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MICROLOCAL PROPERTIES OF SCATTERING MATRICES FOR SCHRÖDINGER EQUATIONS ON SCATTERING MANIFOLDS

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Let M be a scattering manifold, i.e., a Riemannian manifold with an asymptotically conic structure, and let H be a Schrödinger operator on M. One can construct a natural time-dependent scattering theory for H with a suitable reference system, and a scattering matrix is defined accordingly. We show here that the scattering matrices are Fourier integral operators associated to a canonical transform on the boundary manifold generated by the geodesic flow. In particular, we learn that the wave front sets are mapped according to the canonical transform. These results are generalizations of a theorem by Melrose and Zworski, but the framework and the proof are quite different. These results may be considered as generalizations or refinements of the classical off-diagonal smoothness of the scattering matrix for two-body quantum scattering on Euclidean spaces.

1. Introduction

Let M be an n-dimensional smooth noncompact manifold such that $M = M_c \cup M_\infty$, where M_c is relatively compact, and M_∞ is diffeomorphic to $\mathbb{R}_+ \times \partial M$, where ∂M is a compact manifold. In the following, we often identify M_∞ with $\mathbb{R}_+ \times \partial M$, and we also suppose $M_c \cap M_\infty \subset (0, 1) \times \partial M$ under this identification.

We recall the construction of the model introduced in [Ito and Nakamura 2010]. Let $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^{n-1}\}$, $U_\alpha \subset \partial M$, be a local coordinate system of ∂M. We take

$$\{\tilde{\varphi}_\alpha = I \otimes \varphi_\alpha : \tilde{U}_\alpha = \mathbb{R}_+ \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$$

as the local coordinate system for $M_\infty \cong \mathbb{R}_+ \times \partial M$, and we use $(r, \theta) \in \mathbb{R} \times \mathbb{R}^{n-1}$ to represent a point in M_∞.

We suppose ∂M is equipped with a smooth strictly positive density $H = H(\theta)$ and a positive $(2, 0)$-tensor $h = (h^{jk}(\theta))$ on ∂M. We let

$$Q = -\frac{1}{2}\sum_{j,k} H(\theta)^{-1}\frac{\partial}{\partial\theta_j} H(\theta) h^{jk}(\theta)\frac{\partial}{\partial\theta_k} \quad \text{on } \mathcal{H}_b = L^2(\partial M, H(\theta)d\theta).$$

Q is an essentially self-adjoint operator on $\mathcal{H}_b$, and we denote its unique self-adjoint extension by the same symbol Q.

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We let G be a smooth strictly positive density on M such that

$$G(x)\,dx = r^{n-1}H(\theta)\,dr d\theta \quad \text{on } (1,\infty)\times\partial M \subset M_\infty,$$

and we set $\mathscr{H} = L^2(M, G(x)dx)$. Let P be a formally self-adjoint second order elliptic operator on M such that

$$P = -\tfrac{1}{2}G^{-1}(\partial_r, \partial_\theta/r)G\begin{pmatrix} a_1 & a_2 \\ {}^t a_2 & a_3\end{pmatrix}\begin{pmatrix}\partial_r \\ \partial_\theta/r\end{pmatrix} + V \quad \text{on } M_\infty,$$

where $\left(\begin{smallmatrix} a_1 & a_2 \\ {}^t a_2 & a_3\end{smallmatrix}\right)$ defines a real-valued smooth tensor and V is a real-valued smooth function. As in [Ito and Nakamura 2010], we introduce the following assumption:

Assumption A. There is $\mu > 0$ such that for any $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{n-1}$, there is $C_{\ell\alpha} > 0$ and

$$|\partial_r^\ell \partial_\theta^\alpha (a_1(r,\theta) - 1)| \le C_{\ell\alpha} r^{-1-\mu-\ell}, \qquad |\partial_r^\ell \partial_\theta^\alpha a_2(r,\theta)| \le C_{\ell\alpha} r^{-\mu-\ell},$$
$$|\partial_r^\ell \partial_\theta^\alpha (a_3(r,\theta) - h(\theta))| \le C_{\ell\alpha} r^{-\mu-\ell}, \qquad |\partial_r^\ell \partial_\theta^\alpha V(r,\theta)| \le C_{\ell\alpha} r^{-1-\mu-\ell},$$

in each local coordinate of M_∞ described above.

We may consider P as a *short range* perturbation of $-\frac{1}{2}\partial_r^2 + \frac{1}{r^2}Q$, but we will use different operators to construct a scattering theory. It is known that P is essentially self-adjoint, that $\sigma_{ess}(P) = [0,\infty)$, and that P is absolutely continuous except on a countable discrete spectrum, the only possible accumulation point being 0 (see [Ito and Nakamura 2010] and references therein). We construct a time-dependent scattering theory for H as follows: We set

$$M_f = \mathbb{R}\times\partial M, \quad \mathscr{H}_f = L^2(M_f, H(\theta)\,dr d\theta), \quad P_f = -\frac{1}{2}\frac{\partial^2}{\partial r^2} \quad \text{on } M_f.$$

P_f is the one-dimensional free Schrödinger operator, and it is self-adjoint with $\mathscr{D}(P_f) = H^2(\mathbb{R})\otimes\mathscr{H}_b$. Let $j(r)\in C^\infty(\mathbb{R})$ such that $j(r) = 0$ on $(-\infty, \frac{1}{2}]$ and $j(r) = 1$ on $[1,\infty)$. We define $\mathscr{I} : \mathscr{H}_f \to \mathscr{H}$ by

$$(\mathscr{I}\varphi)(r,\theta) = r^{-(n-1)/2} j(r)\varphi(r,\theta) \quad \text{if } (r,\theta)\in M_\infty,$$

and $(\mathscr{I}\varphi)(x) = 0$ if $x \notin M_\infty$. We define the wave operators by

$$W_\pm = W_\pm(P, P_f, \mathscr{I}) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itP}\mathscr{I}e^{-itP_f}.$$

It is shown in [Ito and Nakamura 2010] that these operators exist and are complete in the following sense. Let $\mathscr{F}$ be the Fourier transform in r, i.e.,

$$(\mathscr{F}\varphi)(\rho,\theta) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ir\rho}\varphi(r,\theta)\,dr \quad \text{for } \varphi\in C_0^\infty(M_f),$$

and extend it to a unitary map in $L^2(M_f)$. If we set

$$\mathscr{H}_{f,\pm} = \{\varphi\in\mathscr{H}_f \mid \operatorname{supp}(\mathscr{F}\varphi)\subset\overline{\mathbb{R}_\pm}\times\partial M\},$$

then $\mathscr{H}_f = \mathscr{H}_{f,+}\oplus\mathscr{H}_{f,-}$. We consider $W_\pm$ as maps from $\mathscr{H}_{f,\pm}$ to $\mathscr{H}$; they are asymptotically complete, i.e., unitary operators from $\mathcal{H}_{f,\pm}$ to $\mathcal{H}_{ac}(P)$ [ibid., Theorem 2]. Then the scattering operator defined by

$$S = W_+^* W_- : \mathcal{H}_{f,-} \to \mathcal{H}_{f,+}$$

is unitary. By the intertwining property ($P_f S = S P_f$), there is $S(\lambda) \in \mathcal{B}(\mathcal{H}_b)$ for $\lambda > 0$ such that

$$(\mathcal{F} S \mathcal{F}^{-1}\varphi)(\rho, \cdot) = S(\rho^2/2)\varphi(-\rho, \cdot) \quad \text{for } \rho > 0, \varphi \in \mathcal{F}\mathcal{H}_{f,-}.$$

$S(\lambda)$ is our scattering matrix, and we study its microlocal properties.

Let

$$q(\theta, \omega) = \frac{1}{2}\sum_{j,k} h^{jk}(\theta)\omega_j\omega_k \quad \text{for } (\theta, \omega) \in T^*\partial M$$

be the classical Hamiltonian associated to Q. We denote the Hamilton flow generated by b by $\exp(tH_b)$ for $t \in \mathbb{R}$.

Theorem 1.1. *Suppose Assumption A holds, and let $u \in \mathcal{H}_b$. Then*

$$WF(S(\lambda)u) = \exp(\pi H_{\sqrt{2q}})WF(u),$$

where $WF(u)$ denotes the wave front set of u.

If $\mu = 1$, then we can show $S(\lambda)$ is a Fourier integral operator (FIO). This is a slight extension of a theorem by Melrose and Zworski [1996].

Theorem 1.2. *Suppose Assumption A holds with $\mu = 1$. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to* $\exp(\pi H_{\sqrt{2q}})$.

If $0 < \mu < 1$, then $S(\lambda)$ is not necessarily an FIO in the usual sense, but we can still show it is an FIO in a generalized sense:

Theorem 1.3. *Suppose Assumption A holds, and let $S(\lambda)$ be the scattering matrix defined as above. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to an asymptotically homogeneous canonical transform in $T^*\partial M$, which is asymptotic to* $\exp(\pi H_{\sqrt{2q}})$ *as $\omega \to \infty$.*

The exact definition of the phrase *an FIO associated to an asymptotically homogeneous canonical transform* is given in [Ito and Nakamura 2012], and we discuss it in Section 6.

Remark 1.4. Since we do not introduce a Riemannian metric, our model looks rather different from the scattering metric defined by Melrose [1994; Melrose and Zworski 1996]. However, as explained in [Ito and Nakamura 2010, Appendix A], the Laplacian on scattering manifolds is a special case of our model. Namely, their model corresponds to the case that $\mu = 1$ and that each a_j has asymptotic expansion in r^{-1} as $r \to \infty$ and $V = 0$.

Theorems 1.1 and 1.2 are essentially corollaries of Theorem 1.3, but they can be proved by a simpler argument than Theorem 1.3. We feel the simpler argument is interesting in itself, and we first prove Theorems 1.1 and 1.2, and then we refine the argument to prove Theorem 1.3 later.

The main idea to prove Theorems 1.1–1.3 is to consider the evolution

$$A(t) = e^{itP_f/h}\mathcal{J}^* e^{-itP/h} a(hr, D_r, \theta, hD_\theta) e^{itP/h}\mathcal{J} e^{-itP_f/h}$$

with some symbol a, and use an argument similar to Egorov's theorem for this time-dependent operator. We use a semiclassical argument, i.e., we consider the asymptotic behavior of the operator as $h \to 0$. We consider $W(t) = e^{itP_f/h} \mathcal{J}^* e^{-itP/h}$ as a time-evolution, and then construct an asymptotic solution for $A(t)$ (with slight modifications) as a solution to a Heisenberg equation. The construction of the asymptotic solution relies on the classical Hamilton flow generated by p, the symbol of P. The dominant part of the symbol p is given by the unperturbed conic Hamiltonian: $p_c = \frac{1}{2}\rho^2 + \frac{1}{r^2} q(\theta, \omega)$. The classical scattering operator for the pair p_c and $p_f = \frac{1}{2}\rho^2$ is explicitly computed, and it is $\exp(\pi H_{\sqrt{2q}})$, which appears in the statement of our main theorems. Thus, one may consider our results as a quantization of the classical mechanical scattering on the scattering manifold. More precisely, we show that the canonical transform appearing in Theorem 1.3 is actually the classical scattering map for the pair p and p_f, which is not necessarily homogeneous, and we need to use the method of FIOs with asymptotically conic Lagrangian manifolds.

As mentioned in the beginning, Theorem 1.2 is slight generalization of the Melrose–Zworski theorem [1996] (see also [Vasy 1998] for a simplification of the theory). They used the theory of Legendre distribution and the notion of scattering wave front sets, whereas we use relatively elementary pseudodifferential operator calculus with somewhat nonstandard symbol classes, and a Beals-type characterization of FIOs. We also note that our proof, as well as the setting, are time-dependent-theoretical, and we investigate the scattering phenomena directly to obtain the properties of the wave operators and scattering operators, whereas the Melrose–Zworski paper relies on the stationary, generalized eigenfunction expansion theory.

Our method is closely related to our previous works on the propagation of singularities for Schrödinger evolution equations [Nakamura 2009a; 2009b; Ito and Nakamura 2009; 2012]. In these works, we considered singularities of solutions, which are described by their high energy behavior, whereas in the scattering phenomena we are concerned with the large r behavior (which in turn is related to the high $|\omega|$ behavior, where ω is the conjugate variable to $\theta \in \partial M$). Thus we are forced to use different symbol classes in the calculus, and the corresponding classical mechanics look slightly different, but the general strategy is essentially the same as in these papers.

If $M = \mathbb{R}^n$ and the Hamiltonian P is a short-range perturbation of the Laplacian $-\frac{1}{2}\triangle$, then the canonical map $\exp(\pi H_{\sqrt{2q}})$ is the antipodal map on T^*S^{n-1}. In this case, the off-diagonal smoothness of the scattering cross-section is well-known (see [Isozaki and Kitada 1986], and Section 9.4 and the references of [Yafaev 2000]), and our result (as well as the Melrose–Zworski theorem) may be considered as its generalizations. For such models, our result implies the scattering matrix is an FIO (associated to a canonical map which is asymptotic to the identity map), and if $\mu = 1$ then it is in fact a pseudodifferential operator. It is also not difficult to show from our argument that the scattering matrix is a pseudodifferential operator with symbol in $S^0_{\mu,0}(S^{n-1})$ if $\mu \in (0, 1)$.

The paper is organized as follows. In Section 2, we discuss Hamilton flows generated by p_c and p, and their scattering theory. In Section 3, we prepare the symbol calculus on the scattering manifolds. In Section 4, we discuss an Egorov-type theorem and the construction of asymptotic solutions, which are sufficient to show Theorems 1.1 and 1.2. We prove Theorems 1.1 and 1.2 in Section 5. In Section 6, we discuss the modification of the argument to show Theorem 1.3. We discuss a local decay estimate necessary in the proof in Appendix A. A Beals-type characterization, or an inverse of Egorov's theorem, is discussed in Appendix B, along with a technical lemma on FIOs used in the proof.

Throughout this paper, we use the following notation: For norm spaces X and Y, the space of bounded linear maps is denoted by $B(X, Y)$, and if $X = Y$, we also write $B(X, X) = B(X)$. More generally, if X and Y are topological linear spaces, the space of continuous linear maps is denoted by $\mathscr{L}(X, Y)$. For a symbol g on T^*X with a manifold X, we denote by $\exp(tH_g)$ the Hamilton flow generated by the Hamilton vector field

$$H_g = \frac{\partial g}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial g}{\partial x} \cdot \frac{\partial}{\partial \xi}.$$

We also write $T^*X \setminus 0 = \{(x, \xi) \in T^*X \mid \xi \neq 0\}$.

2. Classical flow and scattering theory

In this section, we consider the classical mechanics, or the Hamilton flow for the Hamiltonian with conic structure on T^*M_∞, where $M_\infty = \mathbb{R}_+ \times \partial M$, and then the Hamilton flow generated by the principal symbol of P.

Exact solutions to the conic Hamilton flow. We set

$$p_c(r, \rho, \theta, \omega) = \frac{1}{2}\rho^2 + \frac{1}{r^2} q(\theta, \omega) \quad \text{and} \quad q(\theta, \omega) = \frac{1}{2} \sum_{j,k} h^{jk}(\theta)\omega_j \omega_k$$

on $T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$. We consider

$$(r(t), \rho(t), \theta(t), \omega(t)) = \exp(tH_{p_c})(r_0, \rho_0, \theta_0, \omega_0),$$

with $(r_0, \rho_0, \theta_0, \omega_0) \in T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$, that is, with $\omega_0 \neq 0$. It satisfies the Hamilton equation

$$\begin{aligned} r'(t) &= \frac{\partial p_c}{\partial \rho} = \rho(t), & \rho'(t) &= -\frac{\partial p_c}{\partial r} = \frac{2}{r(t)^3} q(\theta(t), \omega(t)), \\ \theta'(t) &= \frac{\partial p_c}{\partial \omega} = \frac{1}{r(t)^2} \frac{\partial q}{\partial \omega}(\theta(t), \omega(t)), & \omega'(t) &= \frac{\partial p_c}{\partial \theta} = -\frac{1}{r(t)^2} \frac{\partial q}{\partial \theta}(\theta(t), \omega(t)). \end{aligned}$$

The solution has two invariants: total energy $E_0 = p_c(r_0, \rho_0, \theta_0, \omega_0)$ and angular energy $q_0 = q(\theta_0, \omega_0)$. (The conservation of the total energy follows from $\{p_c, p_c\} = 0$, and of the angular energy from $\{q, p_c\} = \frac{1}{2}\{q, \rho^2\} + \{q, \frac{1}{r^2}\}q + \frac{1}{r^2}\{q, q\} = 0$.) Then $(r(t), \rho(t))$ satisfies

$$r'(t) = \rho(t), \quad \rho'(t) = \frac{2}{r(t)^3} q_0,$$

which is independent of $(\theta(t), \omega(t))$. Noting that $(r^2(t))'' = 4E_0$, we can easily solve this equation to obtain

$$r(t) = \sqrt{2E_0 t^2 + 2r_0\rho_0 t + r_0^2}, \quad \rho(t) = \frac{2E_0 t + r_0\rho_0}{\sqrt{2E_0 t^2 + 2r_0\rho_0 t + r_0^2}}, \quad t \in \mathbb{R}.$$

We now set

$$\tau(t) = \int_0^t \frac{ds}{r(s)^2} = \frac{1}{\sqrt{2q_0}} \left(\tan^{-1} \frac{2E_0 t + r_0\rho_0}{\sqrt{2q_0}} - \tan^{-1} \frac{r_0\rho_0}{\sqrt{2q_0}} \right).$$

Then $(\theta(t),\omega(t))$ satisfies

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial q}{\partial\omega}(\theta,\omega),\qquad \frac{\partial\omega}{\partial\tau}=-\frac{\partial q}{\partial\theta}(\theta,\omega),$$

and hence we learn that

$$(\theta(t),\omega(t))=\exp(\tau(t)H_q)(\theta_0,\omega_0).$$

Moreover, if we set $\sigma(t)=\sqrt{2q_0}\cdot\tau(t)$, then we learn that

$$\frac{\partial\theta}{\partial\sigma}=\frac{1}{\sqrt{2q}}\frac{\partial q}{\partial\omega}=\frac{\partial\sqrt{2q}}{\partial\omega}(\theta,\omega),\qquad \frac{\partial\omega}{\partial\sigma}=-\frac{1}{\sqrt{2q}}\frac{\partial q}{\partial\theta}=-\frac{\partial\sqrt{2q}}{\partial\theta}(\theta,\omega),$$

and hence that

$$(\theta(t),\omega(t))=\exp\bigl(\sigma(t)H_{\sqrt{2q}}\bigr)(\theta_0,\omega_0).$$

Note that $\exp\bigl(tH_{\sqrt{2q}}\bigr)$ is the geodesic flow on ∂M with respect to the (co)metric $(h^{jk}(\theta))$ on $T^*\partial M$.

Classical mechanical wave operators and a scattering operator for the conic Hamilton flow. Now we consider the asymptotics as $t\to\pm\infty$. We set

$$\begin{aligned}
r_\pm&=\lim_{t\to\pm\infty}\tilde r(t)=\lim_{t\to\pm\infty}(r(t)-t\rho(t))=\pm\frac{r_0\rho_0}{\sqrt{2E_0}},\\
\rho_\pm&=\lim_{t\to\pm\infty}\rho(t)=\pm\sqrt{2E_0},\\
(\theta_\pm,\omega_\pm)&=\lim_{t\to\pm\infty}(\theta(t),\omega(t))=\exp(\sigma_\pm H_{\sqrt{2q}})(\theta_0,\omega_0),
\end{aligned}$$

where $\sigma_\pm=\pm\frac{1}{2}\pi-\tan^{-1}(r_0\rho_0/\sqrt{2q_0})$. Note we need a modification only for $r(t)$. $(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm)$ are the scattering data for the trajectory $(r(t),\rho(t),\theta(t),\omega(t))$. We also note the identities

$$E_0=\frac{1}{2}\rho_0^2+\frac{1}{r_0^2}q_0=\frac{1}{2}\rho_\pm^2,\qquad r_0\rho_0=r_\pm\rho_\pm,\qquad q_0=q(\theta_\pm,\omega_\pm).$$

Using these, we can solve $(r_0,\rho_0,\theta_0,\omega_0)$ for given $(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm)$ if $\pm\rho_\pm>0$ and $\omega_\pm\neq 0$:

$$r_0=\sqrt{r_\pm^2+2q_0/\rho_\pm^2},\qquad \rho_0=\frac{r_\pm\rho_\pm}{\sqrt{r_\pm^2+2q_0/\rho_\pm^2}},\qquad (\theta_0,\omega_0)=\exp(-\sigma_\pm H_{\sqrt{2q}})(\theta_\pm,\omega_\pm),$$

where $\sigma_\pm=\pm\frac{1}{2}\pi-\tan^{-1}(r_\pm\rho_\pm/\sqrt{2q})$. We define the classical wave operators (for the pair p_c and $p_f:=\frac{1}{2}\rho^2$) by

$$w_{c,\pm}:(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm)\mapsto(r_0,\rho_0,\theta_0,\omega_0).$$

We can also write

$$w_{c,\pm}(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm)=\lim_{t\to\pm\infty}\exp(-tH_{p_c})\circ\exp(tH_{p_f})(r_\pm,\rho_\pm,\theta_\pm,\omega_\pm).$$

It is easy to check that $w_{c,\pm}$ are diffeomorphisms from $\mathbb{R}\times\mathbb{R}_\pm\times(T^*\partial M\setminus 0)$ to $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$. Hence the classical scattering operator

$$s_c=w_{c,+}^{-1}\circ w_{c,-}:(r_-,\rho_-,\theta_-,\omega_-)\mapsto(r_+,\rho_+,\theta_+,\omega_+)$$

is a diffeomorphism from $\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0)$ to $\mathbb{R}\times\mathbb{R}_+\times(T^*\partial M\setminus 0)$. We can easily compute s_c explicitly, we have

$$s_c(r,\rho,\theta,\omega)=(-r,-\rho,\exp(\pi H_{\sqrt{2q}})(\theta,\rho)),$$

and this is the classical analogue of the Melrose–Zworski theorem.

We write

$$w_c(t)=\exp(-tH_{p_c})\circ\exp(tH_{p_f})\quad\text{so that } w_{c,\pm}=\lim_{t\to\pm\infty}w_c(t).$$

Let $U\subset\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\setminus 0)$ be a relatively compact domain. Then the convergence of $w_c(t)^{-1}$ to $w_{c,\pm}^{-1}$ (as $t\to\pm\infty$) is uniform in U, along with all derivatives. Since the limit is a diffeomorphism, its inverse $w(t)$ also has the same property (on $w_c(t)^{-1}U$). In particular, all the derivatives of $w_c(t)^{-1}$ on U are uniformly bounded in t, and all the derivatives of $w_c(t)$ on $w_c(t)^{-1}U$ are uniformly bounded.

We note that it is easy to check that $w_{c,\pm}$ and hence s_c are homogeneous of order one with respect to the (r,ω)-variables, i.e.,

$$w_{c,\pm}^{-1}(\lambda r_0,\rho_0,\theta_0,\lambda\omega_0)=(\lambda r_\pm,\rho_\pm,\theta_\pm,\lambda\omega)\quad\text{for }\lambda>0.$$

This is consistent with the scaling property of $w_c(t)$:

$$w_c^{-1}(\lambda t)(\lambda r_0,\rho_0,\theta_0,\lambda\omega_0)=(\lambda\tilde r(t),\rho(t),\theta(t),\lambda\omega(t))$$

for any $\lambda>0$, $t\in\mathbb{R}$.

Classical flow generated by the scattering metric. Here we discuss the Hamilton flow generated by the symbol of P:

$$p(r,\rho,\theta,\omega)=\frac{1}{2}\left(a_1(r,\theta)\rho^2+\frac{2\rho a_2(r,\theta)\cdot\omega}{r}+\frac{\omega\cdot a_3(r,\theta)\omega}{r^2}\right)+V \tag{2-1}$$

on T^*M_∞.

We let $\Omega_0\Subset T^*\mathbb{R}_+\times(T^*\partial M\setminus 0)$. For $h\in(0,1]$, we set

$$\Omega_0^h=\{(r,\rho,\theta,\omega)\in T^*\mathbb{R}_+\times(T^*\partial M\setminus 0)\mid(hr,\rho,\theta,h\omega)\in\Omega_0\},$$

and we consider the Hamilton flow with initial conditions in Ω_0^h. We show that if h is sufficiently small then the classical (inverse) wave operators exist on Ω_0^h, and they are very close to $w_{c,\pm}^{-1}$, the (inverse) wave operators for the conic metric.

Theorem 2.1. (i) *Let Ω_0 and Ω_0^h as above. Then there is $h_0>0$ such that if $h\in(0,h_0]$, then*

$$w_\pm^*(r,\rho,\theta,\omega):=\lim_{t\to\pm\infty}\exp(-tH_{p_f})\circ\exp(tH_p)(r,\rho,\theta,\omega)$$

exists for $(r,\rho,\theta,\omega)\in\Omega_0^h$, and the convergence holds in the C^∞-topology on Ω_0^h.

(ii) *We write*

$$(r(t),\rho(t),\theta(t),\omega(t))=\exp(tH_p)(r,\rho,\theta,\omega),$$
$$(r_c(t),\rho_c(t),\theta_c(t),\omega_c(t))=\exp(tH_{p_c})(r,\rho,\theta,\omega),$$

for $(r, \rho, \theta, \omega) \in \Omega_0^h$. *Then for any indices* α, β, γ *and* δ, *there is* $C > 0$ *such that*

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (r(t) - r_c(t))| + |\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\omega(t) - \omega_c(t))| \le C h^{-1+\mu+|\alpha|+|\delta|},$$
$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\rho(t) - \rho_c(t))| + |\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\theta(t) - \theta_c(t))| \le C h^{\mu+|\alpha|+|\delta|},$$

for $(r, \rho, \theta, \omega) \in \Omega_0^h$, $t \in \mathbb{R}$, $0 < h \le h_0$.

(iii) *If we write*

$$w_\pm^*(r, \rho, \theta, \omega) = (r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) \quad \textit{and} \quad w_{c,\pm}^*(r, \rho, \theta, \omega) = (r_{c,\pm}, \rho_{c,\pm}, \theta_{c,\pm}, \omega_{c,\pm}),$$

then

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (r_\pm - r_{c,\pm})| + |\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\omega_\pm - \omega_{c,\pm})| \le C h^{-1+\mu+|\alpha|+|\delta|},$$
$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\rho_\pm - \rho_{c,\pm})| + |\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta (\theta_\pm - \theta_{c,\pm})| \le C h^{\mu+|\alpha|+|\delta|}$$

for $(r, \rho, \theta, \omega) \in \Omega_0^h$, $0 < h \le h_0$.

For $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$, we define $(r^h(t), \rho^h(t), \theta^h(t), \omega^h(t))$ so that

$$(h^{-1} r^h(t), \rho^h(t), \theta^h(t), h^{-1}\omega^h(t)) = \exp(h^{-1} t H_p)(h^{-1} r_0, \rho_0, \theta_0, h^{-1}\omega_0).$$

We also set

$$p^h(r, \rho, \theta, \omega) = p(h^{-1} r, \rho, \theta, h^{-1}\omega), \quad (r, \rho, \theta, \omega) \in T^* M_\infty.$$

Then it is easy to check that

$$(r^h(t), \rho^h(t), \theta^h(t), \omega^h(t)) = \exp(t H_{p^h})(r_0, \rho_0, \theta_0, \omega_0).$$

On the other hand, if we write

$$p^h(r, \rho, \theta, \omega) = p_c(r, \rho, \theta, \omega) + v^h(r, \rho, \theta, \omega),$$

then we learn by Assumption A that for any indices $\alpha, \beta, \gamma, \delta$,

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta v^h(r, \rho, \theta, \omega)\right| \le C_{\alpha\beta\gamma\delta} h^\mu \big(r^{-1}\langle\rho\rangle^2 + r^{-1}\langle\rho\rangle\langle\omega\rangle + r^{-2}\langle\omega\rangle^2\big) r^{-\mu-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}. \quad (2\text{-}2)$$

In order to prove Theorem 2.1, it suffices to show:

Theorem 2.2. (i) *There is* $h_0 > 0$ *such that if* $h \in (0, h_0]$, *then*

$$w_{\pm,h}^*(r_0, \rho_0, \theta_0, \omega_0) := \lim_{t\to\pm\infty} \exp(-t H_{p_f}) \circ \exp(t H_{p^h})(r_0, \rho_0, \theta_0, \omega_0)$$

exists for $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$, *and the convergence holds in the* C^∞*-topology.*

(ii) *For any indices* $\alpha, \beta, \gamma, \delta$, *there is* $C > 0$ *such that*

$$\left|\partial_{r_0}^\alpha \partial_{\rho_0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega_0}^\delta (r^h(t) - r_c(t))\right| + \left|\partial_{r_0}^\alpha \partial_{\rho_0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega_0}^\delta (\rho^h(t) - \rho_c(t))\right|$$
$$+ \left|\partial_{r_0}^\alpha \partial_{\rho_0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega_0}^\delta (\theta^h(t) - \theta_c(t))\right| + \left|\partial_{r_0}^\alpha \partial_{\rho_0}^\beta \partial_{\theta_0}^\gamma \partial_{\omega_0}^\delta (\omega^h(t) - \omega_c(t))\right| \le C h^\mu$$

for $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$, $h \in (0, h_0]$, $t \in \mathbb{R}$, *where*

$$(r_c(t), \rho_c(t), \theta_c(t), \omega_c(t)) = \exp(t H_{p_c})(r_0, \rho_0, \theta_0, \omega_0).$$

(iii) *Writing*

$$(r^h_\pm, \rho^h_\pm, \theta^h_\pm, \omega^h_\pm) = w^*_{\pm,h}(r_0, \rho_0, \theta_0, \omega_0),$$

we have for any indices $\alpha, \beta, \gamma, \delta$ *that*

$$\begin{aligned}\big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(r^h_\pm - r_{c,\pm})\big| + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\rho^h_\pm - \rho_{c,\pm})\big| &\\ + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\theta^h_\pm - \theta_{c,\pm})\big| + \big|\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}(\omega^h_\pm - \omega_{c,\pm})\big| &\le Ch^\mu.\end{aligned}$$

Proof of Theorem 2.2. The proof is analogous to the arguments in [Nakamura 2009a, Section 2; Ito and Nakamura 2009, Section 2]. We only outline the proof, and we omit the details.

Step 1. By the standard virial-type argument, we learn that there is $R > 0$ such that

$$\frac{d^2}{dt^2}(r^h(t)^2) \ge c > 0 \quad \text{if } r^h(t) \ge R,$$

if $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$. Here we use the fact that $|\rho|$ and $|\omega/r|$ are uniformly bounded by the conservation of energy. On the other hand, since $v^h = O(h^\mu)$, we also learn that $r^h(t) \to r_c(t)$ as $h \downarrow 0$, locally uniformly in t. Thus, if t_0 is large and h is small enough, $r^h(t) \ge R$, and combining this with the above observation, we have

$$|r^h(t)| \ge \sqrt{R + c|t - t_0|^2/2} \quad \text{for } t \ge t_0.$$

Hence we learn

$$c_1\langle t\rangle \le r^h(t) \le c_2\langle t\rangle \quad \text{for } h \in (0, h_0],\ t > 0,$$

with some $h_0, c_1, c_2 > 0$. The case $t < 0$ can be handled similarly.

Step 2. We consider the time evolution of $q_0(t) = q(\theta^h(t), \omega^h(t))$. By the Hamilton equation and (2-2), we have

$$\frac{d}{dt}q_0(t) = -\{p^h, q_0\} = -\{v^h, q_0\} = O(h^\mu r^{-1-\mu}\langle\omega\rangle^2) = O(h^\mu\langle t\rangle^{-1-\mu}(1 + q_0(t))).$$

Here we have used the boundedness of $|\rho(t)|$ and $|\omega(t)/r(t)|$ again. Then by the Duhamel formula, we learn that $q_0(t)$ is uniformly bounded for initial conditions in Ω_0 and $h \in (0, h_0]$. This implies $|\omega^h(t)|$ is also uniformly bounded.

Step 3. Combining these observations with the Hamilton equation, we learn that

$$\left|\frac{d\rho^h(t)}{dt}\right| \le C\langle t\rangle^{-2-\mu}, \quad \left|\frac{d\theta^h(t)}{dt}\right| \le C\langle t\rangle^{-1-\mu}, \quad \left|\frac{d}{dt}(r^h(t) - t\rho^h(t))\right| \le C\langle t\rangle^{-1-\mu}, \quad \left|\frac{d\omega^h(t)}{dt}\right| \le C\langle t\rangle^{-1-\mu},$$

uniformly for $(r_0, \rho_0, \theta_0, \omega_0) \in \Omega_0$, $h \in (0, h_0]$ and $t \in \mathbb{R}$. These imply the existence of $w^*_{\pm,h}$ on Ω_0. We can show the similar estimates for the derivatives, i.e.,

$$\left|\frac{d}{dt}\big(\partial^\alpha_{r_0}\partial^\beta_{\rho_0}\partial^\gamma_{\theta_0}\partial^\delta_{\omega_0}\rho^h(t)\big)\right| \le C\langle t\rangle^{-2-\mu-|\alpha|},$$

and so on. These imply the convergence in C^∞-topology, and we conclude that assertion (i) holds.

Step 4. We set

$$g^h(t) = |r^h(t) - r_c(t)| + |\rho^h(t) - \rho_c(t)| + |\theta^h(t) - \theta_c(t)| + |\omega^h(t) - \omega_c(t)|.$$

Then by the Hamilton equation, (2-2), and the estimates in Steps 1 and 2, we learn that

$$\left|\frac{d}{dt} g^h(t)\right| \le C\langle t\rangle^{-1-\mu} g^h(t) + Ch^\mu \langle t\rangle^{-1-\mu}$$

uniformly for initial conditions in Ω_0 and $h \in (0, h_0]$. Then by using the Duhamel formula and noting that $g^h(0) = 0$, we obtain

$$|g^h(t)| \le Ch^\mu, \quad t \in \mathbb{R}.$$

This is assertion (ii) with $\alpha = \beta = \gamma = \delta = 0$. The derivatives can be estimated similarly by induction. For the details of this argument, we refer to [Craig et al. 1995, Section 2; Nakamura 2009a, Section 2]. Assertion (iii) follows immediately from assertion (ii). □

By the above argument, we also learn that $w^*_{\pm,h}$ are invertible for small h. The inverses are uniformly bounded, and their inverses

$$w_{\pm,h} = (w^*_{\pm,h})^{-1}$$

are well-defined for $h \in (0, h_0]$. It follows that

$$w_\pm = (w^*_\pm)^{-1}$$

is well-defined and diffeomorphic on $w^*_\pm[\Omega^h_0]$ with $h \in (0, h_0]$. Thus we can define the classical scattering operator by

$$s = w^*_+ \circ w_-$$

on $w^*_-[\Omega^h_0]$, with sufficiently small h.

3. Symbol classes and their quantization on scattering manifolds

Here we prepare a pseudodifferential operator calculus which is used extensively in the proof of the main theorems. We refer to [Hörmander 1985; Taylor 1981, Chapter XVIII] for the standard theory of microlocal analysis.

In the following, we employ symbol calculus on T^*M, but we always suppose the symbol is supported in T^*M_∞, and we use a local coordinate system as in Section 1. More specifically, we choose a local coordinate system on ∂M: $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^{n-1}\}$, $U_\alpha \subset \partial M$, and we use the coordinate system $\{1 \otimes \varphi_\alpha : \mathbb{R}_+ \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$ on M_∞. We also use a similar local coordinate system on M_f, defined by $\{1 \otimes \varphi_\alpha : \mathbb{R} \times U_\alpha \to \mathbb{R} \times \mathbb{R}^{n-1}\}$. We often identify U_α (or $\mathbb{R}_+ \times U_\alpha$, $\mathbb{R} \times U_\alpha$, respectively) with Ran φ_α (or Ran $(1 \otimes \varphi_\alpha)$, respectively).

Symbol classes. We define a metric either on T^*M_∞ or T^*M_f by

$$g_1 = \frac{dr^2}{\langle r\rangle^2} + d\rho^2 + d\theta^2 + \frac{d\omega^2}{\langle\omega\rangle^2},$$

and consider symbols in $S(m, g_1)$ with a weight function m, i.e., $a \in S(m, g_1)$ if and only if for any indices a, β, γ, δ, there is C such that

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(r, \rho, \theta, \omega)\right| \le C m(r, \rho, \theta, \omega)\langle r\rangle^{-|\alpha|}\langle\omega\rangle^{-|\delta|}.$$

Later, we will consider the calculus of such symbols on sets $\Omega^h = \{(r, \rho, \theta, \omega) \mid (hr, \rho, \theta, h\omega) \in \Omega\}$, where $\Omega \subset T^*M_\infty$ is some compact set (supported away from $\{\omega = 0\}$) and $h > 0$ is small. In such cases, the symbol satisfies

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta a(h; r, \rho, \theta, \omega)\right| \le C m(h) h^{|\alpha|+|\delta|},$$

and we denote such a (h-dependent) symbol as $a \in S_h(m, g_1^h)$, where m is an h-dependent weight. The corresponding metric is naturally

$$g_1^h = h^2 dr^2 + d\rho^2 + d\theta^2 + h^2 d\omega^2.$$

Weyl quantization. Let $\{\chi_\alpha^2\}$ be a partition of unity on ∂M compatible with our coordinate system $\{\varphi_\alpha, U_\alpha\}$, that is, $\chi_\alpha \in C_0^\infty(U_\alpha)$ and $\sum_\alpha \chi_\alpha(\theta)^2 \equiv 1$ on ∂M. We set $\tilde{\chi}_\alpha(r, \theta) = \chi_\alpha(\theta) j(r) \in C^\infty(M_\infty)$. Let $a \in S(m, g_1)$ be a symbol on T^*M_∞, and let $u \in C_0^\infty(T^*M)$. We denote by $a_{(\alpha)}$ and $G_{(\alpha)}$ the representations of a and G in the local coordinate $(1 \otimes \varphi_\alpha, \mathbb{R} \times U_\alpha)$, respectively. We quantize a by

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} \tilde{\chi}_\alpha u,$$

where $a_{(\alpha)}^W(r, D_r, \theta, D_\theta)$ denotes the usual Weyl quantization on the Euclidean space $\mathbb{R}^n$, and we use the identification $\mathbb{R}_+ \times U_\alpha \cong \mathbb{R}_+ \times (\mathrm{Ran}\,\varphi_\alpha)$ for each α. (Strictly speaking, we should have written this as

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha (\tilde{\varphi}_\alpha)^* \big(G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} (\tilde{\varphi}_\alpha)_* (\tilde{\chi}_\alpha u)\big),$$

but we will omit $(\tilde{\varphi}_\alpha)_*$, $(\tilde{\varphi}_\alpha)^*$, ..., when there can be no confusion.) This definition is compatible with the standard definition of pseudodifferential operators on manifolds, but we choose a specific quantization that preserves the asymptotically conic structure of M. Similarly, for a symbol a on T^*M_f, we quantize it by

$$\mathrm{Op}^W(a)u = \sum_\alpha \chi_\alpha H_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) H_{(\alpha)}^{1/2} \chi_\alpha u$$

for $u \in C_0^\infty(M_f)$, where $H_{(\alpha)}$ denotes the representation of H in the local coordinate $(\varphi_\alpha, U_\alpha)$. In this case, the linear structure in r is preserved.

In the above definition, we put weights around the locally defined pseudodifferential operators a_α^W so that $\mathrm{Op}^W(a)$ is symmetric if a is real-valued. Moreover, by virtue of these weights, the symbol corresponding to the operator is unique, including the subprincipal symbol, though we will not take advantage of this fact in this paper.

The above definitions of quantizations also have the convenient property that if we identify a symbol a on T^*M_∞ with a symbol on T^*M_f (by the obvious identification), then we have

$$\mathcal{I}\mathrm{Op}^W(a)\mathcal{I}^* = \mathrm{Op}^W(a) \quad \text{on } \mathcal{H},$$

provided a is supported in $\{r > 1\}$, and we may identify these quantizations by using $\mathcal{I}$. For a symbol supported in $\{r > 1\}$, we may consider $\mathrm{Op}^W(a)$ as an operator from $\mathcal{H}$ to $\mathcal{H}_f$ (or from $\mathcal{H}_f$ to $\mathcal{H}$) also. We define these operators by

$$\mathrm{Op}^W(a)u = \sum_\alpha \chi_\alpha H_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) G_{(\alpha)}^{1/2} \tilde{\chi}_\alpha u$$

for $u \in C_0^\infty(M)$ and

$$\mathrm{Op}^W(a)u = \sum_\alpha \tilde{\chi}_\alpha G_{(\alpha)}^{-1/2} a_{(\alpha)}^W(r, D_r, \theta, D_\theta) H_{(\alpha)}^{1/2} \chi_\alpha u$$

for $u \in C_0^\infty(M_f)$.

If $A = \mathrm{Op}^W(a)$, we denote the (Weyl) symbol of A by $a = \Sigma(A)$.

Hamiltonians. Now we consider properties of our Schrödinger operators and related operators as a preparation for the next section.

We note that, as in the usual Weyl calculus on $\mathbb{R}^n$, if $a(x,\xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k$, then

$$\mathrm{Op}^W(a) = \sum_{j,k} D_j a_{jk}(x) D_k - \frac{1}{4}\sum_{j,k}(\partial_j\partial_k a_{jk}(x)).$$

Hence, if we let p be the symbol of P as in (2-1), we have

$$\mathrm{Op}^W(p) = P + f,$$

where $f \in C^\infty(M_f)$ is such that

$$\left|\partial_r^\alpha \partial_\theta^\beta f(r,\theta)\right| \le C_{\alpha\beta}\langle r\rangle^{-2-|\alpha|}$$

for any α, β. Thus, we can include this error term in V and we may consider $P = \mathrm{Op}^W(p)$. On the other hand, it is easy to see $P_f = \mathrm{Op}^W(p_f)$ on $\mathcal{H}_f$, where $p_f = \frac{1}{2}\rho^2$.

4. An Egorov-type theorem

Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*(\mathbb{R}_+ \times \partial M)$, $\omega_0 \ne 0$, and suppose $a \in C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ is supported in a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ so that a is supported away from $\{\omega = 0\}$. We set

$$a^h(r, \rho, \theta, \omega) = a(h; hr, \rho, \theta, h\omega), \quad h > 0,$$

where a itself may depend on the parameter $h > 0$, but we suppose it is bounded uniformly in the C_0^∞-topology, and supported in the same small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$. The notation here is different from that of Section 2. We set

$$A_0 = \mathrm{Op}^W(a^h) \quad \text{on } M.$$

We set $\varepsilon > 0$ so small that

$$\exp(tH_{p_c})(\operatorname{supp} a) \cap \{r \le \varepsilon\langle t\rangle\} = \varnothing$$

for all $t \in \mathbb{R}$. We choose $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 1$ for $r \geq 1$ and $\eta(r) = 0$ for $r \leq 1/2$, and we set

$$Y = \eta\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big).$$

Then we define

$$A(t) = e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} e^{-itP_f/h}$$

for $t \in \mathbb{R}$. The purpose of this section is to obtain the symbols of $A(t)$ as a pseudodifferential operator, and to study its behavior as $t \to \pm\infty$.

We compute (formally) that

$$\frac{d}{dt}\big(e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\big) = \frac{i}{h} e^{itP/h} T(t) e^{-itP_f/h},$$

where

$$T(t) = PY\mathcal{J} - Y\mathcal{J}P_f - \frac{h(hr)t}{i\varepsilon\langle t\rangle^3}\eta'\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big)\mathcal{J}.$$

We further rewrite this as

$$\begin{aligned}\frac{d}{dt}\big(e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\big) &= \frac{i}{h}\big(e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\big)\big(e^{itP_f/h}\mathcal{J}^* T(t) e^{-itP_f/h}\big) + \frac{i}{h} e^{itP/h}(1 - Y\mathcal{J}\mathcal{J}^*) T(t) e^{-itP_f/h} \\ &= \frac{i}{h}\big(e^{itP/h} Y \mathcal{J} e^{-itP_f/h}\big) L(t) + R_1(t),\end{aligned}$$

where

$$L(t) = e^{itP_f/h}\mathcal{J}^* T(t) e^{-itP_f/h} \quad\text{and}\quad R_1(t) = \frac{i}{h} e^{itP/h}(1 - Y\mathcal{J}\mathcal{J}^*) T(t) e^{-itP_f/h}.$$

We now consider the symbols of $T(t)$ and $L(t)$ as pseudodifferential operators. By direct computations, it is easy to see that for any indices α, β, γ, δ,

$$\begin{aligned}&\big|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \Sigma(T(t))(r, \rho, \theta, \omega)\big| \\ &\qquad\leq C\big(\langle r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-2}\langle\omega\rangle^2\big)\langle r\rangle^{-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}. \qquad (4\text{-}1)\end{aligned}$$

Since $T(t)$ is supported in $\{r \geq \varepsilon\langle t\rangle/2h\}$, we may replace $\langle r\rangle$ by $\langle r\rangle + \varepsilon\langle t\rangle/2h$ in the above estimate. We also have

$$\begin{aligned}&\bigg|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \Big(\Sigma(T(t)) - Y\frac{q(\theta,\omega)}{r^2}\Big)\bigg| \\ &\qquad\leq C\big(\langle r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle r\rangle^{-2-\mu}\langle\omega\rangle^2\big)\langle r\rangle^{-|\alpha|}\langle\rho\rangle^{-|\beta|}\langle\omega\rangle^{-|\delta|}.\end{aligned}$$

In particular, we learn that

$$\bigg|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \Big(\Sigma(T(t)) - Y\frac{q(\theta,\omega)}{r^2}\Big)\bigg| \leq C\langle t\rangle^{-1-\mu-|\alpha|} h^{\mu+|\alpha|+|\delta|} \qquad (4\text{-}2)$$

on $\exp(tH_{p_c})[\operatorname{supp} a^h]$, where the constant is independent of t and h.

Now we note, by virtue of the Weyl calculus (and our choice of the quantization), that

$$\Sigma(L(t))(r, \rho, \theta, \omega) = \Sigma(\mathcal{J}^* T(t))(r + (t/h)\rho, \rho, \theta, \omega).$$

Hence we have, by (4-1),

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \Sigma(L(t))| \le C\big(\langle\tilde r\rangle^{-1-\mu}\langle\rho\rangle^2 + \langle\tilde r\rangle^{-1-\mu}\langle\rho\rangle\langle\omega\rangle + \langle\tilde r\rangle^{-2}\langle\omega\rangle^2\big)\langle\tilde r\rangle^{-|\alpha|}\langle\omega\rangle^{-|\delta|},$$

where $\tilde r = r + (t/h)\rho$. Note that we take advantage of the cut-off function Y in this estimate. We also note, along with (4-2), that

$$\left|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta \left(\Sigma(L(t)) - \frac{q(\theta,\omega)}{\tilde r^2}\right)\right| \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{\mu+|\alpha|+|\delta|} \tag{4-3}$$

on $\exp(-tH_{p_f})\circ\exp(tH_{p_c})[\operatorname{supp} a^h] = \operatorname{supp}(a^h\circ w_c(t))$.

We then construct an asymptotic solution to the Heisenberg equation as

$$\frac{d}{dt}B(t) = -\frac{i}{h}[L(t), B(t)], \quad B(0) = \mathcal{J}^* A_0 \mathcal{J}. \tag{4-4}$$

Lemma 4.1. *There exists $b^h(t; r, \rho, \theta, \omega) \in C_0^\infty(T^*M_f)$ satisfying the following conditions*:

(i) $b^h(0) = a^h$.

(ii) *$b^h(t)$ is supported in $w_c(t/h)^{-1}[\operatorname{supp} a^h]$.*

(iii) *$b^h(t) \in S(1, g_1^h)$, and it is bounded uniformly in $t \in \mathbb{R}$.*

(iv) *$b^h(t) - a^h \circ w_c(t/h) \in S(h^\mu, g_1^h)$, i.e., the principal symbol of $b^h(t)$ is given by $a^h \circ w_c(t/h)$, and the remainder is bounded uniformly in t.*

(v) *If we set $B(t) = \mathrm{Op}^W(b^h(t))$, then*

$$\left\|\frac{d}{dt}B(t) + \frac{i}{h}[L(t), B(t)]\right\| \le C_N\langle t\rangle^{-1-\mu}h^N, \quad h > 0,$$

for any N.

(vi) *$B(t)$ converges to $B_\pm$ as $t \to \pm\infty$ in $B(\mathcal{H}_f)$, and the symbols $b_\pm^h := \Sigma(B_\pm)$ satisfy*

$$b_\pm^h - a^h \circ w_{c,\pm} \in S(h^\mu, g_1^h).$$

Proof. We follow the standard procedure to construct asymptotic solutions to Heisenberg equations (see [Taylor 1981, Chapter 8; Martinez 2002, Chapter 4]). We let

$$\ell_0(t; r, \rho, \theta, \omega) = \frac{q(\theta,\omega)}{(r+t\rho)^2}$$

be the principal symbol of $L(ht)$. If we set

$$b_0(t) = a \circ w_c(t) = a \circ \exp(-tH_{p_c}) \circ \exp(tH_{p_f}),$$

then b_0 satisfies the equation

$$\frac{\partial}{\partial t}b_0(t) = -\{\ell_0(t), b_0(t)\}, \quad b_0(0) = a.$$

We set $b_0^h(t; r, \rho, \theta, \omega) = b_0(t/h; hr, \rho, \theta, h\omega)$, and we also set $B_0(t) = \mathrm{Op}^W(b_0^h(t))$. We note that

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b_0^h(t; r, \rho, \theta, \omega)| \le Ch^{|\alpha|+|\delta|}$$

uniformly in t with any $\alpha, \beta, \gamma, \delta$, since $b_0(t)$ converges to $a \circ w_{c,\pm}$ as $t \to \pm\infty$. We write

$$R_0^0(t) = \frac{d}{dt} B_0(t) + \frac{i}{h}[L(t), B_0(t)], \quad r_0^0(t) = \Sigma(R_0^0(t)).$$

Then by (4-3) and the symbol calculus, $r_0^0(t)$ is supported on $w_c(t/h)^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and

$$\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta r_0^0(t) \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{\mu+|\alpha|+|\delta|} \tag{4-5}$$

for any $\alpha, \beta, \gamma, \delta$. We set $\tilde{r}_0^0(t)$ so that

$$\tilde{r}_0^0(t/h; hr, \rho, \theta, h\omega) = r_0^0(t; r, \rho, \theta, \omega),$$

and solve the transport equation

$$\frac{\partial}{\partial t} b_1(t) + \{\ell_0(t), b_1(t)\} = -\tilde{r}_0^0(t), \quad b_1(0) = 0.$$

By (4-5), it is easy to observe that $|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b_1(t; r, \rho, \theta, \omega)| \le Ch^\mu$ uniformly in t. Moreover, $b_1(t)$ converges to a symbol supported in $w_{c,\pm}^{-1}[\operatorname{supp} a]$ in the C_0^∞-topology as $t \to \pm\infty$. We then set

$$B_1(t) = \mathrm{Op}^W(b_1^h(t)), \quad b_1^h(t; r, \rho, \theta, \omega) = b_1(t/h; hr, \rho, \theta, h\omega).$$

We construct b_j, $j = 1, 2, \ldots$, iteratively, so that $b_j^h \in S(h^{j\mu}, g_1^h)$, and set

$$b^h(t) \sim \sum_{j=0}^\infty b_j^h(t), \quad B(t) = \mathrm{Op}^W(b^h(t)).$$

By construction, $b^h(t)$ and $B(t) = \mathrm{Op}^W(b^h(t))$ satisfy the assertion. □

We then observe that $A(t)$ is very close to $B(t)$ constructed as above.

Lemma 4.2. *For any N, there is $C_N > 0$ such that*

$$\|A(t) - B(t)\| \le C_N h^N, \quad t \in \mathbb{R}.$$

In particular, A_+ and A_-, defined by

$$A_\pm := \operatorname*{w-lim}_{t\to\pm\infty} A(t),$$

have the symbols $b_\pm^h$ as pseudodifferential operators.

Proof. We first observe that

$$\begin{aligned}
\|A(t) - B(t)\| &= \|e^{itP_f/h}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} e^{-itP_f/h} - B(t)\| \\
&= \|\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y \mathcal{J} - e^{-itP_f/h} B(t) e^{itP_f/h}\| \\
&\le \|Y\mathcal{J}\mathcal{J}^* Y e^{-itP/h} A_0 e^{itP/h} Y\mathcal{J}\mathcal{J}^* Y - Y\mathcal{J} e^{-itP_f/h} B(t) e^{itP_f/h} \mathcal{J}^* Y\| \\
&\le \|e^{-itP/h} A_0 e^{itP/h} - Y\mathcal{J} e^{-itP_f/h} B(t) e^{itP_f/h} \mathcal{J}^* Y\| + R_2 \\
&= \|A_0 - \tilde{B}(t)\| + R_2,
\end{aligned}$$

where

$$R_2 = 2\|(1 - Y\mathcal{J}\mathcal{J}^*Y)e^{-itP/h}A_0\| \quad \text{and} \quad \tilde{B}(t) = e^{itP/h}Y\mathcal{J}e^{-itP_f/h}B(t)e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}.$$

By Corollary A.2, we learn that $R_2 = O(\langle t\rangle^{-N}h^N)$ for any N. We then show $\tilde{B}(t)$ is very close to A_0 uniformly in t. We compute

$$\begin{aligned}
\frac{d}{dt}\tilde{B}(t) = \big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)\frac{d}{dt}B(t)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) \\
&+ \frac{i}{h}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)L(t)B(t)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) \\
&- \frac{i}{h}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)B(t)L(t)^*\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) \\
&+ R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) - \big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)B(t)R_1(t)^* \\
= \big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)\Big(\frac{d}{dt}B(t) + \frac{i}{h}[L(t), B(t)]\Big)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) + R_3(t),
\end{aligned}$$

where

$$\begin{aligned}
R_3(t) = R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big) - \big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)B(t)R_1(t)^* \\
+ \frac{i}{h}\big(e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\big)B(t)(L(t) - L(t)^*)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big).
\end{aligned}$$

We can show that $\|R_3(t)\| = O(\langle t\rangle^{-N}h^N)$ for any N. For example,

$$\begin{aligned}
\big\|R_1(t)B(t)\big(e^{itP_f/h}\mathcal{J}^*Ye^{-itP/h}\big)\big\| &\le h^{-1}\|(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}B(t)\| \\
&= h^{-1}\|e^{itP_f/h}(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}B(t)\|.
\end{aligned}$$

As we have seen already, $e^{itP_f/h}(1 - Y\mathcal{J}\mathcal{J}^*)T(t)e^{-itP_f/h}$ is a pseudodifferential operator, and its support is separated from the support of $b^h(t)$ by a distance not less than $c\langle t\rangle h^{-1}$, for some $c > 0$. Thus their product has a vanishing symbol, and its norm is $O(\langle t\rangle^{-N}h^N)$ with any N. The other terms are estimated similarly. Combining this with Lemma 4.1(v), we learn that

$$\Big\|\frac{d}{dt}\tilde{B}(t)\Big\| \le C_N\langle t\rangle^{-1-\mu}h^N$$

for any N, and hence $\|\widetilde{B}(t) - \widetilde{B}(0)\| \le C_N h^N$. We note that

$$\widetilde{B}(0) = \eta\Big(\frac{hr}{\varepsilon}\Big)\mathcal{J}\mathcal{J}^*A_0\mathcal{J}\mathcal{J}^*\eta\Big(\frac{hr}{\varepsilon}\Big) = A_0 + O(h^N)$$

by the choice of $\varepsilon > 0$. Combining these facts, we conclude the assertion holds. □

5. Proofs of Theorems 1.1 and 1.2

Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*(\mathbb{R}_+ \times \partial M)$, and suppose $\omega_0 \neq 0$ as in the last section. Also we let a in $C_0^\infty(T^*(\mathbb{R}_+ \times \partial M))$ be supported in a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ and we set

$$A_0 = \mathrm{Op}^W(a^h), \quad a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega).$$

Let $\varepsilon > 0$ also as in the last section. Write

$$(r_\pm, \rho_\pm, \theta_\pm, \omega_\pm) = w_{c,\pm}^{-1}(r_0, \rho_0, \theta_0, \omega_0)$$

as in Section 2, and recall that $w_{c,\pm}$ are diffeomorphisms from $\mathbb{R}\times\mathbb{R}_\pm\times(T^*\partial M\backslash 0)$ to $\mathbb{R}_+\times\mathbb{R}\times(T^*\partial M\backslash 0)$. We also note that

$$E_0 = p_c(r_0, \rho_0, \theta_0, \omega_0) = \tfrac{1}{2}\rho_\pm^2 > 0$$

by conservation of energy.

Lemma 5.1. *If $\delta > 2\varepsilon^2$, then*

$$\operatorname*{w-lim}_{t\to\pm\infty} \eta(P_f/\delta)A(t)\eta(P_f/\delta) = \eta(P_f/\delta)W_\pm^* A_0 W_\pm \eta(P_f/\delta).$$

Proof. It is easy to show by the stationary phase method that

$$\operatorname*{s-lim}_{t\to\pm\infty} \Big(1-\eta\Big(\frac{hr}{\varepsilon\langle t\rangle}\Big)\Big)\mathcal{J}e^{-itP_f/h}\eta(P_f/\delta) = 0$$

(for fixed h), since the stationary points (in ρ) satisfy $hr = t\rho$. This implies that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itP/h}Y\mathcal{J}e^{-itP_f/h}\eta(P_f/\delta) = W_\pm\eta(P_f/\delta),$$

and the claim follows immediately. □

This implies, combined with Lemmas 4.1 and 4.2:

Lemma 5.2. *Let A_0 as above. Then $W_\pm^* A_0 W_\pm$ are pseudodifferential operators with the symbols $b_\pm^h$ given in Lemma 4.1. In particular, $\Sigma(W_\pm^* A_0 W_\pm)$ are supported in $w_{c,\pm}^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbols (modulo $S(h^\mu, g_1^h)$) are given by $a^h \circ w_{c,\pm}$.*

For the moment, we set

$$\rho_0 = 0 \quad \text{and hence} \quad r_\pm = 0.$$

Then we may take $\varepsilon = \sqrt{E_0}$ provided a is supported in a sufficiently small neighborhood of $(r_0, 0, \theta_0, \omega_0)$.

Now let us suppose $(0, \rho_-, \theta_-, \omega_-)$ (with $\omega_- \neq 0$, $\rho_- > 2\varepsilon$) is given, and $(0, \rho_0, \theta_0, \omega_0)$ is defined by $w_{c,-}(0, \rho_-, \theta_-, \omega_-) = (0, \rho_0, \theta_0, \omega_0)$. The converse of Lemma 5.2 is given as follows:

Lemma 5.3. *Let $\tilde{a} \in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\backslash 0))$ be supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and let*

$$\tilde{A} = \operatorname{Op}^W(\tilde{a}^h), \quad \tilde{a}^h(r, \rho, \theta, \omega) = \tilde{a}(hr, \rho, \theta, h\omega).$$

Then there is a symbol a_0^h supported in $w_{c,-}[\operatorname{supp}\tilde{a}^h]$ such that for any $f \in C_0^\infty(\mathbb{R}_+)$,

$$f(P)A_0 f(P) = W_- f(P_f)\tilde{A} f(P_f)W_-^*,$$

where $A_0 = \operatorname{Op}^W(a_0^h)$. Moreover, the principal symbol (modulo $S(h^\mu, g_1^h)$) is $\tilde{a}^h \circ w_{c,-}^{-1}$.

Proof. We set $a_{0,0}^h = \tilde{a}^h \circ w_{c,-}^{-1}$. Then by Lemma 5.2, we have

$$a_{-,1}^h := \Sigma(\tilde{A} - W_-^* \mathrm{Op}^W(a_{0,0}^h) W_-) \in S(h^\mu, g_1^h),$$

and it is supported in $\mathrm{supp}[\tilde{a}^h]$ modulo $O(h^\infty)$-terms. Then we set $a_{0,1} = a_{-,1}^h \circ w_{c,-}^{-1}$, and set

$$a_{-,2}^h := \Sigma(\tilde{A} - W_-^* \mathrm{Op}^W(a_{0,0}^h + a_{0,1}^h) W_-) \in S(h^{2\mu}, g_1^h).$$

We construct $a_{-,j}^h$, $j = 2, 3, \ldots$, iteratively by

$$a_{-,j}^h := \Sigma(\tilde{A} - W_-^* \mathrm{Op}^W(a_{0,0}^h + \cdots + a_{0,j-1}^h) W_-) \in S(h^{j\mu}, g_1^h),$$

$a_{0,j}^h = a_{-,j}^h \circ w_{c,-}^{-1}$, and we set $a_0^h \sim \sum_{j=0}^\infty a_{0,j}^h$ as an asymptotic sum. Then we have

$$\tilde{A} = W_-^* A_0 W_-$$

modulo $S(h^\infty \langle r\rangle^{-\infty} \langle \omega\rangle^{-\infty}, g_1)$-terms. Since there are no positive eigenvalues [Ito and Skibsted 2011; Melrose and Zworski 1996], we also have $W_\pm f(P_f) W_\pm^* = f(P)$ by virtue of the intertwining property and asymptotic completeness [Ito and Nakamura 2010]. These imply

$$W_- f(P_f) \tilde{A} f(P_f) W_-^* = W_- f(P_f) W_-^* A_0 W_- f(P_f) W_-^* = f(P) A_0 f(P),$$

and this implies the assertion. $\square$

We note Lemma 5.3 naturally holds for $w_{c,+}$ instead of $w_{c,-}$, but we only use the above case. By Lemma 5.3, we learn that

$$S f(P_f) \tilde{A} f(P_f) S^* = W_+^* f(P) A_0 f(P) W_+ = f(P_f)(W_+^* A_0 W_+) f(P_f).$$

By Lemma 5.2, $W_+^* A_0 W_+$ is a pseudodifferential operator. By choosing $f \in C_0^\infty(\mathbb{R}_+)$ so that $f(\rho^2/2) = 1$ in a neighborhood of the support of $\tilde{a}$, we may omit $f(P_f)$ factors up to negligible terms. Thus, $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s_c[\mathrm{supp}\, \tilde{a}^h]$, and the principal symbol is given by $\tilde{a}^h \circ s_c^{-1}$, where $\tilde{a}$ is the symbol given in Lemma 5.3, i.e., $\tilde{a}$ is supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$.

We note that, by the intertwining property of the scattering operator,

$$e^{-itP_f} S = S e^{-itP_f}, \quad \forall t \in \mathbb{R}.$$

This in turn implies

$$T_\tau S = S T_\tau, \quad \forall \tau \in \mathbb{R}, \text{ where } T_\tau = \exp(-i\tau\sqrt{2P_f}).$$

On the other hand, $\sqrt{2P_f} = \mp i \frac{\partial}{\partial r}$ on $\mathscr{H}_{f,\pm}$, and hence T_τ are translations with respect to r. More precisely, we have

$$T_\tau u_\pm(r, \theta) = u_\pm(r \mp \tau, \theta) \quad \text{for } u_\pm \in \mathscr{H}_{f,\pm}.$$

We learn from these facts that

$$S(T_\tau \tilde{A} T_\tau^*) S^* = T_\tau (S\tilde{A}S^*) T_\tau^*,$$

and the symbols of $T_\tau \tilde{A} T_\tau^*$ and $T_\tau(S\tilde{A}S^*)T_\tau^*$ are given by $\tilde{a}^h(r+\tau,\rho,\theta,\omega)$ and $\Sigma(S\tilde{A}S^*)(r+\tau,\rho,\theta,\omega)$, respectively. Using this observation, we may replace $\tilde{a}$ by a symbol supported in a small neighborhood $(r_-,\rho_-,\theta_-,\omega_-)$ with arbitrary $r_- \in \mathbb{R}$. Thus we have proved:

Lemma 5.4. *Let $a \in C_0^\infty(\mathbb{R}\times\mathbb{R}_-\times(T^*\partial M\setminus 0))$ be supported in a small neighborhood of $(r_-,\rho_-,\theta_-,\omega_-)$ with $|\rho_-| \ge 2\varepsilon$, and let*

$$\tilde{A} = \mathrm{Op}^W(a^h), \quad a^h(r,\rho,\theta,\omega) = a(hr,\rho,\theta,h\omega).$$

Then $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s_c[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbol (modulo $S(h^\mu, g_1^h)$) is given by $a^h \circ s_c^{-1}$.

Here we have used the formula

$$s_c(r,\rho,\theta,\omega) = (-r,-\rho,\exp(\pi H_{\sqrt{2q}})(\theta,\omega)).$$

We set $\hat{\mathcal{H}}_{f,\pm} = \mathcal{F}\mathcal{H}_{f,\pm}$. Then $\mathcal{F}S\mathcal{F}^{-1}$ is a unitary map from $\hat{\mathcal{H}}_{f,-}$ to $\hat{\mathcal{H}}_{f,+}$. For notational simplicity, we set

$$\Pi u(r,\theta) = u(-r,\theta) \quad \text{for } u \in \mathcal{H}_{f,\pm},$$

so that $\mathcal{F}(S\Pi)\mathcal{F}^{-1}$ is a unitary map on $\hat{\mathcal{H}}_{f,+}$. By the intertwining property above, $\mathcal{F}(S\Pi)\mathcal{F}^{-1}$ commutes with functions of ρ, and hence is decomposed so that

$$\mathcal{F}(S\Pi)\mathcal{F}^{-1}u(\rho,\omega) = (S(\rho^2/2)u(\rho,\cdot))(\omega) \quad \text{on } \hat{\mathcal{H}}_{f,+} \cong L^2(\mathbb{R}_+; L^2(\partial M)),$$

where $S(\lambda) \in B(L^2(\partial M))$ is the scattering matrix.

Proof of Theorem 1.1. We recall the semiclassical-type characterization of the wave front set: Let $g(\rho,\theta) \in \mathcal{D}'(\mathbb{R}_+\times\partial M)$, and let $(\rho_0,\theta_0,r_0,\omega_0) \in T^*(\mathbb{R}_+\times\partial M)$. $(\rho_0,\theta_0,r_0,\omega_0) \notin WF(g)$ if and only if there is $a \in C_0^\infty(T^*(\mathbb{R}_+\times\partial M))$ such that $a(\rho_0,\theta_0,r_0,\omega_0) \ne 0$ and

$$\|a(\rho,\theta,hD_\rho,hD_\theta)g\| = O(h^\infty) \quad \text{as } h \to +0.$$

We may replace a by an h-dependent symbol with a principal symbol which does not vanish at $(\rho_0,\theta_0,r_0,\omega_0)$.

We fix $\lambda_0 = \rho_0^2/2$ with $\rho_0 > 2\varepsilon$ and consider $S(\lambda)$ where λ is in a small neighborhood of λ_0. Let $u \in L^2(\partial M)$ and let $v \in C_0^\infty(\mathbb{R}_+)$ be supported in a small neighborhood of λ_0. Then it is easy to see that

$$WF(v(\rho)u(\theta)) = \{(\rho,\theta,0,\omega) \mid \rho \in \operatorname{supp} v, (\theta,\omega) \in WF(u)\}.$$

Then, by Lemma 5.4 and the above characterization of the wave front set, we learn that

$$\begin{aligned} WF(\mathcal{F}(S\Pi)\mathcal{F}^{-1}v(\rho)u(\theta)) &= (1\otimes\exp(\pi H_{\sqrt{2q}}))WF(v(\rho)u(\theta)) \\ &= \{(\rho,\theta,0,\omega) \mid \rho \in \operatorname{supp} v, (\theta,\omega) \in \exp(\pi H_{\sqrt{2q}})WF(u)\}; \end{aligned}$$

see [Nakamura 2009b]. By the definition of the scattering matrix, this implies

$$WF(S(\lambda)u) \subset \exp(\pi H_{\sqrt{2q}})WF(u)$$

for $\lambda \in \operatorname{supp} v$. Since this argument works for S^{-1} also, the above inclusion is actually an equality, and we conclude Theorem 1.1. $\square$

Proof of Theorem 1.2. Here we suppose $\mu = 1$. Then by Lemma 5.4 and the Beals-type characterization of FIOs (Theorem B.1), $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$ is an FIO associated to $1 \otimes \exp(\pi H_{\sqrt{2q}})$ on $\{(\rho, \theta, r, \omega) \mid \omega \neq 0\}$. Since $\mathscr{F}(S\Pi)\mathscr{F}^{-1}$ is decomposed as $\{S(\lambda)\}$, this implies $S(\lambda)$ are FIOs on ∂M associated to the canonical transform $\exp(\pi H_{\sqrt{2q}})$ (see Proposition B.4). $\square$

6. Proof of Theorem 1.3

Here we discuss how to generalize the proof of Theorem 1.2 to conclude Theorem 1.3.

We first modify the Egorov-type argument in Section 4. Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*M_\infty$, $\omega_0 \neq 0$, and let Ω_0 be a small neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$. We suppose $a \in C_0^\infty(T^*M_\infty)$ is supported in Ω_0, and we consider the behavior of $A(t)$ as in Section 4. We set

$$w^*(t) = \exp(-itH_{p_f}) \circ \exp(tH_p),$$

which is well-defined for $X \in T^*M_\infty$ as long as $\exp(tH_p)(X) \in T^*M_\infty$. By the discussion in the proof of Theorem 2.2, this condition is always satisfied if $X = (r, \rho, \theta, \omega) \in \Omega_0^h$ and h is sufficiently small. We set

$$w(t) = w^*(t)^{-1} = \exp(-tH_p) \circ \exp(tH_{p_f})$$

on the range of $w(t)$. We note that

$$w^*_\pm = \lim_{t\to\pm\infty} w^*(t)$$

on Ω_0^h with sufficiently small h, and that

$$w_\pm = \lim_{t\to\pm\infty} w(t)$$

on $w_\pm^{-1}[\Omega_0^h]$ with sufficiently small h. Convergence of these maps holds in the C^∞-topology.

We replace Lemma 4.1 by the following slightly different statement:

Lemma 6.1. *There exists $b^h(t; r, \rho, \theta, \omega) \in C_0^\infty(T^*M_f)$ satisfying the following conditions*:

(i) $b^h(0) = a^h$.

(ii) $b^h(t)$ *is supported in* $w^*(t)[\operatorname{supp} a^h]$.

(iii) $b^h(t) \in S(1, g_1^h)$, *and it is bounded uniformly in* $t \in \mathbb{R}$.

(iv) $b^h(t) - a^h \circ w(t) \in S(h, g_1^h)$, *i.e., the principal symbol of* $b^h(t)$ *is given by* $a^h \circ w(t)$, *and the remainder is bounded uniformly in* t.

(v) *If we set* $B(t) = \mathrm{Op}^W(b^h(t))$, *then*

$$\left\| \frac{d}{dt} B(t) + \frac{i}{h}[L(t), B(t)] \right\| \le C_N \langle t\rangle^{-1-\mu} h^N, \quad h > 0,$$

for any N.

(vi) *$B(t)$ converges to $B_\pm$ as $t \to \pm\infty$ in $B(\mathcal{H}_f)$, and the symbols $b_\pm^h := \Sigma(B_\pm)$ satisfy*

$$b_\pm^h - a^h \circ w_\pm \in S(h, g_1^h).$$

We note that $w(t)$ is not homogeneous in the (r, ω)-variables, but very close to a homogeneous map when $|(r, \omega)|$ is very large thanks to Theorem 2.2.

In order to prove Lemma 6.1, we set

$$b_0^h(t) = a^h \circ w(t) = a \circ \exp(-tH_p) \circ \exp(tH_{p_f}),$$

which is supported in $w^*(t)[\Omega_0^h]$. We have $b_0^h(t) \in S(1, g_1^h)$ uniformly in t (for small h) again by Theorem 2.2. Moreover, b_0^h satisfies

$$\frac{\partial}{\partial t} b_0^h(t) = -h^{-1}\{\ell(t), b_0^h(t)\},$$

where $\ell(t) = \Sigma(L(t))$. Hence the first remainder term $r_0^0(t)$ (as defined in Section 4) satisfies

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta r_0^0(t)| \le C\langle t\rangle^{-1-\mu-|\alpha|} h^{1+|\alpha|+|\delta|}$$

for any indices $\alpha, \beta, \gamma, \delta$. Then we construct the asymptotic solution as in the proof of Lemma 4.1 by solving transport equations

$$\frac{\partial}{\partial t} b_j^h(t) + h^{-1}\{\ell(t), b_j^h(t)\} = -r_j^h(t), \quad j = 0, 1, 2, \ldots,$$

and we conclude Lemma 6.1. □

Lemma 4.2 holds when the construction of $B(t)$ is replaced by the one above, with no modifications. Lemmas 5.2 and 5.3 hold in the following form. The proofs are the same.

Lemma 6.2. *Let A_0 as above. Then $W_\pm^* A_0 W_\pm$ are pseudodifferential operators with the symbols $b_\pm^h$ given in Lemma 6.1. In particular, $\Sigma(W_\pm^* A_0 W_\pm)$ are supported in $w_\pm^{-1}[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbols (modulo $S(h, g_1^h)$) are given by $a^h \circ w_\pm$.*

Lemma 6.3. *Let $\tilde{a} \in C_0^\infty(\mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0))$ be supported in a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and let*

$$\tilde{A} = \mathrm{Op}^W(\tilde{a}^h), \quad \tilde{a}^h(r, \rho, \theta, \omega) = \tilde{a}(hr, \rho, \theta, h\omega).$$

Then $W_- \tilde{A} W_-^*$ is a pseudodifferential operator with a symbol supported in $w_-[\operatorname{supp} \tilde{a}^h]$, and the principal symbol (modulo $S(h, g_1^h)$) is give by $\tilde{a}^h \circ w_-^*$.

Combining these, we learn (as in Section 5) the following assertion.

Lemma 6.4. *Let $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_- \times (T^*\partial M \setminus 0))$ be supported in a small neighborhood of $(r_-, \rho_-, \theta_-, \omega_-)$ with $|\rho_-| \ge 2\varepsilon$, $\omega_- \ne 0$, and let*

$$\tilde{A} = \mathrm{Op}^W(a^h), \quad a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega).$$

Then $S\tilde{A}S^*$ is a pseudodifferential operator with a symbol supported in $s[\operatorname{supp} a^h]$ modulo $O(h^\infty)$-terms, and the principal symbol (modulo $S(h, g_1^h)$) is given by $a^h \circ s^{-1}$.

In the following, we consider $(r, \rho, \theta, \omega) \in \Omega_0^h$ with some Ω_0 and sufficiently small h, or equivalently, when $|\omega|$ is sufficiently large. By conservation of energy (or equivalently, by invariance under a shift in r), the classical scattering operator has the form

$$s(r, \rho, \theta, \omega) = (-r + g(\rho, \theta, \omega), -\rho, s(\lambda)(\theta, \omega)), \tag{6-1}$$

where $\lambda = \rho^2/2$ and $s(\lambda)$ is a canonical transform on $T^*\partial M$ for each $\lambda > 0$. (We note that without $g(\rho, \theta, \omega)$, the map s is not necessarily canonical.) Moreover, by Theorem 2.1, we have for any indices α, β, γ that

$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma g(\rho, \theta, \omega)| \le Ch^{-1+\mu+|\gamma|},$$
$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma s_1(\rho, \theta, \omega)| \le Ch^{\mu+|\gamma|},$$
$$|\partial_\rho^\alpha \partial_\theta^\beta \partial_\omega^\gamma s_2(\rho, \theta, \omega)| \le Ch^{-1+\mu+|\gamma|},$$

on Ω_0^h, where Ω_0 is a small neighborhood of $(0, \rho_-, \theta_-, \omega_-)$, and s_1, s_2 are defined by

$$(s_1(\rho, \theta, \omega), s_2(\rho, \theta, \omega)) = s(\lambda)(\theta, \omega) - \exp(\pi H_{\sqrt{2q}})(\theta, \omega),$$

i.e., s_1 denotes the θ-components of the right-hand side terms, and s_2 denotes the ω-components. These estimates imply that s is asymptotically homogeneous (in (r, ω)-variables) in the sense of [Ito and Nakamura 2012, Section 4].

In general, an operator U with distribution kernel u is called an FIO of order m associated to an asymptotically homogeneous canonical transform S if u is a Lagrangian distribution associated to

$$\Sigma_S := \{(x, y, \xi, -\eta) \mid (x, \xi) = S(y, \eta)\},$$

that is, for any $a_1, \dots, a_N \in S^1_{cl}$ such that a_j vanishes on Σ_S for each j, we have that $\mathrm{Op}(a_1) \cdots \mathrm{Op}(a_N)u$ is in $B^{-m-n/2,\infty}_{2,\infty}(\mathbb{R}^{2n})$ [Ito and Nakamura 2012]. The Beals-type characterization of FIOs discussed in Appendix B holds for such FIOs without any change.

By Lemma 6.4 and the analogue of Corollary B.2, we learn that S is an FIO associated to the classical scattering map s. Moreover, by Proposition B.4, we learn that the scattering matrix $S(\lambda)$ is an FIO associated to $s(\lambda)$, where $s(\lambda)$ is defined by (6-1) and it is asymptotic to $\exp(\pi H_{\sqrt{2q}})$. Thus we have proved the following slightly more precise version of Theorem 1.3:

Theorem 6.5. *Suppose Assumption A holds. Then for each $\lambda > 0$, $S(\lambda)$ is an FIO associated to $s(\lambda)$ defined by* (6-1)*. The canonical map $s(\lambda)$ on $T^*\partial M$ is asymptotically homogeneous in ω, asymptotic to $\exp(\pi H_{\sqrt{2q}})$ with the error of $O(|\omega|^{1-\mu})$.*

Appendix A: Local decay estimates

Let P be as in Section 1. For a symbol a, we set $a^h(r, \rho, \theta, \omega) = a(hr, \rho, \theta, h\omega)$. Then we have the following:

Theorem A.1. *Let $(r_0, \rho_0, \theta_0, \omega_0) \in T^*M_\infty \cong T^*\mathbb{R}_+ \times T^*\partial M$, and suppose $\omega_0 \neq 0$. We denote the ε-neighborhood of $(r_0, \rho_0, \theta_0, \omega_0)$ by Ω_ε. We suppose $\varepsilon > 0$ so small that $\Omega_{2\varepsilon} \Subset T^*\mathbb{R}_+ \times (T^*\partial M \setminus 0)$.*

If $a \in C_0^\infty(T^*M_\infty)$ is real-valued, and supported in Ω_ε, then there is an h-dependent symbol $b(t)$ in $C_0^\infty(T^*M_\infty)$ for any $t \in \mathbb{R}$ such that:

(i) $|a(r,\rho,\theta,\omega)| \le c_1 b(0;r,\rho,\theta,\omega)$ *with some* $c_1 > 0$.

(ii) $b(t)$ *is supported in* $\Omega(t) := \exp(tH_{p_c})[\Omega_{2\varepsilon}]$ *for* $t \in \mathbb{R}$.

(iii) *For any indices* α, β, γ *and* δ, *there is* $C_{\alpha\beta\gamma\delta} > 0$ *such that*

$$|\partial_r^\alpha \partial_\rho^\beta \partial_\theta^\gamma \partial_\omega^\delta b(t,r,\rho,\theta,\omega)| \le C_{\alpha\beta\gamma\delta}, \quad (r,\rho,\theta,\omega) \in T^*M_\infty, t \in \mathbb{R}.$$

(iv) *There is* $R(t) \in B(L^2(M))$ *such that* $\|R(t)\| \le C_N h^N$ *for any* N, *and*

$$e^{-itP/h}\mathrm{Op}^W(a^h)e^{itP/h} \le c_1 \mathrm{Op}^W(b^h(t)) + R(t)$$

for $t > 0$, *and the reverse inequality for* $t < 0$. *Moreover*, $R(t)$ *satisfies*

$$\|K^N R(t) K^N\|_{B(L^2)} \le C_N h^N, \quad t \in \mathbb{R},$$

for any N, *where* $K(\cdot) = \langle \mathrm{dist}(\cdot, \Omega^h(t))\rangle$ *with*

$$\mathrm{supp}[b^h(t)] \subset \Omega^h(t) := \{(r,\rho,\theta,\omega) \mid (hr,\rho,\theta,h\omega) \in \Omega(t)\}.$$

Before proving Theorem A.1, we present a corollary which is needed in Section 4.

Corollary A.2. *Let* $\bar\eta \in C^\infty(\mathbb{R})$ *be such that* $\bar\eta(r) = 0$ *if* $r > 2$, *and* $\bar\eta(r) = 1$ *if* $r \le 1$. *We choose* $\varepsilon_1 > 0$ *so small that*

$$\mathrm{dist}\big(\{(r,\rho,\theta,\omega) \mid |r| \le \varepsilon_1\langle t\rangle\}, \Omega(t)\big) \ge \delta\langle t\rangle$$

with some $\delta > 0$. *Then for any* N *there is* $C_N > 0$ *such that*

$$\left\| \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) e^{-itP/h}\mathrm{Op}(a^h) \right\| \le C_N h^N \langle t\rangle^{-N}, \quad t \in \mathbb{R}.$$

We note that if $\varepsilon > 0$ is chosen sufficiently small, then we can find $\varepsilon_1 > 0$ satisfying the property above.

Proof of Corollary A.2. We apply Theorem A.1 with $\tilde a$ such that $\mathrm{Op}^W(\tilde a) = \mathrm{Op}^W(a)\mathrm{Op}^W(a)^*$, which satisfies the same condition. Then we have

$$\begin{aligned}
\left\| \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) e^{-itP/h}\mathrm{Op}(a^h) \right\|^2 &= \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) e^{-itP/h}\mathrm{Op}(\tilde a^h) e^{itP/h} \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) \\
&\le c_1 \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) \mathrm{Op}(b^h(t)) \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) + \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) R(t) \bar\eta\left(\frac{hr}{\varepsilon_1\langle t\rangle}\right) \\
&\le C_N h^N \langle t\rangle^{-N},
\end{aligned}$$

where we used the fact that $\mathrm{supp}[b^h(t)]$ is separated from $\Omega^h(t)$ by a distance not less than $\delta\langle t/h\rangle$. □

Proof of Theorem A.1. The proof is analogous to that of [Nakamura 2009b; Ito 2006; Ito and Nakamura 2009, Section 3], and we only sketch the main steps. We may suppose a is nonnegative without loss of generality. If we set

$$\psi(t) = a \circ \exp(tH_{p_c})^{-1},$$

then it is easy to see that

$$\frac{\partial}{\partial t}\psi = -\{p_c, \psi\}, \quad \psi(0) = a,$$

and this is a good candidate for the principal term of $b(t)$, but ψ does not satisfy the boundedness of the derivatives uniformly in t. We choose $\varphi \in C_0^\infty(\mathbb{R})$ so that

$$\operatorname{supp}\varphi \subset [-1, 1], \quad \varphi(t) \geq 0 \text{ for all } t, \quad \int_{-1}^{1} \varphi(t)\, dt = 1,$$

and moreover, $\pm\varphi'(t) \leq 0$ for $\pm t \geq 0$. We set

$$\varphi_\nu(t) = \varphi(t/\nu), \quad \text{for } \nu > 0,$$

and we denote convolution in t by $\underset{t}{*}$. Then we set

$$b_0(t, \cdot) = \varphi_{\delta\langle t\rangle} \underset{t}{*} \psi = \int \varphi_{\delta\langle t\rangle}(t-s)\psi(s, \cdot)\, ds$$

with sufficiently small $\delta > 0$. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}b_0 = \int \partial_t(\varphi_{\delta\langle t\rangle}(t-s))\psi(s, \cdot)\, ds &= -\int \frac{t(t-s)}{\delta\langle t\rangle^3}\varphi'((t-s)/\delta\langle t\rangle)\psi(s, \cdot)\, ds + \varphi_{\delta\langle t\rangle} \underset{t}{*} (\partial_t\psi) \\ &\geq -\varphi_{\delta\langle t\rangle} \underset{t}{*} \{p_c, \psi\} = -\{p_c, b_0(t, \cdot)\} \qquad \text{(A-1)}\end{aligned}$$

for $t > 0$, by the conditions on φ. We have the reverse inequality for $t < 0$.

We then show the derivatives of b_0 satisfy the required uniform boundedness. We first note that

$$\tilde{\psi}(t; r, \rho, \theta, \omega) := \psi(t; r + t\rho, \rho, \theta, \omega) \to a \circ w_\pm \quad (t \to \pm\infty)$$

in the C_0^∞-topology, by virtue of the existence of the classical scattering for p_c. Thus we have the representation

$$\psi(t; r, \rho, \theta, \omega) = \tilde{\psi}(t; r - t\rho, \rho, \theta, \omega),$$

with $\tilde{\psi}(t)$ uniformly bounded in $C_0^\infty(T^*M)$. Hence we learn that the derivatives in variables other than ρ are uniformly bounded. Then this property applies also to $b_0(t)$. Let us consider the first derivative of $b_0(t)$ in ρ:

$$\partial_\rho b_0(t) = -\int \varphi_{\delta\langle t\rangle}(t-s)s(\partial_r\tilde{\psi})(s, r - s\rho, \rho, \theta, \omega)\, ds + \int \varphi_{\delta\langle t\rangle}(t-s)(\partial_\rho\tilde{\psi})(s, r - s\rho, \rho, \theta, \omega)\, ds.$$

The second term is clearly uniformly bounded. We note that

$$(\partial_r\tilde{\psi})(s; r - s\rho, \rho, \theta, \omega) = -\frac{1}{\rho}\Big\{\frac{\partial}{\partial s}[\tilde{\psi}(s; r - s\rho, \rho, \theta, \omega)] - (\partial_s\tilde{\psi})(s; r - s\rho, \rho, \theta, \omega)\Big\},$$

and then by integration by parts we have

$$\begin{aligned}\int \varphi_{\delta\langle t\rangle}(t-s)s(\partial_r\tilde{\psi})(s,r-s\rho,\rho,\theta,\omega)\,ds \\ &= \frac{1}{\rho}\int \frac{\partial}{\partial s}\Big(\varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds + \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s(\partial_s\tilde{\psi})(s;r-s\rho,\rho,\theta,\omega)\,ds \\ &= \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds - \frac{1}{\rho}\int \frac{s}{\delta\langle t\rangle}\varphi'\Big(\frac{t-s}{\delta\langle t\rangle}\Big)\tilde{\psi}(s,r-s\rho,\rho,\theta,\omega)\,ds \\ &\qquad + \frac{1}{\rho}\int \varphi\Big(\frac{t-s}{\delta\langle t\rangle}\Big)s(\partial_s\tilde{\psi})(s;r-s\rho,\rho,\theta,\omega)\,ds.\end{aligned}$$

Each term in the last expression is bounded uniformly in t since $s\sim t$, and $\partial_s\tilde{\psi}=O(\langle s\rangle^{-2})$. Repeating this procedure, we can show that all the derivatives of b_0 are uniformly bounded. It is also easy to check that b_0 satisfies the required support property provided a is supported in a sufficiently small neighborhood, and $\delta>0$ is chosen sufficiently small.

Now by (A-1) and the sharp Gårding inequality, we have

$$\frac{d}{dt}\mathrm{Op}^W(b_0^h(t)) \ge -\frac{i}{h}[P,\mathrm{Op}^W(b_0^h(t))]+\mathrm{Op}(r_1^h(t))$$

with $r_1(t)=O(h^\mu)$. We set $c_j=7/4-2^{-j}$ for $j=1,2,\dots$, and set

$$a_j(r,\rho,\theta,\omega)=a\Big(\frac{r}{c_j},\frac{\rho}{c_j},\frac{\theta}{c_j},\frac{\omega}{c_j}\Big),\qquad b_j(t)=\varphi_{\delta\langle t\rangle}\underset{t}{*}(a_j\circ\exp(tH_{p_c})).$$

Then we set

$$b(t)\sim b_0(t)+\sum_{j=1}^{\infty}\mu_j b_j(t),$$

with appropriately chosen constants $\mu_j>0$ so that

$$\frac{d}{dt}\mathrm{Op}^W(b^h(t)) \ge -\frac{i}{h}[P,\mathrm{Op}^W(b^h(t))]+O(h^\infty),$$

and $b(t)$ satisfies all the required properties. We refer to [Nakamura 2009b; Ito and Nakamura 2009] for the details of the construction. □

Appendix B: Beals-type characterization of Fourier integral operators

In this appendix, we consider operators on $\mathbb{R}^n$, and we discuss Beals-type characterization of FIOs in terms of h-pseudodifferential operators. We use the result for scattering manifolds, but the generalization is straightforward, and we omit it. Most of the arguments here are similar to those of [Ito and Nakamura 2012, Section 2], and we mainly discuss the modifications necessary to show our results.

We let S be a canonical diffeomorphism on $T^*\mathbb{R}^n$, which is also supposed to be homogeneous in the ξ-variable, i.e.,

$$\text{if } (y,\eta)=S(x,\xi),\ \text{ then } S(x,\lambda\xi)=(y,\lambda\eta) \text{ for } \lambda>0.$$

We also let $U \in \mathcal{L}(\mathcal{S}, \mathcal{S}')$, and let $u \in \mathcal{D}'(\mathbb{R}^{2n})$ be its distribution kernel. For a symbol $a \in C^\infty(T^*\mathbb{R}^n)$, we write

$$a^h(x,\xi) = a(x, h\xi), \quad \mathrm{Op}^W(a^h) = a^W(x, hD_x),$$

for $h > 0$ as before. For $a \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$, we define

$$\mathrm{Ad}_S(a^h)U = \mathrm{Op}^W(a^h \circ S^{-1})U - U\mathrm{Op}^W(a^h) \in \mathcal{L}(\mathcal{S}, \mathcal{S}').$$

We note that $\mathrm{Op}^W(a^h \circ S^{-1}) = \mathrm{Op}^W((a \circ S^{-1})^h)$ since S is homogeneous in ξ.

Theorem B.1. *Let $U \in B(L^2_{cpt}(\mathbb{R}^n), L^2_{loc}(\mathbb{R}^n))$. Suppose for any $a_1, a_2, \dots, a_N \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$, there is $C_N > 0$ such that*

$$\|\mathrm{Ad}_S(a_1^h)\,\mathrm{Ad}_S(a_2^h)\cdots\mathrm{Ad}_S(a_N^h)U\|_{B(L^2)} \le C_N h^N. \tag{B-1}$$

Then U is an FIO of order 0 associated to S.

Corollary B.2. *Let S and U as above. If for any $a \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ there is an h-dependent symbol $b \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta b(h; x, \xi)| \le C_{\alpha\beta} h,$$

for any $\alpha, \beta \in \mathbb{Z}_+^n$, $h \in (0, 1]$, and

$$\mathrm{Ad}_S(a^h)U = \mathrm{Op}^W(b^h)U + R, \quad \|R\|_{B(L^2)} = O(h^\infty),$$

then U is an FIO of order 0 associated to S.

Proof of Corollary B.2. We show (B-1) follows from the above condition. The cases $N = 0, 1$ are obvious. Let $N = 2$ and we write

$$\mathrm{Ad}_S(a_j^h)U = \mathrm{Op}^W(b_j^h)U + R_j, \quad j = 1, 2.$$

Then we have

$$\begin{aligned}\mathrm{Ad}_S(a_1^h)\,\mathrm{Ad}_S(a_2^h)U &= \mathrm{Op}^W(a_1^h \circ S^{-1})\mathrm{Op}^W(b_2^h)U - \mathrm{Op}^W(b_2^W)U\mathrm{Op}^W(a_1^h) + \mathrm{Ad}_S(a_1^h)R_2 \\ &= \big[\mathrm{Op}^W(a_1^h \circ S^{-1}), \mathrm{Op}^W(b_2^h)\big]U + \mathrm{Op}^W(b_2^h)\mathrm{Op}^W(b_1^h)U + \mathrm{Ad}_S(a_1^h)R_2 + \mathrm{Op}^W(b_2^h)R_1 \\ &= \mathrm{Op}^W(b_{12}^h)U + R_{12},\end{aligned}$$

where $R_{12} = O(h^\infty)$ and $b_{12} \in C_0^\infty(T^*\mathbb{R}^n \setminus 0)$ satisfies

$$\big|\partial_x^\alpha \partial_\xi^\beta b_{12}(h; x, \xi)\big| \le C'_{\alpha\beta} h^2, \quad \text{for any } \alpha, \beta \in \mathbb{Z}_+^n,\ h \in (0, 1],$$

and (B-1) for $N = 2$ follows. Iterating this procedure, we obtain (B-1) for any N. □

In order to prove Theorem B.1, we first note the semiclassical-type characterization of Besov spaces. By the standard partition-of-unity argument, it is straightforward to observe that $u \in B^{\sigma,\infty}_{2,loc}(\mathbb{R}^m)$ if and only if for any $(x_0, \xi_0) \in T^*\mathbb{R}^m \setminus 0$ there is $\varphi \in C_0^\infty(T^*\mathbb{R}^m)$ such that $\varphi(x_0, \xi_0) \ne 0$ and

$$\|\mathrm{Op}^W(\varphi^h)u\|_{L^2} \le Ch^\sigma, \quad h > 0.$$

Thus, in turn, we learn that $u \in B^{\sigma,\infty}_{2,loc}(\mathbb{R}^{2n})$ if and only if for any $(x_0, y_0, \xi_0, \eta_0)$, $(\xi_0, \eta_0) \neq (0, 0)$, there are $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_1(x_0, \xi_0) \neq 0$, $\varphi_2(y_0, \eta_0) \neq 0$, and

$$\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{HS} \le C h^\sigma, \quad h > 0,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm in $B(L^2(\mathbb{R}^n))$. Now we choose $\varphi_3 \in C_0^\infty(\mathbb{R}^n)$ so that $\varphi_3 = 1$ in a neighborhood of $\operatorname{supp}\varphi_2$. We note that

$$\|\mathrm{Op}^W(\varphi_3)\|_{HS} = (2\pi)^{-n/2}\left(\int_{\mathbb{R}^n} |\varphi_3(x, h\xi)|^2 dx d\xi\right)^{1/2} = (2\pi h)^{-n/2}\left(\int_{\mathbb{R}^n} |\varphi_3(x, \xi)|^2 dx d\xi\right)^{1/2} = C h^{-n/2}$$

for $h > 0$ with some $C > 0$. Hence we have

$$\begin{aligned}\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{HS} &\le \|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\mathrm{Op}^W(\varphi_3^h)\|_{HS} + R \\ &\le C h^{-n/2}\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{B(L^2)} + R,\end{aligned}$$

where

$$R = \|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)(1 - \mathrm{Op}^W(\varphi_3^h))\|_{HS} = O(h^\infty)$$

by the symbol calculus. Thus we have proved the following lemma:

Lemma B.3. *If for any $(x_0, y_0, \xi_0, \eta_0) \in T^*\mathbb{R}^{2n}$ with $(\xi_0, \eta_0) \neq (0, 0)$ there are $\varphi_1, \varphi_2 \in C_0^\infty(T^*\mathbb{R}^n)$ such that $\varphi_1(x_0, \xi_0) \neq 0$, $\varphi_2(y_0, \eta_0) \neq 0$ and*

$$\|\mathrm{Op}^W(\varphi_1^h) U \mathrm{Op}^W(\varphi_2^h)\|_{B(L^2)} \le C, \quad h > 0,$$

then $u \in B^{-n/2,\infty}_{2,loc}(\mathbb{R}^{2n})$.

Proof of Theorem B.1. We modify the proof of Theorem 2.1 in [Ito and Nakamura 2012], to which we refer for further details.

We first note that

$$WF(u) \subset \Lambda_S = \{(x, y, \xi, -\eta) \in T^*\mathbb{R}^{2n} \mid (x, \xi) = S(y, \eta)\}.$$

We note that if $(x_0, y_0, \xi_0, -\eta_0) \notin \Lambda_S$ with $\eta_0 \neq 0$, it is straightforward to show $(x_0, y_0, \xi_0, -\eta_0) \notin WF(u)$. If $\xi_0 \neq 0$, we consider U^* and we can also conclude $(x_0, y_0, \xi_0, -\eta_0) \notin WF(u)$.

Now we let $a_1, a_2, \dots, a_N \in S^1_{cl}(\mathbb{R}^n)$ and let $(x_0, \xi_0) = S(y_0, \eta_0)$. We may assume a_j are homogeneous of order one in the ξ-variable. By Lemma B.3 and the proof just cited, it suffices to show the following to conclude U is an FIO of order 0 associated to S: There are $\psi_1, \psi_2 \in C_0^\infty(T^*\mathbb{R}^n)$ such that $\psi_1(x_0, \xi_0) \neq 0$, $\psi_2(y_0, \eta_0) \neq 0$ and

$$\|\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(a_1)\cdots \mathrm{Ad}_S(a_N) U]\mathrm{Op}^W(\psi_2^h)\|_{B(L^2)} \le C, \quad h \in (0, 1], \tag{B-2}$$

with some $C > 0$.

We set $\Psi_0, \Psi_1 \in C_0^\infty(T^*\mathbb{R}^n)$ so that they are supported in a small neighborhood of (y_0, η_0), $\Psi_j = 1$ on a neighborhood of (y_0, η_0), and $\Psi_0 = 1$ on $\operatorname{supp}\Psi_1$. We then set

$$\varphi_j(x, \xi) = a_j(x, \xi)\Psi_0(x, \xi) \in C_0^\infty(T^*\mathbb{R}^n).$$

We note, since a_j are homogeneous of order one in ξ, that

$$a_j(x,\xi)\Psi_0(x,h\xi)=h^{-1}a_j(x,h\xi)\Psi_0(x,h\xi)=h^{-1}\varphi_j(x,h\xi).$$

We also set

$$\psi_1=\Psi_1\circ S^{-1}\quad\text{and}\quad\psi_2=\Psi_1$$

so $\psi_1(1-\Psi_0\circ S^{-1})=0$ and $(1-\Psi_0)\psi_2=0$. This implies, in particular, that

$$\psi_1(x,h\xi)(a_j\circ S^{-1})(x,\xi)=h^{-1}\psi_1(x,h\xi)(\varphi_j\circ S^{-1})(x,h\xi),$$
$$a_j(y,\eta)\psi_2(y,h\eta)=h^{-1}\varphi_j(y,h\eta)\psi_2(y,h\eta).$$

Using these, and applying the h-pseudodifferential operator calculus, we learn that

$$\begin{aligned}&\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(a_1)\cdots\mathrm{Ad}_S(a_N)U]\mathrm{Op}^W(\psi_2^h)\\&\qquad=h^{-N}\mathrm{Op}^W(\psi_1^h)[\mathrm{Ad}_S(\varphi_1^h)\cdots\mathrm{Ad}_S(\varphi_N^h)U]\mathrm{Op}^W(\psi_2^h)+O(h^\infty),\end{aligned}$$

and this implies the right-hand side is bounded by the assumption of Theorem B.1. Now (B-2) follows from this observation, and we conclude that the assertion hold. □

We note that the conditions and the assertion of Theorem B.1 are microlocal, and hence the theorem is easily extended to a statement in a conic set in $T^*\mathbb{R}^n$. In the next proposition, we use the extended statement on a conic set.

Proposition B.4. *Let $\mathbb{R}^m=\mathbb{R}^n\times\mathbb{R}^k$, and let U be a bounded operator on $L^2(\mathbb{R}^m)$ and let S be a homogeneous canonical diffeomorphism on $T^*\mathbb{R}^m$. Suppose U commutes with multiplication operators in y so that U is decomposed as*

$$U=\int_{\mathbb{R}^k}^{\oplus}\tilde U(y)dy\quad\text{on } L^2(\mathbb{R}^m)\cong L^2(\mathbb{R}^k_y,L^2(\mathbb{R}^n_x)),$$

where $\{U(y)\}$ is a family of operators on $L^2(\mathbb{R}^n_x)$. Suppose also that S is decomposed as

$$S:(x,\xi,y,\eta)\mapsto(\tilde S(y)(x,\xi),y,\eta+g(x,\xi,y))$$

for $(x,\xi,y,\eta)\in T^*\mathbb{R}^n\cong T^*\mathbb{R}^n_x\times T^*\mathbb{R}^k_y$, where $\{\tilde S(y)\}$ is a family of canonical maps on $T^*\mathbb{R}^n_x$. If U is an FIO associated to S on a conic set $\{(x,\xi,t,\eta)\mid\xi\neq0\}$, then for each $y\in\mathbb{R}^k$, $\tilde U(y)$ is an FIO of order 0 associated to $\tilde S(y)$.

Remark B.5. The assumption on S actually follows from the properties of U. We include it to introduce the notations.

Proof. Let $a\in C_0^\infty(T^*\mathbb{R}^n\setminus0)$, and let $\varphi,\psi\in C_0^\infty(\mathbb{R}^k)$ such that $\varphi,\psi\geq0$ and $\int\psi(\eta)\,d\eta=1$. We also denote $\psi_z(\eta)=\psi(\eta-z)$ for $z\in\mathbb{R}^k$. We consider

$$A_z=a_z(x,hD_x,y,hD_y)=a(x,hD_x)\varphi(y)\psi_z(hD_y).$$

Since U is an FIO, there is b_z, which is bounded in $C_0^\infty(T^*\mathbb{R}^m)$ uniformly in $h \in (0, 1]$, such that

$$UA_z = B_zU + O(h^\infty), \quad B_z = b_z(x, hD_x, y, hD_y),$$

with the principal symbol

$$a_z \circ S^{-1} = (a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y)\psi\big(\eta - g(\tilde{S}(y)^{-1}(x, \xi), y) - z\big).$$

Since U commutes with $\{e^{iy\cdot z} \mid z \in \mathbb{R}^k\}$ (translations in the η-variable), we learn that

$$b_z(x, \xi, y, \eta) = b_0(x, \xi, y, \eta - z),$$

and the remainder term also satisfies this property. Moreover, these symbols decay rapidly outside $S[\operatorname{supp} a_z]$.

It is also easy to see that

$$\int_{|z|\le R} A_z dz \to a(x, hD_x)\varphi(y) \quad \text{and} \quad \int_{|z|\le R} B_z dz \to \tilde{b}(x, hD_x, y)$$

strongly as $R \to \infty$, where $\tilde{b}(x, \xi, y) = \int_{\mathbb{R}^k} b_0(x, \xi, y, \eta)\, d\eta$. The principal symbol of $\tilde{b}$ is given by $(a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y)$. These facts imply that

$$\tilde{U}(y)a(x, hD_x)\varphi(y) = \tilde{b}(x, hD_x, y)\tilde{U}(y) + O(h^\infty),$$

where $\tilde{b}(x, \xi, y) - (a \circ \tilde{S}(y)^{-1})(x, \xi)\varphi(y) = O(h)$. Since $\varphi \in C_0^\infty(\mathbb{R}^k)$ is arbitrary, for a fixed $y \in \mathbb{R}^k$ we may replace $\varphi(y)$ by 1, and we learn $\tilde{U}(y)$ is an FIO of order 0 associated to $\tilde{S}(y)$ by Corollary B.2. □

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