w|^2 + |a|^2 - 2\operatorname{Re}(w\overline{a}) \\ &\le 2\operatorname{Re}(1-w\overline{a}) \le 2|1-w\overline{a}|, \end{aligned} \tag{ES-22}$$

we obtain that $|Q_a w| \le C|1-\overline{w}a|^{\frac{1}{2}}$. Now

$$|1-w\overline{a}| \approx |1-w\overline{z}| \ge \frac{1}{2}\left(1-|z|^2\right) \approx \left(1-|a|^2\right), \quad z \in B_\beta(a,C)$$

shows that

$$\left(1-|a|^2\right)^{\frac{1}{2}}+|1-\overline{w}a|^{\frac{1}{2}}\leq C\,|1-\overline{w}z|^{\frac{1}{2}}\,,\qquad z\in B_\beta\left(a,C\right),$$

and so we see that

$$\frac{\left(1-|a|^2\right)^{\frac{1}{2}}|P_aw|+|Q_aw|}{|1-\overline{w}z|^{n+2}}\leq\frac{C}{|1-\overline{w}z|^{n+\frac{3}{2}}},\qquad z\in B_\beta\left(a,C\right).$$

Plugging this estimate into (ES-21) yields

$$|D_af\left(z\right)|\leq C\left(1-|a|^2\right)^{\frac{1}{2}}\int_{\mathbb{B}_n}\frac{|g\left(w\right)|}{|1-\overline{w}z|^{n+\frac{3}{2}+\sigma}}dV\left(w\right),$$

which is the case $m=1$ of (ES-19).

To obtain the case $m=2$ of (ES-19), we differentiate (ES-20) again to get

$$D_a^2f\left(z\right)=c\int_{\mathbb{B}_n}\left(1-\overline{w}z\right)^{-(n+3+\sigma)}W\overline{W}^t g\left(w\right)dV\left(w\right).$$

where we have written $W=\left\{\left(1-|a|^2\right)P_aw+\left(1-|a|^2\right)^{\frac{1}{2}}Q_aw\right\}$ for convenience. Again taking absolute values inside, we obtain

$$\left|D_a^2f\left(z\right)\right|\leq C\left(1-|a|^2\right)\int_{\mathbb{B}_n}\frac{\left(\left(1-|a|^2\right)^{\frac{1}{2}}|P_aw|+|Q_aw|\right)^2}{|1-\overline{w}z|^{n+3+\sigma}}|g\left(w\right)|\,dV\left(w\right).$$

Once again, using $|Q_aw|\leq C\,|1-\overline{w}a|^{\frac{1}{2}}$ and $\left(1-|a|^2\right)^{\frac{1}{2}}+|1-\overline{w}a|^{\frac{1}{2}}\leq C\,|1-\overline{w}z|^{\frac{1}{2}}$ for $z\in B_\beta\left(a,C\right)$, we see that

$$\frac{\left(\left(1-|a|^2\right)^{\frac{1}{2}}|P_aw|+|Q_aw|\right)^2}{|1-\overline{w}z|^{n+3+\sigma}}\leq\frac{C}{|1-\overline{w}z|^{n+2+\sigma}},\qquad z\in B_\beta\left(a,C\right),$$

which yields the case $m=2$ of (ES-19). The general case of (ES-19) follows by induction on m.

The inequality (ES-19) shows that $\left(1-|z|^2\right)^\sigma\left|D_{c_\alpha}^mf\left(z\right)\right|\leq C_mS\,|g|\left(z\right)$ for $z\in B_\beta\left(c_\alpha,C\right)$, where

$$Sg\left(z\right)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{\frac{m}{2}+\sigma}}{|1-\overline{w}z|^{n+1+\frac{m}{2}+\sigma}}g\left(w\right)dV\left(w\right).$$

We will now use the symbol a differently than before. The operator S is the operator $T_{a,b,c}$ in Lemma 24 above (see also Theorem 2.10 of [Zhu 2005]) with parameters $a=\frac{m}{2}+\sigma$ and $b=c=0$. Now with $t=-n-1$, our assumption that $m>2\left(\frac{n}{p}-\sigma\right)$ yields $-p\left(\frac{m}{2}+\sigma\right)<-n<p\left(0+1\right)$, i.e.

$$-pa<t+1<p\left(b+1\right).$$

Thus the bounded overlap property of the balls $B_\beta(c_\alpha, C_2)$ together with Lemma 24 above yields

$$\begin{aligned}\|f\|^*_{B^\sigma_{p,m}(\mathbb{B}_n)} &= \left(\sum_{\alpha\in\mathcal{T}_n}\int_{B_\beta(c_\alpha,C_2)}\left|\left(1-|z|^2\right)^\sigma D^m_{c_\alpha} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\le C_m\left(\int_{\mathbb{B}_n}|Sg(z)|^p\, d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\le C'_m\left(\int_{\mathbb{B}_n}|g(z)|^p\, d\lambda_n(z)\right)^{\frac{1}{p}}\\ &\le C''_m\left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma} R^{\sigma,m} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}}\end{aligned}$$

by (ES-17). This completes the proof of Lemma ES.3. □

Multilinear inequalities. Proposition 22 is proved by adapting the proof of Theorem 3.5 in [Ortega and Fàbrega 2000] to ℓ^2-valued functions. This argument uses the complex interpolation theorem of [Beatrous 1986] and [Ligocka 1987], which extends to Hilbert space valued functions with the same proof. In order to apply this extension we will need the following operator norm inequality.

If $\varphi \in M_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}$ and $f = \sum_{|I|=\kappa} f_I e_I \in B^\sigma_p\left(\mathbb{B}_n; \otimes^{\kappa-1}\ell^2\right)$, we define

$$\mathbb{M}_\varphi f = \varphi\otimes f = \varphi\otimes\left(\sum_{|I|=\kappa-1} f_I e_I\right) = \sum_{|I|=\kappa-1}(\varphi f_I)\otimes e_I,$$

where $I = (i_1, ..., i_{\kappa-1}) \in \mathbb{N}^{\kappa-1}$ and $e_I = e_{i_1}\otimes ...\otimes e_{i_{\kappa-1}}$.

Proof of Proposition 22 and Lemma 23. We begin with the proof of the case $M = 1$ of Proposition 22. We will show that for $m = \ell + k$,

$$\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right)\right|^p d\lambda_n(z) \le C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)}\|h\|^p_{B^\sigma_p(\mathbb{B}_n)}. \tag{ES-23}$$

Following the proof of Theorem 3.1 in [Ortega and Fàbrega 2000] we first convert the Leibniz formula

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \mathcal{Y}^\ell\left(g\mathcal{Y}^k h\right) - \sum_{\alpha=0}^{\ell-1}\binom{\ell}{\alpha}\left(\mathcal{Y}^\alpha g\right)\left(\mathcal{Y}^{k+\ell-\alpha} h\right)$$

to "divergence form"

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \sum_{\alpha=0}^{\ell}(-1)^\alpha\binom{\ell}{\ell-\alpha}\mathcal{Y}^{\ell-\alpha}\left(g\mathcal{Y}^{k+\alpha} h\right).$$

This is easily established by induction on ℓ with k held fixed and can be stated as

$$\left(\mathcal{Y}^\ell g\right)\left(\mathcal{Y}^k h\right) = \sum_{\alpha=0}^{\ell} c^\ell_\alpha \mathcal{Y}^\alpha\left(g\mathcal{Y}^{k+\ell-\alpha} h\right). \tag{ES-24}$$

Next we note that for $s > \frac{n}{p}$, $B_p^s\left(\mathbb{B}_n;\ell^2\right)$ is a Bergman space, hence $M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)} = H^\infty\left(\mathbb{B}_n;\ell^2\right)$. Thus using (5-2) we have for $s > \frac{n}{p}$,

$$g \in M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \cap H^\infty\left(\mathbb{B}_n;\ell^2\right) = M_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \cap M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}.$$

Then, still following the argument in [Ortega and Fàbrega 2000], we use the complex interpolation theorem of [Beatrous 1986] and [Ligocka 1987], [1]

$$\left(B_p^\sigma\left(\mathbb{B}_n\right), B_p^{\frac{n}{p}+\varepsilon}\left(\mathbb{B}_n\right)\right)_\theta = B_p^{(1-\theta)\sigma+\theta\left(\frac{n}{p}+\varepsilon\right)}\left(\mathbb{B}_n\right), \qquad 0\le\theta\le1,$$
$$\left(B_p^\sigma\left(\mathbb{B}_n;\ell^2\right), B_p^{\frac{n}{p}+\varepsilon}\left(\mathbb{B}_n;\ell^2\right)\right)_\theta = B_p^{(1-\theta)\sigma+\theta\left(\frac{n}{p}+\varepsilon\right)}\left(\mathbb{B}_n;\ell^2\right), \qquad 0\le\theta\le1,$$

to conclude that $g \in M_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}$ for all $s \ge \sigma$, and with multiplier norm $\left\|\mathbb{M}_g\right\|_{B_p^s(\mathbb{B}_n)\to B_p^s(\mathbb{B}_n;\ell^2)}$ bounded by $\left\|\mathbb{M}_g\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)}$. Recall now that

$$\|h\|^p_{B_p^\sigma(\mathbb{B}_n)} = \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^\sigma \mathcal{Y}^m h(z)\right|^p d\lambda_n(z),$$

and similarly for $\|f\|^p_{B_p^\sigma(\mathbb{B}_n;\ell^2)}$, provided m satisfies

$$\left(\sigma+\frac{m}{2}\right)p > n, \tag{ES-25}$$

where $\frac{m}{2}$ appears in the inequality since the derivatives D that can appear in $\mathcal{Y}^m$ only contribute $\left(1-|z|^2\right)^{\frac{1}{2}}$ to the power of $1-|z|^2$ in the integral (see Section 5).

Remark. At this point we recall the convention established in Definitions Definition 18 and Definition 19 that the factors of $1-|z|^2$ that are embedded in the notation for the derivative $\mathcal{Y}^\alpha$ are treated as constants relative to the actual differentiations. In the calculations below, we will adopt **the same convention** for the factors $\left(1-|z|^2\right)^s$ that we introduce into the integrals. Alternatively, the reader may wish to write out all the derivatives explicitly with the appropriate power of $1-|z|^2$ set aside as is done in [Ortega and Fàbrega 2000].

So we have, keeping in mind the remark,

$$\begin{aligned}\int_{\mathbb{B}_n}&\left|\left(1-|z|^2\right)^\sigma \mathcal{Y}^\alpha\left(g(z)\,\mathcal{Y}^{k+\ell-\alpha}h(z)\right)\right|^p d\lambda_n\\ &= \int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^s \mathcal{Y}^\alpha\left\{g(z)\left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h(z)\right\}\right|^p d\lambda_n\\ &= \left\|g(z)\left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}.\end{aligned}$$

Here the function

$$H(z) = \left(1-|z|^2\right)^{\sigma-s}\mathcal{Y}^{k+\ell-\alpha}h(z)$$

is *not* holomorphic, but we have defined the norm $\|\cdot\|_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}$ on smooth functions anyway. Now we would like to apply a multiplier property of g, and for this we must be acting on a Besov–Sobolev space of *holomorphic*

[1] In those references only the scalar-valued version of the theorem is proved, but the Hilbert space valued version has the same proof

functions, since that is what we get from the complex interpolation of Beatrous and Ligocka. Precisely, we get that $\mathbb{M}_g$ is a bounded operator from $B_p^s(\mathbb{B}_n)$ to $B_p^s(\mathbb{B}_n;\ell^2)$ for *all* $s \geq \sigma$.

Now we express $\mathcal{Y}^{k+\ell-\alpha} h(z)$ as a sum of terms that are products of a power of $1-|z|^2$ and a derivative $R^i L^j h(z)$ where $i+j=k+\ell-\alpha$ and R is the radial derivative and L denotes a complex tangential derivative $\frac{\partial}{\partial z_j} - \overline{z_j} R$ as in [Ortega and Fàbrega 2000]. However, the operators $R^i L^j$ have different weights in the sense that the power of $1-|z|^2$ that is associated with $R^i L^j$ is $\left(1-|z|^2\right)^{i+\frac{j}{2}}$, i.e.

$$\mathcal{Y}^{k+\ell-\alpha} h(z) = \sum \left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z).$$

It turns out that to handle the term $\left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z)$ we will use that g is a multiplier on $B_p^s(\mathbb{B}_n)$ with

$$s = \sigma + i + \frac{j}{2},$$

an exponent that depends on $i+\frac{j}{2}$ and *not* on $i+j=k+\ell-\alpha$.

Indeed, we have using our "convention" that

$$\begin{aligned}
&\left\| g(z)\left(1-|z|^2\right)^{\sigma-s}\left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)} \\
&= \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{s} \mathcal{Y}^{\alpha}\left\{ g(z)\left(1-|z|^2\right)^{\sigma-s}\left(1-|z|^2\right)^{i+\frac{j}{2}} R^i L^j h(z)\right\}\right|^p d\lambda_n \\
&= \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma+i+\frac{j}{2}} \mathcal{Y}^{\alpha}\left\{ g(z) R^i L^j h(z)\right\}\right|^p d\lambda_n \\
&= \left\| g(z) R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n;\ell^2)}.
\end{aligned}$$

Now the function $g(z) R^i L^j h(z)$ is holomorphic and $s=\sigma+i+\frac{j}{2}\geq\sigma$ so that we can use that g is a multiplier on $B_p^s(\mathbb{B}_n) = B_{p,\alpha}^s(\mathbb{B}_n)$ (this latter equality holds because $\left(s+\frac{\alpha}{2}\right)p>n$ by (ES-25)). The result is that

$$\begin{aligned}
&\left\| g(z) R^i L^j h(z)\right\|^p_{B^s_{p}(\mathbb{B}_n;\ell^2)} \\
&\leq \left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)} \left\| R^i L^j h(z)\right\|^p_{B^s_{p,\alpha}(\mathbb{B}_n)} \\
&\leq \left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma+i+\frac{j}{2}} \mathcal{Y}^{\alpha} R^i L^j h(z)\right|^p d\lambda_n \\
&= \left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma} \mathcal{Y}^{\alpha}\left[\left(1-|z|^2\right)R\right]^i\left[\sqrt{1-|z|^2}L\right]^j h(z)\right|^p d\lambda_n \\
&\leq \left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma} \mathcal{Y}^{\alpha+i+j} h(z)\right|^p d\lambda_n \\
&= \left\|\mathbb{M}_g\right\|^p_{B^s_p(\mathbb{B}_n)\to B^s_p(\mathbb{B}_n;\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{\sigma} \mathcal{Y}^{m} h(z)\right|^p d\lambda_n \\
&\leq \left\|\mathbb{M}_g\right\|^p_{B^\sigma_p(\mathbb{B}_n)\to B^\sigma_p(\mathbb{B}_n;\ell^2)} \left\| h\right\|^p_{B^\sigma_p(\mathbb{B}_n)},
\end{aligned}$$

and the case $M=1$ of Proposition 22 is proved.

Now we turn to the proof of the operator norm inequality (5-5) in Lemma 23. The case $p=2$ is particularly easy:

$$\begin{aligned}
\left\|\mathbb{M}_\varphi f\right\|^2_{B_2^\sigma(\mathbb{B}_n;\otimes^\kappa\ell^2)} &= \int_{\mathbb{B}_n}\left(1-|z|^2\right)^{2\sigma}\sum_{|I|=\kappa-1}\left|\mathcal{Y}^m\left(\varphi f_I\right)\right|^2 d\lambda_n\\
&= \sum_{|I|=\kappa-1}\left\|\mathbb{M}_\varphi f_I\right\|^2_{B_2^\sigma(\mathbb{B}_n;\ell^2)}\\
&\le \left\|\mathbb{M}_\varphi\right\|^2_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}\sum_{|I|=\kappa-1}\left\|f_I\right\|^2_{B_2^\sigma(\mathbb{B}_n)}\\
&= \left\|\mathbb{M}_\varphi\right\|^2_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}\int_{\mathbb{B}_n}\left(1-|z|^2\right)^{2\sigma}\sum_{|I|=\kappa-1}\left|\mathcal{Y}^m f_I\right|^2 d\lambda_n\\
&= \left\|\mathbb{M}_\varphi\right\|^2_{B_2^\sigma(\mathbb{B}_n)\to B_2^\sigma(\mathbb{B}_n;\ell^2)}\left\|f\right\|^2_{B_2^\sigma(\mathbb{B}_n;\otimes^{\kappa-1}\ell^2)},
\end{aligned}$$

and from this we easily obtain (5-6).

For $p\neq 2$ it suffices to show that

$$\left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\nu)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu\otimes\mathbb{C}^\nu)} \le C_{n,\sigma,p}\left\|\mathbb{M}_\varphi\right\|_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)} \tag{ES-26}$$

for all $\mu,\nu\ge 1$ where the constant $C_{n,\sigma,p}$ is independent of μ,ν. Indeed, both ℓ^2 and $\otimes^{\kappa-1}\ell^2$ are separable Hilbert spaces and so can be appropriately approximated by $\mathbb{C}^\mu$ and $\mathbb{C}^\nu$ respectively. For each $z\in\mathbb{B}_n$ we will view $\varphi(z)\in\mathbb{C}^\mu$ as a column vector and $f(z)\in\mathbb{C}^\nu$ as a row vector so that $\left(\mathbb{M}_\varphi f\right)(z)$ is the rank one $\mu\times\nu$ matrix

$$\left(\mathbb{M}_\varphi f\right)(z) = \begin{bmatrix} (\varphi_1 f_1)(z) & \cdots & (\varphi_1 f_\nu)(z)\\ \vdots & \ddots & \vdots\\ (\varphi_\mu f_1)(z) & \cdots & (\varphi_\mu f_\nu)(z)\end{bmatrix} = \varphi(z)\odot f(z),$$

where we have inserted the symbol $\odot$ simply to remind the reader that this is *not* the dot product $\varphi(z)\cdot f(z) = f(z)\,\varphi(z)$ of the vectors $\varphi(z)$ and $f(z)$.

Now we consider a single component X^m of the vector differential operator $\mathcal{Y}^m$ for some $m>2\left(\frac{n}{p}-\sigma\right)$, which can be chosen independent of μ and ν. The main point in the proof of the lemma is that the matrix $X^m\left(\mathbb{M}_\varphi f\right)(z)$ has rank at most $m+1$ independent of μ and ν. Indeed, the Leibniz formula yields

$$X^m\left(\mathbb{M}_\varphi f\right)(z) = X^m\left(\varphi(z)\odot f(z)\right) = \sum_{\ell=0}^m c_{\ell,m}X^{m-\ell}\varphi(z)\odot X^\ell f(z),$$

where each matrix $X^{m-\ell}\varphi(z)\odot X^\ell f(z)$ is rank one, and where the Hilbert Schmidt norm is multiplicative:

$$\left|X^{m-\ell}\varphi(z)\odot X^\ell f(z)\right| = \left|X^{m-\ell}\varphi(z)\right|\left|X^\ell f(z)\right|.$$

Momentarily fix $0\le\ell\le m$ and define

$$\begin{aligned}
T^\ell h(z) &= X^{m-\ell}\varphi(z)\,h(z), \qquad h(z)\in\mathbb{C},\\
T^\ell g(z) &= X^{m-\ell}\varphi(z)\odot g(z), \qquad g(z)\in\mathbb{C}^\nu.
\end{aligned}$$

For $x\in\partial\mathbb{B}_\mu$, which we view as a row vector, define

$$T_x^\ell g(z) = xT^\ell g(z) = x\left(X^{m-\ell}\varphi\right)(z)\odot g(z).$$

Now choose $x(z) \in \partial\mathbb{B}_\mu$ such that $x(z)\left(X^{m-\ell}\varphi\right)(z) = \left|X^{m-\ell}\varphi(z)\right|$ so that

$$T^\ell_{x(z)} g(z) = x(z)\left(X^{m-\ell}\varphi\right)(z) \odot g(z) = \left|X^{m-\ell}\varphi(z)\right| g(z),$$

and hence

$$\left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\right| = \left|X^{m-\ell}\varphi(z)\right|\left|X^\ell f(z)\right| = \left|X^{m-\ell}\varphi(z) \odot X^\ell f(z)\right| = \left|T^\ell\left(X^\ell f\right)(z)\right|.$$

Now we follow the well known argument on page 451 of [Stein 1993]. For $y \in \partial\mathbb{B}_\nu$, which we view as a column vector, and $g(z) \in \mathbb{C}^\nu$ define the scalars

$$\begin{aligned} g_y(z) &= g(z)\, y, \\ \left(T^\ell_{x(z)} g\right)_y(z) &= T^\ell_{x(z)} g(z)\, y = x(z)\left(X^{m-\ell}\varphi\right)(z) \odot g(z)\, y, \end{aligned}$$

and note that

$$T^\ell_{x(z)}\left(X^\ell f\right)(z)\, y = x(z)\left(X^{m-\ell}\varphi\right)(z) \odot \left(X^\ell f\right)(z)\, y = T^\ell_{x(z)}\left(X^\ell f\right)_y(z).$$

Thus we have with $d\sigma_\nu$ surface measure on $\partial\mathbb{B}_\nu$,

$$\int_{\partial\mathbb{B}_\nu}\left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\, y\right|^p d\sigma_\nu(y) = \left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\right|^p \int_{\partial\mathbb{B}_\nu}\left|\frac{T^\ell_{x(z)}\left(X^\ell f\right)(z)}{\left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\right|}\cdot y\right|^p d\sigma_\nu(y),$$

as well as

$$\int_{\partial\mathbb{B}_\nu}\left|\left(X^\ell f\right)_y(z)\right|^p d\sigma_\nu(y) = \left|X^\ell f(z)\right|^p \int_{\partial\mathbb{B}_\nu}\left|\frac{X^\ell f(z)}{\left|X^\ell f(z)\right|}\cdot y\right|^p d\sigma_\nu(y).$$

The crucial observation now is that

$$\int_{\partial\mathbb{B}_\nu}\left|\frac{T^\ell_{x(z)}\left(X^\ell f\right)(z)}{\left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\right|}\cdot y\right|^p d\sigma_\nu(y) = \int_{\partial\mathbb{B}_\nu}\left|\frac{X^\ell f(z)}{\left|X^\ell f(z)\right|}\cdot y\right|^p d\sigma_\nu(y) = \gamma_{p,\nu}$$

is *independent* of the row vector in $\partial\mathbb{B}_\nu$ that is dotted with y. Thus we have

$$\begin{aligned} \left|T^\ell\left(X^\ell f\right)(z)\right|^p &= \left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\right|^p = \frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\left|T^\ell_{x(z)}\left(X^\ell f\right)(z)\, y\right|^p d\sigma_\nu(y), \\ \left|X^\ell f(z)\right|^p &= \frac{1}{\gamma_{p,\nu}}\int_{\partial\mathbb{B}_\nu}\left|\left(X^\ell f\right)_y(z)\right|^p d\sigma_\nu(y). \end{aligned}$$

So with $d\omega_{p\sigma}(z) = \left(1-|z|^2\right)^{p\sigma} d\lambda_n(z)$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{B}_n} \left|X^m\left(\mathbb{M}_\varphi f\right)\right|^p d\omega_{p\sigma}(z) \\
&\leq C_{n,\sigma,p,m} \sum_{\ell=0}^m \int_{\mathbb{B}_n} \left|T^\ell\left(X^\ell f\right)(z)\right|^p d\omega_{p\sigma}(z) \\
&= C_{n,\sigma,p,m} \sum_{\ell=0}^m \frac{1}{\gamma_{p,\nu}} \int_{\partial\mathbb{B}_\nu} \int_{\mathbb{B}_n} \left|x(z)\left(X^{m-\ell}\varphi\right)(z)\left(X^\ell f_y\right)(z)\right|^p d\omega_{p\sigma}(z)\, d\sigma_\nu(y) \\
&\leq C_{n,\sigma,p,m} \sum_{\ell=0}^m \frac{1}{\gamma_{p,\nu}} \int_{\partial\mathbb{B}_\nu} \int_{\mathbb{B}_n} \left|\left(X^{m-\ell}\varphi\right)(z)\left(X^\ell f\right)_y(z)\right|^p d\omega_{p\sigma}(z)\, d\sigma_\nu(y) \\
&\leq C_{n,\sigma,p,m} \sum_{\ell=0}^m \frac{1}{\gamma_{p,\nu}} \int_{\partial\mathbb{B}_\nu} \left\|\mathbb{M}_\varphi\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)} \int_{\mathbb{B}_n} \left|\left(\mathcal{X}^m f\right)_y(z)\right|^p d\omega_{p\sigma}(z)\, d\sigma_\nu(y)
\end{aligned}$$

by the case $M=1$ of Proposition 22, where ℓ^2 there is replaced by $\mathbb{C}^\nu$, g by φ and h by f_y. Now we use the equality

$$\int_{\partial\mathbb{B}_\nu} \left|\left(\mathcal{X}^m f\right)_y(z)\right|^p d\sigma_\nu(y) = \gamma_{p,\nu}\left|\mathcal{X}^m f(z)\right|^p$$

to obtain

$$\begin{aligned}
\int_{\mathbb{B}_n} \left|X^m\left(\mathbb{M}_\varphi f\right)\right|^p d\omega_{p\sigma}(z) &\leq C_{n,\sigma,p,m} \left\|\mathbb{M}_\varphi\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)} \int_{\mathbb{B}_n} \left|\mathcal{X}^m f(z)\right|^p d\omega_{p\sigma}(z) \\
&\leq C_{n,\sigma,p,m} \left\|\mathbb{M}_\varphi\right\|^p_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)} \|f\|^p_{B_p^\sigma(\mathbb{B}_n;\mathbb{C}^\mu)}.
\end{aligned}$$

Since m depends only on n, σ and p, this completes the proof of (ES-26), and hence that of Lemma 23.

Finally we return to complete the proof of Proposition 22. We have already proved the case $M=1$. Now we sketch a proof of the case $M=2$ using the multiplier norm inequality (5-5) with $\kappa=2$. By multiplicativity of $|\cdot|$ on tensors, it suffices to show that for $m=\ell_1+\ell_2+k$,

$$\begin{aligned}
\int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^\sigma\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left(\mathcal{Y}^{\ell_2}g\right)\left(\mathcal{Y}^k h\right)\right|^p d\lambda_n(z) \\
\leq C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^{2p}_{B_p^\sigma(\mathbb{B}_n)\to B_p^\sigma(\mathbb{B}_n;\ell^2)} \|h\|^p_{B_p^\sigma(\mathbb{B}_n)}.
\end{aligned} \tag{ES-27}$$

This time we write using the divergence form of Leibniz' formula on tensor products (c.f. (ES-24)),

$$\begin{aligned}
\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left(\mathcal{Y}^{\ell_2}g\right)\left(\mathcal{Y}^k h\right) &= \left(\mathcal{Y}^{\ell_1}g\right)\otimes\left\{\sum_{\alpha=0}^{\ell_2} c_\alpha^{\ell_2}\mathcal{Y}^\alpha\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right\} \\
&= \sum_{\alpha=0}^{\ell_2} c_\alpha^{\ell_2}\left(\mathcal{Y}^{\ell_1}g\right)\otimes\left[\mathcal{Y}^\alpha\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right] \\
&= \sum_{\alpha=0}^{\ell_2} c_\alpha^{\ell_2}\left\{\sum_{\beta=0}^{\ell_1} c_\beta^{\ell_1}\mathcal{Y}^\beta\left(g\otimes\mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha}h\right)\right)\right\}.
\end{aligned}$$

We first use the Hilbert space valued interpolation theorem together with the case $\kappa = 2$ of Lemma 23 to show that $g \in M_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)}$ and $g \in M_{B_p^{s_2}(\mathbb{B}_n)\to B_p^{s_2}(\mathbb{B}_n;\ell^2)}$ for appropriate values of s_1 and s_2. Assuming for convenience that $\mathcal{Y} = \left(1-|z|^2\right) R$, and keeping in mind Remark , we obtain

$$\begin{aligned}
&\left\| g(z) \otimes \left(1-|z|^2\right)^{\sigma-s_1} \mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha} h\right)\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)} \\
&\le \left\|\mathbb{M}_g\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)} \left\|\left(1-|z|^2\right)^{\sigma-s_1} \mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha} h\right)\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)} \\
&= \left\|\mathbb{M}_g\right\|^p_{B_p^{s_1}(\mathbb{B}_n;\ell^2)\to B_p^{s_1}(\mathbb{B}_n;\ell^2\otimes\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{s_1} \mathcal{Y}^{\beta}\left(1-|z|^2\right)^{\sigma-s_1} \mathcal{Y}^{\alpha+\ell_1-\beta}\left(g\mathcal{Y}^{k+\ell_2-\alpha} h\right)\right|^p d\lambda_n,
\end{aligned}$$

which by (5-5) is at most

$$\begin{aligned}
&C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^{\sigma}(\mathbb{B}_n)\to B_p^{\sigma}(\mathbb{B}_n;\ell^2)} \int_{\mathbb{B}_n} \left|\left(1-|z|^2\right)^{s_2} \mathcal{Y}^{\alpha+\ell_1}\left(g\left(1-|z|^2\right)^{\sigma-s_2} \mathcal{Y}^{k+\ell_2-\alpha} h\right)\right|^p d\lambda_n \\
&= C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^{\sigma}(\mathbb{B}_n)\to B_p^{\sigma}(\mathbb{B}_n;\ell^2)} \left\|g\left(1-|z|^2\right)^{\sigma-s_2} \mathcal{Y}^{k+\ell_2-\alpha} h\right\|^p_{B_p^{s_2}(\mathbb{B}_n;\ell^2)} \\
&\le C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^p_{B_p^{\sigma}(\mathbb{B}_n)\to B_p^{\sigma}(\mathbb{B}_n;\ell^2)} \left\|\mathbb{M}_g\right\|^p_{B_p^{s_2}(\mathbb{B}_n)\to B_p^{s_2}(\mathbb{B}_n;\ell^2)} \left\|\left(1-|z|^2\right)^{\sigma-s_2} \mathcal{Y}^{k+\ell_2-\alpha} h\right\|^p_{B_p^{s_2}(\mathbb{B}_n)} \\
&\le C_{n,\sigma,p}\left\|\mathbb{M}_g\right\|^{2p}_{B_p^{\sigma}(\mathbb{B}_n)\to B_p^{\sigma}(\mathbb{B}_n;\ell^2)} \|h\|^p_{B_p^{\sigma}(\mathbb{B}_n)}.
\end{aligned}$$

Summing up over α and β gives (ES-27).

Repeating this procedure for $M \ge 3$ and using (5-5) with $\kappa = M$ finishes the proof of Proposition 22. □

Schur's test. We prove Lemma 24 using Schur's Test as given in Theorem 2.9 on page 51 of [Zhu 2005].

Lemma ES.4. *Let (X,μ) be a measure space and $H(x,y)$ be a nonnegative kernel. Let $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Define*

$$\begin{aligned}
Tf(x) &= \int_X H(x,y) f(y)\, d\mu(y), \\
T^*g(y) &= \int_X H(x,y) g(x)\, d\mu(x).
\end{aligned}$$

If there is a positive function h on X and a positive constant A such that

$$\begin{aligned}
Th^q(x) &= \int_X H(x,y) h(y)^q\, d\mu(y) \le A h(x)^q, \qquad \mu-a.e.x \in X, \\
T^*h^p(y) &= \int_X H(x,y) h(x)^p\, d\mu(x) \le A h(y)^p, \qquad \mu-a.e.y \in X,
\end{aligned}$$

then T is bounded on $L^p(\mu)$ with $\|T\| \le A$.

Now we turn to the proof of Lemma 24. The case $c = 0$ of Lemma 24 is Lemma 2.10 in [Zhu 2005]. To minimize the clutter of indices, we first consider the proof for the case $c \neq 0$ when $p = 2$ and $t = -n-1$. Recall that

$$\begin{aligned}
\sqrt{\triangle(w,z)} &= |1-w\overline{z}|\,|\varphi_z(w)|, \\
\psi_\varepsilon(\zeta) &= \left(1-|\zeta|^2\right)^{\varepsilon},
\end{aligned}$$

and

$$T_{a,b,c}f(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b+n+1}\left(\sqrt{\triangle(w,z)}\right)^c}{|1-w\overline{z}|^{n+1+a+b+c}}f(w)\,d\lambda_n(w).$$

We will compute conditions on a, b, c and ε such that we have

$$T_{a,b,c}\psi_\varepsilon(z)\le C\psi_\varepsilon(z)\ \text{ and }\ T^*_{a,b,c}\psi_\varepsilon(w)\le C\psi_\varepsilon(w),\qquad z,w\in\mathbb{B}_n,\tag{ES-28}$$

where $T^*_{a,b,c}$ denotes the dual relative to $L^2(\lambda_n)$. For this we take $\varepsilon\in\mathbb{R}$ and compute

$$T_{a,b,c}\psi_\varepsilon(z)=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{n+1+b+\varepsilon}|\varphi_z(w)|^c}{|1-w\overline{z}|^{n+1+a+b}}d\lambda_n(w).$$

Note that the integral is finite if and only if $\varepsilon>-b-1$. Now make the change of variable $w=\varphi_z(\zeta)$ and use that λ_n is invariant to obtain

$$\begin{aligned}T_{a,b,c}\psi_\varepsilon(z)&=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{n+1+b+\varepsilon}|\varphi_z(w)|^c}{|1-w\overline{z}|^{n+1+a+b}}d\lambda_n(w)\\&=\int_{\mathbb{B}_n}F(w)\,d\lambda_n(w)=\int_{\mathbb{B}_n}F(\varphi_z(\zeta))\,d\lambda_n(\zeta)\\&=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^a\left(1-|\varphi_z(\zeta)|^2\right)^{n+1+b+\varepsilon}|\zeta|^c}{\left|1-\overline{\varphi_z(\zeta)}z\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}}dV(\zeta).\end{aligned}$$

From the identity (Theorem 2.2.2 in [Rudin 1980]),

$$1-\langle\varphi_a(\beta),\varphi_a(\gamma)\rangle=\frac{(1-\langle a,a\rangle)(1-\langle\beta,\gamma\rangle)}{(1-\langle\beta,a\rangle)(1-\langle a,\gamma\rangle)},$$

we obtain the identities

$$\begin{aligned}1-\varphi_z(\zeta)\overline{z}&=1-\langle\varphi_z(\zeta),\varphi_z(0)\rangle=\frac{1-|z|^2}{1-\zeta\overline{z}},\\1-|\varphi_z(\zeta)|^2&=1-\langle\varphi_z(\zeta),\varphi_z(\zeta)\rangle=\frac{\left(1-|z|^2\right)\left(1-|\zeta|^2\right)}{|1-\zeta\overline{z}|^2}.\end{aligned}$$

Plugging these identities into the formula for $T_{a,b,c}\psi_\varepsilon(z)$ we obtain

$$\begin{aligned}T_{a,b,c}\psi_\varepsilon(z)&=\int_{\mathbb{B}_n}\frac{(1-|z|^2)^a\left(\frac{\left(1-|z|^2\right)\left(1-|\zeta|^2\right)}{|1-\zeta\overline{z}|^2}\right)^{n+1+b+\varepsilon}|\zeta|^c}{\left|\frac{1-|z|^2}{1-\zeta\overline{z}}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}}dV(\zeta)\\&=\psi_\varepsilon(z)\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{b+\varepsilon}|\zeta|^c}{|1-\zeta\overline{z}|^{n+1+b-a+2\varepsilon}}dV(\zeta).\end{aligned}\tag{ES-29}$$

Now from Theorem 1.12 in [Zhu 2005] we obtain that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{\alpha}}{|1-\zeta\overline{z}|^{\beta}}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. Provided $c>-2n$ it is now easy to see that we also have

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{\alpha}|\zeta|^c}{|1-\zeta\overline{z}|^{\beta}}dV(\zeta)<\infty$$

if and only if $\beta-\alpha<n+1$. It now follows from the above that

$$T_{a,b,c}\psi_\varepsilon(z)\leq C\psi_\varepsilon(z),\qquad z\in\mathbb{B}_n,$$

if and only if

$$-b-1<\varepsilon<a.$$

Now we turn to the adjoint $T^*_{a,b,c}=T_{b+n+1,a-n-1,c}$ with respect to the space $L^2(\lambda_n)$. With the change of variable $z=\varphi_w(\zeta)$ we have

$$\begin{aligned}
T^*_{a,b,c}\psi_\varepsilon(w)&=\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\varphi_w(z)|^c}{|1-w\overline{z}|^{n+1+a+b}}d\lambda_n(z) \qquad\text{(ES-30)}\\
&=\int_{\mathbb{B}_n}G(z)\,d\lambda_n(z)=\int_{\mathbb{B}_n}G(\varphi_w(\zeta))\,d\lambda n(\zeta)\\
&=\int_{\mathbb{B}_n}\frac{\left(1-|\varphi_w(\zeta)|^2\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\zeta|^c}{\left|1-w\overline{\varphi_w(\zeta)}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}}dV(\zeta)\\
&=\int_{\mathbb{B}_n}\frac{\left(\frac{\left(1-|w|^2\right)\left(1-|\zeta|^2\right)}{|1-\zeta\overline{w}|^2}\right)^{a+\varepsilon}\left(1-|w|^2\right)^{b+n+1}|\zeta|^c}{\left|\frac{1-|w|^2}{1-\zeta\overline{w}}\right|^{n+1+a+b}\left(1-|\zeta|^2\right)^{n+1}}dV(\zeta)\\
&=\psi_\varepsilon(w)\int_{\mathbb{B}_n}\frac{\left(1-|\zeta|^2\right)^{a+\varepsilon-n-1}|\zeta|^c}{|1-\zeta\overline{w}|^{a-b+2\varepsilon-n-1}}dV(\zeta).
\end{aligned}$$

Arguing as above and provided $c>-2n$, we obtain

$$T^*_{a,b,c}\psi_\varepsilon(w)\leq C\psi_\varepsilon(w),\qquad w\in\mathbb{B}_n,$$

if and only if

$$-a+n<\varepsilon<b+n+1.$$

Altogether then there is $\varepsilon\in\mathbb{R}$ such that $h=\sqrt{\psi_\varepsilon}$ is a Schur function for $T_{a,b,c}$ on $L^2(\lambda_n)$ in Lemma ES.4 if and only if

$$\max\{-a+n,-b-1\}<\min\{a,b+n+1\}.$$

This is equivalent to $-2a<-n<2(b+1)$, which is (6-1) in the case $p=2$, $t=-n-1$. Thus Lemma ES.4 completes the proof that this case of (6-1) implies the boundedness of $T_{a,b,c}$ on $L^2(\lambda_n)$. The converse is easy - see for example the argument for the case $c=0$ on page 52 of [Zhu 2005].

We now turn to the general case. The adjoint $T^*_{a,b,c}$ relative to the Banach space $L^p(\nu_t)$ is easily computed to be $T^*_{a,b,c} = T_{b-t,a+t,c}$ (see page 52 of [Zhu 2005] for the case $c = 0$). Then from (ES-29) and (ES-30) we have

$$T_{a,b,c}\psi_\varepsilon(z) = \psi_\varepsilon(z)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{b+\varepsilon}|\zeta|^c}{|1-\zeta\overline{z}|^{n+1+b-a+2\varepsilon}}dV(\zeta),$$

$$T^*_{a,b,c}\psi_\varepsilon(w) = \psi_\varepsilon(w)\int_{\mathbb{B}_n} \frac{\left(1-|\zeta|^2\right)^{a+t+\varepsilon}|\zeta|^c}{|1-\zeta\overline{w}|^{a-b+2\varepsilon+t}}dV(\zeta).$$

Let $\frac{1}{p}+\frac{1}{q}=1$. We apply Schur's Lemma ES.4 with $h(\zeta) = \left(1-|\zeta|^2\right)^s$ and

$$s \in \left(-\frac{b+1}{q}, \frac{a}{q}\right) \cap \left(-\frac{a+1+t}{p}, \frac{b-t}{p}\right). \tag{ES-31}$$

Using Theorem 1.12 in [Zhu 2005] we obtain for h with s as in (ES-31) that

$$T_{a,b,c}h^q \le Ch^q \text{ and } T^*_{a,b,c}h^p \le Ch^p.$$

Lemma ES.4 now shows that $T_{a,b,c}$ is bounded on $L^p(\nu_t)$. Again, the converse follows from the argument for the case $c = 0$ on page 52 of [Zhu 2005].

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