ANALYSIS & PDE

Volume 5 No. 5 2012

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ANALYSIS AND PDE
Vol. 5, No. 5, 2012
dx.doi.org/10.2140/apde.2012.5.1063

THE TWO-PHASE STEFAN PROBLEM: REGULARIZATION NEAR LIPSCHITZ INITIAL DATA BY PHASE DYNAMICS

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In this paper we investigate the regularizing behavior of two-phase Stefan problem near initial Lipschitz data. A description of the regularizing phenomena is given in terms of the corresponding space-time scale.

1. Introduction

Consider $u_0(x) : B_R(0) \to \mathbb{R}$ with $R \gg 1$ and $u_0 \geq -1$, $|\{u_0 = 0\}| = 0$ and $u_0(x) = -1$ on $\partial B_R(0)$ (see Figure 1). The two-phase Stefan problem can be formally written as

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\} \cup \{u < 0\}, \\ u_t/|Du^+| = |Du^+| - |Du^-| & \text{on } \partial\{u > 0\}, \\ u(\cdot, 0) = u_0, & \\ u = -1 & \text{on } \partial B_R(0). \end{cases} \tag{ST2}$$

Here Du denotes the spatial derivative of u, and u^+ and u^- respectively denote the positive and negative parts of u, i.e,

$$u^+ := \max(u, 0) \quad \text{and} \quad u^- := -\min(u, 0).$$

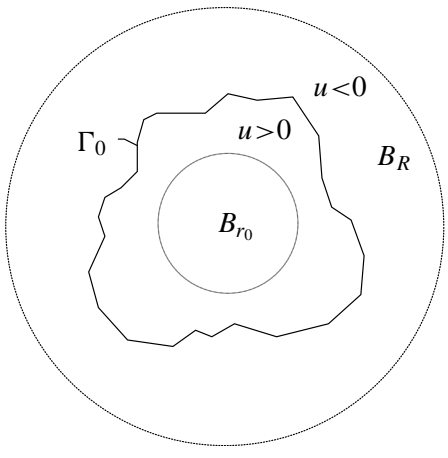

Figure 1. Initial setting of the problem.

I. Kim is partially supported by NSF grant number 0970072.
MSC2010: 35K55, 35R35, 76D27.
Keywords: free boundary problem, Stefan problem, phase transition, regularity near initial data, viscosity solutions.

The classical Stefan problem describes the phase transition between solid/liquid or liquid/liquid interface (see [Meirmanov 1992; Oleinik et al. 1993]). In our setting, we consider a bounded domain $\Omega_0 \subset B_R(0)$ and initial data $u_0(x)$ such that

$$\{u_0 > 0\} = \Omega_0 \quad \text{and} \quad \{u_0 < 0\} = B_R(0) - \overline{\Omega}_0.$$

To avoid complications at infinity, we consider the problem in the domain $Q = B_R(0) \times [0, \infty)$. For simplicity we have set $u = -1$ on $\partial B_R(0)$; our analysis presented in this paper applies to (ST2) with the generalized Dirichlet condition

$$u = f(x, t) < 0 \quad \text{on } \partial B_R(0),$$

where $f(x, t)$ is smooth.

Since our initial data will be only locally Hölder continuous, we employ the notion of viscosity solutions to discuss the evolution of the problem. Viscosity solutions for (ST2) were originally introduced by Athanasopoulos et al. [1996] (see also [Caffarelli and Salsa 2005]). As for existence and uniqueness of viscosity solutions for (ST2), we refer to [Kim and Požár 2011].

Note that the second condition of (ST2) states that the normal velocity $V_{x,t}$ at each free boundary point $(x, t) \in \partial\{u > 0\}$ is given by

$$V_{x,t} = (|Du^+| - |Du^-|)(x, t) = (Du^+(x, t) - Du^-(x, t)) \cdot \nu_{x,t},$$

where $\nu_{x,t}$ denotes the spatial unit normal vector of $\partial\{u > 0\}$ at (x, t), pointing inward with respect to the *positive phase* $\{u > 0\}$.

In this paper we investigate the regularizing behavior of the *free boundary* $\partial\{u > 0\}$. Our main result states that when $\Gamma_0 := \partial\{u_0 > 0\}$ is locally a Lipschitz graph with small Lipschitz constant, then the free boundary immediately regularizes and becomes smooth after $t = 0$. Moreover we provide a natural space-time scale for such regularization. More precisely, for $x_0 \in \Gamma_0$, we show that the free boundary regularizes in $B_d(x_0)$ by the time $t(x_0, d)$ given in (1-3) (see Theorem 1.1, and also the heuristic discussion below (1-3)). Corresponding results have been obtained in recent studies on the one-phase free boundary problems [Choi et al. 2007; 2009; Choi and Kim 2010], but the presence of two phases poses new challenges in the analysis. For example there is no generic class of global solutions other than radial solutions where topological changes are ruled out. In the one-phase setting we relied on the fact that solutions with star-shaped initial data stay star-shaped over time: this is no longer true in the two-phase setting (see Remark 3.2). More importantly, the interface motion is no longer monotone and competition between positive and negative fluxes across the free boundary necessitates additional localization procedure (see the remarks below Theorem 1.1).

The celebrated results of [Athanasopoulos et al. 1996; 1998] state that if the solution of (ST2) stays close to a Lipschitz profile in the unit space-time neighborhood $B_1(0) \times [0, 1]$, then the solution is indeed smooth in half of the neighborhood, that is, in $B_{1/2}(0) \times [1/2, 1]$. The main step in our analysis is to prove that the free boundary $\partial\{u > 0\}$ stays close to a locally Lipschitz profile in any given scale. Proving this step corresponds to the derivation of several Harnack-type inequalities for our problem, which are of independent interest.

Before discussing our result in detail, let us introduce precise conditions on the initial data.

(I-a) Ω_0 and u_0 are star-shaped with respect to a ball $B_{r_0}(0) \subset \Omega_0$.

Observe that then the *Lipschitz constant* L of $\partial\Omega_0$ is determined by r_0 and d_0, where

$$d_0 := \sup\{\operatorname{dist}(x, B_{r_0}(0)) : x \in \partial\Omega_0\}.$$

In other words, there exist $h = h(r_0)$ and $L = L(r_0, d_0)$ such that for any $x_0 \in \partial\Omega_0$, after rotation of coordinates, one has the representation

$$B_h(x_0) \cap \Omega_0 = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n \le f(x)\}, \tag{1-1}$$

where f is a Lipschitz function with $\operatorname{Lip} f \le L$. For simplicity, we set $h = 1$.

For a locally Lipschitz domain such as Ω_0, there exist growth rates $0 < \beta < 1 < \alpha$ such that the following holds: Let H be a positive harmonic function in $\Omega_0 \cap B_2(x)$, $x \in \partial\Omega_0$, with Dirichlet condition on $\partial\Omega_0 \cap B_2(x)$, and with value 1 at $x - e_n$. (Here e_n is the direction of the axis for the Lipschitz graph near x.) Then for $x - se_n \in \Omega_0 \cap B_1(x)$,

$$s^\alpha \le H(x - se_n) \le s^\beta. \tag{1-2}$$

Below we list conditions on the range of the Lipschitz constant L of the initial positive phase Ω_0.

(I-b) $L < L_n$ for a sufficiently small dimensional constant L_n so that

$$5/6 \le \beta < \alpha \le 7/6.$$

The remaining conditions are on the regularity of u_0.

(I-c) $-N_0 \le \Delta u_0 \le N_0$ in $\Omega_0 \cup (B_R(0) - \overline{\Omega}_0)$, where N_0 is some constant.

(I-d) For $x \in \partial\Omega_0$, we may let $e_n = x/|x|$ after a rotation. Then for small $s > 0$ (for $0 < s < 1/10$),

$$|Du_0(x \pm se_n)| \ge Cs^{\alpha-1}.$$

Note that (I-c) and (I-d) hold for u_0 which is smooth in its positive and negative phases and is harmonic near the initial free boundary, that is, $-\Delta u_0 = 0$ in the set $(\{u_0 > 0\} \cup \{u_0 < 0\}) \cap \{x : \operatorname{dist}(x, \partial\Omega_0) \le 1\}$.

We mention that, roughly speaking, the series of the hypothesis (Ia)–(Id) suggests that we have in mind an initial positive phase Ω_0 whose boundary is "almost" C^1 (that is, a small perturbation of a C^1 boundary in its Lipschitz norm), and initial data u_0 whose rescaled profile is "almost" harmonic near $\partial\Omega_0$. The smallness assumption on L given in (I-b) is to avoid waiting time phenomena (see [Athanasopoulos et al. 1996; Choi and Kim 2006]), and is most natural in the spirit of previous results [Athanasopoulos et al. 1996; 1998]. The assumption on u_0 is introduced to ensure that the initial data does not perturb the initial geometry of Ω_0 too much (see the discussion in [Choi and Kim 2010]). We expect that regularization of the interface over time should hold for general continuous initial data u_0.

For a function $u(x, t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, let us write

$$\Omega(u) := \{u > 0\}, \quad \Omega_t(u) := \{u(\cdot, t) > 0\},$$

and

$$\Gamma(u) := \partial\{u > 0\}, \quad \Gamma_t(u) := \partial\{u(\cdot, t) > 0\}.$$

Since $\Gamma_0 = \partial\{u(\cdot, 0) > 0\} = \partial\{u(\cdot, 0) < 0\}$ in our setting, this property is preserved for later times, i.e.,

$$\Gamma_t(u) = \partial\{u(\cdot, t) > 0\} = \partial\{u(\cdot, t) < 0\} \quad \text{for all } t > 0;$$

see [Rogers and Berger 1984; Götz and Zaltzman 1991; Kim and Požár 2011].

For $x_0 \in \Gamma_0 = \Gamma_0(u)$, we may let $e_n = x_0/|x_0|$ after a rotation. Then we define

$$t(x_0, r) := \min\Big\{\frac{r^2}{u^+(x_0 - re_n, 0)}, \frac{r^2}{u^-(x_0 + re_n, 0)}\Big\}. \tag{1-3}$$

Some remarks concerning $t(x_0, r)$ are in order. In one-phase case (where $u^- \equiv 0$), it was shown in [Choi et al. 2007] that

$$t(x_0, r) \sim \sup\{t > 0 : u(x_0 + re_n, t) = 0\},$$

i.e., $t(x_0, r)$ is the time it takes for the free boundary to reach $x_0 + re_n$. In our (two-phase) case $t(x_0, r)$ is the time it takes for the free boundary to reach $x_0 + re_n$ if we evolve the free boundary only according to the dominant phase with bigger size of u. In particular $\Gamma(u)$ moves at most by distance r by the time $t(x_0, r)$. It turns out that $t(x_0, r)$ is the correct time scale for the solutions in r-neighborhood of x_0 to "mix" and regularize the interface (Theorem 1.1(3)). See the paragraph below Theorem 1.1 for further heuristics based on scaling properties of our problem.

Suppose u is a solution of (ST2) with initial data u_0 satisfying (Ia)–(Id) with $\Omega_0(u) \subset B_R(0)$. Due to (Ia)–(Ib), for sufficiently small r and given $x_0 \in \Gamma_0$ the initial free boundary Γ_0 is given by the graph of a Lipschitz function in $B_r(x_0)$. After a rotation if necessary, we may assume that

$$\Omega_0 \cap B_r(x_0) = \{x + x_0 : x = (x', x_n),\ x_n \le f(x')\},$$

where f is a Lipschitz function with Lipschitz constant $L < L_n$. We summarize our main results:

Theorem 1.1 (cf. Theorem 5.6, Theorem 5.7 and Corollary 5.8). *Let u, Ω_0, r and f be as above. There exists $d_0 > 0$ depending only on n and N_0 such that the following holds for $r \le d_0$:*

(1) *In $\Sigma_r := B_{2r}(x_0) \times [t(x_0, r)/2, t(x_0, r)]$, we have*

$$\Gamma(u) = \{(x + x_0, t) : x = (x', x_n),\ x_n \le f(x', t)\},$$

where $f(x', t)$ is a C^1 function of space and time. Moreover, there exists a positive dimensional constant c_0 and $1 < m < 2$ such that

$$|D_{x'} f(x', t) - D_{x'} f(y', t)| \le c_0\Big(-\log\Big|\frac{x'}{r} - \frac{y'}{r}\Big|\Big)^{-m},$$

$$|\partial_t f(x', t) - \partial_t f(x', s)| \le c_0\Big(-\log\Big|\frac{t}{t(x_0, r)} - \frac{s}{t(x_0, r)}\Big|\Big)^{-1/3}.$$

(2) *u is a classical solution of* (ST2) *in Σ_r in the sense that*

(i) *Du^+ exists in $\Omega(u)$ and is continuous up to $\overline{\Omega}(u)$;*

(ii) *Du^- exists in $\Omega(u)$ and is continuous up to $\Sigma_r \cap (\mathbb{R}^n - \Omega(u))$;*

(iii) *the free boundary condition is satisfied in the classical sense, i.e.,*

$$V_{x,t} = (|Du^+| - |Du^-|)(x,t) \quad \text{on } \Gamma(u) \cap \Sigma_r.$$

(3) *There exists a positive dimensional constant M such that*

$$M^{-1}\frac{u^+(x_0 - re_n, 0)}{r} \le |Du^+|(x,t) \le M\frac{u^+(x_0 - re_n, 0)}{r}$$

and

$$M^{-1}\frac{u^-(x_0 + re_n, 0)}{r} \le |Du^-|(x,t) \le M\frac{u^-(x_0 + re_n, 0)}{r}$$

in Σ_r.

Remark 1.2. Our result extends to the case where the star-shaped condition (I-a)–(I-b) is replaced by:

(I-ab) Ω_0 is locally Lipschitz with a sufficiently small Lipschitz constant.

See the discussion in Section 6.

The one phase version of the above result was proved in [Choi and Kim 2010] (see Theorem 2.16 in Section 2). Let us briefly motivate our result below in the context of the existing literature.

For a given reference point $(x_0, t_0) \in \mathbb{R}^n \times [0, \infty)$ and positive constants r and c, one can rescale the solution u of (ST2) as

$$\tilde{u} := \frac{1}{c}u\Big(x_0 + rx, t_0 + \frac{r^2}{c}t\Big). \tag{1-4}$$

Then $\tilde{u}$ satisfies the free boundary problem

$$\begin{cases} r\tilde{u}_t - \Delta\tilde{u} = 0 & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} < 0\}, \\ V = |D\tilde{u}^+| - |D\tilde{u}^-| & \text{on } \partial\{\tilde{u} > 0\}, \end{cases} \tag{$\tilde{P}$}$$

in a corresponding neighborhood of the origin. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, so that $x \in \mathbb{R}^n$ can be written as $x = (x', x_n)$, with $x_n = x \cdot e_n$. Choose $(x_0, t_0) = (x_0, 0)$ with $x_0 \in \Gamma_0(u)$. By our hypothesis, after a change of coordinates if necessary, there exists a Lipschitz function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with a small Lipschitz constant such that

$$\Omega_0(u) \cap B_{2r}(x_0) = \{x : x_n \le f(x')\}.$$

Let us choose

$$c = \max\{u^+(x_0 - re_n, 0), u^-(x_0 + re_n, 0)\} \tag{1-5}$$

so that one of $\tilde{u}^+(-e_n, 0)$ and $\tilde{u}^-(+e_n, 0)$ equals 1, and the other is less than 1.

Now suppose that we can show the following two conditions:

(A) $|\tilde{u}|(x,t) \le C$ in $B_1(0) \times [0,1]$ with a constant $C > 0$ independent of x_0 and r.

(B) The level sets of $\tilde{u}$ are Lipschitz graphs in space and time with small Lipschitz constant in $B_1(0) \times [0,1]$.

Then Theorem 1.1 follows from the results of [Athanasopoulos et al. 1996] applied to $\tilde{u}$. Indeed, (B) can be replaced by a relaxed version (B′) as stated below, which is sufficient to derive Theorem 1.1 due to the results of [Athanasopoulos et al. 1998].

(B′) The level sets of $\tilde{u}$ are ϵ-monotone with respect to cones of directions $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $\nu \in \text{span}(e_n, e_t)$, and $\pi/2 - \theta^x$ and ϵ sufficiently small.

(For the meaning of ϵ-monotonicity and the space and time cones W_x and W_t, see Definition 2.1.)

In our case (A) can be verified using previously known results in the one-phase Stefan problem (Lemma 3.3 and Lemma 3.4). Unfortunately, as shown in [Choi and Kim 2010], verifying (B′) for all scales r turns out to be as difficult as showing (B) or the full regularity of u. Since $\tilde{u}$ no longer satisfies the heat equation, one loses control of the change of u over time. For this reason it is necessary to show (B′) for all level sets of $\tilde{u}$, not just for the free boundary $\Gamma(\tilde{u})$. Indeed, in this article we will first show that $\tilde{u}$ (scaled correspondingly for the two-phase) is ϵ-monotone in the space variable (Lemma 3.1), and then we show that $\Gamma(\tilde{u})$ is ϵ-monotone in the space-time variables (Corollary 4.4 and Lemma 4.7). Then in Section 5 we use the ϵ-monotonicity obtained from previous sections, the almost-harmonicity of $\tilde{u}$ (Lemma 3.6), as well as the iteration methods originating from [Athanasopoulos et al. 1996; 1998] to show directly that $\tilde{u}$ is a classical solution and u satisfies (B) and (B′) (Section 5). The arguments in Section 5 are mostly drawn from [Athanasopoulos et al. 1996; 1998] as well as [Choi et al. 2007; 2009].

Let us now illustrate the underlying ideas in the analysis in Section 4, where we show the ϵ-monotonicity of the solution over time. In terms of the original solution u, verifying (A) and (B′) corresponds to analyzing u over the time interval $[0, t(x_0, r)]$, where $t(x_0, r)$ is given by

$$t(x_0, r) := r^2/c,$$

and c is as given in (1-5). Note that $t(x_0, r)$ coincides with the one given by (1-3).

Heuristically speaking, there are two possible scenarios for interface regularization, depending on its initial configuration in the local neighborhood:

(1) One of the phases has much bigger flux than the other, i.e.,

$$u^+(x_0 - se_n, 0) \gg u^-(x_0 + se_n, 0) \quad \text{or} \quad u^-(x_0 - se_n, 0) \ll u^-(x_0 + se_n, 0)$$

for s comparable to r.

In this case one-phase-like phenomena (regularization by the dominant phase as obtained in Theorem 2.16) are expected. As mentioned above, in this case the time interval for regularization of the free boundary in r-neighborhood is proportional to the distance it has traveled.

(2) Both phases are in balance, i.e.,

$$u^+(x_0 - se_n, 0) \sim u^-(x_0 + se_n, 0) \tag{1-6}$$

for s comparable to r.

In this case one expects regularization due to competition between two phases, resulting in Lipschitz-like behavior over time. Again the corresponding time interval for regularization amounts to $t(x_0, r)$ given in (1-3).

To make the above heuristics rigorous, in Section 4 we will introduce a decomposition procedure based on Harnack-type inequalities, which illustrates local dynamics near the free boundary: roughly speaking, for a given $r > 0$ we divide $B_r(x_0) \times \{t = 0\}$ into regions where (1-6) holds for $0 < s \ll r$ (*balanced* region) and the rest of domain (*unbalanced* region). (See detailed definitions of these regions in Section 4.) Of course the main issue is whether the dynamics of one region affect the other, in particular whether the one-phase-type dynamics of the unbalanced region breaks the property (1-6) in the balanced region for future times. We will show that this does not happen (Proposition 4.3), due to a fast regularization property in the unbalanced region (Proposition 3.7 and Lemma 4.7) as well as Harnack-type inequalities (Lemmas 4.5 and 4.6) in the balanced region.

Let us finish this section with an outline of the paper. In Section 2 we introduce preliminary results and notation, including the regularity results in the one-phase Stefan problem (Theorem 2.16). Sections 3 to 5 consist of the proof of Theorem 1.1; in Section 3 we prove some properties on the evolution of solutions of (ST2) with star-shaped data. In addition to Harnack inequalities, we show that the solution stays near the star-shaped profile for a unit time (Lemma 3.1), which in turn yields that the solution stays very close to harmonic functions (Lemma 3.6). This establishes that (B′) holds in the space variable. Making use of the results in Section 3, we perform a decomposition procedure in Section 4, to show that (A) holds for $\tilde{u}$ (Proposition 4.3) and that (B′) holds for $\Gamma(\tilde{u})$ (Corollary 4.4). This completes the main step in our analysis. In Section 5 we describe the rather technical iteration procedure leading to further regularization, and we complete the proof of Theorem 1.1 by combining arguments from [Athanasopoulos et al. 1996; 1998; Choi et al. 2007; 2009] (Theorem 5.7 and Corollary 5.8). In Section 6 we discuss a generalized proof of the corresponding regularization result (Theorem 6.1) when the star-shapedness of the initial data (I-a) and (I-b) are replaced by the local version (I-ab).

2. Preliminary lemmas and notation

We introduce some notation.

- For $x \in \mathbb{R}^n$, denote by $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $x_n = x \cdot e_n$.
- Let $B_r(x)$ be the space ball of radius r, centered at x.
- Let $Q_r := B_r(0) \times [-r^2, r^2]$ be the parabolic cube and let $K_r := B_r(0) \times [-r, r]$ be the hyperbolic cube.
- A caloric function in $\Omega \cap Q_r$ will denote a nonnegative solution of the heat equation, vanishing along the lateral boundary of Ω.
- For $x_0 \in \Gamma_0$ and $e_n = x_0/|x_0|$, define

$$t(x_0, d) := \min\left\{ \frac{d^2}{u^+(x_0 - de_n, 0)}, \frac{d^2}{u^-(x_0 + de_n, 0)} \right\}.$$

• C is called a universal constant if it depends only on the dimension n and the regularity constant N_0 of u_0.

• We say $a \sim b$ if there exists a dimensional constant $C > 0$ such that $C^{-1}b \le a \le Cb$.

• Lastly let us recall the definition of ϵ-monotonicity. Let $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $e \in \mathbb{R}^n$ and $\nu \in \text{span}(e_n, e_t)$, respectively, denote a spatial circular cone of aperture $2\theta^x$ and axis in the direction of e, and a two-dimensional space-time cone in (e_n, e_t) plane of aperture $2\theta^t$ and axis in the direction of ν.

Definition 2.1. (a) Given $\epsilon > 0$, a function w is called *ϵ-monotone* in the direction τ if

$$u(p + \lambda\tau) \ge u(p) \quad \text{for any } \lambda \ge \epsilon.$$

(b) w is *ϵ-monotone* in a cone of directions $W_x(\theta^x, e)$ or $W_t(\theta^t, \nu)$ if w is ϵ-monotone in every direction in the cone.

Next we state preliminary results that are important in our analysis. The first lemma is a direct consequence of the interior Harnack inequalities proved in [Caffarelli and Cabré 1995].

Lemma 2.2. *Suppose $w(x) : \mathbb{R}^n \to \mathbb{R}$ has bounded Laplacian. Then w is Hölder continuous with its constant depending on the Laplacian bound.*

Lemma 2.3 [Fabes et al. 1984, Theorem 3]. *Let Ω be a domain in $\mathbb{R}^n \times \mathbb{R}$ such that $(0, 0)$ is on its lateral boundary. Suppose Ω is a $\text{Lip}^{1,1/2}$ domain, i.e.,*

$$\Omega = \{(x', x_n, t) : |x'| < 1, |x_n| < 2L, |t| < 1, x_n \le f(x', t)\},$$

where f satisfies $|f(x', t) - f(y', s)| \le L(|x' - y'| + |t - s|^{1/2}.)$ If u is a caloric function in Ω, then there exists $C = C(n, L)$, where L is the Lipschitz constant for Ω, such that

$$\frac{u(x, t)}{v(x, t)} \le C \frac{u(-Le_n, \frac{1}{2})}{v(-Le_n, -\frac{1}{2})}$$

for $(x, t) \in Q_{1/2}$.

Lemma 2.4 [Athanasopoulos et al. 1996, Theorem 1]. *Let Ω be a Lipschitz domain in $\mathbb{R}^n \times \mathbb{R}$, i.e.,*

$$Q_1 \cap \Omega = Q_1 \cap \{(x, t) : x_n \le f(x', t)\},$$

where f satisfies $|f(x, t) - f(y, s)| \le L(|x - y| + |t - s|)$. Let u be a caloric function in $Q_1 \cap \Omega$ with $(0, 0) \in \partial\Omega$ and $u(-e_n, 0) = m > 0$ and $\sup_{Q_1} u = M$. Then there exists a constant C, depending only on $n, L, m/M$ such that

$$u(x, t + \rho^2) \le Cu(x, t - \rho^2)$$

for all $(x, t) \in Q_{1/2} \cap \Omega$ and for $0 \le \rho \le d_{x,t}$.

Lemma 2.5 [Athanasopoulos et al. 1996, Lemma 5]. *Let u and Ω be as in Lemma 2.4. Then there exist $a, \delta > 0$ depending only on $n, L, m/M$ such that*

$$w_+ := u + u^{1+a} \quad \text{and} \quad w_- := u - u^{1+a}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Next we state several properties of harmonic functions:

Lemma 2.6 [Dahlberg 1979]. *Let* u_1, u_2 *be two nonnegative harmonic functions in a domain* D *of* $\mathbb{R}^n$ *of the form*

$$D = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 2, |x_n| < 2L, x_n > f(x')\},$$

with f *a Lipschitz function with constant less than* L *and* $f(0) = 0$. *Assume further that* $u_1 = u_2 = 0$ *along the graph of* f. *Then in*

$$D_{1/2} = \{|x'| < 1, |x_n| < L, x_n > f(x')\}$$

we have

$$0 < C_1 \le \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, L)}{u_1(0, L)} \le C_2,$$

with C_1, C_2 *depending only on* L.

Lemma 2.7 [Jerison and Kenig 1982]. *Let* D, u_1 *and* u_2 *be as in Lemma 2.6. Assume further that*

$$\frac{u_1(0, L/2)}{u_2(0, L/2)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ *is Hölder continuous in* $\overline{D}_{1/2}$ *for some coefficient* α, *both* α *and the* C^α *norm of* u_1/u_2 *depending only on* L.

Lemma 2.8 [Caffarelli 1987]. *Let* u *be as in Lemma 2.6. Then there exists* $c > 0$ *depending only on* L *such that for* $0 < d < c$, $\frac{\partial}{\partial x_n} u(0, d) \ge 0$ *and*

$$C_1 \frac{u(0, d)}{d} \le \frac{\partial u}{\partial x_n}(0, d) \le C_2 \frac{u(0, d)}{d},$$

where $C_i = C_i(M)$.

Lemma 2.9 [Jerison and Kenig 1982, Lemma 4.1]. *Let* Ω *be Lipschitz domain contained in* $B_{10}(0)$. *There exists a dimensional constant* $\beta_n > 0$ *such that for any* $\zeta \in \partial\Omega$, $0 < 2r < 1$ *and positive harmonic function* u *in* $\Omega \cap B_{2r}(\zeta)$, *if* u *vanishes continuously on* $B_{2r}(\zeta) \cap \partial\Omega$, *then for* $x \in \Omega \cap B_r(\zeta)$,

$$u(x) \le C\Big(\frac{|x - \zeta|}{r}\Big)^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\},$$

where C *depends only on the Lipschitz constants of* Ω.

Next, we point out that we use the notion of viscosity solutions for our investigation. When $\{u_0 = 0\}$ is of zero Lebesgue measure, it was proved in [Kim and Požár 2011] that the viscosity solution of (ST2) is unique and coincides with the usual weak solutions. (See [Kim and Požár 2011] for the definition as well

as other properties of viscosity solutions.) Below we state important properties of viscosity solutions for (ST2) that relate our solutions to the one-phase version of our problem,

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u_t/|Du| = |Du| & \text{on } \partial\{u > 0\}, \\ u(\cdot, 0) = u_0 \geq 0. \end{cases} \tag{ST1}$$

Lemma 2.10. *Suppose u is a viscosity solution of* (ST2)*. Then*:

(a) *u is caloric in its positive and negative phases.*

(b) *$-u$ is also a viscosity solution of* (ST2) *with boundary data $-g$.*

(c) *$u^+ = \max(u, 0)$ (or $u^- = -\min(u, 0)$) is a viscosity subsolution of* (ST1) *with initial data u_0^+ (or u_0^-).*

We say that a pair of functions $u_0, v_0 : \overline{D} \to [0, \infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset \mathbb{R}^n$ if:

(i) The support of u_0, $\operatorname{supp}(u_0) = \overline{\{u_0 > 0\}}$, restricted to $\overline{D}$ is compact.

(ii) $u_0(x) < v_0(x)$ in $\operatorname{supp}(u_0) \cap \overline{D}$.

Lemma 2.11 [Kim and Požár 2011, Comparison principle]. *Let u, v be, respectively, viscosity sub- and supersolutions of* (ST2) *in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \overline{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [0, T)$.*

Below we state a distance estimate for the free boundary and Harnack inequality for the one-phase solution u of (ST1).

Lemma 2.12 [Choi and Kim 2010, Lemma 2.2]. *Let u be given as in Theorem 2.16. There exists $t_0 = t_0(N_0, M_0, n) > 0$ such that if $x_0 \in \Gamma_0$ and $t \leq t_0$, then*

$$\frac{1}{C} t^{1/(2-\alpha)} \leq d(x_0, t) \leq C t^{1/(2-\beta)}, \tag{2-1}$$

where α and β are given in (1-2)*, C depends on N_0, M_0 and n, and $d(x_0, t)$ denotes the distance that Γ moved from the point x_0 during the time t, i.e.,*

$$d(x_0, t) := \sup\{d : u(x_0 + d e_n, t) > 0\}.$$

Lemma 2.13 [Choi and Kim 2010, Lemma 2.3]. *Let u be given as in Theorem 2.16. There exists d_0 depending on N_0, M_0 and n such that if $x_0 \in \Gamma_0$ and $d \leq d_0$, then*

$$u(x_0 - d e_n, t) \leq C u(x_0 - d e_n, 0) \quad \text{for } 0 \leq t \leq t(x_0, d),$$

where C depends on N_0, M_0 and n.

The following monotonicity formula by Alt–Caffarelli–Friedman prevents the scenario that both phases compete with large pressure in our problem.

Lemma 2.14 [Alt et al. 1984]. *Let h_+ and h_- be nonnegative continuous functions in $B_1(0)$ such that $\Delta h_\pm \geq 0$ and $h_+ \cdot h_- = 0$ in $B_1(0)$. Then the functional*

$$\phi(r) = \frac{1}{r^4} \int_{B_r(0)} \frac{|Dh_+|^2}{|x|^{n-2}}\, dx \int_{B_r(0)} \frac{|Dh_-|^2}{|x|^{n-2}}\, dx$$

is monotone increasing in r, $0 < r < 1$.

Corollary 2.15. *Let $\partial\Omega_0 \subset \mathbb{R}^n$ be star-shaped with respect to the ball $B_1(0) \subset \Omega_0$ and suppose that $B_{4/3}(0) \subset \Omega_0 \subset B_{5/3}(0)$. Let h_+ be the harmonic function in $\Omega_0 - B_1(0)$ with boundary values $h_+ = 0$ on $\partial\Omega_0$, and $h_+ = 1$ on $\partial B_1(0)$. Let h_- be the harmonic function in $B_2(0) - \Omega_0$ with boundary values $h_- = 0$ on $\partial\Omega_0$, and $h_- = 1$ on $\partial B_2(0)$. Then there exists a sufficiently large dimensional constant $M > 0$ such that*

$$\frac{h_+(x_0 - re_n)}{r} \geq M \ \text{ implies } \ \frac{h_-(x_0 + re_n)}{r} \leq 1$$

for $x_0 \in \partial\Omega_0$, $e_n = x/|x|$ and $0 \leq r \leq 1/6$.

Proof. This follows from Lemma 2.14, since

$$\begin{aligned}
\Big(\frac{h_+(x_0 - re_n)}{r} \cdot \frac{h_-(x_0 + re_n)}{r}\Big)^2 &\sim \frac{1}{(2r)^4} \int_{B_{r/2}(x_0 - re_n)} \frac{|Dh_+|^2}{|x - x_0|^{n-2}}\, dx \cdot \int_{B_{r/2}(x_0 + re_n)} \frac{|Dh_-|^2}{|x - x_0|^{n-2}}\, dx \\
&\leq \frac{1}{(2r)^4} \int_{B_{2r}(x_0)} \frac{|Dh_+|^2}{|x - x_0|^{n-2}}\, dx \cdot \int_{B_{2r}(x_0)} \frac{|Dh_-|^2}{|x - x_0|^{n-2}}\, dx \\
&= \phi(2r) \leq \phi(1/3) \leq C_n. \qquad \square
\end{aligned}$$

Lastly, let us finish this section by stating the results obtained in [Choi and Kim 2010] for the one-phase version of our problem in the local setting:

Theorem 2.16 [Choi and Kim 2010, Theorem 0.1]. *Suppose a nonnegative function $u(x, t)$ is a solution of* (ST1) *in $B_2(0) \times [0, 1]$, $0 \in \Gamma_0(u)$, with the initial data $u_0 \geq 0$ satisfying (I-b), (I-c) and (I-d) in $B_2(0)$. Suppose the initial data satisfies*

$$\{u(x, 0) \geq 0\} = \{x + x_0 : x_n \leq f(x')\}$$

in $B_1(0)$, where f is a Lipschitz function with Lipschitz constant $L < L_n$. Further, suppose $u_0(-e_n) = 1$ and $\sup_{B_2(0)\times[0,1]} u \leq M_0$.

For given $r > 0$, let us define

$$t(x_0, r) := \frac{r^2}{u(x_0 + re_n, 0)}.$$

Then there exists a small $c_0 > 0$ depending on M_0 and n such that in $\Sigma_r = B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$ for $r \leq c_0$, we have:

(1) *Theorem 1.1*(1) *holds for u.*

(2) *u is a classical solution of* (ST1) *in Σ_r in the sense that the spatial derivative Du exists in $\Omega(u)$ and is continuous up to $\overline{\Omega}(u)$, and the free boundary condition is satisfied in the classical sense, i.e.,*

$$V_{x,t} = |Du|(x,t) \quad \text{on } \Gamma(u) \cap \Sigma_r.$$

(3) *There exists a positive constant M depending on M_0 and n such that*

$$M^{-1}\frac{u(x_0 - re_n, 0)}{r} \le |Du|(x,t) \le M\frac{u(x_0 - re_n, 0)}{r}.$$

(4) *If $x \in \Gamma_0(u) \cap B_{c_0}(0)$ and $x + re_n \in \Gamma_t(u) \cap B_{c_0}(0)$, then*

$$M^{-1}\frac{u(x - re_n, 0)}{r} \le |Du(x + re_n, t)| = V_{x+re_n,t} \le M\frac{u(x - re_n, 0)}{r},$$

where M depends on n and M_0. In particular,

$$\frac{r}{t} \sim |Du(x + re_n, t)| \sim \frac{u(x - re_n, 0)}{r}.$$

Theorem 2.16 states that the free boundary regularizes in a scale proportional to the distance it has traveled. Note that the regularity results hold up to the initial time and all the regularity assumptions are imposed only on the initial data.

3. Properties of solutions with star-shaped initial data

Lemma 3.1. *If Ω_0 and u_0 are star-shaped with respect to the ball $B_{r_0}(0) \subset \Omega_0$, then $\Omega_t(u)$ and $u(\cdot, t)$ stays σ-close to star-shaped for all $0 \le t \le \frac{1}{3}\sigma^{1/5}$ (see Figure 2).*

Proof. Step 1. For any $a > 0$, the parabolic scaling $(x,t) \to (ax, a^2t)$ preserves both the heat operator

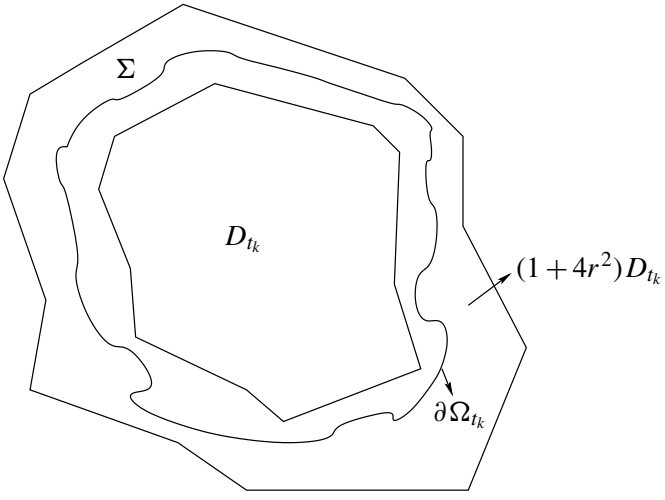

Figure 2. Approximation of the positive phase by a star-shaped domain.

and the boundary motion law in (ST2). Therefore, for any $\sigma > 0$ the function

$$u_1(x,t) := u\big((1+\sigma)(x-x_0)+x_0,\,(1+\sigma)^2 t\big)$$

is also a viscosity solution of (ST2) with corresponding initial data.

Step 2. Choose $x_0 \in B_{r_0}(0)$. Take a small $c_0 > 0$ such that $B_{r_0+c_0}(0) \subset \Omega_0$. We claim that for $0 \le \delta \le \sigma^{6/5}$,

$$u_1(x,0) \le u(x,\delta) \quad \text{in } B_R(0) - B_{r_0+c_0}(0) \tag{3-1}$$

if σ is small enough. To show (3-1), let us introduce another function

$$\tilde{u}(x,0) := u\big((1+\tfrac{1}{2}\sigma)(x-x_0)+x_0,\,0\big).$$

Also let v^* be the solution of the one phase problem (ST1) with initial data u_0^-, and with $v^* = 1$ on $\partial B_R(0)$. Note also that, due to Lemma 2.10, u^- is a subsolution of (ST1) with initial data $v^*(x,0) = u^-(x,0)$.

Thus by Lemma 2.11, $v^* \le u^-$. It follows that $\Omega_t(v^*) \subset \Omega_t(u) \subset \Omega_t(u)$. Hence by Lemma 2.12 applied to $-v^*$,

$$\Omega_0(\tilde{u}) \subset \Omega_t(u) \quad \text{for } 0 \le t \le \sigma^{7/6}.$$

Moreover, due to our assumption,

$$\tilde{u}(x,0) \le u_0(x).$$

Therefore, the maximum principle for caloric functions implies

$$w(x,t) \le u(x,t),$$

where w solves the heat equation in the cylindrical domain $D = \Omega_0(\tilde{u}) \times [0, \sigma^{7/6}]$ with initial data $\tilde{u}(x,0)$ and zero boundary data on $\partial\Omega_0(\tilde{u}) \times [0, \sigma^{7/6}]$.

Now w_t solves the heat equation in D,

$$w_t = \Delta w \ge -C \text{ at } t = 0 \quad \text{and} \quad w_t = 0 \text{ on } \partial\Omega_0(\tilde{u}).$$

Therefore we conclude that $w_t \ge -C$ in D. In particular

$$w(x,\delta) \ge \tilde{u}(x,0) - C\delta. \tag{3-2}$$

Next we compare $u_1(x,0)$ with $w(x,\delta)$. Observe that for $x \in B_R(0) - B_{r_0+c_0}(0)$,

$$\begin{aligned} u_1(x,0) = \tilde{u}(x,0) + \int_{\sigma/2}^{\sigma} \big((x-x_0)\cdot Du((1+s)(x-x_0)+x_0,0)\big)\,ds &\le \tilde{u}(x,0) - c_0\sigma^{7/6} \\ &\le \tilde{u}(x,0) - C\sigma^{6/5} \\ &\le w(x,\delta) \le u(x,\delta) \end{aligned}$$

for $0 \le \delta \le \sigma^{6/5}$, where the first inequality follows from our assumption (I-d) on u_0, the second inequality follows if σ is sufficiently small, and the third inequality follows from (3-2). Hence we conclude (3-1).

Step 3. Our goal is to prove that for $0 \le \delta \le \sigma^{6/5}$,

$$u_1(x,t) \le u_2(x,t) := u(x,t+\delta) \tag{3-3}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0, \sigma^{1/5}]$. Note that the inequality holds at $t = 0$ by Step 2. However, we need a few more arguments since we do not know yet if the lateral boundary data on $\partial B_{r_0+c_0}(0)$ is properly ordered.

Suppose

$$\Omega(u_1) \subset \Omega(u) \quad \text{for } 0 \le t \le t_0$$

and $\Omega(u_1)$ contacts $\partial\Omega(u)$ for the first time at $t = t_0$. Observe then that

$$f(x, t) := u(x, t + \delta) - u_1(x, t)$$

solves the heat equation in $\Omega(u_1)$ with nonnegative boundary data for $0 \le t \le t_0$, with

$$f(x, 0) \ge 0 \quad \text{in } B_R(0) - B_{r_0+c_0}(0).$$

Indeed, following the computation given above, it follows that

$$f(x, 0) \ge c_0\sigma \quad \text{in } B_{r_0+c_0}(0) - B_{r_0+c_0/2}(0).$$

On the other hand, due to the fact that $w_t \ge -C$ and $\delta \le \sigma^{6/5}$, we have

$$f(x, 0) \ge (w(x, \delta) - w(x, 0)) + (w(x, 0) - u_1(x, 0)) \ge -C\sigma^{6/5} \quad \text{in } B_{r_0+c_0/2}(0).$$

Therefore we have

$$f(x, t) > 0 \quad \text{on } \partial B_{r_0+c_0}(0) \times [0, t_0]$$

if $t_0 \ll 1$. But then this contradicts Lemma 2.11 applied to the region $(B_R(0) - B_{r_0+c_0}(0)) \times [0, t_0]$.

Step 4. From (3-3) of Step 3, we obtain

$$u\big((1+\sigma)(x - x_0) + x_0, (1+\sigma)^2 t\big) \le u(x, t + \delta) \tag{3-4}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0, \sigma^{1/5}]$ for any $x_0 \in B_{r_0}(0)$, as long as σ and δ are sufficiently small and satisfy $0 \le \delta \le \sigma^{6/5}$. As a result, for $0 \le t \le \frac{1}{3}\sigma^{1/5}$, we can choose $\delta = \sigma(2+\sigma)t \le \sigma^{6/5}$ such that

$$(1+\sigma)^2 t = t + \delta.$$

It follows then from (3-4) that the function $u(\cdot, t)$ is σ-monotone with respect to the cone of directions W_x in $(B_R(0) - B_{r_0+c_0}(0))$ for $t \in [0, \frac{1}{3}\sigma^{1/5}]$. (Here $W_x = \{\nu \in S^n : \nu = (x - x_0)/|x - x_0| \text{ for some } x_0 \in B_{r_0}(0)\}$.) □

Remark 3.2. For $x \in \Gamma_0$, we may let $e_n = x/|x|$ after a rotation. Then, due to (I-b),

$$t(x, d) := \min\left\{ \frac{r^2}{u^+(x - re_n, 0)}, \frac{r^2}{u^-(x + re_n, 0)} \right\} \in [r^{7/6}, r^{5/6}] \ll r^{4/5}, \tag{3-5}$$

where $t(x, r)$ is the time it takes for the free boundary to regularize in $B_r(0)$. Therefore, we have, for $0 \le t \le t(x_0, r)$,

$$u(\cdot, t) \text{ is } r^4\text{-monotone with respect to } W_x \text{ in } (B_R(0) - B_{r_0+c_0}(0)).$$

This property will ensure that our solution u has its level sets close to Lipschitz graphs in the space variable in an appropriate scale, which serves as the first step towards the regularization argument; see Lemma 3.6.

Lemma 3.3 (Harnack at $t = 0$). *Let u be as in Theorem 1.1. For $x \in \Gamma_0$, we may set $e_n = x/|x|$ after a rotation. Then for all $s > 0$ and for $0 \le t \le t(x, s)$ we have*

$$u^+(x - se_n, t) \le C_1 u^+(x - se_n, 0) \quad \text{and} \quad u^-(x + se_n, t) \le C_1 u^-(x + se_n, 0),$$

where $e_n = x/|x|$.

Proof. Let v^{**} solve the one-phase Stefan problem (ST1) with initial data $v_0^{**}(x) = u_0^+(x)$. Then v^{**} is also a solution of (ST2) with $u_0(x) \le v_0^{**}(x)$, and thus by Lemma 2.11 we have

$$u(x, t) \le v^{**}(x, t).$$

Therefore it follows from one-phase Harnack inequality applied for $v^{**}(x, t)$ that

$$u^+(x - se_n, t) \le v^{**}(x - se_n, t) \le C_1 v^{**}(x - se_n, 0) = C_1 u(x - se_n, 0)$$

for $0 \le t \le t_0$, where $t_0 = s^2/u(x - se_n, 0) \ge t(x, s)$.

As for $u^-(x, t)$, we compare u^- with the solution v^* of (ST1) with initial data $v_0^*(x) = u_0^-(x)$ and with boundary data $v^* = 1$ on $\partial B_R(0)$. The rest of the argument is parallel to the above one. □

Lemma 3.4 (backward Harnack at $t = 0$). *Let u be as in Theorem 1.1. Let $x \in \Gamma_0$ and let $e_n = x/|x|$ after a rotation. Then for $s > 0$ and for $0 \le t \le t(x, s)$,*

$$u^+(x - se_n, 0) \le C_1 u^+(x - se_n, t) \quad \text{and} \quad u^-(x + se_n, 0) \le C_1 u^-(x + se_n, t).$$

Proof. We will only show the lemma for u^+. The other part follows by a parallel argument. Let v^* solve the one phase problem (ST1) with initial data u_0^- and with boundary data 1 on $\partial B_R(0)$. Then $-v^*$ is also a solution of (ST2) with $-v_0^* \le u_0$, and thus by Lemma 2.11, $-v^* \le u$. This inequality implies that

$$\{v^* = 0\} \subset \{u \ge 0\}.$$

Note that $\Omega(v^*)$ moves according to the one-phase dynamics, which have been studied in detail in [Choi and Kim 2006]. In particular we know that $\Omega(v^*)$ will be Lipschitz at each time. Moreover, for a boundary point $(x, t) \in \Gamma(v^*)$ and $d := \operatorname{dist}(x, \Gamma_0(v^*))$, the normal velocity $V_{x,t}$ satisfies

$$V_{x,t} = |Dv^*(x, t)| \sim \frac{v^*(x + 2de_n, 0)}{2d} \le d^{\beta-1} \le t^{(\beta-1)/(2-\alpha)}, \tag{3-6}$$

where the last inequality follows from Lemma 2.12. Let $v_*(x, t)$ solve the heat equation in $\{v^* = 0\}$ with initial data $u_0(x)$ and boundary data 0 on the lateral boundary of $\partial\{v^* = 0\}$, i.e., v_* solves

$$\begin{cases} \partial_t v_* - \Delta v_* = 0 & \text{in } \{v^* = 0\} = B_R(0) \times [0, 1] - \Omega(v^*), \\ v_*(x, 0) = u_0(x) & \text{on } \{v^* = 0\} \cap \{t = 0\}, \\ v_* = 0 & \text{on } \partial\{v^* = 0\} \cap \{t > 0\}. \end{cases}$$

Since

$$\Omega(v_*) = \{v^* = 0\} \subset \{u \geq 0\},$$

we have $v_*(x,t) \leq u(x,t)$ in $\{v^* = 0\}$. Moreover, for any given $t > 0$, $\tilde{v}^-(x,s) := v^*(\sqrt{t}x, ts)$ satisfies the assumptions of Lemma 2.5. Thus it follows that $v^*(\cdot\,,t)$ is t^a-close to a harmonic function in $B_{\sqrt{t}}(x)$ for some $a > 0$, where $x \in \Gamma_0$. Moreover, due to the assumption on the initial data, $(v_*)_t = \Delta v_* \geq -C$ at $t = 0$. Also on $\Gamma(v_*)$,

$$(v_*)_t/|Dv_*| = -(v^*)_t/|Dv^*| = -|Dv^*| \geq -t^{(\beta-1)/(2-\alpha)}.$$

Here the first equality follows since $(v_*)_t/|Dv_*|$ and $-(v^*)_t/|Dv^*|$ are the normal velocities of their respective level sets $\Gamma(v_*)$ and $\Gamma(v^*)$, but $\Gamma(v^*) = \Gamma(v_*)$ by definition. The second equality follows since v^* solves the one phase problem (ST1), and the last inequality follows from (3-6).

Since $\Omega(v_*)$ is Lipschitz and $\Gamma_t(v_*) = \Gamma_t(v^*)$ is regularized in space over time (see Theorem 2.16), (3-6) also holds for $|Dv_*|$.

Hence on $\Gamma(v_*)$,

$$(v_*)_t = -|Dv^*||Dv_*| \geq -t^{2(\beta-1)/(2-\alpha)} > -t^{-2/5},$$

where α and β are the growth rates defined in (1-2), and the last inequality follows from the assumption (I-b). Since $(v_*)_t$ solves a heat equation in $\Omega(v_*)$, it follows that for $x \in \Gamma_0$,

$$(v_*)_t \geq -t^{-2/5} \quad \text{in } B_{\sqrt{t}/2}(x - \sqrt{t}e_n) \times [0,t]. \tag{3-7}$$

Then since $v_*(x - \sqrt{t}e_n, 0) \geq (\sqrt{t})^\alpha \geq (\sqrt{t})^{7/6} = t^{7/12}$, for $x \in \Gamma_0$ we have

$$\begin{aligned} v_*(x - \sqrt{t}e_n, t) = v_*(x - \sqrt{t}e_n, 0) + \int_0^t (v_*)_t(x - \sqrt{s}e_n, s)\,ds &\geq v_*(x - \sqrt{t}e_n, 0) - \tfrac{5}{3}t^{3/5} \\ &\geq \tfrac{1}{2}v_*(x - \sqrt{t}e_n, 0) + \tfrac{1}{2}t^{7/12} - \tfrac{5}{3}t^{3/5} \\ &\geq \tfrac{1}{2}v_*(x - \sqrt{t}e_n, 0) \end{aligned}$$

if t is sufficiently small. It follows that

$$u^+(x - \sqrt{t}e_n, 0) = v_*(x - \sqrt{t}e_n, 0) \leq 2v_*(x - \sqrt{t}e_n, t) \leq 2u^+(x - \sqrt{t}e_n, t),$$

where the first inequality follows from (3-7).

Since $\Gamma(v_*) = \Gamma(v^*)$ is Lipschitz in a parabolic scaling, v_* is almost harmonic. Hence $v_*(\cdot\,,t)$ is bigger than the harmonic function $\omega^t(x)$ in $\Omega_t(v_*) \cap B_{\sqrt{t}}(x)$ with its value

$$\omega^t(x - \sqrt{t}e_n) = (C_1)^{-1}u^+(x - \sqrt{t}e_n, 0).$$

Note that if $0 \leq t \leq t(x,s)$, then $s < \sqrt{t}$. Hence for $0 \leq t \leq t(x,s)$,

$$C_1u^+(x - se_n, t) \geq C_1v_*(x - se_n, t) \geq C_1\omega^t(x - se_n) \geq Cu^+(x - se_n, 0),$$

where the last inequality follows since the one-phase result implies a power law on the movement of $\Gamma(v^*) = \Gamma(v_*)$ (see Lemma 2.5 of [Choi et al. 2007]), and this yields a bound on $u^+(x - se_n, 0)/\omega^t(x - se_n)$.

Similar arguments apply to u^-, if we consider the function v^{**} solving (ST1) with initial data u_0^+, and the function $v^\star$ solving the heat equation in $\{v^{**}=0\}$ with initial data u_0 and with boundary data 0 on $\Gamma(v^{**})$ and -1 on $\partial B_R(0)$. □

Lemma 3.5 (distance estimate at $t=0$). *Let u be as in Theorem 1.1. Let $x\in\Gamma_0$ and let $e_n = x/|x|$ after a rotation. Let s be a sufficiently small positive constant. If*

$$\frac{|u^+(x-se_n,0)|}{s}\le m \quad \textit{and} \quad \frac{|u^-(x+se_n,0)|}{s}\le m,$$

then for $t\in[0,s/m]$,

$$d(x,t)=\sup\{r : x+re_n \text{ or } x-re_n\in\Gamma_t(u)\}\le s.$$

Proof. Let v^{**} solve (ST1) with initial data u_0^+, and let v^* solve (ST1) with initial data u_0^- and with $v^*=1$ on $\partial B_R(0)$. Then by comparison, $-v^*\le u\le v^{**}$ and the lemma follows from the one-phase result of Theorem 2.16. □

In the next lemma, we approximate our solution by harmonic functions.

Note that, due to Lemma 3.1, We know that the rescaled function $\tilde u(x,t)$ as given in (1-4) satisfies the condition (B′) in the space variable. On the other hand, it is not clear if the level sets of u are close to Lipschitz graphs in the time variable. The approximation by harmonic functions given by Lemma 3.6, as well as Harnack-type inequalities obtained at $t=0$ and at future times, will ensure us that $\Gamma(u)$ is almost Lipschitz in the time variable as well (Corollary 4.4). This fact will serve as the first step towards the regularization procedure in Section 5.

Lemma 3.6 (spatial regularity in the whole domain). *Let u be as in Theorem 1.1. Then there exists a positive constant r_0 depending only on n such that for $x_0\in\Gamma_0$ and $0<r<r_0$, there exists a function $\omega(x,t):=\omega^+(x,t)-\omega^-(x,t)$ that satisfies*:

(a) *$\omega(\cdot\,,t)$ is harmonic in its positive and negative phase in $(1+r)\Omega_t(u)-(1-r)\Omega_t(u)$, and $\Omega(\omega^+)$, $\Omega(\omega^-)$ are star-shaped with respect to $B_{r_0}(0)$ given in* (I-a).

(b) *For a dimensional constant $C>0$, we have*

$$\omega^+(x,t)\le u^+(x,t)\le C\omega^+((1-r^{5/4})x,t) \quad \textit{and} \quad \omega^-(x,t)\le u^-(x,t)\le C\omega^-((1+r^{5/4})x,t)$$

in $B_r(x_0)\times[r^2, t(x_0,r)]$.

Note that $t(x_0,r)\ge r^{7/6}\ge r^2$, and $\partial\{\omega^+>0\}$ need not be $\partial\{\omega^->0\}$.

Proof. Step 1. We will only show the lemma for u^+. For a given $x_0\in\Gamma_0$, we may assume that $e_n = x_0/|x_0|$ after a rotation.

First we will construct a barrier function v_1 which will serve as a supersolution of (ST2). For this, let us first consider the viscosity solution $u^\star$ of (ST1) with the initial data u_0^+ for $0\le t\le t_0$. We may assume that for t_0 small compared to R the support of $u^\star$ stays inside $B_R(0)$. Let us define

$$\Omega_+^\star:=\{u^\star>0\},\quad \Gamma^\star:=\partial\{u^\star=0\},\quad \Omega_-^\star:=B_R(0)\times[0,t_0]-\overline{\Omega}_+^\star.$$

Now let v_1 solve the heat equation in $\Omega_+^\star$ and in $\Omega_-^\star$, with initial data u_0 and with $v_1 = -1$ on $\partial B_R(0)$. In other words, $v_1 = v_1^+ - v_1^-$, where

$$\begin{cases} \partial_t v_1^+ - \Delta v_1^+ = 0 & \text{in } \Omega_+^\star, \\ v_1^+(x,0) = u_0^+(x) & \text{on } \{t=0\}, \\ v_1^+ = 0 & \text{on } \Gamma^\star, \end{cases}$$

and

$$\begin{cases} \partial_t v_1^- - \Delta v_1^- = 0 & \text{in } \Omega_-^\star, \\ v_1^-(x,0) = u_0^-(x) & \text{on } \{t=0\}, \\ v_1^- = 0 & \text{on } \Gamma^\star, \\ v_1^- = 1 & \text{on } \partial B_R(0) \times [0,1]. \end{cases}$$

Note that v_1 solves the heat equation in two regions $\Omega_+^\star$ and $\Omega_-^\star$, with free boundary $\Gamma^\star$. Also note that $v_1^+ = u^\star$ and $\partial_t v_1^+ = |Dv_1^+|^2$ on $\Gamma^\star$ since the boundary $\Gamma^\star$ is obtained from the one phase problem with initial data u_0^+. Hence we can observe that v_1 is a supersolution of the two-phase problem (ST2).

Similarly one can construct a subsolution of (ST2): let us consider $\tilde{u}^\star$: the viscosity solution of (ST1) in $B_R(0) \times [0, t_0]$ with the initial data u_0^- and fixed boundary data 1 on $\partial B_R(0) \times [0, t_0]$. Let us define

$$\widetilde{\Omega}_-^\star := \{\tilde{u}^\star > 0\}, \quad \widetilde{\Gamma}^\star := \partial\{\tilde{u}^\star = 0\}, \quad \widetilde{\Omega}_+^\star := B_R(0) \times [0,t_0] - \overline{\widetilde{\Omega}_-^\star}.$$

Now let v_2 solve the heat equation in two regions $\widetilde{\Omega}_-^\star$ and $\widetilde{\Omega}_+^\star$, with boundary data 0 on $\widetilde{\Gamma}^\star$ and -1 on $\partial B_R(0)$, and with initial data u_0. Note that $v_2^- = \tilde{u}^\star$. Then v_2 is a subsolution of (ST2), and by comparison,

$$v_2 \le u \le v_1. \tag{3-8}$$

Hence the free boundary of u is trapped between the free boundaries of v_1 and v_2. Note that the free boundaries $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ of v_1 and v_2 are obtained from the one-phase problem (ST1). Hence by Theorem 2.16(a), $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ are Lipschitz in space in $B_d(x_0)$ for a small constant $d > 0$. Also, Theorem 2.16(c) implies that for $\delta \in [d/2, d]$ and $x_0 + \delta e_n \in \Gamma_t^\star$, the normal velocity $V_{x_0+\delta e_n,t}$ of $\Gamma^\star$ at $(x_0 + \delta e_n, t)$ satisfies

$$V_{x_0+\delta e_n,t} = |Dv_1^+(x_0 + \delta e_n, t)| \sim \frac{d}{t} \sim \frac{u_0^+(x_0 - de_n)}{d} \le d^{\beta-1}.$$

Since $d/t \le d^{\beta-1}$, we obtain

$$t \ge d^{2-\beta} > d^2.$$

Hence the above speed bound of $\Gamma^\star$ implies that $\Omega_+^\star$ and $\Omega_-^\star$ are Lipschitz in space and time, in parabolic scaling. Then by Lemma 2.5, v_1^+ and v_1^- are almost harmonic up to a d-neighborhood of their free boundaries for $t \ge d^2$. Similarly, we obtain that v_2^+ and v_2^- are almost harmonic up to a d-neighborhood of their free boundaries for $t \ge d^2$.

Next we fix $r \le d$. Note that if $t \le t(x_0, r)$, then by Theorem 2.16(c), both of the sets $\Gamma_t(v_1)$ and $\Gamma_t(v_2)$ are within distance r of $\Gamma_0(u)$ in $B_r(x_0)$ during this time. In particular, arguments parallel to the

ones in the proofs of Lemmas 2.1 and 2.3 in [Choi and Kim 2010] yield that

$$\sup\{u(y,s) : (y,s) \in B_d(x_0) \times [0, d^2]\} \sim u(x - de_n, 0).$$

Now using the almost harmonicity of v_1^+ and v_2^+, we conclude that for $0 \le t \le t(x_0, r)$,

$$v_2(x_0 - 2re_n, t) \sim u_0(x_0 - 2re_n, 0) \sim v_1(x_0 - 2re_n, t). \tag{3-9}$$

Step 2. Observe that by the definition of $t(x_0, r)$ and the assumption on the growth rates of u_0,

$$r^{2-\beta} \le t(x_0, r) \le r^{2-\alpha} \le r^{5/6} := \tau. \tag{3-10}$$

Due to Lemma 3.1, we know that at each time, $\Omega_t(u)$ is τ^5-close to a star-shaped domain D_t up to the time $t = \tau$, i.e.,

$$D_t \subset \Omega_t(u) \subset (1 + \tau^5)D_t \subset (1 + r^4)D_t \tag{3-11}$$

for $0 \le t \le \tau$.

Also note that by the first inequality of (3-10) with $\beta \ge 5/6$,

$$t(z, r^{13/20}) \ge r^{13(2-\beta)/20} > \tau \quad \text{for any } z \in \Gamma_0.$$

Hence we can apply Lemma 3.3 for $s = r^{13/20}$ up to the time τ. Then by Lemma 3.3 and (3-11) with $\beta \ge 5/6$,

$$u(x,t) \le r^{(13/20)(5/6)} = r^{13/24}$$

for $x \in \partial(1 - r^{13/20})D_0$ and for $0 \le t \le \tau$. Then by the τ^5-monotonicity of u,

$$u(x,t) \le r^{13/24} \quad \text{on } B_R(0) - (1 - r^{13/20} + r^4)D_0 \tag{3-12}$$

for $0 \le t \le \tau$. Since $\Gamma_t(u)$ is located between the free boundaries $\Gamma^\star$ and $\widetilde{\Gamma}^\star$ of one-phase problem, Lemma 2.12 with $\beta \ge 5/6$ implies that $\Gamma(u)$ stays in the $\tau^{6/7}$-neighborhood of $\Gamma_0(u)$ up to τ. Also (3-11) implies that ∂D_t stays in the τ^5-neighborhood of $\Gamma_t(u)$ up to τ. Hence we obtain that ∂D_t stays in the $\tau^{5/6}$-neighborhood of ∂D_0 up to the time τ. Since $\tau^{5/6} = r^{25/36} < r^{13/20}$, (3-12) implies

$$u(x,t) \le r^{13/24} \quad \text{on } B_R(0) - D_s \tag{3-13}$$

for any $0 \le s, t \le \tau$.

Step 3. Let

$$t_0 = 0 \le t_1 = r^2 \le t_2 = 2r^2 \le \cdots \le t_{k_0} = k_0 r^2 \le \tau$$

and fix a number b such that

$$5/4 \le b < 61/48.$$

We will construct a supersolution of (ST2) in

$$(B_R(0) - (1 + r^b)D_{t_k}) \times [t_k, t_{k+1}].$$

Let $w^k(x)$ be the harmonic function in

$$\Sigma := (1+4r^b)D_{t_k} - D_{t_k},$$

with boundary data zero on $\partial(1+4r^b)D_{t_k}$ and $C_n r^{13/24}$ on ∂D_{t_k}, where C_n is a sufficiently large dimensional constant. Extend $w^k(x)$ by 0 to $\mathbb{R}^n - \Sigma$. Next define

$$\Phi^k(x,t) := \inf\{w^k(y) : |x-y| \le r^b - (t-t_k)\tfrac{1}{2}r^{b-2}\}$$

in $(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$. We claim that the function Φ^k is a supersolution of (ST2) in $(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$, since our constant b satisfies

$$r^{b-2} > r^{13/24-b}. \tag{3-14}$$

For simplicity, write $\Phi = \Phi^k$. To check that Φ is a supersolution, first note that $\Phi(\cdot, t)$ is superharmonic in its positive set and $\Phi_t \ge 0$. Hence we only need to show that

$$\frac{\Phi_t}{|D\Phi|} \ge |D\Phi| \quad \text{on } \Gamma(\Phi). \tag{3-15}$$

Due to the definition of Φ, $\Gamma_t(\Phi)$ has an interior ball of radius at least $r^b/2$ for $t_k \le t \le t_{k+1}$. This and the superharmonicity of Φ in the positive set yield that

$$|D\Phi| \le \frac{Cr^{13/24}}{r^b} \quad \text{on } \Gamma(\Phi)$$

for a dimensional constant $C > 0$. Moreover $\Gamma(\Phi)$ evolves with normal velocity $\frac{1}{2}r^{b-2}$. Since (3-14) holds for our choice of b (i.e., for $5/4 \le b < 61/48$), we conclude that (3-15) holds for r smaller than a dimensional constant $r(n)$. Now we compare u with Φ on

$$(B_R(0) - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}].$$

Note that by (3-13),

$$u^+ \le \Phi \quad \text{on } \partial(1+r^b)D_{t_k}$$

if C_n is chosen sufficiently large. Also at $t = t_k$, (3-11) implies

$$u(\cdot, t_k) \le 0 \le \Phi(\cdot, t_k) \quad \text{on } B_R(0) - (1+r^b)D_{t_k}.$$

Hence we get $u \le \Phi$ in $(\mathbb{R}^n - (1+r^b)D_{t_k}) \times [t_k, t_{k+1}]$. This implies

$$\Omega(u) \subset \Omega(\Phi) \cup ((1+r^b)D_{t_k} \times [t_k, t_{k+1}]) := \widetilde{\Omega}(\Phi) \tag{3-16}$$

for $t_k \le t \le t_{k+1}$.

Step 4. Next we let $v(x,t)$ solve the heat equation in

$$\widetilde{\Omega}(\Phi) - ((1-3r)\Omega_0(u) \times [t_k, t_{k+1}])$$

with initial data $v(\cdot\,, t_k) = u(\cdot\,, t_k)$ and boundary data zero on $\Gamma(\Phi)$ and $v = u$ on $(1-3r)\Gamma_0(u)$. Observe that, due to (3-16), we have

$$u^+ \le v \quad \text{for } t_k \le t \le t_{k+1}. \tag{3-17}$$

Since $\widetilde{\Omega}(\Phi)$ is star-shaped and expands with its normal velocity $< r^{b-2}$, which is less than r^{-1}, Lemma 2.5 applies to $\tilde{v}(x,t) := v(rx, r^2t)$. In particular there exists a constant $C > 0$ such that

$$(1/C)v(x,t) \le h_1(x,t) \le Cv(x,t)$$

for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$, where $h_1(\cdot\,, t)$ is the harmonic function in $\Omega_t(v) - (1-2r)\Omega_0(u)$ with boundary data zero on $\Gamma_t(v)$ and v on $(1-2r)\Gamma_0(u)$.

Hence we conclude that

$$u^+ \le v \le Ch_1$$

in $(B_R(0) - (1-2r)\Omega_0(u)) \times [(t_k + t_{k+1})/2, t_{k+1}]$.

Step 5. Similar arguments, now pushing the boundary purely by the minus phase given by the harmonic function, yield that

$$\Pi_t := \{x \in D_{t_k} : \operatorname{dist}(x, \partial D_{t_k}) \ge 3r^b + \tfrac{1}{2}r^{b-2}(t - t_k)\} \subset \Omega_t(u)$$

for $t_k \le t \le t_{k+1}$. Let $w(x,t)$ solve the heat equation in

$$\Pi - ((1-3r)\Omega_0(u) \times [t_k, t_{k+1}]))$$

with initial data $u(\cdot\,, t_k)$ and boundary data zero on $\partial\Pi$, and u on $(1-3r)\Gamma_0(u)$. Then $u \ge w(x,t)$. Since Π is star-shaped and it shrinks with its normal velocity $< r^{b-2}$, which is less than r^{-1}, Lemma 2.5 applies to $\tilde{w}(x,t) := w(rx, r^2t)$. In particular there exists $C > 0$ such that

$$u^+ \ge w \ge (1/C)h_2$$

for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$, where $h_2(\cdot\,, t)$ is the harmonic function in $\Pi_t - (1-2r)\Omega_0(u)$ with boundary data coinciding with that of w.

Step 6. Lastly we will show that h_1 and h_2 are not too far apart, i.e.,

$$h_1(x,t) \le Ch_2(x - 8r^b e_n, t), \tag{3-18}$$

with a dimensional constant $C > 0$. Since u is between $(1/C)h_2$ and Ch_1, this will conclude our proof for $(t_k + t_{k+1})/2 \le t \le t_{k+1}$. Then by changing the time intervals $[t_k, t_{k+1}]$ to $[t_k + r^2/2, t_{k+1} + r^2/2]$, we obtain the lemma for any $t \in [r^2, t(x_0, r)]$.

To prove (3-18), observe that by the construction of v and w,

$$\Omega_t(w) \subset \Omega_t(v) \subset (1 + 8r^b)\Omega_t(w).$$

Since $t_{k+1} - t_k = r^2$, Lemma 2.12 implies

$$\sup\{d(x, \Gamma_t(u)) : x \in \Gamma_{t_k}(u)\} \le r^{12/7}$$

for $t \in [t_k, t_{k+1}]$. Then by (3-11),

$$\sup\{d(x, \Gamma_t(u)) : x \in \partial D_{t_k}\} \le r^{12/7} + r^4 \ll r^b \tag{3-19}$$

for $t \in [t_k, t_{k+1}]$. Then we obtain

$$v_2(x,t) \le v(x,t) \le v_1\big((1-4r^b)x, (1-4r^b)^2(t-t_k)+t_k\big) \tag{3-20}$$

for $t_k \le t \le t_{k+1}$, where the first inequality follows from (3-8) and (3-17), and the second inequality follows from the comparison principle along with (3-8), $v(\cdot, t_k) = u(\cdot, t_k)$ and (3-19). Similarly,

$$v_2\big((1+4r^b)x, (1+4r^b)^2(t-t_k)+t_k\big) \le w(x,t) \le v_1(x,t). \tag{3-21}$$

Combing (3-20) and (3-21), we get

$$\begin{aligned} v_2\big((1+4r^b)x, (1+4r^b)^2(t-t_k)+t_k\big) &\le w(x,t), v(x,t) \\ &\le v_1\big((1-4r^b)x, (1-4r^b)^2(t-t_k)+t_k\big). \end{aligned}$$

This and (3-9) yield

$$v(x_0 - 2re_n, t) \sim w(x_0 - 2re_n, t) \sim u(x_0 - 2re_n, 0).$$

It follows that

$$w(x,t) \le v(x,t) \le Cw(x - 8r^b e_n, t) \quad \text{on } (1-2r)\Gamma_0 \times [t_k, t_{k+1}].$$

Hence due to Dahlberg's lemma, we conclude that

$$h_1(x,t) \le C_1 v(x,t) \le C_2 w(x - 8r^b e_n, t) \le C_3 h_2(x - 8r^b e_n, t)$$

in $B_r(x_0) \times [(t_k + t_{k+1})/2, t_{k+1}]$. Since the inequality holds for any $5/4 \le b < 61/48$, we obtain the lemma. □

Next we show that in the "unbalanced" region, where one phase has much larger flux than the other, the regularization process occurs similarly to the one in the one-phase problem. This observation will be useful for the analysis in Section 4.

Proposition 3.7 (regularization in unbalanced region I). *Let u be as given in Theorem 1.1. For a fixed $x_0 \in \Gamma_0(u)$, we may let $e_n = x_0/|x_0|$ after a rotation. Suppose that either*

$$u^+(x_0 - re_n, 0) \ge Mu^-(x_0 + re_n, 0) \quad \text{or} \quad u^-(x_0 + re_n, 0) \ge Mu^+(x_0 - re_n, 0)$$

for $M > M_n$, where M_n is a sufficiently large dimensional constant. Then, for $r \le 1/M_n$, there exists a dimensional constant $C > 0$ such that

$$|Du^+(x,t)| \le C\frac{u^+(x_0 - re_n, 0)}{r} \quad \text{and} \quad |Du^-(x,t)| \le C\frac{u^-(x_0 + re_n, 0)}{r}$$

in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$.

Remark 3.8. 1. In the next section, we will extend Proposition 3.7 to later times, i.e., to $x_0 \in \Gamma_{t_0}$ (see Lemma 4.7).

2. The situation given in Proposition 3.7 is essentially a perturbation of the one-phase case in [Choi and Kim 2010]. The main step in the proof is the verification of this observation; by barrier arguments we will show that our solution is very close to a rescaled version of the one-phase solution, for which the regularity of solutions is well-understood (see Theorem 2.16).

Proof of Proposition 3.7. Without loss of generality, we may assume that

$$u^+(x_0 - re_n, 0) \geq Mu^-(x_0 + re_n, 0).$$

Step 1. First we will show that after a small amount of time u becomes almost harmonic near the free boundary. Lemmas 3.3 and 3.4 imply that for $0 \leq t \leq t(x_0, r)$,

$$u^+(x_0 - re_n, t) \sim u^+(x_0 - re_n, 0), \quad u^-(x_0 + re_n, t) \sim u^-(x_0 + re_n, 0). \tag{3-22}$$

Also note that, by the assumption on the initial data u_0, Lemma 3.6 holds at $t = 0$. In other words, there exists a function $\omega(x, 0) = \omega_0(x)$ such that:

(a) ω_0 is harmonic in its positive and negative phases in $(1+r)\Omega_0(u) - (1-r)\Omega_0(u)$.

(b) $\Omega(\omega_0^+)$ and $\Omega(\omega_0^-)$ are star-shaped.

(c) In $B_r(x_0)$, we have

$$\omega_0^+(x) \leq u_0^+(x) \leq C\omega_0^+((1 - r^{5/4})x), \tag{3-23}$$

$$\omega_0^-(x) \leq u_0^-(x) \leq C\omega_0^-((1 + r^{5/4})x). \tag{3-24}$$

Next we will improve (3-23) and (3-24) for later times to obtain the inequalities with $C = (1 + r^a)$ for $t \geq r^{3/2}$. By the distance estimate in Lemma 2.12, the free boundary of u moves less that $r^{9/7} < r^{5/4}$ during the time $t = r^{3/2}$. Then we let v_1 solve

$$\begin{cases} \partial_t v_1 = \Delta v_1 & \text{in } (1 + 2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}], \\ \partial_t v_1 = \Delta v_1 & \text{in } (B_R(0) - (1 + 2r^{5/4})\overline{\Omega}_0(\omega^+)) \times [0, r^{3/2}], \\ v_1(\cdot, 0) = u_0^+ & \text{on } (1 + 2r^{5/4})\Omega_0(\omega^+), \\ v_1(\cdot, 0) = -u_0^- & \text{on } B_R(0) - (1 + 2r^{5/4})\overline{\Omega}_0(\omega^+), \\ v_1 = 0 & \text{on } (1 + 2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}], \\ v_1 = -1 & \text{on } \partial B_R(0) \times [0, r^{3/2}]. \end{cases}$$

Similarly, we let v_2 solve the heat equation in two cylindrical regions,

$$(1 - 2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}], \quad \text{and} \quad (B_R(0) - (1 - 2r^{5/4})\overline{\Omega}_0(\omega^+)) \times [0, r^{3/2}]$$

with initial data u_0^+ and $-u_0^-$, and with lateral boundary data zero on $(1 - 2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}]$ and -1 on $\partial B_R(0) \times [0, r^{3/2}]$. Then, by comparison,

$$v_2 < u < v_1. \tag{3-25}$$

Also by Lemma 2.5 with $\beta \geq 5/6$,

$$|v_1 - v_2| \leq r^{5/4\times 5/6} = r^{25/24}.$$

Note that on $(1 - r^{6/7})\Gamma_0(\omega^+)$,

$$|v_1| \geq r^{(6/7)\alpha} \geq r^{6/7\times 7/6} = r$$

and thus for $a_1 = 1/24$,

$$|v_1 - v_2| \leq r^{a_1}|v_1| \quad \text{on } (1 - r^{6/7})\Gamma_0(\omega^+). \tag{3-26}$$

Similarly,

$$|v_1 - v_2| \leq r^{a_1}|v_2| \quad \text{on } (1 + r^{6/7})\Gamma_0(\omega^+). \tag{3-27}$$

Now, v_1 and v_2 are almost harmonic in the $r^{3/4}$-neighborhood of their boundaries for $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$ by Lemma 2.5. Then the almost harmonicity of v_1 and v_2 with (3-25)–(3-27) imply the following: For $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$, there exist positive harmonic functions $\widetilde{\omega}^+(\cdot, t)$ and $\widetilde{\omega}^-(\cdot, t)$ defined respectively in

$$\Omega_t(v_2^+) \cap (B_R(0) - (1 - r^{1-b})\Omega_0(\omega^+)) \quad \text{and} \quad \Omega_t(v_1^-) \cap (1 + r^{1-b})\Omega_0(\omega^+)),$$

where $b = 1/7$, such that for some $a > 0$,

$$\widetilde{\omega}^+(x, t) \leq u^+(x, t) \leq (1 + r^a)\widetilde{\omega}^+((1 - 4r^{5/4})x, t) \tag{3-28}$$

and

$$\widetilde{\omega}^-(x, t) \leq u^-(x, t) \leq (1 + r^a)\widetilde{\omega}^-((1 + 4r^{5/4})x, t). \tag{3-29}$$

Now on the time interval $[0, r^{3/2}] + \frac{1}{2}kr^{3/2}$, $1 \leq k \leq m$, we construct v_1 and v_2 so that they solve the heat equation in the cylindrical domains with

$$\Gamma(v_1) = (1 + 2r^{5/4})\Gamma_{\frac{1}{2}kr^{3/2}}(\omega^+) \times \left[\tfrac{1}{2}kr^{3/2}, (1 + \tfrac{1}{2}k)r^{3/2}\right],$$
$$\Gamma(v_2) = (1 - 2r^{5/4})\Gamma_{\frac{1}{2}kr^{3/2}}(\omega^+) \times \left[\tfrac{1}{2}kr^{3/2}, (1 + \tfrac{1}{2}k)r^{3/2}\right].$$

By a similar argument to the one above, we then obtain harmonic functions $\widetilde{\omega}^{\pm}(\cdot, t)$ satisfying (3-28) and (3-29) for

$$\tfrac{1}{2}(1 + k)r^{3/2} \leq t \leq (1 + \tfrac{1}{2}k)r^{3/2}.$$

Hence we conclude (3-28) and (3-29) for $r^{3/2} \leq t \leq t(x_0, r)$.

Step 2. Next we rescale $u(x, t)$ as

$$\tilde{u}(x, t) := \alpha^{-1}u(rx + x_0, r^2\alpha^{-1}t) \quad \text{in } 2Q_{x_0},$$

where $\alpha := u^+(x_0 - re_n, 0) \ll r^{1/2}$. Then $\tilde{u}(x,t)$ solves

$$\begin{cases} (\alpha\partial_t - \Delta)\tilde{u} = 0 & \text{in } \Omega(\tilde{u}), \\ V = |D\tilde{u}^+| - |D\tilde{u}^-| & \text{on } \Gamma(\tilde{u}), \\ \tilde{u}(-e_n, 0) = 1, & \\ \tilde{u}(e_n, 0) = -1/N, & \text{where } N \geq M. \end{cases}$$

Furthermore, (3-22) implies that for $0 \leq t \leq 1$,

$$\tilde{u}^+(-e_n, t) \sim 1, \quad \tilde{u}^-(e_n, t) \sim 1/N.$$

Let $\tilde{w}$ be the corresponding rescaled version of $\widetilde{\omega}$ given in (3-28) and (3-29), then in $B_{r^{-b}}(0) \cap \Omega_0(\tilde{u})$ we have

$$(1 - r^a)\tilde{w}^+((1 + 4r^{5/4})x, \alpha r^{-1/2}) \leq \tilde{u}^+(x, \alpha r^{-1/2}) \leq \tilde{w}^+(x, \alpha r^{-1/2}) \tag{3-30}$$

and

$$(1 - r^a)\tilde{w}^-(x, \alpha r^{-1/2}) \leq \tilde{u}^-(x, \alpha r^{-1/2}) \leq \tilde{w}^-((1 + 4r^{5/4})x, \alpha r^{-1/2}). \tag{3-31}$$

Here note that

$$\alpha r^{-1/2} = \sqrt{r} \cdot \frac{u^+(x_0 - re_n, t_0)}{r} \leq r^{1/3}.$$

Lastly, for given $x_0 \in \Gamma(\tilde{u}) \cap B_1(0)$, an argument similar to the one in (3-7) implies that

$$\tilde{u}(x,t) \leq (1 + r^b)\tilde{u}(x, 0) \quad \text{in } \partial B_{(1/2)r^{-b}}(r^{-b}e_n) \times [0, 1]. \tag{3-32}$$

Step 3. We claim that we can construct a supersolution U_1 and a subsolution U_2 of (ST2) such that

$$U_2(x,t) \leq \tilde{u}(x,t) \leq U_1(x,t) \leq U_2(x - \sqrt{\epsilon}e_n, t) \quad \text{in } B_1(0) \times [\alpha r^{-1/2}, 1],$$

and so that U_2 is a smooth solution with uniformly Lipschitz boundary in space and time. Then for sufficiently small $r > 0$ the lemma will follow from analysis parallel to that of [Athanasopoulos et al. 1998].

To illustrate the main ideas, let us first assume that:

(a) (3-30) and (3-31) hold in the entire ring domain $\mathcal{R} \times [0, 1]$, where

$$\mathcal{R} := \{x : d(x, \Gamma_0(\tilde{u})) \leq r^{-b}\}.$$

(b) $\tilde{u}(x,t) \leq (1 + r^b)\tilde{u}(x, 0)$ on $\partial\mathcal{R} \times [0, 1]$.

Let

$$\Sigma := \{x : d(x, \mathbb{R}^n - \Omega_0) \leq r^{-b}\} \times [\alpha r^{-1/2}, 1],$$

and let U_1^+ be the solution of the one-phase Hele-Shaw problem in Σ,

$$\begin{cases} \Delta U_1^+ = 0 & \text{in } \{U_1^+ > 0\} \cap \Sigma, \\ \partial_t U_1^+ = |DU_1^+|^2 & \text{on } \partial\{U_1^+ > 0\} \cap \Sigma, \\ U_1^+(x, \alpha r^{-1/2}) = \tilde{w}^+(x, \alpha r^{-1/2}), & \\ U_1^+(x,t) = (1+r^b)\tilde{u}(x,0) & \text{for } x \in \partial\Sigma. \end{cases} \tag{HS}$$

Let

$$U_1 = U_1^+ - U_1^- \quad \text{in } \mathcal{R} \times [\alpha r^{-1/2}, 1],$$

where $U_1^-(\cdot, t)$ is the harmonic function in $\mathcal{R} - \Omega(U_1^+)$ with boundary data

$$U_1^- = 0 \quad \text{on } \Gamma(U_1^+), \quad U_1^- = C/N \quad \text{on } \partial\mathcal{R} - \Omega(U_1^+).$$

Then U_1 is a supersolution of (ST2) in Σ, and thus by Lemma 2.11 and the assumptions (a)–(b) we have $\tilde{u} \le U_1$ in Σ.

Step 4. The construction of the subsolution U_2 is a bit less straightforward. We use

$$U_2^+(x,t) := (1-\epsilon) \sup_{|y-x| \le \sqrt{\epsilon}(1-c(t))} U_1^+((1+\sqrt{\epsilon})y, t),$$

where $\epsilon = 1/N$ and $c(t) := t^{4/5}$. Then we define

$$U_2 = U_2^+ - U_2^- \quad \text{in } R \times [\alpha r^{-1/2}, 1],$$

where $\mathcal{R}$ is the ring domain as given above and $U_2^-(\cdot, t)$ is the harmonic function in $R - \Omega(U_2^+)$ with fixed boundary data zero on $\Gamma(U_2^+)$ and C/N on $\partial\mathcal{R} - \Omega(U_2^+)$. Then U_2 satisfies the free boundary condition

$$V_{U_2} \le (1+\epsilon)|DU_2^+| - \sqrt{\epsilon}c'(t).$$

Therefore, U_2 is a subsolution of (ST2) if we can show that

$$\sqrt{\epsilon}c'(t) \ge \epsilon|DU_2^+| + |DU_2^-| \quad \text{on } \Gamma(U_2) \tag{3-33}$$

and $\int_0^1 c'(s)\,ds \le 1$.

The analysis performed in [Choi and Kim 2010], as in the proof of (c) of Theorem 2.16, yields that at a fixed time t, $\Gamma(U_1)$ regularizes in the scale of $d := d(t)$ that solves

$$t = \frac{d^2}{U_1(-de_n, 0)}.$$

Therefore,

$$|DU_2^+| \sim \frac{U_2^+(-de_n, 0)}{d} \quad \text{and} \quad |DU_2^-| \sim \frac{U_2^-(de_n, 0)}{d}$$

on

$$\Gamma(U_2) \times [t/2, t].$$

Observe that since $\beta \geq 5/6$,

$$U_2^+(-de_n, 0) \leq d^{5/6} \quad \text{and} \quad U_2^-(de_n, 0) \leq \epsilon d^{5/6},$$

then we have

$$\epsilon \frac{U_2^+(-de_n, 0)}{d} + \frac{U_2^-(de_n, 0)}{d} \leq \epsilon d^{-1/6} \leq \sqrt{\epsilon} t^{-1/5}.$$

where the last inequality follows from

$$t = d^2/U_1(-de_n, 0) \leq d^2/d^\alpha \leq d^{5/6}.$$

Hence $c(t) = t^{4/5}$ satisfies (3-33), and we conclude that U_2 is a subsolution of (ST2) in Σ.

Now we can use the fact that

$$U_2 \leq \tilde{u} \leq U_1 \quad \text{in } B_c(0) \times [\alpha r^{-1/2}, c]$$

to conclude that $\tilde{u}$ is $\sqrt{\epsilon}$- close to a Lipschitz (and smooth) solution U_1 in $B_1(0) \times [1/2, 1]$, confirming (B′). Moreover (A) holds due to Lemma 3.3 and Lemma 3.4. Once we can confirm this, we can conclude our proof by using the results of [Athanasopoulos et al. 1998] with the choice of a sufficiently small ϵ.

Step 5. Now we proceed to the general proof without the simplified assumptions (a) and (b) in Step 3, which are replaced by the local inequalities (3-30)–(3-32). For this we need to perturb the initial data outside of $B_1(0)$ (see Section 4, pages 2781–2783 of [Choi et al. 2009]), to obtain functions $W_1(x)$ and $W_2(x)$ that satisfy:

(a) $\{W_k > 0\}$ with $k = 1, 2$ is star-shaped and coincides with $\Omega_{\alpha r^{-1/2}}(\tilde{w})$ in $B_{r^{-b}}(0)$.

(b) $\{W_2 > 0\} \subset \Omega_{\alpha r^{-1/2}}(\tilde{w}) \subset \{W_1 > 0\}$.

(c) $d(x, \{W_k > 0\}) \geq r^{-b}$ with $k = 1, 2$ for $x \in \Gamma_{\alpha r^{-1/2}}(\tilde{w}) \cap (\mathbb{R}^n - B_{2r^{-b}}(0))$.

(d) W_k is harmonic in $\{W_k > 0\} - K$ with boundary data zero on $\Gamma(W_k)$ and $(1 + r^b)\tilde{w}(x, \alpha r^{-1/2})$ on ∂K, where

$$K = \{x : d(x, \Gamma(W_k)) \geq r^{-b}\}.$$

Let U_k be the solution of Hele-Shaw problem in

$$\mathbb{R}^n - \tfrac{1}{2}\{W_k > 0\} \times [\alpha r^{-1/2}, 1]$$

with initial data W_1 and with lateral boundary data $(1 + r^b)\tilde{w}(x, \alpha r^{-1/2})$. Due to Proposition 4.1 of [Choi et al. 2009], for sufficiently small $r > 0$, the level sets of U_1 are then ϵc-close to those of U_2 in $B_1(0) \times [0, 1]$. Hence we can use U_2 instead of U_1 in Step 4 and proceed as in Step 4 to conclude. □

4. Decomposition based on local phase dynamics

Throughout the rest of the paper, let u be as in Theorem 1.1, and fix $x_0 \in \Gamma_0$ and a sufficiently small constant $r > 0$. We will prove the regularization of the solution u in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$. After a rotation if necessary, we may assume that $x_0/|x_0| = e_n$.

Let us fix a constant $M \geq M_n$, where M_n is a sufficiently large dimensional constant. If the ratio between $u^+(x_0 - re_n, 0)$ and $u^-(x_0 + re_n, 0)$ is bigger than M, then we can directly apply Proposition 3.7 to prove the main theorem. Therefore we assume that

$$M^{-1}u^-(x_0 + re_n, 0) \leq u^+(x_0 - re_n, 0) \leq Mu^-(x_0 + re_n, 0). \tag{4-1}$$

Let

$$C_0 := \max\left\{\frac{u^+(x_0 - re_n, 0)}{r}, \frac{u^-(x_0 + re_n, 0)}{r}\right\}. \tag{4-2}$$

Then since u_0^+ and u_0^- are comparable with harmonic functions, C_0 is less than a constant depending on n and M (See Corollary 2.15). Also note that

$$C_0 \geq r^{\alpha-1} \geq r^{1/6}.$$

Let us now sort out the initial free boundary points where the flux from one phase dominates the flux from the other phase. Let us define

$$A^+ = \left\{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^+(x - se_n, 0)}{s} \geq MC_0 \text{ for some } s \text{ with } r^{5/4} \leq s \leq r\right\},$$
$$A^- = \left\{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^-(x + se_n, 0)}{s} \geq MC_0 \text{ for some } s \text{ with } r^{5/4} \leq s \leq r\right\}.$$

We then write

$$A = A^+ \cup A^-.$$

Throughout the paper we will let $e_n = x/|x|$ for any boundary point x, after a necessary rotation.

Lemma 4.1. *Let u be as given in Theorem 1.1, and let M and C as given above.*

(a) *If* $\dfrac{u^+(x - se_n, 0)}{s} \geq MC_0$ *for some* $s \leq r$, *then* $\dfrac{u^+(x - se_n, 0)}{s} \leq C_0$.

(b) *If* $\dfrac{u^-(x + se_n, 0)}{s} \geq MC_0$ *for some* $s \leq r$, *then* $\dfrac{u^-(x + se_n, 0)}{s} \leq C_0$.

Proof. Since $u_0^\pm$ are comparable with harmonic functions $h^\pm$, we can argue similarly as in Corollary 2.15. Observe that

$$\frac{u_0^+(x - se_n)}{s} \cdot \frac{u_0^-(x + se_n)}{s} \sim \frac{h^+(x - se_n)}{s} \cdot \frac{h^-(x + se_n)}{s} \lesssim \sqrt{\phi(r)} \lesssim C_0^2. \qquad \square$$

Now for $x \in A^+$, there exists a largest constant $r_x < r$ such that

$$\frac{u^+(x - r_x e_n, 0)}{r_x} = MC_0.$$

We then define

$$Q_x = B_{r_x}(x) \times \left[0, \frac{r_x}{MC_0}\right].$$

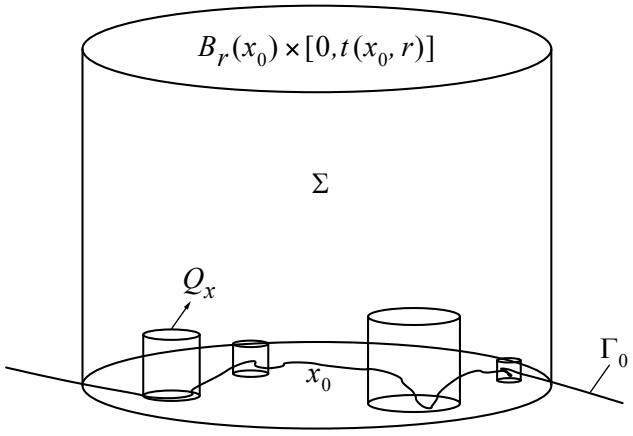

Figure 3. Decomposition of the domain.

Also for $x \in A^-$, we can similarly define r_x and Q_x. Now we define

$$\Sigma := B_r(x_0) \times [0, t(x_0, r)] - \bigcup_{x\in A} Q_x;$$

see Figure 3. Σ is then the region where the fluxes from both sides are initially balanced. Our aim in this section is to prove that the balance is kept over time, so that the interface remains close to a Lipschitz graph over time.

The following statement is a direct consequence of the definition of Σ.

Lemma 4.2. *If $x \in \Gamma_0 \cap \Sigma_0$, then for all $r^{5/4} \le s \le r$,*

$$\frac{u^+(x - se_n, 0)}{s}, \frac{u^-(x + se_n, 0)}{s} \le MC_0.$$

The next proposition, the main result in this section, states that the solution is "well-behaved" in Σ.

Proposition 4.3. *There exists a dimensional constant $K > 0$ such that for all $(x, t) \in \Gamma \cap \Sigma$,*

$$\frac{u^+(x - se_n, t)}{s}, \frac{u^-(x + se_n, t)}{s} < KMC_0 \quad \text{for } r^{5/4} \le s \le r. \tag{4-3}$$

Before proving Proposition 4.3, we show an immediate consequence of it; we are ready to show that $\Gamma(u)$ is close to a Lipschitz graph in time as well as in space.

Corollary 4.4. *For $(x, t) \in \Gamma \cap \Sigma$, suppose $(x + ke_n, t + \tau) \in \Gamma$. Then there exists a dimensional constant $K_1 > 0$ such that*

$$|k| \le r^{5/4} \quad \text{if } \tau \in \left[0, \frac{r^{5/4}}{K_1 MC_0}\right].$$

Proof. Due to Lemma 3.6, at any time $0 \le t \le t(x_0, r)$, we have

$$h^+(x, t) \le u^+(x, t) \le C_1 h^+(x - r^{5/4}e_n, t) \tag{4-4}$$

and

$$h^-(x,t) \le u^-(x,t) \le C_1 h^-(x + r^{5/4} e_n, t), \tag{4-5}$$

in $B_r(x_0)$, where the function $h := h^+(\cdot\,,t) - h^-(\cdot\,,t)$ is harmonic in its positive and negative phases in $(1+r)\Omega_t(u) - (1-r)\Omega_t(u)$, and the domains $\Omega(h^+)$ and $\Omega(h^-)$ are both star-shaped with respect to $B_{r_0}(0)$.

Let us pick $(y_0, t_0) \in \Gamma \cap \Sigma$. Due to Proposition 4.3, (4-4) and the Harnack inequality for harmonic functions, we have

$$\sup_{y \in B_{10r^{5/4}}(y_0)} u(y, t_0) \le C C_1 K M C_0 r^{5/4}, \tag{4-6}$$

where C is a dimensional constant. On the other hand, due to Lemma 3.1 and $t_0^5 \le r^{25/6}$, we have

$$u(\cdot\,, t_0) \le 0 \quad \text{in } B_{(1/2)r^{5/4}}(y_0 + r^{5/4} e_n). \tag{4-7}$$

Let

$$y_1 := y_0 + r^{5/4} e_n, \quad C_2 := C C_1 K M C_0, \quad r(t) := \tfrac{1}{2} r^{5/4} - C_3 (t - t_0),$$

where $C_3 = C C_2$. Next we define $\phi(x,t)$ in the domain

$$\Pi := B_{2r^{5/4}}(y_1) \times \left[t_0, t_0 + \frac{r^{5/4}}{C_3} \right]$$

such that

$$\begin{cases} -\Delta \phi(\cdot\,, t) = 0 & \text{in } B_{2r^{5/4}}(y_1) - B_{r(t)}(y_1), \\ \phi = 2C_2 r^{5/4} & \text{on } \partial B_{2r^{5/4}}(y_1), \\ \phi = 0 & \text{in } B_{r(t)}(y_1). \end{cases}$$

Then by (4-4)–(4-7), $u \prec \phi$ at $t = t_0$ in Π. Let T_0 be the first time when u hits ϕ from below in Π. Since (4-6) also holds for any $(x,t) \in \Gamma \cap \Sigma$ in place of (y_0, t_0), we have $u < \phi$ on the parabolic boundary of $\Pi \cap \{t_0 \le t \le T_0\}$. On the other hand, if C is chosen sufficiently large, then

$$\frac{\phi_t}{|D\phi|} = C_3 \ge |D\phi| \quad \text{on } \partial B_{r(t)}(y_1) \times \left[t_0, t_1 := t_0 + \frac{r^{5/4}}{4C_3} \right],$$

and thus ϕ is a supersolution of (ST1). This and Lemma 2.11 applied to u and ϕ in Π yields a contradiction, and we conclude that $\Gamma(u)$ lies outside of $B_{\frac{1}{4}r^{5/4}}(y_0 + r^{5/4} e_n)$ for $t_0 \le t \le t_1$.

Similarly, by constructing a negative radial barrier and comparing it with u, one can show that $\Gamma(u)$ lies outside of $B_{\frac{1}{4}r^{5/4}}(y_0 - r^{5/4} e_n)$ for $t_0 \le t \le t_1$. This concludes the proof. □

For $x_0 \in \Gamma_{t_0}$, define

$$t(x_0, r) := \min\left\{ \frac{r^2}{u^+(x_0 - r e_n, t_0)}, \frac{r^2}{u^-(x_0 + r e_n, t_0)} \right\}.$$

We now proceed to show our main result, Proposition 4.3. First we show Harnack-type inequalities for positive times.

Lemma 4.5 (Harnack at later times). *Fix $s \in [r^{5/4}, r]$. If $(y_0, t_0) \in \Gamma \cap \Sigma$, then*

$$u^+(y_0 - se_n, t_0) \geq c_1 u^+(y_0 - se_n, t_0 + \tau) \quad \textit{and} \quad u^-(y_0 + se_n, t_0) \geq c_1 u^-(y_0 + se_n, t_0 + \tau)$$

for $0 \leq \tau \leq t(y_0, s)/2$ and $c_1 > 0$.

Proof. We will show the lemma for u^+, the statement for u^- follows via parallel arguments.

Step 1. Let $(y_0, t_0) \in \Gamma \cap \Sigma$ and let $s \in [r^{5/4}, r]$. Let h^+ be given as in (4-4). Due to Lemma 3.3 and Lemma 3.4, we have

$$h^+(y_0 - 2re_n, t_1) \leq u^+(y_0 - 2re_n, t_1) \leq Cu^+(y_0 - 2re_n, t_2) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_2)$$

for $0 \leq t_1, t_2 \leq t_0 + t(y_0, r)/2$. (Here note that $y_0 \in B_r(x_0)$.) In particular

$$u^+(y_0 - 2re_n, t) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_0) \leq C_1 h^+(y_0 - 2re_n, t_0) \tag{4-8}$$

for $t \leq t_0 + t(y_0, s)/2$.

Step 2. Now let v^{**} solve (ST1) in $(\mathbb{R}^n - (1-2r)D_{t_0}) \times [t_0, t_0 + t(y_0, s)/2]$ with initial and boundary data $C_2 h^+(x - 2se_n, t)$. Since $s \geq r^{5/4}$, (4-4) implies

$$\Omega_t(u) \subset \Omega_{t_0}(v^{**}) \subset \Omega_t(v^{**}) \quad \text{in } B_{2s}(y_0) \times [t_0, t_0 + t(y_0, s)/2]. \tag{4-9}$$

Then by (4-9), (4-8) and (4-4),

$$u^+ \leq v^{**} \quad \text{in } B_s(y_0) \times [t_0, t_0 + t(y_0, s)/2]$$

if we choose C_2 as a multiple of C_1 by a dimensional constant. Moreover, due to the Harnack inequality for one-phase (ST1), one can conclude that

$$\begin{aligned} u^+(y_0 - se_n, t_0 + \tau) &\leq v^{**}(y_0 - se_n, t_0 + \tau) \\ &\leq Cv^{**}(y_0 - se_n, t_0) \\ &= CC_2 h^+(y_0 - 3se_n, t_0) \\ &\leq C_3 h^+(y_0 - se_n, t_0) \leq C_3 u^+(y_0 - se_n, t_0) \end{aligned}$$

for

$$0 \leq \tau \leq \frac{s^2}{v^{**}(y_0 - se_n, t_0)} \sim t(y_0, s)/2.$$

Here the first inequality uses the fact $u^+ \leq v^{**}$, the second uses the Harnack inequality for v^{**}, the third one uses the Harnack inequality for harmonic functions and the last one uses (4-4). □

Lemma 4.6 (backward Harnack). *Suppose that* (4-3) *holds up to time $t = T_0 \leq t(x_0, r)$. If $(y_0, t_0) \in \Gamma$ and $t_0 \leq T_0$, then for $0 \leq \tau \leq t(y_0, s)/2$,*

$$u^+(y_0 - se_n, t_0) \leq Cu^+(y_0 - se_n, t_0 + \tau) \quad \textit{and} \quad u^-(y_0 + se_n, t_0) \leq Cu^-(y_0 + se_n, t_0 + \tau),$$

where $0 \leq s \leq r$ and C is a universal constant.

Proof. We will show the argument for u^+, due to the symmetric nature of the claim. The argument here will be similar to that of Lemma 3.4, replacing the initial data u_0^+ and u_0^- (used in the construction of barriers) by $h^+(x,t_0)$ and $h^-(x,t_0)$ given in (4-4)–(4-5).

We consider a solution v_1 of (ST1) in

$$\Pi := (1+r)\Omega_{t_0} \times [t_0, t_0 + t(y_0,s)/2]$$

with initial and lateral boundary data $C_1 h^-$. Then $v_1 \le u$ in Π. Now let v_2 solve the heat equation in $\{v_1 = 0\} \times [t_0, t_0 + t(y_0,s)/2]$ with initial data

$$v_2(\cdot\,,t_0) = \begin{cases} h^+(\cdot\,,t_0) & \text{in } \{v_1(\cdot\,,t_0)=0\} - (1-r)\{h^+(\cdot\,,t_0)>0\}, \\ \tilde{h}(\cdot) & \text{in } (1-r)\{h^+(\cdot\,,t_0)>0\}, \end{cases}$$

where $\tilde{h}(\cdot)$ is a C^2 extension function of $h^+(\cdot\,,t_0)$ chosen so that $\tilde{h}(\cdot) \le u^+(\cdot\,,t_0)$. The rest of the proof is the same as that of Lemma 3.4. □

Next we show that in the unbalanced region, possibly forming at positive times, the fast regularization phenomena still holds. This lemma will be used in the proof of Proposition 4.3 to show that there cannot be a severe unbalance of flux in the initially balanced region Σ.

Lemma 4.7 (regularization in unbalanced region II). *For a fixed $(x_0,t_0) \in \Gamma(u)$, suppose that*

$$u^+(x_0 - re_n, t_0) \ge Mu^-(x_0 + re_n, t_0) \quad \text{or} \quad u^-(x_0 + re_n, t_0) \ge Mu^+(x_0 - re_n, t_0)$$

for $M > M_n$, where M_n is a dimensional constant. Then for $r \le 1/M_n$, there exists a dimensional constant $C > 0$ such that

$$|Du^+| \le C\frac{u^+(x_0 - re_n, t_0)}{r} \quad \text{and} \quad |Du^-| \le C\frac{u^-(x_0 + re_n, t_0)}{r}$$

in $B_r(x_0) \times [t_0 + t(x_0,r)/2, t_0 + t(x_0,r)]$.

Proof. The proof of this lemma is parallel to that of Proposition 3.7. We use the Harnack and backward Harnack inequalities (Lemmas 4.5 and 4.6) instead of Lemmas 3.3 and 3.4. □

We are now ready to prove our main result, Proposition 4.3. Observe that (4-3) holds up to some $T_0 > 0$ by Lemma 4.2 and Lemma 3.3.

Proof of Proposition 4.3. Let K be a sufficiently large dimensional constant such that $K \gg M$. Let us assume that (4-3) breaks down for u^+ for the first time at $t = T_0$. Then

$$\frac{u^+(z_0 - se_n, T_0)}{s} = KMC_0 \tag{4-10}$$

for some $(z_0, T_0) \in \Gamma \cap \Sigma$ and $r^{5/4} \le s \le r$. Let

$$h = \sup\left\{h : \frac{u^+(z_0 - ke_n, T_0)}{k} \ge M^2 C_0 \quad \text{for } s \le k \le h\right\}. \tag{4-11}$$

Note that $h < r/2$ due to Lemma 3.3 and the definition of C_0, and $h > 2s$ due to Lemma 3.6. By the definition of h we have

$$\frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0. \tag{4-12}$$

Let us find the largest time t_0 before T_0 such that for some $(y_0, t_0) \in \Gamma$

$$T_0 - t_0 = \frac{t(y_0, h)}{2} \quad \text{and} \quad \frac{y_0}{|y_0|} = \frac{z_0}{|z_0|}.$$

Then Lemma 4.5 implies

$$\frac{u^+(y_0 - he_n, t_0)}{h} \sim \frac{u^+(y_0 - he_n, T_0)}{h} \sim \frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0.$$

Since $u^+(\cdot, t_0)$ and $u^-(\cdot, t_0)$ are comparable to harmonic functions (Lemma 3.6), a similar argument as in Lemma 4.1 implies that

$$\frac{u^-(y_0 + he_n, t_0)}{h} \lesssim C_0 \lesssim \frac{1}{M^2} \frac{u^+(y_0 - he_n, t_0)}{h}.$$

Hence by Lemma 4.7, we have

$$|Du^+(\cdot, T_0)| \sim M^2 C_0 \quad \text{in } B_h(y_0).$$

Since $B_s(z_0) \subset B_h(y_0)$, this would contradict (4-10) as $K \gg M$. □

Due to Lemma 3.6, Proposition 4.3 and Corollary 4.4, we have shown that condition (A) holds and that the level sets of u are close to a Lipschitz graph, and $\Gamma(u)$ is close to a Lipschitz graph in space and time (see the detailed description of this fact in the next section). However, we do not yet have sufficient control of the change of u over time to verify the condition (B′). We will therefore prove Theorem 1.1 by carrying out a modified argument, combining arguments from [Athanasopoulos et al. 1996; 1998] and [Choi et al. 2007; 2009].

5. Further regularization based on flatness

Let u, Γ_0 be as given in Theorem 1.1. Recall that $x_0 \in \Gamma_0$ and $r > 0$ are fixed, and they satisfy (4-1). Let C_0 be as given in (4-2) and $t(x_0, r)$ as given in (1-3).

Our goal is to prove the regularization of the free boundary after the time $t(x_0, r)/2$ in $B_r(x_0)$. Define

$$\Sigma_r(x_0) := B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)] \subset \Sigma.$$

Let us briefly review the information we have on u so far. As a result of Proposition 4.3, condition (A) holds up to

$$t = t(x_0, r) \le C r^{2-\alpha} < r^{3/4}.$$

Also due to Lemma 3.6, our solution u is ϵ-monotone in $Q_r(x_0)$ with respect to a space cone $W_x(e_n, \theta_0)$ satisfying

$$|\theta_0 - \pi| = O(L),$$

where L is the Lipschitz constant of the initial domain Ω_0 given by (1-1).

Moreover $Q_r(x_0) \subset \Sigma$, and thus Corollary 4.4 and Lemma 3.1 yield that the free boundary $\Gamma(u)$ is $r^{4/3}$-monotone in $Q_r(x_0)$ with respect to the time cone $W_t(e_n, \tan^{-1}(1/K_1MC_0))$ and the space cone $W_x(e_n, \theta_0)$. Here θ_0 is the angle corresponding to the Lipschitz constant of Γ_0, and $t(x_0, r) = r/C_0$.

On the other hand, by Lemma 3.3 and the definition of C_0,

$$\frac{u(x_0 - re_n, \frac{1}{2}t(x_0, r))}{C_0 r} \sim 1.$$

Since $Q_r(x_0) \subset \Sigma$, Proposition 4.3 implies

$$\frac{u(x,t)}{C_0 r} \lesssim KM \quad \text{in } B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)].$$

The main difficulty in applying the method of [Athanasopoulos et al. 1996; 1998] lies in the fact that we cannot guarantee the ϵ-monotonicity of the solution u in the time variable (although we can obtain, as above, the $r^{4/3}$-monotonicity of the free boundary $\Gamma(u)$). To go around this difficulty, we will first use the parabolic scale to improve the regularity of the solution in space. Consider the function

$$\bar{u}(x,t) := \frac{1}{C_0 r} u\big(rx + x_0, r^2 t + \tfrac{1}{2}t(x_0, r)\big). \tag{5-1}$$

In [Athanasopoulos et al. 1996; 1998], it was important that initially the time derivative of the solution was assumed to be controlled by the spatial derivative, i.e.,

$$|u_t| \le C(|Du^+| + |Du^-|). \tag{5-2}$$

Using (5-2) one can prove that the *direction vectors*

$$\frac{Du^+}{|Du^+|}(-le_n, t) \quad \text{and} \quad \frac{Du^-}{|Du^-|}(le_n, t)$$

do not change much for $0 \le t \le l$. This is pivotal in the regularization procedure since then $\Gamma(u)$ regularizes along the direction of the "common gain" obtained by those two direction vectors, the regularity of $\Gamma(u)$ then makes the above two vectors line up better in a smaller scale, which contributes to further regularization of $\Gamma(u)$ in a finer scale. In our case we do not know *a priori* that $\Gamma(u)$ is Lipschitz in either space or in time; in fact the Lipschitz continuity of $\Gamma(u)$ in time will be proved in the very last stage of Section 5 (see Theorem 5.7). Therefore, we do not have (5-2), and thus extra care is required to show that the spatial gradients $Du^{\pm}$ do not change their directions too rapidly.

In the following series of results, we will assume that $\bar{u}$ is given by (5-1). The lemmas and theorems will be proved to in the order they are stated, to improve the regularity of $\bar{u}$ in multiple steps.

Lipschitz continuity in space. First we prove that the ϵ-monotonicity of $\Gamma(\bar{u})$ improves to Lipschitz continuity. Let $a = C_0 r$. Then, in the domain $B_1(0) \times [-1/a, 1/a]$, $\bar{u}(x,t)$ solves

$$\begin{cases} \bar{u}_t - \Delta \bar{u} = 0 & \text{in } \{\bar{u} > 0\}, \\ V = a(|D\bar{u}^+| - |D\bar{u}^-|) & \text{on } \partial\{\bar{u} > 0\}. \end{cases}$$

Here note that

$$r^{7/6} \le r^{\alpha} \le a \le r^{\beta} \le r^{5/6}.$$

In this scale, since $\bar{u}$ is caloric and $\Gamma(\bar{u})$ is $r^{1/3}$-close to a Lipschitz graph in space and time, it follows that so is $\bar{u}$ in $B_{1/2}(0) \times [-1/a + 1, 1/a]$.

Note that in above step we are losing a lot of information over time; $\Gamma(\bar{u})$ is in fact $r^{1/3}$-close to a Lipschitz graph moving very slowly in time, but this does not guarantee that $\bar{u}$ also changes slowly in time.

We then follow the iteration process in Lemma 7.2 of [Athanasopoulos et al. 1996] to show this:

Lemma 5.1. *If r is sufficiently small, then there exist $0 < c, d < 1/2$ such that $\bar{u}$ is $\lambda r^{1/3}$-monotone in the cone of directions $W_x(\theta_x - r^d, e_n)$ and $W_t(\theta_t - r^d, \nu)$ in the domain $B_{1-r^c}(0) \times [(-1 + r^c)/a, 1/a]$.*

One can then iterate above lemma to improve the ϵ-monotonicity to full monotonicity, and state the result in terms of $\bar{u}$:

Lemma 5.2. *$\bar{u}$ is fully monotone in $B_{1/2}(0) \times [0, 1/a]$ for the cone*

$$\mathcal{C}_1 := W_x(\theta_x - r^d, e_n) \cup W_t(\theta_t - r^d, \nu)$$

for some constant $0 < d < 1/2$.

Regularity in time away from the free boundary. Now we suppose that $\bar{u}$ is Lipschitz in space and time. Then in particular, we have the Lipschitz regularity of u in space (and very weak Lipschitz regularity of u in time). We are interested in proving the following type of statement:

Lemma 5.3 (enlargement for the cone of monotonicity). *There exists $\lambda > 0$ such that if $\bar{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(e_n, \theta_0)$ in $B_1(0) \times [-1/a, 1/a]$, then in the half domain $B_{1/2}(0) \times [-1/(2a), 1/(2a)]$, $\bar{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(\nu, (1 + \lambda)\theta_0)$ with some unit vector ν.*

To prove the enlargement of the cone, we take a closer look at the change of $\bar{u}$ over time, in the interior region. More precisely, we need the following lemma, which follows the approach taken in [Choi et al. 2007; 2009].

Lemma 5.4. *We have*

$$|\bar{u}_t| \le a|D\bar{u}|^2 \le Ca \quad \text{in } [B_{1/2}(e_n) \cup B_{1/2}(-e_n)] \times [-1/(2a), 1/(2a)],$$

where C is a dimensional constant.

Proof. Step 1. The proof is similar to that of Lemma 8.3 of [Choi et al. 2009]. Note that $\bar{u}_t$ is a caloric function in $\Omega^+(\bar{u})$ and $\Omega^-(\bar{u})$. Let us prove the lemma for $\bar{u}^+$, since parallel arguments apply to $\bar{u}^-$.

Step 2. We divide $\bar{u}_t$ into two parts. More precisely, let

$$\bar{u}_t = v_1 + v_2,$$

where both functions v_1 and v_2 are caloric in $\Omega^+(\bar{u})$, v_1 has the initial data zero and the boundary data $a|D\bar{u}^+|(|D\bar{u}^+| - |D\bar{u}^-|)$ on $\Gamma(\bar{u})$, and v_2 has the initial data $\bar{u}_t(\cdot, -1/a)$ and the boundary data zero on $\Gamma(\bar{u})$.

Step 3. For v_1, we need to use the absolute continuity of the caloric measure with respect to the harmonic measure, as well as the Lipschitz continuity of the free boundary. We proceed as in Lemma 8.3 of [Choi et al. 2007]. Note that we have

$$|D\bar{u}^+| \sim |D\bar{u}^-| \sim 1$$

in $[B_{1/2}(e_n) \cup B_{1/2}(-e_n)] \times [-1/a, 1/a]$: this follows from the assumption in (4-1), and Lemmas 3.3 and 3.4. Therefore we can proceed as in Lemma 8.3 of [Choi et al. 2007] to obtain

$$v_1(x,t) \le a \int_{\Gamma(\bar{u})\cap\{-1/a\le s\le t\}} |D\bar{u}^+|^2 d\omega^{(x,t)} \le a|D\bar{u}|^2(x,t),$$

where $\omega^{(x,t)}$ is the caloric measure for $\Omega(\bar{u})$, and

$$v_1(x,t) \ge a \int_{\Gamma(\bar{u})\cap\{-1/a\le s\le t\}} -|D\bar{u}^-|^2 d\omega^{(x,t)} \ge -a|D\bar{u}|^2(x,t).$$

Step 4. As for v_2, we conclude that it must be smaller than a caloric function solved in the whole domain with the absolute value of its initial data. The advantage is that then we can use the heat kernel. Note that the initial data is given at $t = -1/a$ and has compact support. The initial data is given by $v_t \le (C/a)v_{e_n}$, where $v_{e_n}(x,t)$ is comparable to the derivative of a harmonic function in a Lipschitz domain.

Therefore the heat kernel representation is given as

$$\frac{1}{(t+1/a)^{n/2+1}} \int |x_n - y_n| \exp^{-|x-y|^2/(t+1/a)} v(y, -1/a)\, dy.$$

Since $t \in [0, 1/a]$ and $k \exp^{-ak^2} \le C \exp^{-(a/2)k^2}$, we get the effect of $O(a)$. □

Further regularity in space. Now that we have sufficient information on the change of u over time, we change the scale following the one introduced in (1-4), and consider the function

$$v(x,t) := \frac{1}{C_0 r} u\Big(rx + x_0, \frac{r}{C_0}t + 1\Big). \tag{5-3}$$

Note that $C_0 = r^{-1}c(x_0, r)$, and thus v coincides with $\tilde{u}$ defined in (1-4) with the choice of $c = rC_0$.

Due to the previous results, this function is Lipschitz continuous, in space and time, *away* from the free boundary. The following lemma suggests that the cone of monotonicity improves away from the free boundary, as we look at smaller scales. The proof is parallel to that of Lemma 8.4 in [Athanasopoulos et al. 1998].

Lemma 5.5. *Let v given by* (5-3). *Suppose that there exist constants $\delta > 0$ and $0 \le A \le B$, $\mu := B - A$, such that*

$$\alpha(Dv, -e_n) \le \delta \quad \text{and} \quad A \le \frac{v_t}{-e_n \cdot Dv} \le B$$

in $B_{1/6}(-\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$ *with* $\delta/\mu < r$. *Then there exist a unit vector* $\nu \in \mathbb{R}^n$ *and positive constants* $r_0, b_0 < 1$ *depending only on A, B and n such that*

$$\alpha(Dv(x,t), \nu) \le b_0\delta \quad \text{in } B_{1/8}(-\tfrac{3}{4}e_n) \times \Big(-r_0\frac{\delta}{\mu}, r_0\frac{\delta}{\mu}\Big).$$

Now we can proceed as in Section 6 of [Choi et al. 2009] to obtain further regularity, using Lemma 5.4 instead of the uniform upper bound on $|Du|$ up to the free boundary.

Theorem 5.6. $\Gamma(v)$ *is* C^1 *in space in* $Q_{1/2}$. *In particular, there exist dimensional constants* $l_0, C_0 > 0$ *such that for a free boundary point* $(x_0, t_0) \in \Gamma(v)$, $\Gamma(v) \cap (B_{2^{-l}}(x_0) \times [t_0 - 2^{-l}, t_0 + 2^{-l}]$ *is a Lipschitz graph in space with Lipschitz constant less than* C_0/l *if* $l \ge l_0$.

Regularity in time up to the free boundary. Lastly, proceeding as in Sections 7–8 of [Choi et al. 2009] yields the differentiability of $\Gamma(v)$ in time. The main step in the argument is the following proposition: the statement and its proof is parallel to those of Theorem 7.2 in [Choi et al. 2009] and the blow-up argument as in Section 8 of [Choi et al. 2009]:

Theorem 5.7. $\Gamma(v)$ *is differentiable in space and time. More precisely there exist dimensional constants* $l_0 > 0$ *and* $1 < \gamma < 2$ *such that for* $(x_0, t_0) \in \Gamma(v) \cap Q_1$, *if* $l > l_0$ *then* $\Gamma(v) \cap (B_{2^{-l}}(x_0) \times [t_0 - 2^{-l}, t_0 + 2^{-l}]$ *is a Lipschitz graph in space with Lipschitz constant less than* $l^{-\gamma}$, *and Lipschitz graph in time with Lipschitz constant less than* $l^{-1/3}$.

Corollary 5.8.

$$C^{-1} \le |Dv^+|(x,t) \le C, \quad C^{-1} \le \frac{|Dv^-|(x,t)}{v(-e_n,t)} \le C$$

in $Q_{1/2}$, *where* $C = C(n)$.

6. General case: solutions with locally Lipschitz initial data

In this section, we present how to extend the result of the main theorem to solutions with locally Lipschitz initial data. Our setting is as follows. Suppose Ω_0 is a bounded region in $B_R(0)$. Suppose u is a solution of (ST2) with $u_0 \ge -1$, $u_0 = -1$ in $B_R(0)$ and $u_0 \le M_0$. Further suppose that Ω_0 is locally Lipschitz, that is, for any $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ is Lipschitz with a Lipschitz constant $L \le L_n$.

Let the initial data u_0 solve $\Delta u_0 = 0$ in $B_1(x_0)$. Then we claim that the parallel statements as in Theorem 1.1 hold in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$, where d_0 is a constant depending on n and M_0. More precisely:

Theorem 6.1. *Suppose u is a solution of* (ST2) *with initial data* u_0 *such that* $-1 \le u_0 \le M_0$. *Further suppose that for* $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ *is Lipschitz with a Lipschitz constant* $L \le L_n$ *and* $\Delta u_0 = 0$ *in the positive and negative phases of* u_0 *in* $B_1(x_0)$. *Then there exists a constant* $d_0 > 0$ *depending on n and* M_0 *such that* (*a*) *and* (*b*) *of Theorem 1.1 hold for u and* $d \le d_0$.

The proof of the above theorem is parallel to that of Theorem 1.1 in Section 5, after proving the following lemma.

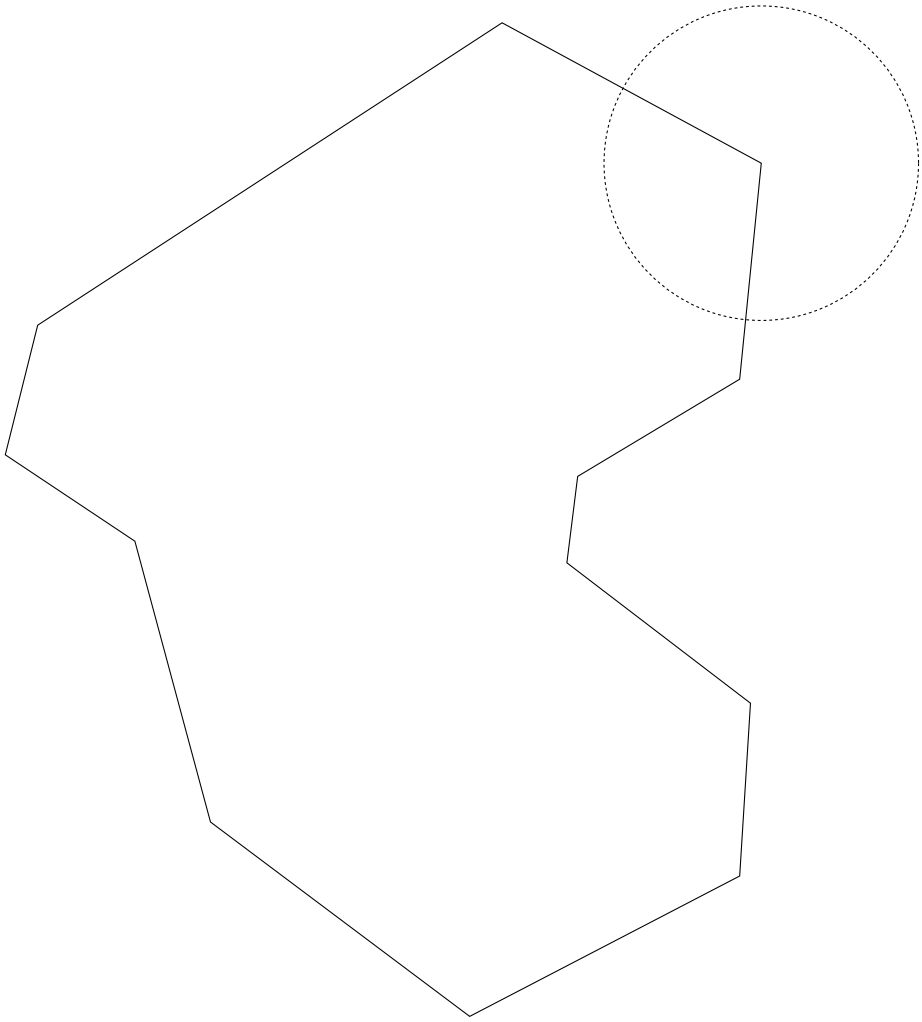

Figure 4. Locally Lipschitz initial domain.

Lemma 6.2. *There exists a solution v of* (ST2) *with star-shaped initial data such that the level sets of u and v are ϵd_0-close to each other in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0; u)$, where $d_0 > 0$ is sufficiently small. In particular, u and $\Gamma(u)$ are ϵ-monotone in a cone of W_x and W_t in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$.*

Even though our equation is nonlocal, the behavior in a far-away region would not affect much the behavior of the solution in the unit ball, if the solution behaves "reasonably" outside the unit ball. For example, in the star-shaped case, we know at least that the free boundary is almost locally Lipschitz at each time. In the locally Lipschitz case, we control the solution by putting an upper bound M_0 on the initial data u_0. We will argue that in a sufficiently small subregion of $B_1(x_0) \times [0, 1]$, the solution is mostly determined by the local initial data in $B_1(x_0)$. The perturbation method in the proof of Lemma 2.4 in [Choi et al. 2007] will be adopted here. Write $B_1(x_0) = B_1$.

Step 1. Construct a star-shaped region $\Omega' \subset B_R(0)$ such that:

(a) $\Omega' \cap B_1 = \Omega_0 \cap B_1$.

(b) Ω' is star-shaped with respect to every $x \in K \subset \Omega'$ for a sufficiently large ball K.

Let v_0^+ be the harmonic function in $\Omega' - K$ with boundary data 1 on ∂K, and 0 on $\partial\Omega'$. Next, let v_0^- be the harmonic function in $B_R(0) - \Omega'$ with boundary data 1 on $\partial B_R(0)$, and 0 on $\partial\Omega'$. Let B_2 be a concentric ball in B_1 with the radius of ϵ^{k_0}, i.e.,

$$B_2 = B_{\epsilon^{k_0}}(x_0) \subset B_1(x_0) = B_1.$$

Let k_0 be sufficiently large. Then by Lemma 2.7, a normalization of $v_0^{\pm}$ by a suitable constant multiple yields that for any $x \in B_2$,

$$1 - \epsilon \le \frac{u_0(x)}{v_0(x)} \le 1 + \epsilon. \tag{6-1}$$

Let v solve (ST2) with initial data $v_0 = v_0^+ - v_0^-$. Then Theorem 1.1 applies for v since v_0 is star-shaped with respect to K.

For the proof of the claim, we will find a sufficiently small d_0 such that v is ϵd_0-close to u in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0)$. More precisely, we will construct a supersolution w_1 and a subsolution w_2 of (ST2) such that in some small ball $B_h(x_0)$, we have

$$w_2 \le u \le w_1$$

and the level sets of w_1 and w_2 are $h\epsilon$ close to the level sets of v.

Step 2. Let k_1 and k_2 be large constants which will be determined later. Define

$$H^{\pm} := (\Gamma_0(v) \pm \epsilon^{k_0+k_1} e_n) \cap B_2.$$

Let

$$d_0 := \epsilon^{k_0+k_1+k_2}$$

and let $t(d_0) := t(x_0, d_0; v) = t(x_0, d_0; u)$. First note that

$$t(d_0) \ge d_0^{2-\beta} \ge \epsilon^{7(k_0+k_1+k_2)/6}.$$

Hence for v to be almost harmonic in a scale much larger than $\epsilon^{k_0+k_1}$, we need $\sqrt{t(d_0)} > \epsilon^{k_0}$, i.e.,

$$7(k_0 + k_1 + k_2)/12 < k_0.$$

Observe that by the construction of $H^{\pm}$ and d_0,

$$\sqrt{t(d_0)} \gg \text{radius}(B_2) \gg \text{dist}(H^{\pm}, \Gamma_0) \gg \max_{x \in \Gamma_t \cap B_2, 0 \le t \le t(d_0)} \text{dist}(x, \Gamma_0), \tag{6-2}$$

where the last inequality follows from Lemma 2.12 if we choose $k_2 \ge 2k_1$. If k_2 is sufficiently large, then one can prove from the last inequality of (6-2) and the bound on v_t that

$$1 - \epsilon \le \frac{|v(x,t)|}{|v_0(x)|} = \frac{|v(x,t)|}{|u_0(x)|} \le 1 + \epsilon \quad \text{on } H^{\pm} \times [0, t(d_0)]. \tag{6-3}$$

Step 3. We do have an estimate, Lemma 2.12, on how far the boundaries move away for the local one-phase case. If we take the one-phase versions with initial data u_0^+ and u_0^-, and compare with u, then we obtain that $\Gamma(u) \cap B_2$ stays in the $d_0^{(2-\alpha)/(2-\beta)}$-neighborhood of $\Gamma_0(u) \cap B_2$ up to the time $t(d_0) = t(x_0, d_0)$. In other words, the free boundary of u moves less than $d_0^{5/7}$ in B_2 up to the time $t(d_0)$.

Now we let S be the region between H^+ and H^-. To construct a subsolution (or supersolution) in S, we take the fixed boundary data $(1-\epsilon)v_0(x)$ on H^- (or H^+), and $(1+\epsilon)v_0(x)$ on H^+ (or H^-). To control the effect from the side $\partial B_2 \cap S$, we bend the free boundary $\Gamma_t(v)$ by $d_0^{5/7}$ on each side of $\partial B_2 \cap S$, using the conformal mapping $\hat{\Phi}$ (or $\breve{\Phi}$). (See Section 4 of for the definitions of $\hat{\Phi}$ and $\breve{\Phi}$.) More precisely, we bend the free boundary of v downward (or upward) using the conformal map $\hat{\Phi}$ (or $\breve{\Phi}$), and solve the heat equation in there. Then similar arguments as in Lemmas 4.1 and 4.3 of [Choi and Kim 2010] yield that the solution is still (almost) a supersolution, and it stays close to the original solution.

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Received 6 Feb 2011. Revised 12 Oct 2011. Accepted 15 Jan 2012.

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Analysis & PDE, at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

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Typeset in LaTeX

ANALYSIS & PDE

Volume 5 No. 5 2012

2157-5045(2012)5:5;1-F