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AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH ONE MEASUREMENT AND THE PSEUDORANDOM SOURCE

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We consider the wave equation $(\partial_t^2 - \Delta_g)u(t,x) = f(t,x)$, in $\mathbb{R}^n$, $u|_{\mathbb{R}_- \times \mathbb{R}^n} = 0$, where the metric $g = (g_{jk}(x))_{j,k=1}^n$ is known outside an open and bounded set $M \subset \mathbb{R}^n$ with smooth boundary ∂M. We define a source as a sum of point sources, $f(t,x) = \sum_{j=1}^\infty a_j \delta_{x_j}(x)\delta(t)$, where the points x_j, $j \in \mathbb{Z}_+$, form a dense set on ∂M. We show that when the weights a_j are chosen appropriately, $u|_{\mathbb{R}\times\partial M}$ determines the scattering relation on ∂M, that is, it determines for all geodesics which pass through M the travel times together with the entering and exit points and directions. The wave $u(t,x)$ contains the singularities produced by all point sources, but when $a_j = \lambda^{-\lambda^j}$ for some $\lambda > 1$, we can trace back the point source that produced a given singularity in the data. This gives us the distance in $(\mathbb{R}^n, g)$ between a source point x_j and an arbitrary point $y \in \partial M$. In particular, if $(\overline{M}, g)$ is a simple Riemannian manifold and g is conformally Euclidian in $\overline{M}$, these distances are known to determine the metric g in M. In the case when $(\overline{M}, g)$ is nonsimple, we present a more detailed analysis of the wave fronts yielding the scattering relation on ∂M.

1. Introduction

In this paper we consider an inverse problem for the wave equation

$$(\partial_t^2 - \Delta_g)u(t,x) = f(t,x) \quad \text{in } (0,\infty)\times\mathbb{R}^n,$$
$$u|_{t=0} = \partial_t u|_{t=0} = 0,$$

where Δ_g is the Laplace–Beltrami operator corresponding to a Riemannian metric $g(x) = [g_{jk}(x)]_{j,k=1}^n$, that is,

$$\Delta_g u = \sum_{j,k=1}^n |g|^{-1/2}\frac{\partial}{\partial x^j}\Big(|g|^{1/2} g^{jk}\frac{\partial}{\partial x^k}u\Big),$$

where $|g| = \det(g_{jk})$ and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$ is the inverse matrix of $[g_{jk}(x)]_{j,k=1}^n$. We assume that $g_{jk} \in C^\infty(\mathbb{R}^n)$ and that there are $c_1, c_2 > 0$ such that

$$c_1|\xi|^2 \le \sum_{j.k=1}^n g_{jk}(x)\xi^j\xi^k \le c_2|\xi|^2, \quad x,\xi \in \mathbb{R}^n. \tag{1}$$

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Moreover, we assume that the metric g is known outside an open and bounded set $M \subset \mathbb{R}^n$ having a C^∞-smooth boundary ∂M.

We choose the origin of the time axis so that the source f is active at time $t = 0$. To ensure compatibility with the initial conditions, we let $T_0 < 0 < T$ and define the measurement map $L = L_g$,

$$L : C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n) \to C^\infty((T_0, T) \times \partial M), \quad Lf = u|_{(T_0,T)\times\partial M}, \tag{2}$$

where u is the solution of the wave equation

$$\begin{aligned} (\partial_t^2 - \Delta_g)u(t, x) &= f(t, x) \quad \text{in } (T_0, T) \times \mathbb{R}^n, \\ u|_{t=T_0} = \partial_t u|_{t=T_0} &= 0. \end{aligned} \tag{3}$$

Above, $C_c^\infty(T_0, T)$ denotes the space of smooth functions having compact support in (T_0, T). Its dual space, the space of generalized functions or distributions, is denoted by $\mathcal{D}'(T_0, T)$. Moreover, for functions $\phi \in C_c^\infty(T_0, T)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, we denote their pointwise product by $(\phi \otimes \psi)(t, x) = \phi(t)\psi(x)$.

We remark that the assumption (1), together with the finite speed of propagation for the wave equation, implies that the measurement Lf does not depend on $g_{jk}(x)$, for $|x| > R$, when R is sufficiently large. Thus we may assume without loss of generality that all the partial derivatives $\partial_x^\alpha g_{jk}$ are bounded on $\mathbb{R}^n$.

Let $x_j \in \partial M$, $j = 1, 2, \dots$, be a dense sequence of points in ∂M, and let us consider point sources

$$f_{x_j}(t, x) := \delta(t)\delta_{x_j}(x), \quad j = 1, 2, \dots.$$

In order to study the measurements Lf_{x_j}, we will use the Sobolev spaces (see [Triebel 1978])

$$\begin{aligned} H_p^s(\mathbb{R}^d) &:= \bigl\{ f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{H_p^s(\mathbb{R}^d)} := \bigl\|(1 - \Delta)^{s/2} f\bigr\|_{L^p(\mathbb{R}^d)} < +\infty \bigr\}, \\ \widetilde{H}_p^s(U) &:= \{ f \in H_p^s(\mathbb{R}^d);\ \operatorname{supp} f \subset \bar{U} \}, \\ H_p^s(U) &:= \{ f \in \mathcal{D}'(U);\ f = h|_U \text{ for some } h \in H_p^s(\mathbb{R}^d) \}, \end{aligned}$$

where $U \subset \mathbb{R}^d$ is open and $s \in \mathbb{R}$. When $p = 2$ we omit the subscript p in our notation, that is, we write $H^s(U) = H_2^s(U)$, etc. Moreover, we use projective topology on the tensor product $X \otimes Y$ of two Banach spaces X and Y, that is, $\|z\|_{X\otimes Y} := \inf \sum_j \|x_j\|_X \|y_j\|_Y$, where the infimum is taken over all representations $z = \sum_j x_j \otimes y_j$. We also use projective topology on tensor products of locally convex spaces; see, e.g., [Trèves 1967, Definition 43.2]. The measurement Lf_{x_j} can be defined in the sense of the following lemma.

Lemma 1.1. *Let $p \in (1, n/(n-1))$ and let $m \in \mathbb{N}$ satisfy $m > (n+1)/4$. Then the measurement operator L defined in (2) has a unique continuous extension*

$$L : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to \mathcal{D}'((T_0, T) \times \partial M).$$

We will prove Lemma 1.1 and other results presented in the introduction in Sections 3–6.

In this paper we study a single measurement Lh_0 that simultaneously combines all the measurements Lf_{x_j} by adding them together with appropriate weights. When the measurements Lf_{x_j} are summed together, to the authors' knowledge, there are no algorithms that can filter the value of a particular measurement from the sum. We will ask, however, whether we can find the essential features given by these measurements, like the travel times between points on ∂M, so that the metric could be determined under certain geometric conditions. Our main result is that Lh_0 determines the *scattering relation* $\Sigma_{M,g}$ for the manifold $(\overline{M}, g)$. Here $h_0(t, x)$ is an explicit source that we call pseudorandom; see Definition 1 in Section 2.

The scattering relation has been efficiently used to solve several geometric inverse problems [Dairbekov and Uhlmann 2010; Pestov and Uhlmann 2006; Stefanov and Uhlmann 2008; 2009]. To define the scattering relation, let TM denote the tangent space of M and let $\dot\gamma$ denote the tangent vector of a smooth curve $\gamma : [a, b] \to M$. Let $SM = \{(x, \xi) \in TM;\ \|\xi\|_g = 1\}$ be the unit sphere bundle on M and define

$$\partial_\pm SM = \big\{(x, \xi) \in SM;\ x \in \partial M,\ \mp(\nu, \xi)_g > 0\big\},$$

where ν is the exterior normal vector of ∂M. Moreover, let $\tau_{M,g}(x, \xi)$ be the infimum of the set

$$\big\{t \in (0, \infty];\ \gamma_{x,\xi}(t) \in \partial M\big\},$$

where $\gamma_{x,\xi}$ denotes the geodesic on (M, g) with initial data $(x, \xi) \in TM$. We write $\tau = \tau_{M,g}$ when the manifold (M, g) is clear from the context. We define the infimum of the empty set to be $+\infty$.

The scattering relation is the map $\Sigma = \Sigma_{M,g}$,

$$\Sigma : \mathcal{D}(\Sigma) \to \overline{\partial_+ SM} \times \mathbb{R}, \quad \mathcal{D}(\Sigma) = \big\{(x, \xi) \in \partial_- SM;\ \tau(x, \xi) < \infty\big\},$$

defined by $\Sigma(x, \xi) = \big(\gamma_{x,\xi}(\tau(x, \xi)), \dot\gamma_{x,\xi}(\tau(x, \xi)), \tau(x, \xi)\big)$.

Our main result is the following.

Theorem 1.2. *Let $M \subset \mathbb{R}^n$, $n \ge 2$ be an open and bounded set having a C^∞-smooth boundary. Then there is a generalized function $h_0(t, x)$ supported on $\{0\} \times \partial M$ and having the following properties: Assume that $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$ are two Riemannian metric tensors satisfying (1). Moreover, assume that $g_{jk}(x) = g'_{jk}(x)$ for $x \in \mathbb{R}^n \setminus M$. Let*

$$T > \max\Big(\sup_{(x,\xi)\in\partial_- SM} \tau_{M,g}(x, \xi),\ \sup_{(x,\xi)\in\partial_- SM} \tau_{M,g'}(x, \xi)\Big),$$

and assume that

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0, T) \times \partial M.$$

Then the scattering relations $\Sigma_{M,g}$ and $\Sigma_{M,g'}$ of Riemannian manifolds (M, g) and (M, g') are the same. In particular, if $(\overline{M}, g)$ and $(\overline{M}, g')$ are simple, the restrictions of the distance functions on the boundary satisfy $d_{\overline{M},g}(x, y) = d_{\overline{M},g'}(x, y)$ for $x, y \in \partial M$.

We remark that if $\sup_{\partial_- SM} \tau$ is infinite, then we prove the above result with measurements on an infinite time interval, that is, we prove that the measurement $u|_{(T_0,\infty)\times\partial M}$ determines $\mathcal{D}(\Sigma)$ and Σ.

Recall that a compact Riemannian manifold $(\overline{M}, g)$ with boundary is simple if it is simply connected, any geodesic has no conjugate points, and ∂M is strictly convex with respect to the metric g. Any two points of a simple manifold can be joined by a unique geodesic.

The key idea of the proof of Theorem 1.2 is to use source $h_0(t,x)=\sum_{j=1}^{\infty} a_j f_{x_j}$. The point source $a_{j_0} f_{x_{j_0}}$ produces a singularity, which is observed at a point $y \in \mathbb{R}^n \setminus M$ at time $t_0 = d(x_{j_0}, y)$ with a magnitude $a_{j_0}\beta(x_{j_0}, y)$, where β is an unknown nonvanishing smooth function. Appropriate choice of the weights a_j allows us find the index j_0 by looking at nearby singularities. Indeed, when $x_{j_k} \to x_{j_0}$ and $j_k \to \infty$, we see that the asymptotic behavior of the magnitude $a_{j_k}\beta(x_{j_k}, y)$ as $k \to \infty$ will be that of the weights a_{j_k}. Thus it is possible to factor out a_{j_k} in the magnitude and determine a_{j_0}. This argument is presented in Section 7 and gives us the distances $d(x_j, y)$ in $(\mathbb{R}^n, g)$ for arbitrary point $y \in \mathbb{R}^n \setminus M$ and a source point x_j.

Theorem 1.2 and boundary rigidity results for simple manifolds imply the following:

Corollary 1.3. *Let $M \subset \mathbb{R}^n$ and let $g_{jk}, g'_{jk} \in C^\infty(\mathbb{R}^n)$ be two Riemannian metric tensors satisfying the assumptions of Theorem 1.2. Let $(\overline{M}, g)$ and $(\overline{M}, g')$ be simple Riemannian manifolds. Then*:

(i) *If $n = 2$ and*

$$L_g h_0 = L_{g'} h_0 \quad \text{on } (T_0, T) \times \partial M, \tag{4}$$

then there is a diffeomorphism $\Phi : M \to M$ such that $\Phi|_{\partial M} = \mathrm{Id}$ and $g = \Phi_* g'$.

(ii) *For $n \geq 3$, there is $\epsilon = \epsilon_{n,M} > 0$ such that if $\|g_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$, $\|g'_{jk} - \delta_{jk}\|_{C^2(M)} < \epsilon_n$, and (4) holds, then there is a diffeomorphism $\Phi : M \to M$ such that $\Phi|_{\partial M} = \mathrm{Id}$ and $g = \Phi_* g'$.*

(iii) *If $g_{jk}(x) = a(x)\delta_{jk}$ and $g'_{jk}(x) = a'(x)\delta_{jk}$, that is, the metric tensors are conformally Euclidian, and (4) holds, then $g_{jk}(x) = g'_{jk}(x)$ for $x \in M$.*

Indeed, by Theorem 1.2, case (i) follows from [Pestov and Uhlmann 2005], case (ii) follows from [Burago and Ivanov 2010], and (iii) from [Muhometov 1977; 1981; Muhometov and Romanov 1978].

If Uhlmann's conjecture [2003], that the scattering relation determines the isometry type of nontrapping compact manifolds with nonempty boundary, can be proven, then Corollary 1.3 holds for a more general class of manifolds.

The problem of determining the metric g (possibly up to a diffeomorphism) given the measurement Lh_0 with only one function $h_0(t,x)$ is a formally determined inverse problem. Indeed, the formally computed "dimension of the data," that is, the dimension of $(T_0, T) \times \partial M$, is n and coincides with the dimension of the set M on which the unknown functions $g_{jk}(x)$ are defined.

The formally determined inverse problems have been studied in many cases. For instance, the two-dimensional Calderón inverse problem [Astala and Päivärinta 2006; Astala et al. 2005; Imanuvilov et al. 2010; Nachman 1996; Sylvester 1990] is formally determined. The same is true for the related inverse problem for the Schrödinger equation in two dimensions [Bukhgeim 2008]. The corresponding inverse problems in dimension $n \geq 3$ — see [Calderón 1980; Kenig et al. 2007; Lassas et al. 2003; Nachman 1988; Sylvester and Uhlmann 1987] and references in [Greenleaf et al. 2009a; 2009b], are overdetermined — that is, the dimension of the data is larger than the dimension of the unknown object. Similar classification holds for the elliptic inverse problems on Riemannian manifolds [Guillarmou and Tzou 2010; 2011; Lassas et al. 2003; Lee and Uhlmann 1989; 2001]. Moreover, the boundary rigidity problem [Kurylev et al. 2010; Michel 1981; Muhometov 1977; 1981; Muhometov and Romanov 1978; Romanov 1987; Stefanov and Uhlmann 2005] is formally determined in dimension $n = 2$ and overdetermined for $n \geq 3$.

Inverse problems in the time domain related to the Laplace–Beltrami operator Δ_g, namely the inverse boundary value problem for the wave, heat, and dynamical Schrödinger equations with Dirichlet-to-Neumann as data — see [Anderson et al. 2004; Belishev and Kurylev 1992; Katchalov and Kurylev 1998; Katchalov et al. 2001] — are overdetermined in dimensions $n \geq 2$. However, these problems are equivalent to the inverse boundary spectral problem (see [Katchalov et al. 2004]), and assuming that the eigenvalues are simple, the Dirichlet-to-Neumann map at a generic Dirichlet boundary value determines the boundary spectral data [Lassas 1995; 1998; Ramm 2001]. Thus, under generic conditions on the spectrum and on the boundary value (that is, under the condition that the these data belong in some open and dense set), it is possible to solve a formally determined inverse problem in time domain.

We point out that in this paper we do not impose any generic conditions on the geometry, and we give an explicit construction of the boundary source. The boundary source considered in this paper is based on the idea of imitating a realization of white noise, and due to the many useful properties of the white noise process, we hope that the constructed source may be useful in the study of other inverse problems requiring generic assumptions on the source.

Another formally determined hyperbolic inverse problem, namely measuring Neumann data when the initial data $(u|_{t=0}, \partial_t u|_{t=0})$ is nonzero and satisfies subharmonicity or positivity conditions, has been studied using Carleman estimates [Bellassoued and Yamamoto 2008; Imanuvilov and Yamamoto 2003; Isakov 2006; Klibanov 1992; Stefanov and Uhlmann 2011]. The present paper is closely related to these studies, but we emphasize that we assume that the initial data for u vanishes.

Moreover, there are two approaches to solving the formally determined hyperbolic inverse problem to determine a potential from a single boundary measurement. The first one uses Carleman estimates analogous to the estimates mentioned above and assumes similar conditions on the initial data [Bukhgeim and Klibanov 1981]. The second one relies on an adaptation of the Gelfand–Levitan method to multidimensional problems [Rakesh and Sacks 2011; Rakesh 2003; 2008; Romanov 2002; Sacks and Symes 1985].

2. Pseudorandom source

In this section we define a special source $h_0(t,x)$ which we call *pseudorandom*. The specific assumptions on the amplitudes are explained in Section 7. An important feature of the pseudorandom source is that it is supported only on a single point in time.

Definition 1. Let $x_j \in \partial M$, $j = 1, 2, \ldots$, be a dense sequence of distinct points in ∂M, and let $a_j \in \mathbb{R}$, $j = 1, 2, \ldots$, with $\sum_{j=1}^{\infty} |a_j| < \infty$, be a sequence of distinct numbers.

We define the pseudorandom source on $(x_j)_{j=1}^{\infty} \subset \partial M$ with coefficients $(a_j)_{j=1}^{\infty} \subset \mathbb{R}$ as the following generalized function on $\mathbb{R} \times \mathbb{R}^n$:

$$h_0(t,x) := \sum_{j=1}^{\infty} a_j \delta(t) \delta_{x_j}(x), \qquad (x,t) \in \mathbb{R}^{n+1},$$

where $\delta(t)$ and $\delta_{x_j}(x)$ are Dirac delta distributions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

It is rather straightforward to show that h_0 is well-defined. First, it is well known that $\delta(t) \in H^{-1}(\mathbb{R})$ and $\delta_{x_j}(x) \in C(\mathbb{R}^n)'$. Next, we have $H^1_{p'}(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ when $1 > n/p'$, due to [Triebel 1978, Theorem 2.8.1]. According to [ibid., Theorem 2.6.1], the dual space satisfies $(H^1_{p'}(\mathbb{R}^n))' = H^{-1}_p(\mathbb{R}^n)$ with $1/p' = 1 - 1/p$, and hence $C(\mathbb{R}^n)' \subset H^{-1}_p(\mathbb{R}^n)$ for $1 < p < n/(n-1)$. Since $\sum_{j=1}^\infty |a_j| < \infty$, we have

$$\sum_{j=1}^{\infty} a_j \delta_{x_j}(x) \in H^{-1}_p(\mathbb{R}^n).$$

This yields that for any $p \in \left(1, \dfrac{n}{n-1}\right)$ and $\epsilon > 0$, the pseudorandom source h_0 satisfies

$$h_0 \in \widetilde{H}^{-1}(-\epsilon, \epsilon) \otimes \widetilde{H}^{-1}_p(M). \tag{5}$$

The spatial structure of the pseudorandom source can be motivated by the structure of the white noise. In the 1-dimensional radar imaging models, white noise signals are considered to be optimal sources when imaging a stationary scatterer [Toomay and Hannen 2004]. This is due to the fact that different translations of the white noise signal are uncorrelated. In a similar fashion, we have the following property for the pseudorandom source h_0: for each x_{j_0} and each sequence $(x_{j_k})_{k=1}^\infty$ converging to x_{j_0} and satisfying $x_{j_k} \neq x_{j_0}$ for all $k \in \mathbb{Z}_+$, it holds that $a_{j_k} \to 0$. This property will be crucial in what follows.

A natural strategy to choose the points x_j is by random sampling. The term pseudorandom refers to the fact that algorithmic generators of random numbers use, in fact, a deterministic function to produce a sequence of numbers, but the mixingness of the process is such that the user of the algorithm can consider the numbers to be analogous to independent samples of a random variable. In this manner, the pseudorandom source can be seen as an imitation of a realization of a noise process.

Another source of inspiration for us was a rather new measurement paradigm called compressed sensing [Candès et al. 2006; Donoho 2006], where one aims for a sparse reconstruction of a linear problem using a small number of noisy measurements. We point out that by using the pseudorandom source, one can compress the measurements Lf_{x_j} with point sources f_{x_j} into a single measurement Lh_0.

3. Measurement map

In this section we prove that the measurement Lh_0 is well-defined. Let us consider the operator $W : f \mapsto u$ mapping f to the solution of (3). We call such an operator the solution operator for (3). First, we note that by [Hörmander 1985, Theorem 23.2.2], the operator $W : f \mapsto u$ extends in a unique way to a continuous linear operator

$$W : L^1\big((T_0, T); H^s(\mathbb{R}^n)\big) \to C\big([T_0, T]; H^{s+1}(\mathbb{R}^n)\big), \quad s \in \mathbb{R}. \tag{6}$$

Moreover, if $f \in C^\infty([T_0, T] \times \mathbb{R}^n)$ and $\operatorname{supp}(f) \Subset (T_0, T] \times \mathbb{R}^n$, that is, $\operatorname{supp}(f)$ is a compact subset of $(T_0, T] \times \mathbb{R}^n$, then $Wf \in C^\infty([T_0, T] \times \mathbb{R}^n)$.

We will compose the operator W with the one-sided inverse $\mathcal{I}$ of the derivative ∂_t, which is given by

$$\mathcal{I}u(t) := \int_{T_0}^t u(t')dt', \quad u \in C_c^\infty(T_0, T).$$

One sees easily that this operator has a unique continuous linear extension $\mathcal{I} : \widetilde{H}^{-1}(T_0, T) \to L^2(T_0, T)$.

Next we prove Lemma 1.1 formulated in the introduction, that is, we prove that the measurement map L has a unique continuous extension

$$\widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to \mathcal{D}'((T_0, T) \times \partial M). \tag{7}$$

Proof of Lemma 1.1. For sufficiently large $z \in \mathbb{R}_+$, the operator $z - \Delta_g$ is an isomorphism between spaces $H^{s+2}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ as well as between spaces $H_p^{s+2}(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n)$ for all integers s by [Shubin 1992].

By the definition of L, we have that $L = \mathrm{Tr} \circ W$, where Tr is the trace operator

$$\mathrm{Tr}(u) = u|_{(T_0,T)\times\partial M}, \quad u \in C^\infty((T_0, T) \times \mathbb{R}^n).$$

Let $f \in C_c^\infty((T_0, T) \times \mathbb{R}^n)$. Then the solution $u = Wf$ of the wave equation $(\partial_t^2 - \Delta_g)u = f$ can be written in the form

$$Wf = (z - \partial_t^2)^m (z - \Delta_g)^{-m} Wf + \sum_{j=0}^{m-1} (z - \partial_t^2)^j (z - \Delta_g)^{-1-j} f. \tag{8}$$

Now $f = \partial_t \mathcal{I} f$, where $\mathcal{I} f$ is C^∞-smooth and satisfies $\mathrm{supp}(\mathcal{I} f) \Subset (T_0, T] \times \mathbb{R}^n$. By (6), $W\mathcal{I} f$ is C^∞-smooth and $\partial_t W \mathcal{I} f = W \partial_t \mathcal{I} f = Wf$. Hence

$$Lf = \partial_t (z - \partial_t^2)^m \,\mathrm{Tr}(z - \Delta_g)^{-m} W\mathcal{I} f + \sum_{j=0}^{m-1} (z - \partial_t^2)^j \,\mathrm{Tr}(z - \Delta_g)^{-1-j} f. \tag{9}$$

Let us next consider terms appearing in (9). First we consider extension of the operator

$$\begin{aligned} \sum_{k=1}^N \phi_k \otimes \psi_k \mapsto & \sum_{k=1}^N (z - \partial_t^2)^j \,\mathrm{Tr}(z - \Delta_g)^{-1-j} (\phi_k \otimes \psi_k) \\ &= \sum_{k=1}^N \big((z - \partial_t^2)^j \phi_k\big) \otimes \big(\mathrm{Tr}(z - \Delta_g)^{-1-j} \psi_k\big), \quad j = 0, \dots, m-1, \end{aligned} \tag{10}$$

mapping $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$ to $C^\infty((T_0, T) \times \partial M)$. By [Triebel 1978, Theorem 4.7.1], the maps

$$H_p^{-1}(\mathbb{R}^n) \xrightarrow{(z-\Delta_g)^{-1-j}} H_p^1(\mathbb{R}^n) \xrightarrow{\mathrm{Tr}} B_{p,p}^{1-1/p}(\partial M)$$

are continuous, where $B_{p,p}^{1-1/p}(\partial M)$ is the Besov space on ∂M. Thus the operator (10) has a continuous extension in spaces (7).

Next, consider extension of the operator

$$\sum_{k=1}^N \phi_k \otimes \psi_k \mapsto \sum_{k=1}^N \partial_t (z - \partial_t^2)^m \,\mathrm{Tr}(z - \Delta_g)^{-m} W((\mathcal{I}\phi_k) \otimes \psi_k) \tag{11}$$

mapping $C_c^\infty(T_0, T) \otimes C_c^\infty(\mathbb{R}^n)$ to $C^\infty((T_0, T) \times \partial M)$. As $-1 - n/p > -1 - n$, we have by [Triebel 1978, Theorem 2.8.1] a continuous embedding $H_p^{-1}(\mathbb{R}^n) \hookrightarrow H^{-1-n/2}(\mathbb{R}^n)$. Moreover, the operator

$\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \to L^2(T_0, T)$ and the embedding $L^2(T_0, T) \otimes H^{-1-n/2}(\mathbb{R}^n) \hookrightarrow L^2\big((T_0, T); H^{-1-n/2}(\mathbb{R}^n)\big)$ are continuous. Thus, by (6),

$$W\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to C\big([T_0, T]; H^{-n/2}(\mathbb{R}^n)\big)$$

is continuous.

As $(1-\Delta_g)^{-m} : C\big([T_0, T]; H^{-n/2}(\mathbb{R}^n)\big) \to C\big([T_0, T]; H^{-n/2+2m}(\mathbb{R}^n)\big)$ is continuous and $-\frac{n}{2}+2m > \frac{1}{2}$, we see that the operator

$$\mathrm{Tr}(1-\Delta_g)^{-m} W\mathcal{J} : \widetilde{H}^{-1}(T_0, T) \otimes H_p^{-1}(\mathbb{R}^n) \to C\big([T_0, T]; L^2(\partial M)\big)$$

is continuous.

Combining the above results, we see that the operator (9) has a continuous extension to the spaces (7). As the spaces $C_c^\infty(T_0, T)$ and $C_c^\infty(\mathbb{R}^n)$ are dense in $\widetilde{H}^{-1}(T_0, T)$ and $H_p^{-1}(\mathbb{R}^n)$, respectively, we see that the continuous extension of L is unique. □

4. Inner product of a solution and a source

Lemma 4.1. *Let $f \in C_c^\infty((T_0, T) \times M)$ and $t_0 \in (T_0, T)$ and let $w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$ satisfy*

$$(\partial_t^2 - \Delta_g)w = 0, \quad \text{in } (T_0, t_0) \times \mathbb{R}^n.$$

Then

$$\int_{T_0}^{t_0} \int_{\mathbb{R}^n} f(t, x) w(t, x)\, dt\, dV(x) = \int_{\mathbb{R}^n} \big((\partial_t Wf)(t_0, x) w(t_0, x) - (Wf)(t_0, x)(\partial_t w)(t_0, x)\big)\, dV(x),$$

where $dV(x) = |g|^{1/2}\,dx$ is the Riemannian volume measure of $(\mathbb{R}^n, g)$ and $W : f \mapsto u$ is the solution operator of the wave equation (3).

Proof. By finite speed of propagation of waves [Ladyzhenskaya 1985, pp. 150–156], $\mathrm{supp}(Wf(t))$ is compact in $\mathbb{R}^n$. The claim follows by integration by parts:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big((\partial_t u)(t_0, x) w(t_0, x) - u(t_0, x)(\partial_t w)(t_0, x)\big)\, dV(x) \\
&\qquad - \int_{\mathbb{R}^n} \big((\partial_t u)(T_0, x) w(T_0, x) - u(T_0, x)(\partial_t w)(T_0, x)\big)\, dV(x) \\
&\qquad\qquad = \int_{(T_0,t_0)\times\mathbb{R}^n} \big((\partial_t^2 - \Delta_g)u(t, x)\, w(t, x) - u(t, x)\, (\partial_t^2 - \Delta_g)w(t, x)\big)\, dt\, dV(x) \\
&\qquad\qquad = \int_{(T_0,t_0)\times\mathbb{R}^n} f(t, x) w(t, x)\, dt\, dV(x).
\end{aligned}$$
□

Next, we will prove a generalization of the previous lemma for nonsmooth sources f. Denote by $B(0, R) = \{x \in \mathbb{R}^n;\ |x| < R\}$ the Euclidean ball. The finite speed of propagation for the wave equation yields that there is $R > 0$ such that all $f \in C_c^\infty((T_0, T) \times M)$ satisfy $\mathrm{supp}(Wf) \Subset (T_0, T] \times B(0, R)$. We define

$$\Omega := B(0, R) \setminus \overline{M}. \tag{12}$$

Below, we use the fact (see [Evans 1998, Theorems 7.2.3/6 and 5.6.3/6]) that the operator $W_\Omega : h \mapsto v$ mapping h to the solution of the equation

$$\begin{aligned}(\partial_t^2 - \Delta_g)v(t,x) &= 0 \quad \text{in } (T_0,T)\times\Omega,\\ v|_{(T_0,T)\times\partial\Omega} &= h,\\ v|_{t=T_0} &= 0, \quad \partial_t v|_{t=T_0} = 0,\end{aligned} \tag{13}$$

is continuous as a map $W_\Omega : C_c^\infty((T_0,T)\times\partial\Omega) \to C^\infty([T_0,T]\times\bar\Omega)$.

We let $t_0 \in (T_0,T)$ and write

$$\Sigma := \{t_0\}\times\Omega. \tag{14}$$

We denote the trace on Σ by Tr_Σ, that is, we define $(\mathrm{Tr}_\Sigma\, u)(x) := u(t_0,x)$. Let $\nu = \nu(z)$ denote the exterior unit normal vector of ∂M at z.

Moreover, let U be an open subset (or a submanifold) of $\mathbb{R}^n$, and let us denote by dV (or dS) the Riemannian volume measure of (U,g). We embed the test functions into the spaces of distribution by using the inner product of the space $L^2(U;dV)$, that is, we identify $u \in C_0^\infty(U)$ with the distribution

$$\psi \mapsto \int_U u(x)\psi(x)\,dV(x). \tag{15}$$

We will denote the distribution pairing of $u \in \mathscr{D}'(U)$ and $\psi \in C_0^\infty(U)$ by $(u,\psi)_{\mathscr{D}'(U)}$ and use analogous notations for other distribution pairings.

Lemma 4.2. *Let $t_0 \in (T_0,T)$ and define Σ by (14). Then operators $\mathrm{Tr}_\Sigma\, W_\Omega$ and $\mathrm{Tr}_\Sigma\,\partial_t W_\Omega$ have unique continuous extensions $\mathscr{E}'((T_0,t_0)\times\partial\Omega) \to \mathscr{D}'(\Omega)$.*

Proof. Let v satisfy (13). Consider a function $w \in C^\infty([T_0,t_0]\times\bar\Omega)$ such that $(\partial_t - \Delta_g)w = 0$ in $(T_0,t_0)\times\Omega$ and $w|_{(T_0,t_0)\times\partial\Omega} = 0$. Then

$$\begin{aligned}0 &= \int_{\Omega\times(T_0,t_0)} \big((\partial_t-\Delta_g)v\big)w - v\big((\partial_t-\Delta_g)w\big)\,dV(x)\,dt\\ &= \left[\int_\Omega \big((\partial_t v)w - v(\partial_t w)\big)\,dV(x)\right]_{t=T_0}^{t=t_0} + \int_{\partial\Omega\times(T_0,t_0)} \big((\partial_\nu v)w - v(\partial_\nu w)\big)\,dS(x)\,dt\\ &= \int_\Omega \big((\partial_t v)w - v(\partial_t w)\big)\,dV(x)\Big|_{t=t_0} - \int_{\partial\Omega\times(T_0,t_0)} h(\partial_\nu w)\,dS(x)\,dt,\end{aligned}$$

where ∂_ν is the normal derivative on $\partial\Omega$,

Denote by $W_1 : f_1 \mapsto w$ the solution operator of the equation

$$\begin{aligned}(\partial_t - \Delta_g)w(t,x) &= 0 \quad \text{in } (T_0,t_0)\times\Omega,\\ w|_{(T_0,t_0)\times\partial\Omega} &= 0,\\ w|_{t=t_0} &= f_1, \quad \partial_t w|_{t=t_0} = 0.\end{aligned}$$

The operator $W_1 : C_c^\infty(\Omega) \to C^\infty([T_0,t_0]\times\bar\Omega)$ is continuous, as can be seen using Theorems 7.2.3/6 and 5.6.3/6 of [Evans 1998]. Hence, the operator

$$\partial_\nu W_1 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega), \quad f \mapsto \partial_\nu W_1 f|_{\partial\Omega}$$

is continuous. Moreover,

$$(\mathrm{Tr}_\Sigma\, \partial_t W_\Omega h, f_1)_{L^2(\Omega;dV)} = (h, \partial_\nu W_1 f_1)_{L^2((T_0,t_0)\times\partial\Omega;dt\otimes dS)},$$

where ∂_ν is the normal derivative on $\partial\Omega$. We define the extension of $\mathrm{Tr}_\Sigma\, \partial_t W_\Omega$ by identifying it with the transpose $(\partial_\nu W_1)^t : \mathscr{E}'((T_0, t_0) \times \partial\Omega) \to \mathscr{D}'(\Omega)$ of the operator $\partial_\nu W_1 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega)$.

Similarly, we define the extension of $\mathrm{Tr}_\Sigma\, W_\Omega$ by the transpose $(\partial_\nu W_2)^t : \mathscr{E}'((T_0, t_0) \times \partial\Omega) \to \mathscr{D}'(\Omega)$ of $\partial_\nu W_2 : C_c^\infty(\Omega) \to C^\infty([T_0, t_0] \times \partial\Omega)$, where $W_2 : f_2 \mapsto w$ is the solution operator of the equation

$$\begin{aligned}(\partial_t - \Delta_g) w(t, x) &= 0 \quad \text{in } (T_0, t_0) \times \Omega,\\ w|_{(T_0,t_0)\times\partial\Omega} &= 0,\\ w|_{t=t_0} = 0, \quad \partial_t w|_{t=t_0} &= -f_2.\end{aligned}$$ □

Denote by $d_\Omega(x, y)$, $x, y \in \overline{\Omega}$, the distance function of Riemannian manifold $(\overline{\Omega}, g|_{\overline{\Omega}})$. Next we generalize the result of Lemma 4.1 for a larger class of functions.

Lemma 4.3. *Let $t_0 \in (0, T)$ and $\epsilon > 0$ satisfy $[-\epsilon, \epsilon] \subset (T_0, t_0)$. Define Σ by (14). Let*

$$f \in \widetilde{H}^{-1}(-\epsilon, \epsilon) \otimes \widetilde{H}_p^{-1}(M) \quad \textit{and} \quad w \in C^\infty([T_0, t_0] \times \mathbb{R}^n)$$

satisfy

$$(\partial_t^2 - \Delta_g) w = 0, \quad \textit{in } (T_0, t_0) \times \mathbb{R}^n.$$

Suppose that $w(t_0), \partial_t w(t_0) \in C_c^\infty(\Omega)$, and let $\chi \in C_c^\infty(T_0, t_0)$ satisfy $\chi = 1$ in a neighborhood of $[-\epsilon, t_0 - r]$, where

$$r := d_\Omega\big(\operatorname{supp}(w(t_0)) \cup \operatorname{supp}(\partial_t w(t_0)), \partial\Omega\big).$$

Then

$$(f, w)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} = (\mathrm{Tr}_\Sigma\, \partial_t W_\Omega \chi L f, w)_{\mathscr{D}'(\Omega)} - (\mathrm{Tr}_\Sigma\, W_\Omega \chi L f, \partial_t w)_{\mathscr{D}'(\Omega)}, \tag{16}$$

where we have defined $Lf = 0$ on $\partial B(0, R)$. Here we regard Ω as a Riemannian manifold $(\Omega, g|_\Omega)$.

Proof. We suppose first that $f \in C_c^\infty((-\epsilon, \epsilon) \times M)$. Recall that W is solution operator of wave equation (3). Then $Wf(\cdot, t) = 0$ if $t < -\epsilon$, and

$$Lf = \mathrm{Tr}_{\partial\Omega}\, Wf = \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf, \quad \text{in } (T_0, t_0 - r) \times \partial\Omega,$$

where $\mathrm{Tr}_{\partial\Omega}$ is the trace on $(T_0, T) \times \partial\Omega$. As $\Omega \cap \overline{M} = \varnothing$, we have that $(\partial_t^2 - \Delta_g) Wf = 0$ in $(T_0, T) \times \Omega$. By uniqueness of the solution of (13),

$$W_\Omega \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf = Wf, \quad \text{in } (T_0, t_0 - r) \times \Omega.$$

By finite speed of propagation,

$$\mathrm{Tr}_\Sigma\, \partial_t^j W_\Omega \chi\, \mathrm{Tr}_{\partial\Omega}\, Wf = \mathrm{Tr}_\Sigma\, \partial_t^j Wf, \quad j = 0, 1,$$

on $\{t_0\} \times \operatorname{supp}(w(t_0)) \cup \operatorname{supp}(\partial_t w(t_0))$. By Lemma 4.1, (16) holds.

Then the claim follows, as the embeddings

$$C_c^\infty(-\epsilon,\epsilon) \hookrightarrow \widetilde{H}^{-1}(-\epsilon,\epsilon), \quad C_c^\infty(M) \hookrightarrow \widetilde{H}_p^{-1}(M)$$

are dense and operators $(\mathrm{Tr}_\Sigma\, \partial_t^j W_\Omega)\chi L : \widetilde{H}^{-1}(-\epsilon,\epsilon) \otimes \widetilde{H}_p^{-1}(M) \to \mathscr{D}'((T_0,t_0)\times\partial\Omega)$, $j=0,1$, are continuous. □

5. Gaussian beams

We consider solutions of the wave equation which are known as Gaussian beams [Babich et al. 1985; Babich and Ulin 1981; Ralston 1982]. These solutions have been constructed to analyze the propagation of singularities for the wave equation in the presence of caustics. Here we use Gaussian beams as an auxiliary technical tool to analyze singularities in the measurements.

Definition 2. Let $\epsilon > 0$, $N \in \mathbb{N}$ and let γ be a unit speed geodesic on $(\mathbb{R}^n, g)$. A formal Gaussian beam of order N propagating along geodesic γ is a function U_ϵ^N of form

$$U_\epsilon^N(t,x) = \epsilon^{-n/4} \exp\bigl\{-(i\epsilon)^{-1}\theta(t,x)\bigr\} \sum_{m=0}^{N} u_m(t,x)(i\epsilon)^m, \quad t\in\mathbb{R},\ x\in\mathbb{R}^n,$$

satisfying the following properties: The phase function θ and the amplitude functions u_m, with $m = 0, 1, \dots, N$, are complex-valued smooth functions. The phase function θ satisfies the conditions

$$\theta(t,\gamma(t)) = 0, \quad \operatorname{Im}\theta(t,x) \ge C_0(t) d(x,\gamma(t))^2,$$

where $C_0(t)$ is a continuous strictly positive function. The amplitude function u_0 satisfies $u_0(t,\gamma(t)) \neq 0$. Finally, for any compact set $K \Subset \mathbb{R}\times\mathbb{R}^n$, there is a constant $C > 0$ such that the inequality

$$\bigl|(\partial_t^2 - \Delta_g) U_\epsilon^N(t,x)\bigr| \le C\epsilon^{N-n/4}$$

is satisfied uniformly for $(t,x) \in K$.

The construction of a formal Gaussian beam $U_\epsilon^N(t,x)$ is considered in detail in [Katchalov et al. 2001, Section 2.4]. Next, we recall the construction and pay attention to the properties of Gaussian beams which we need later.

Let us write the geodesic γ in the usual coordinates of $\mathbb{R}^n$ as $\gamma(t) = \bigl(\gamma^1(t),\dots,\gamma^n(t)\bigr)$. We construct the phase function $\theta(t,x)$ at each time $t\in\mathbb{R}$ in terms of a finite Taylor expansion in the x variable centered at $\gamma(t)$,

$$\theta(t,x) = \sum_{|\alpha|\le N} \frac{\theta_\alpha(t)}{\alpha!}(x-\gamma(t))^\alpha,$$

where θ_α are smooth functions and $N\in\mathbb{N}$.

Let $e_j = (\delta_{1j},\dots,\delta_{nj})$ be multi-indices with the value 1 at the j-th place. For clarity, we use the notation $p_j(t) = \theta_{e_j}(t)$ for the first-order coefficients and the notation $H_{jk}(t) = \theta_\alpha(t)$, $\alpha = e_j + e_k$, for the second-order coefficients in the expansion of θ.

The construction of a formal Gaussian beam consists of the following steps.

(1) We define $\theta_0(t)=0$ and $p_j(t)=\sum_{k=1}^n g_{jk}(\gamma(t))\dot\gamma^k(t)$, that is, the first-order coefficients $p_j(t)$ are the covariant representation of the velocity vector $\dot\gamma$.

(2) The symmetric matrix $H(t)=[H_{jk}(t)]_{j,k=1}^n$ of the second-order coefficients is obtained by solving a Riccati equation, or an equivalent system of ordinary differential equations. We write $H(t)=Z(t)Y(t)^{-1}$, where the pair of complex $n\times n$ matrices $(Z(t),Y(t))$ is the solution of the system of ordinary differential equations

$$\begin{aligned}\frac{d}{dt}Y(t)&=B(t)^*Y(t)+C(t)Z(t), &\quad Y|_{t=0}&=Y^0,\\ \frac{d}{dt}Z(t)&=-D(t)Y(t)-B(t)Z(t), &\quad Z|_{t=0}&=Z^0.\end{aligned}$$

Here we choose the initial values to be $Z^0=iI$ and $Y^0=I$, where I is the identity matrix and i is the imaginary unit. The matrices $B(t)$, $C(t)$, and $D(t)$ in $\mathbb{R}^{n\times n}$ have components given by the second derivatives of the Hamiltonian $h(x,p)=\bigl(\sum_{j,k=1}^n g_{jk}(x)p^jp^k\bigr)^{1/2}$ evaluated in the point $(x,p)=(\gamma(t),p(t))$:

$$B_l^j=\frac{\partial^2 h}{\partial x^l\partial p_j};\quad C^{jl}=\frac{\partial^2 h}{\partial p_j\partial p_l};\quad D_{jl}=\frac{\partial^2 h}{\partial x^j\partial x^l}.$$

The fact that the complex matrix $Y(t)$ is invertible for all $t\in\mathbb{R}$ is crucial for the construction, and is discussed in detail in [Katchalov et al. 2001, Section 2.4].

(3) The coefficients $\theta_\alpha(t)$ of order $|\alpha|=m\ge 3$ are solved inductively, with respect to m. The coefficients $\theta_\alpha(t)$ are constructed using the coefficients $\tilde\theta_\alpha(t)$ defined so that

$$\sum_{|\alpha|=m}\tilde\theta_\alpha(t)\tilde y^\alpha=\sum_{|\alpha|=m}\theta_\alpha(t)(x-\gamma(t))^\alpha,$$

for all $\tilde y=Y^{-1}(t)(x-\gamma(t))$, $y\in\mathbb{C}^n$. We obtain the coefficients $\tilde\theta_\alpha(t)$ by solving successive linear systems of ordinary differential equations

$$\frac{d}{dt}\tilde\theta_\alpha(t)=K_\alpha(t),\quad \tilde\theta_\alpha(0)=0,$$

where $K_\alpha(t)$ depend on $\theta_\beta(t)$ with $|\beta|\le m-1$, the matrix $H(t)$, the vector $p(t)$, and the metric g_{jk} and its derivatives at $\gamma(t)$.

(4) When the phase function $\theta(t,x)$ is constructed, the amplitude functions $u_n(t,x)$ are solved using the transport equations, or equivalently, the following ordinary differential equations. Let

$$u_m(t,x)=\sum_{|\alpha|\le N}\tilde u_{m,\alpha}(t)\tilde y^\alpha,\quad \tilde y=Y^{-1}(t)(x-\gamma(t)),$$

where the coefficients $\tilde u_{m,\alpha}(t)$ are obtained by solving the successive equations

$$\frac{d}{dt}\tilde u_{m,\alpha}(t)+r(t)\tilde u_{m,\alpha}(t)=\mathcal{F}_{m,\alpha}(t),\quad \tilde u_{m,\alpha}(0)=\delta_{m,0}\delta_{|\alpha|,0},$$

where $r(t)$ and $\mathcal{F}_{m,\alpha}(t)$ depend on $\tilde u_{m',\beta}$ with $|\beta|\le|\alpha|+2$ and $m'\le m-1$, the function $\theta(t,x)$, the metric g_{jk}, and their derivatives at (t,x), $x=\gamma(t)$.

By the above construction, we have the following remark.

Remark 1. The phase function $\theta(t,x)$ and the amplitude functions $u_m(t,x)$ at time $t=0$ have the form

$$\theta(0,x)=\sum_{j,k=1}^{n} g_{jk}(y)\eta^k(x^j-y^j)+i|x-y|^2,$$

where $(y,\eta)=(\gamma(0),\dot\gamma(0))$ is the initial data of the geodesic γ, $u_0(0,x)=1$, and $u_m(0,x)=0$ for $m>0$. Hence, $U_\epsilon^N(0,x)$ is dependent on the metric g_{jk} only via $g_{jk}(y)$. Moreover, $\partial_t U_\epsilon^N(0,x)$, although of more complex form, is dependent on the metric g_{jk} only via $\partial^\alpha g_{jk}(y)$ for a certain finite collection of multi-indices $\alpha\in\mathbb{N}^n$.

If the coefficients of an ordinary differential equation depend smoothly on some parameter, so does the solution [Amann 1990], and thus we see using an induction that the phase function θ and the amplitude functions u_m depend smoothly on the initial data $(y,\eta)=(\gamma(0),\dot\gamma(0))$ of the geodesic γ. In particular, the amplitude function $u_0(t,x;y,\eta)$ satisfies

$$u_0\in C^\infty(\mathbb{R}\times\mathbb{R}^n\times S\mathbb{R}^n). \tag{17}$$

Thus far we have considered a formal Gaussian beam. By using continuous dependency of the solution of the wave equation on the source term, one obtains the following results [Katchalov et al. 2001]:

Let γ be a unit speed geodesic, $N\in\mathbb{N}$, $\epsilon>0$, and let U_ϵ^N be a formal Gaussian beam of order N propagating along geodesic γ. Let $\chi\in C_0^\infty(\mathbb{R}^n)$ be a function which is identically one in a neighborhood of $\gamma(0)$ and let $t_0>0$ and let R be the radius in (12). Then for $j\in\mathbb{N}$ and $\alpha\in\mathbb{N}^n$ satisfying $j+|\alpha|<N-n/4$, there is $C>0$ such that the solution w_ϵ of the wave equation,

$$\begin{aligned}(\partial_t^2-\Delta_g)w_\epsilon(t,x)&=0,\qquad (t,x)\in(T_0,t_0)\times\mathbb{R}^n,\\ w_\epsilon(t_0,x)&=\chi(x)U_\epsilon^N(0,x),\\ \partial_t w_\epsilon(t_0,x)&=-\chi(x)\partial_t U_\epsilon^N(0,x),\end{aligned}\tag{18}$$

satisfies

$$\sup_{x\in B(0,R),t\in(T_0,t_0)}\left|\partial_t^j\partial_x^\alpha\big(w_\epsilon(t_0-t,x)-U_\epsilon^N(t,x)\big)\right|\le C\epsilon^{N-(j+|\alpha|)-n/4}. \tag{19}$$

We call w_ϵ a Gaussian beam of order N propagating along geodesic γ backwards on time interval (T_0,t_0).

6. Determination of the travel times

Lemma 6.1. *Let w_ϵ be a Gaussian beam of order $N\ge 1+n/4$ propagating along geodesic γ backwards on time interval (T_0,t_0), that is, let w_ϵ be the solution of (18). Let h_0 be the pseudorandom source*

$$h_0(t,x)=\sum_{j=1}^{\infty} a_j\delta(t)\delta_{x_j}(x). \tag{20}$$

If $\gamma(t_0)\ne x_j$ for all $j=1,2,\dots,$ then

$$\lim_{\epsilon\to 0}\epsilon^{n/4}(h_0,w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))}=0.$$

Moreover, if $\gamma(t_0) = x_j$, *then*

$$\lim_{\epsilon\to 0} \epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} = a_j u_0(t_0, x_j)|g|^{1/2}(x_j),$$

where $u_0(t, x)$ *is the first amplitude function of a formal Gaussian beam propagating along geodesic* γ.

We remind the reader that the test functions are embedded in $\mathscr{E}'(\mathbb{R}^n \times (T_0, T))$ using (15).

Proof. By (19), we have that

$$\begin{aligned}\epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} &= \epsilon^{n/4}\sum_{j=1}^{\infty} a_j U_\epsilon^N(t_0, x_j)|g|^{1/2}(x_j) + O(\epsilon) \\ &= \sum_{j=1}^{\infty} a_j u_0(t_0, x_j)\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\}|g|^{1/2}(x_j) + O(\epsilon).\end{aligned}$$

As $\operatorname{Im}\theta(t_0, x_j) \geq C_0(t_0)d(x_j, \gamma(t_0))$, we have that

$$\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\}\right| = O(\epsilon), \quad \text{if } \gamma(t_0) \neq x_j.$$

Suppose that $\gamma(t_0) = x_j$. Then $\exp\{-(i\epsilon)^{-1}\theta(t_0, x_j)\} = 1$ and there is a constant $C > 0$ depending on γ and t_0 such that

$$\begin{aligned}&\left|\epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))} - a_j u_0(t_0, x_j)|g|^{1/2}(x_j)\right| \\ &\quad\leq C\sum_{k=1}^{j-1}|a_k|\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_k)\}\right| + C\sum_{k=j+1}^{l}|a_k|\left|\exp\{-(i\epsilon)^{-1}\theta(t_0, x_k)\}\right| + C\sum_{l+1}^{\infty}|a_l| + O(\epsilon).\end{aligned}$$

We may first choose large $l \in \mathbb{N}$ and then small $\epsilon > 0$ so that the above three sums are arbitrarily small. The case $\gamma(t_0) \neq x_j$ for all $j = 1, 2, \ldots,$ is similar. □

Next we define an auxiliary function $S(y_0, \eta_0, t_0)$ which is nonzero if and only if there is $j \in \mathbb{Z}_+$ such that $\gamma_{y_0,\eta_0}(t_0) = x_j$.

Definition 3. Let $(y_0, \eta_0) \in T\mathbb{R}^n$ be such that $y_0 \in \Omega^{\text{int}}$ and $\|\eta_0\|_g = 1$. We denote by $\gamma(t; y_0, \eta_0) = \gamma_{y_0,\eta_0}(t)$ the geodesic on $(\mathbb{R}^n, g)$ with $\gamma(0) = y_0$, $\dot\gamma(0) = \eta_0$. Moreover, let w_ϵ be a Gaussian beam of order $N \geq 1 + n/4$ propagating along $\gamma(t; y, \eta)$ backwards on time interval (T_0, t_0). We define

$$S(y_0, \eta_0, t_0) := \lim_{\epsilon\to 0} \epsilon^{n/4}(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n\times(T_0,t_0))}.$$

Lemma 6.2. *Let* $(y_0, \eta) \in S\Omega$ *and* $t_0 \in (0, T)$. *Then* Lh_0, *for pseudorandom source* h_0, *and* $(\Omega, g|_\Omega)$, *given as a Riemannian manifold, determine* $S(y_0, \eta_0, t_0)$.

Proof. Let w_ϵ be a Gaussian beam of order $N \geq 1 + n/4$ propagating along the geodesic $\gamma(\,\cdot\,; y_0, \eta_0)$ backwards on time interval (T_0, t_0). We may choose the cut-off function χ in (18) so that $w_\epsilon(t_0)$ and $\partial_t w_\epsilon(t_0)$ lie in $C_c^\infty(\Omega)$. As $g|_\Omega$ is known, we have by Remark 1 that the initial data $w_\epsilon(t_0)$, $\partial_t w_\epsilon(t_0)$ are known. Moreover, operators $\operatorname{Tr}_\Sigma \partial_t^j W_\Omega$, $j = 0, 1$, $\Sigma := \{t_0\} \times \Omega$, are known. After choosing a suitable cut-off function χ in Lemma 4.3, we have that the measurement Lh_0 determines the distributional pairing $(h_0, w_\epsilon)_{\mathscr{E}'(\mathbb{R}^n \times (T_0,t))}$. Hence $S(y_0, \eta_0, t_0)$ is determined. $\square$

The implicit function theorem yields the following remark. Note that $t_0 \in \mathbb{R}$ in the remark is not necessarily the first intersection time.

Remark 2. Let $(y_0, \eta_0) \in S\mathbb{R}^n$ and $t_0 \in \mathbb{R}$ satisfy

$$\big(\gamma(t_0; y_0, \eta_0), \dot{\gamma}(t_0; y_0, \eta_0)\big) \in \partial_\pm SM.$$

Then there are neighborhoods $I \subset \mathbb{R}$ and $U \subset S\mathbb{R}^n$ of t_0 and (y_0, η_0) and a smooth map $\ell : U \to I$ such that for $t \in I$ and $(y, \eta) \in U$,

$$\gamma(t; y, \eta) \in \begin{cases} M, & \text{for } \pm t < \pm\ell(y, \eta), \\ \partial M, & \text{for } t = \ell(y, \eta), \\ \Omega, & \text{for } \pm t > \pm\ell(y, \eta). \end{cases}$$

We remind the reader that $\tau(x, \xi)$, $(x, \xi) \in T\mathbb{R}^n$, is defined as the first intersection time with ∂M:

$$\tau(y_0, \eta_0) := \inf\big\{t \in (0, \infty];\ \gamma(t; y_0, \eta_0) \in \partial M\big\}.$$

In the following, we use the Sasaki metric on the tangent bundle TM.

Lemma 6.3. *The first intersection times $\tau : S\Omega \to (0, \infty]$ and $\tau : \partial_- SM \to (0, \infty]$ are lower semicontinuous.*

Proof. Let us consider τ on $S\Omega$. Let a sequence $((y_j, \eta_j))_{j=1}^\infty \subset S\Omega$ converge to $(y_0, \eta_0) \in S\Omega$ as $j \to \infty$. We write $\gamma_j(t) := \gamma(t; y_j, \eta_j)$ and $\tau_j := \tau(y_j, \eta_j)$.

We will show next that $\liminf_{j\to\infty} \tau_j \notin (0, \tau_0)$. Let $t \in (0, \tau_0)$. Then $\gamma_0(t) \notin \partial M$ and

$$d(\gamma_0(t), \partial M) > 0.$$

Let $j \in \mathbb{Z}_+$. Suppose for a moment that $\tau_j < \infty$. Noting that γ_j is unit speed and $\gamma_j(\tau_j) \in \partial M$, we have

$$|t - \tau_j| \geq d\big(\gamma_j(t), \gamma_j(\tau_j)\big) \geq d(\gamma_j(t), \partial M).$$

If $\tau_j = \infty$, then $|t - \tau_j| = \infty > d(\gamma_j(t), \partial M)$.

The convergence $\gamma_j(t) \to \gamma_0(t)$, as $j \to \infty$, implies that for large j,

$$|t - \tau_j| \geq \frac{d(\gamma_0(t), \partial M)}{2} > 0.$$

Hence, $\liminf_{j\to\infty} \tau_j \neq t$ for all $t \in (0, \tau_0)$.

Clearly $\liminf_{j\to\infty} \tau_j \geq 0$, and there is $J \in \mathbb{Z}_+$ such that

$$\tau_j \geq d(y_j, \partial M) \geq \frac{d(y_0, \partial M)}{2} > 0, \quad j \geq J.$$

Hence, $\liminf_{j\to\infty} \tau_j \neq 0$ and $\liminf_{j\to\infty} \tau_j \geq \tau_0$.

Let us consider τ on $\partial_- SM$. Let a sequence $((y_j, \eta_j))_{j=1}^\infty \subset \partial_- SM$ converge to $(y_0, \eta_0) \in \partial_- SM$ as $j \to \infty$. We write $\gamma_j(t) := \gamma(t; y_j, \eta_j)$ and $\tau_j := \tau(y_j, \eta_j)$.

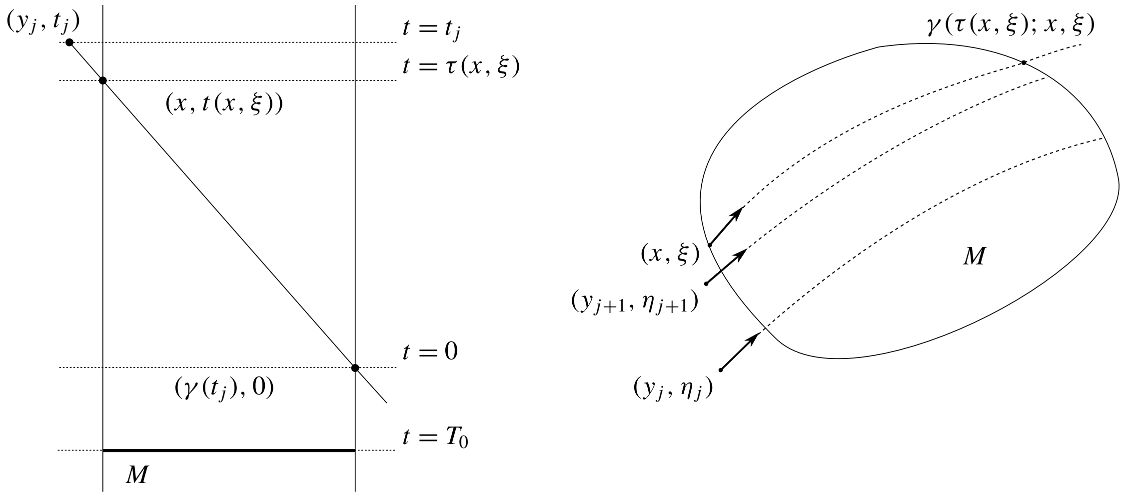

Figure 1. On the left, the trajectory of a Gaussian beam propagating along geodesic $\gamma(t) := \gamma(t; y_j, \eta_j)$ backwards on time interval (T_0, t_j). If $S(y_j, \eta_j, t_j) \neq 0$, then there is a point source at $\gamma(t_j)$. On the right, a sequence $(y_j, \eta_j) \in S\Omega$ converging to $(x, \xi) \in \partial_- SM$ and trajectories of the corresponding geodesics.

Repeating the above argument, we see that $\liminf_{j\to\infty} \tau_j \notin (0, \tau_0)$. Thus it is enough to show that $\liminf_{j\to\infty} \tau_j \neq 0$.

Remark 2 gives neighborhoods $I \subset \mathbb{R}$ and $U \subset S\mathbb{R}^n$ of zero and (y_0, η_0) and a map $\ell : U \to I$ of boundary intersection times. We write $V := U \cap \partial_- SM$. As $\gamma(0; x, \xi) \in \partial M$ for $(x, \xi) \in V$, we have $\ell = 0$ in V. In particular, $r := d\big(\ell(V), \mathbb{R} \setminus I\big) > 0$. For large j, $(\gamma_j(0), \dot\gamma_j(0)) \in V$, and thus

$$\gamma_j(t) \in M, \quad t \in (0, r).$$

Hence, $\tau_j \geq r > 0$ for large j, and $\liminf_{j\to\infty} \tau_j \geq \tau_0$. □

We easily see the following continuity result for τ.

Lemma 6.4. *Let $((y_j, \eta_j))_{j=1}^{\infty} \subset S\Omega$ converge to $(x, \xi) \in \partial_- SM$ in the Sasaki metric. Then*

$$\lim_{j\to\infty} \tau(y_j, \eta_j) = 0.$$

Theorem 6.5. *Let $(x, \xi) \in \partial_- SM$, and denote by $J(x, \xi)$ the set of sequences*

$$((t_j; y_j, \eta_j))_{j=1}^{\infty} \subset (0, \infty) \times S\Omega$$

for which

$$\lim_{j\to\infty} (y_j, \eta_j) = (x, \xi), \quad \lim_{j\to\infty} t_j \in (0, \infty), \quad S(y_j, \eta_j, t_j) \neq 0.$$

The function $S : S\Omega \times (0,\infty) \to \mathbb{C}$ determines $\tau : \partial_- SM \to (0,\infty]$ by the formula

$$\tau(x,\xi) = \inf\Big\{ \lim_{j\to\infty} t_j;\ ((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi) \text{ for some } ((y_j,\eta_j))_{j=1}^{\infty} \subset S\Omega \Big\}.$$

Moreover, if $\tau(x,\xi) < \infty$, then there is a sequence $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ satisfying

$$\tau(x,\xi) = \lim_{j\to\infty} t_j.$$

Proof. Let $(x,\xi) \in \partial_- SM$ and $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$. Let us show that $\tau(x,\xi) \le \lim_{j\to\infty} t_j$. By Lemma 6.4, $\tau_j := \tau(y_j,\eta_j) \to 0$ as $j \to \infty$. We define

$$\tilde{y}_j := \gamma(\tau_j; y_j, \eta_j), \quad \xi_j := \dot{\gamma}(\tau_j; y_j, \eta_j).$$

As $S(y_j, \eta_j, t_j) \neq 0$, we have

$$\gamma(t_j - \tau_j; \tilde{y}_j, \xi_j) = \gamma(t_j; y_j, \eta_j) \in \partial M.$$

As $\lim_{j\to\infty} t_j > 0$ and $\lim_{j\to\infty} \tau_j = 0$, we have $t_j - \tau_j > 0$ for large j. Thus $\tau(\tilde{y}_j, \xi_j) \le t_j - \tau_j$ for large j. Moreover,

$$\lim_{j\to\infty} (\tilde{y}_j, \xi_j) = \big(\gamma(0; x, \xi), \dot{\gamma}(0; x, \xi)\big) = (x, \xi).$$

In particular, $(\tilde{y}_j, \xi_j) \in \partial_- SM$ for large j. Hence, Lemma 6.3 gives

$$\lim_{j\to\infty} t_j = \lim_{j\to\infty} (t_j - \tau_j) \ge \liminf_{j\to\infty} \tau(\tilde{y}_j, \xi_j) \ge \tau(x,\xi).$$

In particular, we have proved the claim in the case $\tau(x,\xi) = \infty$.

Let us assume that $\tau(x,\xi) < \infty$. It is enough to show that there is a sequence $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ satisfying $\tau(x,\xi) = \lim_{j\to\infty} t_j$. We write

$$t_0 := \tau(x,\xi), \quad z := \gamma(t_0; x, \xi), \quad \zeta := -\dot{\gamma}(t_0; x, \xi).$$

We have

$$(x,\xi) = \big(\gamma(t_0; z, \zeta), -\dot{\gamma}(t_0; z, \zeta)\big).$$

As $(x,\xi) \in \partial_- SM$, Remark 2 gives neighborhoods I and U of t_0 and (z,ζ) and a map $\ell : U \to I$ of boundary intersection times. After choosing local coordinates around z, we may define

$$(y_j, \eta_j) := \big(\gamma(t_j; x_{k_j}, \zeta), -\dot{\gamma}(t_j; x_{k_j}, \zeta)\big),$$

where $(x_{k_j})_{j=1}^{\infty} \subset U$ is a subsequence of the dense sequence of source points in (20) satisfying $\lim_{j\to\infty} x_{k_j} = z$ and $(t_j)_{j=1}^{\infty} \subset I$ satisfies

$$t_j > \ell(x_{k_j}, \zeta), \quad \lim_{j\to\infty} t_j = \ell(z, \zeta) = t_0.$$

Clearly, $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x,\xi)$ and

$$\lim_{j\to\infty} t_j = t_0 = \tau(x,\xi). \qquad \square$$

7. Determination of the scattering relation

In the next theorem, we consider the pseudorandom source $h_0(t,x)$ with coefficients

$$a_j = \lambda^{-\lambda^j},$$

with some $\lambda > 1$, and make computations "modulo an error in A", where

$$A = \{-\lambda^j : j \in \mathbb{N}\}.$$

To this end, let $m_A(s)$ be the real number r such that $s = r + a$, where $a \in A$ and r has the smallest possible absolute value. In the case when both r and $-r$ satisfy this condition, we choose the positive value.

Lemma 7.1. *Let $(y_0, \eta_0) \in S\Omega$, $t_0 \in (0, T)$, and suppose that $S(y_0, \eta_0, t_0) \neq 0$. Then there is a sequence $((y_j, \eta_j))_{j=1}^{\infty} \subset S\Omega$ and $(t_j)_{j=1}^{\infty} \subset (0, T)$ such that*

$$(y_j, \eta_j) \to (y_0, \eta_0), \quad t_j \to t_0, \quad S(y_j, \eta_j, t_j) \to 0, \quad \textit{as } j \to \infty, \quad S(y_j, \eta_j, t_j) \neq 0. \tag{21}$$

Suppose, moreover, that the coefficients of the pseudorandom source h_0 are $a_j = \lambda^{-\lambda^j}$. Then for any sequences $((y_j, \eta_j))_{j=1}^{\infty} \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^{\infty} \subset (0, T)$ satisfying (21), we have that

$$\lim_{j\to\infty} m_A\bigl(\log_\lambda |S(y_j, \eta_j, t_j)|\bigr) = \log_\lambda \bigl| u_0\bigl(t_0, \gamma(t_0); y_0, \eta_0\bigr) |g|^{1/2}(\gamma(t_0)) \bigr|,$$

where $\gamma(t) = \gamma(t; y_0, \eta_0)$ and u_0 is defined as in (17).

Proof. We will use the notation

$$\gamma_j(t) := \gamma(t; y_j, \eta_j), \quad z_j := \gamma_j(t_j), \quad S_j := S(y_j, \eta_j, t_j), \quad \beta_j := \bigl| u_0(t_j, z_j; y_j, \eta_j) |g|^{1/2}(z_j) \bigr|.$$

As $S_0 \neq 0$, we have that $z_0 = x_j$ for some $j = 1, 2, \dots$. By continuity of the geodesic flow and density of $(x_j)_{j=1}^{\infty} \subset \partial M$, there exist a subsequence $(x_{k_j})_{j=1}^{\infty} \subset (x_j)_{j=1}^{\infty}$ and sequences $((y_j, \eta_j))_{j=1}^{\infty} \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^{\infty} \subset (0, T)$ such that

$$x_{k_j} \to z_0, \quad (y_j, \eta_j) \to (y_0, \eta_0), \quad t_j \to t_0, \quad \text{as } j \to \infty,$$

and $z_j = x_{k_j} \neq z_0$. Then $|S_j| = |a_{k_j}|\beta_j \neq 0$. As $x_{k_j} \neq z_0$ and $x_{k_j} \to z_0$, we have that $k_j \to \infty$ and thus $a_{k_j} \to 0$. By (17) and continuity of the geodesic flow, it holds that $\beta_j \to \beta_0 > 0$. Hence $S_j \to 0$.

Next we use the assumption that $a_j = \lambda^{-\lambda^j}$. Let $((y_j, \eta_j))_{j=1}^{\infty} \subset T\mathbb{R}^n$ and $(t_j)_{j=1}^{\infty} \subset (0, T)$ satisfy (21). As $S_j \neq 0$, we have that $|S_j| = a_{k_j}\beta_j$ for some subsequence $(a_{k_j})_{j=1}^{\infty} \subset (a_j)_{j=1}^{\infty}$. As $S_j \to 0$, we have that $a_{k_j} \to 0$. Moreover, sequence $(\log_2 \beta_j)_{j=1}^{\infty}$ is bounded. This boundedness, together with $\log_\lambda a_{k_j} \in A$ and $\log_\lambda a_{k_j} \to -\infty$, yields

$$m_A(\log_\lambda a_{k_j} + \log_\lambda \beta_j) = \log_\lambda \beta_j$$

for large $j \in \mathbb{N}$. Hence,

$$\lim_{j\to\infty} m_A(\log_\lambda |S_j|) = \lim_{j\to\infty} \log_\lambda \beta_j = \log_\lambda \beta_0. \qquad \square$$

Theorem 7.2. *If the coefficients of the pseudorandom source h_0 are $a_j = \lambda^{-\lambda^j}$, then the functions $S : S\Omega \times (0, \infty) \to \mathbb{C}$ and $\tau : \partial_- SM \to (0, \infty]$ determine $D(\Sigma)$ and*

$$\gamma\big(\tau(x, \xi); x, \xi\big), \quad (x, \xi) \in D(\Sigma).$$

Proof. Clearly τ on $\partial_- SM$ determines $D(\Sigma)$. Let $(x, \xi) \in D(\Sigma)$. By Theorem 6.5, we may choose $((t_j; y_j, \eta_j))_{j=1}^{\infty} \in J(x, \xi)$ such that $\lim_{j\to\infty} t_j = \tau(x, \xi)$. As $S(y_j, \eta_j, t_j) \neq 0$, we have $\gamma(t_j; y_j, \eta_j) = x_{k_j}$ for some subsequence $(x_{k_j})_{j=1}^{\infty}$ of the sequence of source points. By Lemma 7.1, the function S determines

$$\frac{|S(y_j, \eta_j, t_j)|}{\big|u_0(t_j, x_{k_j}; y_j, \eta_j)|g|^{1/2}(x_{k_j})\big|} = a_{k_j}.$$

As a_j, $j \in \mathbb{Z}_+$, are disjoint, this determines the index k_j and thus also the point x_{k_j}. Moreover,

$$\lim_{j\to\infty} x_{k_j} = \lim_{j\to\infty} \gamma(t_j; y_j, \eta_j) = \gamma\big(\tau(x, \xi); x, \xi\big). \qquad \square$$

The following result follows from Remark 2.

Lemma 7.3. *Let us denote by X either $S\Omega$ or $\partial_- SM$. Let $(y_0, \eta_0) \in X$ satisfy*

$$\tau(y_0, \eta_0) < \infty, \quad \dot\gamma\big(\tau(y_0, \eta_0); y_0, \eta_0\big) \notin T_z\partial M,$$

where $z = \gamma\big(\tau(y_0, \eta_0); y_0, \eta_0\big)$. Then there is a neighborhood $V \subset X$ of (y_0, η_0) such that $\tau = \ell$ in V, where $\ell : U \to I$ is the map of boundary intersection times defined in Remark 2 for neighborhoods $U \subset X$ and $I \subset \mathbb{R}$ of (y_0, η_0) and $\tau(y_0, \eta_0)$. In particular, τ is smooth in V.

Lemma 7.4. *The set of (x, ξ) such that $\gamma(\,\cdot\,; x, \xi)$ is transverse to ∂M is open and dense in*

$$\partial SM := \{(x, \xi) \in SM; x \in \partial M\}.$$

Proof. As $\partial_- SM \cup \partial_+ SM$ is open and dense in ∂SM, it is enough to show that the set of (x, ξ) such that $\gamma(\,\cdot\,; x, \xi)$ is transverse to ∂M is open and dense in $\partial_\pm SM$. By the parametric transversality theorem [Hirsch 1976, Theorem 3.2.7], the claim follows from the fact that the evaluation map

$$F^{\mathrm{ev}} : \partial_\pm SM \times \mathbb{R} \to \mathbb{R}^n,$$
$$F^{\mathrm{ev}} : (x, \xi, t) \mapsto \gamma(t; x, \xi)$$

is transverse to ∂M. $\square$

Lemma 7.5. *Let $(x_0, \xi_0) \in D(\Sigma)$. Then there is a sequence $((x_j, \xi_j))_{j=1}^{\infty} \subset D(\Sigma)$ such that $\gamma(\,\cdot\,; x_j, \xi_j)$ is transverse to ∂M and*

$$\lim_{j\to\infty} (x_j, \xi_j) = (x_0, \xi_0), \quad \lim_{j\to\infty} \tau(x_j, \xi_j) = \tau(x_0, \xi_0).$$

Proof. We write $\tau_0 := \tau(x_0, \xi_0)$ and

$$(z_0, \zeta_0) := \big(\gamma(\tau_0; x_0, \xi_0), -\dot\gamma(\tau_0; x_0, \xi_0)\big).$$

Remark 2 gives a map of boundary intersection times $\ell : U \to I$ for neighborhoods $U \subset S\mathbb{R}^n$ and $I \subset \mathbb{R}$ of (z_0, ζ_0) and τ_0. By Lemma 7.4, there is a sequence $((z_j, \zeta_j))_{j=1}^{\infty} \subset SM \cap U$ converging to (z_0, ζ_0) such that $\gamma(\cdot\,; z_j, \zeta_j)$ is transverse to ∂M.

We define $t_j := \ell(z_j, \zeta_j)$ and

$$(x_j, \xi_j) := \big(\gamma(t_j; z_j, \zeta_j), -\dot{\gamma}(t_j; z_j, \zeta_j)\big).$$

Then $(x_j, \xi_j) \to (x_0, \xi_0)$ as $j \to \infty$. In particular, there is $J \geq 1$ such that $(x_j, \xi_j) \in \partial_- SM$ for $j \geq J$. By Lemma 6.3,

$$\begin{aligned}\tau(x_0, \xi_0) &\leq \liminf_{j\to\infty} \tau(x_j, \xi_j) \leq \limsup_{j\to\infty} \tau(x_j, \xi_j)\\ &\leq \lim_{j\to\infty} \ell(z_j, \zeta_j) = \ell(z_0, \zeta_0) = \tau(x_0, \xi_0).\end{aligned}$$

□

Lemma 7.6. *Let $(x_0, \xi_0) \in D(\Sigma)$ be such that $\gamma(\cdot\,; x_0, \xi_0)$ is transverse to ∂M. Then there is $(y_0, \eta_0) \in S\Omega$ lying on the geodesic $\gamma(\cdot\,; x_0, \xi_0)$ and a neighborhood $V \subset S_{y_0}\Omega$ of η_0 such that the following conditions hold.*

(C1) *The map $\eta \mapsto \tau(y_0, \eta)$ is smooth $V \to (0, \infty)$.*

(C2) *The map*

$$(x(\eta), \xi(\eta)) := \big(\gamma(\tau(y_0, \eta); y_0, \eta), \dot{\gamma}(\tau(y_0, \eta); y_0, \eta)\big) \tag{22}$$

is smooth $V \to D(\Sigma)$ and $(x(\eta_0), \xi(\eta_0)) = (x_0, \xi_0)$.

(C3) *The map*

$$\tilde{\ell}(\eta) := \tau(x(\eta), \xi(\eta)) + \tau(y_0, \eta) \tag{23}$$

is smooth $V \to (0, \infty)$.

(C4) *There is a neighborhood $W \subset \partial M$ of $\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$ such that*

$$\eta \mapsto \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big) \tag{24}$$

is a diffeomorphism $V \to W$.

Proof. We write $\gamma(t) := \gamma(t; x_0, \xi_0)$ and $z_0 := \gamma(\tau(x_0, \xi_0))$. By Remark 2, $\gamma(-t) \in \Omega$ for small $t > 0$. Moreover, the points that are conjugate to z_0 along γ are discrete on γ [Jost 2008].

Thus there is $\tau_0 > 0$ such that

$$(y_0, \eta_0) := \big(\gamma(-\tau_0), \dot{\gamma}(-\tau_0)\big)$$

is in $S\Omega$, y_0 is not conjugate to z_0 along γ, $\tau(y_0, \eta_0) = \tau_0$, and

$$\big(\gamma(\tau_0; y_0, \eta_0), \dot{\gamma}(\tau_0; y_0, \eta_0)\big) = (x_0, \xi_0).$$

By Lemma 7.3, there is a neighborhood $V_0 \subset S_{y_0}\Omega$ of η_0 such that $\eta \mapsto \tau(y_0, \eta)$ is smooth in V_0. Hence, the function $\eta \mapsto (x(\eta), \xi(\eta))$ maps η_0 to (x_0, ξ_0) and is smooth in V_0. Moreover, this smoothness, transversality of $\gamma(\cdot\,, x_0, \xi_0)$, and Lemma 7.3 imply that there is a neighborhood $V_1 \subset V_0$ of η_0 such that

$(x(\eta), \xi(\eta)) \in \partial_- SM$ and $\eta \mapsto \tau(x(\eta), \xi(\eta))$ is smooth $V_1 \to (0, \infty)$. In particular, $(x(\eta), \xi(\eta)) \in D(\Sigma)$ for all $\eta \in V_1$. We have shown that (y_0, η_0) and V_1 satisfy (C1)–(C3).

We have

$$\big(\gamma(s; y_0, \eta), \dot\gamma(s; y_0, \eta)\big)\big|_{s=t+\tau(y_0,\eta)} = \big(\gamma(t; x(\eta), \xi(\eta)), \dot\gamma(t; x(\eta), \xi(\eta))\big). \tag{25}$$

In particular, $\gamma(\tilde\ell(\eta_0); y_0, \eta_0) = z_0$ and

$$\gamma(\tilde\ell(\eta); y_0, \eta) = \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big) \in \partial M.$$

Moreover, as y_0 is not conjugate to z_0 along γ, there are neighborhoods $V_2 \subset V_1$, $I_0 \subset (0, \infty)$ and $U_0 \subset \mathbb{R}^n$ of η_0, $\tilde\ell(\eta_0)$ and z_0 such that $(t, \eta) \mapsto \gamma(t; y_0, \eta)$ is a diffeomorphism $V_2 \times I_0 \to U_0$.

There is a neighborhood $V \subset V_2$ of η_0 such that $\tilde\ell(V) \subset I_0$. The graph of $\eta \mapsto \tilde\ell(\eta)$ is an $(n-1)$-dimensional submanifold on $V \times I_0$. Hence, the diffeomorphism $(t, \eta) \mapsto \gamma(t; y_0, \eta)$ maps it onto an $(n-1)$-dimensional submanifold W of U_0. Moreover, $z_0 \in W$ and $W \subset \partial M$. Thus W is a neighborhood of z_0 in ∂M. $\square$

Lemma 7.7. *Let $(x_0, \xi_0) \in D(\Sigma)$ and $(y_0, \eta_0) \in S\Omega$ satisfy conditions (C1)–(C4) of Lemma 7.6 for neighborhoods $V \subset S_{y_0}\Omega$ and $W \subset \partial M$ of η_0 and $z_0 := \gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$. We denote by $F : W \to V$ the inverse map of (24). Then*

$$\operatorname{grad}_{\partial M}(\tilde\ell \circ F)|_{z=z_0} = \dot\gamma_{z_0}^{\top}, \tag{26}$$

where $\tilde\ell : V \to (0, \infty)$ is the function (23) and $\dot\gamma_{z_0}^{\top}$ is the orthogonal projection of $\dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$ into $T_{z_0}\partial M$.

Proof. Let $\sigma : (-\epsilon, \epsilon) \to W$ be a smooth curve such that $\sigma(0) = z_0$. We define

$$\Gamma : (-\epsilon, \epsilon) \times \mathbb{R} \to \mathbb{R}^n, \quad \Gamma(s, t) := \gamma\big(t; y_0, F(\sigma(s))\big).$$

We write $\lambda := \tilde\ell \circ F \circ \sigma$ and $\tilde\ell_0 := \tilde\ell(\eta_0)$. By (25),

$$\Gamma(s, \lambda(s)) = \gamma\big(\tau(x(\eta), \xi(\eta)); x(\eta), \xi(\eta)\big)\big|_{\eta = F(\sigma(s))} = \sigma(s),$$
$$(\partial_t\Gamma)(0, \tilde\ell_0) = \dot\gamma(\tilde\ell_0; y_0, \eta_0) = \dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big).$$

Hence

$$\dot\sigma(0) = \partial_s\Gamma(s, \lambda(s))|_{s=0} = (\partial_s\Gamma)(0, \tilde\ell_0) + (\partial_t\Gamma)(0, \tilde\ell_0)\lambda'(0).$$

The curve $t \mapsto \Gamma(s, t)$ is a unit speed geodesic for all $s \in (-\epsilon, \epsilon)$. Hence

$$\begin{aligned}\big(\dot\sigma(0), \dot\gamma(\tau(x_0, \xi_0); x_0, \xi_0)\big)_g &= \big((\partial_s\Gamma, \partial_t\Gamma)_g + \lambda'(0)(\partial_t\Gamma, \partial_t\Gamma)_g\big)\big|_{s=0, t=\tilde\ell_0} \\ &= (\partial_s\Gamma, \partial_t\Gamma)_g|_{s=0, t=\tilde\ell_0} + \lambda'(0).\end{aligned} \tag{27}$$

We define

$$\mathscr{L}(s, l) := \int_0^l |\partial_t\Gamma(s, t)|_g dt, \quad (s, l) \in (-\epsilon, \epsilon) \times (0, \infty).$$

Then $\mathscr{L}(s, l)$, $s \in (-\epsilon, \epsilon)$ is the length of a unit speed geodesic on the interval $[0, l]$. Thus $\mathscr{L}(s, l) = l$ for all $s \in (-\epsilon, \epsilon)$. We may derive an expression for $\partial_s \mathscr{L}(s, l)|_{s=0}$ as in [Lee 1997, Proposition 6.5]:

$$\partial_s \mathscr{L}(s, l)|_{s=0} = \int_0^l (D_t \partial_s \Gamma, \partial_t \Gamma)_g \, dt|_{s=0}.$$

As $t \mapsto \Gamma(s, t)$ is a geodesic, $D_t \partial_t \Gamma(s, t) = 0$, and thus

$$\partial_t (\partial_s \Gamma, \partial_t \Gamma)_g = (D_t \partial_s \Gamma, \partial_t \Gamma)_g.$$

Moreover, $\Gamma(s, 0) = y_0$ for all $s \in (-\epsilon, \epsilon)$, and thus $\partial_s \Gamma(s, 0) = 0$. Hence

$$0 = \partial_s \mathscr{L}(s, l)|_{s=0} = \int_0^l \partial_t (\partial_s \Gamma, \partial_t \Gamma)_g \, dt|_{s=0} = (\partial_s \Gamma, \partial_t \Gamma)_g|_{s=0, t=l}, \quad l \in (0, \infty).$$

By (27), we have

$$\begin{aligned}(\dot\sigma(0), \gamma_{z_0}^{\top})_g &= \big(\dot\sigma(0), \dot\gamma(\tau(x_0, \xi_0); x_0, \xi_0)\big)_g \\ &= \lambda'(0) = \big\langle d(\tilde{\ell} \circ F)|_{z=z_0}, \dot\sigma(0)\big\rangle_{T^*_{z_0}\partial M \times T_{z_0}\partial M} = \big(\dot\sigma(0), \operatorname{grad}_{\partial M}(\tilde{\ell} \circ F)|_{z=z_0}\big)_g,\end{aligned}$$

for all smooth curves σ in W such that $\sigma(0) = z_0$, which proves the claim. ☐

Theorem 7.8. *The functions* $\tau : \partial_- SM \to (0, \infty]$ *and*

$$z : D(\Sigma) \to \partial M, \quad z(x, \xi) := \gamma\big(\tau(x, \xi); x, \xi\big),$$

together with the Riemannian manifold $(\Omega, g|_\Omega)$, *determine*

$$\dot\gamma\big(\tau(x, \xi); x, \xi\big), \quad (x, \xi) \in D(\Sigma).$$

Proof. The functions τ and z on $D(\Sigma)$ determine the set B of points $(x_0, \xi_0) \in D(\Sigma)$ such that the conditions (C1)–(C4) of Lemma 7.6 hold for some $(y_0, \eta_0) \in S\Omega$.

Let $(x_0, \xi_0) \in B$. We write $\zeta_0 := \dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big)$. The map

$$\eta \mapsto z(x(\eta), \xi(\eta))$$

determines its local inverse. Hence τ and z determine the function F of Lemma 7.7, and thus they determine $\dot\gamma_{z_0}^{\top}$ by the formula (26). As ζ_0 is a unit vector,

$$\zeta_0 = \dot\gamma_{z_0}^{\top} + (1 - |\dot\gamma_{z_0}^{\top}|^2)^{1/2} \nu_{z_0},$$

where ν_{z_0} is the unit exterior normal vector of ∂M. Hence τ and z determine ζ_0 for all $(x_0, \xi_0) \in B$.

Let $(x_0, \xi_0) \in D(\Sigma)$. By Lemmata 7.5 and 7.6, there is a sequence $((x_j, \xi_j))_{j=1}^{\infty} \subset B$ such that

$$\lim_{j\to\infty} (x_j, \xi_j) = (x_0, \xi_0), \quad \lim_{j\to\infty} \tau(x_j, \xi_j) = \tau(x_0, \xi_0).$$

Moreover, the functions τ and z determine the set of such sequences, and thus they determine

$$\lim_{j\to\infty} \dot\gamma\big(\tau(x_j, \xi_j); x_j, \xi_j\big) = \dot\gamma\big(\tau(x_0, \xi_0); x_0, \xi_0\big).$$

☐

Theorems 6.5, 7.2 and 7.8 prove Theorem 1.2 formulated in the introduction.

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