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A VARIATIONAL PRINCIPLE FOR CORRELATION FUNCTIONS FOR UNITARY ENSEMBLES, WITH APPLICATIONS

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In the theory of random matrices for unitary ensembles associated with Hermitian matrices, m-point correlation functions play an important role. We show that they possess a useful variational principle. Let μ be a measure with support in the real line, and K_n be the n-th reproducing kernel for the associated orthonormal polynomials. We prove that, for $m \geq 1$,

$$\det\left[K_n(\mu, x_i, x_j)\right]_{1\leq i,j\leq m} = m! \sup_P \frac{P^2(\underline{x})}{\int P^2(\underline{t})\, d\mu^{\times m}(\underline{t})}$$

where the supremum is taken over all alternating polynomials P of degree at most $n-1$ in m variables $\underline{x} = (x_1, x_2, \ldots, x_m)$. Moreover, $\mu^{\times m}$ is the m-fold Cartesian product of μ. As a consequence, the suitably normalized m-point correlation functions are *monotone decreasing in the underlying measure μ*. We deduce pointwise one-sided universality for arbitrary compactly supported measures, and other limits.

1. Introduction

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j\, d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

satisfying

$$\int p_n p_m\, d\mu = \delta_{mn}.$$

The n-th *reproducing kernel* is

$$K_n(\mu, x, t) = \sum_{j=0}^{n-1} p_j(x) p_j(t)$$

and the n-th *Christoffel function* is

$$\lambda_n(\mu, x) = 1/K_n(\mu, x, x) = 1\Big/\sum_{j=0}^{n-1} p_j^2(x). \tag{1-1}$$

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It admits an extremal property that is very useful in investigating asymptotics of orthogonal polynomials [Nevai 1986; Simon 2011]:

$$\lambda_n(\mu, x) = \inf_{\deg(P)<n} \frac{\int P(t)^2 \, d\mu(t)}{P^2(x)}.$$

Equivalently,

$$K_n(\mu, x, x) = \sup_{\deg(P)<n} \frac{P^2(x)}{\int P(t)^2 \, d\mu(t)}. \tag{1-2}$$

We shall prove a direct generalization for $\det[K_n(\mu, x_i, x_j)]_{1\le i,j\le m}$, a determinant that plays a key role in analysis of random matrices.

Random Hermitian matrices rose to prominence with the work of Eugene Wigner, who used their eigenvalues as a model for scattering theory of heavy nuclei. One places a probability distribution on the entries of an n by n Hermitian matrix. When expressed in "spectral form", that is, as a probability distribution on the (real) eigenvalues $x_1, x_2, \ldots, x_n$, it has the form

$$\mathcal{P}^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\left(\prod_{1\le j<k\le n}(x_k - x_j)^2\right) d\mu(x_1)\, d\mu(x_2) \cdots d\mu(x_n)}{\int \cdots \int \left(\prod_{1\le j<k\le n}(t_k - t_j)^2\right) d\mu(t_1) \cdots d\mu(t_n)};$$

see [Deift 1999, p. 102]. Given $1 \le m \le n$, we define the m-point correlation function

$$R_m^n(\mu; x_1, \ldots, x_m) = \frac{n!}{(n-m)!} \frac{\int \cdots \int \left(\prod_{1\le j<k\le n}(x_k - x_j)^2\right) d\mu(x_{m+1}) \cdots d\mu(x_n)}{\int \cdots \int \left(\prod_{1\le j<k\le n}(t_k - t_j)^2\right) d\mu(t_1) \cdots d\mu(t_n)}. \tag{1-3}$$

Thus R_m^n is, up to normalization, a marginal distribution, where we integrate out $x_{m+1}, x_{m+2}, \ldots, x_n$. Note that we exclude from R_m^n a factor of $\mu'(x_1)\mu'(x_2)\cdots\mu'(x_m)$, which is used by Deift. It is a well established fact [Deift 1999, p. 112] that

$$R_m^n(\mu; x_1, x_2, \ldots, x_m) = \det\left[K_n(\mu, x_i, x_j)\right]_{1\le i,j\le m}. \tag{1-4}$$

Again, we emphasize that in [Deift 1999], as distinct from this paper, μ' is absorbed into K_n. Since much of the interest lies in asymptotics as $n \to \infty$, for fixed m, it is obviously easier to handle asymptotics of this fixed size determinant, than to deal with the $(n-m)$-fold integral in (1-3).

R_m^n can be used to describe the local spacing of m-tuples of eigenvalues. For example, if $m = 2$, and $B \subset \mathbb{R}$ is measurable, then [Deift 1999, p. 117]

$$\int_B \int_B R_2^n(\mu; t_1, t_2)\, d\mu(t_1)\, d\mu(t_2)$$

is the expected number of pairs (t_1, t_2) of eigenvalues, with both $t_1, t_2 \in B$.

Of course there are other settings for random matrices that do not involve orthogonal polynomials. There one considers a class of matrices (such as normal matrices or symmetric matrices) where the elements of the matrix are independently distributed, or there are appropriate bounds on the dependence. The methods are quite different, but remarkably, similar limiting results arise [Erdős 2011; Erdős et al. 2010; 2011; Forrester 2010; Tao and Vu 2011].

The formulation of our main result involves $\mathcal{AL}_n^m$, the alternating polynomials of degree at most n in m variables. We say that $P \in \mathcal{AL}_n^m$ if

$$P(x_1, x_2, \ldots, x_m) = \sum_{0 \le j_1, j_2, \ldots, j_m \le n} c_{j_1 j_2 \cdots j_m} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}, \tag{1-5}$$

so that P is a polynomial of degree less than or equal to n in each of its m variables, and in addition is *alternating*, so that for every pair (i, j) with $1 \le i < j \le m$,

$$P(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) = -P(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_m). \tag{1-6}$$

Thus swapping variables changes the sign. Sometimes, these are called *skew-symmetric* polynomials.

Observe that if P_i is a univariate polynomial of degree less than or equal to n for each $i = 1, 2, \ldots, m$, then

$$P(t_1, t_2, \ldots, t_m) = \det\big[P_i(t_j)\big]_{1 \le i,j \le m} \in \mathcal{AL}_n^m. \tag{1-7}$$

The set of such determinants of polynomials is a proper subset of $\mathcal{AL}_n^m$. It is well known, and easy to see, that every alternating polynomial is the product of a Vandermonde determinant and a symmetric polynomial. Thus $P \in \mathcal{AL}_n^m$ if and only if

$$P(t_1, t_2, \ldots, t_m) = \Bigg(\prod_{1 \le i < j \le m} (t_j - t_i) \Bigg) S(t_1, t_2, \ldots, t_m),$$

where S is symmetric, and of degree less than or equal to $n - m + 1$ in each variable.

Given a fixed m, we shall use the notation

$$\underline{x} = (x_1, x_2, \ldots, x_m), \quad \underline{t} = (t_1, t_2, \ldots, t_m)$$

while $\mu^{\times m}$ denotes the m-fold Cartesian product of μ, so that

$$d\mu^{\times m}(\underline{t}) = d\mu(t_1)\, d\mu(t_2) \cdots d\mu(t_m). \tag{1-8}$$

We prove:

Theorem 1.1. *Let $m \ge 1$, $n \ge m + 1$. Let $\underline{x} = (x_1, x_2, \ldots, x_m)$ be an m-tuple of real numbers. Then*

$$\det\big[K_n(\mu, x_i, x_j)\big]_{1 \le i,j \le m} = m! \sup_{P \in \mathcal{AL}_{n-1}^m} \frac{(P(\underline{x}))^2}{\int (P(\underline{t}))^2\, d\mu^{\times m}(\underline{t})}. \tag{1-9}$$

The supremum is attained for

$$P(\underline{t}) = \det\big[K_n(\mu, x_i, t_j)\big]_{1 \le i,j \le m}. \tag{1-10}$$

We could also just take the supremum in (1-9) over the strictly smaller class of determinants of the form (1-7). An immediate, but important, consequence is:

Corollary 1.2. *$R_m^n(\mu; x_1, x_2, \ldots, x_m)$ is a monotone decreasing function of μ, and a monotone increasing function of n.*

Despite an extensive literature search, I have not found Theorem 1.1 or Corollary 1.2 in the rich literature on random matrices. At the very least, they must be new to those interested in universality limits, because of the applications they have there. We shall present some in Section 2.

The proof of Theorem 1.1 is based on multivariate orthogonal polynomials built from μ. Given $m \geq 1$, and nonnegative integers $j_1, j_2, \ldots, j_m$, we define

$$T_{j_1,j_2,\ldots,j_m}(x_1, x_2, \ldots, x_m) = \det(p_{j_i}(x_k))_{1 \leq i,k \leq m} = \det \begin{bmatrix} p_{j_1}(x_1) & p_{j_1}(x_2) & \ldots & p_{j_1}(x_m) \\ p_{j_2}(x_1) & p_{j_2}(x_2) & \ldots & p_{j_2}(x_m) \\ \vdots & \vdots & \ddots & \vdots \\ p_{j_m}(x_1) & p_{j_m}(x_2) & \ldots & p_{j_m}(x_m) \end{bmatrix}. \tag{1-11}$$

We show that the $\{T_{j_1,j_2,\ldots,j_m}\}_{j_1<j_2<\cdots<j_m}$ form an orthogonal family with respect to $\mu^{\times m}$, and moreover, the m-point correlation function admits an expansion as a sum of squares of $\{T_{j_1,j_2,\ldots,j_m}\}$, just as does K_n in terms of squares of the orthonormal polynomials. We shall need an associated reproducing kernel,

$$K_n^m(\mu, \underline{x}, \underline{t}) = \frac{1}{m!} \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} T_{j_1,j_2,\ldots,j_m}(\underline{x}) T_{j_1,j_2,\ldots,j_m}(\underline{t}). \tag{1-12}$$

Theorem 1.3. (a) *Let $0 \leq j_1 < j_2 < \cdots < j_m$ and $0 \leq k_1 < k_2 < \cdots < k_m$. Then*

$$\int T_{j_1,j_2,\ldots,j_m}(\underline{t}) T_{k_1,k_2,\ldots,k_m}(\underline{t})\, d\mu^{\times m}(\underline{t}) = m!\, \delta_{j_1k_1} \delta_{j_2k_2} \cdots \delta_{j_mk_m}. \tag{1-13}$$

(b) *For $P \in \mathcal{AL}_{n-1}^m$, and $\underline{x} \in \mathbb{R}^n$,*

$$P(\underline{x}) = \int P(\underline{t}) K_n^m(\mu, \underline{x}, \underline{t})\, d\mu^{\times m}(\underline{t}). \tag{1-14}$$

(c) *For $\underline{x}, \underline{t} \in \mathbb{R}^n$,*

$$\det\big[K_n(\mu, x_i, t_j)\big]_{1 \leq i,j \leq m} = m!\, K_n^m(\mu, \underline{x}, \underline{t}). \tag{1-15}$$

In particular,

$$\det\big[K_n(\mu, x_i, x_j)\big]_{1 \leq i,j \leq m} = \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq n} (T_{j_1,j_2,\ldots,j_m}(\underline{x}))^2. \tag{1-16}$$

Remarks. (a) In the case $m = 1$, (1-16) reduces to (1-1) for $K_n(\mu, x, x)$. After an extensive literature search, we found that (1-16) already appears for general m in [Erdős 2011, Section 1.5.3]. We may also express it as

$$\det\big[K_n(\mu, x_i, x_j)\big]_{1 \leq i,j \leq m} = \frac{1}{m!} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq n} (T_{j_1,j_2,\ldots,j_m}(\underline{x}))^2, \tag{1-17}$$

as $T_{j_1,j_2,\ldots,j_m}$ vanishes if any two indices j_i are equal.

(b) The expression (1-15) may also be thought of as a Christoffel–Darboux formula, for it expresses the sum (1-12) in a compact form involving an $m \times m$ determinant.

One consequence of the variational principle is a lower bound for ratios of correlation functions:

Theorem 1.4. *Let $m \geq 2$, $n \geq m+1$, and $x_1, x_2, \ldots, x_m$ be distinct real numbers. Define a measure ν by*

$$d\nu(t) = d\mu(t) \prod_{j=2}^{m} (t - x_j)^2.$$

Then

$$K_n(\mu, x_1, x_1) \geq \frac{\det\left[K_n(\mu, x_i, x_j)\right]_{1 \leq i,j \leq m}}{\det\left[K_n(\mu, x_i, x_j)\right]_{2 \leq i,j \leq m}} \geq \frac{1}{m} K_{n-m+1}(\nu, x_1, x_1) \prod_{j=2}^{m} (x_1 - x_j)^2. \tag{1-18}$$

The upper bound is a well known consequence of inequalities for positive definite matrices. It is the lower bound that is new.

This paper is organized as follows: in Section 2, we state some applications of Theorem 1.1 to asymptotics and universality limits. In Section 3, we first prove Theorem 1.3, and then deduce Theorem 1.1 and Corollary 1.2, followed by Theorem 1.4. Theorems 2.1, 2.2, and 2.3 are proved in Section 4. Theorem 2.4 is proved in Section 5, and Theorem 2.5 and Corollary 2.6 in Section 6.

2. Applications to asymptotics and universality limits

The extremal property (1-2) is essential in proving the following: if μ is any measure with support in $[-1, 1]$, then at every Lebesgue point x of μ in $(-1, 1)$,

$$\liminf_{n\to\infty} \frac{1}{n} K_n(\mu, x, x)\mu'(x) \geq \frac{1}{\pi\sqrt{1-x^2}}. \tag{2-1}$$

Here μ' is understood as the Radon–Nikodym derivative of the absolutely continuous part of μ. This is more commonly formulated for Christoffel functions as

$$\limsup_{n\to\infty} n\lambda_n(\mu, x) \leq \mu'(x)\pi\sqrt{1-x^2}.$$

Barry Simon calls this the *Máté–Nevai–Totik upper bound*. See, for example, [Máté et al. 1991; Simon 2011, Theorem 5.11.1, p. 334; Totik 2000].

Under additional conditions, including regularity of μ, there is equality in (2-1), with a full limit. We say that μ is *regular in the sense of Stahl, Totik, and Ullman*, or just *regular*, if the leading coefficients $\{\gamma_n\}$ of its orthonormal polynomials satisfy

$$\lim_{n\to\infty} \gamma_n{}^{1/n} = \frac{1}{\operatorname{cap}(\operatorname{supp}[\mu])}. \tag{2-2}$$

Here $\operatorname{cap}(\operatorname{supp}[\mu])$ is the logarithmic capacity of the support of μ. We shall need only a very simple criterion for regularity, namely a version of the Erdős–Turán criterion: if the support of μ consists of finitely many intervals, and $\mu' > 0$ a.e. with respect to Lebesgue measure in that support, then μ is regular [Stahl and Totik 1992, p. 102].

Máté, Nevai and Totik [Máté et al. 1991] showed that if μ is a regular measure with support $[-1,1]$, and in some subinterval I of $(-1,1)$, we have

$$\int_I \log \mu' > -\infty, \tag{2-3}$$

then for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, x)\mu'(x) = \frac{1}{\pi\sqrt{1-x^2}}. \tag{2-4}$$

Totik gave a far-reaching extension of this to measures with compact support J [Totik 2000; 2009]. Here one needs the equilibrium measure ν_J for the compact set J, as well as its Radon–Nikodym derivative, which we denote by ω_J. Thus ν_J is the unique probability measure that minimizes the energy integral

$$\iint \log \frac{1}{|s-t|}\, d\nu(s)\, d\nu(t)$$

amongst all probability measures ν with support in J [Ransford 1995; Saff and Totik 1997]. If I is some subinterval of J, then ν_J is absolutely continuous in I, and moreover, $\omega_J > 0$ in the interior I^o of I. In the special case $J = [-1,1]$, we have

$$d\nu_J(x) = \omega_J(x)\, dx = \frac{dx}{\pi\sqrt{1-x^2}}.$$

Totik showed that if μ is regular, and in some subinterval I of J, we have (2-3), then

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, x)\mu'(x) = \omega_J(x) \quad \text{for a.e. } x \in I. \tag{2-5}$$

Further developments are explored in [Simon 2011].

It is a fairly straightforward consequence of this last relation, and the Christoffel–Darboux formula, that, for $m \geq 2$ and a.e. $(x_1, x_2, \ldots, x_m) \in I^m$,

$$\lim_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1\leq i,j\leq m} = \prod_{j=1}^{m} \frac{\omega_J(x_j)}{\mu'(x_j)}. \tag{2-6}$$

The right-hand side is interpreted as ∞ if any $\mu'(x_j) = 0$. Thus, the matrix $[K_n(\mu, x_i, x_j)]_{1\leq i,j\leq m}$ behaves essentially like its diagonal. We shall prove this in Section 4. Without having to assume regularity, or (2-3), we can use Theorem 1.1 to prove one-sided versions of (2-6).

For measures μ with compact support J, and $x \in J$, we let

$$\omega_\mu(x) = \inf\{\omega_L(x) : L \subset J \text{ is compact, } \mu_{|L} \text{ is regular, } x \in L\}. \tag{2-7}$$

Since ν_L decreases as L increases, one can roughly think of ω_μ as the density of the equilibrium measure of the largest set to whose restriction μ is regular. In the sequel, J^o denotes the interior of J.

Theorem 2.1. *Let μ have compact support J, of positive Lebesgue measure, and let ω_J denote the equilibrium density of J. Let $m \geq 1$.*

(a) *For Lebesgue a.e.* $(x_1, x_2, \ldots, x_m) \in (J^o)^m$,

$$\liminf_{n\to\infty} \frac{1}{n^m} \det\big[K_n(\mu, x_i, x_j)\big]_{1\le i,j\le m} \ge \prod_{j=1}^{m} \frac{\omega_J(x_j)}{\mu'(x_j)}. \tag{2-8}$$

The right-hand side is interpreted as ∞ *if any* $\mu'(x_j) = 0$.

(b) *Suppose that* I *is a compact subset of* J *consisting of finitely many intervals, for which* (2-3) *holds. Then, for Lebesgue a.e.* $(x_1, x_2, \ldots, x_m) \in I^m$,

$$\limsup_{m\to\infty} \frac{1}{n^m} \det\big[K_n(\mu, x_i, x_j)\big]_{1\le i,j\le m} \le \prod_{j=1}^{m} \frac{\omega_\mu(x_j)}{\mu'(x_j)}. \tag{2-9}$$

A perhaps more impressive application of Theorem 1.1 is to universality limits in the bulk, which describe local spacing of eigenvalues of random Hermitian matrices [Deift 1999; Deift and Gioev 2009; Forrester 2010; Mehta 1991]. One of the more standard formulations, for a measure μ supported on $[-1, 1]$, is

$$\begin{aligned}\lim_{n\to\infty}&\left(\frac{\mu'(x)\pi\sqrt{1-x^2}}{n}\right)^m R_m^n\left(\mu; x + a_1\frac{\pi\sqrt{1-x^2}}{n}, \ldots, x + a_m\frac{\pi\sqrt{1-x^2}}{n}\right)\\ &= \lim_{n\to\infty}\left(\frac{\mu'(x)\pi\sqrt{1-x^2}}{n}\right)^m \det\left[K_n\left(\mu; x + a_i\frac{\pi\sqrt{1-x^2}}{n}, x + a_j\frac{\pi\sqrt{1-x^2}}{n}\right)\right]_{1\le i,j\le m}\\ &= \det(S(a_i - a_j))_{1\le i,j\le m},\end{aligned}$$

where

$$S(t) = \frac{\sin \pi t}{\pi t} \tag{2-10}$$

is the sine (or sinc) kernel. There is a vast literature for universality limits, especially in the case where μ is replaced by varying weights. A great many methods have been applied, including classical asymptotics for orthonormal polynomials, Riemann Hilbert techniques, and theory of entire functions of exponential type [Baik et al. 2003; 2008; Deift 1999; Deift and Gioev 2009; Deift et al. 1999; Findley 2008; Forrester 2010; Levin and Lubinsky 2008; Lubinsky 2009a; Simon 2008a; 2011; Totik 2009].

For fixed measures μ with compact support J, the most general pointwise result is due to Totik [2009]. It asserts that if μ is regular, while (2-3) holds in some interval I in the support, then, for a.e. $x \in I$, and all real $a_1, a_2, \ldots, a_m$, there are limits for the scaled reproducing kernels that immediately yield

$$\lim_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x + \frac{a_1}{n\omega_J(x)}, \ldots, x + \frac{a_m}{n\omega_J(x)}\right) = \det(S(a_i - a_j))_{1\le i,j\le m}.$$

Simon [2008a; 2008b] had a similar result, proved using Jost functions. Totik used the comparison method of [Lubinsky 2009a], together with "polynomial pullbacks". Without any local or global restrictions on μ, we showed in [Lubinsky 2012] that universality holds in measure in $\{\mu' > 0\} = \{x : \mu'(x) > 0\}$.

We prove pointwise, almost everywhere, one-sided universality, without any local or global restrictions on μ:

Theorem 2.2. *Let μ have compact support J, and let ω_J denote the equilibrium density of J. Let $m \geq 1$.*

(a) *For a.e. $x \in J^o \cap \{\mu' > 0\}$, and for all real $a_1, a_2, \dots, a_m$,*

$$\liminf_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x+\frac{a_1}{n\omega_J(x)}, \dots, x+\frac{a_m}{n\omega_J(x)}\right) \geq \det(S(a_i-a_j))_{1\leq i,j\leq m}. \tag{2-11}$$

(b) *Suppose that I is a compact subset of J consisting of finitely many intervals, for which (2-3) holds. Then for a.e. $x \in I$, and for all real $a_1, a_2, \dots, a_m$,*

$$\limsup_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_\mu(x)}\right)^m R_m^n\left(\mu; x+\frac{a_1}{n\omega_\mu(x)}, \dots, x+\frac{a_m}{n\omega_\mu(x)}\right) \leq \det(S(a_i-a_j))_{1\leq i,j\leq m}. \tag{2-12}$$

Pointwise universality at a given point x seems to usually require at least something like μ' being continuous at x, or x being a Lebesgue point of μ. Indeed, when μ' has a jump discontinuity, the universality limit is different from the sine kernel [Foulquié Moreno et al. 2011], and involves de Branges spaces [Lubinsky 2009b]. In our next result, we show that one can still bound the behavior of the correlation function above and below near such a given x. It is noteworthy, though, that pure singularly continuous measures can exhibit sine kernel behavior [Breuer 2011].

Theorem 2.3. *Let μ have compact support J, be regular, and let ω_J denote the equilibrium density of J. Assume that the singular part μ_s of μ satisfies, at a given x in the interior of J,*

$$\lim_{h\to 0+} \mu_s[x-h, x+h]/h = 0. \tag{2-13}$$

Assume moreover that the derivative μ' of the absolutely continuous part of μ satisfies

$$0 < C_1 = \liminf_{t\to x} \mu'(t) \leq \limsup_{t\to x} \mu'(t) = C_2 < \infty. \tag{2-14}$$

Then, for all real $a_1, a_2, \dots, a_m$,

$$\begin{aligned} C_2^{-m}\det(S(a_i-a_j))_{1\leq i,j\leq m} &\leq \liminf_{n\to\infty}\left(\frac{1}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x+\frac{a_1}{n\omega_J(x)}, \dots, x+\frac{a_m}{n\omega_J(x)}\right) \\ &\leq \limsup_{n\to\infty}\left(\frac{1}{n\omega_J(x)}\right)^m R_m^n\left(\mu; x+\frac{a_1}{n\omega_J(x)}, \dots, x+\frac{a_m}{n\omega_J(x)}\right) \\ &\leq C_1^{-m}\det(S(a_i-a_j))_{1\leq i,j\leq m}. \end{aligned} \tag{2-15}$$

At the boundary of the support of the measure (referred to as the edge of the spectrum in random matrix theory), the universality limit takes a different form [Forrester 2010; Kuijlaars and Vanlessen 2002]. For fixed measures that behave like Jacobi weights near the endpoints, they involve the Bessel kernel of order $\alpha > -1$:

$$\mathbb{J}_\alpha(u,v) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J_\alpha'(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J_\alpha'(\sqrt{u})}{2(u-v)}.$$

Here J_α is the usual Bessel function of the first kind and order α. Using a comparison method, the author proved [Lubinsky 2008] that if μ is a regular measure on $[-1, 1]$, and μ is absolutely continuous in some left neighborhood $(1-\eta, 1]$ of 1, and there $\mu'(t) = h(t)(1-t)^\alpha$, where $h(1) > 0$ and h is continuous at 1, then

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n\left(\mu, 1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a, b), \tag{2-16}$$

uniformly for a, b in compact subsets of $(0, \infty)$. Here, and in the sequel,

$$\tilde{K}_n(\mu, x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(\mu, x, y).$$

When $\alpha \geq 0$, we may allow also $a, b = 0$. This has the immediate consequence that, for $m \geq 2$, and $a_1, a_2, \ldots, a_m > 0$,

$$\lim_{n\to\infty} \left(\frac{1}{2n^2}\right)^m R_m^n\left(\mu; 1-\frac{a_1}{2n^2}, \ldots, 1-\frac{a_m}{2n^2}\right)\left(\prod_{j=1}^m \mu'\left(1-\frac{a_j}{2n^2}\right)\right) = \det(\mathbb{J}_\alpha(a_i, a_j))_{1\leq i,j\leq m}. \tag{2-17}$$

Under weak conditions at the edge, we can prove one-sided universality:

Theorem 2.4. *Let μ have support contained in $[-1, 1]$ and let 1 be the right endpoint of that support. Assume that μ is absolutely continuous near 1, and, for some $\alpha > -1$,*

$$0 < C_1 = \liminf_{t\to 1-} \mu'(t)(1-t)^{-\alpha} \leq \limsup_{t\to 1-} \mu'(t)(1-t)^{-\alpha} = C_2 < \infty. \tag{2-18}$$

Then, for $a_1, a_2, \ldots, a_m > 0$,

$$\liminf_{n\to\infty} \left(\frac{1}{2n^2}\right)^m R_m^n\left(\mu; 1-\frac{a_1}{2n^2}, \ldots, 1-\frac{a_m}{2n^2}\right) \prod_{j=1}^m \mu'\left(1-\frac{a_j}{2n^2}\right) \geq \left(\frac{C_1}{C_2}\right)^m \det(\mathbb{J}_\alpha(a_i, a_j))_{1\leq i,j\leq m}. \tag{2-19}$$

If $\alpha \geq 0$, we may also allow $a_1, a_2, \ldots, a_m \geq 0$.

We note that if, in addition, μ has support $[-1, 1]$ and is regular, then we may replace the lim inf by lim sup, the asymptotic lower bound by an upper bound, provided we replace $(C_1/C_2)^m$ by $(C_2/C_1)^m$.

Our final result has a comparison or "localization" flavor, generalizing similar results for Christoffel functions. Recall that a set $J \subset \mathbb{R}$ is said to be regular for the Dirichlet problem [Ransford 1995; Stahl and Totik 1992] if, for every function f continuous on J, there exists a function harmonic in $\bar{\mathbb{C}} \backslash J$, continuous on $\mathbb{C}$, whose restriction to J is f. Of course, this is confusing when juxtaposed with the notion of a regular measure!

Theorem 2.5. *Let μ, ν have compact support J and both be regular. Assume that J is regular with respect to the Dirichlet problem. Let $\xi \in J$ and $\mu'(\xi)$, $\nu'(\xi)$ be finite and positive, with*

$$\lim_{\mathrm{dist}(I,\xi)\to 0} \frac{\mu(I)}{\nu(I)} = \frac{\mu'(\xi)}{\nu'(\xi)}, \tag{2-20}$$

where the limit is taken over intervals I of length $|I|$, and $\operatorname{dist}(I,\xi)=\sup\{|x-\xi|:x\in I\}$. *Let* $m\geq 1$. *Assume that, for* $n\geq 1$,

$$\underline{y}_n=(y_{1n},y_{2n},\dots,y_{mn})$$

is a vector of real numbers satisfying

$$\lim_{n\to\infty}\Big(\max_{1\leq j\leq m}|y_{mj}-\xi|\Big)=0, \tag{2-21}$$

and

$$\lim_{\varepsilon\to 0+}\left(\limsup_{n\to\infty}\left|\frac{K^m_{[n(1\pm\varepsilon)]}(\nu,\underline{y}_n,\underline{y}_n)}{K^m_n(\nu,\underline{y}_n,\underline{y}_n)}-1\right|\right)=0. \tag{2-22}$$

Then

$$\lim_{n\to\infty}\frac{K^m_n(\mu,\underline{y}_n,\underline{y}_n)}{K^m_n(\nu,\underline{y}_n,\underline{y}_n)}=\left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m. \tag{2-23}$$

Of course, in (2-22), $[n(1\pm\varepsilon)]$ denotes the integer part of $n(1\pm\varepsilon)$. As an immediate consequence, we obtain:

Corollary 2.6. *Let μ, ν have compact support J and be regular. Assume that J is regular with respect to the Dirichlet problem. Let $x\in J$ and $\mu'(x)$, $\nu'(x)$ be finite and positive, with (2-20) holding at $\xi=x$. Assume that, for given $m\geq 2$ and all real $a_1,a_2,\dots,a_m$,*

$$\lim_{n\to\infty}\left(\frac{\nu'(x)}{n\omega_J(x)}\right)^m R^n_m\left(\nu;x+\frac{a_1}{n\omega_J(x)},\dots,x+\frac{a_m}{n\omega_J(x)}\right)=\det(S(a_i-a_j))_{1\leq i,j\leq m}. \tag{2-24}$$

Then, for all real $a_1,a_2,\dots,a_m$,

$$\lim_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_J(x)}\right)^m R^n_m\left(\mu;x+\frac{a_1}{n\omega_J(x)},\dots,x+\frac{a_m}{n\omega_J(x)}\right)=\det(S(a_i-a_j))_{1\leq i,j\leq m}. \tag{2-25}$$

3. Proofs of Theorems 1.1, 1.3, 1.4 and Corollary 1.2

Proof of Theorem 1.3(a). We use σ and η to denote permutations of $(1,2,\dots,m)$ with respective signs ε_σ and ε_η. We see that

$$\begin{aligned}I&=\int\cdots\int T_{j_1,j_2,\dots,j_m}(t_1,t_2,\dots,t_m)T_{k_1,k_2,\dots,k_m}(t_1,t_2,\dots,t_m)\,d\mu(t_1)\cdots d\mu(t_m)\\&=\sum_{\sigma,\eta}\varepsilon_\sigma\varepsilon_\eta\int\cdots\int\left(\prod_{i=1}^m p_{j_{\sigma(i)}}(t_i)\right)\left(\prod_{i=1}^m p_{k_{\eta(i)}}(t_i)\right)d\mu(t_1)\cdots d\mu(t_m)\\&=\sum_{\sigma,\eta}\varepsilon_\sigma\varepsilon_\eta\prod_{i=1}^m\delta_{j_{\sigma(i)}k_{\eta(i)}}=\sum_{\sigma,\eta}\varepsilon_\sigma\varepsilon_\eta\prod_{\ell=1}^m\delta_{j_\ell k_{\eta(\sigma^{-1}(\ell))}},\end{aligned} \tag{3-1}$$

where σ^{-1} is the inverse of the permutation σ. For a term in this last sum to be nonzero, we need

$$j_\ell=k_{\eta(\sigma^{-1}(\ell))}\quad\text{for all }1\leq\ell\leq m. \tag{3-2}$$

Since $j_1 < j_2 < \cdots < j_m$ and $k_1 < k_2 < \cdots < k_m$, we see that this will fail unless

$$\eta(\sigma^{-1}(\ell)) = \ell \quad \text{for all } 1 \le \ell \le m.$$

Indeed, if $\eta(\sigma^{-1}(i)) \ne i$ for some smallest i, then $j_{i-1} = k_{i-1}$ but either $j_i = k_{\eta(\sigma^{-1}(i))} \ge k_{i+1}$ or $j_i = k_{\eta(\sigma^{-1}(i))} \le k_{i-1}$. In the former case, all of $j_i, j_{i+1}, \ldots, j_m > k_i$, and k_i is omitted from the equalities in (3-2), a contradiction. In the latter case, we obtain $j_i \le j_{i-1}$, contradicting the strict monotonicity of the j's. Thus necessarily $\eta = \sigma$, so (3-1) becomes, under (3-2),

$$I = \sum_{\sigma} \varepsilon_\sigma^2 = m!. \qquad \square$$

Proof of Theorem 1.3(b). We first show that every $P \in \mathcal{AL}_{n-1}^m$ is a linear combination of the T polynomials. We can write

$$P(x_1, x_2, \ldots, x_m) = \sum_{0 \le j_1, j_2, \ldots, j_m < n} c_{j_1 j_2 \cdots j_m} p_{j_1}(x_1) p_{j_2}(x_2) \cdots p_{j_m}(x_m).$$

Because of the alternating property (1-6), and the linear independence of

$$\{p_{j_1}(x_1) p_{j_2}(x_2) \cdots p_{j_m}(x_m)\}_{1 \le j_1, j_2, \ldots, j_m \le n},$$

necessarily, when we swap indices j_k and j_ℓ, the coefficients change sign; that is,

$$c_{j_1 \cdots j_k \cdots j_\ell \cdots j_m} = -c_{j_1 \cdots j_\ell \cdots j_k \cdots j_m}.$$

In particular, coefficients vanish if any two subscripts coincide. More generally, this implies that if σ is a permutation of $\{1, 2, \ldots, m\}$ with sign ε_σ, then

$$c_{j_{\sigma(1)} j_{\sigma(2)} \cdots j_{\sigma(m)}} = \varepsilon_\sigma c_{j_1 j_2 \cdots j_m}.$$

Next, given distinct $0 \le j_1, j_2, \ldots, j_m < n$, let $\tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_m$ denote these indices in increasing order. We can write, for some permutation σ,

$$j_i = \tilde{j}_{\sigma(i)}, \quad 1 \le i \le m.$$

Conversely, for the given $\{\tilde{j}_i\}$, every such permutation σ defines indices $\{j_i\}$ with $0 \le j_1, j_2, \ldots, j_m < n$. Thus

$$\begin{aligned} P(x_1, x_2, \ldots, x_m) &= \sum_{0 \le \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m} \sum_{\sigma} \varepsilon_\sigma p_{\tilde{j}_{\sigma(1)}}(x_1) p_{\tilde{j}_{\sigma(2)}}(x_2) \cdots p_{\tilde{j}_{\sigma(m)}}(x_m) \\ &= \sum_{0 \le \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m} \det\big[p_{\tilde{j}_i}(x_k)\big]_{1 \le i,k \le m} \\ &= \sum_{0 \le \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_m < n} c_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m} T_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m}(x_1, x_2, \ldots, x_m). \end{aligned} \tag{3-3}$$

Inasmuch as each $T_{\tilde{j}_1 \tilde{j}_2 \cdots \tilde{j}_m}$ lies in $\mathcal{AL}_{n-1}^m$, we have shown that $\mathcal{AL}_{n-1}^m$ is the linear span of the T

polynomials, and (3-3) is an orthogonal expansion. Orthogonality in the form (1-13) gives
$$c_{\tilde{j}_1\tilde{j}_2\cdots\tilde{j}_m} = \frac{1}{m!}\int P(\underline{t})T_{\tilde{j}_1\tilde{j}_2\cdots\tilde{j}_m}(\underline{t})\,d\mu^{\times m}(\underline{t}).$$
Now our definition (1-12) of the reproducing kernel gives (1-14). □

Proof of Theorem 1.3(c). Fix $\underline{x} = (x_1, x_2, \dots, x_m)$. Let
$$P(\underline{t}) = P(t_1, t_2, \dots, t_m) = \det\left[K_n(\mu, x_i, t_j)\right]_{1\le i,j\le m}. \tag{3-4}$$
By successively extracting the sums from the 1st, 2nd, …, m-th rows, we see that
$$\begin{aligned} P(\underline{t}) &= \det\begin{bmatrix} \sum_{j_1=0}^{n-1} p_{j_1}(x_1)p_{j_1}(t_1) & \dots & \sum_{j_1=0}^{n-1} p_{j_1}(x_1)p_{j_1}(t_m) \\ \vdots & \ddots & \vdots \\ \sum_{j_m=0}^{n-1} p_{j_m}(x_m)p_{j_m}(t_1) & \dots & \sum_{j_m=0}^{n-1} p_{j_m}(x_m)p_{j_1}(t_m) \end{bmatrix} \\ &= \sum_{j_1=0}^{n-1}\cdots\sum_{j_m=0}^{n-1}\big(p_{j_1}(x_1)\cdots p_{j_m}(x_m)\big)T_{j_1j_2\cdots j_m}(t_1,t_2,\dots,t_m). \end{aligned}$$
When $j_i = j_k$ for distinct i, k, then $T_{j_1j_2\cdots j_m} = 0$. Thus only terms with $j_1, j_2, \dots, j_m$ distinct are nonzero. As in the proof of Theorem 1.3(b), given distinct $0 \le j_1, j_2, \dots, j_m < n$, we can write, for some permutation σ uniquely determined by these indices,
$$j_i = \tilde{j}_{\sigma(i)}$$
where $0 \le \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_m < n$. As there, this yields
$$\begin{aligned} P(\underline{t}) &= \sum_{0\le\tilde{j}_1<\tilde{j}_2<\cdots<\tilde{j}_m<n}\ \sum_{\sigma}\varepsilon_\sigma\big(p_{\tilde{j}_{\sigma(1)}}(x_1)\cdots p_{\tilde{j}_{\sigma(m)}}(x_m)\big)T_{\tilde{j}_1\tilde{j}_2\cdots\tilde{j}_m}(t_1,t_2,\dots,t_m) \\ &= \sum_{0\le\tilde{j}_1<\tilde{j}_2<\cdots<\tilde{j}_m<n} T_{\tilde{j}_1\tilde{j}_2\cdots\tilde{j}_m}(x_1,x_2,\dots,x)T_{\tilde{j}_1\tilde{j}_2\cdots\tilde{j}_m}(t_1,t_2,\dots,t_m). \end{aligned}$$
So
$$\det\left[K_n(\mu, x_i, t_j)\right]_{1\le i,j\le m} = P(\underline{t}) = m!\,K_n^m(\mu, \underline{x}, \underline{t}),$$
and we have (1-15). Then (1-16) follows from (1-12). □

Proof of Theorem 1.1. By the reproducing kernel relation (1-14), and Cauchy–Schwarz, for all $P \in \mathscr{AL}_{n-1}^m$,
$$P(\underline{x})^2 \le \left(\int P(\underline{t})^2\,d\mu^{\times m}(\underline{t})\right)\left(\int K_n^m(\mu,\underline{x},\underline{t})^2\,d\mu^{\times m}(\underline{t})\right) = \left(\int P(\underline{t})^2\,d\mu^{\times m}(\underline{t})\right)K_n^m(\mu,\underline{x},\underline{x}).$$
Thus
$$K_n^m(\mu,\underline{x},\underline{x}) \ge \sup_{P\in\mathscr{AL}_{n-1}^m}\frac{(P(\underline{x}))^2}{\int(P(\underline{t}))^2\,d\mu^{\times m}(\underline{t})}. \tag{3-5}$$
By choosing P as in (3-4), we obtain equality in (3-5). Now (1-9) follows from (1-15). □

Proof of Corollary 1.2. This follows immediately from (1-9) and the positivity of all the terms there. □

Proof of Theorem 1.4. The upper bound in (1-18) is a standard inequality for determinants involving symmetric positive definite matrices. See, for example, [Beckenbach and Bellman 1961, Theorem 7, p. 63]. For the lower bound, let $R(t_2, t_3, \ldots, t_m) \in \mathcal{AL}_{m-1}^{n-1}$. Let P be a univariate polynomial of degree less than or equal to $n-1$ satisfying $P(x_j) = 0$, $2 \le j \le m$. Let

$$S(t_1, t_2, \ldots, t_m) = \sum_{j=1}^{m} P(t_j)(-1)^j R(t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m).$$

We claim that $S \in \mathcal{AL}_m^{n-1}$. Suppose we swap the variables t_k and t_ℓ, where $1 \le k < \ell \le m$. The terms involving $P(t_k)$ and $P(t_\ell)$ before the variable swap are

$$P(t_k)(-1)^k R(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{\ell-1}, t_\ell, t_{\ell+1}, \ldots, t_m) \\ + P(t_\ell)(-1)^\ell R(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_{\ell-1}, t_{\ell+1}, \ldots, t_m)$$

and become, after swapping t_k, t_ℓ,

$$P(t_\ell)(-1)^k R(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{\ell-1}, t_k, t_{\ell+1}, \ldots, t_m) \\ + P(t_k)(-1)^\ell R(t_1, \ldots, t_{k-1}, t_\ell, t_{k+1}, \ldots, t_{\ell-1}, t_{\ell+1}, \ldots, t_m).$$

Using $\ell - k - 1$ swaps of adjacent variables in each R term, the alternating property of R gives

$$-\{P(t_\ell)(-1)^\ell R(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_{\ell-1}, t_{\ell+1}, \ldots, t_m) \\ + P(t_k)(-1)^k R(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{\ell-1}, t_\ell, t_{\ell+1}, \ldots, t_m)\}.$$

In the remaining terms $P(t_j)(-1)^j R(t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m)$ with $j \ne k, \ell$, we swap t_k and t_ℓ, and use the alternating property to obtain $-P(t_j)(-1)^j R(t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m)$. So we have proved that $S \in \mathcal{AL}_m^n$. Moreover, as P has zeros at $x_2, x_3, \ldots, x_m$, we have

$$S(x_1, x_2, \ldots, x_m) = -P(x_1) R(x_2, x_3, \ldots, x_m).$$

Next, by Cauchy–Schwarz,

$$\begin{aligned} \int S^2 \, d\mu^{\times m} &\le m \int \sum_{j=1}^{m} P^2(t_j) R^2(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m) \, d\mu(t_1) \cdots d\mu(t_m) \\ &= m^2 \Bigl(\int P^2 \, d\mu\Bigr)\Bigl(\int R^2 \, d\mu^{\times(m-1)}\Bigr). \end{aligned}$$

Then (1-9) gives

$$\det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1 \le i,j \le m} \ge m! \frac{S^2(x_1, x_2, \ldots, x_m)}{\int S^2 \, d\mu^{\times m}} \ge \frac{m!}{m^2} \frac{P^2(x_1)}{\int P^2 \, d\mu} \frac{R^2(x_2, \ldots, x_m)}{\int R^2 \, d\mu^{\times(m-1)}}.$$

Write

$$P(t) = P_1(t) \prod_{j=2}^{m} (t - x_j),$$

where P_1 is any polynomial of degree at most $n-m$. Next, take the supremum over P_1 of degree at most $n-m$ and $R \in \mathcal{AL}_{m-1}^{n-1}$. Recalling the definition of ν and (1-2) gives

$$\det\bigl[K_n(\mu,x_i,x_j)\bigr]_{1\le i,j\le m} \ge \frac{m!}{m^2} K_{n-m+1}(\nu,x_1,x_1)\left(\prod_{j=2}^{m}(x_1-x_j)^2\right)\frac{1}{(m-1)!}\det\bigl[K_n(\mu,x_i,x_j)\bigr]_{2\le i,j\le m}.$$

This gives the lower bound in (1-18). □

4. Proofs of Theorems 2.1, 2.2, and 2.3

Lemma 4.1. *Let μ have compact support J, let μ be regular, and assume that I is a subset of the support consisting of finitely many intervals in which (2-3) holds. Let $m \ge 2$. Then, for Lebesgue a.e. $(x_1, x_2, \dots, x_m) \in I^m$,*

$$\lim_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1\le i,j\le m} = \prod_{j=1}^{m} \frac{\omega_J(x_j)}{\mu'(x_j)}. \tag{4-1}$$

Proof. We already know that, for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, x) \frac{\mu'(x)}{\omega_J(x)} = 1, \tag{4-2}$$

by Totik's result (2-5). (Formally, the integral condition (2-3) follows in each of the intervals whose union is I, and hence (2-5) does.) We next show that there is a set $\mathscr{E}$ of Lebesgue measure 0 such that for distinct $x, y \in I\backslash\mathscr{E}$, both (4-2) holds, and

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, y)\left(\frac{\mu'(x)\mu'(y)}{\omega_J(x)\omega_J(y)}\right)^{1/2} = 0. \tag{4-3}$$

These last two assertions give the result. Indeed for distinct $x_1, x_2 \cdots x_m \in I\backslash\mathscr{E}$, we have

$$\begin{aligned}\frac{1}{n^m} \det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1\le i,j\le m} \prod_{j=1}^{m} \frac{\mu'(x_j)}{\omega_J(x_j)} &= \sum_{\sigma} \varepsilon_\sigma \prod_{i=1}^{m}\left(\frac{1}{n} K_n(\mu, x_i, x_{\sigma(i)})\left(\frac{\mu'(x_i)\mu'(x_{\sigma(i)})}{\omega_J(x_i)\omega_J(x_{\sigma(i)})}\right)^{1/2}\right)\\ &= \prod_{i=1}^{m}\left(\frac{1}{n} K_n(\mu, x_i, x_i)\frac{\mu'(x_i)}{\omega_J(x_i)}\right) + o(1) = 1 + o(1),\end{aligned}$$

by (4-2) and (4-3). Of course the set of $x_1, x_2, \dots, x_m$ where any two $x_i = x_j$ with $i \ne j$ has Lebesgue measure 0 in I^m.

We turn to the proof of (4-3). It follows from (4-2) that there is a set $\mathscr{E}$ of measure 0 such that, for $x \in I\backslash\mathscr{E}$, we have

$$\lim_{n\to\infty} \frac{1}{n} p_n^2(x) = \lim_{n\to\infty} \frac{1}{n}(K_{n+1}(\mu, x, x) - K_n(\mu, x, x)) = 0.$$

Then, for distinct x, y, the Christoffel–Darboux formula gives, for $x, y \in I\backslash\mathscr{E}$,

$$\frac{1}{n} K_n(\mu, x, y) = \frac{1}{n} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y} = o(1).$$

Here we are also using the fact that $\{\gamma_{n-1}/\gamma_n\}$ is bounded as μ has compact support. □

Proof of Theorem 2.1(a). Since $J = \operatorname{supp}[\mu]$ is compact, we can find a decreasing sequence of compact sets $\{J_\ell\}_{\ell=1}^{\infty}$ such that each J_ℓ consists of finitely many disjoint closed intervals, and

$$J = \bigcap_{\ell=1}^{\infty} J_\ell.$$

(This follows by a straightforward covering of J by open intervals, and using compactness, then closing them up; at the $(\ell+1)$-st stage, we ensure that $J_{\ell+1} \subset J_\ell$ by intersecting those intervals in $J_{\ell+1}$ with those in J_ℓ.) For $\ell \geq 1$, let

$$d\mu_\ell(x) = d\mu(x) + \frac{1}{\ell}\omega_{J_\ell}(x)\,dx, \tag{4-4}$$

so that we are adding a (small) multiple of the equilibrium measure for J_ℓ to μ. Because $\omega_{J_\ell} > 0$ in the interior of each J_ℓ, we have $\mu_\ell' > 0$ a.e. in J_ℓ, so μ_ℓ is a regular measure [Stahl and Totik 1992, p. 102]. Moreover, ω_{J_ℓ} is positive and continuous in each compact subinterval I of the interior of J_ℓ, so

$$\int_I \log \mu_\ell' > -\infty. \tag{4-5}$$

By Lemma 4.1, for a.e. $(x_1, x_2, \ldots, x_m) \in I^m$,

$$\lim_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\mu_\ell, x_i, x_j)\bigr]_{1\leq i,j\leq m} = \prod_{j=1}^{m} \frac{\omega_{J_\ell}(x_j)}{\mu_\ell'(x_j)}.$$

As $\mu_\ell \geq \mu$, Corollary 1.2 gives

$$\liminf_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1\leq i,j\leq m} \geq \prod_{j=1}^{m} \frac{\omega_{J_\ell}(x_j)}{\mu_\ell'(x_j)}. \tag{4-6}$$

Since a countable union of sets of the form I^m exhausts J_ℓ^m, this last relation actually holds for a.e. $(x_1, x_2, \ldots, x_m) \in J_\ell^m$. Now, by [Totik 2009, Lemma 4.2], uniformly for x in compact subsets of an open set contained in J,

$$\lim_{\ell\to\infty} \omega_{J_\ell}(x) = \omega_J(x). \tag{4-7}$$

Moreover, ω_J is positive and continuous in that open set. We can now let $\ell \to \infty$ in (4-6) and use the fact that the left-hand side in (4-6) is independent of ℓ to obtain (2-8). □

Proof of Theorem 2.1(b). Let L be a compact subset of $\operatorname{supp}[\mu]$ such that $\mu_{|L}$ is regular. $L = I$ is one such choice, because of the Szegő condition (2-3). We may assume that $I \subset L$, since ω_L decreases as L increases. Let

$$d\nu(x) = \mu'(x)_{|L}\,dx, \tag{4-8}$$

so that $d\nu$ is the restriction to L of the absolutely continuous part of μ. Here $\int_I \log \nu' > -\infty$, so ν satisfies the hypotheses of Lemma 4.1, while $\mu \geq \nu$, so Corollary 1.2, followed by Lemma 4.1, gives, for a.e. $(x_1, x_2, \ldots, x_m) \in I^m$,

$$\limsup_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\mu, x_i, x_j)\bigr]_{1\le i,j\le m} \le \limsup_{n\to\infty} \frac{1}{n^m} \det\bigl[K_n(\nu, x_i, x_j)\bigr]_{1\le i,j\le m} = \prod_{j=1}^{m} \frac{\omega_L(x_j)}{\mu'(x_j)};$$

recall that $\nu' = \mu'$ in $I \subset L$. Now take the infimum over all such L and use the fact that the left-hand side is independent of L. □

We turn to:

Proof of Theorem 2.2(a). Let μ_ℓ and J_ℓ be as in the proof of Theorem 2.1(a). It then follows from results of Totik [2009, Theorem 2.3] and/or Simon [2011, Theorem 5.11.13, p. 344] that, for a.e. $x \in J_\ell$, and all real $a_1, a_2, \ldots a_m$, and $1 \le i, j \le m$,

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\mu_\ell, x + \frac{a_i}{n}, x + \frac{a_j}{n}\right) = \frac{\omega_{J_\ell}(x)}{\mu'_\ell(x)} S((a_i - a_j)\omega_{J_\ell}(x)).$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n^m} R^n_m\left(\mu_\ell; x + \frac{a_1}{n}, \ldots, x + \frac{a_m}{n}\right) = \left(\frac{\omega_{J_\ell}(x)}{\mu'_\ell(x)}\right)^m \det\bigl(S((a_i - a_j)\omega_{J_\ell}(x))\bigr)_{1\le i,j\le m}.$$

Now we use the fact that $\mu \le \mu_\ell$, and Corollary 1.2: for a.e. $x \in J$, and all $a_1, a_2, \ldots, a_m$,

$$\liminf_{n\to\infty} \frac{1}{n^m} R^n_m\left(\mu; x + \frac{a_1}{n}, \ldots, x + \frac{a_m}{n}\right) \ge \left(\frac{\omega_{J_\ell}(x)}{\mu'_\ell(x)}\right)^m \det\bigl(S((a_i - a_j)\omega_{J_\ell}(x))\bigr)_{1\le i,j\le m}. \tag{4-9}$$

Moreover we have (4-7). We can now let $\ell \to \infty$ in (4-9), and use the fact that the left-hand side in (4-9) is independent of ℓ to obtain (2-11), with a scale change. □

Proof of Theorem 2.2(b). Let L and ν be as in the proof of Theorem 2.1(b). We can use the aforementioned results of Totik applied to ν, to obtain, for a.e. $x \in I$, and real $a_1, a_2, \ldots, a_m$,

$$\lim_{n\to\infty} \frac{1}{n^m} R^n_m\left(\nu; x + \frac{a_1}{n}, \ldots, x + \frac{a_m}{n}\right) = \left(\frac{\omega_L(x)}{\nu'(x)}\right)^m \det\bigl(S((a_i - a_j)\omega_L(x))\bigr)_{1\le i,j\le m}. \tag{4-10}$$

Now we use the fact that $\mu \ge \nu$, and that $\mu' = \nu'$ in $I \subset L$ and Corollary 1.2: for a.e. $x \in I$, and real $a_1, a_2, \ldots, a_m$,

$$\limsup_{n\to\infty} \frac{1}{n^m} R^n_m\left(\mu; x + \frac{a_1}{n}, \ldots, x + \frac{a_m}{n}\right) \le \left(\frac{\omega_L(x)}{\mu'(x)}\right)^m \det\bigl(S((a_i - a_j)\omega_L(x))\bigr)_{1\le i,j\le m}.$$

Now choose a sequence of compact subsets L of $\mathrm{supp}[\mu]$ such that $\omega_L(x)$ converges to the infimum $\omega_\mu(x)$. □

Proof of Theorem 2.3. Let $\eta \in (0, C_1)$, and choose $\delta > 0$ such that, in $(x - \delta, x + \delta)$,

$$C_1 - \eta \le \mu' \le C_2 + \eta.$$

Here μ' denotes the derivative of the absolutely continuous component of μ. Define

$$d\nu = d\mu \quad \text{in } J\backslash(x-\delta, x+\delta)$$

and

$$d\nu(t) = d\mu_s(t) + (C_1 - \eta)\, dt \quad \text{in } (x-\delta, x+\delta).$$

Then $d\nu \le d\mu$, and ν is regular on J (see [Stahl and Totik 1992, Theorem 5.3.3, p. 148]). Moreover, the derivative ν' of the absolutely continuous part of ν exists and equals $C_1 - \eta$ in $(x-\delta, x+\delta)$, while (2-13) implies that

$$\lim_{h\to 0} \nu_s[x-h, x+h]/h = 0.$$

By a theorem of Totik [2009, Theorem 2.3], we obtain, for the given x and real $a_1, a_2, \dots, a_m$, that

$$\lim_{n\to\infty} \frac{1}{n^m} R_m^n\left(\nu; x+\frac{a_1}{n}, \dots, x+\frac{a_m}{n}\right) = \left(\frac{\omega_J(x)}{C_1-\eta}\right)^m \det\bigl(S((a_i-a_j)\omega_J(x))\bigr)_{1\le i,j\le m}. \tag{4-11}$$

Note that the Lebesgue condition for the local Szegő function required by Totik is satisfied because ν' is smooth (even constant) near x. Then Corollary 1.2 gives

$$\limsup_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu; x+\frac{a_1}{n}, \dots, x+\frac{a_m}{n}\right) \le \left(\frac{\omega_J(x)}{C_1-\eta}\right)^m \det\bigl(S((a_i-a_j)\omega_J(x))\bigr)_{1\le i,j\le m}.$$

As the left-hand side is independent of η, we obtain

$$\limsup_{n\to\infty} \frac{1}{n^m} R_m^n\left(\mu; x+\frac{a_1}{n}, \dots, x+\frac{a_m}{n}\right) \le \left(\frac{\omega_J(x)}{C_1}\right)^m \det\bigl(S((a_i-a_j)\omega_J(x))\bigr)_{1\le i,j\le m}.$$

The lower bound is similar. □

5. Proof of Theorem 2.4

Let

$$w(t) = (1-t)^\alpha, \quad t \in (-1, 1).$$

Choose $\delta > 0$ such that μ is absolutely continuous in $(1-\delta, 1)$, satisfying there

$$(C_1 - \delta) w(t) \le \mu'(t) \le (C_2 + \delta) w(t).$$

Here C_1, C_2 are as in (2-18). Let

$$d\nu(t) = d\mu(t) + (C_2 + \delta) w(t)\, dt \quad \text{in } (-1, 1-\delta]$$

and

$$d\nu(t) = (C_2 + \delta) w(t)\, dt \quad \text{in } (1-\delta, 1].$$

Then

$$d\nu \ge d\mu \quad \text{in } [-1, 1].$$

Note too that, in $(1-\delta, 1)$, the derivative μ' of the absolutely continuous component of μ satisfies

$$\frac{\mu'(t)}{\nu'(t)} \geq \frac{C_1-\delta}{C_2+\delta}. \tag{5-1}$$

Inasmuch as $w > 0$ in $(-1, 1)$, ν is a regular measure in the sense of Stahl, Totik and Ullman, while $\nu'(t)(1-t)^{-\alpha}$ is continuous and positive at 1. By a result of the author [Lubinsky 2008, Theorem 1.2],

$$\lim_{n\to\infty} \frac{1}{2n^2}\tilde{K}_n\left(\nu, 1-\frac{a}{2n^2}, 1-\frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a, b),$$

uniformly for a, b in compact subsets of $(0, \infty)$. If $\alpha \geq 0$, we may also allow a, b to lie in compact subsets of $[0, \infty)$. Then, for $m \geq 2$, Corollary 1.2 and (5-1) give, for $a_1, a_2, \ldots, a_m > 0$,

$$\begin{aligned}
\liminf_{n\to\infty}&\left(\frac{1}{2n^2}\right)^m R_m^n\left(\mu; 1-\frac{a_1}{2n^2}, \ldots, 1-\frac{a_m}{2n^2}\right)\prod_{j=1}^m \mu'\left(1-\frac{a_j}{2n^2}\right)\\
&\geq \left(\frac{C_1-\delta}{C_2+\delta}\right)^m \liminf_{n\to\infty}\left(\frac{1}{2n^2}\right)^m R_m^n\left(\nu; 1-\frac{a_1}{2n^2}, \ldots, 1-\frac{a_m}{2n^2}\right)\prod_{j=1}^m \nu'\left(1-\frac{a_j}{2n^2}\right)\\
&= \left(\frac{C_1-\delta}{C_2+\delta}\right)^m \det(\mathbb{J}_\alpha(a_i, a_j))_{1\leq i,j\leq m}.
\end{aligned}$$

Now let $\delta \to 0+$. □

6. Proofs of Theorem 2.5 and Corollary 2.6

We begin with a lemma that uses the by now classical technique of Totik involving fast decreasing polynomials:

Lemma 6.1. *Assume the hypotheses of Theorem 2.5, except that we do not assume (2-22), nor that μ is regular. Let $\varepsilon \in (0, 1)$. Then*

$$\liminf_{n\to\infty} \frac{K_n^m(\mu, \underline{y}_n, \underline{y}_n)}{K_{[n(1-\varepsilon)]}^m(\nu, \underline{y}_n, \underline{y}_n)} \geq \left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m. \tag{6-1}$$

Proof. We may assume that the common support J of μ and ν is contained in $[-1, 1]$, as a linear transformation of the variable changes the limits in a trivial way. Let $\eta > 0$, and

$$c = \frac{\mu'(\xi)}{\nu'(\xi)}.$$

Our hypothesis (2-20) ensures that we can choose $\delta > 0$ such that

$$\frac{\mu(I)}{\nu(I)} \leq (c+\eta) \quad \text{for } I \subset [\xi-\delta, \xi+\delta]. \tag{6-2}$$

Let $n \geq 4/\varepsilon$ and $\ell = \ell(n) = \left[\frac{1}{2}\varepsilon n\right]$, so that $n - \ell \geq [n(1-\varepsilon)]$. We may choose a polynomial R_ℓ of degree less than or equal to ℓ and $\kappa \in (0, 1)$ such that

$$0 \leq R_\ell \leq 1 \quad \text{in } [-2, 2],$$

$$|R_\ell(t) - 1| \leq \kappa^\ell \quad \text{in } [-\delta/2, \delta/2], \tag{6-3}$$

$$|R_\ell(t)| \leq \kappa^\ell \quad \text{in } [-2, -\delta] \cup [\delta, 2]. \tag{6-4}$$

The crucial thing here is that κ is independent of ℓ, depending only on δ. These polynomials are easily constructed from the approximations to the sign function of Ivanov and Totik [1990, Theorem 3, p. 3]. For the given ξ and n, we let

$$\Psi_n(\underline{t}) = \Psi_n(t_1, t_2, \ldots, t_m) = \prod_{j=1}^{m} R_\ell(\xi - t_j).$$

Observe that this is a symmetric polynomial in $t_1, t_2, \ldots, t_m$. Moreover, for large enough n, we have from (2-21), (6-3), and (6-4),

$$\Psi_n(\underline{y}_n) \geq (1 - \kappa^\ell)^m; \tag{6-5}$$

$$|\Psi_n(\underline{t})| \leq \kappa^l \quad \text{in } [-1, 1]^m \backslash \mathbb{Q}, \tag{6-6}$$

where

$$\mathbb{Q} = \left\{(t_1, t_2, \ldots, t_m) : \max_{1 \leq j \leq m} |\xi - t_j| \leq \delta\right\}.$$

Next, let $P_1 \in \mathscr{AL}^m_{n-\ell-1}$, and set $P = P_1 \Psi_n$. We see that $P \in \mathscr{AL}^m_{n-1}$. Using (6-2), (6-6), we see that

$$\int P^2 \, d\mu^{\times m} \leq (c + \eta)^m \int_{\mathbb{Q}} P_1^2 \, d\nu^{\times m} + \|P_1\|^2_{L_\infty(J^m)} \kappa^{2\ell} \int_{J^m \backslash \mathbb{Q}} d\mu^{\times m}. \tag{6-7}$$

Now we use the regularity of ν, and the fact that J is regular for the Dirichlet problem. These properties imply that [Stahl and Totik 1992, Theorem 3.2.3(v), p. 68]

$$\lim_{n \to \infty} \left(\sup_{\deg(T) \leq n} \frac{\|T\|^2_{L_\infty(J)}}{\int |T^2| \, d\nu} \right)^{1/n} = 1.$$

The supremum is taken over all univariate polynomials T of degree at most n. By successively applying this in each of the m variables, we see that

$$\|P_1\|^2_{L_\infty(J^m)} \leq (1 + o(1))^n \int P_1^2 \, d\nu^{\times m},$$

where the $o(1)$ term is crucially independent of P_1. Thus we may continue (6-7) as

$$\int P^2 \, d\mu^{\times m} \leq (c + \eta)^m \left(\int P_1^2 \, d\nu^{\times m} \right) \left(1 + (1 + o(1))^n \kappa^{n\varepsilon}\right).$$

Since also

$$P^2(\underline{y}_n) \geq P_1^2(\underline{y}_n)(1 + O(\kappa^{\varepsilon n})),$$

we see from (3-5), with an appropriate choice of P_1, that

$$K_n^m(\mu, \underline{y}_n, \underline{y}_n) \geq \frac{P^2(\underline{y}_n)}{\int P^2\, d\mu^{\times m}} \geq \sup_{P_1 \in \mathscr{AL}^m_{n-\ell-1}} \frac{P_1^2(\underline{y}_n)(1 + O(\kappa^{\varepsilon n}))}{(c+\eta)^m (\int P_1^2\, d\nu^{\times m})(1 + (1+o(1))^n \kappa^{n\varepsilon})}$$
$$= \frac{1+o(1)}{(c+\eta)^m} K^m_{n-\ell}(\nu, \underline{y}_n, \underline{y}_n).$$

Thus

$$\liminf_{n\to\infty} \frac{K_n^m(\mu, \underline{y}_n, \underline{y}_n)}{K^m_{[n(1-\varepsilon)]}(\nu, \underline{y}_n, \underline{y}_n)} \geq (c+\eta)^{-m}.$$

As the left-hand side is independent of η, we obtain (6-1). □

Proof of Theorem 2.5. Lemma 6.1 asserts that

$$\liminf_{n\to\infty} \frac{K_n^m(\mu, \underline{y}_n, \underline{y}_n)}{K^m_{[n(1-\varepsilon)]}(\nu, \underline{y}_n, \underline{y}_n)} \geq \left(\frac{\nu'(\xi)}{\mu'(\xi)}\right)^m.$$

Swapping the roles of μ and ν, Lemma 6.1 also gives

$$\liminf_{n\to\infty} \frac{K^m_{[n(1+\varepsilon)]}(\nu, \underline{y}_n, \underline{y}_n)}{K_n^m(\mu, \underline{y}_n, \underline{y}_n)} \geq \left(\frac{\mu'(\xi)}{\nu'(\xi)}\right)^m.$$

Now we apply our hypothesis (2-22) and let $\varepsilon \to 0+$. □

Proof of Corollary 2.6. We apply Theorem 2.5 with $\xi = x$ and, for $n \geq 1$,

$$\underline{y}_n = \left(x + \frac{a_1}{n\omega_J(x)}, \dots, x + \frac{a_m}{n\omega_J(x)}\right).$$

This satisfies (2-21) with $\xi = x$. Now $\det[S(a_i - a_j)]_{1\leq i,j\leq m} > 0$, so our hypothesis (2-24) easily implies (2-22). Then (1-4) and Theorem 2.5 give the result. □

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