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# NONLINEAR SCHRÖDINGER EQUATION AND FREQUENCY SATURATION

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We propose an approach that permits to avoid instability phenomena for the nonlinear Schrödinger equations. We show that by approximating the solution in a suitable way, relying on a frequency cut-off, global well-posedness is obtained in any Sobolev space with nonnegative regularity. The error between the exact solution and its approximation can be measured according to the regularity of the exact solution, with different accuracy according to the cases considered.

## 1. Introduction

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \epsilon |u|^{2\sigma} u, \quad (t,x) \in I \times \mathbb{R}^d, \quad u_{|t=0} = u_0, \tag{1-1}$$

for some time interval $I \ni 0$, with $\epsilon = 1$ (defocusing case) or $\epsilon = -1$ (focusing case). The aim of this paper is to propose an approach to overcome the lack of local well-posedness in Sobolev spaces with nonnegative regularity.

Recall two important invariances associated to (1-1):

- Scaling: if $u$ solves (1-1), then for $\lambda > 0$, so does $u_\lambda(t,x) := \lambda^{1/\sigma} u(\lambda^2 t, \lambda x)$. This scaling leaves the $\dot{H}^{s_c}_x$-norm invariant, with $s_c = d/2 - 1/\sigma$.
- Galilean: if $u$ solves (1-1), then for $v \in \mathbb{R}^d$, so does $e^{iv\cdot x - i|v|^2 t/2} u(t, x - vt)$. This transform leaves the $L^2_x$-norm invariant.

These two arguments suggest that the critical Sobolev regularity to solve (1-1) is $\max(s_c, 0)$. Indeed, if $s_c \geqslant 0$, local well-posedness from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for $s \geqslant s_c$ has been established in [Cazenave and Weissler 1990], and if $s_c < 0$, local well-posedness from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for $s \geqslant 0$ has been established in [Tsutsumi 1987].

If $s_c > 0$, pathological phenomena have been exhibited for initial data in $H^s(\mathbb{R}^d)$ with $0 < s < s_c$; Gilles Lebeau has proved a "norm inflation" phenomenon for the wave equation $\partial_t^2 u - \Delta u + u^p = 0$, $x \in \mathbb{R}^3$, $p \in 2\mathbb{N}+1$, $p \geqslant 7$ [Lebeau 2001; Métivier 2004]. The analogous result for (1-1) is this:

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**Theorem 1.1** [Christ et al. 2003; Burq et al. 2005]. *Let $\sigma \geqslant 1$. Assume that $s_c = d/2 - 1/\sigma > 0$, and let $0 < s < s_c$. There exists a family $(u_0^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^d)$ with*

$$\|u_0^h\|_{H^s(\mathbb{R}^d)} \to 0 \quad \text{as } h \to 0,$$

*a solution $u^h$ to (1-1) and $0 < t^h \to 0$, such that*

$$\|u^h(t^h)\|_{H^s(\mathbb{R}^d)} \to +\infty \quad \text{as } h \to 0.$$

The argument of the proof consists in considering concentrated initial data

$$u_0(x) = h^{s-d/2}(\log 1/h)^{-\alpha} a_0\Big(\frac{x}{h}\Big), \quad \text{with } h \to 0,$$

and showing that for very short time, the Laplacian can be neglected in (1-1). The above result then stems from its (easy) counterpart in the ODE case, by choosing a suitable $\alpha > 0$. In the spirit of [Lebeau 2005], the above result has been strengthened to a "loss of regularity" in [Alazard and Carles 2009; Carles 2007; Thomann 2008]. The assumptions and conclusion are similar to that in Theorem 1.1; the only difference is that $u^h(t^h, \cdot)$ is measured in $H^k(\mathbb{R}^d)$ for any $k > s/(1+\sigma(s_c - s))$, so $k$ is allowed to be smaller than $s$. In all the cases mentioned here, the lack of uniform continuity of the nonlinear flow map near the origin is due to the appearance of higher and higher frequencies on a very short time scale. If $s_c < 0$, similar pathological phenomena have been established in $H^s(\mathbb{R}^d)$ with $s < 0$, where on the contrary, low frequencies are ignited; see [Bejenaru and Tao 2006; Carles et al. 2012; Christ et al. 2003; Kenig et al. 2001]. In the rest of this paper, we focus on nonnegative regularity, $s \geqslant 0$.

The goal of this paper is twofold. First, we want to investigate a method to remove the pathology mentioned above, causing a lack of well-posedness for (1-1), in a deterministic way, as opposed to the probabilistic approach initiated in [Burq and Tzvetkov 2008a; 2008b] for the wave equation. The other motivation is related to numerical simulations for (1-1), where high frequencies may be a source of important errors; see for instance [Ignat and Zuazua 2012], a reference which will be discussed in further detail in Sections 3 and 4.

We show that with a suitable cut-off on the high frequencies of the nonlinearity, the obstructions to local well-posedness vanish, and the problem becomes globally well-posed: the nonlinear evolution of any initial datum in $L^2(\mathbb{R}^d)$ can be controlled *a priori*, an information which may be useful for numerics, since we do not have to decide if the initial datum belongs to a full measure set or not. This strategy is validated inasmuch as this procedure yields a good approximation of the solution to (1-1) as the cut-off tends to the identity. Note that this approach can be viewed as a deterministic counterpart of the one presented in [Burq et al. 2012] (see also [Burq 2011]). There, for the one-dimensional $L^2$-supercritical defocusing nonlinear Schrödinger equation, the authors construct a Gibbs measure such that, among other features, the pathological phenomenon described in Theorem 1.1 occurs for a set of initial data whose measure is zero: on the support of the Gibbs measure, the Cauchy problem is globally well-posed, and a scattering theory is available. Both points of view aim at showing that norm inflation in the sense of Theorem 1.1 is a rare phenomenon: in [Burq et al. 2012], the authors give a rigorous meaning to this statement in an abstract way, while we are rather interested in a recipe to avoid instabilities for sure, by a suitable approximation of the equation, which can be used typically for numerical simulations.

Our choice of cutting off the high frequencies instead of, for instance, the values of the function itself is indeed motivated by numerics, where it is standard to filter out high frequencies (sometimes without even saying so). In an appendix, we discuss another approach, consisting in saturating the values of the nonlinearity. One could of course combine both approaches, frequency and physical saturations.

***Notation.*** We define the Fourier transform by the formula

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

We write $a \lesssim b$ if there exists $C$ such that $a \leqslant Cb$. In the presence of a small parameter $h$, the notation indicates that $C$ is independent of $h \in (0, 1]$.

## 2. From instability to global well-posedness

Let $\chi : \mathbb{R}^d \to [0, 1]$ be a smooth function, equal to one on the unit ball, and even: $\chi(-x) = \chi(x)$ for all $x \in \mathbb{R}^d$. It may be compactly supported, in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, or with a slower decay at infinity. For simplicity, we will not discuss sharp assumptions on $\chi$. We define the frequency "cut-off" $\Pi$ as the Fourier multiplier

$$\widehat{\Pi(f)}(\xi) = \chi(\xi)\widehat{f}(\xi).$$

As pointed out in the introduction, in the examples constructed to prove the lack of local well-posedness, the mechanism of high frequencies amplification occurs at the level of the ordinary differential equation. We discuss some strategies to saturate high frequencies at the ODE level first, with $\epsilon = 1$ for simplicity.

**2A.** ***Candidates at the ODE level.*** The first possibility to prevent the appearance of high frequencies by nonlinear self-interaction consists in saturating the whole nonlinearity:

$$i\partial_t v = \Pi(|v|^{2\sigma} v). \tag{2-1}$$

This can be viewed as an extremely simplified version of the $I$-method (see [Colliander et al. 2002]). Another choice consists in saturating the high frequencies in the "nonlinear multiplicative potential" only, that is $|v|^{2\sigma}$. For $\sigma \in \mathbb{N}$, we propose two possibilities,

$$i\partial_t v = \Pi(|v|^{2\sigma})v, \tag{2-2}$$

$$i\partial_t v = \big(\Pi(|v|^2)\big)^{\sigma} v. \tag{2-3}$$

In the cubic case $\sigma = 1$, the last two approaches obviously coincide. These approaches have two advantages over (2-1):

- They preserve gauge invariance. If $v$ solves the equation, then so does $ve^{i\theta}$ for any constant $\theta \in \mathbb{R}$.
- They preserve conservation of mass.

To see the second point, rewrite $\Pi(f)$ as $K * f$, with $K(x) = (2\pi)^{-d/2}\widehat{\chi}(-x)$. Since $\chi$ is even and real-valued, so is $K$, and therefore $\partial_t |v|^2 = 0$ in (2-2) and (2-3). This identity leads to the conservation of the $L^2$-norm at the PDE level.

Before passing to the PDE case, we conclude this section by showing that even at the ODE level, cutting off high frequencies in the initial data does not suffice to prevent the appearance of higher frequencies in the solution for positive time. For $a \in \mathcal{S}(\mathbb{R}^d)$ and $s > 0$, consider the solution $v^h$ to

$$i\partial_t v^h = |v^h|^{2\sigma} v^h, \quad v^h(0,x) = h^{s-d/2} a\left(\frac{x}{h}\right).$$

Then $v^h{}_{|t=0}$ is bounded in $H^s(\mathbb{R}^d)$, uniformly in $h \in (0,1]$, and if $\widehat{a}$ is compactly supported (in $B(0,R)$), then $\widehat{v^h}{}_{|t=0}$ is compactly supported (in $B(0,R/h)$). Since $\partial_t |v^h|^2 = 0$, we have the explicit formula

$$v^h(t,x) = h^{s-d/2} a\left(\frac{x}{h}\right) \exp\left(-ith^{2\sigma(s-d/2)} \left|a\left(\frac{x}{h}\right)\right|^{2\sigma}\right).$$

We check that for $t > 0$, as $h \to 0$, the homogeneous Sobolev norms behave like

$$\|v^h(t)\|_{\dot{H}^k} \approx h^{s-2k\sigma(s-d/2)-k} t^k,$$

at least for $k \in \mathbb{N}$. The above quantity is unbounded as $h \to 0$ if

$$k > \frac{s}{1+2\sigma(s-d/2)}.$$

Therefore, if $s < d/2$, $v^h(t,\cdot)$ is unbounded in $H^s(\mathbb{R}^d)$ for $t > 0$, as $h \to 0$: cutting off the high frequencies in the initial data does not suffice to control the frequency support of the solution. On the other hand, the models (2-2) and (2-3) prevent the appearance of high frequencies by nonlinear self-interaction. The above mechanism is essentially the one that leads to the norm inflation phenomenon in [Burq et al. 2005; Christ et al. 2003; Lebeau 2001], except that in those papers, the approximation by an ODE is used only on a time interval where the $H^s$-norm becomes unbounded, but not the $H^k$-norm for any $k < s$. The above mechanism at the PDE level leads to the loss of regularity [Alazard and Carles 2009; Carles 2007; Lebeau 2005; Thomann 2008], where indeed $k$ is allowed to be smaller than $s$, as recalled in the introduction. Roughly speaking, the appearance of oscillations is quite similar to the above ODE example; in the PDE case, the numerology is different, and the proof is more intricate.

**2B.** ***Choice at the PDE level.*** We consider now the equations

$$i\partial_t u + P(D)u = \epsilon \Pi(|u|^{2\sigma})u, \tag{2-4}$$

$$i\partial_t u + P(D)u = \epsilon \left(\Pi(|u|^2)\right)^{\sigma} u, \tag{2-5}$$

where $P(D)$ is a Fourier multiplier with a real-valued symbol $P : \mathbb{R}^d \to \mathbb{R}$,

$$\widehat{P(D)f} = P(\xi)\widehat{f}(\xi).$$

The $L^2$-norm of $u$ is formally independent of time:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2\, dx = 0. \tag{2-6}$$

In view of this conservation and of the Young inequality

$$\|\Pi(f)\|_{L^\infty} \leqslant \|K\|_{L^\infty}\|f\|_{L^1}, \tag{2-7}$$

the option (2-5) seems more interesting than (2-4), and we have the following result.

**Theorem 2.1.** *Let $\sigma \in \mathbb{N}$, $\epsilon \in \{\pm 1\}$, $P : \mathbb{R}^d \to \mathbb{R}$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ even and real-valued.*

- *For any $u_0 \in L^2(\mathbb{R}^d)$, (2-5) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ such that $u_{|t=0} = u_0$. Its $L^2$-norm is independent of time; (2-6) holds.*
- *If in addition $u_0 \in H^s(\mathbb{R}^d)$, $s \in \mathbb{N}$, then $u \in C(\mathbb{R}; H^s(\mathbb{R}^d))$.*
- *The flow map $u_0 \mapsto u$ is uniformly continuous from the balls in $L^2(\mathbb{R}^d)$ to $C(\mathbb{R}; L^2(\mathbb{R}^d))$. More precisely, for all $u_0, v_0 \in L^2(\mathbb{R}^d)$, there exists $C$ depending on $\sigma$, $\|K\|_{L^\infty}$, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$ such that, for all $T > 0$,*

$$\|u - v\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} e^{CT}, \tag{2-8}$$

  *where $u$ and $v$ denote the solutions to (2-5) with initial data $u_0$ and $v_0$, respectively.*
- *More generally, let $s \in \mathbb{N}$. For all $u_0, v_0 \in H^s(\mathbb{R}^d)$, there exists $C$ depending on $\sigma$, $\|K\|_{W^{s,\infty}}$, $\|u_0\|_{H^s}$ and $\|v_0\|_{H^s}$ such that for all $T > 0$,*

$$\|u - v\|_{L^\infty([-T,T];H^s(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{H^s(\mathbb{R}^d)} e^{CT}. \tag{2-9}$$

**Remark 2.2.** As pointed out in [Cazenave et al. 2011], even if the solution is constructed by a fixed point argument, the continuity of the flow map is not trivial in general. In the case of Schrödinger equations (1-1), continuity of the flow map in $H^s(\mathbb{R}^d)$ is known only in a limited number of cases: see [Tsutsumi 1987] for $s = 0$, [Kato 1987] for $s = 1$ and $s = 2$, and [Cazenave et al. 2011] for $0 < s < 1$.

*Proof.* First, recall that $S(t) = e^{-itP(D)}$ is a unitary group on $\dot{H}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. Duhamel's formula associated to (2-5) reads

$$u(t) = S(t)u_0 - i\epsilon \int_0^t S(t-\tau)\big((K * |u|^2)^\sigma u\big)(\tau)\, d\tau. \tag{2-10}$$

The local existence in $L^2$ stems from a standard fixed point argument in

$$X(T) = \big\{u \in C([-T,T]; L^2(\mathbb{R}^d)) \;\big|\; \|u\|_{L^\infty([-T,T];L^2)} \leqslant 2\|u_0\|_{L^2}\big\}.$$

Denote by $\Phi(u)(t)$ the right hand side of (2-10). In view of (2-7), for $t \in [-T, T]$,

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^2} &\leqslant \|u_0\|_{L^2} + \int_{-T}^{T} \big\|\big((K * |u|^2)^\sigma u\big)(\tau)\big\|_{L^2}\, d\tau \\
&\leqslant \|u_0\|_{L^2} + \int_{-T}^{T} \big\|K * |u(\tau)|^2\big\|_{L^\infty}^\sigma \|u(\tau)\|_{L^2}\, d\tau \\
&\leqslant \|u_0\|_{L^2} + \|K\|_{L^\infty}^\sigma \int_{-T}^{T} \|u(\tau)\|_{L^2}^{2\sigma+1}\, d\tau.
\end{aligned}$$

By choosing $T > 0$ sufficiently small, we see that $X(T)$ is stable under the action of $\Phi$. Note that in the case of the model (2-4), the above estimate would have to be adapted, forcing us to work in a space smaller than $X(T)$ ($L^2$ regularity in space would no longer be sufficient in general). Contraction is established in the same way:

$$\begin{aligned}\|\Phi(u)(t)-\Phi(v)(t)\|_{L^2} &\leqslant \int_{-T}^{T}\left\|\big((K*|u|^2)^\sigma u\big)(\tau)-\big((K*|v|^2)^\sigma v\big)(\tau)\right\|_{L^2}d\tau\\ &\leqslant \int_{-T}^{T}\left\|\big((K*|u|^2)^\sigma-(K*|v|^2)^\sigma\big)u\right\|_{L^2}d\tau+\int_{-T}^{T}\left\|\big((K*|v|^2)^\sigma\big)(u-v)\right\|_{L^2}d\tau.\end{aligned}$$

Using the estimate $|a^\sigma-b^\sigma|\lesssim(|a|^{\sigma-1}+|b|^{\sigma-1})|a-b|$, and (2-7) again, we infer

$$\begin{aligned}&\|\Phi(u)(t)-\Phi(v)(t)\|_{L^2}\\ &\quad\lesssim \|K\|_{L^\infty}^\sigma\int_{-T}^{T}\big(\|u\|_{L^2}^{2\sigma-1}+\|v\|_{L^2}^{2\sigma-1}\big)\|u-v\|_{L^2}\|u\|_{L^2}\,d\tau+\|K\|_{L^\infty}^\sigma\int_{-T}^{T}\|v\|_{L^2}^{2\sigma}\|u-v\|_{L^2}\,d\tau,\end{aligned}$$

where all the functions inside the integrals are implicitly evaluated at time $\tau$. Choosing $T>0$ possibly smaller, $\Phi$ is a contraction on $X(T)$. Note that this small time $T$ depends only on $\sigma$, $\|K\|_{L^\infty}$ and $\|u_0\|_{L^2}$. Since the $L^2$-norm of $u$ is preserved (see [Cazenave 2003] for a rigorous justification), the construction of a local solution can be repeated indefinitely, hence global existence and uniqueness at the $L^2$ level.

Global existence in $H^s(\mathbb{R}^d)$ for $s\in\mathbb{N}$ then follows easily, thanks to the estimate

$$\left\|(K*|u|^2)^\sigma u\right\|_{H^s}\lesssim\sum_{|\alpha|+|\beta|=s}\left\|\partial^\alpha(K*|u|^2)^\sigma\partial^\beta u\right\|_{L^2}\lesssim\|K\|_{W^{s,\infty}}^\sigma\|u\|_{L^2}^\sigma\|u\|_{H^s}.$$

The continuity of the flow map in $L^2$ is obtained by resuming the estimate written to establish the contraction of $\Phi$: For $t>0$,

$$\begin{aligned}\|u(t)-v(t)\|_{L^2}&\leqslant\|u_0-v_0\|_{L^2}+\|K\|_{L^\infty}^\sigma\int_0^t\big(\|u\|_{L^2}^{2\sigma}+\|v\|_{L^2}^{2\sigma}\big)\|u-v\|_{L^2}\,d\tau\\ &\leqslant\|u_0-v_0\|_{L^2}+\|K\|_{L^\infty}^\sigma\big(\|u_0\|_{L^2}^{2\sigma}+\|v_0\|_{L^2}^{2\sigma}\big)\int_0^t\|u-v\|_{L^2}\,d\tau,\end{aligned}$$

where we have used the conservation of the $L^2$-norm. Proceeding similarly for $t<0$, Gronwall's lemma then yields (2-8) for $C$ depending only of $\sigma$, $\|K\|_{L^\infty}$, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$. Finally, (2-9) is obtained in a similar fashion. □

**Remark 2.3.** The proof of continuity of the flow map is easy. This is in sharp contrast with the case of the equation without frequency cut-off. In the case of Schrödinger equations ($P(\xi)=-|\xi|^2$), continuity is more intricate to establish (see [Tsutsumi 1987]), and is true only for $L^2$-subcritical nonlinearities, $\sigma\leqslant 2/d$, from [Christ et al. 2003].

We note that even for large $\sigma$, global well-posedness in $L^2$ is available, in sharp contrast with the nonlinear Schrödinger equation (1-1). Even in the focusing case $\epsilon=-1$, the high frequency cut-off prevents finite time blow-up. In (2-9), consider $v_0=v=0$ and $s=1$ for instance: by comparison with the case of (1-1), we see that the constant $C$ necessarily depends on $K$ (or equivalently on $\chi$), and is unbounded as $\chi$ converges to the Dirac mass. The frequency cut-off $\Pi$ removes the instabilities, and prevents finite time blow-up.

**Remark 2.4** (Hamiltonian structure in the cubic case). If $\sigma = 1$, (2-4) and (2-5) coincide. We have the equivalence

$$\chi \text{ even and real-valued } \iff K \text{ even and real-valued.}$$

This implies that under the assumption of Theorem 2.1, (2-5) has an Hamiltonian structure, and the conserved energy is

$$H(u) = \int_{\mathbb{R}^d} \overline{u}(x) P(D) u(x)\, dx + \frac{\epsilon}{2} \iint K(x-y)|u(y)|^2 |u(x)|^2 \, dx dy.$$

## 3. Convergence in the smooth case

Suppose that $P(D)$ converges to $\Delta$ and that $\Pi$ converges to Id, does the solution to (2-5) then converge to the solution of NLS? We show that this is the case under suitable assumptions on these convergences, at least in the case where the solution to the limiting Equation (1-1) is very smooth. In the sequel, the convergence is indexed by $h \in (0, 1]$.

**Proposition 3.1.** *Let $\sigma \in \mathbb{N}$. We assume that $P$ and $\Pi$ verify the following properties*:

- *There exist $\alpha, \beta \geqslant 0$ such that $P_h(\xi) = -|\xi|^2 + \mathcal{O}(h^\alpha \langle \xi \rangle^\beta)$.*
- *$\chi_h(\xi) = \chi(h\xi)$, with $\chi \in \mathcal{S}(\mathbb{R}^d; [0, 1])$ even, real-valued, $\chi = 1$ on the unit ball.*

*Denote by $u^h$ the solution to (2-5) with $P_h$ and $\chi_h$, such that $u^h_{|t=0} = u_{|t=0}$. Suppose that the solution to (1-1) satisfies $u \in L^\infty([0, T]; H^{s+\beta})$, for some $s > d/2$. Then*

$$\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,\beta)}.$$

**Example 3.2.** The above assumption on $P_h$ is satisfied with $\alpha = 1$ and $\beta = 2$ in the following cases:

- $P_h(\xi) = \dfrac{-|\xi|^2}{1+h|\xi|^2}$.
- $P_h(\xi) = -\dfrac{1}{h} \arctan(h|\xi|^2)$.

The second example is borrowed from [Debussche and Faou 2009], where this truncated operator appears naturally when discretizing the Laplacian for numerical schemes.

**Remark 3.3.** In this result, no assumption is needed on the possible decay of $\chi$ at infinity.

*Proof.* Let $w^h = u - u^h$. It satisfies $w^h_{|t=0} = 0$ and

$$\begin{aligned} i\partial_t w^h + P_h(D) w^h = \epsilon (\Pi_h(|u|^2))^\sigma u - \epsilon (\Pi_h(|u^h|^2))^\sigma u^h \\ + (P_h(D) - \Delta)\, u + \epsilon \big(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\big) u, \end{aligned}$$

where we have denoted by $\Pi_h$ the Fourier multiplier of symbol $\chi_h$. Denote by $R^h(u)$ the second line, which corresponds to a source term. In view of the assumption on $P_h$, there exists $C$ independent of $h \in (0, 1]$ such that

$$\|P_h(D) f - \Delta f\|_{H^s} \leqslant C h^\alpha \|f\|_{H^{s+\beta}} \quad \text{for all } f \in H^{s+\beta}(\mathbb{R}^d).$$

We also have, by the Plancherel formula,

$$\begin{aligned}\|(1-\Pi_h)f\|_{H^s}^2 &= \int_{\mathbb{R}^d}(1-\chi(h\xi))^2\langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2 d\xi\\ &\leqslant \int_{|\xi|>1/h}\langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2 d\xi \leqslant h^{2\beta}\int_{|\xi|>1/h}\langle\xi\rangle^{2s+2\beta}|\widehat{f}(\xi)|^2 d\xi \leqslant h^{2\beta}\|f\|_{H^{s+\beta}}^2.\end{aligned}$$

Therefore,

$$\|R^h(u)\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,\beta)}\|u\|_{L^\infty([0,T];H^{s+\beta})}.$$

Now since $s>d/2$, $H^s(\mathbb{R}^d)$ is an algebra, and there exists $C$ independent of $h$ such that

$$\left\|(\Pi_h(|u|^2))^\sigma u-(\Pi_h(|u^h|^2))^\sigma u^h\right\|_{H^s} \leqslant C\|\widehat{\chi}\|_{L^1}^\sigma\left(\|u\|_{H^s}^{2\sigma}+\|u^h\|_{H^s}^{2\sigma}\right)\|u-u^h\|_{H^s},$$

where the Young inequality that we have used is not the same as in Section 2:

$$\|K*f\|_{L^2}\leqslant\|K\|_{L^1}\|f\|_{L^2}.$$

This is essentially the only way to obtain an estimate independent of $h\in(0,1]$. Indeed, $\Pi_h(f)=K_h*f$, with

$$K_h(x)=\frac{1}{(2\pi)^{d/2}h^d}\widehat{\chi}\left(\frac{-x}{h}\right).$$

The result then stems from a bootstrap argument: so long as

$$\|u^h\|_{L^\infty([0,t];H^s)}\leqslant 1+\|u\|_{L^\infty([0,T];H^s)},$$

Gronwall's lemma yields

$$\|u-u^h\|_{L^\infty([0,t];H^s)} \lesssim h^{\min(\alpha,\beta)}\|u\|_{L^\infty([0,T];H^{s+\beta})}.$$

Therefore, up to choosing $h$ sufficiently small, this estimate is valid up to $t=T$. □

Such a convergence result can be compared to the one proved in [Ignat and Zuazua 2012] to prove the convergence of numerical approximations. The approach there is a bit different though, inasmuch as the frequency cut-off does not affect the nonlinearity (as in (2-5)), but the initial data: consider a solution $v^h$ to

$$i\partial_t v^h+P_h(D)v^h=\epsilon|v^h|^{2\sigma}v^h,\quad v^h_{|t=0}=\Pi_h u_0.$$

Ignat and Zuazua proved that the discrete analogue of $\Pi_h u-v_h$ is small. Proposition 3.1 differs from their results in several aspects:

- The context in [Ignat and Zuazua 2012] is discrete.
- Only the low frequency part of $u$, $\Pi_h u$, is shown to be well approximated.
- The regularity assumption on $u$ may be much weaker.

As mentioned above, the second point is due to the choice of the model. However, as discussed in Section 2A, controlling the high frequencies of the initial data must not be expected to ensure a control of high frequencies of the solution $v^h$ for positive time.

The third point is due to the use of Strichartz estimates in [Ignat and Zuazua 2012]. In the next section, we show that in the presence of dispersion (with $P_h(\xi) = -|\xi|^2$), Proposition 3.1 can be adapted to rougher data.

## 4. Convergence using dispersive estimates

We first recall a standard definition.

**Definition 4.1.** A pair $(p,q) \neq (2,\infty)$ is admissible if $p \geqslant 2$, $q \geqslant 2$, and

$$\frac{2}{p} = d\Big(\frac{1}{2} - \frac{1}{q}\Big).$$

We shall consider (2-5) when $P(D)$ is exactly the Laplacian, and not an approximation as in Proposition 3.1. The reason is that when $P$ is bounded, then no Strichartz estimate is available, as we now recall. Let $S(\cdot)$ be bounded on $H^s$ for all $s \geqslant 0$. By the Sobolev embedding, for all $(p,q)$ (not necessarily admissible) with $2 \leqslant q < \infty$, there exists $C > 0$ such that for all $u_0 \in H^{d/2-d/q}(\mathbb{R}^d)$, and all finite time interval $I$,

$$\|S(\cdot)u_0\|_{L^p(I;L^q(\mathbb{R}^d))} \leqslant C\|S(\cdot)u_0\|_{L^p(I;H^{d/2-d/q}(\mathbb{R}^d))} \leqslant C\|u_0\|_{L^p(I;H^{d/2-d/q}(\mathbb{R}^d))} = C|I|^{1/p}\|u_0\|_{H^{d/2-d/q}(\mathbb{R}^d)}.$$

If the Fourier multiplier $P$ is bounded, the above estimate cannot be improved, in sharp contrast with the result provided by the Strichartz estimates.

**Proposition 4.2** [Carles 2011]. *Let $d \geqslant 1$, and $P \in L^\infty(\mathbb{R}^d;\mathbb{R})$. Write $S(t) = e^{-itP(D)}$. Suppose that there exist an admissible pair $(p,q)$, an index $k \in \mathbb{R}$, a time interval $I \ni 0$, $|I| > 0$, and a constant $C > 0$ such that*

$$\|S(\cdot)u_0\|_{L^p(I;L^q(\mathbb{R}^d))} \leqslant C\|u_0\|_{H^k(\mathbb{R}^d)} \quad \textit{for all } u_0 \in H^k(\mathbb{R}^d).$$

*Then necessarily $k \geqslant 2/p = d/2 - d/q$.*

We now state the main result of this section.

**Theorem 4.3.** *Let $\sigma \in \mathbb{N}$ and $T > 0$. We assume that $\chi_h(\xi) = \chi(h\xi)$, with $\chi \in \mathcal{S}(\mathbb{R}^d)$ even, real-valued, $\chi = 1$ on $B(0,1)$. Let $u$ solve (1-1), and consider the solution $u^h$ to*

$$i\partial_t u^h + \Delta u^h = \epsilon\Big(\Pi_h\big(|u^h|^2\big)\Big)^\sigma u^h, \quad u^h_{|t=0} = u_0.$$

1. *Suppose that $\sigma = 1$ and $d \leqslant 2$. If $u \in L^\infty([0,T];L^2) \cap L^{8/d}([0,T];L^4)$, then*

$$\|u - u^h\|_{L^\infty([0,T];L^2)} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that $\sigma = 1$ and $d = 3$.*

- *If* $u, \nabla u \in L^\infty([0,T];L^2)\cap L^{8/d}([0,T];L^4)$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

- *If* $u\in L^\infty([0,T];H^s)$, *with* $s>3/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h^{s-1}.$$

3. *Suppose that* $\sigma\geqslant 1$ *and* $d\leqslant 2$. *If* $u\in L^\infty([0,T];H^s)$, *with* $s\geqslant 1$ *and* $s>d/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

*If in addition* $s>1$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h^{s-1}.$$

**Remark 4.4.** Suppose $u_0$ sufficiently smooth. If $\epsilon=+1$ (defocusing case), the bounds for $u$ are known in several cases, with $T>0$ arbitrarily large. On the contrary, if $\epsilon=-1$ (focusing case), $T$ may have to be finite, bounded by a blow-up time. See [Cazenave 2003; Ginibre and Velo 1984]. Typically, if $\sigma=d=1$, then the assumption of the first point is fulfilled for all $T>0$ as soon as $u_0\in L^2(\mathbb{R})$, for $\epsilon\in\{\pm 1\}$, from [Tsutsumi 1987], and if $\sigma\geqslant 1$, $d\leqslant 2$, the assumption of the third point is fulfilled for all $T>0$ as soon as $u_0\in H^s(\mathbb{R}^d)$, for $\epsilon=+1$, from [Ginibre and Velo 1984].

*Proof.* For fixed $h>0$, Theorem 2.1 shows that $u^h\in C(\mathbb{R};H^k)$, with $k=0$, 1 or $s$ according to the cases considered in the assumptions of the theorem. Of course, the bounds provided by Theorem 2.1 blow up as $h\to 0$ if $k>0$.

As in the proof of Proposition 3.1, let $w^h=u-u^h$. The equation satisfied by $w^h$ is simpler than in the proof of Proposition 3.1, since $P_h(D)=\Delta$:

$$i\partial_t w^h+\Delta w^h=\epsilon\left(\Pi_h(|u|^2)\right)^\sigma u-\epsilon\left(\Pi_h(|u^h|^2)\right)^\sigma u^h+\epsilon\big(|u|^{2\sigma}-\left(\Pi_h(|u|^2)\right)^\sigma\big)u.$$

We resume the notation

$$R^h(u)=\epsilon\big(|u|^{2\sigma}-\left(\Pi_h(|u|^2)\right)^\sigma\big)u \quad \text{and} \quad \Pi_h(f)=K_h * f,$$

with $K_h(x)=(2\pi)^{-d/2}h^{-d}\widehat{\chi}(-x/h)$. From the Young inequality, we have, for all $q\in[1,\infty]$,

$$\|\Pi_h(f)\|_{L^q}\leqslant \|K_h\|_{L^1}\|f\|_{L^q}\leqslant \|\widehat{\chi}\|_{L^1}\|f\|_{L^q}, \tag{4-1}$$

an estimate which is uniform in $h>0$. Introduce the Lebesgue exponents

$$q=2\sigma+2,\qquad p=\frac{4\sigma+4}{d\sigma},\qquad \theta=\frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma}.$$

The pair $(p,q)$ is admissible, and

$$\frac{1}{q'}=\frac{2\sigma}{q}+\frac{1}{q},\qquad \frac{1}{p'}=\frac{2\sigma}{\theta}+\frac{1}{p}. \tag{4-2}$$

For $t>0$, write $L^j_tL^k=L^j([0,t];L^k(\mathbb{R}^d))$. From the Strichartz estimates (see [Cazenave 2003]),

$$\begin{aligned}\|w^h\|_{L^p_tL^q\cap L^\infty_tL^2} &\lesssim \left\|\left(\Pi_h(|u|^2)\right)^\sigma u-\left(\Pi_h(|u^h|^2)\right)^\sigma u^h\right\|_{L^{p'}_tL^{q'}}+\|R^h(u)\|_{L^{p'_1}_tL^{q'_1}}\\ &\lesssim \left(\|u\|^{2\sigma}_{L^\theta_tL^q}+\|u^h\|^{2\sigma}_{L^\theta_tL^q}\right)\|w^h\|_{L^p_tL^q}+\|R^h(u)\|_{L^{p'_1}_tL^{q'_1}},\end{aligned}$$

where we have used the Hölder inequality and (4-1), and where $(p_1,q_1)$ is an admissible pair whose value will be given later.

If $\sigma=1$ and $d\leqslant 2$, then $\theta\leqslant p$, and we infer

$$\|w^h\|_{L^p_tL^q\cap L^\infty_tL^2}\lesssim t^{1/\theta-1/p}\left(\|u\|^{2\sigma}_{L^p_tL^q}+\|u^h\|^{2\sigma}_{L^p_tL^q}\right)\|w^h\|_{L^p_tL^q}+\|R^h(u)\|_{L^{p'_1}_tL^{q'_1}}.$$

In the first case of the theorem, we assume $u\in L^p([0,T];L^q)$, since $p=8/d$ and $q=4$ for $\sigma=1$. We use again a bootstrap argument: so long as $\|u^h\|_{L^p_tL^q}\leqslant 2\|u\|_{L^p_tL^q}$, we divide the interval $[0,T]$ into finitely many small intervals so the first term of the right hand side is absorbed by the left hand side (recall that $p$ is finite), and we have

$$\|w^h\|_{L^p_tL^q\cap L^\infty_tL^2}\lesssim\|R^h(u)\|_{L^{p'_1}_tL^{q'_1}}.$$

The bootstrap argument is validated provided that $\|R^h(u)\|_{L^{p'_1}_TL^{q'_1}}\to 0$ as $h\to 0$.

If we have only $\sigma<2/(d-2)$, then by the Sobolev embedding,

$$\|u\|_{L^\theta_tL^q}\leqslant t^{1/\theta}\|u\|_{L^\infty_tH^1}.$$

In the same way as above,

$$\|\nabla w^h\|_{L^p_tL^q\cap L^\infty_tL^2}\lesssim\left\|\nabla\left(\left(\Pi_h(|u|^2)\right)^\sigma u-\left(\Pi_h(|u^h|^2)\right)^\sigma u^h\right)\right\|_{L^{p'}_tL^{q'}}+\|\nabla R^h(u)\|_{L^{p'_1}_tL^{q'_1}}.$$

The first term of the right hand side is controlled by

$$\left\|\left(\Pi_h(|u|^2)\right)^\sigma\nabla u-\left(\Pi_h(|u^h|^2)\right)^\sigma\nabla u^h\right\|_{L^{p'}_tL^{q'}}+\left\|u\nabla\left(\Pi_h(|u|^2)\right)^\sigma-u^h\nabla\left(\Pi_h(|u^h|^2)\right)^\sigma\right\|_{L^{p'}_tL^{q'}}. \tag{4-3}$$

Introducing the factor $(\Pi_h(|u|^2))^\sigma\nabla u^h$, the first term is estimated by

$$\begin{aligned}&\left\|\left(\Pi_h(|u|^2)\right)^\sigma\nabla w^h\right\|_{L^{p'}_tL^{q'}}+\left\|\left(\left(\Pi_h(|u|^2)\right)^\sigma-\left(\Pi_h(|u^h|^2)\right)^\sigma\right)\nabla u^h\right\|_{L^{p'}_tL^{q'}}\\ &\quad\lesssim\left\|\Pi_h(|u|^2)\right\|^\sigma_{L^{\theta/2}_tL^{q/2}}\|\nabla w^h\|_{L^p_tL^q}+\left(\|u\|^{2\sigma-2}_{L^\theta_tL^q}+\|u^h\|^{2\sigma-2}_{L^\theta_tL^q}\right)\left\||u|^2-|u^h|^2\right\|_{L^{\theta/2}_tL^{q/2}}\|\nabla u^h\|_{L^p_tL^q}\\ &\quad\lesssim\|u\|^{2\sigma}_{L^\theta_tL^q}\|\nabla w^h\|_{L^p_tL^q}+\left(\|u\|^{2\sigma-1}_{L^\theta_tL^q}+\|u^h\|^{2\sigma-1}_{L^\theta_tL^q}\right)\left\|w^h\right\|_{L^\theta_tL^q}\|\nabla u^h\|_{L^p_tL^q}\\ &\quad\lesssim t^{2\sigma/\theta}\|u\|^{2\sigma}_{L^\infty_tH^1}\|\nabla w^h\|_{L^p_tL^q}+t^{2\sigma/\theta}\left(\|u\|^{2\sigma-1}_{L^\infty_tH^1}+\|u^h\|^{2\sigma-1}_{L^\infty_tH^1}\right)\left\|w^h\right\|_{L^\infty_tH^1}\|\nabla u^h\|_{L^p_tL^q}.\end{aligned}$$

Proceeding similarly for the other term in (4-3), splitting $[0,T]$ into finitely many time intervals where the terms containing $w^h$ on the right hand side can be absorbed by the left hand side, and using a bootstrap argument, we end up with

$$\|w^h\|_{L^p_tW^{1,q}\cap L^\infty_tH^1}\lesssim\|R^h(u)\|_{L^{p'_1}_tW^{1,q'_1}}.$$

Therefore, it suffices to show that for some admissible pair $(p_1, q_1)$, the source term converges to 0 in $L^{p_1'}([0,T]; L^{q_1'})$ (if $\sigma = 1$ and $d \leqslant 2$) or in $L^{p_1'}([0,T]; W^{1,q_1'})$ (in the other cases), so the bootstrap argument is completed. In addition, the rate of converge of the source term, if any, yields a rate of convergence for $w^h$. The theorem then stems from the following lemma, in which $(p,q)$ is given by (4-2).

**Lemma 4.5.** *Let $T > 0$. The source term $R^h(u)$ can be controlled as follows.*

1. *Suppose that $\sigma = 1$ and $d \leqslant 2$. If $u \in L^\infty([0,T]; L^2) \cap L^{8/d}([0,T]; L^4)$, then*

$$\|R^h(u)\|_{L^{p'}([0,T];L^{q'})} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that $\sigma = 1$ and $d = 3$.*

  - *If $u, \nabla u \in L^\infty([0,T]; L^2) \cap L^{8/d}([0,T]; L^4)$, then*

$$\|R^h(u)\|_{L^{p'}([0,T];W^{1,q'})} \underset{h\to 0}{\longrightarrow} 0.$$

  - *If $u \in L^\infty([0,T]; H^s)$, with $s > 3/2$, then*

$$\|R^h(u)\|_{L^1([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|R^h(u)\|_{L^1([0,T];H^1)} \lesssim h^{s-1}.$$

3. *Suppose that $\sigma \geqslant 1$ and $d \leqslant 2$. If $u \in L^\infty([0,T]; H^s)$, with $s \geqslant 1$ and $s > d/2$, then*

$$\|R^h(u)\|_{L^1([0,T];L^2)} \lesssim h^s \quad \textit{and} \quad \|R^h(u)\|_{L^1([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

*If in addition $s > 1$, then*

$$\|R^h(u)\|_{L^1([0,T];H^1)} \lesssim h^{s-1}.$$

*Proof of Lemma 4.5.* For the first case, we use the Hölder inequality, in view of (4-2):

$$\|R^h(u)\|_{L^{p'}_T L^{q'}} = \|(1-\Pi_h)(|u|^2)u\|_{L^{p'}_T L^{q'}} \leqslant \|(1-\Pi_h)(|u|^2)\|_{L^{\theta/2}_T L^{q/2}} \|u\|_{L^p_T L^q}.$$

We note that for $\sigma = 1$, $q = 4$, so by the Plancherel theorem,

$$\|(1-\Pi_h)(|u|^2)\|^2_{L^2} = \int_{\mathbb{R}^d} (1-\chi(h\xi))^2 |\mathcal{F}(|u|^2)(\xi)|^2 d\xi \leqslant \int_{|\xi|>1/h} |\mathcal{F}(|u|^2)(\xi)|^2 d\xi.$$

By assumption, $u \in L^p([0,T]; L^4) \subset L^\theta([0,T]; L^4)$, thus $|u|^2 \in L^{\theta/2}([0,T]; L^2)$, and by the Plancherel theorem, $\mathcal{F}(|u|^2) \in L^{\theta/2}([0,T]; L^2)$. The first point of the lemma then stems from the dominated convergence theorem.

For the first case of the second point, we note that now $\theta > p$, so the above argument must be adapted, and we have to estimate the gradient of $R^h(u)$ in the same space as above. Since we have $L^\infty([0,T]; H^1(\mathbb{R}^3)) \subset L^\theta([0,T]; L^4(\mathbb{R}^3))$, the dominated convergence theorem yields

$$\|R^h(u)\|_{L^{p'}_T L^{q'}} \underset{h\to 0}{\longrightarrow} 0.$$

We now estimate $\nabla R^h(u)$. Write

$$\begin{aligned}\|\nabla R^h(u)\|_{L_T^{p'}L^{q'}} &\leqslant \|(1-\Pi_h)(|u|^2)\|_{L_T^{\theta/2}L^2}\|\nabla u\|_{L_T^pL^2} + \|(1-\Pi_h)\nabla\left(|u|^2\right)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L_T^\theta L^2}\\ &\lesssim \|(1-\Pi_h)(|u|^2)\|_{L_T^\infty L^2}\|\nabla u\|_{L_T^pL^2} + \|(1-\Pi_h)\nabla(|u|^2)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L_T^\infty L^2}.\end{aligned}$$

By the same argument as above,

$$\|(1-\Pi_h)(|u|^2)\|_{L_T^\infty L^2}\|\nabla u\|_{L_T^pL^2} \underset{h\to 0}{\longrightarrow} 0.$$

We note that $u$ bounded in $L^\infty([0,T];H^1(\mathbb{R}^3)) \subset L^\theta([0,T];L^4(\mathbb{R}^3))$, and $\nabla u$ bounded in $L_T^pL^4$, so $\nabla|u|^2$ is bounded in $L_T^{(1/\theta+1/p)^{-1}}L^2$. Invoking Plancherel theorem and the dominated convergence theorem like above, we infer that

$$\|(1-\Pi_h)\nabla(|u|^2)\|_{L_T^{(1/\theta+1/p)^{-1}}L^2}\|u\|_{L_T^\infty L^2} \underset{h\to 0}{\longrightarrow} 0.$$

This completes the proof for the first case of the second point.

For the remaining cases, we use that $H^s(\mathbb{R}^d)$ is embedded into $L^\infty(\mathbb{R}^d)$: for fixed $t$,

$$\begin{aligned}\|R^h(u)(t)\|_{L^2} &\lesssim \left(\|u(t)\|_{L^\infty}^{2\sigma-2} + \|\Pi_h(|u(t)|^2)\|_{L^\infty}^{\sigma-1}\right)\|(1-\Pi_h)(|u(t)|^2)\|_{L^2}\|u(t)\|_{L^\infty}\\ &\lesssim \|u(t)\|_{L^\infty}^{2\sigma-1}\|(1-\Pi_h)(|u(t)|^2)\|_{L^2} \lesssim \|u(t)\|_{H^s}^{2\sigma-1}\|(1-\Pi_h)(|u(t)|^2)\|_{L^2}.\end{aligned}$$

Like in the proof of Proposition 3.1, we use the estimate

$$\|(1-\Pi_h)f\|_{L^2} \leqslant h^s\|f\|_{H^s}, \tag{4-4}$$

and since $H^s(\mathbb{R}^d)$ is an algebra,

$$\|R^h(u)\|_{L^\infty([0,T];L^2)} \lesssim h^s\|u\|_{L^\infty([0,T];H^s)}^{2\sigma+1}.$$

To conclude the proof, we estimate $\nabla R^h(u)$ in $L^2(\mathbb{R}^d)$. We compute

$$\begin{aligned}\nabla R^h(u) = \sigma|u|^{2\sigma-2}\big((1-\Pi_h)(\nabla(|u|^2))\big)u + \left(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\right)\nabla u,&\\ +\sigma\left(|u|^{2\sigma-2} - (\Pi_h(|u|^2))^{\sigma-1}\right)\Pi_h(\nabla(|u|^2))u,&\end{aligned}$$

where the second line is zero if $\sigma = 1$. We estimate successively, thanks to (4-1),

$$\begin{aligned}\left\||u|^{2\sigma-2}\left((1-\Pi_h)\left(\nabla(|u|^2)\right)\right)u\right\|_{L^2} &\leqslant \|u\|_{L^\infty}^{2\sigma-1}\left\|(1-\Pi_h)(|u|^2)\right\|_{H^1},\\ \left\|\left(|u|^{2\sigma} - (\Pi_h(|u|^2))^\sigma\right)\nabla u\right\|_{L^2} &\leqslant \|u\|_{L^\infty}^{2\sigma-2}\left\|(1-\Pi_h)\left(|u|^2\right)\right\|_{L^\infty}\|\nabla u\|_{L^2},\end{aligned}$$

and, if $\sigma \geqslant 2$,

$$\begin{aligned}\left\|\left(|u|^{2\sigma-2} - (\Pi_h(|u|^2))^{\sigma-1}\right)\Pi_h(\nabla(|u|^2))u\right\|_{L^2} &\lesssim \|u\|_{L^\infty}^{2\sigma-4}\|(1-\Pi_h)(|u|^2)\|_{L^2}\|\nabla(|u|^2)\|_{L^2}\|u\|_{L^\infty}\\ &\lesssim \|u\|_{L^\infty}^{2\sigma-2}\|(1-\Pi_h)(|u|^2)\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

Since we have $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, we end up with

$$\|\nabla R^h(u)\|_{L^2} \lesssim \|u\|_{H^s}^{2\sigma-2}\|(1-\Pi_h)(|u|^2)\|_{H^1}.$$

If $s>1$, (4-4) yields, since in addition $s>d/2$,

$$\|(1-\Pi_h)(|u|^2)\|_{H^1}\lesssim h^{s-1}\||u|^2\|_{H^s}\lesssim h^{s-1}\|u\|_{H^s}^2.$$

If $s=1$ (a case which may occur only if $d=1$, since $s>d/2$), we write

$$\|\nabla(1-\Pi_h)(|u|^2)\|_{L^2}^2\leqslant\int_{|\xi|>1/h}|\mathcal{F}(\nabla(|u|^2))(\xi)|^2d\xi.$$

Now since $\nabla(|u|^2)=2\operatorname{Re}\bar{u}\nabla u$ and $u\in H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$, $\nabla u\in L^2(\mathbb{R})$, we conclude thanks to the dominated convergence theorem. □

This completes the proof of Theorem 4.3, by choosing $(p_1,q_1)=(p,q)$ or $(\infty,2)$. □

## Appendix: Physical saturation of the nonlinearity

Instead of cutting off the high frequencies, one may be tempted to saturate the nonlinear potential, by replacing $|u|^2$ not by $\Pi(|u|^2)$ but by $f(|u|^2)$ where $f$ is smooth, equal to the identity near the origin, and constant at infinity. Note also that a saturated nonlinearity may be in better agreement with physical models (recall however that (1-1) appears in rather different physical contexts, such as quantum mechanics, optics, and even fluid mechanics), since typically the power-like nonlinearity in (1-1) may stem from a Taylor expansion; see [Lannes 2011; Sulem and Sulem 1999]. More precisely, let $f\in C^\infty(\mathbb{R};\mathbb{R})$ such that

$$f(s)=\begin{cases}s & \text{if } 0\leqslant s\leqslant 1,\\ 2 & \text{if } s\geqslant 2.\end{cases}\tag{A-1}$$

The analogue of the Fourier multiplier $\Pi_h$ is defined as

$$f_h(|u|^2)=\frac{1}{h}f(h|u|^2),$$

and we replace (2-5) with

$$i\partial_t u^h+P_h(D)u^h=\epsilon\left(f_h(|u^h|^2)\right)^\sigma u^h,\tag{A-2}$$

so the formal conservation of the $L^2$-norm still holds. We could also consider

$$f_h(|u|^2)=\frac{|u|^2}{1+h|u|^2}.\tag{A-3}$$

In both cases, the main aspect to notice is that $f_h$ is bounded and $z\mapsto f_h(|z|^2)^\sigma z$ is globally Lipschitzian. We infer the analogue of Theorem 2.1, at least in the $L^2$ case.

**Proposition A.1.** *Let $\sigma\in\mathbb{N}$, $\epsilon\in\{\pm1\}$, $P:\mathbb{R}^d\to\mathbb{R}$ and $f$ given either by (A-1) or by (A-3).*

- *For any $u_0\in L^2(\mathbb{R}^d)$, (A-2) has a unique solution $u^h\in C(\mathbb{R};L^2(\mathbb{R}^d))$ such that $u^h_{|t=0}=u_0$. Its $L^2$-norm is independent of time.*

- *The flow map $u_0 \mapsto u^h$ is uniformly continuous from the balls in $L^2(\mathbb{R}^d)$ to $C(\mathbb{R}; L^2(\mathbb{R}^d))$. More precisely, for all $u_0, v_0 \in L^2(\mathbb{R}^d)$, there exists $C$ depending on $\sigma$, $h$, $\|u_0\|_{L^2}$ and $\|v_0\|_{L^2}$ such that for all $T > 0$,*
$$\|u^h - v^h\|_{L^\infty([-T,T];L^2(\mathbb{R}^d))} \leqslant \|u_0 - v_0\|_{L^2(\mathbb{R}^d)} e^{CT},$$
*where $u^h$ and $v^h$ denote the solutions to (A-2) with data $u_0$ and $v_0$, respectively.*

Introduce
$$F_h(s) = \int_0^s f_h(y)^\sigma \, dy.$$
We check that the following conservation of energy holds:
$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \bar{u}^h(t,x) P_h(D) u^h(t,x)\, dx + \epsilon \int_{\mathbb{R}^d} F_h(|u(t,x)|^2)\, dx\right) = 0.$$
Proving the analogue of Proposition 3.1 is easy in the case (A-1), since the last source term for the error $w^h$ is now
$$R^h(u) = \left(|u|^{2\sigma} - f_h(|u|^2)^\sigma\right)u,$$
and under the assumptions of Proposition 3.1, $u \in L^\infty([0,T] \times \mathbb{R}^d)$, so there exists $h_0 > 0$ such that for $0 < h \leqslant h_0$,
$$|u(t,x)|^{2\sigma} = f_h(|u(t,x)|^2)^\sigma \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$
Therefore, this source term simply vanishes for $h$ sufficiently small. In the case (A-3), we can use the relation
$$|R_h(u)| = \left|\left(|u|^{2\sigma} - f_h(|u|^2)^\sigma\right)u\right| \lesssim \frac{h|u|^2}{1+h|u|^2}|u|^{2\sigma+1}, \tag{A-4}$$
and the Schauder lemma to get a source term which is $\mathcal{O}(h)$ in $H^s(\mathbb{R}^d)$, for $s > d/2$.

**Proposition A.2.** *Let $\sigma \in \mathbb{N}$. We assume that $P$ is such that $P_h(\xi) = -|\xi|^2 + \mathcal{O}(h^\alpha \langle \xi \rangle^\beta)$ for some $\alpha, \beta \geqslant 0$. Denote by $u^h$ the solution to (A-2) with $P_h$ and $f_h$, such that $u^h_{|t=0} = u_{|t=0}$. Suppose that the solution to (1-1) satisfies $u \in L^\infty([0,T]; H^{s+\beta})$, for some $s > d/2$.*

- *In the case (A-1), $\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^\alpha$.*
- *In the case (A-3), $\|u - u^h\|_{L^\infty([0,T];H^s)} \lesssim h^{\min(\alpha,1)}$.*

In the case (A-1), proving an analogue to Theorem 4.3 seems to be more delicate though, and we choose not to investigate this aspect here. On the other hand, in the case (A-3), using the estimate (A-4), Strichartz estimates and Hölder inequalities with the "standard" Lebesgue exponents (in the same fashion as in the proof of Theorem 4.3, see [Cazenave 2003]), we have, with steps similar to those presented in the proof of Theorem 4.3:

**Theorem A.3.** *Let $\sigma \in \mathbb{N}$ and $T > 0$. Let $u$ solve (1-1), and consider a solution $u^h$ to*
$$i\partial_t u^h + \Delta u^h = \epsilon\left(\frac{|u^h|^2}{1+h|u^h|^2}\right)^\sigma u^h, \quad u^h_{|t=0} = u_0.$$

1. *If* $\sigma \leqslant 2/d$, *and* $u \in L^\infty([0,T];L^2) \cap L^{(4\sigma+4)/d\sigma}([0,T];L^{2\sigma+2})$, *then*

$$\|u-u^h\|_{L^\infty([0,T];L^2)} \underset{h\to 0}{\longrightarrow} 0.$$

2. *Suppose that* $\sigma = 1$ *and* $d = 3$.

   - *If* $u, \nabla u \in L^\infty([0,T];L^2) \cap L^{8/d}([0,T];L^4)$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \underset{h\to 0}{\longrightarrow} 0.$$

   - *If* $u \in L^\infty([0,T];H^s)$, *with* $s > 3/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h.$$

3. *Suppose that* $\sigma \geqslant 1$ *and* $d \leqslant 2$. *If* $u \in L^\infty([0,T];H^s)$, *with* $s \geqslant 1$ *and* $s > d/2$, *then*

$$\|u-u^h\|_{L^\infty([0,T];H^1)} \lesssim h.$$

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