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TWO-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH RANDOM RADIAL DATA

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We study radial solutions of a certain two-dimensional nonlinear Schrödinger (NLS) equation with harmonic potential, which is supercritical with respect to the initial data. By combining the nonlinear smoothing effect of Schrödinger equation with L^p estimates of Laguerre functions, we are able to prove an almost-sure global well-posedness result and the invariance of the Gibbs measure. We also discuss an application to the NLS equation without harmonic potential.

1. Introduction

Burq, Thomann and Tzvetkov [Burq et al. 2010] studied the nonlinear Schrödinger (NLS) equation on $\mathbb{R} \times \mathbb{R}^d$ with harmonic potential

$$i\partial_t u + (\Delta - |x|^2)u = \pm |u|^{p-1}u, \quad (1-1)$$

where the space dimension was one. The purpose of this paper is to extend their results to two space dimensions. We will prove global well-posedness almost surely with respect to a Gaussian measure supported on $\bigcap_{\delta>0} \mathcal{H}^{-\delta}$ (see Section 1.2 for the definition), and we construct the Gibbs measure, absolutely continuous with respect to this Gaussian, which we prove to be invariant.

We also study the NLS equation on $\mathbb{R} \times \mathbb{R}^d$ without harmonic potential, namely

$$i\partial_t u + \Delta u = \pm |u|^{p-1}u. \quad (1-2)$$

In [Burq et al. 2010], it was noticed that using an explicit transform (referred to as the lens transform in [Tao 2009]), we can obtain local and global well-posedness results of (1-2) from the corresponding results of (1-1). This issue is also pursued here.

Like most earlier papers on random data theory of NLS equations in two or more dimensions, ours considers only radial solutions. In the defocusing case in two dimensions, we can prove, when $p \geq 3$ is an odd integer, almost-sure global well-posedness and measure invariance for (1-1) and almost-sure global well-posedness and scattering for (1-2); in the focusing case, we have the same results only for (1-1), when $1 < p < 3$.

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1.1. NLS equation and probabilistic methods. The nonlinear Schrödinger equation (1-2) and its periodic variant (which is solved on $\mathbb{R} \times \mathbb{T}^d$) have been extensively studied over the last several decades. Beginning with [Lebowitz et al. 1988; Bourgain 1994; 1996], it has been observed that low regularity local and global solutions to (1-2) on $\mathbb{R} \times \mathbb{T}^d$ can be obtained via randomization of initial data and construction of Gibbs measure. This idea was later developed in a number of papers, for instance [Burq and Tzvetkov 2008a; 2008b; Nahmod et al. 2012; Oh 2009; 2010; Thomann and Tzvetkov 2010; Tzvetkov 2006; 2008]. In [Burq et al. 2010], the method mentioned above was first used to study (1-1).

There are three reasons why (1-1) is worth studying. First, the spectrum of the harmonic oscillator $H = -\Delta + |x|^2$ is discrete, so (1-1) can be approximated by ODEs, and the current techniques of constructing Gibbs measure apply at least formally. Second, (1-1) is solved on $\mathbb{R} \times \mathbb{R}^d$, where the space domain is noncompact, while previous works usually involve a compact manifold. Also (1-1) is related to (1-2) via the lens transform, so results about (1-1) may shed some light on the study of (1-2), where probabilistic methods have not yet entered. Finally, (1-1) also arises naturally from the theory of Bose–Einstein condensates, as noted in [Burq et al. 2010].

The major difficulty in the study of (1-1) is that the support of the Gaussian part of the Gibbs measure contains functions with very low regularity. With radial symmetry the typical element in the support of the Gibbs measure belongs to $\bigcap_{\delta>0} \mathcal{H}^{-\delta}$ but not L^2 ; without it the typical element does not even belong to \mathcal{H}^{1-d} (the spaces \mathcal{H}^σ , as defined in Section 1.2, are Sobolev spaces associated to H ; see Section 3 for more details). A consequence of this is that we cannot expect even local well-posedness in the deterministic sense for initial data of such low regularity. In fact in [Thomann 2008] local ill-posedness for \mathcal{H}^σ initial data was shown¹, provided $\sigma < \sigma_c := d/2 - 2/(p-1)$. In particular, we have $\sigma_c \rightarrow 1$ as $p \rightarrow \infty$ for the two-dimensional defocusing equation; thus deterministic local well-posedness fails completely for regularity below L^2 .

In [Burq et al. 2010], the problem was resolved by a probabilistic improvement of (weighted) Strichartz estimate, and it was shown that $H^{\delta/2} e^{-itH} f(\omega)$ almost surely belongs to some weighted Lebesgue space for $\delta < \frac{1}{2}$ (see [Burq et al. 2010, Lemma 6.2] for more details). Since $\sigma_c < \frac{1}{2}$ in one dimension, local well-posedness in this space could be proved. In two dimensions, however, it will be shown in the Appendix that the distribution $H^{\sigma/2} f(\omega)$ is almost surely not a locally integrable function (thus cannot belong to any weighted space) when $\sigma \geq \frac{1}{2}$. Since $\frac{1}{2}$ fails to reach the σ_c threshold when p is large, we have to use different tools to get local well-posedness. Fortunately, the nonlinear smoothing effect of the NLS equation provides such a tool. To fully exploit this effect, we will work in $\mathcal{X}^{\sigma,b}$ spaces (see Section 1.2 for definitions) and use multilinear eigenfunction estimates. This requires p to be an odd integer, but we believe that by more delicate treatment we can remove this restriction and allow for all $1 < p < \infty$.

When there is no radial symmetry, the support of the Gaussian will have such low regularity that we cannot even define the Gibbs measure. It would be possible to use alternative Gaussians to get local results, but then we do not have an invariant measure, so global results still seem out of reach. One

¹The counterexample constructed in [Thomann 2008] was for (1-2), but it could be easily adapted to (1-1) as noted in [Thomann 2009]; also one can check the proof there that the initial data could be made radial.

possible way is to combine the probabilistic local result with the high-low analysis of Bourgain or the I -method of Colliander, Keel, Staffilani, Takaoka and Tao. For progress in this direction, see [Colliander and Oh 2012].

Finally, as we mentioned above, the study of (1-1) is closely related to the study of (1-2). The result we obtain for (1-2) (see Theorem 1.2 below) is an almost-sure global well-posedness and scattering result with supercritical initial data (the critical index of (1-2) is $d/2 - 2/(p - 1) \rightarrow 1$ as $p \rightarrow \infty$ in two dimensions, while the initial data is below L^2), but due to the use of the lens transform, our result is unsatisfactory in the sense that (i) the space in which uniqueness holds cannot be described in a simple way, and (ii) the Gaussian measure in Theorem 1.2 does not arise naturally from (1-2), and we do not know how to construct the Gibbs measure of (1-2). This may be an interesting problem for further study.

1.2. Notations and preliminaries. From now on we assume the spacial dimension $d = 2$, and all the functions we consider are radial. Define the Hermite operator $\mathbf{H} = -\Delta + |x|^2$. It has a complete series of real L^2_{rad} eigenfunctions

$$e_k(x) = \frac{1}{\sqrt{\pi}} \mathcal{L}_k^0(|x|^2) \quad \text{for } k \geq 0 \tag{1-3}$$

with eigenvalue $4k + 2$. Here \mathcal{L}_k^0 are Laguerre functions

$$\mathcal{L}_k^0(z) = \frac{e^{z/2}}{k!} \frac{d^k}{dz^k} (z^k e^{-z}).$$

Concerning these functions we have the basic pointwise estimates

$$|\mathcal{L}_k^\alpha(z)| \leq \begin{cases} C & \text{if } 0 \leq z \leq 1/\nu, \\ C(z\nu)^{-1/4} & \text{if } 1/\nu \leq z \leq \nu/2, \\ C\nu^{-1/4}(\nu^{1/3} + |\nu - z|)^{-1/4} & \text{if } \nu/2 \leq z \leq 3\nu/2, \\ Ce^{-cz} & \text{if } z \geq 3\nu/2. \end{cases} \tag{1-4}$$

Here $\nu = 4k + 2$, C and c (possibly with subscripts) are positive constants varying from line to line, and will be used in this way throughout this paper. For an introduction to Laguerre functions, see [Szegő 1975] or [Thangavelu 1993, Chapter 1]. The proof of (1-4) is also contained in [Erdélyi 1960; Askey and Wainger 1965].

For $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define the Sobolev spaces associated to \mathbf{H} :

$$\mathcal{W}_{\text{rad}}^{\sigma,p} = \{u \in \mathcal{S}'_{\text{rad}} : \|u\|_{\mathcal{W}^{\sigma,p}} = \|\mathbf{H}^{\sigma/2}u\|_{L^p} < \infty\}. \tag{1-5}$$

We also write $\mathcal{W}_{\text{rad}}^{\sigma,2} = \mathcal{H}_{\text{rad}}^\sigma$.

We also define a class of spacetime Hilbert spaces associated to \mathbf{H} , as

$$\mathcal{X}_{\text{rad}}^{\sigma,b} = \{u \in \mathcal{S}'_{\text{rad}}(\mathbb{R} \times \mathbb{R}^2) : \|u\|_{\mathcal{X}^{\sigma,b}} = \|\mathbf{H}^{\sigma/2} \langle i\partial_t - \mathbf{H} \rangle^b u\|_{L^2_{t,x}} < \infty\}, \tag{1-6}$$

or use the radial Hermite expansion and Fourier transform to write

$$\|u\|_{\mathcal{X}^{\sigma,b}}^2 = \left(\sum_{k=0}^{\infty} (4k + 2)^\sigma \int_{\mathbb{R}} (1 + (\tau + 4k + 2)^2)^b |\mathcal{F}_t \langle u, e_k \rangle(\tau)|^2 d\tau \right)^{1/2},$$

where as usual $\langle t \rangle = (|t|^2 + 1)^{1/2}$, \mathcal{F}_t denotes the Fourier transform $(2\pi)^{-1/2} \int_{\mathbb{R}} e^{-it\tau} f(\tau) d\tau$ in t , and $\langle f, g \rangle$ denotes the $L^2(\mathbb{R}^n)$ inner product of f and g . For an interval I we define a localized version of this space by

$$\|u\|_{\mathcal{X}^{\sigma,b,I}} = \inf\{\|v\|_{\mathcal{X}^{\sigma,b}} : v(t) = u(t), t \in I\}, \tag{1-7}$$

and denote it by $\mathcal{X}_{\text{rad}}^{\sigma,b,I}$. When $I = [-T, T]$, we simply write $\mathcal{X}_{\text{rad}}^{\sigma,b,T}$. Since all the functions will be radial, the ‘‘rad’’ subscript will be dropped from now on. Trivially $\mathcal{X}^{\sigma,b,I}$ is a separable Banach space (simply restrict a countable dense subset of $\mathcal{X}^{\sigma,b}$ to I).

We fix a smooth, nonincreasing function η such that $1 = \eta(1) \geq \eta(x) \geq \eta(2) = 0$ for all x . Using this cutoff, we define Littlewood–Paley projections

$$\Delta_N = \eta\left(\frac{2\mathbf{H}}{N^2}\right) - \eta\left(\frac{4\mathbf{H}}{N^2}\right) \tag{1-8}$$

for dyadic N . Then $\Delta_N = 0$ for $N \leq 1$, since the first eigenvalue of \mathbf{H} is 2. Thus whenever we talk about Δ_N , we always assume $N \geq 2$.

We shall denote by $\#M$ the cardinality of a finite set M and by $|E|$ the Lebesgue measure of a subset set E of Euclidean space. We define $A \lesssim B$ by $A \leq CB$ and define \gtrsim and \sim similarly. The constants C_j and c_j will also be used freely, as indicated above. All these constants will ultimately depend on the only parameter p in (1-1) and (1-2). Finally, we define the finite-dimensional subspace V_k to be the span of $\{e_j\}_{0 \leq j \leq k}$. For a function g on \mathbb{R}^2 or $I \times \mathbb{R}^2$, where I is an interval, we define g_k° and g_k^\perp to be the projection of g on V_k and V_k^\perp .

1.3. Statement of main results and plan for this paper. Fix a probability space $(\Omega, \Sigma, \mathbb{P})$ with a sequence of independent normalized complex Gaussians $\{g_k\}$ on Ω (which has density $\pi^{-1} e^{-|z|^2} dx dy$, so g_k has mean 0 and variance 1), so that $\omega \mapsto (g_k(\omega))_{k \geq 0}$ is injective, and the series

$$f(\omega) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{4k+2}} g_k(\omega) e_k \tag{1-9}$$

converges² in $\mathcal{S}'(\mathbb{R}^2)$ for all $\omega \in \Omega$. Then $f = f(\omega)$ is an $\mathcal{S}'(\mathbb{R}^2)$ -valued random variable, and is a bijection between Ω and its range. Our main results can then be stated as follows.

Theorem 1.1. *Consider the Cauchy problem*

$$\begin{cases} i\partial_t u + (\Delta - |x|^2)u = \pm |u|^{p-1}u, \\ u(0) = f(\omega) \end{cases} \tag{1-10}$$

and distinguish two cases: the sign is $-$ and $1 < p < 3$, or the sign is $+$ and $p \geq 3$ is an odd integer. In the former let $\sigma = 0$, and in the latter let $0 < \sigma < 1$ be sufficiently close to 1, depending on p . In both cases let $1 > b > \frac{1}{2}$ be sufficiently close to $\frac{1}{2}$, depending on σ and p .

²For example, we may take the usual product space \mathbb{C}^∞ equipped with the product of complex Gaussian measures, and coordinate functions g_j , and choose the (full-measure) subset where $|g_k(\omega)| = O(\langle k \rangle^{10})$ as Ω , this can easily guarantee the convergence of (1-9).

Then almost surely in \mathbb{P} , we have a unique global (strong) solution u in the affine space

$$\mathfrak{y} = e^{-it\mathbf{H}} f(\omega) + \bigcap_{T>0} \mathfrak{X}^{\sigma,b,T}, \quad (1-11)$$

and we have continuous embeddings

$$\mathfrak{y} \subset e^{-it\mathbf{H}} f(\omega) + \mathcal{C}(\mathbb{R}, \mathfrak{H}^\sigma(\mathbb{R}^2)) \subset \mathcal{C}(\mathbb{R}, \bigcap_{\delta>0} \mathfrak{H}^{-\delta}(\mathbb{R}^2)).$$

We also have a Gibbs measure on $\mathcal{S}'(\mathbb{R}^2)$, which is absolutely continuous with respect to the push forward of \mathbb{P} under f , and is invariant under the flow defined by (1-10).

Theorem 1.2. *Let σ and b be as in Theorem 1.1. Consider the (defocusing) Cauchy problem*

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \\ u(0) = f(\omega) \end{cases} \quad (1-12)$$

with $p \geq 3$ an odd integer. Then almost surely in \mathbb{P} , we have a global (strong) solution u in the affine space

$$\mathfrak{X} = e^{it\Delta} f(\omega) + \bigcap_{T>0} X^{\sigma,b,T}, \quad (1-13)$$

and we have a continuous embedding

$$\mathfrak{X} \subset e^{it\Delta} f(\omega) + \mathcal{C}(\mathbb{R}, H^\sigma(\mathbb{R}^2)).$$

Here $X^{\sigma,b,T}$ is defined in the same way as in (1-6) and (1-7), but with \mathbf{H} replaced by $-\Delta$. We also have an appropriate affine subspace \mathfrak{X}' of \mathfrak{X} containing the solution u , in which uniqueness holds. Finally we have a scattering result: There exist functions $g_\pm \in H^\sigma$ such that

$$\lim_{t \rightarrow \pm\infty} \|u - e^{it\Delta}(f(\omega) + g_\pm)\|_{H^\sigma} = 0. \quad (1-14)$$

The rest of this paper is devoted to the proof of Theorems 1.1 and 1.2. In Section 2 we recall the linear Strichartz and L^2 -based estimates with respect to the propagator $e^{-it\mathbf{H}}$. We will rely on the functional calculus of \mathbf{H} (thus the results hold for more general Schrödinger operators, though we do not discuss this here). Some results in this section are standard and can be found for example in [Colliander and Oh 2012; Tao 2006]. In Section 3, we prove some large deviation bounds for Gaussian random variables, and use these to construct the Gibbs measure of (1-1). In Section 4, which is the core of this paper, we use a Littlewood–Paley decomposition and hypercontractivity of Gaussians to prove a multilinear estimate in $\mathfrak{X}^{\sigma,b}$ spaces, which shows the nonlinear smoothing effect. In Section 5, we put these estimates together to develop a local Cauchy theory. Then in Section 6 we extend this to a global well-posedness result by exploiting the invariance of truncated Gibbs measure under the flow of approximating ODEs. In Section 7 we introduce the lens transform and convert the result on (1-1) to one on (1-2), proving Theorem 1.2. In Section 8, we show the invariance of the Gibbs measure, completing the proof of Theorem 1.1. Finally in the Appendix, we discuss the typical regularity (in terms of \mathbf{H}) on the support of the Gibbs measure.

2. Functional calculus and Strichartz estimates

We begin with the following kernel estimate about the harmonic oscillator H .

Proposition 2.1. *Let ψ be a Schwarz function; then for $t > 0$ the operator $\psi(tH)$ is an integral operator with kernel $K_t(x, y)$, where*

$$|K_t(x, y)| \lesssim t^{-1}(1 + t^{-1/2}|x - y|)^{-N}. \tag{2-1}$$

The implicit constants in \lesssim depend only on N and ψ . In particular, these operators K_t are bounded uniformly in t on $W^{\sigma,p}$ for all $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$.

Proof. It was proved in [Dziubański 1998, Corollary 3.14] that, for any fixed N , the inequality (2-1) holds, provided

$$\psi \in \mathcal{S}_0^m([0, +\infty)) = \{\psi \in \mathcal{S}([0, +\infty)) : \psi^{(k)}(0) = 0, 0 \leq k \leq m\}, \tag{2-2}$$

where m is large enough depending on N (actually the same result was proved for any Schrödinger operator with nonnegative polynomial potential). On the other hand, when $\psi(z) = e^{-\sigma z}$ with $\sigma > 0$, we have from Mehler’s formula that

$$K_t(x, y) = \frac{e^{-2\sigma t}}{\pi(1 - e^{-4\sigma t})} \exp\left(-\frac{1}{2} \frac{1 + e^{-4\sigma t}}{1 - e^{-4\sigma t}} (|x|^2 + |y|^2) + \frac{2e^{-2\sigma t}}{1 - e^{-4\sigma t}} x \cdot y\right). \tag{2-3}$$

Writing $2\sigma t = \delta$, we know

$$-\frac{1}{2} \frac{1 + e^{-2\delta}}{1 - e^{-2\delta}} (|x|^2 + |y|^2) + \frac{2e^{-\delta}}{1 - e^{-2\delta}} x \cdot y \leq -\frac{c}{\delta} |x - y|^2,$$

thus the kernel satisfies

$$0 \leq K_t(x, y) \leq \frac{c_1}{\delta} e^{-(c_2/\delta)|x-y|^2} \lesssim t^{-1}(1 + t^{-1/2}|x - y|)^{-N} \tag{2-4}$$

for any N . Now for any fixed m , there exists l such that any function $f \in \mathcal{S}([0, +\infty))$ can be written as

$$f(z) = f_0(z) + \sum_{j=1}^l c_j e^{-\sigma_j z}, \tag{2-5}$$

where $f_0 \in \mathcal{S}_0^m([0, +\infty))$ and $\sigma_j > 0$. Combining the two results above, we have proved (2-1). The uniform boundedness now follows from (2-1), Schur’s test, and commutativity of $\psi(tH)$ and $H^{\sigma/2}$. \square

Remark 2.2. The constants in Proposition 2.1 certainly depend on ψ and the Lebesgue or Sobolev exponents, but this dependence can be safely ignored since throughout this paper we only use a finite number of fixed cutoff functions ψ and a finite number of fixed exponents.

Corollary 2.3. *Suppose $1 \leq p \leq \infty$, $\sigma_{1,2} \in \mathbb{R}$, $R > 0$ and g is a function.*

- (1) *If $\sigma_1 \geq \sigma_2$, and $\langle g, e_k \rangle \neq 0$ only if $4k + 2 \gtrsim R^2$ (for example, when $g = \sum_{N>R} \Delta_N h$ for some h), then $\|g\|_{W^{\sigma_1,p}} \gtrsim R^{\sigma_1 - \sigma_2} \|g\|_{W^{\sigma_2,p}}$.*

- (2) If $\sigma_1 \leq \sigma_2$, and $\langle g, e_k \rangle \neq 0$ only if $4k + 2 \lesssim R^2$ (for example, when $g = \sum_{N \leq R} \Delta_N h$ for some h), then $\|g\|_{\mathfrak{W}^{\sigma_1, p}} \gtrsim R^{\sigma_1 - \sigma_2} \|g\|_{\mathfrak{W}^{\sigma_2, p}}$.
- (3) If $\langle g, e_k \rangle \neq 0$ only if $4k + 2 \sim R^2$ (for example, when $R = N$ is dyadic and $g = \Delta_N h$ for some h), then $\|g\|_{\mathfrak{W}^{\sigma_1, p}} \sim R^{\sigma_1 - \sigma_2} \|g\|_{\mathfrak{W}^{\sigma_2, p}}$.
- (4) All the operators $\sum_{N > R} \Delta_N$, $\sum_{N \leq R} \Delta_N$ and Δ_N are uniformly bounded from $\mathfrak{W}^{\sigma_1, p}$ to itself.

Proof. First (4) is obvious, since $\sum_{N < R} \Delta_N = \eta(t\mathbf{H})$ and $\Delta_N = \eta(t'\mathbf{H}) - \eta(2t'\mathbf{H})$ for some t and t' , and $\sum_{N > R} \Delta_N = \text{Id} - \sum_{N \leq R} \Delta_N$. Also it is clear that (1) and (2) implies (3). In proving these we may assume $\min\{\sigma_1, \sigma_2\} = 0$, since $\mathbf{H}^{\sigma/2}g$ satisfies the same properties as g .

To prove (1), choose a smooth cutoff ψ_1 that equals 1 for $x \gtrsim 1$ and equals 0 for very small x . Then in (1) we have $g = \psi_1(R^{-2}\mathbf{H})g$. Therefore we need to prove that

$$\mathbf{H}^{-\sigma/2} R^\sigma \psi_1(R^{-2}\mathbf{H}) = \sum_{k \geq 0} 2^{-k\sigma/2} \psi_2(2^{-k} R^{-2}\mathbf{H}) \tag{2-6}$$

is uniformly bounded on L^p for $\sigma > 0$, where $\psi_2(x) = x^{-\sigma/2}(\psi_1(x) - \psi_1(2^{-1}x))$ is a fixed smooth compactly supported function. Using (2-1), we can estimate the kernel $K(x, y)$ of $\mathbf{H}^{-\sigma/2} R^\sigma \psi_1(R^{-2}\mathbf{H})$ as

$$|K(x, y)| \lesssim \sum_{k \geq 0} 2^{-k\sigma/2} 2^k R^2 \langle 2^{k/2} R |x - y| \rangle^{-N} = R^2 \psi_3(R|x - y|), \tag{2-7}$$

where

$$\psi_3(x) = \sum_{k \geq 0} 2^{(1-\sigma/2)k} \langle 2^{k/2} x \rangle^{-N} \lesssim (1 + |x|^{\sigma-2}) \langle x \rangle^{-N}.$$

The last inequality is easily verified by considering $|x| \geq 1$ and $|x| < 1$ separately. Therefore by Schur’s test we have proved the uniform boundedness of the operator, thus proving (1). The proof of (2) is similar and is left as an exercise. \square

To get Sobolev and product estimates, we next need a lemma.

Lemma 2.4. *For all $1 < p < \infty$ and $\sigma > 0$, we have*

$$\|g\|_{\mathfrak{W}^{\sigma, p}} \sim \|(\nabla)^\sigma g\|_{L^p} + \langle x \rangle^\sigma g\|_{L^p}. \tag{2-8}$$

In particular we have $\|g\|_{\mathfrak{W}^{\sigma_1, p}} \lesssim \|g\|_{\mathfrak{W}^{\sigma_2, p}}$ for $\sigma_1 \leq \sigma_2$.

Proof. See Dziubański and Głowacki [2009], who proved the same result for any Schrödinger operator with nonnegative polynomial potential (the latter inequality also follows from Corollary 2.3). \square

Proposition 2.5. *We have the estimate*

$$\|g\|_{\mathfrak{W}^{\sigma_1, q}} \lesssim \|g\|_{\mathfrak{W}^{\sigma_2, q'}} \tag{2-9}$$

if $1 < q, q' < \infty$ and $\sigma_2 - \sigma_1 \geq 2(1/q' - 1/q) \geq 0$, and the estimate

$$\left\| \prod_{j=1}^k g_j \right\|_{\mathfrak{W}^{\sigma, p}} \lesssim \sum_{j=1}^k \|g_j\|_{\mathfrak{W}^{\sigma, q_j}} \prod_{i \neq j} \|g_i\|_{L^{q_i}} \tag{2-10}$$

if $\sigma > 0$ and $1 < p, q_j < \infty$ with $1 \leq j \leq k$ and $\sum_{j=1}^k 1/q_j = 1/p$.

Proof. In considering the first estimate we may assume $\sigma_1 = 0$, and the inequality follows immediately from Lemma 2.4 and the usual Sobolev inequality.

As for the second estimate, if the $\mathcal{W}^{\sigma,p}$ norm is replaced by the usual Sobolev $W^{\sigma,p}$ norm, then (2-10) is a well-known result in Fourier analysis (for $k = 2$, but the general case easily follows from induction). Now using Lemma 2.4, we only need to show

$$\|\langle x \rangle^\sigma g_1 \cdots g_k\|_{L^p} \lesssim \|\langle x \rangle^\sigma g_1\|_{L^{q_1}} \prod_{j=2}^k \|g_j\|_{L^{q_j}},$$

which is simply Hölder’s inequality. □

Before proving Strichartz and other estimates, we need a lemma, which gives a representation formula of $\mathcal{X}^{\sigma,b}$ functions.

Lemma 2.6. *Suppose $\sigma, b \in \mathbb{R}$. Then for every u , if $\|u\|_{\mathcal{X}^{\sigma,b}} \lesssim 1$, we have*

$$u(t, x) = \int_{\mathbb{R}} \phi(\lambda) e^{i\lambda t} \sum_k a_\lambda(k) e^{-i(4k+2)t} e_k(x) \, d\lambda, \tag{2-11}$$

where $\sum_k (4k + 2)^\sigma |a_\lambda(k)|^2 = 1$ for all $\lambda \in \mathbb{R}$. Furthermore, if $b > \frac{1}{2}$, then we also have $\int_{\mathbb{R}} |\phi(\lambda)| \, d\lambda \lesssim 1$. If $b < \frac{1}{2}$ and $\mathcal{F}_t \langle u, e_k \rangle(\lambda)$ is supported in $\{|\lambda + 4k + 2| \leq K\}$ for each k , where $K \gtrsim 1$, then we also have $\int_{\mathbb{R}} |\phi(\lambda)| \, d\lambda \lesssim K^{1/2-b}$.

Proof. Using radial Hermite expansion and Fourier transform, we can write

$$\begin{aligned} u(t, x) &= (2\pi)^{-1/2} \sum_k \int_{\mathbb{R}} \mathcal{F}_t \langle u, e_k \rangle(\tau) e^{it\tau} e_k(x) \, d\tau \\ &= (2\pi)^{-1/2} \sum_k \int_{\mathbb{R}} \mathcal{F}_t \langle u, e_k \rangle(\lambda - 4k - 2) e^{-i(4k-2)t} e_k(x) e^{it\lambda} \, d\lambda, \end{aligned}$$

so we may choose

$$a_\lambda(k) = (\mathcal{F}_t \langle u, e_k \rangle)(\lambda - 4k - 2) \cdot \left(\sum_l (4l + 2)^\sigma |\mathcal{F}_t \langle u, e_l \rangle(\lambda - 4l - 2)|^2 \right)^{-1/2}, \tag{2-12}$$

and

$$\phi(\lambda) = (2\pi)^{-1/2} \left(\sum_l (4l + 2)^\sigma |\mathcal{F}_t \langle u, e_l \rangle(\lambda - 4l - 2)|^2 \right)^{1/2}. \tag{2-13}$$

Then we clearly have $\sum_k (4k + 2)^\sigma |a_\lambda(k)|^2 = 1$ for each λ , and from the definition of $\mathcal{X}^{\sigma,b}$ norm we see that

$$\int_{\mathbb{R}} \langle \lambda \rangle^{2b} |\phi(\lambda)|^2 \, d\lambda = \frac{1}{2\pi} \|u\|_{\mathcal{X}^{\sigma,b}}^2 \lesssim 1. \tag{2-14}$$

If $b > \frac{1}{2}$, then $\langle \lambda \rangle^{-b} \in L^2(\mathbb{R})$, and it follows from Cauchy–Schwartz that $\|\phi\|_{L^1} \leq \|\langle \lambda \rangle^b \phi\|_{L^2} \cdot \|\langle \lambda \rangle^{-b}\|_{L^2} \lesssim 1$. If instead $b < \frac{1}{2}$ and u satisfies the support condition, then $\phi(\lambda) = 0$ if $|\lambda| > K$. Again from Cauchy–Schwartz,

$$\|\phi\|_{L^1} \lesssim \left(\int_{|\lambda| \leq K} \langle \lambda \rangle^{-2b} d\lambda \right)^{1/2} \sim K^{1/2-b}. \quad \square$$

Proposition 2.7. *Suppose $b > \frac{1}{2}$, $\sigma_{1,2} \in \mathbb{R}$, and $1 < q_2, r_2 < 2 < q, r, q_1, r_1 < \infty$. We have the following estimates:*

$$\|e^{-it\mathbf{H}}g\|_{L_t^r L_x^{q'}([-T, T] \times \mathbb{R}^2)} \lesssim \langle T \rangle^{1/r} \|g\|_{L^2} \quad (2-15)$$

if $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and g is defined on \mathbb{R}^2 ;

$$\left\| \int_0^t e^{-i(t-s)\mathbf{H}}u(s) ds \right\|_{L_t^{r_1} L_x^{q_1}([-T, T] \times \mathbb{R}^2)} \lesssim \langle T \rangle^{1+\frac{1}{r_1}-\frac{1}{r_2}} \|u\|_{L_t^{r_2} L_x^{q_2}([-T, T] \times \mathbb{R}^2)} \quad (2-16)$$

if $\frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{2}$, $\frac{1}{q_2} + \frac{1}{r_2} = \frac{3}{2}$, and u is defined on $[-T, T] \times \mathbb{R}^2$;

$$\|u\|_{L_t^{r_1} W_x^{\sigma_1, q}([-T, T] \times \mathbb{R}^2)} \lesssim \langle T \rangle^{1/r} \|u\|_{\mathcal{X}^{\sigma_2, b, T}} \quad (2-17)$$

if $\sigma_2 - \sigma_1 \geq 1 - \frac{2}{q} - \frac{2}{r} \geq 0$, and either u is defined on $[-T, T] \times \mathbb{R}^2$, or u is defined on $\mathbb{R} \times \mathbb{R}^2$ and the right side is replaced by $\|u\|_{\mathcal{X}^{\sigma_2, b}}$;

$$\|u\|_{\mathcal{X}^{\sigma_1, b-1, T}} \lesssim \langle T \rangle^{\frac{1}{q_2}-\frac{1}{2}} \|u\|_{L_t^{q_2} W_x^{\sigma_1, q_2}([-T, T] \times \mathbb{R}^2)}, \quad (2-18)$$

if $b < 1$, $q_2 > \frac{2}{2-b}$, and either u is defined on $[-T, T] \times \mathbb{R}^2$, or u is defined on $\mathbb{R} \times \mathbb{R}^2$, supported on $[-T, T]$, and the left side is replaced by $\|u\|_{\mathcal{X}^{\sigma_1, b-1}}$; and finally

$$\|u\|_{\mathcal{C}([-T, T], \mathcal{H}^{\sigma_1}(\mathbb{R}^2))} \lesssim \|u\|_{\mathcal{X}^{\sigma_1, b, T}} \quad (2-19)$$

if u is defined on $[-T, T] \times \mathbb{R}^2$. In particular if $T \leq 1$, all the implicit constants can be taken 1.

Proof. For (2-15), since $e^{-it\mathbf{H}}$ is periodic, we may assume $T \lesssim 1$; thus $\langle T \rangle \sim 1$. In addition, by subdividing the interval $[-T, T]$, we may assume T is small enough. Substituting $\sigma = i$ in Mehler’s formula (2-3), we can easily see the integral kernel of $e^{-it\mathbf{H}}$ is an L^∞ function in the space variables with norm $\lesssim |t|^{-1}$ for $|t| \lesssim T$. Now using the TT^* method we reduce (2-15) to

$$\left\| \int_{-T}^T e^{-i(t-s)\mathbf{H}}u(s) ds \right\|_{L_t^r L_x^{q'}([-T, T] \times \mathbb{R}^2)} \lesssim \|u\|_{L_t^{r'} L_x^{q'}([-T, T] \times \mathbb{R}^2)}. \quad (2-20)$$

Now we interpolate between L^2 conservation and the $L^1 \rightarrow L^\infty$ inequality deduced from the L^∞ bound of the integral kernel, to get $\|e^{-i\delta\mathbf{H}}g\|_{L^q} \lesssim |\delta|^{\frac{2}{q}-1} \|u\|_{L^{q'}}$ for $|t| \lesssim T$. Using this and the usual Hardy–Littlewood–Sobolev fractional integral inequality, we immediately get (2-20).

Now from (2-15) and duality we easily get

$$\left\| \int_0^T e^{-i(t-s)\mathbf{H}}u(s) ds \right\|_{L_t^{r_1} L_x^{q_1}([-T, T] \times \mathbb{R}^2)} \lesssim \langle T \rangle^{1+\frac{1}{r_1}-\frac{1}{r_2}} \|u\|_{L_t^{r_2} L_x^{q_2}([-T, T] \times \mathbb{R}^2)}$$

for the exponents q_1, r_1, q_2, r_2 ; thus from the Christ–Kiselev lemma we get (2-16).

We now prove (2-19) and (2-17), under the assumption $\sigma_2 - \sigma_1 = 1 - \frac{2}{q} - \frac{2}{r} = 0$. Here we may assume $\sigma_1 = 0$. By the definition of $\mathcal{X}^{0,b,T}$ we can assume that u is defined for all $t \in \mathbb{R}$, and only need to prove that the left side of each equation is controlled by $\|u\|_{\mathcal{X}^{0,b}}$. We shall use $\|\cdot\|_{\mathfrak{X}}$ to denote either the norm $\langle T \rangle^{-1/r} \|\cdot\|_{L_t^r L_x^q([-T, T] \times \mathbb{R}^2)}$ or $\|\cdot\|_{\mathcal{C}([-T, T], L^2(\mathbb{R}^2))}$, and from what we just proved, we know $\|e^{-itH} g\|_{\mathfrak{X}} \lesssim \|g\|_{L^2}$. Assume $\|u\|_{\mathcal{X}^{0,b}} \lesssim 1$; by Lemma 2.6 we may write

$$u(t, x) = \int_{\mathbb{R}} \phi(\lambda) e^{i\lambda t} \sum_k a_\lambda(k) e^{-i(4k+2)t} e_k(x) d\lambda \tag{2-21}$$

with $\|\phi\|_{L^1} \lesssim 1$ and $\sum_k |a_\lambda(k)|^2 = 1$ for each λ . Then we have

$$u = \int_{\mathbb{R}} \phi(\lambda) e^{i\lambda t} e^{-itH} \left(\sum_k a_\lambda(k) e_k \right) d\lambda.$$

From Minkowski and Cauchy–Schwartz we see that

$$\begin{aligned} \|u\|_{\mathfrak{X}} &\lesssim \|\phi\|_{L^1} \cdot \sup_\lambda \left\| e^{i\lambda t} e^{-itH} \left(\sum_k a_\lambda(k) e_k \right) \right\|_{\mathfrak{X}} \\ &\lesssim \|\phi\|_{L^1} \cdot \sup_\lambda \left\| \sum_k a_\lambda(k) e_k \right\|_{L^2} \lesssim 1, \end{aligned} \tag{2-22}$$

proving (2-19) and this special case of (2-17). To prove (2-17) in general, we use Proposition 2.5 to deduce

$$\|u\|_{L_t^r W_x^{\sigma_1, q}([-T, T] \times \mathbb{R}^2)} \lesssim \|u\|_{L_t^r W_x^{\sigma_2, q'}([-T, T] \times \mathbb{R}^2)} \lesssim \langle T \rangle^{1/r} \|u\|_{\mathcal{X}^{\sigma_2, b, T}},$$

where $\frac{1}{q'} + \frac{1}{r} = \frac{1}{2}$ (so that $2 < q, q', r < \infty$ and $\sigma_2 - \sigma_1 \geq 2(\frac{1}{q'} - \frac{1}{q}) \geq 0$), and with obvious modifications when u is globally defined.

Finally we prove (2-18). Again we may assume $\sigma_1 = 0$. For $v = u$ on $[-T, T]$ and $v = 0$ elsewhere, we need to show

$$\|v\|_{\mathcal{X}^{0,b-1}} \lesssim \langle T \rangle^{\frac{1}{q_2} - \frac{1}{2}} \|u\|_{L_{t,x}^{q_2}([-T, T] \times \mathbb{R}^2)}. \tag{2-23}$$

For any w with $\|w\|_{\mathcal{X}^{0,1-b}} \lesssim 1$, we have

$$\left| \int_{\mathbb{R} \times \mathbb{R}^2} v \bar{w} dt dx \right| = \left| \int_{[-T, T] \times \mathbb{R}^2} u \bar{w} dt dx \right| \lesssim \|w\|_{L_{t,x}^{q_3}([-T, T] \times \mathbb{R}^2)} \cdot \|u\|_{L_{t,x}^{q_2}([-T, T] \times \mathbb{R}^2)}, \tag{2-24}$$

where $q_3 = q_2/(q_2 - 1)$. Thus by duality, we only need to prove $\|w\|_{L_{t,x}^{q_3}} \lesssim \langle T \rangle^{\frac{1}{2} - \frac{1}{q_3}} \|w\|_{\mathcal{X}^{0,1-b}}$ for all $2 < q_3 < \frac{2}{b}$. Since the imaginary power $\langle i\partial_t - H \rangle^{it}$ is an isometry on $L_{t,x}^{2,q}$, we can use Stein’s complex interpolation to reduce to the cases $(b, q_3) = (1, 2)$ and $(b_1, 4)$, where $b_1 = (q_3 - 4 + bq_3)/(2q_3 - 4) < \frac{1}{2}$. The former is trivial by definition, and the latter is a special case of (2-17). \square

Lemma 2.8. Fix $\sigma, b \in \mathbb{R}, 0 < T \leq 1$ and a cutoff function ψ .

(1) If $-\frac{1}{2} < b' \leq b < \frac{1}{2}$, then for $u \in \mathcal{X}^{\sigma,b}$ we have

$$\|\psi(T^{-1}t)u\|_{\mathcal{X}^{\sigma,b'}} \lesssim T^{b-b'} \|u\|_{\mathcal{X}^{\sigma,b}}. \tag{2-25}$$

Also for $u \in \mathcal{X}^{\sigma,b,T}$ we have

$$\|u\|_{\mathcal{X}^{\sigma,b',T}} \lesssim T^{b-b'} \|u\|_{\mathcal{X}^{\sigma,b,T}}. \tag{2-26}$$

(2) If $\frac{1}{2} < b' = b < 1$, then for $u \in \mathcal{X}^{\sigma,b}$ with $u(0) = 0$, (2-25) holds, as well as the limit

$$\lim_{T \rightarrow 0} \|\psi(T^{-1}t)u\|_{\mathcal{X}^{\sigma,b}} = 0. \tag{2-27}$$

Proof. (1) If (2-25) is true, then for any $u \in \mathcal{X}^{\sigma,b,T}$ and any extension $v \in \mathcal{X}^{\sigma,b}$ of u , we have

$$\|u\|_{\mathcal{X}^{\sigma,b',T}} \leq \|\psi(T^{-1}t)v\|_{\mathcal{X}^{\sigma,b'}} \lesssim T^{b-b'} \|v\|_{\mathcal{X}^{\sigma,b}},$$

provided $\psi \equiv 1$ on $[-1, 1]$. Taking the infimum over v , we get (2-26). Now we prove (2-25). Define the operator $Mu(t, x) := e^{itH}u(t, \cdot)(x)$. We have

$$i\partial_t(Mu) = e^{itH}(i\partial_t - H)u,$$

and therefore we get $\|u\|_{\mathcal{X}^{\sigma,b}} = \|Mu\|_{H_t^b \mathcal{X}_x^{\sigma}}$. Since M also commutes with multiplication of functions of time, we can reduce to $\|\psi(T^{-1}t)v\|_{H_t^{b'} \mathcal{X}_x^{\sigma}} \lesssim T^{b-b'} \|v\|_{H_t^b \mathcal{X}_x^{\sigma}}$. By eigenfunction expansion, we can further reduce to

$$\|\psi(T^{-1}t)g\|_{H^{b'}} \lesssim T^{b-b'} \|g\|_{H^b}. \tag{2-28}$$

By composition we may assume $0 \leq b' \leq b$ or $b' \leq b \leq 0$, by duality we may assume $0 \leq b' \leq b$, and by interpolation we may assume $b' \in \{0, b\}$.

First suppose $b' = b$; we want to prove that multiplication by $\psi(T^{-1}t)$ is bounded, independent of $T > 0$, on H^b . Since it is bounded on L^2 , we only need to show that it is also bounded on \dot{H}^b . By rescaling we may set $T = 1$. For each $g \in \dot{H}^b$, we split $g = g_1 + g_2$, where \hat{g}_1 is supported on $\{|\xi| \leq 1\}$ and \hat{g}_2 supported on $\{|\xi| \geq 1\}$. Multiplication by ψ is obviously bounded from H^1 to \dot{H}^1 , and from L^2 to L^2 . So it is bounded from H^b to \dot{H}^b ; thus $\|\psi g_2\|_{\dot{H}^b} \lesssim \|g_2\|_{H^b} \lesssim \|g\|_{\dot{H}^b}$. Since $b < \frac{1}{2}$, we also know

$$\int_{|\tau| \leq 1} |\hat{g}_1(\tau)| d\tau \lesssim \|\tau|^b \hat{g}_1(\tau)\|_{L^2([-1,1])} \cdot \|\tau|^{-b}\|_{L^2([-1,1])} \lesssim \|g_1\|_{\dot{H}^b} \lesssim \|g\|_{\dot{H}^b}.$$

Thus $(\psi g_1)^\wedge(\tau) = (\psi * \hat{g}_1)(\tau)$ is bounded pointwise by $\langle \tau \rangle^{-N} \|g\|_{\dot{H}^b}$, since ψ is Schwartz, and the result follows.

Next suppose $b' = 0$, we only need to prove the stronger result

$$\|\psi(T^{-1}t)g\|_{L^2} \lesssim T^b \|g\|_{\dot{H}^b}.$$

By rescaling we can set $T = 1$. Using the same splitting $g = g_1 + g_2$, we have $\|\psi g_2\|_{L^2} \lesssim \|g_2\|_{L^2} \lesssim \|g\|_{\dot{H}^b}$, and $|\psi g_1(\tau)| \lesssim \langle \tau \rangle^{-N} \|g\|_{\dot{H}^b}$. This proves (2-28) and hence (2-25).

(2) We want to prove (2-25), and again we can reduce to (2-28), where we also have $g(0) = 0$. Using the same arguments as in (1), we can further reduce to the boundedness on \dot{H}^b and assume $T = 1$. Split $g = g_1 + g_2$ so that (though we are considering \dot{H}^b norm here, we still assume $g \in H^b$, so $\hat{g} \in L^1$)

$$\hat{g}_2(\tau) = \chi_{|\tau| \geq 1} \cdot \hat{g}(\tau) - \frac{1}{2} \int_{|\lambda| \geq 1} \hat{g}(\lambda) d\lambda \cdot \chi_{1 \leq |\tau| \leq 2};$$

then \hat{g}_1 is supported in $\{|\tau| \leq 2\}$, \hat{g}_2 is supported in $\{|\tau| \geq 1\}$, both the \hat{g}_i have integral zero (since \hat{g} has integral zero), and $\|g_i\|_{\dot{H}^b} \lesssim \|g\|_{\dot{H}^b}$ (since $b > \frac{1}{2}$, we have $\|\hat{g}\|_{L^1(\{|\tau| \geq 1\})} \lesssim \|\tau\|^b \|\hat{g}\|_{L^2} = \|g\|_{\dot{H}^b}$). For g_2 we have $\|\psi g_2\|_{\dot{H}^b} \lesssim \|g_2\|_{H^b} \lesssim \|g\|_{\dot{H}^b}$ as in (1); for g_1 we have

$$(\widehat{\psi g_1})(\tau) = \int_{-2}^2 (\psi(\tau - \xi) - \psi(\tau)) \hat{g}_1(\xi) d\xi.$$

By Cauchy–Schwartz

$$|(\psi g_1)^\wedge(\tau)| \lesssim \|g_1\|_{\dot{H}^b} \left(\int_{-2}^2 |\xi|^{-2b} |\psi(\tau - \xi) - \psi(\tau)|^2 d\xi \right)^{1/2} \lesssim \langle \tau \rangle^{-N} \|g\|_{\dot{H}^b},$$

and (2-28) follows. Finally, to prove (2-27), we first use the operator M and approximation by a finite linear combination of eigenfunctions to reduce to $\|\psi(T^{-1}t)g\|_{H^b} \rightarrow 0$ ($T \rightarrow 0$). Since this is easily verified for Schwartz g , we only need to check any $g \in H^b$ with $g(0) = 0$ can be approximated by Schwartz h also with $h(0) = 0$. But this easily follows since H^b is embedded in L^∞ . \square

Proposition 2.9. *Suppose $\frac{1}{2} < b < 1$. We have*

$$\left\| \int_0^t e^{-i(t-s)\mathbf{H}} u(s) ds \right\|_{\mathcal{X}^{\sigma,b,T}} \lesssim \|u\|_{\mathcal{X}^{\sigma,b-1,T}} \tag{2-29}$$

for $T \leq 1$. Also for $u \in \mathcal{X}^{\sigma,b,T}$, the function $\|u\|_{\mathcal{X}^{\sigma,b,\delta}}$ is continuous for $T \geq \delta > 0$, and if $u(0) = 0$, it tends to 0 as $\delta \rightarrow 0$. Moreover, if $p > \frac{1}{2}$ and

$$\|u - e^{-i(t-k\delta)\mathbf{H}} u(k\delta)\|_{\mathcal{X}^{\sigma,b,[(k-1)\delta,(k+1)\delta]}} \leq C \tag{2-30}$$

for $|k| \leq K$, then

$$\|u - e^{-it\mathbf{H}} u(0)\|_{\mathcal{X}^{\sigma,b,K\delta}} \lesssim c_1 K^2 \delta^{-b/2}. \tag{2-31}$$

Proof. For the operator M defined in the proof of Lemma 2.8 we have

$$M\left(\int_0^t e^{-i(t-s)\mathbf{H}} u(s) ds\right) = \int_0^t M u(s) ds; \tag{2-32}$$

therefore we can again use an eigenfunction expansion to reduce the problem and see that (2-29) will follow if the operator

$$g(t) \mapsto \mathcal{F}g(t) := \eta(t) \int_0^t g(s) ds \tag{2-33}$$

is bounded from H_t^{b-1} to H_t^b , where η is a fixed smooth function supported on $[-3, 3]$ that equals 1 on $[-2, 2]$. Choose a smooth compactly supported function ψ that equals 1 on $[-10, 10]$, and choose ϕ supported on $[-5, 5]$ that equals 1 on $[-4, 4]$. Then we have

$$\mathcal{I}g(t) = \eta(t) \int_{-\infty}^t \psi(t-s)\phi(s)g(s) ds - \eta(t) \int_{-5}^0 \phi(s)g(s) ds. \tag{2-34}$$

We know multiplication by η is bounded on H^b , multiplication by ϕ is bounded on H^{1-b} (to prove these, we first prove them in L^2 and H^1 explicitly, then interpolate), and convolution with $\psi \cdot \chi_{[0,\infty)}$ is bounded from H^{b-1} to H^b , since its Fourier transform is controlled by $\langle \tau \rangle^{-1}$. Thus the first term is bounded. For the second term, we only need to prove $|\langle g, \phi_0 \rangle| \lesssim \|g\|_{H^{b-1}}$, where $\phi_0 = \phi \cdot \chi_{[0,5]}$ with $|\hat{\phi}_0(\tau)| \lesssim \langle \tau \rangle^{-1}$. But this follows from Plancherel, Cauchy–Schwartz, and the assumption $b > \frac{1}{2}$. This proves (2-29).

Next we consider the function $M(\delta) := \|u\|_{\mathcal{X}^{\sigma,b,\delta}}$, which is clearly nondecreasing. Since we only consider $0 < \delta \leq T$, we may assume u is defined for $t \in \mathbb{R}$ and belongs to $\mathcal{X}^{\sigma,b}$. For each $\delta > 0$, denote by M_0 the left limit of the function M at point δ , and choose a sequence $\delta_n \uparrow \delta$, and (by definition) a sequence of v_n such that $v_n \equiv u$ on $[-\delta_n, \delta_n]$ and $\lim_{n \rightarrow \infty} \|v_n\|_{\mathcal{X}^{\sigma,b}} \leq M_0$. These v_n have a subsequence converging weakly to some v with $\|v\|_{\mathcal{X}^{\sigma,b}} \leq M_0$. Using the embedding $L_t^\infty \mathcal{H}_x^\sigma \supset \mathcal{X}^{\sigma,b}$, we easily see $v \equiv u$ on $[-\delta, \delta]$. This proves left continuity. To prove right continuity at δ , write $M(\delta) = M_1$. For any ϵ , we choose $v \equiv u$ on $[-\delta, \delta]$ and $\|v\|_{\mathcal{X}^{\sigma,b}} < M_1 + \epsilon$. Let $u - v = w$ with $w \equiv 0$ on $[-\delta, \delta]$, and define

$$w_\tau = (\psi(\tau^{-1}(t - \delta)) + \psi(\tau^{-1}(t + \delta)))w,$$

for some suitable cutoff that equals 1 on a small neighborhood of 0. From the definition of w_τ , we see that for small τ , we have $v + w_\tau \equiv u$ on a neighborhood of $[-\delta, \delta]$. From Lemma 2.8 we know $\|w_\tau\|_{\mathcal{X}^{\sigma,b}} \rightarrow 0$ as $\tau \rightarrow 0$, thus $\|v + w_\tau\|_{\mathcal{X}^{\sigma,b}} < M_1 + 2\epsilon$ if τ is small enough. This proves right continuity. Finally, if $u(0) = 0$, then

$$\lim_{\delta \rightarrow 0} \|u\|_{\mathcal{X}^{\sigma,b,\delta}} \leq \lim_{\tau \rightarrow 0} \|\psi(\tau^{-1}t)u\|_{\mathcal{X}^{\sigma,b}} = 0$$

for the same cutoff ψ .

Finally we prove (2-31). From (2-30) and the embedding $\|g\|_{L_t^\infty \mathcal{H}_x^\sigma} \lesssim \|g\|_{\mathcal{X}^{\sigma,b,\delta}}$ we see in particular $\|u(k\delta) - e^{-ik\delta \mathbf{H}} u(0)\|_{\mathcal{H}^\sigma} \lesssim K$. Now choose w_k so that $w_k \equiv u - e^{-i(t-k\delta)\mathbf{H}} u(k\delta)$ on $[(k-1)\delta, (k+1)\delta]$ and $\|w_k\|_{\mathcal{X}^{\sigma,b}} \leq C$, and choose a partition of unity ψ_k subordinate to the covering $\{((k-1)\delta, (k+1)\delta)\}$ of $[-K\delta, K\delta]$, so that $\psi_k(t) = \tilde{\psi}_k(t/\delta - k)$ and $\tilde{\psi}_k$ have bounded Schwartz norms (this is well known). We then have

$$w = \sum_k \psi_k w_k + \sum_k \psi_k e^{-i(t-k\delta)\mathbf{H}} (u(k\delta) - e^{-ik\delta \mathbf{H}} u(0)) \equiv v \text{ on } [-K\delta, K\delta], \tag{2-35}$$

and $\|w\|_{\sigma,b} \lesssim K^2 \delta^{-b/2}$, since it is easy to check (by reducing to estimates of functions of t and interpolating between L^2 and H^1) that multiplication by ψ_k is bounded from $\mathcal{X}^{\sigma,b}$ to itself with norm $\lesssim \delta^{-b/2}$, and that by definition

$$\|\psi_k e^{-i(t-k\delta)\mathbf{H}} (u(k\delta) - e^{-ik\delta \mathbf{H}} u(0))\|_{\mathcal{X}^{\sigma,b}} = \|u(k\delta) - e^{-ik\delta \mathbf{H}} u(0)\|_{\mathcal{H}^\sigma} \|\psi_k\|_{H^b} \lesssim K \delta^{1/2-b}. \quad \square$$

3. Construction of Gibbs measure

We will construct the Gibbs measure of (1-1) for $1 < p < \infty$ (defocusing case) and $1 < p < 3$ (focusing case). From the definition (1-9) of f , it is obvious that

$$\|f(\omega)\|_{\mathcal{H}^\tau}^2 = \sum_{k=0}^{\infty} (4k+2)^{-1+\tau} |g_k(\omega)|^2. \tag{3-1}$$

This expression is almost surely finite if $\tau < 0$, and is almost surely infinite if $\tau \geq 0$. Thus we have

$$f(\omega) \in \mathcal{H}^{0-} := \bigcap_{\delta>0} \mathcal{H}^{-\delta}, \tag{3-2}$$

almost surely in \mathbb{P} . Define $\mu = \mathbb{P} \circ f^{-1}$ to be the push-forward of \mathbb{P} under f ; then we see that the typical element in the support of μ belongs to any $\mathcal{H}^{-\delta}$ for all $\delta > 0$, but does not belong to L^2 . We also define $\mu_{2^k}^\circ = \mathbb{P} \circ (f_{2^k}^\circ)^{-1}$, and $\mu_{2^k}^\perp = \mathbb{P} \circ (f_{2^k}^\perp)^{-1}$. Now we prove two lemmas concerning linear and multilinear estimates of the eigenfunctions $e_k(x)$ as defined in (1-3).

Lemma 3.1. *For any $2 \leq q \leq \infty$ and $q \neq 4$, write $v = 4k + 2$ for $k \geq 0$; then we have*

$$\|e_k\|_{L^q(\mathbb{R}^2)} \lesssim v^{-\rho(q)}, \tag{3-3}$$

where $\rho(q) = \min\{\frac{1}{2} - \frac{1}{q}, \frac{1}{q}\}$. If $q = 4$ we have

$$\|e_k\|_{L^4(\mathbb{R}^2)} \lesssim v^{-\frac{1}{4}} \log^{\frac{1}{4}} v. \tag{3-4}$$

Proof. Since $e_k(x) = \pi^{-\frac{1}{2}} \mathcal{L}_k^0(|x|^2)$, we easily see $\|e_k\|_{L^q(\mathbb{R}^2)} \sim \|\mathcal{L}_k^0\|_{L^q(\mathbb{R}^+)}$. Then we can use (1-4) to compute

$$\begin{aligned} \|\mathcal{L}_k^0\|_{L^4(\mathbb{R}^+)}^4 &\lesssim \int_0^{1/v} dz + \int_{1/v}^{v/2} (zv)^{-1} dz + v^{-1} \int_{v/2}^{3v/2} (v^{1/3} + |v-z|)^{-1} dz + \int_{3v/2}^{\infty} e^{-cz} dz \\ &\lesssim v^{-1} \log v. \end{aligned} \tag{3-5}$$

This proves (3-4). As for (3-3) we have

$$\begin{aligned} \|\mathcal{L}_k^0\|_{L^q(\mathbb{R}^+)}^q &\lesssim \int_0^{1/v} dz + \int_{1/v}^{v/2} (zv)^{-q/4} dz + v^{-q/4} \int_{v/2}^{3v/2} (v^{1/3} + |v-z|)^{-q/4} dz + \int_{3v/2}^{\infty} e^{-cz} dz \\ &\lesssim v^{-q/4+|1-q/4|} + v^{1-q/3} + v^{\max(1-q/2, 1-q/3)} \\ &\lesssim v^{-q\rho(q)}. \end{aligned} \quad \square$$

Lemma 3.2. *Suppose $l \geq 4$ and $n_1, \dots, n_l \geq 0$. Let $v_j = 4n_j + 2$ for $1 \leq j \leq l$, and assume $v_1 \gtrsim \dots \gtrsim v_l$. Then we have*

$$\left| \int_{\mathbb{R}^2} e_{n_1}(x) \cdots e_{n_l}(x) \right| \lesssim v_1^{-1/2} v_3^{-1/4} \log v_1. \tag{3-6}$$

Moreover, if $v_1 \gtrsim v_2^{1+\epsilon}$ for some $\epsilon > 0$, then

$$\left| \int_{\mathbb{R}^2} e_{n_1}(x) \cdots e_{n_l}(x) \right| \lesssim v_1^{-N} \quad \text{for all } N > 0. \tag{3-7}$$

Proof. Recalling that $\mathbf{H}e_{n_j} = v_j e_{n_j}$ and \mathbf{H} is self-adjoint on $L^2(\mathbb{R}^2)$, we can compute using [Proposition 2.5](#) and [Lemma 3.1](#) that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} e_{n_1}(x) \cdots e_{n_l}(x) \right| &\leq v_1^{-m} \|\mathbf{H}^m(e_{n_2} \cdots e_{n_l}) \cdot e_{n_1}\|_{L^1} \\ &\lesssim v_1^{-m} \|e_{n_2} \cdots e_{n_l}\|_{\mathfrak{H}^m} \lesssim v_1^{-m} \sum_{j=2}^l \|e_{n_j}\|_{\mathfrak{W}^{2m, 2(l-1)}} \prod_{2 \leq i \neq j} \|e_{n_i}\|_{L^{2(l-1)}} \lesssim (v_1^{-1} v_2)^m. \end{aligned}$$

If $v_1 \gtrsim v_2^{1+\epsilon}$, we can choose m large enough and prove (3-7). As for (3-6), we choose $m = 1$ and estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^2} e_{n_1}(x) \cdots e_{n_l}(x) \right| &\leq v_1^{-1} \|\mathbf{H}(e_{n_2} \cdots e_{n_l}) \cdot e_{n_1}\|_{L^1} \\ &\lesssim v_1^{-\frac{5}{4}} \log^{\frac{1}{4}} v_1 \cdot \|e_{n_2} \cdots e_{n_l}\|_{\mathfrak{W}^{2, \frac{4}{3}}} \\ &\lesssim v_1^{-\frac{5}{4}} \log^{\frac{1}{4}} v_1 \cdot v_2 \|e_{n_2}\|_{L^4} \|e_{n_3}\|_{L^4} \prod_{j \geq 4} \|e_{n_j}\|_{L^{4(l-3)}} \\ &\lesssim v_1^{-\frac{5}{4}} v_2^{\frac{3}{4}} v_3^{-\frac{1}{4}} \log^{\frac{3}{4}} v_1 \lesssim v_1^{-\frac{1}{2}} v_3^{-\frac{1}{4}} \log v_1. \quad \square \end{aligned}$$

To state and prove the probabilistic L^p estimates for our \mathcal{S}^l -valued random variable f , we first need a result proved by Fernique.

Lemma 3.3 (Fernique). *There exist absolute constants c, C such that for any finite-dimensional normed vector space $(V, \|\cdot\|)$, any centered Gaussian random variable $f(\omega)$ taking its value in V , and any positive constant A , we have*

$$\mathbb{E}(e^{cA^{-2}\|f(\omega)\|^2}) \leq C \tag{3-8}$$

if $\mathbb{P}(\|f(\omega)\| > A) < \frac{1}{10}$.

Proof. See [\[Fernique 1975\]](#) or [\[Da Prato and Zabczyk 1992, Theorem 2.6\]](#). □

Proposition 3.4. *Fix $2 < q < \infty$, $1 < r < \infty$, $0 < \alpha < \min(\frac{2}{q}, 1 - \frac{2}{q})$, and two positive integers $M > 10N$. For any g , we define*

$$\Pi g = \sum_{j=N-1}^M \langle g, e_j \rangle e_j. \tag{3-9}$$

Then, for the random variable f as defined in (1-9), we have the large deviation estimates

$$\mathbb{P}(\|\Pi f(\omega)\|_{\mathfrak{W}^{\alpha, q}} > AN^{-\delta}) \leq Ce^{-cA^2}, \tag{3-10}$$

$$\mathbb{P}(\|e^{-it\mathbf{H}}\Pi f(\omega)\|_{L_t^r \mathfrak{W}_x^{\alpha, q}([-T, T] \times \mathbb{R}^2)} > AN^{-\delta} T^{1/r}) \leq Ce^{-cA^2}, \tag{3-11}$$

where $\delta > 0$ is some small positive exponent.

Proof. We compute for each $t \in [-\pi, \pi]$

$$\mathbb{E}(\|e^{-it\mathbf{H}}\Pi f(\omega)\|_{\mathfrak{W}_x^{\alpha,q}}^q) = \int_{\mathbb{R}^2} \mathbb{E} \left| \sum_{j=N-1}^M (4j+2)^{(\alpha-1)/2} g_j(\omega) e_j(x) \right|^q dx. \tag{3-12}$$

Now by Khintchine’s inequality (the variant for Gaussians), we have

$$\mathbb{E} \left| \sum_{j=N-1}^M (4j+2)^{(\alpha-1)/2} g_j(\omega) e_j(x) \right|^q \lesssim \left(\sum_{j=N-1}^M \frac{e_j(x)^2}{(4j+2)^{1-\alpha}} \right)^{q/2}. \tag{3-13}$$

Then integrating in x , using Minkowski’s inequality (since $q > 2$), we get

$$\mathbb{E}(\|e^{-it\mathbf{H}}\Pi f(\omega)\|_{\mathfrak{W}_x^{\alpha,q}}^q) \lesssim \left(\sum_{j=N-1}^M \frac{\|e_j\|_{L^q}^2}{(4j+2)^{1-\alpha}} \right)^{q/2} \leq CN^{-q\delta}, \tag{3-14}$$

due to Lemma 3.1 and the assumption $\alpha < 2\rho(q)$. Now we can take $t = 0$ in (3-14) and use Markov’s inequality and Lemma 3.3, and immediately get (3-10).

As for (3-11), we need a little more work. What we need is

$$\mathbb{P}(\|e^{-it\mathbf{H}}\Pi f(\omega)\|_{L_t^r \mathfrak{W}_x^{\alpha,q}} > CN^{-\delta} T^{1/r}) < \frac{1}{10} \tag{3-15}$$

for large C . If the event in (3-15) happens, then there exists an integer $l \geq 0$ such that

$$|\{t \in [-T, T] : \|e^{-it\mathbf{H}}\Pi f(\omega)\|_{\mathfrak{W}_x^{\alpha,q}} > 2^l N^{-\delta}\}| > K 2^{-2rl} T. \tag{3-16}$$

For fixed t , due to (3-14) and Lemma 3.3, the probability that $\|e^{-it\mathbf{H}}\Pi f(\omega)\|_{\mathfrak{W}_x^{\alpha,q}} > 2^l N^{-\delta}$ is less than $c_1 \exp(-c_2 2^{2l})$. We then use Fubini’s theorem to conclude that the probability that (3-16) happens is less than $K^{-1} c_1 2^{2rl} \exp(-c_2 2^{2l})$. Then we sum over $l \geq 0$ and choose K large enough so that this sum is less than $\frac{1}{10}$. \square

Corollary 3.5. *For the same parameters q, r, α as in Proposition 3.4, we have*

$$\mathbb{P}(\|f(\omega)\|_{\mathfrak{W}^{\alpha,q}} > A) \leq Ce^{-cA^2}, \tag{3-17}$$

$$\mathbb{P}(\sup_{k \geq 0} \|f_{2^k}^\circ(\omega)\|_{\mathfrak{W}^{\alpha,q}} > A) \leq Ce^{-cA^2}, \tag{3-18}$$

$$\mathbb{P}(\|e^{-it\mathbf{H}} f(\omega)\|_{L_t^r \mathfrak{W}_x^{\alpha,q}([-T, T] \times \mathbb{R}^2)} > AT^{1/r}) \leq Ce^{-cA^2}, \tag{3-19}$$

$$\mathbb{P}(\sup_{k \geq 0} \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_t^r \mathfrak{W}_x^{\alpha,q}([-T, T] \times \mathbb{R}^2)} > AT^{1/r}) \leq Ce^{-cA^2}, \tag{3-20}$$

$$\lim_{k \rightarrow \infty} \|f_{2^k}^\circ(\omega) - f(\omega)\|_{\mathfrak{W}^{\alpha,q}} + \|e^{-it\mathbf{H}}(f_{2^k}^\circ(\omega) - f(\omega))\|_{L_t^r \mathfrak{W}_x^{\alpha,q}([-T, T] \times \mathbb{R}^2)} = 0 \text{ almost surely in } \mathbb{P}. \tag{3-21}$$

Proof. We know $f_{2^k}^\circ(\omega) \rightarrow f(\omega)$ and $e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) \rightarrow e^{-it\mathbf{H}} f(\omega)$ in \mathcal{S}' . If we can prove (3-18) and (3-20), then almost surely in \mathbb{P} , we have

$$\sup_{k \geq 0} \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_t^r \mathfrak{W}_x^{\alpha,q}} < \infty, \tag{3-22}$$

and there must be a subsequence of $\{e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\}$ converging weakly in $L_t^{r,q}W_x^{\alpha,q}$. This weak limit must be $e^{-it\mathbf{H}} f(\omega)$, so we know that

$$\|e^{-it\mathbf{H}} f(\omega)\|_{L_t^{r,q}W_x^{\alpha,q}} \leq \sup_{k \geq 0} \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_t^{r,q}W_x^{\alpha,q}} < \infty, \tag{3-23}$$

almost surely in \mathbb{P} . Thus (3-19) also holds true, with the same constants as in (3-20). Clearly (3-17) also follows from (3-18) in the same way.

To prove (3-18) and (3-20), we use (3-10) and (3-11). For any k , the difference $f_{2^k}^\circ(\omega) - f_{2^{k-1}}^\circ(\omega)$ is of the form $\Pi f(\omega)$ as defined in Proposition 3.4, with the parameter $N \sim 2^k$. We then have, for some $\delta > 0$,

$$\mathbb{P}(\|e^{-it\mathbf{H}}(f_{2^k}^\circ(\omega) - f_{2^{k-1}}^\circ(\omega))\|_{L_t^{r,q}W_x^{\alpha,q}} > A2^{-k\delta/2}T^{1/r}) \leq c_1e^{-c_22^{k\delta}A^2}. \tag{3-24}$$

Choose c small enough; then

$$\sup_{k \geq 0} \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_t^{r,q}W_x^{\alpha,q}} > AT^{1/r} \tag{3-25}$$

implies

$$\|e^{-it\mathbf{H}}(f_{2^k}^\circ(\omega) - f_{2^{k-1}}^\circ(\omega))\|_{L_t^{r,q}W_x^{\alpha,q}} > cA2^{-k\delta/2}T^{1/r} \quad \text{for some } k \geq 0. \tag{3-26}$$

Now we can combine this with (3-24) to get

$$\mathbb{P}(\sup_{k \geq 0} \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_t^{r,q}W_x^{\alpha,q}} > AT^{1/r}) \leq \sum_{k=0}^{\infty} c_3e^{-c_42^{k\delta}A^2} \leq c_5e^{-c_6A^2}. \tag{3-27}$$

This proves (3-20). Clearly (3-18) also follows from (3-10) in the same way.

Finally we prove (3-21). From the discussion above we see

$$\mathbb{P}(\sup_{k \geq 0} 2^{k\delta/2} \|e^{-it\mathbf{H}}(f_{2^k}^\circ(\omega) - f_{2^{k-1}}^\circ(\omega))\|_{L_t^{r,q}W_x^{\alpha,q}} < \infty) = 1; \tag{3-28}$$

thus with probability 1, the series

$$\sum_{k=0}^{\infty} e^{-it\mathbf{H}}(f_{2^k}^\circ(\omega) - f_{2^{k-1}}^\circ(\omega)) \tag{3-29}$$

converges in $L_t^{r,q}W_x^{\alpha,q}$. This can only converge to $e^{-it\mathbf{H}} f(\omega)$, and the same argument works for the space ${}^qW^{\alpha,q}$. □

Equation (1-1) is a hamiltonian PDE with formally conserved mass $\|u\|_{L^2}^2$ and Hamiltonian

$$E(u) = \langle \mathbf{H}u, u \rangle \pm \frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1} = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + |xu|^2 \pm \frac{2}{p+1} |u|^{p+1} \right) dx. \tag{3-30}$$

Recall that $\mu = \mathbb{P} \circ f^{-1}$ is a probability measure on $\mathcal{S}'(\mathbb{R}^2)$, the push-forward of \mathbb{P} under f . In the defocusing case, for all $1 < p < \infty$, we define the Gibbs measure of (1-1) to be

$$d\nu = \exp\left(-\frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1}\right) d\mu. \tag{3-31}$$

Since the integrand in (3-31) is well defined, bounded and positive, by Corollary 3.5, we know ν is finite and mutually absolutely continuous with μ . We also define the truncated measures

$$d\nu_{2^k} = \exp\left(-\frac{2}{p+1} \|u_{2^k}^\circ\|_{L^{p+1}}^{p+1}\right) d\mu. \tag{3-32}$$

Since $\|u_{2^k}^\circ\|_{L^{p+1}} \rightarrow \|u\|_{L^{p+1}}$ almost everywhere in μ , thanks to Corollary 3.5, we know $\nu_{2^k} \rightarrow \nu$ in the strong sense that the total variation of $\nu_{2^k} - \nu$ tends to 0.

In the focusing case, for $1 < p < 3$, we define the truncated measures $d\nu_{2^k} = \rho_{2^k} d\mu$, where

$$\rho_{2^k}(u) = \chi(\|u_{2^k}^\circ\|_{L^2}^2 - \alpha_{2^k}) \exp\left(\frac{2}{p+1} \|u_{2^k}^\circ\|_{L^{p+1}}^{p+1}\right). \tag{3-33}$$

Here χ is some compactly supported continuous function on \mathbb{R} that equals 1 on a neighborhood of 0, and

$$\alpha_{2^k} = \mathbb{E}(\|f_{2^k}^\circ(\omega)\|_{L^2}^2) = \sum_{j=0}^{2^k} \frac{1}{4j+2}. \tag{3-34}$$

Clearly $\alpha_{2^k} \lesssim k$ for $k \geq 1$. We define the Gibbs measure ν as the limit of these ν_{2^k} . More precisely:

Proposition 3.6. *The functions ρ_{2^k} converge to a function ρ in $L^r(\mu)$ for all $1 \leq r < \infty$. The measure $d\nu = \rho d\mu$ is finite and absolutely continuous with respect to μ . We also know $\nu_{2^k} \rightarrow \nu$ in the strong sense that the total variation of $\nu_{2^k} - \nu$ tends to 0. Finally, we can choose a countable number of $\chi_{(m)}$ so that the union of the supports of the corresponding Radon–Nikodym derivatives $\rho_{(m)}$ has full μ measure in $\mathcal{S}'(\mathbb{R}^2)$. If we have fixed χ , we will define ν to be the Gibbs measure of equation (1-1).*

Proof. First we prove that ρ_{2^k} converges almost everywhere in μ , or equivalently, that $\rho_{2^k}(f(\omega))$ converges almost surely in \mathbb{P} . Consider

$$\|f_{2^k}^\circ(\omega)\|_{L^2}^2 - \alpha_{2^k} = \sum_{j=0}^k \frac{|g_j(\omega)|^2 - 1}{4j+2}, \tag{3-35}$$

and see that it is a (partial) independent sum of random variables with zero mean and summable variance (the variance of j -th term is $\sim (j+1)^{-2}$), so it converges almost surely. Thus by the continuity of χ , the first factor $\chi(\|f_{2^k}^\circ(\omega)\|_{L^2}^2 - \alpha_{2^k})$ in $\rho_{2^k}(f(\omega))$ converges almost surely. Next, since $f_{2^k}^\circ(\omega) \rightarrow f(\omega)$ in L^{p+1} for almost surely $\omega \in \Omega$, we know that the second factor also converges almost surely. Therefore, ρ_{2^k} converges almost everywhere in μ , say to some ρ .

To prove $\rho_{2^k}(f) \rightarrow \rho(f)$ in $L^r(\mathbb{P})$, we need some uniform integrability conditions. This is provided by the large deviation estimate

$$\mathbb{P}(\|f_{2^k}^\circ(\omega)\|_{L^2}^2 - \alpha_{2^k} \leq \beta, \|f_{2^k}^\circ(\omega)\|_{L^{p+1}} > A) \leq C e^{-cA^\delta} \tag{3-36}$$

for some $\delta > p+1$ and all large enough A , where β is such that $\chi(z) = 0$ for $|z| \geq \beta$. To prove (3-36) we may assume A is sufficiently large, and set $k_0 \in \mathbb{N}$ so that $2^{k_0} \sim e^{A^\delta}$ for some $\delta > 0$ to be determined later.

First we prove (3-36) is true for $k \leq k_0 + 1$, with β and A on the left side replaced by 2β and $A/2$. In fact, by Hölder's inequality, if

$$\|f_{2^k}^\circ(\omega)\|_{L^2}^2 \leq \alpha_{2^k} + 2\beta \lesssim k \lesssim A^\delta \quad \text{and} \quad \|f_{2^k}^\circ(\omega)\|_{L^{p+1}} > A/2, \tag{3-37}$$

then

$$\|f_{2^k}^\circ(\omega)\|_{L^q} \gtrsim A^\sigma \quad \text{and} \quad \sigma = \frac{(q-2)(p+1) - \delta(q-p-1)}{(p-1)q}, \tag{3-38}$$

under the assumption $p+1 \leq q < \infty$. Since $2 < q < \infty$, we know from Corollary 3.5 that

$$\mathbb{P}(\|f_{2^k}^\circ(\omega)\|_{L^q} > A^\sigma) \leq C e^{-cA^{2\sigma}}. \tag{3-39}$$

If $1 < p < 3$, then for q sufficiently large and δ sufficiently small, we have $2\sigma > p+1$, so (3-36) is true in this case.

Next we assume $k \geq k_0 + 2$. In this case we can prove

$$\mathbb{P}(\|f_{2^k}^\circ(\omega) - f_{2^{k_0}}^\circ(\omega)\|_{L^{p+1}} > A/2) \leq c_1 \exp(-c_2 e^{c_3 A^{c_4}}). \tag{3-40}$$

In fact, since $f_{2^k}^\circ(\omega) - f_{2^{k_0}}^\circ(\omega)$ is of the form $\Pi f(\omega)$ as defined in Proposition 3.4, with the parameter $N \sim 2^{k_0}$, by Proposition 3.4 we immediately get (3-40) (notice $N \sim e^{A^\delta}$).

Now if $\|f_{2^k}^\circ(\omega)\|_{L^2}^2 \leq \alpha_k + \beta$ and $\|f_{2^k}^\circ(\omega)\|_{L^{p+1}} > A$, then we have three possibilities.

- (1) If $\|f_{2^k}^\circ(\omega) - f_{2^{k_0}}^\circ(\omega)\|_{L^{p+1}} > A/2$, then we are already done, since this probability is controlled due to (3-40).
- (2) If $\|f_{2^{k_0}}^\circ(\omega)\|_{L^{p+1}} > A/2$ and $\|f_{2^{k_0}}^\circ(\omega)\|_{L^2}^2 \leq \alpha_{k_0} + 2\beta$, then we may set $k = k_0$ in the arguments from (3-37) to (3-39), and again get the desired bound.
- (3) If $\|f_{2^k}^\circ(\omega)\|_{L^2}^2 \leq \alpha_{2^k} + \beta$ as well as $\|f_{2^{k_0}}^\circ(\omega)\|_{L^2}^2 > \alpha_{2^{k_0}} + 2\beta$, then

$$\|f_{2^k}^\circ(\omega)\|_{L^2}^2 - \|f_{2^{k_0}}^\circ(\omega)\|_{L^2}^2 - (\alpha_{2^k} - \alpha_{2^{k_0}}) \leq -\beta \tag{3-41}$$

or equivalently

$$Y = \sum_{j=2^{k_0}+1}^{2^k} \frac{1 - |g_j|^2}{4j+2} \geq \beta. \tag{3-42}$$

Noticing that Y is an independent sum with standard deviation

$$\kappa = \left(\sum_{j=2^{k_0}+1}^{2^k} \frac{1}{(4j+2)^2} \right)^{1/2} \lesssim 2^{-k_0/2} \leq c_1 e^{-c_2 A^{c_3}}, \tag{3-43}$$

we can compute

$$\begin{aligned} \mathbb{E}(\exp(Y/2\kappa)) &= \prod_{j=2^{k_0}+1}^{2^k} \mathbb{E}\left(\exp\left(\frac{\kappa(1-|g_j|^2)}{2(4j+2)}\right)\right) \\ &= \prod_{j=2^{k_0}+1}^{2^k} \left(e^{\kappa/(2(4j+2))}\left(1+\frac{\kappa}{2(4j+2)}\right)^{-1}\right) \leq \prod_{j=2^{k_0}+1}^{2^k} e^{c\theta_j^2/(4(4j+2)^2\kappa^2)} = e^{c/4}. \end{aligned} \quad (3-44)$$

Here we have used the fact that $\mathbb{E}(e^{-\lambda|g|^2}) = (1+\lambda)^{-1}$ when $\lambda > -1$, and g is a normalized complex Gaussian; and that $e^x(1+x)^{-1} \leq e^{cx^2}$ for large c , and $0 \leq x \leq \frac{1}{2}$. Therefore we have obtained

$$\mathbb{P}(Y > \beta) \leq e^{-c\kappa^{-1}} \leq c_1 \exp(-c_2 e^{c_3 A^{c_4}}). \quad (3-45)$$

This completes the proof of (3-36). The other conclusions now follow easily from this large deviation estimate, except the one regarding the support of ρ . We choose a sequence of cutoff functions $\chi_{(m)}$ so that $\chi_{(m)} \equiv 1$ on $[-\gamma_m, \gamma_m]$ with $\gamma_m \uparrow \infty$. By our previous discussions, after discarding null sets, the function $\rho_{(m)}$ will be nonzero wherever

$$\lim_{k \rightarrow \infty} \left| \|f_{2^k}^\circ(\omega)\|_{L^2}^2 - \alpha_{2^k} \right| \leq \gamma_m. \quad (3-46)$$

Since this limit exists almost surely, and $\gamma_m \uparrow \infty$, we know almost surely, (3-46) will hold for at least one m . So the union of support of these ρ_j will have full μ measure. □

Now in both defocusing and focusing case we have defined the Gibbs measure ν and the approximating measure ν_{2^k} . They will be used in Section 6 to obtain global well-posedness, and the invariance of ν will be proved in Section 8.

4. Multilinear analysis in $\mathcal{X}^{\sigma,b}$ spaces

First let us recall the hypercontractivity property of complex Gaussians. To make equations easier to write, we introduce the notation in which u^- represents some element in $\{u, \bar{u}\}$ for any complex number u . This will be used throughout the rest of the paper. The first result about hypercontractivity was proved in [Nelson 1973]. Here we use a formulation of this property taken from [Thomann and Tzvetkov 2010].

Proposition 4.1. *Suppose $l, d \geq 1$, and a random variable S has the form*

$$S = \sum_{0 \leq n_1, \dots, n_l \leq d} c_{n_1, \dots, n_l} \cdot g_{n_1}^-(\omega) \cdots g_{n_l}^-(\omega), \quad (4-1)$$

where $c_{n_1, \dots, n_l} \in \mathbb{C}$, and the $(g_n)_{0 \leq n \leq d}$ are independent normalized complex Gaussians; then we have the estimate

$$(\mathbb{E}|S|^p)^{1/p} \leq \sqrt{l+1}(p-1)^{l/2}(\mathbb{E}|S|^2)^{1/2} \quad \text{for all } p \geq 2.$$

Proof. This is basically a restatement of [Thomann and Tzvetkov 2010, Proposition 2.4]. There the authors required $n_j \geq 1$ and $n_1 \leq \dots \leq n_l$, but an easy modification will immediately settle this. The

only difference is that here we may have g_{n_j} or \bar{g}_{n_j} , but if we write $g_n = (\gamma_n + i\tilde{\gamma}_n)/\sqrt{2}$ where γ_n and $\tilde{\gamma}_n$ are mutually independent normalized real Gaussians, then $\bar{g}_n = (\gamma_n - i\tilde{\gamma}_n)/\sqrt{2}$. So S is again written as a linear combination of products of independent normalized real Gaussians. Then the result follows in the same way as in [Thomann and Tzvetkov 2010]. \square

Next we want to adapt the result in Proposition 4.1 to our specific case to yield a large deviation bound on appropriate multilinear expressions of Gaussians.

Proposition 4.2. *Let $N_1 \geq \dots \geq N_l \geq 2$ be dyadic numbers such that $N_1 \geq 10^3 N_2$. Assume for $n \geq 0$ and $4n + 2 \leq 10N_1^2$ that we have independent normalized complex Gaussians $\{w_n\}$. Also let ϱ be any integer, and let δ_{n_1, \dots, n_l} be arbitrary complex numbers with absolute value ≤ 1 . Define*

$$\Xi = \left\{ (n_1, \dots, n_l) : n_j \geq 0, \frac{1}{10} \leq \frac{4n_j + 2}{N_j^2} \leq 10 \ (1 \leq j \leq l), \sum_{j=1}^l \epsilon_j (4n_j + 2) = \varrho \right\} \tag{4-2}$$

with $\epsilon_j = \pm 1$; then we have

$$\mathbb{P} \left(\left\{ \left| \sum_{(n_1, \dots, n_l) \in \Xi} \delta_{n_1, \dots, n_l} w_{n_1}^-(\omega) \cdots w_{n_l}^-(\omega) \right| > K \prod_{j=2}^l N_j \right\} \right) \leq c_1 \exp(-c_2 K^{c_3}). \tag{4-3}$$

Here all the constants depend only on l .

Proof. We denote the sum on the left side of (4-3) by S . Using Proposition 4.1, we can get

$$(\mathbb{E}|S|^p)^{1/p} \leq \sqrt{l+1} (p-1)^{l/2} A,$$

where we denote $A = (\mathbb{E}|S|^2)^{1/2}$. By Markov's inequality, we in particular have

$$\mathbb{P}(|S| > KA) \leq (KA)^{-p} \cdot \mathbb{E}|S|^p \leq K^{-p} (l+1)^{p/2} (p-1)^{lp/2} \quad \text{for all } p \geq 2.$$

If $K \geq 2\sqrt{l+1}$, we may choose $p = 1 + K^{2/l} 2^{-2/l} (l+1)^{-1/l} \geq 2$ in the inequality above to obtain

$$\mathbb{P}(|S| > KA) \leq 2^{-p} \leq c_1 e^{-c_2 K^{c_3}}.$$

By choosing the constants appropriately, we can guarantee that this also hold for $K < 2\sqrt{l+1}$. Now what remains is to prove that $A \lesssim \prod_{j=2}^l N_j$, or equivalently

$$\mathbb{E}|S|^2 \lesssim \prod_{j=2}^l N_j^2.$$

Now we expand the square to get

$$\mathbb{E}|S|^2 = \sum \delta_{n_1, \dots, n_l} \bar{\delta}_{m_1, \dots, m_l} \Delta_{n_1, \dots, n_l, m_1, \dots, m_l},$$

where the sum is taken over all $(n_1, \dots, n_l, m_1, \dots, m_l) \in \Xi \times \Xi$, and

$$\Delta_{n_1, \dots, n_l, m_1, \dots, m_l} = \mathbb{E} \left(\prod_{j=1}^l w_{n_j}^- w_{m_j}^- \right).$$

Since each of the δ and Δ has absolute value $\lesssim 1$ (depending on l) in any possible case, we will be done once we establish that

$$\#\{(n_1, \dots, n_l, m_1, \dots, m_l) \in \mathfrak{E} \times \mathfrak{E} : \Delta_{n_1, \dots, n_l, m_1, \dots, m_l} \neq 0\} \lesssim \prod_{j=2}^l N_j^2. \tag{4-4}$$

The crucial observation is that, due to the independence assumption, if the expectation Δ is nonzero, then any integer that appears in $(n_1, \dots, n_l, m_1, \dots, m_l)$ must appear at least twice. Next, due to our assumption $N_1 \geq 10^3 N_2$, we know $n_1 = m_1$, and any integer that appears in $(n_2, \dots, n_l, m_2, \dots, m_l)$ must appear at least twice. If we permute all the different integers appearing in this $(2l - 2)$ -tuple as $\sigma_1 > \sigma_2 > \dots > \sigma_r$, then with r and all σ_i fixed, we have at most $(2l - 2)^{2l-2}$ choices for the $(2l - 2)$ -tuple; also due to the linear relation enjoyed by both (n_1, \dots, n_l) and (m_1, \dots, m_l) , the $(2l - 2)$ -tuple will uniquely determine n_1 and m_1 . Thus we only need to show for each possible $1 \leq r \leq 2l$, there are $\lesssim \prod_{j=2}^l N_j^2$ choices for $(\sigma_1, \dots, \sigma_r)$. Now for each $1 \leq i \leq r$, since each $\sigma_j (1 \leq j \leq i)$ appear in the $(2l - 2)$ -tuple at least twice (and different σ_j cannot appear at the same place), there must exist $1 \leq j_1 \leq i < i + 1 \leq j_2$ such that $\sigma_{j_1} \in \{n_{j_2}, m_{j_2}\}$. This implies

$$4\sigma_i + 2 \leq 4\sigma_{j_1} + 2 \lesssim N_{j_2}^2 \lesssim N_{i+1}^2,$$

so for each $1 \leq i \leq r$, there are at most N_{i+1}^2 choices for σ_i , and necessarily $1 \leq r \leq l - 1$. Therefore, for each $r \leq l - 1$, we have at most

$$\prod_{i=1}^r N_{i+1}^2 \lesssim \prod_{j=2}^l N_j^2$$

choices for $(\sigma_1, \dots, \sigma_r)$. □

Proposition 4.3. *Suppose $p \geq 3$ is an odd integer. We choose σ and b so that $0 < \sigma < 1$ is sufficiently close to 1 depending on p , and $1 > b > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$ depending on σ and p . Let T be small enough depending on b, σ and p . Then we can find a set $\Omega_T \subset \Omega$ and a positive number θ that only depends on σ, b and T , so that $\mathbb{P}(\Omega_T) \leq c_1 e^{-c_2 T^{-c_3}}$, and that the following holds: For any $t_0 \in \mathbb{R}$ and $\omega \in \Omega_T^c$, if for each $1 \leq j \leq p$, a function u_j on $[-T, T] \times \mathbb{R}^2$ is given by either*

$$u_j = e^{-i(t+t_0)\mathbf{H}} f(\omega), \tag{4-5}$$

or

$$\|u_j\|_{\mathcal{X}^{\sigma,b,T}} \lesssim 1, \tag{4-6}$$

then we have

$$\|u_1^- \cdots u_p^-\|_{\mathcal{X}^{\sigma,b-1,T}} \lesssim T^\theta. \tag{4-7}$$

Here all the constants will depend on σ, b and p .

Proof. In what follows, if an estimate holds for ω outside a set with measure ϵ , we simply say it holds “with exceptional probability ϵ ”. We will use various exponents q_j , and each of them will remain the

same throughout the proof. First we can use [Lemma 2.8](#) to estimate

$$\|u_1^- \cdots u_p^-\|_{\mathcal{X}^{\sigma, b-1, T}} \lesssim T^{2b-1} \|u_1^- \cdots u_p^-\|_{\mathcal{X}^{\sigma, 3b-2, T}},$$

since $-\frac{1}{2} < b - 1 < 3b - 2 < \frac{1}{2}$. Thus we only need to prove

$$\|u_1^- \cdots u_p^-\|_{\mathcal{X}^{\sigma, 3b-2, T}} \lesssim T^{1/2-b}, \tag{4-8}$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3})$. Recalling the Littlewood–Paley projections [\(1-8\)](#), we have

$$u = \sum_{N \geq 2} u_N, \tag{4-9}$$

where for simplicity we write $u_N = \Delta_N u$. Thus we only need to estimate the terms (note $(u_N)^- = (u^-)_N$ since the Littlewood–Paley projectors are real)

$$\prod_{j=1}^p (u_j^-)_{N_j},$$

where we have fixed a choice between u_j and \bar{u}_j , and between [\(4-5\)](#) and [\(4-6\)](#), for each u_j . Define

$$A = \{1 \leq j \leq p : u_j \text{ given by (4-5)}\}, \quad \text{and} \quad B = \{1 \leq j \leq p : u_j \text{ given by (4-6)}\}.$$

Let

$$\mathcal{A} = \left\{ (N_1, \dots, N_p) : N_j > 10^3 \sum_{i \neq j} N_i \text{ for some } j \in B \right\}. \tag{4-10}$$

We first consider the sum of terms with $(N_1, \dots, N_p) \in \mathcal{A}$, and rewrite it as

$$\sum_{j \in B} \sum_{(N_i)_{i \neq j}} \prod_{i \neq j} (u_i^-)_{N_i} \cdot \left(\sum_{N_j > 10^3 \sum_{i \neq j} N_i} (u_j^-)_{N_j} \right). \tag{4-11}$$

To bound this expression we only need to consider a fixed $j_0 \in B$, and without loss of generality, we may assume $j_0 = p$. For each (N_1, \dots, N_{p-1}) if we write

$$u_p^{hi} = \sum_{N_p > 10^3 \sum_{i=1}^{p-1} N_i} (u_p)_{N_p}, \tag{4-12}$$

then we only need to prove

$$\mathfrak{S} := \|(u_1^-)_{N_1} \cdots (u_{p-1}^-)_{N_{p-1}} (u_p^{hi})^-\|_{\mathcal{X}^{\sigma, 3b-2, T}} \lesssim T^{1/2-b} (\max_{j < p} N_j)^{-\theta} \tag{4-13}$$

for some $\theta > 0$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j < p} N_j)^{c_4})$ (note that when we take the sum over all (N_1, \dots, N_{p-1}) , we still get an expression $\leq c_1 e^{-c_2 T^{-c_3}}$).

To prove [\(4-13\)](#), we use [Propositions 2.5](#) and [2.7](#) to estimate (for simplicity, we shall omit the spacetime domain $[-T, T] \times \mathbb{R}^2$ in the following estimates, but one should keep in mind that we are working on a

very short time

$$\mathfrak{S} \lesssim \|(u_1)_{N_1}^- \cdots (u_{p-1})_{N_{p-1}}^- (u_p^{hi})^- \|_{L_t^{q_1} W_x^{\sigma, q_1}} \tag{4-14}$$

$$\begin{aligned} &\lesssim \|(u_p^{hi})^- \|_{L_t^{4q} W_x^{\sigma, 4}} \prod_{j=1}^{p-1} \|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} + \sum_{j=1}^{p-1} \|(u_p^{hi})^- \|_{L_{t,x}^4} \|(u_j)_{N_j}^- \|_{L_t^{q_2} W_x^{\sigma, q_2}} \prod_{j \neq i < p} \|(u_i)_{N_i}^- \|_{L_{t,x}^{q_2}} \\ &\lesssim \|(u_p^{hi})^- \|_{L_t^{4q} W_x^{\sigma, 4}} \prod_{j=1}^{p-1} \|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} + \sum_{j=1}^{p-1} N_j^\sigma \|(u_p^{hi})^- \|_{L_{t,x}^4} \prod_{i=1}^{p-1} \|(u_i)_{N_i}^- \|_{L_{t,x}^{q_2}} \end{aligned} \tag{4-15}$$

$$\lesssim \|u_p^-\|_{L_t^{4q} W_x^{\sigma, 4}} \prod_{j=1}^{p-1} \|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} \tag{4-16}$$

$$\lesssim \prod_{j=2}^p \|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}}, \tag{4-17}$$

where in (4-15) and (4-16) we have used [Corollary 2.3](#) (recall the definition of u_p^{hi}). In (4-17) we have used [Proposition 2.7](#) and the assumption that $p \in B$. For the parameters, we choose $q_1 > \frac{4}{3}$ and sufficiently close to $\frac{4}{3}$ depending on p , and $\frac{p-1}{q_2} = \frac{1}{q_1} - \frac{1}{4}$, and check that (4-14) indeed hold, provided b is sufficiently close to $\frac{1}{2}$, depending on q_1 (see [Proposition 2.7](#), with b there replaced by $3b - 1$).

Now we proceed to analyze the expression (4-17). Choose $1 \leq j \leq p - 1$ so that $N_j = \max_{i < p} N_i$. If $j \in B$, then from [Corollary 2.3](#) and [Proposition 2.7](#) we have

$$\|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} \lesssim N_j^{-\epsilon} \|u_j\|_{L_t^{q_2} W_x^{\epsilon, q_2}} \lesssim N_j^{-\epsilon} \|u_j\|_{\mathcal{X}^{\sigma, b, T}} \lesssim N_j^{-\epsilon}, \tag{4-18}$$

provided $\sigma - \epsilon > 1 - \frac{4}{q_2}$ (note $q_2 > 4$ from our choice of exponents above). This can be achieved if ϵ is small enough depending on q_2 , and σ is sufficiently close to 1 depending on q_2 and ϵ . If instead $j \in A$, then from [Corollary 2.3](#) we have

$$\|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} \lesssim N_j^{-\epsilon} \|u_j\|_{L_t^{q_2} W_x^{\epsilon, q_2}} = N_j^{-\epsilon} \|e^{-i(t+t_0)\mathbf{H}} f(\omega)\|_{L_t^{q_2} W_x^{\epsilon, q_2}}. \tag{4-19}$$

The norm in the last expression equals the $L_t^{q_2} W_x^{\epsilon, q_2}$ norm of $e^{-it\mathbf{H}} f(\omega)$ on the interval $[t_0 - T, t_0 + T]$. Since $T < 1$, we may expand this interval to an interval with length 2π . Since $e^{-it\mathbf{H}} f(\omega)$ has period 2π in t , we may replace the enlarged norm by the norm on $[-\pi, \pi]$. Then we could use [Corollary 3.5](#) to bound

$$N_j^{-\epsilon} \|e^{-i(t+t_0)\mathbf{H}} f(\omega)\|_{L_t^{q_2} W_x^{\epsilon, q_2}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_j^{-\frac{\epsilon}{2}} \tag{4-20}$$

for all t_0 , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_j^{c_4})$, provided $0 < \epsilon < \frac{2}{q_2}$. Therefore in each case we have

$$\|(u_j)_{N_j}^- \|_{L_{t,x}^{q_2}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_j^{-\theta}, \tag{4-21}$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_j^{c_4})$, for some $\theta > 0$.

Then we treat the terms with $i \neq j$. If $i \in B$, we can use [Proposition 2.7](#) to bound $\|(u_i)_{N_i}^-\|_{L_{t,x}^{q_2}} \lesssim 1$; if $i \in A$, we can use [Corollary 3.5](#) to bound

$$\|(u_i)_{N_i}^-\|_{L_{t,x}^{q_2}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_j^{\frac{\theta}{10p}}$$

for all t_0 , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_j^{c_4})$. Putting these together, we have shown

$$(4-17) \lesssim T^{1/2-b} (\max_{j < p} N_j)^{-\theta} \tag{4-22}$$

for some $\theta > 0$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j < p} N_j)^{c_4})$. This takes care of the sum of terms with $(N_1, \dots, N_p) \in \mathcal{A}$.

For $(N_1, \dots, N_p) \notin \mathcal{A}$, we are going to prove

$$J = \|v_1^- \cdots v_p^-\|_{\mathcal{X}^{\sigma,3b-2,T}} \lesssim T^{1/2-b} (\max_{j \geq 1} N_j)^{-\theta}, \tag{4-23}$$

where $v_j = (u_j)_{N_j}$, with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} (\max_{j \geq 1} N_j)^{c_4})$. This, together with the analysis above, clearly implies (4-7). Now without loss of generality, assume $N_1 = \max_{j \geq 1} N_j$. If $1 \in B$, then we have $N_1 \sim \max_{j \geq 2} N_j$. By switching the role of 1 and p in the argument above and replacing u_1^{hi} by v_1 (note v_1 also satisfy the estimates about u_1^{hi} that we would use), we can prove (4-22) with the role of 1 and p switched. Since $N_1 \sim \max_{j \geq 2} N_j$, this also proves (4-23).

Now we assume that $N_1 = \max_{j \geq 1} N_j$ and $1 \in A$. If $N_1 \lesssim N_{j_0}^{(1+\sigma)/(3\sigma-1)}$ (note this exponent is > 1) for some $j_0 \geq 2$, then we may assume $j_0 = 2$. Now use the same arguments as in (4-14) (but with different exponents), we have

$$J \lesssim \|v_1^- v_2^- \cdots v_p^-\|_{L_t^{q_1} W_x^{\sigma, q_1}} \tag{4-24}$$

$$\begin{aligned} &\lesssim (\|v_1^-\|_{L_t^{4q} W_x^{\sigma, 4}} \|v_2^-\|_{L_{t,x}^4} + \|v_1^-\|_{L_{t,x}^2} \|v_2^-\|_{L_t^{4q} W_x^{\sigma, 4}}) \prod_{j=3}^p \|v_j^-\|_{L_{t,x}^{q_4}} \\ &\quad + \sum_{j=3}^p \|v_1^-\|_{L_{t,x}^4} \|v_2^-\|_{L_{t,x}^4} \|v_j^-\|_{L_t^{q_4} W_x^{\sigma, q_4}} \prod_{3 \leq i \neq j} \|v_i^-\|_{L_{t,x}^{q_4}} \\ &\lesssim \left(\sum_{j=1}^p N_j^\sigma \right) \|v_1^-\|_{L_{t,x}^4} \|v_2^-\|_{L_{t,x}^4} \prod_{j=3}^p \|v_j^-\|_{L_{t,x}^{q_4}} \end{aligned} \tag{4-25}$$

$$\lesssim N_1^{\frac{1+\sigma}{4}} N_2^{\frac{1+\sigma}{4}} \|v_1^-\|_{L_{t,x}^4} \|v_2^-\|_{L_{t,x}^4} \prod_{j=3}^p \|v_j^-\|_{L_{t,x}^{q_4}} \tag{4-26}$$

$$\lesssim \|v_1^-\|_{L_t^{4q} W_x^{\frac{1+\sigma}{4}, 4}} \|v_2^-\|_{L_t^{4q} W_x^{\frac{1+\sigma}{4}, 4}} \prod_{j=3}^p \|v_j^-\|_{L_{t,x}^{q_4}}, \tag{4-27}$$

where $\frac{p-2}{q_4} = \frac{1}{q_1} - \frac{1}{2}$ and $q_4 > 4$. Here in (4-25) and (4-27) we have used [Corollary 2.3](#) and the fact that $v_j = (u_j)_{N_j}$, while in (4-26) we have used $N_j \lesssim N_1 \lesssim N_2^{(1+\sigma)/(3\sigma-1)}$ for all j .

Now we analyze the expression (4-27). If $2 \in B$, then by Corollary 2.3 and Proposition 2.7 we have (note $N_1 \lesssim N_2^2$ when $\sigma > \frac{3}{5}$)

$$\|v_2^-\|_{L_t^{4q}W_x^{\frac{1+\sigma}{4},4}} \lesssim N_1^{-\frac{1}{24}} \|v_2^-\|_{L_t^{6q}W_x^{\frac{2\sigma}{3},4}} \lesssim N_1^{-\frac{1}{24}} \|u_2\|_{\mathcal{X}^{\sigma,b,T}} \lesssim N_1^{-\frac{1}{24}}, \tag{4-28}$$

provided $\frac{2\sigma}{3} > \frac{1+\sigma}{4} + \frac{1}{12}$ and $\sigma > \frac{2\sigma}{3} + \frac{1}{6}$, which is true for $\sigma > \frac{4}{5}$. If $2 \in A$ (which is the case for 1), we can use the arguments from (4-19) to (4-20) to get

$$\|v_2^-\|_{L_t^{4q}W_x^{\frac{1+\sigma}{4},4}} \lesssim N_1^{-\frac{1-\sigma}{16}} \|u_2^-\|_{L_t^{4q}W_x^{\frac{3+\sigma}{8},4}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{-\frac{1-\sigma}{32}} \tag{4-29}$$

for all t_0 , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, thanks to Corollary 3.5, and the hypothesis $\sigma < 1$ (hence $\frac{3+\sigma}{8} < \frac{1}{2}$).

Then we treat the terms with $j \geq 3$. If $j \in B$, we can use Proposition 2.7 to bound $\|v_j^-\|_{L_{t,x}^{q_4}} \lesssim 1$; if $j \in A$, we can use Corollary 3.5 to bound

$$\|v_j^-\|_{L_{t,x}^{q_4}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{\frac{1-\sigma}{100p}}$$

for all t_0 , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. Putting these together, we have proved

$$(4-27) \lesssim T^{1/2-b} N_1^{-\theta}, \tag{4-30}$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$ for some $\theta > 0$. Thus we have proved (4-23) in this case.

In the final case, we assume that $N_1 > (10p)^3 (\max_{j \geq 2} N_j)^{\frac{1+\sigma}{3\sigma-1}}$, which in particular implies $N_1 > 10^3 \sum_{j \geq 1} N_j$, and that $1 \in A$. For each $j \in B$, by definition we can extend u_j to be a function on $\mathbb{R} \times \mathbb{R}^2$ (still denoted by u_j) with $\mathcal{X}^{\sigma,b}$ norm $\lesssim 1$. The relation $v_j = (u_j)_{N_j}$ also extends to $t \in \mathbb{R}$, giving an extension of v_j also. Choose ζ_0 smooth, supported on $[-2, 2]$ and equaling 1 on $[-1, 1]$ and define $\zeta(t) = \zeta_0(T^{-1}t)$. We are to prove

$$\|\zeta \cdot v_1^- \cdots v_p^-\|_{\mathcal{X}^{\sigma,3b-2}} \lesssim T^{1/2-b} N_1^{-\theta} \tag{4-31}$$

for the extended v_j , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. For a function w on $\mathbb{R} \times \mathbb{R}^2$ radial in x , we split $w = w_{ne} + w_{fa}$, with

$$\mathcal{F}_t \langle w_{ne}, e_k \rangle(\tau) = \chi_{\{|\tau+4k+2| \leq N_1^\gamma\}} \cdot \mathcal{F}_t \langle w, e_k \rangle(\tau), \tag{4-32}$$

and w_{fa} by replacing the \leq by $>$. We now split the product in (4-31) into fa and ne parts and estimate them separately.

We first estimate the *fa* part of product as (due to the presence of ζ , we can work on time interval $[-2T, 2T]$ in the time-Lebesgue norms below, thus gaining powers in T)

$$\|(\zeta \cdot v_1^- \cdots v_p^-)_{fa}\|_{\mathcal{X}^{\sigma, 3b-2}} \lesssim N_1^{-\gamma/36} \|\zeta \cdot v_1^- \cdots v_p^-\|_{\mathcal{X}^{\sigma, -4/9}} \tag{4-33}$$

$$\lesssim N_1^{-\gamma/36} \|\zeta \cdot v_1^- \cdots v_p^-\|_{L_t^{3/2} W_x^{\sigma, 3/2}} \tag{4-34}$$

$$\lesssim N_1^{\sigma-\gamma/36} \prod_i \|v_i^-\|_{L_{t,x}^{3p/2}}. \tag{4-35}$$

Here in (4-33) we have used the definition of the *fa*-projection and that b is close to $\frac{1}{2}$ (in particular, $b < \frac{1}{2} + \frac{1}{108}$); in (4-34) we have used Proposition 2.7; in (4-35) we have combined Corollary 2.3 and Proposition 2.5. Now for each i , if $i \in B$ then (provided σ is close to 1 depending on p)

$$\|v_i^-\|_{L_{t,x}^{3p/2}} \lesssim \|v_i\|_{\mathcal{X}^{\sigma, b}} \lesssim 1.$$

If $i \in A$ (such as $i = 1$) we have

$$\|v_i^-\|_{L_{t,x}^{3p/2}} \lesssim T^{\frac{1}{10p}(\frac{1}{2}-b)} N_1^{\frac{1}{p}}$$

for all t_0 , with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$. Therefore, we have (4-35) $\lesssim T^{1/2-b} N_1^{-\theta}$ with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, provided $\gamma > 108$.

Now we estimate the *ne* part of the product. Choose v_0 so that $\|v_0\|_{\mathcal{X}^{0, 2-3b}} \lesssim 1$. Since we are taking the *ne* part, we may assume $v_0 = v_{0,ne}$. The aim is to estimate $|\mathfrak{J}|$ (recall H is self-adjoint), where

$$\mathfrak{J} = \int_{\mathbb{R} \times \mathbb{R}^2} v_1^- \cdots v_p^- \cdot (\zeta H^{\sigma/2} \bar{v}_0). \tag{4-36}$$

We use Lemma 2.6 to write down

$$v_j(x, t) = \int_{\mathbb{R}} \phi_j(\lambda_j) e^{i\lambda_j t} \sum_k a_{\lambda_j}^j(k) e^{-i(4k+2)t} e_k(x) d\lambda_j \tag{4-37}$$

for $j \in B \cup \{0\}$, where the parameters satisfy

$$\sum_k |a_{\lambda_0}^0(k)|^2 \lesssim 1 \tag{4-38}$$

for each λ_0 . Since $v_0 = v_{0,ne}$, we also have $\|\phi_0\|_{L^1} \lesssim N_1^{3\gamma(b-\frac{1}{2})}$. For $j \in B$, since $v_j = (u_j)_{N_j}$, we know $a_{\lambda_j}^j(n_j) = 0$ unless $\frac{1}{10} \leq (4n_j + 2)/N_j^2 \leq 10$, and hence

$$\sum_{4n_j+2 \sim N_j^2} |a_{\lambda_j}^j(k)|^2 \lesssim N_j^{-2\sigma}. \tag{4-39}$$

Also since $b > \frac{1}{2}$, we have $\|\phi_j\|_{L^1} \lesssim 1$.

For the sake of convenience, in the following proof, we shall use $v^\sim(n, \tau)$ to denote $\mathcal{F}_t\langle v, e_n \rangle(\tau)$. Thus from (4-37) we have

$$v_j^\sim(n_j, \tau_j) = (2\pi)^{1/2} a_{\tau_j+4n_j+2}^j(n_j) \phi_j(\tau_j + 4n_j + 2) \tag{4-40}$$

for $j \in B$. If $j \in A$ we have

$$v_j^\sim(n_j, \tau_j) = (2\pi)^{1/2} e^{-i(4n_j+2)t_0} \frac{\theta_j(n_j) g_{n_j}(\omega)}{\sqrt{4n_j + 2}} \delta(\tau_j + 4n_j + 2), \tag{4-41}$$

where

$$\theta_j(n_j) = \eta\left(\frac{2(4n_j + 2)}{N_j^2}\right) - \eta\left(\frac{4(4n_j + 2)}{N_j^2}\right).$$

Clearly $|\theta_j| \leq 2$, and $\theta_j \neq 0$ only when $\frac{1}{10} \leq (4n_j + 2)/N_j^2 \leq 10$ (note we have fixed N_j). Finally, for $j = 0$ we have (we may assume ζ is real)

$$(\zeta \mathbf{H}^{\sigma/2} v_0)^\sim(n_0, \tau_0) = (4n_0 + 2)^{\sigma/2} \cdot \int_{\mathbb{R}} a_{\varrho_0+4n_0+2}^0(n_0) \phi_0(\varrho_0 + 4n_0 + 2) \hat{\zeta}(\tau_0 - \varrho_0) d\varrho_0. \tag{4-42}$$

We write $\gamma_j = v_j^\sim$ for $j \geq 1$, and $\gamma_0 = (\zeta \mathbf{H}^{\sigma/2} v_0)^\sim$. From the rules of Fourier transform and orthogonality of e_k , we have

$$\mathfrak{J} = (2\pi)^{-(p-2)/2} \sum_{n_1, \dots, n_p, n_0} \kappa_{n_1, \dots, n_p}^{n_0} \int_{\mathbb{D}} \prod_{j=0}^p (\gamma_j(n_j, \tau_j))^- d\tau_1 \cdots d\tau_p, \tag{4-43}$$

where

$$\kappa_{n_1, \dots, n_p}^{n_0} = \int_{\mathbb{R}^2} e_{n_1}(x) \cdots e_{n_p}(x) e_{n_0}(x) dx, \tag{4-44}$$

and

$$\mathbb{D} = \left\{ (\tau_1, \dots, \tau_p, \tau_0) : \tau_0 = \sum_{j=1}^p \epsilon_j \tau_j \right\}, \tag{4-45}$$

with $\epsilon_j = \pm 1$ depending on the choice of v_j or \bar{v}_j . We notice that $\epsilon_j = 1$ if and only if the corresponding γ_j^- equals γ_j . Now plug in (4-40), (4-41), and (4-42), and use the change of variables $\lambda_j = \tau_j + 4n_j + 2$ for $j \in B$, $\lambda_0 = \varrho_0 + 4n_0 + 2$; we get

$$\begin{aligned} \mathfrak{J} = 2\pi \sum_{n_1, \dots, n_p, n_0} \kappa_{n_1, \dots, n_p}^{n_0} \int \prod_{j \in B \cup \{0\}} d\lambda_j \prod_{j \in B} \phi_j(\lambda_j) a_{\lambda_j}^j(n_j)^- \prod_{j \in A} \frac{\theta_j(n_j) g_{n_j}^-(\omega)}{\sqrt{4n_j + 2}} \cdot a_{\lambda_0}^0(n_0)^- \phi_0(\lambda_0) \\ \times \hat{\zeta} \left(\sum_{j \in B} \epsilon_j \lambda_j - \lambda_0 - \sum_{j=1}^p \epsilon_j (4n_j + 2) + (4n_0 + 2) \right)^- \\ \times (4n_0 + 2)^{\sigma/2} \exp \left(-it_0 \sum_{j \in A} (4n_j + 2) \epsilon_j \right). \end{aligned} \tag{4-46}$$

Here the terms corresponding to $j \in A$ are delta functions and have already been incorporated in the final expression. Letting $\varrho = (4n_0 + 2) - \sum_{j=1}^p \epsilon_j(4n_j + 2)$, we can further reduce the expression to

$$\begin{aligned} \mathfrak{J} &= (2\pi)^{p+\frac{1}{2}} \sum_{\varrho \in \mathbb{Z}} \int \prod_{j \in B \cup \{0\}} \phi_j(\lambda_j) d\lambda_j \cdot \hat{\zeta} \left(\sum_{j \in B} \epsilon_j \lambda_j - \lambda_0 + \varrho \right)^{-} \\ &\quad \times \sum_{\mathbb{S}_\varrho} \kappa_{n_1, \dots, n_p}^{n_0} (4n_0 + 2)^{\sigma/2} \prod_{j \in B \cup \{0\}} a_{\lambda_j}^j(n_j)^{-} \prod_{j \in A} \frac{\theta_j(n_j) g_{n_j}^-(\omega)}{\sqrt{4n_j + 2}} \\ &\quad \times \exp \left(-it_0 \sum_{j \in A} (4n_j + 2) \epsilon_j \right), \\ \mathbb{S}_\varrho &= \left\{ (n_0, \dots, n_p) : \frac{1}{10} \leq \frac{4n_j + 2}{N_j^2} \leq 10 \ (j \geq 1), \ (4n_0 + 2) - \sum_{j=1}^p \epsilon_j(4n_j + 2) = \varrho \right\}. \end{aligned} \tag{4-47}$$

Noticing that $\hat{\zeta} = T\hat{\zeta}_0(T \cdot)$, and that $\hat{\zeta}_0$ is a Schwartz function, we have

$$\sum_{\varrho \in \mathbb{Z}} |\hat{\zeta}(\lambda + \varrho)| \lesssim \sum_{\varrho \in \mathbb{Z}} T \langle T(\lambda + \varrho) \rangle^{-2} \lesssim 1 \tag{4-48}$$

for all $\lambda \in [0, 1]$, and by periodicity, for all $\lambda \in \mathbb{R}$. Therefore

$$\sum_{\varrho \in \mathbb{Z}} \int \prod_{j \in B \cup \{0\}} |\phi_j(\lambda_j)| d\lambda_j \cdot \hat{\zeta} \left| \left(\sum_{j \in B} \epsilon_j \lambda_j - \lambda_0 + \varrho \right) \right| \lesssim N_1^{3\gamma(b-\frac{1}{2})}. \tag{4-49}$$

Since we choose b close enough to $\frac{1}{2}$ depending on σ and p , and γ does not have any dependence on b whatsoever (we may simply take $\gamma = 200$), (4-31) will follow if

$$\left| \sum_{\mathbb{S}_\varrho} \kappa_{n_1, \dots, n_p}^{n_0} (4n_0 + 2)^{\sigma/2} \times \prod_{j \in B \cup \{0\}} a_{\lambda_j}^j(n_j)^{-} \prod_{j \in A} \frac{\theta_j(n_j) g_{n_j}^-(\omega)}{\sqrt{4n_j + 2}} \times \exp \left(-it_0 \sum_{j \in A} (4n_j + 2) \epsilon_j \right) \right| \lesssim T^{1/2-b} N_1^{-\delta} \tag{4-50}$$

for all possible choices of $t_0 \in \mathbb{R}$, $\varrho \in \mathbb{Z}$, $\lambda_j \in \mathbb{R} (j \in B \cup \{0\})$, $\{a_{\lambda_j}^j(k)\}$ satisfying (4-38) and (4-39), with $\delta > 0$ depending on σ and p , but not on b .

Next, by Cauchy–Schwartz in the sum with respect to n_0 , we can further estimate the left side of (4-50) by

$$\begin{aligned} &\left(\sum_{n_0} (4n_0 + 2)^\sigma \times \left| \sum_{\mathbb{S}_{\varrho, n_0}} \kappa_{n_1, \dots, n_p}^{n_0} \prod_{j \in B} b_j(n_j)^{-} \prod_{j \in A} \frac{\theta_j(n_j) g_{n_j}^-(\omega)}{\sqrt{4n_j + 2}} \right. \right. \\ &\quad \left. \left. \times \exp \left(-it_0 \sum_{j \in A} (4n_j + 2) \epsilon_j \right) \right|^2 \right)^{1/2}, \end{aligned} \tag{4-51}$$

where $\mathbb{S}_{\varrho, n_0} = \{(n_1, \dots, n_p) : (n_0, \dots, n_p) \in \mathbb{S}_\varrho\}$, and $b_j(k) = a_{\lambda_j}^j(k)$.

Concerning the inner sum of (4-51), we have (recall that $\frac{1}{10} \leq (4n_j + 2)/N_j^2 \leq 10$ for each $1 \leq j \leq p$)

$$\left| \sum_{\mathbb{S}_{\varrho, n_0}} \kappa_{n_1, \dots, n_p}^{n_0} \prod_{j \in B} b_j(n_j)^{-} \prod_{j \in A} \frac{\theta_j(n_j) g_{n_j}^-(\omega)}{\sqrt{4n_j + 2}} \cdot e^{-it_0 \sum_{j \in A} (4n_j + 2)\epsilon_j} \right| \tag{4-52}$$

$$\lesssim \sum_{(n_j)_{j \in B}} \left| \sum_{\Theta} \tau_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} g_{n_j}^-(\omega) \right| \prod_{j \in B} |b_j(n_j)| \tag{4-53}$$

$$\lesssim \sup_{(n_j)_{j \in B}} \left| \sum_{\Theta} \tau_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} g_{n_j}^-(\omega) \right| \prod_{j \in B} \sum_{4n_j + 2 \sim N_j^2} |b_j(n_j)|$$

$$\lesssim \sup_{(n_j)_{j \in B}} \left| \sum_{\Theta} \tau_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} g_{n_j}^-(\omega) \right| \prod_{j \in B} (N_j^2 N_j^{-2\sigma})^{1/2} \tag{4-54}$$

$$\lesssim \sup_{(n_j)_{j \in B}} \left| \sum_{\Theta} \tau_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} g_{n_j}^-(\omega) \right| \prod_{j \in B} N_j^{1-\sigma},$$

where in (4-53) we write $\Theta = \{(n_j)_{j \in A} : (n_1, \dots, n_p) \in \mathbb{S}_{\varrho, n_0}\}$ for fixed $(n_j)_{j \in B}$, and $\tau_{n_1, \dots, n_p}^{n_0} = \kappa_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} \theta_j(n_j) (4n_j + 2)^{-1/2}$. One should notice that for all $(n_j)_{j \in A} \in \Theta$, by definition the expression $\exp(-it_0 \sum_{j \in A} (4n_j + 2)\epsilon_j)$ is a fixed constant with absolute value 1, which can be extracted. In (4-54) we have used Cauchy–Schwartz and (4-39).

Let us fix ϱ and n_0 , and $(n_j)_{j \in B}$. We also assume $|4n_0 + 2 - \varrho| \lesssim N_1^2$ (otherwise $\mathbb{S}_{\varrho, n_0}$ would be empty). Since the set Θ has the form of Ξ in (4-2) and $N_1 > 10^3 \sum_{j \in A - \{1\}} N_j$, we can use Proposition 4.2 to get

$$\left| \sum_{\Theta} \tau_{n_1, \dots, n_p}^{n_0} \prod_{j \in A} g_{n_j}^-(\omega) \right| \leq K \prod_{j \in A - \{1\}} N_j \cdot \sup_{\Theta} |\tau_{n_1, \dots, n_p}^{n_0}|, \tag{4-55}$$

with exceptional probability $\leq c_1 \exp(-c_2 K^{c_3})$. We choose $K = T^{1/2-b} N_1^{(1-\sigma)/200} (4n_0 + 2)^{(1-\sigma)/400}$, then the corresponding exceptional probability is $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4} (4n_0 + 2)^{c_5})$. If we add up these probabilities with respect to *all* possible choices of ϱ and $(n_j)_{j \in B \cup \{0\}}$, we still get an expression $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$ (there are $\lesssim N_1^2$ choices for each $n_j (j \in B)$, and for fixed n_0 , there are $\lesssim N_1^2$ choices of ϱ). Therefore with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3} N_1^{c_4})$, we have

$$(4-51) \lesssim T^{1/2-b} N_1^{\frac{1-\sigma}{200}} \prod_{j \in B} N_j^{1-\sigma} \prod_{j \in A - \{1\}} N_j \left(\sum_{n_0} (4n_0 + 2)^{\sigma + \frac{1-\sigma}{200}} \sup_{\mathbb{S}_{\mu, n_0}} |\tau_{n_1, \dots, n_p}^{n_0}|^2 \right)^{1/2} \tag{4-56}$$

$$\lesssim T^{1/2-b} N_1^{\frac{1-\sigma}{200}-1} \prod_{j \in B} N_j^{1-\sigma} \left(\sum_{n_0} (4n_0 + 2)^{\sigma + \frac{1-\sigma}{200}} \sup_{\mathbb{S}_{\mu, n_0}} |\kappa_{n_1, \dots, n_p}^{n_0}|^2 \right)^{1/2}. \tag{4-57}$$

To complete the proof of Proposition 4.3, we are going to estimate $\kappa_{n_1, \dots, n_p}^{n_0}$. Let $\nu_{(0)} \geq \dots \geq \nu_{(p)}$ be the nonincreasing permutation of $\nu_j = 4n_j + 2$ (where $0 \leq j \leq p$). If $\nu_0 \geq N_1^{2(1+(1-\sigma)/200)}$, from Lemma 3.2 we have $|\kappa_{n_1, \dots, n_p}^{n_0}| \lesssim \nu_0^{-100}$. If $\nu_0 < N_1^{2(1+(1-\sigma)/200)}$, since $N_1 \gtrsim (\max_{j \geq 2} N_j)^{(\sigma+1)/(3\sigma-1)}$, we see that if $\nu_0 \leq \max_{j \geq 2} \nu_j$, then $\nu_1 \gtrsim \max_{j \neq 1} \nu_j^{(\sigma+1)/(3\sigma-1)}$ and $|\kappa_{n_1, \dots, n_p}^{n_0}| \lesssim N_1^{-100}$; if $\nu_0 > \max_{j \geq 2} \nu_j$,

then $v_{(2)} \geq \max_{j \geq 2} v_j$ and from [Lemma 3.2](#) we have

$$|\kappa_{n_1, \dots, n_p}^{n_0}| \lesssim v_{(0)}^{-\frac{1}{2}} v_{(2)}^{-\frac{1}{4}} \log v_{(0)} \lesssim N_1^{-1} (\max_{j \geq 2} N_j)^{-\frac{1}{2}} \log N_1. \tag{4-58}$$

Therefore we have

$$\begin{aligned} (4-57) \lesssim & T^{1/2-b} N_1^{\frac{1-\sigma}{200}-1} \prod_{j \in B} N_j^{1-\sigma} \\ & \times \left(\sum_{v_0 < N_1^{2(1+\frac{1-\sigma}{200})}} (N_1)^{2(\sigma+\frac{1-\sigma}{200})(1+\frac{1-\sigma}{200})} N_1^{-2} (\max_{j \geq 2} N_j)^{-1} \log^2 N_1 + \sum_{v_0 \geq N_1^{2(1+\frac{1-\sigma}{200})}} (4n_0 + 2)^{-198} \right)^{1/2} \\ \lesssim & T^{1/2-b} N_1^{-\theta_0} \log N_1 \cdot (\max_{j \geq 2} N_j)^{-\frac{1}{2}} \prod_{j \in B} N_j^{1-\sigma} \\ \lesssim & T^{1/2-b} N_1^{-\frac{\theta_0}{2}} (\max_{j \geq 2} N_j)^{-\frac{1}{2}} \prod_{j \in B} N_j^{1-\sigma}, \end{aligned}$$

where

$$\theta_0 = 1 - \frac{1-\sigma}{100} - \sigma - \frac{(1+\sigma)(1-\sigma)}{200} - \frac{(1-\sigma)^2}{40000} > \frac{1-\sigma}{2} > 0. \tag{4-59}$$

Finally, since $1 \in A$, we have

$$(\max_{j \geq 2} N_j)^{-\frac{1}{2}} \prod_{j \in B} N_j^{1-\sigma} \lesssim (\max_{j \geq 2} N_j)^{-\frac{1}{2}+(p-1)(1-\sigma)} \lesssim 1, \tag{4-60}$$

provided $\sigma > 1 - 1/(2(p-1))$.

Having considered all the different cases, we have now finished the proof of [Proposition 4.3](#). □

From now on we will fix σ and b as stated in [Proposition 4.3](#). We have an easy corollary:

Corollary 4.4. *There exist some $\theta > 0$ and $T_0 > 0$, such that the following holds: For all $0 < T < T_0$, there exists a set $\Omega_T \subset \Omega$ such that $\mathbb{P}(\Omega_T) \leq c_1 e^{-c_2 T^{-c_3}}$ and for all $\omega \notin \Omega_T$, the mapping*

$$u \mapsto e^{-it\mathbf{H}} f(\omega) \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|u(s)|^{p-1} u(s)) ds \tag{4-61}$$

is a contraction mapping from the affine ball

$$e^{-it\mathbf{H}} f(\omega) + \{v : \|v\|_{\mathcal{X}^{\sigma,b,T}} \leq T^\theta\} \tag{4-62}$$

to itself.

Proof. Suppose $u = e^{-it\mathbf{H}} f(\omega) + v$, where $\|v\|_{\mathcal{X}^{\sigma,b,T}} \leq T^\theta \leq 1$. From [Proposition 2.9](#) we have

$$\begin{aligned} \mathfrak{M} &:= \left\| \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|u(s)|^{p-1} u(s)) ds \right\|_{\mathcal{X}^{\sigma,b,T}} \\ &\lesssim \| |u|^{p-1} u \|_{\mathcal{X}^{\sigma,b-1,T}} \\ &= \| (e^{-it\mathbf{H}} f(\omega) + v)^{\frac{p+1}{2}} \cdot \overline{(e^{-it\mathbf{H}} f(\omega) + v)^{\frac{p-1}{2}}} \|_{\mathcal{X}^{\sigma,b-1,T}}. \end{aligned}$$

If we expand the product, then each term has the form as in [Proposition 4.3](#) (namely, $u_1^- \cdots u_p^-$ with each u_j either equal to $e^{-it\mathbf{H}} f(\omega)$ or has $\mathcal{X}^{\sigma,b,T}$ norm $\lesssim 1$); thus we have $\mathfrak{M} \lesssim T^{\theta_0}$ for some θ_0 depending only on σ, b and p ; thus if we choose $\theta < \theta_0$ and T_0 small enough, then the mapping does map the affine ball to itself.

In addition, if $u_i = e^{-it\mathbf{H}} f + v_i$ with $\|v_i\|_{\mathcal{X}^{\sigma,b,T}} \leq T^\theta$ for $i \in \{1, 2\}$, then

$$\begin{aligned} \mathfrak{D} &:= \left\| \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|u_1(s)|^{p-1}u_1(s) - |u_2(s)|^{p-1}u_2(s)) \, ds \right\|_{\mathcal{X}^{\sigma,b,T}} \\ &\lesssim \left\| |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 \right\|_{\mathcal{X}^{\sigma,b-1,T}} \\ &\lesssim \sum_{\mathbb{F}} \|(u_1 - u_2)^- \prod_{k=1}^{p-1} u_{j_k}^- \|_{\mathcal{X}^{\sigma,b-1,T}}, \end{aligned}$$

where \mathbb{F} is some finite set, and each $j_k \in \{1, 2\}$. Since $u_1 - u_2 = v_1 - v_2 \in \mathcal{X}^{\sigma,b,T}$, and each u_j is the sum of two terms, one being $e^{-it\mathbf{H}} f(\omega)$, the other having $\mathcal{X}^{\sigma,b,T}$ norm $\lesssim 1$, we can use [Proposition 4.3](#) to estimate $\mathfrak{D} \lesssim T^{\theta_0} \|v_1 - v_2\|_{\mathcal{X}^{\sigma,b,T}}$ for all $\omega \notin \Omega_T$. Thus the result follows if we choose T small enough. \square

5. Local well-posedness results

In proving local in time results, we will not care about the \pm sign in [\(1-10\)](#). First we define the truncated Cauchy problem

$$\begin{cases} i\partial_t u + (\Delta - |x|^2)u = (\pm|u|^{p-1}u)_{2^k}^\circ \\ u(0) = f_{2^k}^\circ(\omega) \end{cases} \tag{5-1}$$

for each $k \geq 1$. When $k = \infty$, we understand that $v_{2^\infty}^\circ = v$, so this is just the original equation [\(1-10\)](#). If $k < \infty$, we solve [\(5-1\)](#) in the finite-dimensional space V_{2^k} . We will consider two cases depending on whether $p \geq 3$ odd or $1 < p < 3$.

5.1. The algebraic case. Here we assume $p \geq 3$ is an odd integer, so we can use the estimates is [Section 4](#).

Proposition 5.1. *Suppose $T > 0$ is sufficiently small. There exists a set Ω_T (possibly different from the one in [Proposition 4.3](#)), such that $\mathbb{P}(\Omega_T) \leq c_1 \exp(-c_2 T^{-c_3})$, and when $\omega \notin \Omega_T$, for each $1 \leq k \leq \infty$, [\(5-1\)](#) has a unique solution*

$$u \in e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + \mathcal{X}^{\sigma,b,T} \tag{5-2}$$

on $[-T, T]$, satisfying

$$\|u - e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}} \leq T^\theta. \tag{5-3}$$

Proof. When $k = \infty$, the existence and uniqueness directly follows from [4.4](#) via Picard iteration. Now we assume $1 \leq k < \infty$, then the equation [\(5-1\)](#) is just an ODE, so the solution is unique, and exists until its norm approaches infinity. Thus we only need to obtain the control on each of these solutions, uniformly in k . To this end we need the following modification of [Proposition 4.3](#).

Lemma 5.2. *For each T sufficiently small, we can find a set (still denoted by Ω_T), such that $\mathbb{P}(\Omega_T) \leq c_1 \exp(-c_2 T^{-c_3})$, and in Proposition 4.3, if one replaces some u_j by any $(u_j)_{2^{k_j}}^\circ$ or $(u_j)_{2^{k_j}}^\perp$, the result still holds true. Moreover, if there is at least one $(u_j)_{k_j}^\perp$, then the left side of (4-7) tends to zero (uniformly in all choices of u_j) as this $k_j \rightarrow \infty$.*

Proof. We use the notations as in Proposition 4.3. Noting that the projections $u_{2^k}^\circ$ and $u_{2^k}^\perp$ are uniformly bounded on $X^{\sigma,b,T}$, we may assume the modification is only for $j \in A$. Since $f_{2^k}^\circ(\omega) = f(\omega) - f_{2^k}^\perp(\omega)$ and the result is true when all terms are still u_j , we may assume each term is either u_j or $(u_j)_{2^{k_j}}^\perp$, with at least one $(u_j)_{2^{k_j}}^\perp$.

For each (k_j) , we follow exactly the proof of Proposition 4.3. Suppose $L = \max_j 2^{k_j}$; then in the dyadic decomposition we only need to consider the terms $\max_{j \in A} N_j \gtrsim L$ (for example, if $(N_1, \dots, N_p) \in \mathcal{A}$ with the largest being N_1 , then $\max_{j \geq 2} N_j \gtrsim L$; otherwise we have $\max_j N_j \gtrsim L$). On the other hand, all the probabilistic Lebesgue/Sobolev estimates of $f(\omega)$ we used in Proposition 4.3 come from Corollary 3.5; thus they also hold for $f_{2^k}^\perp(\omega) = f(\omega) - f_{2^k}^\circ(\omega)$ uniformly in k . As for the multilinear estimates of Gaussians (Proposition 4.2), they indeed hold for fixed k_j , because fixing k_j (and replacing $f(\omega)$ by $f(\omega)_{2^{k_j}}^\circ$) corresponds to adding constraints $n_j \leq 2^{k_j}$ in the set Ξ in (4-2), which does not affect the estimates in (4-4) (which is based on upper bounds of the cardinals of some sets). Therefore for fixed k_j , the estimates about each individual term (including the “grouped” terms in \mathcal{A}) in the proof of Proposition 4.3 still hold, with constants independent of k_j . Therefore, we have

$$\|\text{Modified}(u_1^- \cdots u_p^-)\|_{X^{\sigma,b,T}} \lesssim \sum_{\max_j N_j \gtrsim L} T^\theta (\max_j N_j)^{-\theta} \lesssim T^\theta L^{-\theta/2},$$

with exceptional probability not exceeding

$$\sum_{\max_j N_j \gtrsim L} c_1 \exp(-c_2 T^{-c_3} (\max_j N_j)^{c_4}) \leq c_5 \exp(-c_6 T^{-c_7} L^{c_8}),$$

which implies

$$\|\text{Modified}(u_1^- \cdots u_p^-)\|_{X^{\sigma,b,T}} \lesssim T^\theta (\max_j 2^{k_j})^{-\theta/2},$$

for all possible choices of k_j , with exceptional probability not exceeding

$$\sum_{(k_j)} c_5 \exp(-c_6 T^{-c_7} (\max_j 2^{k_j})^{c_8}) \lesssim c_9 \exp(-c_{10} T^{-c_{11}}).$$

If we choose this final exceptional set as our Ω_T , we easily see that all requirements are satisfied. □

Remark 5.3. In Proposition 4.3 and Lemma 5.2, the estimates still hold when the $\mathcal{X}^{\sigma,b,T}$ norm is replaced by $\mathcal{X}^{\sigma,b,I}$ (but with T^θ on the right side of (4-7) unchanged), for any interval $I \subset [-T, T]$, and ω outside a single Ω_T . One can check the proof that all estimates do not become worse with $[-T, T]$ replaced by I . In particular we can get a contraction mapping as in Corollary 4.4 for interval $[-T, 0]$ or $[-T_1, T_1]$ for $T_1 \leq T$.

Using Lemma 5.2, we can now proceed with the proof of Proposition 5.1. Suppose for some k that $u = e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v$ is a maximal solution to (5-1) (strictly speaking the T below should be another T' denoting the lifespan of u , but we will ignore this, in view of Remark 5.3). Then outside the Ω_T constructed in Lemma 5.2 we have

$$\begin{aligned} \|v\|_{\mathcal{X}^{\sigma,b,T}} &= \left\| \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|u(s)|^{p-1} u(s))_{2^k}^\circ ds \right\|_{\mathcal{X}^{\sigma,b,T}} \\ &\lesssim \| |u|^{p-1} u \|_{\mathcal{X}^{\sigma,b-1,T}} \\ &= \left\| (e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v)^{\frac{p+1}{2}} \cdot \overline{(e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v)^{\frac{p-1}{2}}} \right\|_{\mathcal{X}^{\sigma,b-1,T}}. \end{aligned}$$

Each term in the expansion of the final product has the form as in Lemma 5.2 (namely $\prod_j (u_j^-)_{2^{k_j}}^\circ$ with $1 \leq k_j \leq \infty$, and each u_j either equal to $e^{-it\mathbf{H}} f(\omega)$ or has $\mathcal{X}^{\sigma,b,T}$ norm $\lesssim \|v\|_{\mathcal{X}^{\sigma,b,T}}$). Therefore for some $\theta > 0$ we get

$$\|v\|_{\mathcal{X}^{\sigma,b,T}} \lesssim T^\theta (1 + \|v\|_{\mathcal{X}^{\sigma,b,T}})^p;$$

since $v \in \mathcal{X}^{\sigma,b,T}$ and $v(0) = 0$, we know $\|v\|_{\mathcal{X}^{\sigma,b,t}} \rightarrow 0$ as $t \rightarrow 0$. The local norm is continuous in t ; thus we can use a bootstrap argument to get $\|v\|_{\mathcal{X}^{\sigma,b,T}} \leq T^{\theta/2}$. Note this also works for the original equation, showing that (5-3) holds for the solution of (1-10) with any k . The uniqueness of (1-10) now follows from Corollary 4.4. □

5.2. The subcubic case. Here we assume $1 < p < 3$, and we do not need any multilinear estimate to solve the local problem.

Proposition 5.4. *Suppose $T > 0$ is sufficiently small. There exists a set Ω_T (possibly different from the one in Proposition 4.3), such that $\mathbb{P}(\Omega_T) \leq c_1 \exp(-c_2 T^{-c_3})$, and when $\omega \notin \Omega_T$, for each $1 \leq k \leq \infty$, (5-1) has a unique solution*

$$u \in e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + \mathcal{X}^{0,b,T} \tag{5-4}$$

on $[-T, T]$, satisfying

$$\|u - e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{0,b,T}} \leq T^\theta. \tag{5-5}$$

Proof. The proof here is almost the same as Proposition 5.1. In fact, once we can obtain

$$\| |e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v|^{p-1} (e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v) \|_{\mathcal{X}^{0,b-1,T}} \lesssim T^\theta (1 + \|v\|_{\mathcal{X}^{0,b,T}}^p) \tag{5-6}$$

and

$$\| |u|^{p-1} u - |u'|^{p-1} u' \|_{\mathcal{X}^{0,b-1,T}} \lesssim T^\theta \|v - v'\|_{\mathcal{X}^{0,b,T}} \cdot (1 + \|v\|_{\mathcal{X}^{0,b,T}} + \|v'\|_{\mathcal{X}^{0,b,T}})^{p-1} \tag{5-7}$$

for all $1 \leq k \leq \infty$ and $\omega \notin \Omega_T$, where $u = e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v$ and $u' = e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) + v'$, we can use Proposition 2.9 and argue as in the proof of Corollary 4.4 to show that for $\omega \notin \Omega_T$,

$$u \mapsto e^{-it\mathbf{H}} f(\omega) \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|u(s)|^{p-1} u(s)) ds$$

is a contraction mapping from

$$e^{-it\mathbf{H}} f(\omega) + \{v : \|v\|_{\mathcal{X}^{0,b,T}} \leq T^\theta\}$$

to itself, for some $\theta > 0$. Also we will have the same estimates on solutions to (5-1) as in Proposition 5.1, which is enough for the proof.

To prove (5-7), we simply compute (again we omit the time domain $[-T, T]$ here)

$$\begin{aligned} & \| |u|^{p-1}u - |u'|^{p-1}u' \|_{\mathcal{X}^{0,b-1,T}} \\ & \lesssim T^{2b-1} \| (u - u') (|u| + |u'|)^{p-1} \|_{L_{t,x}^q} \end{aligned} \tag{5-8}$$

$$\lesssim T^{2b-1} \| v - v' \|_{L_{t,x}^{pq}} \cdot (\|u\|_{L_{t,x}^{pq}} + \|u'\|_{L_{t,x}^{pq}})^{p-1} \tag{5-9}$$

$$\lesssim T^{2b-1} \| v - v' \|_{\mathcal{X}^{0,b,T}} (\|v\|_{\mathcal{X}^{0,b,T}} + \|v'\|_{\mathcal{X}^{0,b,T}} + \|e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{L_{t,x}^{pq}})^{p-1} \tag{5-10}$$

$$\lesssim T^{b-\frac{1}{2}} \| v - v' \|_{\mathcal{X}^{0,b,T}} \cdot (\|v\|_{\mathcal{X}^{0,b,T}} + \|v'\|_{\mathcal{X}^{0,b,T}} + 1)^{p-1}, \tag{5-11}$$

outside Ω_T , where $\mathbb{P}(\Omega_T) \leq c_1 \exp(-c_2 T^{-c_3})$. In (5-8) we have used Proposition 2.7 and Lemma 2.8, and required $\frac{1}{2} < b < \frac{2}{3}$, $2 > q > \frac{2}{3-3b}$. In (5-9) we have used Hölder and required $1 < pq < \infty$. In (5-10) we have used (Hölder in time and) Proposition 2.7 and required $2 < pq < 4$. In (5-11) we have used Corollary 3.5 to bound

$$\| e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) \|_{L_t^{r_2} L_x^{q_2}} \lesssim T^{-\frac{2b-1}{100p}},$$

with exceptional probability $\leq c_1 \exp(-c_2 T^{-c_3})$. Therefore, we may choose q so that $\frac{4}{3} < q < 2$ and $2 < pq < 4$ (such q exists because $1 < p < 3$). Then we may choose $\frac{1}{2} < b < 1 - \frac{2}{3q}$, and see that all the requirements indeed hold. This completes the proof of (5-7).

The estimate (5-6) follows from the same choice of exponents and similar arguments. The only difference is that we will have a term $\| e^{-it\mathbf{H}} f_{2^k}^\circ(\omega) \|_{L_t^{r_1} L_x^{q_1}}$, which is fine as long as $2 < q_1 < \infty$. \square

5.3. Approximating by ODEs. Here we will prove that almost surely, uniform global bounds on the solutions to the truncated equations (5-1) for infinitely many $k < \infty$ implies the global existence and uniqueness for the original equation (1-10).

Proposition 5.5. *Let $[-T, T]$ be a time interval, where we assume T is large. Suppose for ω belonging to some set E , there exists a subsequence $\{k_j\}_{j \geq 0} \uparrow \infty$ (possibly depending on ω) such that each of the equations (5-1) with $k = k_j$ has a unique solution u_j on $[-T, T]$ and that*

$$\sup_j \| u_j - e^{-it\mathbf{H}} f_{2^{k_j}}^\circ(\omega) \|_{\mathcal{X}^{\sigma,b,T}} < \infty. \tag{5-12}$$

Then almost surely $\omega \in E$, the equation (1-10) possesses a unique solution u on $[-T, T]$ such that $u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{X}^{\sigma,b,T}$. Moreover for this subsequence we have

$$\lim_{j \rightarrow \infty} \| u_j - e^{-it\mathbf{H}} f_{2^{k_j}}^\circ(\omega) - (u - e^{-it\mathbf{H}} f(\omega)) \|_{\mathcal{X}^{\sigma,b,T}} = 0. \tag{5-13}$$

Proof. For $\omega \in E$, with small exceptional probability (tending to 0 as $A \rightarrow \infty$), we may choose a sequence u_j solving (5-1) with $k = k_j \uparrow \infty$, and

$$\| u_j - e^{-it\mathbf{H}} f_{2^{k_j}}^\circ(\omega) \|_{\mathcal{X}^{\sigma,b,T}} \leq A \tag{5-14}$$

for all j . Then we choose an integer M large enough depending on T and A . We are going to prove for each $1 \leq m \leq M$ that (1-10) has a unique solution $u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{H}^{\sigma,b,mT/M}$ on the interval $[-\frac{mT}{M}, \frac{mT}{M}]$, and

$$\lim_{j \rightarrow \infty} \|u_j - e^{-it\mathbf{H}} f_{2^{k_j}}^\circ(\omega) - (u - e^{-it\mathbf{H}} f(\omega))\|_{\mathcal{H}^{\sigma,b,mT/M}} \rightarrow 0, \tag{5-15}$$

for ω outside the fixed set $\Omega_{T/M}$ that is constructed in the proof of Lemma 5.2. Since $\mathbb{P}(\Omega_{T/M}) \rightarrow 0$ as $M \rightarrow \infty$, this clearly contains the conclusion we need.

Now we proceed by induction on m . First assume $p \geq 3$ is odd. Supposing the conclusion holds for $m - 1$ (including $m = 1$), we will prove it for m . Write $\delta = M^{-1}T$ and $t_0 = (m - 1)\delta$, we know the solution u exists and is unique on $[-t_0, t_0]$, and we want to extend it to $[-(t_0 + \delta), t_0 + \delta]$. Without loss of generality we consider the half-line $t > 0$.

From (5-14) and (5-15) we have

$$\lim_{j \rightarrow \infty} \|u_j(t_0) - u(t_0) + e^{-it_0\mathbf{H}} f_{2^{k_j}}^\perp(\omega)\|_{\mathcal{H}^\sigma} = 0, \tag{5-16}$$

and $\|u(t_0) - e^{-it_0\mathbf{H}} f(\omega)\|_{\mathcal{H}^\sigma} \leq A$. We would like to solve the equation (1-1) with initial data $u(t_0)$ on $[-\delta, \delta]$, and argue as in Corollary 4.4. Here the linear term is not $e^{-it\mathbf{H}} f(\omega)$, but

$$e^{-it\mathbf{H}} u(t_0) = e^{-i(t+t_0)\mathbf{H}} f(\omega) + v,$$

where v is the linear evolution of some function with \mathcal{H}^σ norm $\lesssim A$; thus $\|v\|_{\mathcal{H}^{\sigma,b,\delta}} \lesssim A$ (this is easily proved by introducing a cutoff and using $\delta \leq 1$). Since $\omega \notin \Omega_{T/M}$, we can use the full strength of Proposition 4.3 and Lemma 5.2. In particular we can proceed as in the proof of Corollary 4.4 and obtain

$$\mathfrak{M} := \left\| \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|w_1(s)|^{p-1} w_1(s)) ds \right\|_{\mathcal{H}^{\sigma,b,\delta}} \lesssim \delta^{\theta_0} A^p \leq \delta^\theta,$$

and

$$\begin{aligned} \mathfrak{D} &= \left\| \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|w_1(s)|^{p-1} w_1(s) - |w_2(s)|^{p-1} w_2(s)) ds \right\|_{\mathcal{H}^{\sigma,b,\delta}} \\ &\lesssim \delta^{\theta_0} A^{p-1} \|h_1 - h_2\|_{\mathcal{H}^{\sigma,b,\delta}} < \frac{1}{2} \|h_1 - h_2\|_{\mathcal{H}^{\sigma,b,\delta}}, \end{aligned}$$

for all $w_i = e^{-it\mathbf{H}} u(t_0) + h_i$ with $\|h_i\|_{\mathcal{H}^{\sigma,b,\delta}} \leq \delta^\theta$, provided M is large enough (δ is small enough) depending on T and A . Then we can use Picard iteration and the same bootstrap argument to prove that the original solution u can be uniquely extended to $[t_0, t_0 + \delta]$ (and by symmetry, to the other side).

It remains to prove (5-15) for m . First we know

$$\lim_{j \rightarrow \infty} \|e^{-i(t-t_0)\mathbf{H}} u_j(t_0) - e^{-i(t-t_0)\mathbf{H}} u(t_0) + e^{-it\mathbf{H}} f_{2^{k_j}}^\perp(\omega)\|_{\mathcal{H}^{\sigma,b,[t_0-\delta,t_0+\delta]}} = 0,$$

which is a consequence of (5-16). In view of the induction hypothesis, we only need to prove³

$$\lim_{j \rightarrow \infty} \|u_j - e^{-i(t-t_0)\mathbf{H}} u_j(t_0) - (u - e^{-i(t-t_0)\mathbf{H}} u(t_0))\|_{\mathcal{H}^{\sigma,b,[t_0-\delta,t_0+\delta]}} = 0,$$

³Here we have used the following fact: Given two intervals $[x, y]$ and $[z, w]$ with $x < z < y < w$, for some constant C we have $\|u\|_{\mathcal{H}^{\sigma,b,[x,w]}} \leq C(\|u\|_{\mathcal{H}^{\sigma,b,[x,y]}} + \|u\|_{\mathcal{H}^{\sigma,b,[z,w]}})$. This is easily proved by using a partition of unity.

which, after a translation of time, is equivalent to

$$\lim_{j \rightarrow \infty} \|w_j - e^{-it\mathbf{H}} w_j(0) - (w - e^{-it\mathbf{H}} w(0))\|_{\mathcal{X}^{\sigma,b,\delta}} = 0, \tag{5-17}$$

where w_j is a solution of the truncated equation with $k = k_j$, and $w_j(0) = u_j(t_0)$; and w is a solution of the original equation with $w(0) = u(t_0)$. Write $w_j - w = h = h_{li} + h_{no}$, where

$$h_{li} = e^{-it\mathbf{H}}(u_j(t_0) - u(t_0)) = -e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\perp(\omega) + e^{-it\mathbf{H}} \lambda_j, \tag{5-18}$$

with $\|\lambda_j\|_{\mathcal{X}^\sigma} \rightarrow 0$, and

$$\begin{aligned} h_{no} &= \mp i \int_0^t e^{-i(t-s)\mathbf{H}} ((|w_j|^{p-1} w_j - |w|^{p-1} w)_{2^{k_j}}^\circ - (|w|^{p-1} w)_{2^{k_j}}^\perp(s)) ds \\ &= \mp i \int_0^t e^{-i(t-s)\mathbf{H}} (|w_j|^{p-1} w_j - |w|^{p-1} w)_{2^{k_j}}^\circ(s) ds - (w - e^{-it\mathbf{H}} w(0))_{2^{k_j}}^\perp. \end{aligned} \tag{5-19}$$

Now we need to prove $\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} \rightarrow 0$. Since $w - e^{-it\mathbf{H}} w(0) \in \mathcal{X}^{\sigma,b,\delta}$, the second term in (5-19) tends to zero in $\mathcal{X}^{\sigma,b,\delta}$ as $j \rightarrow \infty$. For the first term, we estimate the norm without the final projection. The expression in parentheses can be written as a linear combination of terms like $z_1^- \cdots z_p^-$, where z_1 is either h_{no} , or $e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\perp(\omega)$, or $e^{-it\mathbf{H}} \lambda_j$ which has $\mathcal{X}^{\sigma,b,\delta}$ norm $\rightarrow 0$. For $i \geq 2$, each z_i is one of the following:

- (1) $e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\circ(\omega)$. This is within the applicability of Lemma 5.2 since $\omega \notin \Omega_{T/M}$.
- (2) $w_j - e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\circ(\omega)$. This has $\mathcal{X}^{\sigma,b,\delta}$ norm $\lesssim A$ since $w_j(t) = u_j(t + t_0)$, due to (5-14).
- (3) One of the components of $w_j - w$. These include h_{no} and $e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\perp(\omega)$, as well as another term with $\mathcal{X}^{\sigma,b,\delta}$ norm $\lesssim A$. Since $\omega \notin \Omega_{T/M}$, these terms are controllable using Lemma 5.2.

If $z_1 = e^{-i(t+t_0)\mathbf{H}} f_{2^{k_j}}^\perp(\omega)$, then from Proposition 2.9 and Lemma 5.2, the corresponding term tends to 0 as $j \rightarrow \infty$ (since h_{no} is bounded in $\mathcal{X}^{\sigma,b,\delta}$ independent of j ; see below). If z_1 is the term with $\mathcal{X}^{\sigma,b,\delta}$ tending to 0, the same conclusion holds. If $z_1 = h_{no}$, then the norm of the corresponding term is bounded by $\delta^\theta \|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} (\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} + A)^{p-1}$. Therefore we have

$$\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} \lesssim \delta^\theta \|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} (\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} + A)^{p-1} + o(1),$$

as $j \rightarrow \infty$. By (5-14) and the Picard argument above, we know $\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} \lesssim A$ independent of j . Therefore, if we choose δ small enough (M large enough), we must have $\|h_{no}\|_{\mathcal{X}^{\sigma,b,\delta}} = o(1)$.

The proof when $1 < p < 3$ is basically the same, using linear estimates (Corollary 3.5) instead of Proposition 4.3. We will also need a variant of Lemma 5.2, but the proof of this is not hard and is essentially contained in Proposition 3.4 and Corollary 3.5. □

6. Global well-posedness

In what follows, we fix a sufficiently large T and a positive integer M such that $M \gtrsim T^2$.

First let us consider the truncated equation (5-1), which is an ODE on the finite-dimensional space V_{2^k} . If we identify V_{2^k} with $\mathbb{R}^{2^{k+1}+2}$ by the coordinates

$$g = \sum_{j=0}^{2^k} (a_j + ib_j)e_j, \tag{6-1}$$

then it is easy to check that (5-1) becomes

$$\partial_t a_j = \frac{\partial E_0}{\partial b_j}, \quad \partial_t b_j = -\frac{\partial E_0}{\partial a_j}, \tag{6-2}$$

with Hamiltonian

$$E_0(a_j, b_j) = \sum_{j=0}^{2^k} (2j + 1)(a_j^2 + b_j^2) \pm \frac{1}{p + 1} \left\| \sum_{j=0}^{2^k} (a_j + ib_j)e_j \right\|_{L^{p+1}}^{p+1}. \tag{6-3}$$

If we denote the solution flow of this equation by $\Phi_{2^k,t}$, then the following is true by the theory of Hamiltonian ODEs and straightforward computation: The map $(t, x) \mapsto \Phi_{2^k,t}(x)$ is defined on the whole spacetime domain $\mathbb{R} \times V_{2^k}$ (this is a consequence of the conservation of L^2 norm; see (6-4) below). For each $t \in \mathbb{R}$, $\Phi_{2^k,t}$ is a homeomorphism from V_{2^k} to itself. If $p \geq 3$ is odd, it is a diffeomorphism and preserves the quantities

$$\|g\|_{L^2}^2 = \sum_{j=0}^{2^k} (a_j^2 + b_j^2) \quad \text{and} \quad E = 2E_0 \tag{6-4}$$

and the Lebesgue measure. If $1 < p < 3$, it (and its inverse) can be approximated, uniformly on each compact subset of V_{2^k} , by a sequence of pairs of diffeomorphisms that preserve the quantities (6-4) and the Lebesgue measure. Therefore $\Phi_{2^k,t}$ itself also preserves (6-4) and the Lebesgue measure.

From above we know that $\Phi_{2^k,t}$ preserves the measure

$$\nu_{2^k}^\circ = \pi^{-1-2^k} \zeta \cdot e^{-E} \prod_{j=0}^{2^k} da_j db_j \tag{6-5}$$

on V_{2^k} , where $\zeta = 1$ in the defocusing case, and $\zeta = \chi(\|g\|_{L^2}^2 - \alpha_{2^k})$ in the focusing case as in (3-33). By the definition of μ and ν_{2^k} (see Section 3) we have

$$\nu_{2^k} = (\rho_{2^k} \cdot \mu_{2^k}^\circ) \otimes \mu_{2^k}^\perp = \nu_{2^k}^\circ \otimes \mu_{2^k}^\perp, \tag{6-6}$$

in both cases, where we understand that $\mu_{2^k}^\circ$ and $\mu_{2^k}^\perp$ are measures on V_{2^k} and $V_{2^k}^\perp$ respectively, and identify V with⁴ $V_{2^k} \times V_{2^k}^\perp$. From this we immediately see, for each Borel set J of V_{2^k} , that

$$\nu_{2^k}(\{g : g_{2^k}^\circ \in J\}) = \nu_{2^k}(\{g : g_{2^k}^\circ \in (\Phi_{2^k,t})^{-1}(J)\}). \tag{6-7}$$

⁴Here V is some space on which μ is supported. The exact choice of V is unimportant; for example, we may choose $V = \mathcal{S}'(\mathbb{R}^2)$, or $V = \bigcap_{\delta>0} \mathcal{H}^{-\delta}(\mathbb{R}^2)$.

Now we fix the choice

$$J = J_M = \{g_{2^k}^\circ : g \in f(\Omega_{T/M}^c)\}^c,$$

where $\Omega_{T/M}$ is constructed in the proof of Lemma 5.2. Consider the maximal $m_0 \leq M + 1$ so that the solution u of equation (5-1) satisfies

$$\|u - e^{-i(t-mT/M)\mathbf{H}}u(mT/M)\|_{\mathcal{X}^{\sigma,b,[(m-1)T/M,(m+1)T/M]}} \leq 1 \tag{6-8}$$

for all $|m| \leq m_0 - 1$. If $m_0 = M + 1$, from Proposition 2.9 we know that u is defined on $[-T, T]$ and

$$\|u - e^{-it\mathbf{H}}f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}} \lesssim M^3. \tag{6-9}$$

If $m_0 \leq M$, then for some choice of \pm sign, we have $\Phi_{2^k, \pm m_0 T/M}(f_{2^k}^\circ(\omega)) \in J_M$. In fact, if this fails, then we can use Propositions 5.1 and 5.4 to extend the solution to $[-(m_0 + 1)T/M, (m_0 + 1)T/M]$ with (6-8) remaining true, thus contradicting the definition of m_0 . Now we use (6-7) and sum over $m_0 \leq M$ to get

$$(v_{2^k} \circ f)(\{\omega : (6-9) \text{ fails}\}) \lesssim M \cdot v_{2^k}(\{g : g_{2^k}^\circ \in J_M\}). \tag{6-10}$$

In the defocusing case we have $v_{2^k} \leq \mu$. Using Fubini's theorem we get

$$\mu(\{g : g_{2^k}^\circ \in J_M^c\}) \geq \mu(f(\Omega_{T/M}^c)) \geq 1 - c_1 \exp(-c_2 T^{-c_3} M^{c_3}), \tag{6-11}$$

hence

$$(v_{2^k} \circ f)(\{\omega : (6-9) \text{ fails}\}) \lesssim c_1 M \exp(-c_2 T^{-c_3} M^{c_3}). \tag{6-12}$$

In the focusing case we have

$$\frac{dv_{2^k}}{d\mu}(g) = \rho_{2^k}(g) = \chi(\|g_{2^k}^\circ\|_{L^2}^2 - \alpha_{2^k}) \exp\left(\frac{2}{p+1} \|g_{2^k}^\circ\|_{L^{p+1}}^{p+1}\right). \tag{6-13}$$

This function, by Proposition 3.6, has bounded $L^2(\mu)$ norm, so by Cauchy–Schwartz we get

$$v_{2^k}(\{g : g_{2^k}^\circ \in J_M\}) \lesssim (\mu(\{g : g_{2^k}^\circ \in J_M\}))^{1/2} \leq c_1 \exp(-c_2 T^{-c_3} M^{c_3}), \tag{6-14}$$

which again implies (6-12). We summarize our results in the following proposition.

Proposition 6.1. *For fixed T and k , there exists a subset⁵ $\Omega_k \subset \Omega$ such that $(v_{2^k} \circ f)(\Omega_k^c) = 0$, and for $\omega \in \Omega_k$, equation (5-1) has a unique solution u_k on $[-T, T]$, and that*

$$\sup_k \int_{\Omega_k} \exp(\|u_k - e^{-it\mathbf{H}}f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}}^\theta) d(v_{2^k} \circ f)(\omega) < \infty \tag{6-15}$$

for some $\theta > 0$.

Proof. We choose

$$\Omega_k = \bigcap_{M \gtrsim T^2} Z_M := \bigcap_{M \gtrsim T^2} \{\omega : (6-9) \text{ fails for } M\}.$$

⁵This should not be confused with the Ω_T notation defined above, since our Ω_k is for $k \geq 1$ here!

From the discussion above we easily see $v_{2^k}(f(\Omega_k^c)) \leq \lim_{M \rightarrow \infty} v_{2^k}(f(Z_M)) = 0$. Also for $\omega \notin Z_M$ the solution u_k to (5-1) exists and is unique, and satisfies

$$\|u_k - e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}} \lesssim M^3.$$

In other words we have

$$(v_{2^k} \circ f)(\omega \in \Omega_k : \|u_k - e^{-it\mathbf{H}} f_{2^k}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}} > A) \leq v_{2^k}(f(Z_M)) \leq c_1 \exp(-c_2 A^{c_3}),$$

for all $A > T^{100}$, where $M \sim A^{1/3}$ is an integer. Since $v_{2^k} \circ f$ is uniformly integrable, the part with small A is also under control. The claim then follows. \square

With Propositions 5.5 and 6.1, we are ready to prove the global well-posedness part of Theorem 1.1. Denote the integrand in (6-15) by $\eta_k(\omega)$, understanding $\eta_k(\omega) = 0$ when $\omega \notin \Omega_k$. Since $v_{2^k} \rightarrow v$ in the strong sense and $(v_{2^k} \circ f)(\Omega_k^c) = 0$, we have $(v \circ f)(\Omega_k^c) \rightarrow 0$, and we fix a subsequence $\{k_l\}$ such that $\sum_l (v \circ f)(\Omega_{k_l}^c) < \infty$ and hence $(v \circ f)(\limsup_{l \rightarrow \infty} \Omega_{k_l}^c) = 0$. From Proposition 6.1, we get

$$\sup_l \int_{\Omega} \rho_{2^{k_l}}(f(\omega)) \eta_{k_l}(\omega) d\mathbb{P}(\omega) < \infty. \tag{6-16}$$

From the proof of Proposition 3.6, we see $\rho_{2^{k_l}} \circ f \rightarrow \rho \circ f$ almost surely, so by Fatou’s lemma we get

$$\liminf_{l \rightarrow \infty} \eta_{k_l}(\omega) < \infty, \tag{6-17}$$

almost surely in \mathbb{P} , on the set where $\rho(f(\omega)) \neq 0$. By the definition of η_k , if (6-17) holds, then either $\omega \in \Omega_{k_l}$ for infinitely many l , or there exists a subsequence $\{k_{l_j}\}_{j \geq 0} \uparrow \infty$, such that (5-1) has a unique solution $u_{k_{l_j}}$ for $k = k_j$ on $[-T, T]$, and

$$\sup_j \|u_{k_{l_j}} - e^{-it\mathbf{H}} f_{2^{k_{l_j}}}^\circ(\omega)\|_{\mathcal{X}^{\sigma,b,T}} < \infty.$$

In the former case we get a null set (actually a set with null $v \circ f$ measure, but $v \circ f$ is mutually absolutely continuous with \mathbb{P} on the set where $\rho(f(\omega)) \neq 0$), while in the latter case we can use Proposition 5.5 to deduce that, except for another null set, (1-10) also has a unique solution u on $[-T, T]$ such that $u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{X}^{\sigma,b,T}$.

Therefore, for each $T > 0$, except for a null set, the equation (1-10) has a unique solution $u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{X}^{\sigma,b,T}$ for ω in the support of $\rho \circ f$. In the defocusing case, this support itself has full probability in Ω ; in the focusing case, it follows from Proposition 3.6 that we can choose a countable number of cutoff χ so that the (countable) union of the support of the corresponding $\rho \circ f$ has full probability. In any case we have found a subset of Ω having full probability, such that when ω does belong to this set, (1-10) has a unique solution $u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{X}^{\sigma,b,T}$. We then take another countable union to get that, almost surely in \mathbb{P} , equation (1-10) has a unique solution u on $\mathbb{R} \times \mathbb{R}^2$ such that

$$\begin{aligned} u \in e^{-it\mathbf{H}} f(\omega) + \mathcal{X}^{\sigma,b,T} &\subset e^{-it\mathbf{H}} f(\omega) + \mathcal{C}([-T, T], \mathcal{H}^\sigma(\mathbb{R}^2)) \\ &\subset \mathcal{C}([-T, T], \bigcap_{\delta > 0} \mathcal{H}^{-\delta}(\mathbb{R}^2)) \end{aligned}$$

for all $T > 0$. This completes the proof.

Remark 6.2. In fact, from the argument we can extract a polynomial bound on the solution; namely we can prove that for each large A , with exceptional probability $\leq c_1 \exp(-c_2 A^{c_3})$ we have

$$\|u - e^{-it\mathbf{H}} f(\omega)\|_{\mathcal{X}^{\sigma,b,T}} \leq A \langle T \rangle^C$$

for all $T > 0$, with some constant C . We omit the details.

7. Transforming into NLS without harmonic potential

As we have mentioned before, the idea of introducing the lens transform and reducing (1-2) to (1-1) is inspired by the arguments in [Burq et al. 2010]. First we define the lens transform [Tao 2009, Section 2; Burq et al. 2010, Section 10]:

$$\mathcal{L}u(t, x) = \frac{1}{\cos(2t)} u\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) e^{-i|x|^2 \tan(2t)/2}, \tag{7-1}$$

where u is defined on $\mathbb{R} \times \mathbb{R}^2$, and $\mathcal{L}u$ is defined on $(-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^2$. By a simple computation we deduce

$$(i\partial_t - \mathbf{H})(\mathcal{L}u)(t, x) = (\cos(2t))^{-2} \mathcal{L}((i\partial_t + \Delta)u)(t, x). \tag{7-2}$$

For the inverse transform

$$\mathcal{L}^{-1}u(t, x) = (1 + 4t^2)^{-\frac{1}{2}} u\left(\frac{\tan^{-1}(2t)}{2}, (1 + 4t^2)^{-\frac{1}{2}}x\right) e^{i|x|^2 t/(1+4t^2)}, \tag{7-3}$$

we have

$$(i\partial_t + \Delta)(\mathcal{L}^{-1}u)(t, x) = \frac{1}{1 + 4t^2} \mathcal{L}^{-1}((i\partial_t - \mathbf{H})u)(t, x). \tag{7-4}$$

Next we prove that the transform \mathcal{L}^{-1} maps the space $\mathcal{X}^{\sigma,b,\delta}$ to $X^{\sigma,b,T}$, where $0 \leq \sigma, b \leq 1, 0 < \delta < \frac{\pi}{4}$, and $T = \frac{1}{2} \tan(2\delta)$. First by using a cutoff, we are reduced to proving that $u \mapsto \mathcal{L}^{-1}(\chi \cdot u)$ is bounded from $\mathcal{X}^{\sigma,b}$ to $X^{\sigma,b}$, where $\chi = \chi(t)$ is any smooth function having compact support in $|t| < \frac{\pi}{4}$. First we fix σ . By interpolation, we can assume $b \in \{0, 1\}$. If we can prove the result in the case $b = 0$, then using the identity

$$\|u\|_{\mathcal{X}^{\sigma,1}}^2 = \|u\|_{\mathcal{X}^{\sigma,0}}^2 + \|(i\partial_t - \mathbf{H})u\|_{\mathcal{X}^{\sigma,0}}^2$$

(which remains true with \mathcal{X} replaced by X and $-\mathbf{H}$ replaced by Δ) and (7-4), we see

$$\|\mathcal{L}^{-1}(\chi \cdot u)\|_{X^{\sigma,1}} \lesssim \|u\|_{\mathcal{X}^{\sigma,0}} + \|(i\partial_t - \mathbf{H})(\chi u)\|_{\mathcal{X}^{\sigma,0}}, \tag{7-5}$$

because $v = (i\partial_t - \mathbf{H})(\chi u)$ has compact support in $|t| < \frac{\pi}{4}$, and hence equals $\chi_1 v$ for some other χ_1 . Since the last term in (7-5) is clearly controlled by $\|u\|_{\mathcal{X}^{\sigma,1}}$, we can conclude the proof for $b = 1$. Therefore we may only consider $b = 0$. Here it is easily seen that we only need to prove that multiplication by $e^{i\lambda|x|^2}$ is uniformly bounded from \mathcal{H}^σ to H^σ for $0 \leq \sigma \leq 1$ and $|\lambda| \leq 1$. By another interpolation we may further reduce to $\sigma \in \{0, 1\}$. The $\sigma = 0$ case is obvious; the $\sigma = 1$ case follows from the observation

$$\nabla(e^{i\lambda|x|^2} \cdot f) = e^{i\lambda|x|^2} (\nabla + 2i\lambda x) \cdot f.$$

Thus we have the desired bound for all $0 \leq \sigma, b \leq 1$.

Using (7-2) or (7-4) we can compute that u is a solution for the Cauchy problem (1-12) on \mathbb{R} , if and only if $v = \mathcal{L}u$ is a solution for the Cauchy problem

$$\begin{cases} i\partial_t v + (\Delta - |x|^2)v = (\cos(2t))^{p-3} \cdot |v|^{p-1}v, \\ v(0) = f(\omega) \end{cases} \tag{7-6}$$

on $|t| < \frac{\pi}{4}$. Moreover, if

$$v - e^{-it\mathbf{H}} f(\omega) \in \mathcal{X}^{\sigma,b,\delta} \tag{7-7}$$

with $\delta < \frac{\pi}{4}$, then from the discussion above we see that

$$u - \mathcal{L}^{-1}(e^{-it\mathbf{H}} f(\omega)) \in X^{\sigma,b,T},$$

with $T = \frac{1}{2} \tan(2\delta) \rightarrow \infty$ as $\delta \rightarrow \frac{\pi}{4}$. From (7-4) we see that $\mathcal{L}^{-1}(e^{-it\mathbf{H}} f(\omega))$ has initial value $f(\omega)$ and annihilates $i\partial_t + \Delta$; thus it must be $e^{it\Delta} f(\omega)$. Thus (1-13) will follow⁶ if (7-7) holds for all $\delta < \frac{\pi}{4}$. Also from (7-3), the constants in the $\mathcal{H}_x^\sigma \rightarrow H_x^\sigma$ boundedness remains under control even near the boundary points $\pm \frac{\pi}{4}$. Thus (1-14) will follow if

$$\lim_{t \rightarrow \pm\pi/4} (v(t) - e^{-it\mathbf{H}} f(\omega)) \text{ exists in } \mathcal{H}^\sigma. \tag{7-8}$$

What we will prove is that almost surely in \mathbb{P} , (7-6) has a unique (strong) solution v for $|t| \leq \frac{\pi}{4}$ such that $v - e^{-it\mathbf{H}} f(\omega) \in \mathcal{X}^{\sigma,b,\pi/4}$. As is demonstrated above, this implies (7-7) and (7-8), and hence Theorem 1.2.

The proof is basically the same as (1-10). Noticing $m(t) = (\cos(2t))^{p-3}$ has all its derivatives bounded on \mathbb{R} , we see that multiplication by $m(t)$ is bounded from any $\mathcal{X}^{\sigma,b}$ (and hence any $\mathcal{X}^{\sigma,b,T}$) to itself. Therefore, the proof from Proposition 4.3 to Lemma 5.2 goes without any difficulty, as if this additional factor were not present. In the proof of Proposition 5.5, when we extend the solution to a larger interval, we must solve another Cauchy problem, which is no longer (7-6), since this equation is not autonomous. This, however, is not a problem; since we just replace $m(t)$ by some $m(t - t_0)$ that obeys the same derivative estimates as $m(t)$, we can use the *same* exceptional set as in Proposition 4.3, Lemma 5.2 and Proposition 5.5, and the other discussions remain unchanged.

The only difficulty we face is the lack of a (formally) invariant measure. This is compensated, however, by a monotonicity property, which was first observed in [Burq et al. 2010].

Lemma 7.1. *Consider the truncated Cauchy problem*

$$\begin{cases} i\partial_t v + (\Delta - |x|^2)v = (\cos(2t))^{p-3} \cdot (|v|^{p-1}v)_{2^k}^\circ, \\ v(0) = f_{2^k}^\circ(\omega); \end{cases} \tag{7-9}$$

then for its solution v , the quantity

$$\mathcal{E}(t, v(t)) = \langle \mathbf{H}v, v \rangle + \frac{2(\cos(2t))^{p-3}}{p+1} \|v\|_{L^{p+1}}^{p+1}$$

⁶The space $\mathcal{X}' \subset \mathcal{X}$ in which we have uniqueness will be the image of the space \mathcal{Y} defined in (1-11) under \mathcal{L}^{-1} ; as we have said before, we do not have a simple characterization for this.

is monotonically nonincreasing in $|t|$ for $|t| \leq \frac{\pi}{4}$.

Proof. We directly compute

$$\frac{d\mathcal{E}}{dt} = -\frac{2(p-3)(\cos(2t))^{p-4} \sin(2t)}{p+1} \|v(t)\|_{L^{p+1}}^{p+1},$$

which is nonpositive for $0 \leq t \leq \frac{\pi}{4}$, and nonnegative for $-\frac{\pi}{4} \leq t \leq 0$. □

We argue as in Section 6, but we fix $T = \frac{\pi}{4}$ here. If we could prove

$$\mu(\{g : g_{2^k}^\circ \in J\}) \geq \nu_{2^k}(\{g : \Phi_{2^k,t}(g_{2^k}^\circ) \in J\}) \tag{7-10}$$

for $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$, where, of course, $\Phi_{2^k,t}$ is now the solution flow of (7-6), then combining this inequality with (6-11) we can get (6-12). Starting from this point, we can follow the argument in Section 6 word by word to get almost surely global well-posedness of (7-6) on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

The proof of (7-10) is also simple. By Lemma 7.1

$$\begin{aligned} \nu_{2^k}(\{g : \Phi_{2^k,t}(g_{2^k}^\circ) \in J\}) &= \pi^{-1-2^k} \int_{J_1} e^{-E(g)} \prod_{j=0}^{2^k} da_j db_j \\ &\leq \pi^{-1-2^k} \int_{J_1} e^{-\mathcal{E}(t,g(t))} \prod_{j=0}^{2^k} da_j db_j \leq \int_J d\mu_{2^k}^\circ = \mu(\{g : g_{2^k}^\circ \in J\}), \end{aligned}$$

where $J_1 = \{h \in V_{2^k} : \Phi_{2^k,t}(h) \in J\}$. Here we have used the invariance of the Lebesgue measure under $\Phi_{2^k,t}$, which can be directly verified; see [Burq et al. 2010, Lemma 8.3]. Therefore we have completed the proof of Theorem 1.2.

8. Invariance of Gibbs measure

Now we return to the final assertion of Theorem 1.1, and prove the invariance of the Gibbs measure ν under the solution flow of (1-10). More precisely:

Proposition 8.1. *Denote the solution flow of (1-10) by Φ_t . There exists a subset $\Sigma \subset \mathcal{S}'(\mathbb{R}^2)$ such that it has full μ measure, and Φ_t becomes a one-parameter group from Σ to Σ preserving the measure ν (in the focusing case, for each choice of cutoff function χ).*

Proof. We only consider the defocusing case. In the focusing case we need to take another countable intersection corresponding to the cutoff χ chosen, but otherwise the proof is completely analogous. Clearly the set Ω_T in Proposition 4.3 and Lemma 5.2 can be chosen so that $e^{-it\mathbf{H}} f(\Omega_T^c) = f(\Omega_T^c)$.

We define $\Sigma = \Sigma_1 \cap \Sigma_2$, where Σ_1 is the set of all $g \in \mathcal{S}'(\mathbb{R}^2)$ so that (1-1) (with initial data $u(0) = g$) has a unique solution⁷ u on \mathbb{R} that belongs to $e^{-it\mathbf{H}} g + \mathcal{X}^{\sigma,b,T}$ for all $T > 0$. This has full μ measure due to the global well-posedness part of Theorem 1.1. Also Σ_2 is defined to be $\Sigma_2 = f(\liminf_{i \rightarrow \infty} \Omega_{\gamma 2^{-i}}^c) + \mathcal{H}^\sigma$, and this also has full μ measure for small enough γ due to our control on $\mathbb{P}(\Omega_T)$. Clearly Σ has full μ measure,

⁷It is a bit vague to say u is a “solution” when g is only a distribution; but since we are considering Σ_2 also, we can assume here $e^{-it\mathbf{H}} g \in L_{t,x}^q$ on any finite time interval, for appropriate q , and then the definition of Σ_1 becomes rigorous.

and Φ_t is uniquely defined on Σ . If we can prove $\Phi_t(\Sigma) \subset \Sigma$, then they obviously form a (measurable) one-parameter group. Clearly $\Phi_t(\Sigma) \subset \Sigma_2$ since for a solution u we have $u(t) \in e^{-itH}u(0) + \mathcal{H}^\sigma$. To prove $\Phi_t(\Sigma) \subset \Sigma_1$, we only need to prove that if u is a solution of (1-10) with $u(0) \in \Sigma_2$, then it is automatically unique. Since all $u(t) \in \Sigma_2$, by bootstrap arguments we only need to prove short time uniqueness. Write $u(0) = f(\omega) + h$ with $\|h\|_{\mathcal{H}^\sigma} = A$ and $\omega \notin \Omega_{c2^{-i}}$ for all large enough i . Repeating the extension argument in Proposition 5.5, we see for i large enough depending on A that $\omega \notin \Omega_{c2^{-i}}$ and the solution is unique for $|t| \leq c2^{-i}$. This proves the existence of Σ .

Now we only need to prove that for each measurable set $E \subset \Sigma$ and $t \in \mathbb{R}$, we have

$$\nu(\Phi_t(E)) \geq \nu(E). \tag{8-1}$$

We may assume $|t| \leq 1$. Write

$$\Pi_{i_0,A} = \Sigma_1 \cap \left(\{h : \|h\|_{\mathcal{H}^\sigma} \leq A\} + \bigcap_{i \geq i_0} f(\Omega_{c2^{-i}}^c) \right)$$

and

$$\Pi'_A = \{g \in \Sigma : \|u - e^{-itH}g\|_{\mathcal{H}^{\sigma,b,2}} \leq A\}.$$

By a limiting argument we can further assume $E \subset \Pi_{i_0,A} \cap \Pi'_A$ for some i_0 and A . Note that this implies $\Phi_t(E) \subset \Pi_{i_0,CA}$ for $|t| \leq 2$ and some constant C .

Let T be small enough depending on i_0 and A , we only need to prove (8-1) for $|t| \leq T$ and $E \subset \Pi_{i_0,CA}$ (since we can iterate to get the result for $|t| \leq 1$). Write $\Pi = \Pi_{i_0,CA}$ and define $\Psi(g) = u - e^{-itH}g$, where u is the solution to (1-1) with initial data g , and consider the mapping

$$\Psi_1 : \Pi \rightarrow \mathcal{H}^{\sigma,b,T} \times \mathbb{C}^\infty, \quad g \mapsto (\Psi(g), ((g, e_k))_{k \geq 0}),$$

where in \mathbb{C}^∞ we use the standard metric. This mapping is clearly injective (thus it induces a metric on Π) and, as will be explained in Remark 8.2, its image is a Borel set of the product space (denoted by Y). By a theorem in measure theory [Halmos 1950], the finite Borel measure $\nu \circ \Psi_1^{-1}$ on the complete separable metric space Y is regular. For each measurable set $E \subset \Pi$ we can find a compact set $K \subset \Psi_1(E)$ such that $(\nu \circ \Psi_1^{-1})(\Psi_1(E) - K) < \epsilon$; thus $\Psi_1^{-1}(K) \subset E$ is compact in the induced metric and $\nu(E - \Psi_1^{-1}(K)) < \epsilon$. Therefore, we only need to prove (8-1) for compact sets $E \subset \Pi$. Due to Propositions 5.1 and 5.4, when T is small enough depending on i_0 and A , the map $\Phi_{2^k,t}$ will be defined on $E \subset \Pi$ for each k and $|t| \leq T$. Thus by the invariance of $\nu_{2^k}^\circ$ under the solution flow $\Phi_{2^k,t}$, we have

$$\nu_{2^k} \left(\{g : g_{2^k}^\circ = \Phi_{2^k,t}(h_{2^k}^\circ), h \in E\} \right) \geq \nu_{2^k}(E).$$

Let $k \rightarrow \infty$, noticing that the total variation of $\nu_{2^k} - \nu$ tends to zero, we only need to prove that

$$\limsup_{k \rightarrow \infty} \{g : g_{2^k}^\circ = \Phi_{2^k,t}(h_{2^k}^\circ), h \in E\} \subset \Phi_t(E).$$

Now suppose that for a subsequence $k_j \uparrow \infty$, we have $g_{2^{k_j}}^\circ = \Phi_{2^{k_j},t}((h_{k_j})_{2^{k_j}}^\circ)$, and by compactness, assume $h_{k_j} \rightarrow h$ with respect to the induced metric. We are going to prove $g = \Phi_t(h)$.

First of all, we have

$$\lim_{k \rightarrow \infty} \|\Phi_{2^k,t}(h_{2^k}^\circ) - \Phi_t(h) + e^{-it\mathbf{H}}h_{2^k}^\perp\|_{\mathcal{H}^\sigma} = 0, \tag{8-2}$$

uniformly for $|t| \leq T$ and $h \in E$. In fact, if T is small enough, we may assume $h = h_1 + h_2$, where $h_1 \in f(\Omega_{T'}^c)$, $2T \leq T' \leq 4T$, and $\|h_2\|_{\mathcal{H}^\sigma} \leq CA$. Since T' is small enough depending on A , we can almost repeat⁸ the proof of Proposition 5.5 to get that the $\mathcal{H}^{\sigma,b,T'}$ norm tends to 0. Since the $\mathcal{H}^{\sigma,b,T'}$ norm is not less than the spacial \mathcal{H}^σ norm at time t , (8-2) follows.

From (8-2) we get

$$\lim_{j \rightarrow \infty} \|g_{2^{k_j}}^\circ - (\Phi_t(h_{k_j}))_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} = 0,$$

and we only need to prove

$$\lim_{k \rightarrow \infty} \|(\Phi_t(h_{k_j}))_{2^{k_j}}^\circ - (\Phi_t(h))_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} = 0. \tag{8-3}$$

But since $h_{k_j} \rightarrow h$ with respect to the induced metric, we only need to prove that $\|(h_{k_j})_{2^{k_j}}^\circ - h_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} \rightarrow 0$.

For $i \geq j$ we have

$$(\Phi_{2^{k_i},t}((h_{k_i})_{2^{k_i}}^\circ))_{2^{k_j}}^\circ = g_{2^{k_j}}^\circ = \Phi_{2^{k_j},t}((h_{k_j})_{2^{k_j}}^\circ),$$

and by using (8-2) once more we see that

$$(\Phi_t(h_{k_i}))_{2^{k_j}}^\circ = (\Phi_t(h_{k_j}))_{2^{k_j}}^\circ + o(1)$$

as $i \geq j \rightarrow \infty$. Again using that $h_{k_j} \rightarrow h$, we deduce

$$\lim_{i \geq j \rightarrow \infty} \|(h_{k_i} - h_{k_j})_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} = 0. \tag{8-4}$$

In particular, we see that $\lim_{i \rightarrow \infty} (h_{k_i})_{2^{k_j}}^\circ$ exists in \mathcal{H}^σ for each j . By the definition of the metric, this limit must be $h_{2^{k_j}}^\circ$. Therefore we get

$$\lim_{i \rightarrow \infty} \|(h - h_{k_i})_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} = 0. \tag{8-5}$$

Combining (8-4) with (8-5), we finally see that $\lim_{j \rightarrow \infty} \|(h_{k_j})_{2^{k_j}}^\circ - h_{2^{k_j}}^\circ\|_{\mathcal{H}^\sigma} = 0$. This completes the proof of Theorem 1.1. □

Remark 8.2. To show that $\Psi_1(\Pi)$ is a Borel set in the product metric space, we only need to show that Ψ is injective, $\Psi(\Pi)$ is a Borel set in $\mathcal{H}^{\sigma,b,T}$, and the map $\Psi^{-1} : \Psi(\Pi) \rightarrow \Pi$ is Borelian. To this end we notice

$$\Psi(g) = -i \int_0^t e^{-i(t-s)\mathbf{H}} (|u(s)|^{p-1}u(s)) ds, \tag{8-6}$$

where $u = u(g)$ is the solution map of (1-2), and $g = u(0)$.⁹ Then we can decompose Ψ as

$$\Psi : g \mapsto u(g) \mapsto |u(g)|^{p-1}u(g) \mapsto \Psi(g),$$

⁸Actually we do not have the *a priori* bound on the nonlinear part of truncated equations, but since $h_1 \in \Omega_{T'}$ with T' small depending on A , it is not hard to get this from scratch.

⁹Here we also require $g \in \mathcal{H}^{-\epsilon}$ for appropriate ϵ , so that $u \in \mathcal{C}(\mathbb{R}, \mathcal{H}^{-\epsilon})$ in which $u(0)$ makes sense.

and see that at each step the mapping is injective, and the image of any Borel set is again Borelian (for example, the set $u(\Pi)$ can be characterized as the set of all u such that $u - e^{-it\mathbf{H}}u(0) \in \mathcal{X}^{\sigma,b}$, that u satisfies equation (1-2), and that $u(0) \in \Pi$, so it is Borelian). Hence the claim.

Appendix. Typical regularity on the support of μ

In this appendix we shall prove that if $\sigma \geq \frac{1}{2}$, then almost surely $\mathbf{H}^{\sigma/2}f(\omega)$ is not a (locally integrable) function. More precisely, almost surely in \mathbb{P} , we have

$$\psi \cdot \mathbf{H}^{\sigma/2}f(\omega) \notin L^1(\mathbb{R}^2) \tag{A-1}$$

for any smooth compactly supported ψ that is not identically zero.

To prove this, first notice that we can find a countable number of ψ_j such that each is compactly supported and equals 1 on some annular region $a < |x| < b$, and for any other ψ there exists $\eta \in L^\infty$ and j such that $\psi_j = \psi \cdot \eta$. So we only need to consider a fixed ψ_j (which we write ψ below) and assume it equals 1 for $a < |x| < b$. Here we use an asymptotic formula of \mathcal{L}_k^0 proved in [Erdélyi 1960]:

$$\mathcal{L}_k^0(z) = \frac{1}{\sqrt{2\pi}}(\nu z)^{-1/4} \cos \theta + O(\nu^{-3/4}), \tag{A-2}$$

where $a^2 < z < b^2$ and $\nu = 4k + 2$ is large, and

$$\theta = \frac{\nu(\phi + \sin \phi) - \pi}{4}, \quad \phi = \cos^{-1} \frac{\nu - 2z}{\nu}.$$

From (A-2) we easily deduce that

$$\mathcal{L}_k^0(z) = 1/\sqrt{2\pi}(\nu z)^{-1/4} \cos(\sqrt{\nu z} - \pi/4) + O(\nu^{-3/4}),$$

and hence for each k

$$\|e_k \psi\|_{L^1} \gtrsim \int_{a^2 < z < b^2} |\mathcal{L}_k^0(z)| \, dz \gtrsim \nu^{-1/4}. \tag{A-3}$$

Now we define the Gaussian random variable

$$h_{M,N}(\omega) = \sum_{k=0}^M (4k + 2)^{(\sigma-1)/2} g_k(\omega) \eta(\mathbf{H}/N^2)(e_k \psi),$$

whose range lies in a finite-dimensional space, and use Lemma 3.3 to get the lower bound

$$\mathbb{P}(\|h_{M,N}(\omega)\|_{L^1} \geq cE_{M,N}) \geq \frac{1}{10}$$

with some absolute constant c , where

$$\begin{aligned} E_{M,N} &= \mathbb{E}(\|h_{M,N}(\omega)\|_{L^1}) = \int_{\mathbb{R}^2} \mathbb{E} \left(\left| \sum_{k=0}^M (4k + 2)^{(\sigma-1)/2} g_k(\omega) e_{k,N}(x) \right| \right) \, dx \\ &\sim \int_{\mathbb{R}^2} \left(\sum_{k=0}^M (4k + 2)^{\sigma-1} |e_{k,N}(x)|^2 \right)^{1/2} \, dx \gtrsim \left(\sum_{k=0}^M (4k + 2)^{\sigma-1} \|e_{k,N}\|_{L^1}^2 \right)^{1/2}, \end{aligned}$$

and $e_{k,N} = \eta(N^{-2}\mathbf{H})(e_k\psi)$. Now for fixed N , we let $M \rightarrow \infty$ to get

$$h_{M,N} \rightarrow \eta(N^{-2}\mathbf{H})(\mathbf{H}^{\sigma/2}f(\omega) \cdot \psi)$$

in \mathcal{S} almost surely, since for fixed n (say $n \leq 3N$), the inner product $\langle e_n, e_k\psi \rangle$ is rapidly decreasing in k (using integration by parts). In particular we have almost surely L^1 convergence and hence (by taking upper limit of a sequence of sets)

$$\mathbb{P}(\|\eta(N^{-2}\mathbf{H})(\mathbf{H}^{\sigma/2}f(\omega) \cdot \psi)\|_{L^1} \geq cE_N) \geq \frac{1}{10},$$

where

$$E_N = \liminf_{M \rightarrow \infty} E_{M,N} \gtrsim \left(\sum_{k=0}^{\infty} (4k+2)^{\sigma-1} \|e_{k,N}\|_{L^1}^2 \right)^{1/2}.$$

By the uniform boundedness of $\eta(N^{-2}\mathbf{H})$, we know $\eta(N^{-2}\mathbf{H})g \rightarrow g$ in L^1 for any $g \in L^1$, so we have

$$\liminf_{N \rightarrow \infty} E_N \gtrsim \left(\sum_{k=0}^{\infty} (4k+2)^{\sigma-1} \|e_k\psi\|_{L^1}^2 \right)^{1/2} \gtrsim \left(\sum_{k=0}^{\infty} (4k+2)^{\sigma-3/2} \right)^{1/2} = \infty,$$

due to (A-3). Now we take another upper limit, and see that with probability $\geq \frac{1}{10}$, we have

$$\limsup_{N \rightarrow \infty} \|\eta(N^{-2}\mathbf{H})(\mathbf{H}^{\sigma/2}f(\omega) \cdot \psi)\|_{L^1} = \infty. \quad (\text{A-4})$$

Now (A-4) implies (A-1), again because of the uniform boundedness of $\eta(N^{-2}\mathbf{H})$ on L^1 . Therefore we have proved that (A-1) holds with positive probability. Since it is clearly a tail event (because $e_k \cdot \psi$ themselves are Schwartz functions), it must hold with probability one.

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ANALYSIS & PDE

Volume 5 No. 5 2012

An inverse problem for the wave equation with one measurement and the pseudorandom source	887
TAPIO HELIN, MATTI LASSAS and LAURI OKSANEN	
Two-dimensional nonlinear Schrödinger equation with random radial data	913
YU DENG	
Schrödinger operators and the distribution of resonances in sectors	961
TANYA J. CHRISTIANSEN	
Weighted maximal regularity estimates and solvability of nonsmooth elliptic systems, II	983
PASCAL AUSCHER and ANDREAS ROSÉN	
The two-phase Stefan problem: regularization near Lipschitz initial data by phase dynamics	1063
SUNHI CHOI and INWON KIM	
C^∞ spectral rigidity of the ellipse	1105
HAMID HEZARI and STEVE ZELDITCH	
A natural lower bound for the size of nodal sets	1133
HAMID HEZARI and CHRISTOPHER D. SOGGE	
Effective integrable dynamics for a certain nonlinear wave equation	1139
PATRICK GÉRARD and SANDRINE GRELLIER	
Nonlinear Schrödinger equation and frequency saturation	1157
RÉMI CARLES	