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A REMARK ON BARELY $\dot{H}^{s_p}$-SUPERCRITICAL WAVE EQUATIONS

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We prove that a good $\dot{H}^{s_p}$ critical theory for the $3D$ wave equation $\partial_{tt}u - \triangle u = -|u|^{p-1}u$ can be extended to prove global well-posedness of smooth solutions of at least one $3D$ barely $\dot{H}^{s_p}$-supercritical wave equation $\partial_{tt}u - \triangle u = -|u|^{p-1}ug(|u|)$, with g growing slowly to infinity, provided that a Kenig-Merle type condition is satisfied. This result is related to those obtained by Tao and the author for the particular case $s_p = 1$, showing global regularity for g growing logarithmically with radial data and for g growing doubly logarithmically with general data.

1. Introduction

For fixed $p > 3$, let $\tilde{H}^2 := \dot{H}^2(\mathbb{R}^3) \cap \dot{H}^{s_p}(\mathbb{R}^3)$ and $\tilde{H}^1 := \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{s_p-1}(\mathbb{R}^3)$, where $s_p := \frac{3}{2} - \frac{2}{p-1}$. We consider the wave equation

$$\begin{cases} \partial_{tt}u - \triangle u = -|u|^{p-1}ug(|u|), \\ u(0) := u_0 \in \tilde{H}^2, \\ \partial_t u(0) := u_1 \in \tilde{H}^1, \end{cases} \tag{1-1}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex-valued scalar field and g is a smooth, real-valued positive function defined on the set of nonnegative numbers and satisfying

$$0 \le g'(x) \lesssim \frac{1}{x}. \tag{1-2}$$

This condition says that g grows more slowly than any positive power of u.

We shall see that (1-1) has many connections with the defocusing power-type wave equation

$$\begin{cases} \partial_{tt}u - \triangle u = -|u|^{p-1}u, \\ u(0) := u_0 \in \dot{H}^{s_p}(\mathbb{R}^3), \\ \partial_t u(0) := u_1 \in \dot{H}^{s_p-1}(\mathbb{R}^3). \end{cases} \tag{1-3}$$

It is known that if u satisfies (1-3), then u_λ defined by

$$u_\lambda(t,x) := \frac{1}{\lambda^{2/(p-1)}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \tag{1-4}$$

satisfies the same equation, but with data

$$u_\lambda(0,x) = \frac{1}{\lambda^{2/(p-1)}} u_0\left(\frac{x}{\lambda}\right) \quad \text{and} \quad \partial_t u_\lambda(0,x) = \frac{1}{\lambda^{2/(p-1)+1}} u_1\left(\frac{x}{\lambda}\right).$$

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Notice that (1-3) is $\dot{H}^{s_p}(\mathbb{R}^3)$ critical, which means that the $\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)$-norm of $(u(0),\partial_t u(0))$ is invariant under the scaling defined above.

We recall the local existence theory. From [Ginibre and Velo 1989; Lindblad and Sogge 1995], we know that there exists a positive constant $\delta := \delta\big(\|(u_0,u_1)\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)}\big) > 0$ and a time of local existence $T_l > 0$ such that if

$$\left\|\cos(tD)u_0 + \frac{\sin(tD)}{D}\right\|_{L_t^{2(p-1)}L_x^{2(p-1)}([0,T_l]\times\mathbb{R}^3)} \le \delta \tag{1-5}$$

then there exists a unique solution $(u,\partial_t u)$ in

$$\mathscr{C}\big([0,T_l],\ \dot{H}^{s_p}(\mathbb{R}^3)\big)\cap L_t^{2(p-1)}L_x^{2(p-1)}([0,T_l]\times\mathbb{R}^3)\cap D^{\frac{1}{2}-s_p}L_t^4L_x^4([0,T_l]\times\mathbb{R}^3)\times\mathscr{C}\big([0,T_l],\ \dot{H}^{s_p-1}(\mathbb{R}^3)\big)$$

of (1-3)[1] in the integral equation sense, i.e., u satisfies the Duhamel formula

$$u(t) := \cos(tD)u_0 + \frac{\sin(tD)}{D}u_1 - \int_0^t \frac{\sin(t-t')D}{D}\big(|u|^{p-1}u\big)(t')\,dt'. \tag{1-6}$$

It follows that we can define a maximal time interval of existence $I_{\max} = (-T_-, T_+)$. Moreover,

$$\|u\|_{L_t^{2(p-1)}L_x^{2(p-1)}(J)} < \infty, \quad \|D^{s_p-\frac{1}{2}}u\|_{L_t^4L_x^4(J)} < \infty, \quad \text{and} \quad \|(u,\partial_t u)\|_{L_t^\infty\dot{H}^{s_p}\times L_t^\infty\dot{H}^{s_p-1}(J)} < \infty$$

for any compact subinterval $J\subset I_{\max}$. See [Kenig and Merle 2006] or [Tao 2006a] for more explanations.

Now we turn to the global well-posedness theory of "(1-3)". In view of the local well-posedness theory, one can prove (see [Kenig and Merle 2011] and references), after some effort, that it is enough to find a finite upper bound of $\|u\|_{L_t^{2(p-1)}L_x^{2(p-1)}(I\times\mathbb{R}^3)}$ on arbitrary long time intervals I, and, if this is the case, then the solution scatters to a solution of the linear wave equation. No blow-up has been observed for (1-3). Therefore it is believed that the following scattering conjecture is true:

Conjecture 1.1 (scattering conjecture). *Assume that u is the solution of* (1-3) *with data* $(u_0,u_1)\in\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)$. *Then u exists for all time t and there exists* $C_1 := C_1\big(\|(u_0,u_1)\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)}\big)$ *such that*

$$\|u\|_{L_t^{2(p-1)}L_x^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)} \le C_1. \tag{1-7}$$

The case $s_p = 1$ (equivalently, $p = 5$) is particular. Indeed the solution

$$(u,\partial_t u)\in\mathscr{C}\big([0,T_l],\ \dot{H}^1(\mathbb{R}^3)\big)\times\mathscr{C}\big([0,T_l],\ L^2(\mathbb{R}^3)\big)$$

satisfies the conservation of the energy $E(t)$ defined by

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^3}|\partial_t u(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(t,x)|^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3}|u|^6(t,x)\,dx. \tag{1-8}$$

[1]The $L_t^{2(p-1)}L_x^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)$-norm of u is invariant under the scaling (1-4). The choice of the space $L_t^{2(p-1)}L_x^{2(p-1)}$ in which we place the solution u is not unique. There exists an infinite number of spaces of the form $L_t^qL_x^r$ scale invariant in which we can establish a local well-posedness theory.

In other words, $E(t) = E(0)$. This is why this equation is often called energy-critical: the exponent $s_p = 1$ corresponds precisely to the minimal regularity required for (1-8) to be defined. The global well-posedness of (1-8) in the energy class and in higher regularity spaces is now understood. Rauch [1981] proved the global existence of smooth solutions of this equation with small data. Struwe [1988] showed that the result still holds for large data but with the additional assumption of spherical symmetry of the data. The general case (large data, no symmetry assumption) was finally settled by Grillakis [1990; 1992]. Shatah and Struwe [1994] and independently Kapitanski [1994] proved global existence of solutions in the energy class. Bahouri and Gérard [1999] reproved this result by using a compactness method and results from Bahouri and Shatah [1998]. In particular, they showed that the $L_t^{2(5-1)} L_x^{2(5-1)}(\mathbb{R}\times\mathbb{R}^3)$-norm of the solution is bounded by an unspecified finite quantity. Lately Tao [2006b] found an exponential tower type bound of this norm. All these proofs of global existence of solutions of the energy-critical wave equation have as a common key point the conservation of energy, which leads, in particular, to the control of the $\dot{H}^1 \times L^2$-norm of the solution $(\partial_t u(t), u(t))$.

If $s_p < 1$, or equivalently, $p < 5$, we are in the energy-subcritical equation. The scattering conjecture is an open problem. Nevertheless, some partial results are known if we consider the same problem (1-3), but with data $(u_0, u_1) \in H^s \times H^{s-1}$, $s_p < s$. More precisely, it is proved in [Kenig et al. 2000; Gallagher and Planchon 2003; Bahouri and Chemin 2006; Roy 2007; Roy 2009a] that there exists $s_0 := s_0(p)$ such that $s_p < s_0 < 1$ and such that (1-3) is globally well-posed in $H^s \times H^{s-1}$, for $s > s_0$.

If $s_p > 1$, or, equivalently, $p > 5$, we are in the energy-supercritical regime. The global behavior of the solution is, in this regime, very poorly understood. Indeed, following the theory of the energy-critical wave equation, the first step would be to prove that the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$-norm of the solution is bounded for all time by a finite quantity depending only on the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$-norm of the initial data. Unfortunately, the control of this norm is a very challenging problem, since there are no known conservation laws in high regularity Sobolev spaces. Kenig and Merle [2011] recently proved, at least for radial data, that this step would be the last, by using their concentration compactness/rigidity theorem method [Kenig and Merle 2006]. More precisely, they showed that if $\sup_{t\in I_{\max}} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)} < \infty$, then Conjecture 1.1 is true.

As mentioned before, the energy supercritical regime is almost *terra incognita*. Nevertheless, Tao [2007] observed that the technology used to prove global well-posedness of smooth solutions of (1-3) can be extended, after some effort, to some equations of the type (1-1), with $p = 5$ and radial data. More precisely, he proved global regularity of (1-1) with $g(x) := \log(2 + x^2)$. This phenomenon, in fact, does not depend on the symmetry of the data: it was proved in [Roy 2009b] that there exists a unique global smooth solution of (1-1) with $g(x) := \log^c \log(10 + x^2)$ and $0 < c < \frac{8}{225}$.

Equations of the type (1-1) are called *barely $\dot{H}^{s_p}$-supercritical* wave equations. Indeed, the condition (1-2) basically says that for every $\epsilon > 0$, there exist two constants $c_1 := c_1(p)$ and $c_2 := c_2(p,\epsilon)$ such that

$$c_1(p) \le g(|u|) \le c_2(p,\epsilon)|u|^{\epsilon} \quad \text{for } |u| \text{ large.} \tag{1-9}$$

Since the critical exponent of the equation $\partial_{tt}u - \triangle u = -|u|^{p-1+\epsilon}u$ is $s_{p+\epsilon} = s_p + O(\epsilon)$, the nonlinearity of (1-1) is barely $\dot{H}^{s_p}$-supercritical.

The goal of this paper is to check that this phenomenon, observed for $s_p = 1$, still holds for other values of s_p. The standard local well-posedness theory shows us that it is enough to control the pointwise-in-time $\tilde{H}^2 \times \tilde{H}^1$-norm of the solution. In this paper, we will use an alternative local well-posedness theory. We shall prove:

Proposition 1.2 (local existence for barely $\dot{H}^{s_p}$-supercritical wave equation). *Assume that g satisfies* (1-2) *and*

$$g''(x) = O\left(\frac{1}{x^2}\right). \tag{1-10}$$

Let M be such that $\|(u_0,\ u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \le M$. Then there exists $\delta := \delta(M) > 0$ small such that, if T_l satisfies

$$\left\| \cos(tD) u_0 + \frac{\sin tD}{D} u_1 \right\|_{L_t^{2(p-1)} L_x^{2(p-1)}([0,T_l] \times \mathbb{R}^3)} \le \delta, \tag{1-11}$$

then there exists a unique $(u, \partial_t u)$ in

$$\mathscr{C}\big([0,\ T_l],\ \tilde{H}^2\big) \cap L_t^{2(p-1)} L_x^{2(p-1)}\big([0,\ T_l]\big) \cap D^{\frac{1}{2}-s_p} L_t^4 L_x^4\big([0,\ T_l]\big) \cap D^{\frac{1}{2}-2} L_t^4 L_x^4\big([0,\ T_l]\big) \times \mathscr{C}\big([0,\ T_l],\ \tilde{H}^1\big)$$

that solves (1-1) *in the integral equation sense*; *i.e., u satisfies the Duhamel formula*

$$u(t) := \cos(tD) u_0 + \frac{\sin tD}{D} u_1 - \int_0^t \frac{\sin(t-t')D}{D} \big(|u(t')|^{p-1} u(t') g(|u(t')|)\big)\, dt'. \tag{1-12}$$

Notice the many similarities between Proposition 1.2 and the local well-posedness theory for (1-3).

This allows us to define a maximum time interval of existence $I_{\max,g} = [-T_{-,g},\ T_{+,g}]$ such that, for any compact subinterval $J \subset I_{\max,g}$, the quantities

$$\|u\|_{L_t^{2(p-1)} L_x^{2(p-1)}(J)}, \quad \|D^{s_p - \frac{1}{2}} u\|_{L_t^4 L_x^4(J)}, \quad \|D^{2-\frac{1}{2}} u\|_{L_t^4 L_x^4(J)}, \quad \|(u,\ \partial_t u)\|_{L_t^\infty \tilde{H}^2(J) \times L_t^\infty \tilde{H}^1(J)}$$

are all finite. Again, see [Kenig and Merle 2006] or [Tao 2006a] for more explanations.

Now we set up the problem. In view of the comments above for $s_p = 1$, we need to make two assumptions. First we will work with a "good" $\dot{H}^{s_p}(\mathbb{R}^3)$ theory: therefore we will assume that Conjecture 1.1 is true. Then, we also would like to work with $\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ bounded solutions $(u(t), \partial_t u(t))$; more precisely, we will assume this:

Condition 1.3 (of Kenig–Merle type). *Let g be a function that satisfies* (1-2) *and that is constant for x large. Then there exists $C_2 := C_2\big(\|(u_0,\ u_1)\|_{\tilde{H}^2 \times \tilde{H}^1},\ g\big)$ such that*

$$\sup_{t \in I_{\max,g}} \|\ (u(t),\ \partial_t u(t))\ \|_{\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)} \le C_2. \tag{1-13}$$

Remark 1.4. In the particular case $s_p = 1$, it is not difficult to see that Condition 1.3 is satisfied. Indeed, u satisfies the energy conservation law

$$E_b(t) := \frac{1}{2} \int_{\mathbb{R}^3} \big(\partial_t u(t,x)\big)^2\, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t,x)|^2\, dx + \int_{\mathbb{R}^3} F(u(t,x), \bar{u}(t,x))\, dx, \tag{1-14}$$

with

$$F(z,\bar{z}) = |z|^{5+1}\int_0^1 t^5\,\mathrm{Re}\,(g(t|z|))\,dt = |z|^{5+1}\int_0^1 t^5\,g(t|z|)\,dt. \tag{1-15}$$

Since g is bounded, we have $|F(z,\bar{z})| \lesssim |z|^6$. By using the Sobolev embeddings $\|u_0\|_{L_x^6} \lesssim \|u_0\|_{\tilde{H}^2}$ and $\|u(t)\|_{L_x^6} \lesssim \|u(t)\|_{\tilde{H}^2}$, we easily conclude that Condition 1.3 holds. The energy conservation law was often in [Tao 2007; Roy 2009b].

Here is the main result of this paper:

Theorem 1.5. *Let p be fixed.*

(1) *There exists a function $\tilde{g}$ satisfying* (1-2) *and*

$$\lim_{x\to\infty}\tilde{g}(x) = \infty \tag{1-16}$$

and such that the solution of (1-1) (*with $g := \tilde{g}$*) *exists for all time, provided that the scattering conjecture and Condition 1.3 are satisfied.*

(2) *There exists a function f depending on T and $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}$ such that*

$$\|u\|_{L_t^\infty\tilde{H}^2([-T,T])} + \|\partial_t u\|_{L_t^\infty\tilde{H}^1([-T,T])} \leq f\big(T,\,\|(u_0,u_1\|)_{\tilde{H}^2\times\tilde{H}^1}\big). \tag{1-17}$$

Theorem 1.5 shows that a "good" $\dot{H}^{s_p}(\mathbb{R}^3)$ theory for (1-3) can be extended, at least, to one barely $\dot{H}^{s_p}(\mathbb{R}^3)$-supercritical equation, with $\tilde{g}$ going to infinity.

Remark 1.6. Apart from its dependence on p, the function $\tilde{g}$ is universal: it does not depend on an upper bound of the initial data. Moreover, $\tilde{g}$ is unbounded: it goes to infinity with as x.

Remark 1.7. In fact, Theorem 1.5 holds for a weaker version of Condition 1.3: there exists a function C_2 such that for all subinterval $I \subset I_{\max,g}$

$$\sup_{t\in I}\big\|(u(t),\,\partial_t u(t))\big\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times\dot{H}^{s_p-1}(\mathbb{R}^3)} \leq C_2, \tag{1-18}$$

with $C_2 := C_2\big(\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1},\ g,\ |I|\big)$. See the proof of Theorem 1.5 and, in particular, (5-21), (5-33) and (5-48).

We recall some basic properties and estimates. If $t_0 \in [t_1,t_2]$, if $F \in L_t^{\tilde{q}}L_x^{\tilde{r}}([t_1,\ t_2])$ and if $(u,\partial_t u) \in C\big([t_1,t_2],\ \dot{H}^m(\mathbb{R}^3)\big)\times C\big([t_1,t_2],\ \dot{H}^{m-1}(\mathbb{R}^3)\big)$ satisfy

$$u(t):\quad \cos(tD)u_0 + \frac{\sin tD}{D}u_1 - \int_{t_0}^{t}\frac{\sin(t-t')D}{D}F(t')\,dt', \tag{1-19}$$

with data $(u(t_0),\partial_t u(t_0)) \in \dot{H}^m(\mathbb{R}^3)\times\dot{H}^{m-1}(\mathbb{R}^3)$, then we have the Strichartz estimates [Ginibre and Velo 1995; Lindblad and Sogge 1995]

$$\begin{aligned}\|u\|_{L_t^qL_x^r([t_1,\,t_2])} + \|u\|_{L_t^\infty\dot{H}^m(\mathbb{R}^3)([t_1,\,t_2])} + \|\partial_t u\|_{L_t^\infty\dot{H}^{m-1}(\mathbb{R}^3)([t_1,\,t_2])}&\\ \lesssim \|(u(t_0),\ \partial_t u(t_0))\|_{\dot{H}^m(\mathbb{R}^3)\times\dot{H}^{m-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}([t_1,\,t_2])}.&\end{aligned} \tag{1-20}$$

Here (q, r) is m-wave admissible, i.e.,

$$(q,r) \in (2,\infty] \times [2,\infty] \quad \text{and} \quad \frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m; \tag{1-21}$$

moreover,

$$\frac{1}{q} + \frac{3}{r} = \frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2. \tag{1-22}$$

We set some notation that will appear throughout the paper.

We write $A \lesssim B$ if there exists a universal nonnegative constant $C' > 0$ such that $A \leq C'B$. The notation $A = O(B)$ means $A \lesssim B$. More generally, we write $A \lesssim_{a_1,\ldots,a_n} B$ if there exists a nonnegative constant $C' = C(a_1, \ldots, a_n)$ such that $A \leq C'B$. We say that C'' is the constant determined by $\lesssim$ in $A \lesssim_{a_1,\ldots,a_n} B$ if C'' is the smallest possible C' such that $A \leq C'B$. We write $A \ll_{a_1,\ldots,a_n} B$ if there exists a universal small nonnegative constant $c = c(a_1, \ldots, a_n)$ such that $A \leq cB$. Following [Kenig and Merle 2011], we define, on an interval I,

$$\|u\|_{S(I)} := \|u\|_{L_t^{2(p-1)} L_x^{2(p-1)}(I)}, \quad \|u\|_{W(I)} := \|u\|_{L_t^4 L_x^4(I)}, \quad \|u\|_{\tilde{W}(I)} := \|u\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(I)}. \tag{1-23}$$

We also define the quantity

$$Q(I,u) := \|D^{s_p - \frac{1}{2}} u\|_{W(I)} + \|D^{2-\frac{1}{2}} u\|_{W(I)} + \|u\|_{L_t^\infty \tilde{H}^2(I)} + \|\partial_t u\|_{L_t^\infty \tilde{H}^1(I)} \tag{1-24}$$

Let X be a Banach space and $r \geq 0$. Then

$$\boldsymbol{B}(X,r) := \{f \in X : \|f\|_X \leq r\} \tag{1-25}$$

We recall also the well-known Sobolev embeddings. We have

$$\|h\|_{L^\infty(\mathbb{R}^3)} \lesssim \|h\|_{\tilde{H}^2}, \tag{1-26}$$

$$\|h\|_{S(I)} \lesssim \|D^{s_p - \frac{1}{2}} h\|_{L_t^{2(p-1)} L_x^{\frac{6(p-1)}{2p-3}}(I)}. \tag{1-27}$$

We shall combine (1-27) with the Strichartz estimates, since $\bigl(2(p-1), \frac{6(p-1)}{2p-3}\bigr)$ is $\frac{1}{2}$- wave admissible.

We also recall some Leibnitz rules [Christ and Weinstein 1991; Kenig et al. 1993]. We have

$$\|D^\alpha F(u)\|_{L_t^q L_x^r(I)} \lesssim \|F'(u)\|_{L_t^{q_1} L_x^{r_1}(I)} \|D^\alpha u\|_{L_t^{q_2} L_x^{r_2}(I)}, \tag{1-28}$$

with $\alpha > 0$, r, r_1, r_2 lying in $[1, \infty]$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$.

The Leibnitz rule for products is

$$\|D^\alpha (uv)\|_{L_t^q L_x^r(I)} \lesssim \|D^\alpha u\|_{L_t^{q_1} L_x^{r_1}(I)} \|v\|_{L_t^{q_2} L_x^{r_2}(I)} + \|D^\alpha u\|_{L_t^{q_3} L_x^{r_3}(I)} \|v\|_{L_t^{q_4} L_x^{r_4}(I)}, \tag{1-29}$$

with $\alpha > 0$, r, r_1, r_2 lying in $[1, \infty]$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{q} = \frac{1}{q_3} + \frac{1}{q_4}$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, and $\frac{1}{r} = \frac{1}{r_3} + \frac{1}{r_4}$.

If $F \in C^2$, we can write

$$F(x) - F(y) = \int_0^1 F'\bigl(tx + (1-t)y\bigr)(x-y)\,dt. \tag{1-30}$$

By using (1-28) and (1-29) the Leibnitz rule for differences can be formulated as

$$\begin{aligned}\|D^\alpha(F(u)-F(v))\|_{L_t^q L_x^r(I)} \lesssim & \sup_{t\in[0,1]} \|F'(tu+(1-t)v)\|_{L_t^{q_1} L_x^{r_1}(I)} \|D^\alpha(u-v)\|_{L_t^{q_2} L_x^{r_2}(I)} \\ & + \sup_{t\in[0,1]} \|F''(tu+(1-t)v)\|_{L_t^{q_1'} L_x^{r_1'}(I)} \big(\|D^\alpha u\|_{L_t^{q_2'} L_x^{r_2'}(I)} + \|D^\alpha v\|_{L_t^{q_2'} L_x^{r_2'}(I)}\big) \|u-v\|_{L_t^{q_3'} L_x^{r_3'}(I)}, \end{aligned} \tag{1-31}$$

with $\alpha > 0$, $r_1, r_2, r_1', r_2', r_3'$ lying in $[1,\ \infty]$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{1}{q} = \frac{1}{q_1'} + \frac{1}{q_2'} + \frac{1}{q_3'}$, and $\frac{1}{r} = \frac{1}{r_1'} + \frac{1}{r_2'} + \frac{1}{r_3'}$.

We shall apply these formulas to several formulas of $F(u)$, and, in particular, to $F(u) := |u|^{p-1}ug(|u|)$. Notice that, by (1-2) and (1-10), we have $F'(x) \sim |x|^{p-1}g(|x|)$ and $F''(x) \sim |x|^{p-2}g(|x|)$. Notice also that, by (1-2) again, we have, for $t \in [0,1]$,

$$g\,(|tx+(1-t)y|) \le g\,(2\max(|x|,|y|)) \le g\big(\max(|x|,|y|)+\log 2\big) \lesssim g(|x|)+g(|y|). \tag{1-32}$$

This will allow us to estimate easily

$$\sup_{t\in[0,1]} \|F'(tu+(1-t)v)\|_{L_t^{q_1} L_x^{r_1}(I)} \quad \text{and} \quad \sup_{t\in[0,1]} \|F''(tu+(1-t)v)\|_{L_t^{q_1} L_x^{r_1}(I)}.$$

Now we explain the main ideas of this paper. We shall prove, in Section 3, that very many values functions g, a special property for the solution of (1-1) holds.

Proposition 1.8 (control of $S(I)$-norm and of norm of initial data imply control of $L_t^\infty \tilde{H}^2(I) \times L_t^\infty \tilde{H}^1(I)$ norm). *Let I be a compact subinterval of $I_{\max,g}$ (so $\|u\|_{S(I)} < \infty$) and assume that* $0 \in I$. *Assume that g satisfies* (1-2), (1-10) *and*[2]

$$\int_1^\infty \frac{1}{yg^2(y)}\,dy = \infty. \tag{1-33}$$

Let $A \ge 0$ such that $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1} \le A$. Let u be the solution of (1-1). *There exists a constant $C > 0$ such that*

$$\|(u,\partial_t u)\|_{L_t^\infty \tilde{H}^2(I)\times L_t^\infty \tilde{H}^1(I)} \le (2C)^N A, \tag{1-34}$$

with $N := N(I)$, such that

$$\int_{2CA}^{(2C)^N A} \frac{1}{yg^2(y)}\,dy \gg \|u\|_{S(I)}^{2(p-1)}. \tag{1-35}$$

Moreover we shall give a criterion of global well-posedness (proved in Section 4):

Proposition 1.9 (criterion of global well-posedness). *Assume that $|I_{\max,g}| < \infty$. Assume that g satisfies* (1-2), (1-10) *and* (1-33). *Then*

$$\|u\|_{S(I_{\max,g})} = \infty. \tag{1-36}$$

The first step is to prove global well-posedness of (1-1), with $g := g_1$ a nondecreasing function that is constant for x large (say $x \ge C_1'$, with C_1' to be determined). By Proposition 1.9, it is enough to find an upper bound of the $S([-T,T])$-norm of the solution $u_{[1]}$ for T arbitrarily large. This can indeed be done, by proving that g_1 can be considered as a subcritical perturbation of the nonlinearity. In other words, $g_1(|u|)|u|^{p-1}u$ will play the same role as that of $|u|^{p-1}u(1-|u|^{-\alpha})$ for some $\alpha > 0$. Once we have noticed

[2] Condition (1-33) basically says that g grows slowly on average.

that this comparison is possible, we shall estimate the relevant norms (in particular, $\|u_{[1]}\|_{S([-T,T])})$) using perturbation theory, Conjecture 1.1 and Condition 1.3, in the spirit of [Zhang 2006]. We expect to find a bound of the form

$$\|u_{[1]}\|_{S([-T,T])} \leq C_3\big(\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1},\ T\big), \tag{1-37}$$

with C_3 increasing as T or $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}$ grows. Notice that if we restrict $[-T,T]$ to the interval $[-1,1]$ and if the $\tilde{H}^2\times\tilde{H}^1$-norm of the initial data (u_0,u_1) is bounded by 1, then we can prove, using (1-37), (1-26) and Proposition 1.8, that the $L_t^\infty L_x^\infty([-T,T])$-norm of the solution $u_{[1]}$ is bounded by a constant (denoted by C_1) on $[-1,1]$. Therefore, if h is a smooth extension of g_1 outside $[0,C_1]$, and if u is the solution of (1-1) (with $g:=h$), we expect to prove that $u=u_{[1]}$ on $[-1,1]$ and for data $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\leq 1$. This implies in particular, by (1-37), that we have a finite upper bound $\|u\|_{S([-1,1])}$.

We are not done yet. There are two problems. First, g_1 does not go to infinity. Second, we only control $\|u\|_{S([-1,1])}$ for data $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\leq 1$: we would like to control $\|u\|_{S(\mathbb{R})}$ for arbitrary data. In order to overcome these difficulties we iterate the procedure described above. More precisely, given a function g_{i-1} that is constant for $x\geq C_{i-1}$ and such that $u_{[i-1]}$, a solution of (1-1) with $g=g_{i-1}$, satisfies $\|u_{[i-1]}\|_{S([-(i-1),i-1])}<\infty$, we construct a function g_i that

- is an extension of g_{i-1} outside $[0,C_{i-1}]$, and
- is increasing and constant (say equal to $i+1$) for $x\geq C_i'$, with C_i' to be determined.

Again, we shall prove that the g_i may be regarded as a subcritical perturbation of the nonlinearity $(i+1)|u|^{p-1}u$. This allow us to control $\|u_{[i]}\|_{S([-i,i])}$, by using perturbation theory, Conjecture 1.1, and Condition 1.3. Using Proposition 1.8 and (1-26), we can find a finite upper bound for $\|u_{[i]}\|_{L_t^\infty L_x^\infty([-i,i])}$. We assign the value of this upper bound to C_i. To conclude the argument we let $\tilde{g}=\lim_{i\to\infty} g_i$. Given $T>0$, we can find a j such that $[-T,T]\subset[-j,j]$ and $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\leq j$. We prove that $u=u_{[j]}$ on $[-j,j]$, where u is a solution of (1-1) with $g:=\tilde{g}$. Since we have a finite upper bound of $\|u_{[j]}\|_{S([-j,j])}$, we also control $\|u\|_{S([-j,j])}$ and $\|u\|_{S([-T,T])}$. Theorem 1.5 follows from Proposition 1.9.

2. Proof of Proposition 1.2

In this section we prove Proposition 1.2 for barely $\dot{H}^{s_p}(\mathbb{R}^3)$-supercritical wave equations (1-1). The proof is based upon standard arguments. Here we have chosen to modify an argument in [Kenig and Merle 2011].

For δ, T_l, C, M to be chosen and such that (1-11) holds we define

$$\begin{aligned} B_1 &:= \boldsymbol{B}\big(\mathscr{C}([0,T_l],\ \tilde{H}^2)\cap D^{\frac12-s_p}W([0,T_l])\cap D^{\frac12-2}W([0,T_l]),\,2CM\big),\\ B_2 &:= \boldsymbol{B}\big(S([0,T_l]),\ 2\delta\big),\\ B' &:= \boldsymbol{B}\big(\mathscr{C}([0,T_l],\ \tilde{H}^1),\,2CM\big),\end{aligned} \tag{2-1}$$

and

$$X := \big\{(u,\partial_t u) : u\in B_1\cap B_2,\ \partial_t u\in B'\big\}. \tag{2-2}$$

Let

$$\Psi(u,\ \partial_t u) := \begin{pmatrix} \cos(tD)u_0 + \dfrac{\sin(tD)}{D}u_1 - \displaystyle\int_0^t \frac{\sin(t-t')D}{D}\left(|u(t')|^{p-1}u(t')g(|u(t')|)\right)\,dt' \\ -D\sin(tD)u_0 + \cos(tD)u_1 - \displaystyle\int_0^t \cos(t-t')D\left(|u(t')|^{p-1}u(t')g(|u(t')|)\right)\,dt' \end{pmatrix}. \tag{2-3}$$

<u>Ψ *maps* X *to* X.</u> Indeed, in view of (1-11), (1-20), and the fractional Leibnitz rule (1-28) applied to $\alpha \in \left\{s_p - \frac{1}{2},\ 2 - \frac{1}{2}\right\}$ and

$$F(u) := |u|^{p-1}ug(|u|)$$

and by applying the multipliers $D^{2-\frac{1}{2}}$ and $D^{s_p-\frac{1}{2}}$ to the Strichartz estimates with $m = \frac{1}{2}$, we have

$$\begin{aligned} &Q([0,\ T_l]) \\ &\quad\lesssim \left\|(u_0,\ u_1)\right\|_{\tilde{H}^2(\mathbb{R}^3)\times\tilde{H}^1(\mathbb{R}^3)} + \left\|D^{s_p-\frac{1}{2}}\left(|u|^{p-1}ug(|u|)\right)\right\|_{\tilde{W}([0,T_l])} + \left\|D^{2-\frac{1}{2}}\left(|u|^{p-1}ug(|u|)\right)\right\|_{\tilde{W}([0,T_l])} \\ &\quad\le CM + C\left(\|D^{s_p-\frac{1}{2}}u\|_{W([0,T_l])} + \|D^{2-\frac{1}{2}}u\|_{W([0,T_l])}\right)\|u\|^{p-1}_{S([0,T_l])}g(\|u\|_{L_t^\infty L_x^\infty([0,T_l])}) \\ &\quad\le CM + (2\delta)^{p-1}C(2CM)g(2CM) \end{aligned} \tag{2-4}$$

for some $C > 0$ and

$$\begin{aligned} \|u\|_{S([0,T_l])} - \delta &\lesssim \left\|D^{s_p-\frac{1}{2}}\left(|u|^{p-1}ug(|u|)\right)\right\|_{\tilde{W}([0,T_l])} \\ &\lesssim \|u\|^{p-1}_{S([0,T_l])}\|D^{s_p-\frac{1}{2}}u\|_{W([0,T_l])}\, g(\|u\|_{L_t^\infty L_x^\infty([0,T_l])}) \lesssim (2\delta)^{p-1}(2CM)g(2CM). \end{aligned} \tag{2-5}$$

Choosing $\delta = \delta(M) > 0$ small enough we see that $\Psi(X) \subset X$.

<u>Ψ *is a contraction.*</u> Indeed we have

$$\begin{aligned} &\|\Psi(u) - \Psi(v)\|_X \\ &\lesssim \left\|D^{s_p-\frac{1}{2}}(|u|^{p-1}ug(|u|) - |v|^{p-1}vg(|v|))\right\|_{\tilde{W}([0,T_l])} + \left\|D^{2-\frac{1}{2}}(|u|^{p-1}ug(|u|) - |v|^{p-1}vg(|v|))\right\|_{\tilde{W}([0,T_l])} \\ &\lesssim \left(g(\|u\|_{L_t^\infty L_x^\infty([0,T_l])}) + g(\|v\|_{L_t^\infty L_x^\infty([0,T_l])})\right) \\ &\quad\times\Bigg(\left(\|u\|^{p-1}_{S([0,T_l])} + \|v\|^{p-1}_{S([0,T_l])}\right)\left(\|D^{s_p-\frac{1}{2}}(u-v)\|_{W([0,T_l])} + \|D^{2-\frac{1}{2}}(u-v)\|_{W([0,T_l])}\right) \\ &\qquad + \left(\|u\|^{p-2}_{S([0,T_l])} + \|v\|^{p-2}_{S([0,T_l])}\right)\|u-v\|_{S([0,T_l])} \\ &\qquad\quad \times\left(\|D^{s_p-\frac{1}{2}}u\|_{W([0,T_l])} + \|D^{2-\frac{1}{2}}u\|_{W([0,T_l])} + \|D^{s_p-\frac{1}{2}}v\|_{W([0,T_l])} + \|D^{2-\frac{1}{2}}v\|_{W([0,T_l])}\right)\Bigg) \\ &\lesssim \left(g(2CM)(2\delta)^{p-1} + (2\delta)^{p-2}(2CM)\right)\|u-v\|_X. \end{aligned} \tag{2-6}$$

In these computations, we applied the Leibnitz rule for differences to $\alpha \in \left\{s_p - \frac{1}{2}, 2 - \frac{1}{2}\right\}$ and

$$F(u) := |u|^{p-1}ug(|u|).$$

Therefore, if $\delta = \delta(M) > 0$ is small enough, Ψ is a contraction.

3. Proof of Proposition 1.8

To show Proposition 1.8, it is enough to prove that $Q(I) < \infty$. Without loss of generality we can assume that $A \gg 1$. Then we divide I into subintervals $(I_i)_{1 \leq i \leq N}$ such that

$$\|u\|_{S(I_i)} = \frac{\eta}{g^{1/(p-1)}((2C)^i A)} \tag{3-1}$$

for some $C \gtrsim 1$ and $\eta > 0$ constants to be chosen later, except maybe the last one. Notice that such a partition always exists since by (1-33) we get, for $N := N(I)$ large enough,

$$\sum_{i=1}^{N} \frac{1}{g^2((2C)^i A)} \geq \int_1^N \frac{1}{g^2((2C)^x A)}\, dx \gtrsim \int_{2CA}^{(2C)^N A} \frac{1}{y g^2(y)}\, dy \gg \|u\|_{S(I)}^{2(p-1)}. \tag{3-2}$$

We get, by a similar reasoning as used in Section 2

$$\begin{aligned}
Q(I_1, u) &\lesssim \|(u_0, u_1)\|_{\tilde{H}^2(\mathbb{R}^3) \times \tilde{H}^1(\mathbb{R}^3)} + \left\| D^{s_p - \frac{1}{2}} \left(|u|^{p-1} u g(|u|)\right)\right\|_{\tilde{W}(I_1)} + \left\| D^{2 - \frac{1}{2}} \left(|u|^{p-1} u g(|u|)\right)\right\|_{\tilde{W}(I_1)} \\
&\lesssim A + \left(\|D^{s_p - \frac{1}{2}} u\|_{W(I_1)} + \|D^{2-\frac{1}{2}} u\|_{W(I_1)}\right) \|u\|_{S(I_1)}^{p-1} g(\|u\|_{L_t^\infty L_x^\infty(I_1)}) \\
&\lesssim A + \|u\|_{S(I_1)}^{p-1} Q(I_1, u) g(Q(I_1, u)).
\end{aligned} \tag{3-3}$$

We choose C to be equal to the constant determined by $\lesssim$ in (3-3). Without loss of generality we can assume that $C > 1$. By a continuity argument, iteration on i, we get, for $\eta \ll 1$, (1-34).

4. Proof of Proposition 1.9

To prove Proposition 1.9, we argue as follows: by time reversal symmetry it is enough to prove that $T_{+,g} < \infty$. If $\|u\|_{S(I_{\max,g})} < \infty$ then we have $Q([0, T_{+,g}], u) < \infty$: this follows by slightly adapting the proof of Proposition 1.8. Consequently, by the dominated convergence theorem, there would exist a sequence $t_n \to T_{+,g}$ such that $\|u\|_{S([t_n, T_{+,g}])} \ll \delta$ and $\|D^{s_p - \frac{1}{2}} u\|_{W([t_n, T_{+,g}])} \ll \delta$ if n is large enough, with δ defined in Proposition 1.2. But, by (1-19) and (1-20),

$$\begin{aligned}
&\left\| \cos((t - t_n)D) u(t_n) + \frac{\sin(t - t_n)D}{D} u_1 \right\|_{S([t_n, T_{+,g}])} \\
&\qquad \lesssim \|u\|_{S([t_n, T_{+,g}])} + \|u\|_{S([t_n, T_{+,g}])}^{p-1} \|D^{s_p - \frac{1}{2}} u\|_{W([t_n, T_{+,g}])} g(Q([0, T_{+,g}, u])) \ll \delta,
\end{aligned} \tag{4-1}$$

and consequently, by continuity, there would exist $\tilde{T} > T_{+,g}$ such that

$$\left\| \cos((t - t_n)D) u(t_n) + \frac{\sin(t - t_n)D}{D} \partial_t u(t_n) \right\|_{S([t_n, \tilde{T}])} \leq \delta, \tag{4-2}$$

which would contradict the definition of $T_{+,g}$.

Remark 4.1. Notice that if we have the stronger bound $\|u\|_{S(I_{\max,g})} \leq C$ with $C := C\left(\|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1}\right) < \infty$, then not only $I_{\max,g} = (-\infty, +\infty)$ but also u scatters as $t \to \pm\infty$. Indeed, by Proposition 1.9, $I_{\max,g} = \mathbb{R}$. Then by time reversal symmetry it is enough to assume that $t \to \infty$. Let $v(t) := (u(t), \partial_t u(t))$.

We are looking for $v_{+,0} := \big(u_{+,0},\ u_{+,1}\big)$ such that

$$\big\| v(t) - K(t)v_{+,0} \big\|_{\tilde{H}^2 \times \tilde{H}^1} \to 0 \tag{4-3}$$

as $t \to \infty$. Here

$$K(t) := \begin{pmatrix} \cos tD & (\sin tD)/D \\ -D \sin tD & \cos tD \end{pmatrix} \tag{4-4}$$

We have

$$K^{-1}(t) = \begin{pmatrix} \cos tD & -(\sin tD)/D \\ D \sin tD & \cos tD \end{pmatrix}. \tag{4-5}$$

Notice that $K^{-1}(t)$ and $K(t)$ are bounded in $\tilde{H}^2 \times \tilde{H}^1$. Therefore it is enough to prove that $K^{-1}(t)v(t)$ has a limit as $t \to \infty$. But since $K^{-1}(t)v(t) = (u_0, u_1) - K^{-1}(t)\,(u_{\mathrm{nl}}(t),\ \partial_t u_{\mathrm{nl}}(t))$ — where

$$u_{\mathrm{nl}}(t) := -\int_0^t \frac{\sin(t-t')D}{D} \big(|u(t')|^{p-1} u(t') g(|u(t')|)\big)\, dt'$$

denotes the nonlinear part of the solution (1-12) — it suffices to prove that $K^{-1}(t)\,(u_{\mathrm{nl}}(t), \partial_t u_{\mathrm{nl}}(t))$ has a limit. But

$$\begin{aligned} &\big\| K^{-1}(t_1) u_{\mathrm{nl}}(t_1) - K^{-1}(t_2) u_{\mathrm{nl}}(t_2) \big\|_{\tilde{H}^2 \times \tilde{H}^1} \\ &\qquad \lesssim \big\| (u_{\mathrm{nl}}, \partial_t u_{\mathrm{nl}}) \big\|_{L_t^\infty \tilde{H}^2([t_1,t_2]) \times L_t^\infty \tilde{H}^1([t_1,t_2])} \\ &\qquad \lesssim \Big(\big\| D^{s_p - \frac12} (|u|^{p-1} u g(|u|)) \big\|_{\tilde{W}([t_1,t_2])} + \big\| D^{2-\frac12} (|u|^{p-1} u g(|u|)) \big\|_{\tilde{W}([t_1,t_2])} \Big) \\ &\qquad \lesssim \Big(\big\| D^{s_p - \frac12} u \big\|_{W([t_1,t_2])} + \big\| D^{2-\frac12} u \big\|_{W([t_1,t_2])} \Big) \|u\|^{p-1}_{S([t_1,t_2])} g\big(\|u\|_{L_t^\infty L_x^\infty(\mathbb{R})}\big). \end{aligned} \tag{4-6}$$

It remains to prove that $Q(\mathbb{R}) < \infty$ in order to conclude that the Cauchy criterion is satisfied, which would imply scattering. This follows from $\|u\|_{S(\mathbb{R})} < \infty$ and a slight modification of the proof of Proposition 1.8.

5. Construction of the function g

In this section we prove Theorem 1.5. Let

$$\mathrm{Up}(i) := \big\{ (T,\ (u_0, u_1)) : 0 \le T \le i,\ \|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \le i \big\} \tag{5-1}$$

As i ranges over $\{1, 2, \dots\}$ we construct, for each set $\mathrm{Up}(i)$, a function g_i satisfying (1-2) and (1-10). Moreover it is constant for large values of $|x|$. The function g_{i+1} depends on g_i; the construction of g_i is made by induction on i. More precisely:

Lemma 5.1. *Let $A \gg 1$. There exist two sequences of numbers $\{C_i\}_{i \ge 0}$, $\{C'_i\}_{i \ge 0}$ and a sequence of functions $\{g_i\}_{i \ge 0}$ such that, for all $(T,\ (u_0, u_1)) \in \mathrm{Up}(i)$, we have*

- $g_0 := 1$, $C_0 := 0$, $C'_0 = 0$;
- $\{C_i\}_{i \ge 0}$ *and* $\{C'_i\}_{i \ge 0}$ *are positive, nondecreasing, and satisfy*

$$A C_{i-1} < C'_i < A C_i \tag{5-2}$$

for $i \geq 1$ *and*

$$C_i \geq i; \tag{5-3}$$

- g_i *is smooth, nondecreasing, and satisfies* (1-2), (1-10),

$$\int_1^{C_i'} \frac{1}{t g_i^2(t)}\, dt \to \infty \quad \text{as } i \to \infty, \tag{5-4}$$

and

$$g_i(|x|) = \begin{cases} g_{i-1}(|x|) & \text{if } |x| \leq A C_{i-1}, \\ i+1 & \text{if } |x| \geq C_i'; \end{cases} \tag{5-5}$$

- *the solution* $u_{[i]}$ *of the wave equation*

$$\begin{cases} \partial_{tt} u_{[i]} - \Delta u_{[i]} = -|u_{[i]}|^{p-1} u_{[i]} g_i(|u_{[i]}|), \\ u_{[i]}(0) = u_0 \in \tilde{H}^2, \\ \partial_t u_{[i]}(0) = u_1 \in \tilde{H}^1 \end{cases} \tag{5-6}$$

satisfies

$$\max\left(\|u_{[i]}\|_{S([-i,i])}, \left\| (u_{[i]}, \partial_t u_{[i]}) \right\|_{L_t^\infty \tilde{H}^2([-T,T]) \times L_t^\infty \tilde{H}^1([-T,T])} \right) \leq C_i. \tag{5-7}$$

We postpone the proof until page 212. Assume the lemma is true and let $\tilde{g} = \lim_{i \to \infty} g_i$. Clearly $\tilde{g}$ is smooth; it satisfies (1-2) and (1-10). It also goes to infinity. Moreover let u be the solution of (1-1) with $g := \tilde{g}$. We want to prove that the solution u exists for all time. Let $T_0 \geq 0$ be a fixed time. Let $j := j(T_0, \|u_0\|_{\tilde{H}^2}, \|u_1\|_{\tilde{H}^1}) > 0$ be the smallest positive integer such that $[-T_0, T_0] \subset [-j, j]$ and $\|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \leq j$. We claim that

$$\|(u, \partial_t u)\|_{L_t^\infty \tilde{H}^2([-T_0,T_0]) \times L_t^\infty \tilde{H}^1([-T_0,T_0])} \leq C_j \quad \text{and} \quad \|u\|_{S([-T_0,T_0])} \leq C_j.$$

Indeed, let

$$F_j := \left\{ t \in [0, j] : \|(u, \partial_t u)\|_{L_t^\infty \tilde{H}^2([-t,t]) \times L_t^\infty \tilde{H}^1([-t,t])} \leq C_j \text{ and } \|u\|_{S([-t,t])} \leq C_j \right\}. \tag{5-8}$$

We must show that F_j coincides with $[0, j]$. Certainly F_j is nonempty, since it contains 0; see (5-3).

F_j is closed. Indeed, let $\tilde{t} \in \bar{F}_j$. There exists a sequence $(t_n)_{n \geq 1}$ in $[0, j]$ such that $t_n \to \tilde{t}$, $\|u\|_{S([-t_n,t_n])} \leq C_j$, and $\|(u, \partial_t u)\|_{L_t^\infty \tilde{H}^2([-t_n,t_n]) \times L_t^\infty \tilde{H}^1([-t_n,t_n])} \leq C_j$. It is enough to prove that $\|u\|_{S([-\tilde{t},\tilde{t}])}$ is finite and then apply dominated convergence. There are two cases:

- If $\operatorname{card}\{t_n : t_n \leq \tilde{t}\} < \infty$, there exists n_0 large enough such that $t_n \geq \tilde{t}$ for $n \geq n_0$ and

$$\|u\|_{S([-\tilde{t},\tilde{t}])} \leq \|u\|_{S([-t_n,t_n])} < \infty. \tag{5-9}$$

- If $\operatorname{card}\{t_n : t_n \leq \tilde{t}\} = \infty$, we can assume by passing to a subsequence that $t_n \leq \tilde{t}$. Let $n_0 \geq 1$ be fixed. Since

$$\left\| \cos(t - t_{n_0}) D u(t_{n_0}) + \frac{\sin(t - t_{n_0}) D}{D} \partial_t u(t_{n_0}) \right\|_{S([t_{n_0},\tilde{t}])} \lesssim \left\| (u(t_{n_0}), \partial_t u(t_{n_0})) \right\|_{\tilde{H}^2 \times \tilde{H}^1} \lesssim C_j, \tag{5-10}$$

we conclude from the dominated convergence theorem that there is $n_1 := n_1(n_0)$ large enough that

$$\| \cos(t - t_{n_0}) D u(t_{n_0}) + \frac{\sin(t - t_{n_0}) D}{D} \partial_t u(t_{n_0}) \|_{S([t_{n_1},\tilde{t}])} \leq \delta, \tag{5-11}$$

with $\delta := \delta(C_j)$ defined in Proposition 1.2. Therefore, by Proposition 1.2, we have $\|u\|_{S([t_{n_1},\tilde{t}])} < \infty$. Similarly, $\|u\|_{S([-\tilde{t},-t_{n_1}])} < \infty$. Combining these inequalities with $\|u\|_{S([-t_{n_1},t_{n_1}])} \le C_j$, we eventually get $\|u\|_{S([-\tilde{t},\tilde{t}])} < \infty$, as desired.

F_j is open. Indeed, let $\bar{t} \in F_j$. By Proposition 1.2 there exists $\alpha > 0$ such that if $t \in (\bar{t}-\alpha, \bar{t}+\alpha) \cap [0, j]$ then $[-t, t] \subset I_{\max,\tilde{g}}$ and $\|u\|_{L_t^\infty L_x^\infty([-t,t])} \lesssim \|u\|_{L_t^\infty \tilde{H}^2([-t,t])} \lesssim C_j$. Also, by (5-7), $[-t, t] \subset I_{\max,g_j}$. In view of these remarks, we conclude, after slightly adapting the proof of Proposition 1.8, that $Q([-t, t], u) \lesssim_j 1$ and $Q([-t, t], u_{[j]}) \lesssim_j 1$. We divide $[-t,\ t]$ into a finite number of subintervals $(I_i)_{i\le k} = ([a_i, b_i])_{1\le i\le k}$ that satisfy, for $\eta \ll 1$ to be defined later, the following properties:

(1) $1 \le i \le k$: $\|u_{[j]}\|_{S(I_i)} \le \eta$, $\|u\|_{S(I_i)} \le \eta$, $\|D^{s_p-\frac{1}{2}} u_{[j]}\|_{W(I_i)} \le \eta$, $\|D^{s_p-\frac{1}{2}} u\|_{W(I_i)} \le \eta$, $\|D^{2-\frac{1}{2}} u\|_{W(I_i)} \le \eta$, and $\|D^{2-\frac{1}{2}} u_{[j]}\|_{W(I_i)} \le \eta$.

(2) $1 \le i < k$: $\|u_{[j]}\|_{S(I_i)} = \eta$ or $\|u\|_{S(I_i)} = \eta$ or $\|D^{s_p-\frac{1}{2}} u_{[j]}\|_{W(I_i)} = \eta$ or $\|D^{s_p-\frac{1}{2}} u\|_{W(I_i)} = \eta$ or $\|D^{2-\frac{1}{2}} u\|_{W(I_i)} = \eta$, or $\|D^{2-\frac{1}{2}} u_{[j]}\|_{W(I_i)} = \eta$.

Notice that, by (1-2), we have

$$\|g_j(|u|) - g_j(|u_{[j]}|)\|_{L_t^\infty L_x^\infty(I_i)} \lesssim \|u - u_{[j]}\|_{L_t^\infty L_x^\infty(I_i)} \lesssim \|u - u_{[j]}\|_{L_t^\infty \tilde{H}^2(I_i)}. \tag{5-12}$$

Consider $w = u - u_{[j]}$. Applying the Leibnitz rules (1-28), (1-31), and (1-29), together with (5-12), we have

$$\begin{aligned}
&Q(I_1, w) \\
&\lesssim \big\|D^{s_p-\frac{1}{2}}(|u|^{p-1}u(\tilde{g}-g_j)(|u|))\big\|_{\tilde{W}(I_1)} + \big\|D^{2-\frac{1}{2}}(|u|^{p-1}u(\tilde{g}-g_j)(|u|))\big\|_{\tilde{W}(I_1)} \\
&\quad + \big\|D^{s_p-\frac{1}{2}}(|u|^{p-1}u - |u_{[j]}|^{p-1}u_{[j]})g_j(|u|)\big\|_{\tilde{W}(I_1)} + \big\|D^{2-\frac{1}{2}}(|u|^{p-1}u - |u_{[j]}|^{p-1}u_{[j]})g_j(|u|)\big\|_{\tilde{W}(I_1)} \\
&\quad + \big\|D^{s_p-\frac{1}{2}}(|u_{[j]}|^{p-1}u_{[j]}(g_j(|u|)-g_j(|u_{[j]}|)))\big\|_{\tilde{W}(I_1)} + \big\|D^{2-\frac{1}{2}}(|u_{[j]}|^{p-1}u_{[j]}(g_j(|u|)-g_j(|u_{[j]}|)))\big\|_{\tilde{W}(I_1)} \\
&\lesssim (\tilde{g}-g_j)(\|u\|_{L_t^\infty \tilde{H}^2(I_1)})\big(\|D^{s_p-\frac{1}{2}}u\|_{W(I_1)} + \|D^{2-\frac{1}{2}}u\|_{W(I_1)}\big)\|u\|_{S(I_1)}^{p-1} \\
&\quad + g_j\big(\|u\|_{L_t^\infty \tilde{H}^2(I_1)}\big)\Big(\big(\|u_{[j]}\|_{S(I_1)}^{p-1} + \|u\|_{S(I_1)}^{p-1}\big)\big(\|D^{s_p-\frac{1}{2}}w\|_{W(I_1)} + \|D^{2-\frac{1}{2}}w\|_{W(I_1)}\big) \\
&\qquad\qquad + \big(\|u_{[j]}\|_{S(I_1)}^{p-2} + \|u\|_{S(I_1)}^{p-2}\big)\,\|w\|_{S(I_1)} \\
&\qquad\qquad\quad \times\big(\|D^{s_p-\frac{1}{2}}u\|_{W(I_1)} + \|D^{2-\frac{1}{2}}u\|_{W(I_1)} + \|D^{s_p-\frac{1}{2}}u_{[j]}\|_{W(I_1)} + \|D^{2-\frac{1}{2}}u_{[j]}\|_{W(I_1)}\big)\Big) \\
&\quad + \|g_j'(|u|)\|_{L_t^\infty L_x^\infty(I_1)}\big(\|D^{s_p-\frac{1}{2}}u\|_{W(I_1)} + \|D^{2-\frac{1}{2}}u\|_{W(I_1)}\big)\big(\|u\|_{S(I_1)}^{p-2} + \|u_{[j]}\|_{S(I_1)}^{p-2}\big)\|w\|_{S(I_1)} \\
&\quad + \big\|g_j(|u|) - g_j(|u_{[j]}|)\big\|_{L_t^\infty L_x^\infty(I_1)}\,\|D^{s_p-\frac{1}{2}}u_{[j]}\|_{W(I_1)}\,\|u_{[j]}\|_{S(I_1)}^{p-1} \\
&\quad + \|u_{[j]}\|_{S(I_1)}^{p-1}\,\|u_{[j]}\|_{L_t^\infty \tilde{H}^2(I_1)}\big(\|w\|_{L_t^\infty \tilde{H}^2(I_1)}\big(\|D^{s_p-\frac{1}{2}}u\|_{W(I_1)} + \|D^{s_p-\frac{1}{2}}u_{[j]}\|_{W(I_1)}\big) + \|D^{s_p-\frac{1}{2}}w\|_{W(I_1)}\big) \\
&\lesssim g_j(C_j)\eta^{p-1}Q(I_1, w) + \eta^{p-1}Q(I_1, w) + \eta^p Q(I_1, w) + C_j\eta^{p-1}\big(\eta Q(I_1, w) + Q(I_1, w)\big),
\end{aligned} \tag{5-13}$$

since, by choosing A large enough and by the construction of $\tilde{g}$, we have

$$(\tilde{g} - g_j)(\|u\|_{L_t^\infty \tilde{H}^2(I_1)}) = 0. \tag{5-14}$$

We conclude via a continuity argument that $Q(I_1, w)=0$, so $u=u_{[j]}$ on I_1. In particular, $u(b_1)=u_{[j]}(b_1)$. By iteration on i, it is not difficult to see that $u=u_{[j]}$ on $[-t,t]$. Hence $(\bar{t}-\alpha, \bar{t}+\alpha)\cap[0,j]\subset F_j$, by (5-7). Thus F_j is open.

The upshot is that $F_j=[0,j]$, so $\|u\|_{S([-T_0,T_0])}\le C_j$. This proves global well-posedness. Moreover, since j depends on T_0 and $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}$, we get (1-17).

Proof of Lemma 5.1. The proof extends to the end of the paper. We must establish *a priori* bounds.

Step 1: Construction of g_1.

Basically, g_1 is a nonnegative function that increases and is equal to 2 for x large. Recall that $[-T,T]\subset[-1,1]$ and $\|(u_0,\ u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\le 1$. Let $I\subset[-T,T]$.

Observe that the point $(\infty-,3+):=\left(\frac{3+\epsilon}{\epsilon}, 3+\epsilon\right)$ with $\epsilon\ll 1$ is $\frac{1}{2}$-wave admissible.

We would like to chop I (satisfying $\|\cdot\|_{L_t^\infty L_x^3(I)}<\infty$) into subintervals I_j such that $\|\cdot\|_{L_t^\infty L_x^3(I_j)}$ is as small as wanted. Unfortunately this is impossible because the L_t^∞-norm is pathological. Instead we will apply this process to $\|\cdot\|_{L_t^{\infty-}L_x^{3+}}$. This creates slight variations almost everywhere in the process of the construction of g_i. Details with respect to these slight perturbations have been omitted for the sake of readability: they are left to the reader, who should ignore the $+$ and $-$ signs at the first reading.

We define

$$X(I):=D^{\frac{1}{2}-s_p}L_t^{\infty-}L_x^{3+}(I)\cap D^{\frac{1}{2}-s_p}W(I)\cap S(I)\cap L_t^\infty\dot{H}^{s_p}(I)\times L_t^\infty\dot{H}^{s_p-1}(I). \tag{5-15}$$

Let g_1 be a smooth function, defined on the set of nonnegative real numbers, nondecreasing, and such that $h_1:=g_1-2$ satisfies the following properties: $h_1(0)=-1$, h is nondecreasing, and $h_1(x)=0$ if $|x|\ge 1$. It is not difficult to see that (1-2) and (1-10) are satisfied.

Observe that

$$|h_1(x)|\lesssim\frac{1}{|x|^{\frac{p-1}{2}-}} \tag{5-16}$$

and

$$|h_1'(x)|\lesssim\frac{1}{|x|^{\frac{p+1}{2}-}}. \tag{5-17}$$

Let $u_{[1]}$ and $v_{[1]}$ be solutions to the equations

$$\begin{cases}\partial_{tt}u_{[1]}-\triangle u_{[1]}=-|u_{[1]}|^{p-1}u_{[1]}g_1(|u_{[1]}|),\\ u_{[1]}(0)=u_0\in\tilde{H}^2,\\ \partial_t u_{[1]}(0)=u_1\in\tilde{H}^1\end{cases} \tag{5-18}$$

and

$$\begin{cases}\partial_{tt}v_{[1]}-\triangle v_{[1]}=-2\left|v_{[1]}\right|^{p-1}v_{[1]},\\ v_{[1]}(0)=u_0,\\ \partial_t v_{[1]}(0)=u_1.\end{cases} \tag{5-19}$$

Step 1a. We claim that $\|v_{[1]}\|_{X(\mathbb{R})}<\infty$. Indeed, since we assumed that Conjecture 1.1 is true, we can divide $\mathbb{R}$ into subintervals $(I_j=[t_j,t_{j+1}])_{1\le j\le l}$ such that

$$\|v_{[1]}\|_{S(I_j)}=\eta\quad\text{and}\quad\|v_{[1]}\|_{S(I_l)}\le\eta,$$

with $\eta \ll 1$. Then

$$\begin{aligned}\|v_{[1]}\|_{X(I_{j+1})} &\lesssim \big\|(v_{[1]}(t_j), \partial_t v_{[1]}(t_j))\big\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} + \big\|D^{s_p-\frac{1}{2}}(|v_{[1]}|^{p-1}v_{[1]})\big\|_{\tilde{W}(I_{j+1})}\\ &\lesssim \big\|(v_{[1]}(t_j), \partial_t v_{[1]}(t_j))\big\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} + \big\|D^{s_p-\frac{1}{2}}v_{[1]}\big\|_{W(I_{j+1})}\|v_{[1]}\|^{p-1}_{S(I_{j+1})}\\ &\lesssim \|v_{[1]}\|_{X(I_j)} + \eta^{p-1}\|v_{[1]}\|_{X(I_{j+1})}. \end{aligned} \tag{5-20}$$

Notice that $l \lesssim 1$: this follows from Conjecture 1.1, Condition 1.3 and the inequality

$$\begin{aligned}\|(u_0,u_1)\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} \le \sup_{t\in I_{\max,g_1}} \|(u(t),\partial_t u(t))\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} &\le C_2\big(\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\big)\\ &\lesssim 1, \end{aligned} \tag{5-21}$$

following from Condition 1.3 and the assumption $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1} \le 1$. (At this stage, we only need to know that $\|(u_0,u_1)\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} \le \|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1} \le 1$ and apply Conjecture 1.1. Therefore the introduction of $\sup_{t\in I_{\max,g_1}} \|(u(t),\partial_t u(t))\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)}$ in (5-21) is redundant. This is done on purpose. Indeed, we will use Condition 1.3 in other parts of the argument: see (5-33).)

Now by a standard continuity argument and iteration on j we have

$$\|v_{[1]}\|_{X(\mathbb{R})} \lesssim 1 \tag{5-22}$$

Step 1b. We control $\|u_{[1]}-v_{[1]}\|_{X([-\tilde{t},\tilde{t}])}$, for $\tilde{t} \ll 1$ to be chosen later. By time reversal symmetry it is enough to control $\|u_{[1]}-v_{[1]}\|_{X([0,\tilde{t}])}$. To this end we consider $w_{[1]} := u_{[1]}-v_{[1]}$. We get

$$\partial_{tt} w_{[1]} - \triangle w_{[1]} = -|w_{[1]}+v_{[1]}|^{p-1}(v_{[1]}+w_{[1]})g_1(v_{[1]}+w_{[1]}) + 2|v_{[1]}|^{p-1}v_{[1]}.$$

Let $\eta' \ll 1$. By (5-22), we can divide $[0,\tilde{t}]$ into subintervals $(J_k = [t'_k, t'_{k+1}])_{1\le k\le m}$ that satisfy

$$\|D^{s_p-\frac{1}{2}}v_{[1]}\|_{L_t^{\infty-}L_x^{3+}(J_k)} = \eta' \ \text{ or } \ \|D^{s_p-\frac{1}{2}}v_{[1]}\|_{W(J_k)} = \eta' \qquad \text{for } 1\le k<m, \tag{5-23}$$

$$\|D^{s_p-\frac{1}{2}}v_{[1]}\|_{W(J_k)} \le \eta' \text{ and } \|D^{s_p-\frac{1}{2}}v_{[1]}\|_{L_t^{\infty-}L_x^{3+}(J_k)} \le \eta' \qquad \text{for } 1\le k\le m. \tag{5-24}$$

We have

$$\|w_{[1]}\|_{X(J_{k+1})} \lesssim \big\|\big(w_{[1]}(t'_k), \partial_t w_{[1]}(t'_k)\big)\big\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} + A_1 + A_2,$$

where

$$\begin{aligned} A_1 &:= \|D^{s_p-\frac{1}{2}}\big(2|v_{[1]}|^{p-1}v_{[1]} - 2|v_{[1]}+w_{[1]}|^{p-1}(v_{[1]}+w_{[1]})\big)\|_{\tilde{W}(J_{k+1})},\\ A_2 &:= \|D^{s_p-\frac{1}{2}}\big(h_1(|v_{[1]}+w_{[1]}|)|v_{[1]}+w_{[1]}|^{p-1}(v_{[1]}+w_{[1]})\big)\big\|_{L_t^1 L_x^{\frac{3}{2}}(J_{k+1})}. \end{aligned} \tag{5-25}$$

By the fractional Leibnitz rule applied to $q(x) := |x|^{p-1}xh(x)$, (5-16), (5-17), Sobolev embedding and Hölder in time we have

$$\begin{aligned} A_2 &\lesssim \left\| |v_{[1]}+w_{[1]}|^{\frac{p-1}{2}+} \right\|_{L_t^{1+}L_x^{3-}(J_{k+1})} \left\| D^{s_p-\frac{1}{2}}(v_{[1]}+w_{[1]}) \right\|_{L_t^{\infty-}L_x^{3+}(J_{k+1})} \\ &\lesssim \|v_{[1]}+w_{[1]}\|^{\frac{p-1}{2}+}_{L_t^{\frac{p-1}{2}+}L_x^{\frac{3(p-1)+}{2}}(J_{k+1})} \left\| D^{s_p-\frac{1}{2}}(v_{[1]}+w_{[1]}) \right\|_{L_t^{\infty-}L_x^{3+}(J_{k+1})} \\ &\lesssim \tilde{t} \left\| D^{s_p-\frac{1}{2}}(v_{[1]}+w_{[1]}) \right\|^{\frac{p+1}{2}+}_{L_t^{\infty-}L_x^{3+}(J_{k+1})} \lesssim \tilde{t}(\eta')^{\frac{p+1}{2}+} + \tilde{t} \left\| D^{s_p-\frac{1}{2}} w_{[1]} \right\|^{\frac{p+1}{2}+}_{L_t^{\infty-}L_x^{3+}(J_{k+1})} \\ &\lesssim \tilde{t}(\eta')^{\frac{p+1}{2}+} + \tilde{t}\|w_{[1]}\|^{\frac{p+1}{2}+}_{X(J_{k+1})}. \end{aligned} \tag{5-26}$$

For A_1 we follow [Kenig and Merle 2011, p. 9]:

$$\begin{aligned} A_1 &\lesssim \left(\|v_{[1]}\|^{p-1}_{S(J_{k+1})} + \|w_{[1]}\|^{p-1}_{S(J_{k+1})}\right) \left\| D^{s_p-\frac{1}{2}} w_{[1]} \right\|_{W(J_{k+1})} \\ &\qquad + \left(\|v_{[1]}\|^{p-2}_{S(J_{k+1})} + \|w_{[1]}\|^{p-2}_{S(J_{k+1})}\right)\left(\left\| D^{s_p-\frac{1}{2}} v_{[1]} \right\|_{W(J_{k+1})} + \left\| D^{s_p-\frac{1}{2}} w_{[1]} \right\|_{W(J_{k+1})}\right) \|w_{[1]}\|_{S(J_{k+1})} \\ &\lesssim (\eta')^{p-1}\|w_{[1]}\|_{X(J_{k+1})} + \|w_{[1]}\|^p_{X(J_{k+1})} + (\eta')^{p-2}\|w_{[1]}\|^2_{X(J_{k+1})} + \eta'\|w_{[1]}\|^{p-1}_{X(J_{k+1})}. \end{aligned} \tag{5-27}$$

This follows from (1-31) and (1-27). Therefore we have

$$\begin{aligned} \|w_{[1]}\|_{X(J_{k+1})} &\lesssim \|w_{[1]}\|_{X(J_k)} + (\eta')^{\frac{p+1}{2}+}\tilde{t} + \tilde{t}\|w_{[1]}\|^{\frac{p+1}{2}+}_{X(J_{k+1})} \\ &\qquad + (\eta')^{p-1}\|w_{[1]}\|_{X(J_{k+1})} + \|w_{[1]}\|^p_{X(J_{k+1})} + (\eta')^{p-2}\|w_{[1]}\|^2_{Y(J_{k+1})} + \eta'\|w_{[1]}\|^{p-1}_{X(J_{k+1})}. \end{aligned} \tag{5-28}$$

Let C be the constant determined by (5-28). By induction, we have

$$\|w_{[1]}\|_{X(J_k)} \leq (2C)^k \tilde{t}, \tag{5-29}$$

provided that for $1 \leq k \leq m-1$ we have

$$\begin{aligned} C(\eta')^{\frac{p+1}{2}}\tilde{t} &\ll C(2C)^k\tilde{t}, \\ C\tilde{t}\left((2C)^k\tilde{t}\right)^{\frac{p+1}{2}+} &\ll (2C)^k\tilde{t}, \\ C(\eta')^{p-1}(2C)^{k+1}\tilde{t} &\ll C(2C)^k\tilde{t}, \\ C\left((2C)^k\tilde{t}\right)^p &\ll C(2C)^k\tilde{t}, \\ C(\eta')^{p-2}\left((2C)^{k+1}\tilde{t}\right)^2 &\ll C(2C)^k\tilde{t}, \\ \eta'\left((2C)^{k+1}\tilde{t}\right)^{p-1} &\ll C(2C)^k. \end{aligned} \tag{5-30}$$

These inequalities are satisfied if $\eta' \ll 1$ and

$$\tilde{t} \ll 1 \tag{5-31}$$

since $k \leq m-1$ and, by (5-22), $m \lesssim 1$. We conclude that

$$\|w_{[1]}\|_{X([0,\tilde{t}])} \lesssim 1. \tag{5-32}$$

Step 1c. We control $\|u_{[1]}\|_{X([-T,T])}$. By time reversal symmetry, it is enough to control $\|u_{[1]}\|_{X([0,T])}$. Recall that $T \le 1$. We chop $T \le 1$ into subintervals $(J_{k'} = [a_{k'}, b_{k'}])_{1\le k'\le l'}$ such that $|J_{k'}| = \tilde{t}$ for $1 \le k' < l'$ and $|J_{l'}| \le \tilde{t}$. Notice that, by Condition 1.3, we have

$$\begin{aligned} \|(u(a_{k'}), \partial_t u(a_{k'}))\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} &\le \sup_{t\in I_{\max,g_1}} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}(\mathbb{R}^3)\times \dot{H}^{s_p-1}(\mathbb{R}^3)} \\ &\le C_2\left(\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1}\right) \lesssim 1, \end{aligned} \tag{5-33}$$

taking advantage of the assumption $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1} \le 1$. For each k' let $v_{[1,k']}$ be the solution of

$$\begin{cases} \partial_{tt} v_{[1,k']} - \triangle v_{[1,k']} = -|v_{[1,k']}|^{p-1} v_{[1,k']}, \\ v_{[1,k']}(a_{k'}) = u_{[1]}(a_{k'}), \\ \partial_t v_{[1,k']}(a_{k'}) = \partial_t u_{[1]}(a_{k'}); \end{cases} \tag{5-34}$$

in particular, $v_{[1,k']} = v_{[1]}$. By slightly modifying the proof of Step 1b and letting $v_{[1,k']}$ play the role of $v_{[1]}$, this leads, by (5-33), to

$$\|v_{[1,k']}\|_{X(\mathbb{R})} \lesssim 1 \tag{5-35}$$

and

$$\|w_{[1,k']}\|_{X(J_{k'})} \lesssim 1, \tag{5-36}$$

with $w_{[1,k']} := u_{[1]} - v_{[1,k']}$. Therefore $\|u_{[1]}\|_{X(J_{k'})} \lesssim 1$, and summing over $J_{k'}$ we have

$$\|u_{[1]}\|_{X([0,T])} \lesssim 1. \tag{5-37}$$

Step 1d. We control $\|(u_{[1]}, \partial_t u_{[1]})\|_{L_t^\infty \tilde{H}^2([-1,1])\times L_t^\infty \tilde{H}^1([-1,1])}$ and $\|u_{[1]}\|_{S([-1,1])}$. We get from (5-37)

$$\|u_{[1]}\|_{S([-1,1])} \lesssim 1. \tag{5-38}$$

To conclude Step 1: By Proposition 1.8 and (5-38) we have

$$\|(u_{[1]}, \partial_t u_{[1]})\|_{L_t^\infty \tilde{H}^2([-1,1])\times L_t^\infty \tilde{H}^1([-1,1])} \lesssim 1. \tag{5-39}$$

Therefore

$$\max\left(\|u_{[1]}\|_{S([-1,1])},\ \|(u_{[1]}, \partial_t u_{[1]})\|_{L_t^\infty \tilde{H}^2([-1,1])\times L_t^\infty \tilde{H}^1([-1,1])}\right) \lesssim 1. \tag{5-40}$$

We let C_1' in the statement of Lemma 5.1 be equal to 1. We can assume without the loss of generality that the constant implicit in $\lesssim$ in (5-40) is larger than 1; let C_1 in the statement of Lemma 5.1 be this constant. Then C_1' and C_1 satisfy (5-2) and (5-3).

Step 2: Construction of g_i from g_{i-1}.

Recall that $[-T,T] \subset [-i,i]$ and $\|(u_0,u_1)\|_{\tilde{H}^2\times\tilde{H}^1} \le i$. In view of (5-5) it is enough to construct g_i for $|x| > AC_{i-1}$. It is clear that, by choosing C_i' large enough, we can construct find a function $\tilde{g}_i$ defined on $[AC_{i-1},\ C_i']$ such that g_i, defined by

$$g_i(x) := \begin{cases} g_{i-1}(x) & \text{if } |x| \le AC_{i-1}, \\ \tilde{g}_i(x) & \text{if } C_i' \ge |x| \ge AC_{i-1}, \\ i+1 & \text{if } |x| \ge C_i' \end{cases} \tag{5-41}$$

is smooth and slowly increasing; also it satisfies (1-2), (1-10), and

$$\int_{AC_{i-1}}^{C_i'} \frac{1}{y g_i^2(y)}\, dy \geq i. \tag{5-42}$$

It remains to determine C_i in the statement of Lemma 5.1. To do that we slightly modify the reasoning in Step 1.

We sketch the argument. Let $h_i(x) := g_i(x) - (i+1)$. Then $h_i(x) = 0$ if $|x| > C_i'$. It is not difficult to see that

$$|h_i(x)| \lesssim_i \frac{1}{|x|^{\frac{p-1}{2}+}}, \tag{5-43}$$

$$|h_i'(x)| \lesssim_i \frac{1}{|x|^{\frac{p+1}{2}+}}. \tag{5-44}$$

Let $u_{[i]}$ and $v_{[i]}$ be the solutions of the equations

$$\begin{cases} \partial_{tt} u_{[i]} - \triangle u_{[i]} = -|u_{[i]}|^{p-1} u_{[i]} g_i(|u_{[i]}|), \\ u_{[i]}(0) := u_0, \\ \partial_t u_{[i]}(0) := u_1 \end{cases} \tag{5-45}$$

and

$$\begin{cases} \partial_{tt} v_{[i]} - \triangle v_{[i]} = -(i+1)|v_{[i]}|^{p-1} v_{[i]}, \\ v_{[i]}(0) := u_0, \\ \partial_t v_{[i]}(0) := u_1 \end{cases} \tag{5-46}$$

Step 2a. We have

$$\|v_{[i]}\|_{X(\mathbb{R})} \lesssim_i 1, \tag{5-47}$$

by adapting the proof of Step 1a. Notice, in particular, that we can use Conjecture 1.1 and control $\|v_{[i]}\|_{S(\mathbb{R})}$ since $w_{[i]} := (i+1)^{\frac{1}{p-1}} v_{[i]}$ satisfies $\partial_{tt} w_{[i]} - \triangle w_{[i]} = -|w_{[i]}|^{p-1} w_{[i]}$.

Step 2b. We have $\|u_{[i]} - v_{[i]}\|_{X([0,\tilde{t}])} \lesssim_i 1$ for $\tilde{t} \ll_i 1$, by adapting the proof of Step 1b. The dependance on i basically comes from (5-43), (5-44) and (5-46).

Step 2c. We prove that $\|u_{[i]}\|_{X([-T,T])} \lesssim_{i,p} 1$. By time reversal symmetry, it is enough to control $\|u_{[i]}\|_{X([0,T])}$. Recall that $T \leq i$. We chop $[0, T]$ into subintervals $(J_{k'} = [a_{k'}, b_{k'}])_{1 \leq k' \leq l'}$ such that $|J_{k'}| = \tilde{t}$ for $1 \leq k' < l'$ and $|J_{l'}| \leq \tilde{t}$ (with $\tilde{t}$ defined in Step 2b). By Condition 1.3 and the assumption $\|(u_0, u_1)\|_{\tilde{H}^2 \times \tilde{H}^1} \leq i$, we have

$$\|(u_{[i]}(a_{k'}), \partial_t u_{[i]}(a_{k'}))\|_{\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p - 1}(\mathbb{R}^3)} \leq \sup_{t \in I_{\max, g_i}} \left\|(u_{[i]}(t),\ \partial_t u_{[i]}(t))\right\|_{\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)} \lesssim_i 1. \tag{5-48}$$

We introduce

$$\begin{cases} \partial_{tt} v_{[i,k']} - \triangle v_{[i,k']} = -(i+1)|v_{[i,k']}|^{p-1} v_{[i,k']}, \\ v_{[i,k']}(a_{k'}) = u_{[i]}(a_{k'}), \\ \partial_t v_{[i,k']}(a_{k'}) = \partial_t u_{[i]}(a_{k'}) \end{cases} \tag{5-49}$$

and, by using (5-48), we can prove that

$$\|u_{[i]}\|_{S([-i,i])} \lesssim_i 1. \tag{5-50}$$

Step 2d. By using Proposition 1.8 and (5-50) we get

$$\max\left(\|u_{[i]}\|_{S([-i,i])},\ \|(u_{[i]}, \partial_t u_{[i]})\|_{L_t^\infty \tilde{H}^2([-i,i]) \times L_t^\infty \tilde{H}^1([-i,i])}\right) \lesssim_i 1. \tag{5-51}$$

We can assume without loss of generality that the constant implicit in $\lesssim$ is larger than i and C_i'. Let C_i be this constant; (5-2) and (5-3) are satisfied.

This concludes Step 2, and the proof of Lemma 5.1. □

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