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POROUS MEDIA: THE MUSKAT PROBLEM IN THREE DIMENSIONS

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The Muskat problem involves filtration of two incompressible fluids through a porous medium. We consider the problem in three dimensions, discussing the relevance of the Rayleigh–Taylor condition and the topology of the initial interface, in order to prove the local existence of solutions in Sobolev spaces.

1. Introduction

The Muskat problem [Muskat and Wickoff 1937; Bear 1972] involves filtration of two incompressible fluids through a porous medium, characterized by a positive constant κ quantifying its porosity and permeability. The two fluids, having velocity fields v^1 and v^2, occupy disjoint regions D^1 and $D^2 = \mathbb{R}^3 - D^1$, with a common boundary (interface) given by the surface $S = \partial D^1 = \partial D^2$. Naturally, those domains change with time, as does the interface. We denote by p^j $(j = 1, 2)$ the corresponding pressures, and we will also assume that the dynamical viscosities μ^j and the densities ρ^j are constants with $\mu^1 \neq \mu^2$, $\rho^1 \neq \rho^2$.

Conservation of mass in this setting is given by the equation $\nabla \cdot v = 0$ (in the distribution sense), where $v = v^1 \chi_{D^1} + v^2 \chi_{D^2}$.

The momentum equation, obtained experimentally by Darcy [1856] (see also [Bear 1972]), is

$$\frac{\mu^j}{\kappa} v^j = -\nabla p^j - (0, 0, \rho^j g), \quad j = 1, 2,$$

where g is the acceleration due to gravity.

One can find in the literature several attempts to derive Darcy's law from Navier–Stokes [Tartar 1980; Sánchez-Palencia and Zaoui 1987] through the process of homogenization under the hypothesis of a periodic, or almost periodic, porosity. In any case, the presence of the porous medium justifies the elimination of the inertial terms in the motion, leaving friction (viscosity) and gravity as the only relevant forces, to which one has to add pressure as it appears in the formulation of Darcy's law. There are three scales involved in the analysis: the macroscopic or bulk mass, the microscopic size of the fluid particle, and the mesoscopic scale corresponding to the pores. In the references above, one finds descriptions of the velocity v as an average over the mesoscopic cells of the fluid particle velocities. Taking into account that each cell contains a solid part where the particle velocity vanishes, it is then natural to get the viscous

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forces associated to that average velocity, which is a scaled approximation of the laplacian term appearing in the Navier–Stokes equation.

In this paper, we shall consider the case of a homogeneous and isotropic porous material. Porosity is the fraction of the volume occupied by pores or void space. But it is important to distinguish between two kinds of pores — the kind that forms a continuous interconnected phase within the medium, and the kind that is isolated — because non-interconnected pores cannot contribute to fluid transport. Permeability is the term used to describe the conductivity of a porous medium with respect to a newtonian fluid, and it depends upon the properties of the medium and the fluid. Darcy's law indicates this dependence, allowing us to define the notion of specific permeability κ and its units. In the case of an anisotropic material, κ will be a symmetric and positive definite tensor, and the methods of our proof can be modified to get local existence; but for a nonhomogeneous medium, the properties of the tensor $\kappa(x)$ will have to be specified in a very precise manner in order to allow an interesting theory.

The Muskat problem and related problems [Saffman and Taylor 1958] have been studied recently [Constantin and Pugh 1993; Siegel et al. 2004; Córdoba and Gancedo 2007; 2009; Córdoba et al. 2011]. The first natural question is about the evolution of the system (existence of solutions), at least for a short time $t > 0$, and the persistence of smoothness of the interface $S(t)$ if we begin with a smooth enough surface at time $t = 0$. One can easily deduce from this formulation that in the event of smooth evolution, both pressures can be taken to be equal at the interface:

$$p^1|_{S(t)} = p^2|_{S(t)}.$$

Therefore, we look at the case without surface tension (see [Escher and Simonett 1997], where the regularizing effect of surface tension is considered). The normal component of the velocity fields must also agree at the free boundary; that is, if ν^j is the unit normal to S pointing into D^j, we have

$$(v^1 - v^2) \cdot \nu^j = 0 \quad \text{at } S(t), \qquad j = 1, 2$$

(note that $\nu^2 = -\nu^1$). Furthermore, the vorticity will be concentrated at the interface, having the form

$$\operatorname{curl}(v) = \omega(z)\, dS(z),$$

where ω is tangent to S at the point z and $dS(z)$ is surface measure.

This paper extends to the three-dimensional case the results obtained in [Córdoba et al. 2011] for the case of two dimensions, by proving local existence in the scale of Sobolev spaces of the initial value problem if the Rayleigh–Taylor (R-T) condition is initially satisfied (see [Saffman and Taylor 1958], where this issue is studied from a physical point of view). In our case, that condition amounts to the positivity of the function

$$\sigma = (\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1)$$

at the interface S. The R-T property also appears in other fluid interface problems, such as water waves [Cordoba et al. 2009].

Together with that hypothesis, one also assumes that the initial surface S is connected and simply connected. In the presence of a global parametrization $X : \mathbb{R}^2 \to S$, the preservation of that character will be controlled by the gauge

$$F(X)(\alpha,\beta)=\frac{|\alpha-\beta|}{|X(\alpha)-X(\beta)|},\qquad \|F(X)\|_{L^\infty}=\sup_{\alpha\neq\beta}\frac{|\alpha-\beta|}{|X(\alpha)-X(\beta)|}<\infty.$$

Section 2 of this paper contains the derivation of the evolution equations for the interface S. In Section 3, we prove the existence of global isothermal parametrization as a consequence of the Koebe–Poincaré uniformization theorem of Riemann surfaces in the geometric scenarios considered in our work, namely, double periodicity in the horizontal variables and asymptotic flatness. Let us add that given the nonlocal character of the operator involved, to obtain a global isothermal parametrization is an important step in the proof, whose main components are sketched in Section 4.

In closing our system (Section 2), we need to control the norm of the inverse operator $(I+\lambda\mathcal{D})^{-1}$, where $\mathcal{D}$ is the double-layer potential and $|\lambda|\leq 1$. It is well-known from Fredholm's theory that those operators are bounded on $L^2(S)$. However, since the surface $S=S(t)$ is moving, a precise control of its norm is needed in order to proceed with our proof. That is the purpose of Section 5, where the estimates for the double-layer potential are revisited.

In Sections 6 and 7, we develop the energy estimates needed to conclude local existence. Let us mention that at a crucial point (more precisely, just at that step where the positivity of $\sigma(\alpha,t)$ (R-T) plays its role), we use the pointwise estimate $\theta(x)\Lambda\theta(x)\geq\frac{1}{2}\Lambda\theta^2(x)$ of [Córdoba and Córdoba 2003], with $\Lambda=\sqrt{-\Delta}$.

In the strategy of our proof, it is crucial to analyze the evolution of both quantities σ and F (Section 8) at the same time as the interface X and vorticity ω. There are several publications (see, for example, [Ambrose 2007]) where different authors have treated these problems assuming that the Rayleigh–Taylor condition is preserved during the evolution. Under such a hypothesis the proof can be considerably simplified, especially if one also assumes the appropriate bounds for the resolvent of the double-layer potential with respect to a moving domain, or the existence of global isothermal coordinates, etc. It is our purpose to carefully go over such items, which are responsible for the more delicate and intricate parts of this paper.

2. The contour equation

We consider the following evolution problem for the active scalars $\rho=\rho(x,t)$ and $\mu=\mu(x,t)$, with $x\in\mathbb{R}^3$ and $t\geq 0$:

$$\rho_t+v\cdot\nabla\rho=0,$$
$$\mu_t+v\cdot\nabla\mu=0,$$

with a velocity $v=(v_1,v_2,v_3)$ satisfying the momentum equation

$$\mu v=-\nabla p-(0,0,\ \rho) \tag{2-1}$$

and the incompressibility condition $\nabla\cdot v=0$, where, without loss of generality, we have prescribed the values $\kappa=g=1$.

The vector (μ, ρ) is defined by

$$(\mu, \rho)(x_1, x_2, x_3, t) = \begin{cases} (\mu^1, \rho^1) & x \in D^1(t), \\ (\mu^2, \rho^2) & x \in D^2(t) = \mathbb{R}^3 \setminus D^1(t), \end{cases}$$

where $\mu^1 \neq \mu^2$ and $\rho^1 \neq \rho^2$. Darcy's law (2-1) implies that the fluid is irrotational in the interior of each domain D^j, and because of the jump of densities and viscosities on the free boundary, we may assume a velocity field such that

$$\operatorname{curl} v = \omega(\alpha, t)\delta(x - X(\alpha, t)),$$

where $\partial D^j(t) = \{X(\alpha, t) \in \mathbb{R}^3 : \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2\}$; that is,

$$\langle \operatorname{curl} v, \varphi \rangle = \int_{\mathbb{R}^2} \omega(\alpha, t) \cdot \varphi(X(\alpha, t))\, d\alpha, \tag{2-2}$$

for any $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ vector field in $C_c^\infty(\mathbb{R}^3)$.

The incompressibility hypothesis ($\langle \nabla \cdot v, \varphi \rangle \equiv -\langle v, \nabla\varphi \rangle = 0$, for any $\varphi \in C_c^\infty(\mathbb{R}^3)$), yields

$$v^1(X(\alpha, t), t) \cdot N(\alpha, t) = v^2(X(\alpha, t), t) \cdot N(\alpha, t),$$

with $N(\alpha, t) = \partial_{\alpha_1} X(\alpha, t) \wedge \partial_{\alpha_2} X(\alpha, t)$, and Equation (2-2) gives us the identity

$$\omega(\alpha, t) = \big(v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)\big) \wedge N(\alpha, t).$$

Defining the potential ϕ by $v(x, t) = \nabla\phi(x, t)$ for $x \in \mathbb{R}^2 \setminus \partial D^j(t)$, we get

$$\begin{aligned} \Omega(\alpha, t) &= \phi^2(X(\alpha, t), t) - \phi^1(X(\alpha, t), t), \\ \partial_{\alpha_1}\Omega(\alpha, t) &= \big(v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)\big) \cdot \partial_{\alpha_1} X, \\ \partial_{\alpha_2}\Omega(\alpha, t) &= \big(v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)\big) \cdot \partial_{\alpha_2} X. \end{aligned}$$

Then one has the equality

$$\omega(\alpha, t) = \big(v^2(X(\alpha, t), t) - v^1(X(\alpha, t), t)\big) \wedge \big(\partial_{\alpha_1} X(\alpha, t) \wedge \partial_{\alpha_2} X(\alpha, t)\big),$$

and therefore

$$\omega(\alpha, t) = \partial_{\alpha_2}\Omega(\alpha, t)\partial_{\alpha_1} X(\alpha, t) - \partial_{\alpha_1}\Omega(\alpha, t)\partial_{\alpha_2} X(\alpha, t), \tag{2-3}$$

implying that $\nabla \cdot \operatorname{curl} v = 0$ in a weak sense.

Using the law of Biot–Savart, we have for x not lying in the free surface ($x \neq X(\alpha, t)$) the following expression for the velocity:

$$v(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{x - X(\beta, t)}{|x - X(\beta, t)|^3} \wedge \omega(\beta)\, d\beta.$$

It follows that

$$X_t(\alpha) = \mathrm{BR}(X, \omega)(\alpha, t) + C_1(\alpha)\partial_{\alpha_1} X(\alpha) + C_2(\alpha)\partial_{\alpha_2} X(\alpha), \tag{2-4}$$

where BR is the well-known Birkhoff–Rott integral:

$$\mathrm{BR}(X, \omega)(\alpha, t) = -\frac{1}{4\pi}\, \mathrm{PV} \int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \omega(\beta)\, d\beta. \tag{2-5}$$

Next we will close the system using Darcy's law. Since

$$\nabla\phi = v(x,t) - \Omega(\alpha,t)N(\alpha,t)\delta(x - X(\alpha,t)),$$

we have

$$\langle\Delta\phi,\varphi\rangle = -\langle\nabla\phi,\nabla\varphi\rangle = \int_{\mathbb{R}^2}\Omega(\alpha,t)N(\alpha,t)\cdot\nabla\varphi(X(\alpha,t))\,d\alpha,$$

and taking $\varphi(y) = -1/(4\pi|x-y|)$, one obtains ϕ in terms of the double layer potential:

$$\phi(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{x - X(\alpha)}{|x - X(\alpha)|^3}\cdot N(\alpha)\Omega(\alpha)\,d\alpha.$$

Darcy's law yields

$$\Delta p(x,t) = -\operatorname{div}(\mu(x,t)v(x,t)) - \partial_{x_3}\rho(x,t),$$

that is,

$$\Delta p(x,t) = P(\alpha,t)\delta(x - X(\alpha,t)),$$

where $P(\alpha,t)$ is given by

$$P(\alpha,t) = (\mu^2-\mu^1)v(X(\alpha,t),t)\cdot N(\alpha,t) + (\rho^2-\rho^1)N_3(\alpha,t),$$

implying the continuity of the pressure at the free boundary.

Next, if $x \neq X(\alpha,t)$, i.e., x is not placed at the interface, we can write Darcy's law in the form

$$\mu\phi(x,t) = -p(x,t) - \rho x_3,$$

and taking limits in both domains D^j, we get at S the equality

$$\big(\mu^2\phi^2(X(\alpha,t),t) - \mu^1\phi^1(X(\alpha,t),t)\big) = -(\rho^2-\rho^1)X_3(\alpha,t).$$

Then the formula for the double-layer potential gives

$$\frac{\mu^2+\mu^1}{2}\Omega(\alpha,t) - (\mu^2-\mu^1)\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\Omega(\beta)\,d\beta = -(\rho^2-\rho^1)X_3(\alpha,t),$$

that is,

$$\Omega(\alpha,t) - A_\mu\mathcal{D}(\Omega)(\alpha,t) = -2A_\rho X_3(\alpha,t), \tag{2-6}$$

where

$$\mathcal{D}(\Omega)(\alpha) = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\Omega(\beta)\,d\beta, \quad A_\mu = \frac{\mu^2-\mu^1}{\mu^2+\mu^1}, \quad A_\rho = \frac{\rho^2-\rho^1}{\mu^2+\mu^1}. \tag{2-7}$$

The evolution equations are then given by (2-3)–(2-7), where the functions C_1 and C_2 will be chosen in the next section.

Furthermore, taking limits, we get from Darcy's law the following two formulas:

$$\partial_{\alpha_1}\Omega(\alpha,t) + 2A_\mu\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot\partial_{\alpha_1}X(\alpha,t) = -2A_\rho\partial_{\alpha_1}X_3(\alpha,t), \tag{2-8}$$

$$\partial_{\alpha_2}\Omega(\alpha,t) + 2A_\mu\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot\partial_{\alpha_2}X(\alpha,t) = -2A_\rho\partial_{\alpha_2}X_3(\alpha,t). \tag{2-9}$$

3. Isothermal parametrization: choosing the tangential terms

Although the normal component of the velocity vector field is the relevant one in the evolution of the interface, it is however very important to choose an adequate parametrization in order to uncover and handle properly the cancellations contained in the equations of motion. Fortunately for our task, we can rely upon the ideas of H. Lewy [1951], and many other authors, who discovered the convenience of using isothermal coordinates in different PDEs for understanding how a minimal surface leaves an obstacle and also in several fluid mechanical problems.

Let us recall that an isothermal parametrization must satisfy

$$|X_{\alpha_1}(\alpha,t)|^2=|X_{\alpha_2}(\alpha,t)|^2,\qquad X_{\alpha_1}(\alpha,t)\cdot X_{\alpha_2}(\alpha,t)=0,$$

for $t\geq 0$.

Next we define

$$C_1(\alpha)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\alpha_1-\beta_1}{|\alpha-\beta|^2}\frac{\mathrm{BR}_{\beta_2}\cdot X_{\beta_2}-\mathrm{BR}_{\beta_1}\cdot X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\alpha_2-\beta_2}{|\alpha-\beta|^2}\frac{\mathrm{BR}_{\beta_1}\cdot X_{\beta_2}+\mathrm{BR}_{\beta_2}\cdot X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta \tag{3-1}$$

and

$$C_2(\alpha)=-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\alpha_2-\beta_2}{|\alpha-\beta|^2}\frac{\mathrm{BR}_{\beta_2}\cdot X_{\beta_2}-\mathrm{BR}_{\beta_1}\cdot X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\alpha_1-\beta_1}{|\alpha-\beta|^2}\frac{\mathrm{BR}_{\beta_1}\cdot X_{\beta_2}+\mathrm{BR}_{\beta_2}\cdot X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta. \tag{3-2}$$

That is, $X_t=\mathrm{BR}+C_1X_{\alpha_1}+C_2X_{\alpha_2}$ and

$$X_{\alpha_1 t}=\mathrm{BR}_{\alpha_1}+C_1X_{\alpha_1\alpha_1}+C_2X_{\alpha_1\alpha_2}+C_{1\alpha_1}X_{\alpha_1}+C_{2\alpha_1}X_{\alpha_2},$$
$$X_{\alpha_2 t}=\mathrm{BR}_{\alpha_2}+C_1X_{\alpha_1\alpha_2}+C_2X_{\alpha_2\alpha_2}+C_{1\alpha_2}X_{\alpha_1}+C_{2\alpha_2}X_{\alpha_2}.$$

Writing $f=\big(|X_{\alpha_1}|^2-|X_{\alpha_2}|^2\big)/2$ and $g=X_{\alpha_1}\cdot X_{\alpha_2}$, we have

$$f_t=\big(\mathrm{BR}_{\alpha_1}\cdot X_{\alpha_1}-\mathrm{BR}_{\alpha_2}\cdot X_{\alpha_2}\big)+C_1f_{\alpha_1}+C_2f_{\alpha_2}+(C_{2\alpha_1}-C_{1\alpha_2})g+2C_{1\alpha_1}f+(C_{1\alpha_1}-C_{2\alpha_2})|X_{\alpha_2}|^2.$$

The expressions for C_1 and C_2 yield the vanishing of the sum of the first and the last terms in the identity above. Therefore, we get

$$f_t=C_1f_{\alpha_1}+C_2f_{\alpha_2}+(C_{2\alpha_1}-C_{1\alpha_2})g+2C_{1\alpha_1}f. \tag{3-3}$$

Similarly, we have

$$g_t=\big(\mathrm{BR}_{\alpha_2}\cdot X_{\alpha_1}+\mathrm{BR}_{\alpha_1}\cdot X_{\alpha_2}\big)+C_1g_{\alpha_1}+C_2g_{\alpha_2}+(C_{1\alpha_1}+C_{2\alpha_2})g-2C_{2\alpha_1}f+(C_{1\alpha_2}+C_{2\alpha_1})|X_{\alpha_1}|^2$$

and

$$g_t=C_1g_{\alpha_1}+C_2g_{\alpha_2}+(C_{1\alpha_1}+C_{2\alpha_2})g-2C_{2\alpha_1}f. \tag{3-4}$$

The linear character of equations (3-3) and (3-4) allows us to conclude that if there is a solution of the system $X_t=\mathrm{BR}+C_1X_{\alpha_1}+C_2X_{\alpha_2}$ and we start with isothermal coordinates at time $t=0$, then they will continue to be isothermal so long as the evolution equations provide us with a smooth enough interface.

The fact that one can always prescribe such coordinates at time $t = 0$ follows from the following argument: in the double periodic setting we have a C^2 simply connected surface, homeomorphic to the euclidean plane $\mathbb{R}^2$, which, by the Riemann–Koebe–Poincaré uniformization theorem, is conformally equivalent to either the Riemann sphere, the plane, or the unit disc. The sphere is easily eliminated by compactness, but we can also rule out the unit disc because the assumption of double periodicity in the horizontal variables implies the existence of a discrete abelian subgroup of rank two in the group of conformal transformations, and that cannot happen in the case of the unit disc.

Therefore, we have an orientation-preserving conformal (isothermal) equivalence

$$\phi : \mathbb{R}^2 \longrightarrow S.$$

Since S is invariant under translations $\tau_\nu(x) = x + 2\pi\nu$, where $\nu \in \mathbb{Z}^2 \times \{0\}$, it follows that $f_\nu(z) = \phi^{-1} \circ \tau_\nu \circ \phi(z)$ must be a diffeoholomorphism of $\mathbb{C} = \mathbb{R}^2$, and therefore it has to be of the form

$$f_\nu(z) = a_\nu z + b_\nu,$$

for certain $a_\nu, b_\nu \in \mathbb{C}$. Clearly, the family f_ν is generated by $f_1 = f_{(1,0,0)}$, $f_2 = f_{(0,1,0)}$. Let

$$f_1(z) = a_1 z + b_1, \quad f_2(z) = a_2 z + b_2.$$

We claim that $a_1 = a_2 = 1$. Suppose that $|a_1| < 1$; then we get $f_1^n(z) = a_1^n z + b_1(1 + a_1 + \cdots + a_1^{n-1})$, a sequence converging to $b_1/(1 - a_1)$, contradicting the discrete character of the group action. On the other hand, if $|a_1| > 1$, then since

$$f_1^{-1}(z) = f_{(-1,0,0)}(z) = \frac{z}{a_1} - \frac{b_1}{a_1},$$

we get a contradiction with the sequence $f_1^{-n}(z)$. Therefore, we must have $a_1 = e^{2\pi i\theta}$ for some $0 \le \theta < 1$. Assume that $0 < \theta < 1$; then

$$f_1^{(n)}(z) = e^{2\pi i n\theta} z + b_1\big(1 + e^{2\pi i\theta} + \cdots + e^{2\pi i(n-1)\theta}\big) = e^{2\pi i n\theta} z + b_1 \frac{1 - e^{2\pi i n\theta}}{1 - e^{2\pi i\theta}},$$

so the sequence $f^n(z)$ is bounded and satisfies $|f^n(z)| \le |z| + |b_1|/\sin\pi\theta$. Therefore it contains a converging subsequence, again contradicting discreteness. It follows that $f_1(z) = z + b_1$ and, similarly, $f_2(z) = z + b_2$, which leads easily to the double periodicity of the isothermal parametrization ϕ.

In the asymptotically flat case, we start with an orientable simply connected surface S that, outside a ball B in $\mathbb{R}^3$, is the graph of a C^2-function $x_3 = \varphi(x_1, x_2)$ such that $|D^\alpha\varphi(x)| = o(|x|^{-N})$ for every N and $|\alpha| \le 2$. In particular, the normal vector $\nu(x) = (-\nabla\varphi, 1)/\sqrt{1 + |\nabla\varphi|^2}$ is roughly vertical and $1/\sqrt{1 + |\nabla\varphi|^2}$ is close to 1 for $|x|$ big enough.

Then one can find isothermal coordinates whose first fundamental form $\lambda(\alpha, \beta)(d\alpha^2 + d\beta^2)$ converges asymptotically to the identity.

Again by the uniformization theorem, S must be conformally equivalent to either $\mathbb{C}$ or the unit disc. But since outside B, the surface S is conformally equivalent to $\mathbb{C} - B \cap \{x_3 = 0\}$, it cannot be also conformally equivalent to $D - K$, for any regular compact set K contained in the unit disc D, because the harmonic measure of the ideal boundary is 1 in the case of D and 0 for $\mathbb{R}^2$.

4. Main theorem and outline of the proof

The proof of local existence requires the following:

(1) A connected and simply connected surface $S = S(t)$ parametrized by isothermal coordinates

$$X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \qquad X = X(\alpha, t),$$

with normal vector $N(\alpha, t) = X_{\alpha_1} \wedge X_{\alpha_2}$ and gauge

$$F(X)(\alpha, \beta) = \frac{|\beta|}{|X(\alpha) - X(\alpha - \beta)|},$$

such that $\|F(X)\|_{L^\infty} < \infty$ and $\||N|^{-1}\|_{L^\infty} < \infty$.

(2) The positivity of

$$\begin{aligned} \sigma(\alpha, t) &= -\big(\nabla p^2(X(\alpha, t), t) - \nabla p^1(X(\alpha, t), t)\big) \cdot N(\alpha, t) \\ &= (\mu^2 - \mu^1)\,\mathrm{BR}(X, \omega)(\alpha, t) \cdot N(\alpha, t) + (\rho^2 - \rho^1) N_3(\alpha, t), \end{aligned} \tag{4-1}$$

where the last equality is a consequence of Darcy's law after taking limits in both domains D^j. This is the Rayleigh–Taylor condition to be imposed at time $t = 0$, it being a part of the problem to prove that it remains true as time passes.

(3) The estimates on the norm of $(I - \lambda \mathfrak{D})^{-1}$, $|\lambda| < 1$, $\mathfrak{D} =$ double-layer potential (see Section 5), allowing us to obtain the inequalities

$$\begin{aligned} \|\Omega\|_{H^{k+1}} &\le P\big(\|X\|^2_{k+1} + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big), \\ \|\omega\|_{H^k} &\le P\big(\|X\|^2_{k+1} + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big), \end{aligned}$$

for $k \ge 3$, where P is a polynomial function and the norm $\|\cdot\|_k$ is given by

$$\|X\|_k = \|X_1 - \alpha_1\|_{L^3} + \|X_2 - \alpha_2\|_{L^3} + \|X_3\|_{L^2} + \|\nabla(X - (\alpha, 0))\|^2_{H^{k-1}},$$

as in (7-1) below, and $\|\cdot\|_{H^j}$ denotes the norm in the Sobolev space H^j.

(4) A control of the Birkhoff–Rott integral $\mathrm{BR}(X, \omega)$:

$$\|\mathrm{BR}(X, \omega)\|_{H^k} \le P\big(\|X\|^2_{k+1} + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big),$$

for $k \ge 3$.

(5) Energy estimates: the properties of isothermal parametrizations help us to reorganize the terms in such a way that

$$\begin{aligned} \frac{d}{dt}\|X\|^2_k(t) &\le P\big(\|X\|^2_k(t) + \|F(X)\|^2_{L^\infty}(t) + \||N|^{-1}\|_{L^\infty}(t)\big) \\ &\quad - \sum_{i=1,2} \frac{2^{3/2}}{(\mu_1 + \mu_2)} \int_{\mathbb{R}^2} \frac{\sigma(\alpha, t)}{|\nabla X(\alpha, t)|^3}\, \partial^k_{\alpha_i} X(\alpha, t) \cdot \Lambda(\partial^k_{\alpha_i} X)(\alpha, t)\, d\alpha, \end{aligned}$$

where $k \geq 4$, $|\nabla X(\alpha)|^3 = \big(|\partial_{\alpha_1} X(\alpha)|^2 + |\partial_{\alpha_2} X(\alpha)|^2\big)^{3/2}$, and $\Lambda = (-\Delta)^{1/2} = R_1(\partial_{\alpha_1}) + R_2(\partial_{\alpha_2})$. Then the pointwise inequality

$$\theta\Lambda(\theta) - \tfrac{1}{2}\Lambda(\theta^2) \geq 0,$$

together with the condition $\sigma > 0$, allows us to get rid of the dangerous terms in the inequality above (those involving $(k+1)$-derivatives of X) to obtain the estimate

$$\frac{d}{dt}\|X\|_k^2(t) \leq P\big(\|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

(6) Finally, we need to control the evolution of $\|F(X)\|_{L^\infty}(t)$ and $\inf(t) = \inf_{\alpha\in\mathbb{R}^2} \sigma(\alpha, t)$, which is obtained via the estimates

$$\frac{d}{dt}\|F(X)\|_{L^\infty}^2(t) \leq P\big(\|X\|_4^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big),$$
$$\frac{d}{dt}\frac{1}{\inf(t)} \leq \frac{1}{\inf(t)^2} P\big(\|X\|_4^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

(7) All those facts together yield the inequality

$$\frac{d}{dt}E(t) \leq CP(E(t))$$

for the energy

$$E(t) = \|X\|_k^2(t) + \|F(X)\|_{L^\infty}^2(t) + \||N|^{-1}\|_{L^\infty}(t) + \inf(t)^{-1},$$

where $k \geq 4$, C is a universal constant, and P has polynomial growth (depending upon k).

At this point it is not difficult to prove the existence of a solution, locally in time, so long as the initial data $X(0)$ is in the appropriate Sobolev space of order $k \geq 4$, and the Rayleigh–Taylor and no-self-intersection conditions ($\sigma_0 > c > 0$, $\|F(X(0))\|_{L^\infty} < \infty$) are satisfied.

The main theorem presented in this paper is the following:

Theorem 4.1. *Let $X(0)$ with $\|X(0)\|_k < \infty$ for $k \geq 4$, $\|F(X(0))\|_{L^\infty} < \infty$, $\||N(\alpha, 0)|^{-1}\|_{L^\infty} < \infty$, and*

$$\sigma(\alpha, 0) = -\big(\nabla p^2(X(0), 0) - \nabla p^1(X(0), 0)\big)\cdot N(\alpha, 0) > 0.$$

Then there exists a time $\tau > 0$ such that there is a solution to (2-3), (2-4), (2-6) *in* $C([0, \tau]; H^k)$ *with* $X(\alpha, 0) = X(0)$.

Finally, let us point out that since our existence proof is based upon energy inequalities, an extra argument is needed to prove uniqueness. Nevertheless, that task is much easier than proving existence. (The interested reader may consult [Córdoba et al. ≥ 2013], where the details of the proof have been written out for some important cases, such as Muskat and SQG patches.)

Let us remark that, at the end, we have to work with a coupled system involving the evolution of the surface X, the "vorticity density" ω, the Rayleigh–Taylor condition σ, the non-self-intersecting character of S quantified by the gauge $F(X)$, and the tangential parts $C_1 X_{\alpha_1} + C_2 X_{\alpha_2}$ of the velocity field.

Remark. This paper is a continuation of [Córdoba et al. 2011], where the two-dimensional case was considered. Many of the needed estimates can be obtained following exactly the same methods that were used in [Córdoba et al. 2011] for the lower-dimensional case. Therefore, in order to simplify our presentation, we shall avoid here many details which were carefully proven there. This is especially the case in Section 6 (control of the Birkhoff–Rott integral) and Section 8 (energy estimates), and also for the approximation schemes which are identical to those developed in [Córdoba et al. 2011]. Therefore, in the following, we shall focus our attention on the more innovative parts of the proof, namely the evolution of the Rayleigh–Taylor condition, the non-self-intersecting property of the free boundary, and the needed estimates for double-layer potentials.

5. Inverting the operator: the single- and double-layer potentials revisited

In this proof, we need to consider the properties of single- and double-layer potentials, which are well-known characters in finding solutions to the Dirichlet and Neumann problems in domains D of $\mathbb{R}^n$.

For our purposes, these domains will be of three different types, namely: bounded, periodic in the "horizontal" variables, and asymptotically flat. We shall also assume that their boundaries are smooth enough (say C^2) and do not present self-intersections. Therefore, one has tangent balls at every point of the boundary, one completely contained in D and the other in D^c. We shall denote by $\nu(x)$ the unit inner normal at the point $x \in \partial D$; then under our hypothesis we have that, for $r > 0$ small enough, the parallel surfaces $\partial D_r = \{x + r\nu(x) \mid x \in \partial D\}$ are also C^2 surfaces with curvatures controlled by those of ∂D. Furthermore, the vector field ν can be extended smoothly up to a collar neighborhood of ∂D, allowing us to write the formula

$$\Delta u(x) = \frac{\partial^2 u}{\partial \nu^2}(x) - h(x)\frac{\partial u}{\partial \nu}(x) + \Delta_s u(x),$$

where Δ denotes the ordinary laplacian in $\mathbb{R}^n$, Δ_s is the Laplace–Beltrami operator in ∂D, $h(x)$ is the mean curvature of ∂D at the point x, and u is any C^2-function defined in a neighborhood of ∂D.

For convenience, we will use the notation $D_1 = D$, $D_2 = D^c$, $S = \partial D_j$, and $\nu_j(x)$ (for $j = 1, 2$) the inner normal at $x \in S$ pointing inside D_j. Let dS be the surface measure in S induced by Lebesgue measure in ambient space. Given integrable functions φ, ψ on S, we call

$$V(x) = c_n \int_S \psi(y) \frac{1}{\|x-y\|^{n-2}}\, dS(y)$$

the single-layer potential of ψ, and we call

$$W(x) = c_n \int_S \varphi(y) \frac{\partial}{\partial \nu_x}\left(\frac{1}{\|x-y\|^{n-2}}\right) dS(y)$$

the double-layer potential of φ. In both cases, c_n is a normalizing constant chosen so that $\frac{c_n}{\|x\|^{n-2}}$ is a fundamental solution of Δ in $\mathbb{R}^n$, $n \geq 3$.

For $x \in S$ and $j = 1, 2$, denote by $W_j(x)$ and $V_j(x)$ the corresponding limits of the potentials in D_j. We have

$$W_1(x) = \frac{1}{2}\left(\varphi(x) - \int_S \varphi(y)K(x, y)\,d\sigma(y)\right) = \frac{1}{2}(\varphi(x) - \mathcal{D}\varphi(x)),$$

$$W_2(x) = \frac{1}{2}\left(\varphi(x) + \int_S \varphi(y)K(x, y)\,d\sigma(y)\right) = \frac{1}{2}(\varphi(x) + \mathcal{D}\varphi(x)),$$

$$\frac{\partial V}{\partial \nu_1}(x) = -\frac{1}{2}\left(\psi(x) + \int_S \psi(y)K(y, x)\,d\sigma(y)\right) = -\frac{1}{2}(\psi(x) + \mathcal{D}^*\psi(x)),$$

$$\frac{\partial V}{\partial \nu_2}(x) = -\frac{1}{2}\left(\psi(x) - \int_S \psi(y)K(y, x)\,d\sigma(y)\right) = -\frac{1}{2}(\psi(x) - \mathcal{D}^*\psi(x)),$$

where

$$K(x, y) = 2c_n \frac{\partial}{\partial \nu_y}\left(\frac{1}{\|x-y\|^{n-2}}\right) = \tilde{c}_n \frac{\langle x - y, \nu(y)\rangle}{|x - y|^n}.$$

It is well-known that in the scenarios considered above, the boundary operators $\mathcal{D}$ (and $\mathcal{D}^*$) are smoothing of order -1, and therefore compact. Furthermore, all their eigenvalues are real numbers having absolute value strictly less than 1. Therefore, by the standard Fredholm theory, the operators $I - \lambda\mathcal{D}$, $I - \lambda\mathcal{D}^*$ are invertible when $|\lambda| \leq 1$. However, in our case, the domains are moving, and the evolution of their common boundary S involves the inverse operators, making it necessary to estimate their norms in terms of the geometry and smoothness of S.

Although there is a vast literature about single- and double-layer potentials, we have not been able to point out a precise statement giving the information needed for our results. Therefore, in this section, we provide arguments to prove that the norms of such inverse operators grow at most polynomially: $P(|||S|||)$, where $|||S|||$ is just $\|S\|_{C^2}$ plus a term of chord-arc type controlling the non-self-intersecting character of the boundary. The term has the form $r(S)^{-1}$, where $r(S)$ is the sup over all the positive r such that S admits tangent balls of radius r in both domains D_j:

$$|||S||| = \|S\|_{C^2} + (r(S))^{-1}.$$

We shall write $P(|||S|||)$ to denote $\leq C(|||S|||^p)$ for certain positive constants C, p which are independent of the characters whose evolution is being controlled, but the size of both constants may change during the proof and we shall make no effort to obtain their best values.

We will consider the case of bounded domains in $\mathbb{R}^n$, $n \geq 3$, because the needed modifications when $n = 2$, namely taking $\log|x|$ as fundamental solution for the laplacian, as well as the changes for the periodic or asymptotically flat domains, are left to the reader.

Let $\mathcal{D}$ and $\mathcal{D}^*$ be the potential defined above, with kernel

$$K(x, y) = c_n \frac{\partial}{\partial \nu(y)} \frac{1}{\|x-y\|^{n-2}} = c_n \frac{\langle x - y, \nu(y)\rangle}{|x - y|^n}$$

and $K(y, x)$ respectively. In the study of the inverse operators $(I - \lambda\mathcal{D})^{-1}$, $|\lambda| \leq 1$, it is convenient to consider first the particular values $\lambda = \pm 1$.

Proposition 5.1. *The following estimate holds, where P is a polynomial function*:

$$\|(I\pm\mathfrak{D})^{-1}\|_{L^2(S)} = P(|||S|||).$$

Since the boundedness of $(I\pm\mathfrak{D})^{-1}$ in $L^2(S)$ is well-known from the general theory, we can simplify the proof, considering only functions $f\in L^2(S)$ whose support lies inside a region of S where the normal $\nu(x)$ is close enough to a fixed direction. Then for a general f, an appropriate partition of unity would allow us to add the local estimates, so long as the number of pieces is controlled by $|||S|||$. We shall use the following observation, whose proof is immediate.

Lemma 5.2 (Rellich). *Let u be a harmonic function and h a smooth vector field in the domain D; then we have*

(i) $\operatorname{div}(|\nabla u|^2 h) = 2\operatorname{div}((\nabla u\cdot h)\nabla u) + O(|\nabla u|^2|\nabla h|)$,

(ii) $\int_{\partial D}\langle \nu, h\rangle|\nabla u|^2 d\sigma = 2\int_{\partial D}(\partial u/\partial\nu)(\nabla u\cdot h)d\sigma + O\big(\int_D |\nabla u|^2|\nabla h|\big)$.

Given a function $f\in C^1(S)$, we may define $\nabla_\tau f$, choosing at each point $x\in S$ an orthonormal basis $\{e_1,\dots,e_{n-1}\}$ of the tangent space $T_x(S)$ (we can consider also $\nabla_\tau f$ to be the gradient naturally associated to the induced Riemannian metric by the ambient space). In both ways, although different, we have that $|\nabla_\tau f|\equiv\Lambda_\tau f$ is an elliptic pseudodifferential operator of order 1 in S. Solving the Dirichlet problem $\Delta u = 0$ in D, $u|_S = f$, we obtain the operator $D_\nu\equiv(\partial u/\partial\nu)|_S$, which is also a pseudodifferential operator of order 1 in S.

Lemma 5.3. *Let $f\in L^2(S)$ having support on the region $\frac{1}{2}\le\langle\nu(x),\eta\rangle\le 1$ (for a fixed unit vector η); then we have*

$$\int_S |D_\nu f|^2 d\sigma \simeq \int_S |\nabla_\tau f|^2 d\sigma,$$

where the constants involved in the stated equivalence $\simeq$ are $P(|||S|||)$.

Proof. Let u be harmonic in D so that $u|_S = f$. Under our hypothesis about f, and since $|\nabla u|^2 = |D_\nu u|^2 + |\nabla_\tau u|^2$ and $\nabla_\tau u$ is a local operator ($\operatorname{supp}_S(\nabla_\tau f)\subset\operatorname{supp}(f)$), Lemma 5.2 yields:

$$\frac{1}{2}\int_S |\nabla_\tau f|^2 d\sigma \le \int_S \langle\nu(x),\eta\rangle|\nabla_\tau u|^2 d\sigma \le 3\int_S |D_\nu u|^2 d\sigma + 2\int_S |\nabla_\tau u||D_\nu u| d\sigma,$$

from which we easily obtain

$$\int_S |\nabla_\tau f|^2 d\sigma \le P(|||S|||)\int_S |D_\nu f|^2 d\sigma.$$

To get the opposite inequality we proceed as before, but since $D_\nu f$ is not local, an extra argument is needed to control the contribution of the region outside $\operatorname{supp}(f)$. Let us introduce surface discs $B_r(x) = \{y\in S \mid \|x-y\|\le r\}$, $x\in S$, $0\le r\le |||S|||^{-1}$ and domains $\Delta_r(x) = \{y+\rho\nu(x) \mid y\in B_r(x),\ \rho\le r\}$. Given $R = \frac{1}{2}|||S|||^{-1}$, there exists a fixed unit vector η so that $\frac{1}{2}\le\langle\nu(y),\eta\rangle\le 1$ for every $y\in B_R(x)$, and also a smooth vector field h such that $h|_{\Delta_R(x)}\equiv\eta$, $\operatorname{supp}(h)\subset\Delta_{2R}(x)$, and $\frac{1}{2}|h(x)|\le\langle h(x),\nu(x)\rangle$, $\|\nabla h\|^2\le P(|||S|||)\|h\|$.

In order to obtain the estimate

$$\int_S |D_\nu f|^2 d\sigma \le P(|||S|||)\int_S |\nabla_\tau f|^2 d\sigma,$$

we may assume, without loss of generality, that $\mathrm{supp}(f)\subset B_R(x)$, for some $x\in S$, and then prove that

$$\int_{B_R(y_0)} |D_\nu f|^2 d\sigma \le P(|||S|||)\int_S |\nabla_\tau f|^2 d\sigma$$

uniformly on $y_0\in S$.

With the vector field h defined above in $\Delta_{2R}(y)$, let us apply Rellich's estimate to get

$$\int_S |D_\nu f|^2 \langle h,\nu(x)\rangle\, d\sigma(x) = \int_S \langle \nu,h\rangle |\nabla_\tau f|^2 d\sigma - 2\int_S D_\nu f \nabla_\tau f\cdot h\, d\sigma + O\Big(\int_D |\nabla u|^2|\nabla h|\Big),$$

where u satisfies $\Delta u=0$ in D, $u|_S=f$. We get easily

$$\int_{B_R(y_0)} |D_\nu f|^2 \langle h,\nu(x)\rangle\, d\sigma(x) = O\Big(\int_S |\nabla_\tau f|^2 d\sigma + \int_D |\nabla u|^2|\nabla h|\, dx\Big).$$

Then the proof will be finished if we can show that

$$\int_D |\nabla u|^2|\nabla h|\, dx \le P(|||S|||)\int_S |\nabla_\tau f|^2 d\sigma.$$

To see this, let us consider the parallel surfaces $S_r=\{x+r\nu(x)\mid x\in S\}$ $(0\le r\le |||S|||)$ and observe that

$$\int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x+r\nu(x))\, d\sigma$$

and

$$\begin{aligned}\int_S \big[u^2(x+r\nu(x))-u^2(x)\big]\, d\sigma(x) &= \int_S\int_0^r \nabla u^2(x+t\nu(x))\cdot\nu(x)\, dt\, d\sigma\\ &= 2\int_{L_r} u(y)\nabla u(y)\cdot\nu(y) \le 2\Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_{L_r} |\nabla u|^2(y)\Big)^{1/2},\end{aligned}$$

where $L_r=\{x+\rho\nu(x)\mid x\in S, 0\le\rho\le r\}$.

Let $\mathscr{X}$ be a smooth cut-off function. Taking

$$F(x+r\nu(x)) = f(x)\mathscr{X}(x),$$

as a comparison function, Dirichlet's principle and Poincaré's inequality give us the estimate

$$\int_D |\nabla u|^2 \le \int_D |\nabla F|^2 \le C\Big(\int_S |\nabla_\tau f|^2 + \int_S |f|^2\Big) = O\Big(\int_S |\nabla_\tau f|^2 d\sigma\Big).$$

Therefore

$$\int_{S_r} u^2 d\sigma_r \simeq \int_S u^2(x+r\nu(x))\, d\sigma \le \int_S f^2(x)\, d\sigma + \Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_S |\nabla_\tau f|^2\Big)^{1/2}.$$

Integration in r in the range $0 \le r \le R = |||S|||^{-1}$ yields

$$\int_{L_r} u^2 dx \le R\Big(\int_S f^2(x)\, d\sigma + \Big(\int_{L_r} u^2(y)\Big)^{1/2}\Big(\int_S |\nabla_\tau f|^2\Big)^{1/2}\Big).$$

That is,

$$\int_{L_r} u^2 dx \le CR \int_S |\nabla_\tau f|^2\, d\sigma.$$

To conclude, let us observe that

$$\begin{aligned}\int_D |\nabla u|^2 |\nabla h| &= \frac{1}{2}\int_D \Delta u^2 |\nabla h| = \frac{1}{2}\int_D \big(\Delta u^2 |\nabla h| - u^2 \Delta(|\nabla h|)\big) + \frac{1}{2}\int_D u^2(|\nabla h|)\\ &= \frac{1}{2}\int_S u \frac{\partial u}{\partial \nu}\cdot |\nabla h|\, d\sigma - \frac{1}{2}\int_S f^2 \frac{(|\nabla h|)}{\partial \nu}\, d\sigma + \frac{1}{2}\int_D u^2 \nabla |h|\\ &\le \Big(\int_S f^2 d\sigma\Big)^{1/2}\Big(\int \Big|\frac{\partial u}{\partial \nu}\Big|^2 |\nabla h|^2 d\sigma\Big)^{1/2} + C\int_S f^2 d\sigma + C\int_{L_R} u^2. \qquad \square\end{aligned}$$

Proof of Proposition 5.1. As before, let $f \in C^1(S)$, $\text{supp}(f) \subset \text{U}_0$, and let u be its single-layer potential:

$$u(x) = c_n \int_S \frac{f(y)}{\|x-y\|^{n-2}}\, dS(y).$$

Taking derivatives on each domain D_j with respect to the normal direction and evaluating at S, we get

$$\frac{\partial u}{\partial \nu_1} = -\tfrac{1}{2}(f(x) + \mathcal{D}^* f(x)), \qquad \frac{\partial v}{\partial \nu_2} = -\tfrac{1}{2}(f(x) - \mathcal{D}^* f(x)).$$

By Lemma 5.3, we know that

$$\int_S \Big|\frac{\partial v}{\partial \nu_1}\Big|^2 d\sigma \simeq \int_S |\nabla_\tau v|^2 d\sigma \simeq \int_S \Big|\frac{\partial v}{\partial \nu_2}\Big|^2 d\sigma,$$

where the constants involved in the equivalences are all controlled by above by $P(|||S|||)$ and below by $1/P(|||S|||)$.

Since $\partial v/\partial \nu_1 + \partial v/\partial \nu_2 = -f$, these estimates imply that

$$\min\big(\|f - \mathcal{D}^* f\|_2, \|f + \mathcal{D}^* f\|_2\big) \ge \frac{1}{P(|||S|||)},$$

that is, $\|(I \pm \mathcal{D})^{-1}\| = P(|||S|||)$. Then using an appropriate partition of unity, that estimate extends to a general $f \in L^2(S)$. $\square$

Next we shall consider Sobolev spaces $H^s(S)$, $0 \le s \le 1$, defined in the usual manner throughout local coordinate charts. We have also the elliptic pseudodifferential operator $\Lambda^s = (-\Delta)^{s/2}$ in such a way that

$$\|f\|_{H^s(S)} \simeq \|f\|_{L^2} + \|\Lambda^s f\|_{L^2}.$$

Then $H^{-s}(S) \equiv (H^s(S))^*$ allows us to consider the negative case by duality, under the pairing

$$\int_S \phi\psi\, d\sigma, \quad \phi \in H^{-s},\ \psi \in H^s,$$

and we have

$$\|\phi\|_{H^{-s}} = \sup_{\|\psi\|_{H^s}=1} \int_S \phi\psi \, d\sigma.$$

Since both $\mathfrak{D}$ and $\mathfrak{D}^*$ are compact and smoothing operators of degree -1, the commutators $[\Lambda^s, \mathfrak{D}]$, $[\Lambda^s, \mathfrak{D}^*]$ are then bounded in $L^2(S)$ $(0 \le s \le 1)$ with norms controlled by $|||S|||$, allowing us to extend Proposition 5.1 to the chain of Sobolev spaces:

Corollary 5.4. *The norm of the operators* $(I \pm \mathfrak{D})^{-1}$, $(I \pm \mathfrak{D}^*)^{-1}$ *in the space* $H^s(S)$, $-1 \le s \le 1$, *is bounded by* $P(|||S|||)$.

Estimates for $\boldsymbol{(I + \lambda\mathfrak{D})^{-1}, |\lambda| \le 1}$**.** With the same notation used before, we have

$$\frac{1-\lambda}{2}\frac{\partial V}{\partial \nu_1} + \frac{1+\lambda}{2}\frac{\partial V}{\partial \nu_2} = -\tfrac{1}{2}(\phi(x) - \lambda\mathfrak{D}^*\phi(x)) \quad \text{and} \quad \frac{1+\lambda}{2}\frac{\partial V}{\partial \nu_1} + \frac{1-\lambda}{2}\frac{\partial V}{\partial \nu_2} = -\tfrac{1}{2}(\phi(x) + \lambda\mathfrak{D}^*\phi(x)),$$

where

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x-y\|^{n-2}} \, dS(y).$$

Then the identity $\phi - \lambda\mathfrak{D}^*\phi = 0$ yields

$$0 = (1-\lambda)\int_{\partial D_1} V \frac{\partial V}{\partial \nu_1} \, dS + (1+\lambda)\int_{\partial D_2} V \frac{\partial V}{\partial \nu_2} \, dS = (1-\lambda)\int_{D_1} |\nabla V|^2 + (1+\lambda)\int_{D_2} |\nabla V|^2,$$

which implies $\phi \equiv 0$. Similarly for $\phi + \lambda\mathfrak{D}^*\phi = 0$, $-1 \le \lambda \le 1$.

Remark. This observation can be improved applying the following fact (whose proof we skip because it will not be used in our theorem):

$$\int_{D_1} |\nabla u|^2 \simeq \int_{D_2} |\nabla u|^2,$$

where, again, the $\simeq$ is controlled by $P(|||S|||)$. In particular, it implies that the spectral radius of the operators $\mathfrak{D}$, $\mathfrak{D}^*$ is less than $1 - (P(|||S|||))^{-1}$.

Theorem 5.5. *The operator norms* $\|(I + \lambda\mathfrak{D})^{-1}\|_{H^s(S)}$, $\|(I + \lambda\mathfrak{D}^*)^{-1}\|_{H^s(S)}$, $|s| \le 1$, $|\lambda| \le 1$, *are* $P(|||S|||)$ (*growth at most polynomially with* $|||S|||$).

Proof. The identity $(I - \mathfrak{D})^{-1}(I - \lambda\mathfrak{D}) = I + (1-\lambda)(I - \mathfrak{D})^{-1}\mathfrak{D}$ shows that the conclusion of the theorem follows easily when $|1-\lambda| \le 1/P(|||S|||)$, and similarly when $|1+\lambda| \le 1/P(|||S|||)$.

Therefore, without loss of generality, we may assume that

$$1 - |\lambda| \ge \frac{1}{P(|||S|||)}.$$

Assume now that $\phi \in H^{-1/2}(S)$ satisfies $\|\phi\|_{H^{-1/2}} = 1$ and

$$\|\phi - \lambda\mathfrak{D}^*\phi\|_{H^{-1/2}} \le \frac{1}{P(|||S|||)}.$$

Then the single-layer potential

$$V(x) = c_n \int_S \frac{\phi(y)}{\|x-y\|^{n-2}} \, dS(y)$$

satisfies the inequality

$$\left|\int_S V(\phi - \lambda \mathcal{D}^*\phi)\, dS\right| \leq \frac{1}{P(|||S|||)}.$$

On the other hand, one has

$$\int_S V(\phi - \lambda \mathcal{D}^*\phi)\, dS = (1-\lambda)\int_{D_1} |\nabla V|^2 + (1+\lambda)\int_{D_2} |\nabla V|^2,$$

implying the estimate

$$\int_S V(\phi + \lambda \mathcal{D}^*\phi)\, dS = (1+\lambda)\int_{D_1} |\nabla V|^2 + (1-\lambda)\int_{D_2} |\nabla V|^2 \leq \frac{1}{P(|||S|||)}.$$

Adding both inequalities together, we would obtain

$$\int_S V\phi\, d\sigma \leq \frac{1}{P(|||S|||)},$$

which is impossible because of the following:

Lemma 5.6. *If V is the single-layer potential of ϕ, then*

$$\int_S V(x)\phi(x)\, dS(x) = \int_S \int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) \geq \frac{1}{P(|||S|||)} \|\phi\|^2_{H^{-1/2}(S)}.$$

Let us first observe that

$$\int_S \int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, d\sigma(x)\, d\sigma(y) = \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} \left|\widehat{\phi\, d\sigma(\xi)}\right|^2 d\xi \geq 0,$$

where $\widehat{\phi\, dS}$ denotes the Fourier transform of the measure $\phi\, dS$ supported on S. This implies that

$$\langle \phi, \psi \rangle = \int_S \int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y)$$

is an inner product satisfying

$$|\langle \phi, \psi \rangle| \leq \langle \phi, \phi \rangle^{1/2} \langle \psi, \psi \rangle^{1/2},$$

and we wish to show that

$$\langle \phi, \phi \rangle \backsimeq \|\phi\|^2_{H^{-1/2}(S)},$$

where $\backsimeq$ denotes equivalence modulo a factor $P(|||S|||)$. To see this, observe first that given $\phi \in H^{-1/2}(S)$, its single-layer potential $u|S$ belongs to the space $H^{1/2}(S)$, satisfying

$$\|u\|_{H^{1/2}(S)} \leq P(|||S|||)\|\phi\|_{H^{-1/2}(S)},$$

which can be proved easily using local coordinates. As a consequence, we have

$$\int_S \int_S \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) \leq P(|||S|||)\|\phi\|^2_{H^{-1/2}(S)}.$$

In the opposite direction, since $H^{-s}=(H^s)^*$, we have

$$\|\phi\|_{H^{-s}}=\sup_{f\in H^s}\int_S \phi(x)f(x)\,d\sigma(x).$$

Let us assume, for the moment, that given $f\in H^s$, there exists $g\in H^{s-1}$ such that

$$f(x)=c_n\int_S \frac{g(y)}{\|x-y\|^{n-2}}\,dS(y)\quad\text{and}\quad \|f\|_{H^s}\backsimeq\|g\|_{H^{s-1}}.$$

Then

$$\|\phi\|_{H^{-s}}\backsimeq\sup_{\|g\|_{H^{s-1}}=1}\langle\phi,g\rangle,$$

and taking $s=\frac{1}{2}$, $s-1=-\frac{1}{2}$, we get

$$\|\phi\|_{H^{-1/2}}\le P(|||S|||)\langle\phi,\phi\rangle^{1/2}\langle g,g\rangle^{1/2}\le P(|||S|||)\langle\phi,\phi\rangle^{1/2}\|g\|_{H^{-1/2}}\le P(|||S|||)\langle\phi,\phi\rangle^{1/2}.$$

To close our argument, it remains to solve the equation

$$f(x)=c_n\int_S \frac{g(y)}{\|x-y\|^{n-2}}\,dS(y),$$

that is, to prove that given $f\in H^s$, there exists $g\in H^{s-1}$ satisfying the this equation.

To see that, let us consider the solution of the Dirichlet problem

$$\begin{cases}\Delta u=0 & \text{in } D_1,\\ u|_S=f\end{cases}$$

and the equation

$$-2\frac{\partial u}{\partial\nu_1}=g-\mathcal{D}^*g,$$

that is, $g=(I-\mathcal{D}^*)^{-1}(-2\partial u/\partial\nu_1)$. Then we claim that such g verifies the identity

$$f(x)=c_n\int_S \frac{g(y)}{\|x-y\|^{n-2}}\,dS(y).$$

This is because the function

$$V(x)=c_n\int_S \frac{g(y)}{\|x-y\|^{n-2}}\,dS(y)$$

is harmonic in D_1 and satisfies

$$-2\frac{\partial V}{\partial\nu_1}=g-\mathcal{D}^*g=-2\frac{\partial u}{\partial\nu_1},$$

which implies that $V=u$ in D_1, and therefore, taking limits up to the boundary, we obtain

$$f(x)=c_n\int_S \frac{g(y)}{\|x-y\|^{n-2}}\,dS(y).$$

To finish the proof of Theorem 5.5, let us consider, for every $0\le\tau\le 1$, the identity

$$(I-\lambda\mathcal{D})^{-1}\Lambda^\tau=\Lambda^\tau(I-\lambda\mathcal{D})^{-1}+(I-\lambda\mathcal{D})^{-1}C_\tau(I-\lambda\mathcal{D})^{-1},$$

where the commutator $C_\tau = [\mathcal{D}\Lambda^\tau - \Lambda^\tau\mathcal{D}]$ is a pseudodifferential operator of order $\tau - 2$ whose bounds are controlled by $|||S|||$. Then

$$\begin{aligned}\|(I-\lambda\mathcal{D})^{-1}f\|_{H^s} &\le \|(I-\lambda\mathcal{D})^{-1}f\|_{H^{-1/2}} + \|\Lambda^{s+1/2}(I-\lambda\mathcal{D})^{-1}f\|_{H^{-1/2}}\\ &\lesssim \|f\|_{H^{-1/2}} + \|(I-\lambda\mathcal{D})^{-1}\Lambda^{s+1/2}f\|_{H^{-1/2}}\\ &\lesssim \|f\|_{L^2} + \|\Lambda^{s+1/2}f\|_{H^{-1/2}} \le P(|||S|||)\|f\|_{H^s}. \qquad \square\end{aligned}$$

Remark 5.7. In the particular case of the sphere $S = S^{n-1}$ ($n \ge 2$), the estimate of Lemma 5.6 becomes an identity:

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{\phi(x)\phi(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = c_n\|\phi\|^2_{H^{-1/2}(S^{n-1})}$$

for $n \ge 3$, and

$$-\int_{S^1}\int_{S^1} \log\|x-y\|\phi(x)\phi(y)\, dS(x)\, dS(y) = c_2\|\phi\|^2_{H^{-1/2}(S^1)}$$

for $n = 2$.

Proof. We present the details when $n \ge 3$. The case $n = 2$ follows similarly. Let $\phi(x) = \sum a_k Y_k(x)$, where Y_k is a spherical harmonic of degree k, normalized so that $\|Y_k\|_{L^2(S^{n-1})} = 1$; then we have

$$|a_0|^2 + \sum_{k\ge 1} \frac{|a_k|^2}{2k+n-2} = \|\phi\|^2_{H^{-1/2}(S)} < \infty.$$

Claim: if $k \ne j$, then

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = 0.$$

Taking the Fourier transform and using Plancherel, we get

$$\int_{S^{n-1}}\int_{S^{n-1}} \frac{Y_k(x)Y_j(y)}{\|x-y\|^{n-2}}\, dS(x)\, dS(y) = \int_{\mathbb{R}^n} \frac{1}{|\xi|^2}\widehat{Y_k\, dS(\xi)}\overline{\widehat{Y_j\, dS(\xi)}}\, d\xi.$$

But it turns out that

$$\widehat{Y_k\, dS(\xi)} = 2\pi i^{-k}|\xi|^{(n-2)/2} J_{(n+2k-2)/2}(|\xi|)Y_k\Big(\frac{\xi}{|\xi|}\Big),$$

where J_ν designates Bessel's function of order ν, implying the claim.

Therefore our estimate diagonalizes:

$$\int_{\mathbb{R}^n} \frac{1}{|\xi|^2}|\widehat{Y_k\, dS(\xi)}|^2 d\xi = c\int_0^\infty \frac{1}{r}|J_{k+(n-2)/2}(r)|^2 dr,$$

and the well-known identity for Bessel's functions

$$\int_0^\infty \frac{J^2_\mu(r)}{r}\, dr = \frac{1}{2\mu}$$

allows us to finish the proof. $\square$

Estimates for Ω and ω. In the following, we shall consider asymptotically flat domains, leaving to the reader the details of the periodic case. Since we have controlled the norms of the operator relating Ω and X, we are in a position to obtain the inequality

$$\|\Omega\|_{H^k} \leq P\big(\|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{5-1}$$

for $k \geq 4$, with P a polynomial function. Then Sobolev's embedding implies

$$\|\omega\|_{H^k} \leq P\big(\|X\|_{k+1}^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{5-2}$$

for $k \geq 3$. We will present the proof of (5-1) when $k = 4$, because the case $k > 4$ can be obtained with the same method.

Theorem 5.5 applied to (2-6) yields

$$\|\Omega\|_{H^1} = \|(I - A_\mu \mathscr{D})^{-1}(-2A_\rho X_3)\|_{H^1} \leq C\|(I - A_\mu \mathscr{D})^{-1}\|_{H^1}\, \|X_3\|_{H^1} \leq P(\|\!|S|\!\|)\|X_3\|_{H^1},$$

implying that

$$\|\Omega\|_{H^1} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big).$$

Next we will show that

$$\|\partial_{\alpha_1}^2 \Omega\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1}, \tag{5-3}$$

which together with the estimate for $\|\Omega\|_{H^1}$ above, will allow us to control $\partial_{\alpha_1}^2 \Omega$ in terms of the free boundary.

In order to do that, we start with formula (2-8) to get $\partial_{\alpha_1}^2 \Omega = I_1 + I_2 + I_3 + I_4 - 2A_\rho \partial_{\alpha_1}^2 X_3$, where

$$I_1 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1}^2 X(\alpha),$$

$$I_2 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$I_3 = -\frac{3A_\mu}{4\pi} \operatorname{PV}\!\int_{\mathbb{R}^2} A(\alpha,\beta) \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^5} \wedge \omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $A(\alpha,\beta) = (X(\alpha) - X(\alpha-\beta)) \cdot (\partial_{\alpha_1} X(\alpha) - \partial_{\alpha_1} X(\alpha-\beta))$, and

$$I_4 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\omega(\alpha-\beta)d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

Our next objective is to introduce the operators $\mathscr{T}_k$ (A-5) defined in the Appendix in the analysis of the integrals I_j. Formula (2-3) gives us $\omega = \partial_{\alpha_2}(\Omega\partial_{\alpha_1} X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2} X)$, and from standard Sobolev's estimates we get

$$\|I_j\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1}, \quad j = 1, 2,$$

and similarly with I_3.

Regarding

$$I_4 = \int_{|\beta|>1} d\beta + \int_{|\beta|<1} d\beta = J_1 + J_2,$$

we integrate by parts in J_1 to obtain

$$J_1 = \frac{A_\mu}{2\pi} \int_{|\beta|>1} \partial_{\beta_1}\Big(\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\Big) \wedge \omega(\alpha-\beta) d\beta \cdot \partial_{\alpha_1} X(\alpha)$$
$$- \frac{A_\mu}{2\pi} \int_{|\beta|=1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, dl(\beta) \cdot \partial_{\alpha_1} X(\alpha).$$

From this last expression, it is easy to deduce the inequality

$$J_1 \le C \|F(X)\|^3_{L^\infty} \|X - (\alpha, 0)\|^2_{C^1} \bigg(\int_{|\beta|>1} \frac{|\omega(\alpha-\beta)|}{|\beta|^3} d\beta + \int_{|\beta|=1} |\omega(\alpha-\beta)|\, dl(\beta), \bigg)$$

providing us with an appropriate control (see the Appendix for more details).

Next let us consider $J_2 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_2}\Omega(\alpha-\beta) \partial^2_{\alpha_1} X(\alpha-\beta) d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$K_2 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta) \partial_{\alpha_1} X(\alpha-\beta) d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$K_3 = -\frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\Omega(\alpha-\beta) \partial_{\alpha_1}\partial_{\alpha_2} X(\alpha-\beta) d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$K_4 = -\frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial^2_{\alpha_1}\Omega(\alpha-\beta) \partial_{\alpha_2} X(\alpha-\beta) d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

Then the terms K_1 and K_3 are handled with the same approach used for I_2 — see (A-13) in the Appendix — and we rewrite K_2 in the form

$$K_2 = \frac{A_\mu}{2\pi} \int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta) (\partial_{\alpha_1} X(\alpha-\beta) - \partial_{\alpha_1} X(\alpha)) d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

to show that it can be estimated via an integration by parts in the variable β_1, using the identity

$$\partial_{\alpha_1}\partial_{\alpha_2}\Omega(\alpha-\beta) = -\partial_{\beta_1}(\partial_{\alpha_2}\Omega(\alpha-\beta))$$

and the fact that the kernel in the integral K_2 has degree -1.

It remains to deal with K_4: to do that, let us consider $K_4 = L_1 + L_2$, where

$$L_1 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial^2_{\alpha_1}\Omega(\alpha-\beta) \big(\partial_{\alpha_2} X(\alpha) - \partial_{\alpha_2} X(\alpha-\beta)\big) d\beta \cdot \partial_{\alpha_1} X(\alpha)$$

and

$$L_2 = \frac{A_\mu}{2\pi} \operatorname{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\, \partial^2_{\alpha_1}\Omega(\alpha-\beta) d\beta \cdot N(\alpha).$$

The term L_1 can be controlled like K_2, and L_2 can be rewritten in the form

$$L_2 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \left(\frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} - \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \right) \partial^2_{\alpha_1} \Omega(\alpha-\beta)\, d\beta \cdot N(\alpha),$$

showing that it can be estimated as we did with $\mathcal{T}_4$ (A-8), that is, we obtain (5-3). Similarly, Equation (2-9) yields

$$\|\partial^2_{\alpha_2}\Omega\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^1},$$

and then the inequality $2\|\partial_{\alpha_1}\partial_{\alpha_2}\Omega\|_{L^2} \le \|\partial^2_{\alpha_1}\Omega\|_{L^2} + \|\partial^2_{\alpha_2}\Omega\|_{L^2}$ gives us the desired control upon $\|\Omega\|_{H^2}$.

Next we will show that

$$\|\partial^3_{\alpha_1}\Omega\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^2}, \tag{5-4}$$

allowing us to use the estimates for $\|\Omega\|_{H^2}$ above. In order to do that, we start with formula (2-8), to get $\partial^3_{\alpha_1}\Omega = \partial_{\alpha_1} I_1 + \partial_{\alpha_1} I_2 + \partial_{\alpha_1} I_3 + \partial_{\alpha_1} I_4 - 2A_\rho \partial^3_{\alpha_1} X_3$, where the most singular terms are given by

$$J_3 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial^3_{\alpha_1} X(\alpha),$$

$$J_4 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^2_{\alpha_1}X(\alpha) - \partial^2_{\alpha_1}X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$J_5 = -\frac{3A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} B(\alpha,\beta)\frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^5} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $B(\alpha,\beta) = (X(\alpha) - X(\alpha-\beta))\cdot(\partial^2_{\alpha_1}X(\alpha) - \partial^2_{\alpha_1}X(\alpha-\beta))$, and

$$J_6 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \partial^2_{\alpha_1}\omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

and where the remainder terms can be estimated with the same method used before.

Now we write

$$J_3 = \frac{A_\mu}{2\pi}\mathcal{T}_1\big(\partial_{\alpha_2}(\Omega\partial_{\alpha_1}X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2}X)\big)\cdot\partial^3_{\alpha_1}X$$

to obtain

$$\|J_3\|_{L^2} \le C\big\|\mathcal{T}_1\big(\partial_{\alpha_2}(\Omega\partial_{\alpha_1}X) - \partial_{\alpha_1}(\Omega\partial_{\alpha_2}X)\big)\big\|_{L^4}\|\partial^3_{\alpha_1}X\|_{L^4}.$$

Next observe that in the proof of estimate (A-9), one can replace L^2 by L^p for $1 < p < \infty$ [Stein 1993]. In particular, we have

$$\|J_3\|_{L^2} \le P\big(\|X-(\alpha,0)\|_{C^{1,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\big(\|\Omega\partial_{\alpha_1}X\|_{L^4} + \|\Omega\partial_{\alpha_2}X\|_{L^4} + \|\omega\|_{L^4}\big)\|\partial^3_{\alpha_1}X\|_{L^4},$$

and then Sobolev's embedding in dimension two, $\|g\|_{L^4} \le C\|g\|_{H^1}$, yields the desired control. Regarding J_4, we follow the approach taken before for $\mathcal{T}_3$, but now using the L^4 norm. That is, we split

$$J_4 = \int_{|\beta|>1} d\beta + \int_{|\beta|<1} d\beta = K_5 + K_6,$$

and since

$$K_5 \leq \|X-(\alpha,0)\|_{C^2}^2 \|F(X)\|_{L^\infty}^3 \int_{|\beta|>1} \frac{|\omega(\alpha-\beta)|}{|\beta|^3}\, d\beta,$$

that term can be estimated as above.

Next we introduce the splitting $K_6 = L_3 + L_4$, where

$$L_3 = \frac{A_\mu}{2\pi} \int_{|\beta|<1} \big(\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)\big) \Big[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\Big] \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$L_4 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

We have

$$L_3 \leq C\|X-(\alpha,0)\|_{C^{2,\delta}}^3 \big(\|F(X)\|_{L^\infty}^4 + \|X-(\alpha,0)\|_{C^1}^4 \||N|^{-1}\|_{L^\infty}^4\big) \int_{|\beta|<1} \frac{|\omega(\alpha-\beta)|}{|\beta|^{2-\delta}}\, d\beta$$

(see the Appendix for more details), giving us the appropriate estimate. Regarding L_4, we use identity (A-16), which, after a careful integration by parts, yields

$$L_4 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{\beta\cdot\nabla_\beta\big((\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)) \wedge \omega(\alpha-\beta)\cdot\partial_{\alpha_1} X(\alpha)\big)}{|\nabla X(\alpha)\cdot\beta|^3}\, d\beta - \frac{A_\mu}{2\pi} \int_{|\beta|=1} \frac{|\beta|\big(\partial_{\alpha_1}^2 X(\alpha) - \partial_{\alpha_1}^2 X(\alpha-\beta)\big) \wedge \omega(\alpha-\beta)\cdot\partial_{\alpha_1} X(\alpha)}{|\nabla X(\alpha)\cdot\beta|^3}\, dl(\beta),$$

helping us to prove the inequality

$$\|L_4\|_{L^2} \leq P\big(\|X-(\alpha,0)\|_{C^2} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)(\|\partial_{\alpha_1}^3 X\|_{L^4}\|\omega\|_{L^4} + \|\omega\|_{L^2}).$$

Clearly, J_5 can be approached with the same method used for J_4. Regarding the term J_6, we have to decompose further: first, its most singular terms, which are given by

$$L_5 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_2}\Omega(\alpha-\beta)\partial_{\alpha_1}^3 X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$L_6 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}^2\partial_{\alpha_2}\Omega(\alpha-\beta)\partial_{\alpha_1} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$L_7 = -\frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_1}^2\partial_{\alpha_2} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$L_8 = -\frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}^3\Omega(\alpha-\beta)\partial_{\alpha_2} X(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

Second, let us observe that the remainder is easy to handle: the terms L_5 and L_7 can be estimated as we did with K_1 and K_3, using the L^4 norm and, finally, L_6 and L_8 are like K_2 and K_4, respectively. Putting all these facts together, we obtain (5-4).

Similarly to the case of lower derivatives, Equation (2-9) yields

$$\|\Omega\|_{H^3} \le P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^2}.$$

To finish, it remains to show the corresponding inequality for derivatives of fourth order:

$$\|\Omega\|_{H^4} \le P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big)\|\Omega\|_{H^3}. \tag{5-5}$$

Identity (2-8) allows us to point out the most singular terms in $\partial_{\alpha_1}^4 \Omega$:

$$M_1 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1}^4 X(\alpha),$$

$$M_2 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial_{\alpha_1}^3 X(\alpha)-\partial_{\alpha_1}^3 X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

$$M_3 = -\frac{3A_\mu}{4\pi} \,\mathrm{PV}\!\int_{\mathbb{R}^2} C(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha),$$

with $C(\alpha,\beta) = (X(\alpha)-X(\alpha-\beta)) \cdot (\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha-\beta))$, and

$$M_4 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial_{\alpha_1}^3 \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha).$$

Then, in order to estimate M_1, we start with $\|M_1\|_{L^2} \le CK\|\partial_{\alpha_1}^4 X\|_{L^2}$, where

$$K = \sup_\alpha \left| \mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \right|.$$

Following [Córdoba and Gancedo 2007], we have

$$K \le O_1 + O_2 + O_3 + O_4 + O_5,$$

where

$$O_1 = \sup_\alpha \left| \mathrm{PV}\!\int_{|\beta|>1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \right|,$$

$$O_2 = \sup_\alpha \left| \int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)-\nabla X(\alpha)\cdot\beta}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \right|,$$

$$O_3 = \sup_\alpha \left| \int_{|\beta|<1} \nabla X(\alpha)\cdot\beta \left[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3} \right] \wedge \omega(\alpha-\beta)\, d\beta \right|,$$

$$O_4 = \sup_\alpha \left| \int_{|\beta|<1} \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \big(\omega(\alpha-\beta)-\omega(\alpha)\big)\, d\beta \right|,$$

$$O_5 = \sup_\alpha \left| \mathrm{PV}\!\int_{|\beta|<1} \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha)\, d\beta \right|.$$

An integration by parts in O_1 yields

$$\begin{aligned} O_1 &\le C\|\nabla X\|_{L^\infty}^2 \|F(X)\|_{L^\infty}^3 \sup_\alpha \Big(\int_{|\beta|>1} \frac{|\Omega(\alpha-\beta)|}{|\beta|^3}\, d\beta + \int_{|\beta|=1} |\Omega(\alpha-\beta)|\, dl(\beta) \Big) \\ &\le C\|\nabla X\|_{L^\infty}^2 \|F(X)\|_{L^\infty}^3 \|\Omega\|_{L^\infty}, \end{aligned}$$

and Sobolev's embedding allows us to conclude.

Regarding O_2, we have

$$O_2 \le \|X-(\alpha,0)\|_{C^{2,\delta}} \|F(X)\|_{L^\infty}^3 \|\omega\|_{L^\infty} \Big| \int_{|\beta|<1} |\beta|^{2-\delta} d\beta \Big|,$$

and the estimate $\|\omega\|_{C^\delta} \le C\|\omega\|_{H^2}$, for $0<\delta<1$, gives the desired control. Using (A-15) and some straightforward algebraic manipulations, we get a similar inequality for O_3. Next, we have

$$O_4 \le C\|X-(\alpha,0)\|_{C^1}^4 \||N|^{-1}\|_{L^\infty}^3 \|\omega\|_{C^\delta} \Big| \int_{|\beta|<1} |\beta|^{2-\delta} d\beta \Big|,$$

giving us also the same estimate. Furthermore, it is easy to prove that $O_5 = 0$.

Next we consider the term M_2 with the splitting $M_2 = Q_1 + Q_2 + Q_3$, where

$$\begin{aligned} Q_1 &= \frac{A_\mu}{2\pi} \int_{|\beta|>1} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta \cdot \partial_{\alpha_1} X(\alpha), \\ Q_2 &= \frac{A_\mu}{2\pi} \int_{|\beta|<1} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge (\omega(\alpha-\beta)-\omega(\alpha))\, d\beta \cdot \partial_{\alpha_1} X(\alpha), \\ Q_3 &= \frac{A_\mu}{2\pi} \mathrm{PV}\int_{|\beta|<1} \frac{\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\, d\beta \wedge \omega(\alpha) \cdot \partial_{\alpha_1} X(\alpha). \end{aligned}$$

The term Q_1 can be estimated as before; regarding Q_2, we can use the identity

$$\partial_{\alpha_1}^3 X(\alpha) - \partial_{\alpha_1}^3 X(\alpha-\beta) = \int_0^1 \nabla \partial_{\alpha_1}^3 X(\alpha + (s-1)\beta)\, ds \cdot \beta,$$

and the control of Q_3 can be approached as we did with the operator in (A-7). Similarly with M_3, while M_4 is analogous to J_6, and all these observations together allow us to obtain (5-5).

6. Controlling the Birkhoff–Rott integral

Here we consider estimates for the Birkhoff–Rott integral along a non-self-intersecting surface. Let us assume that $\nabla(X(\alpha)-(\alpha,0)) \in H^k(\mathbb{R}^2)$ for $k \ge 3$, and that both $F(X)$ and $|N|^{-1}$ are in L^∞, where

$$F(X)(\alpha,\beta) = |\beta| \big/ |X(\alpha)-X(\alpha-\beta)| \quad \text{and} \quad N(\alpha) = \partial_{\alpha_1} X(\alpha) \wedge \partial_{\alpha_2} X(\alpha).$$

The main purpose of this section is to prove the estimate

$$\|\mathrm{BR}(X,\omega)\|_{H^{k-1}} \le P\big(\|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big), \tag{6-1}$$

for $k \geq 4$. Here we shall show it when $k = 4$, because the other cases, $k > 4$, follow by similar arguments. We rewrite BR in the following manner:

$$\mathrm{BR}(X, \omega)(\alpha, t) = -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge (\partial_{\beta_2}(\Omega\partial_{\beta_1} X) - \partial_{\beta_1}(\Omega\partial_{\beta_2} X))(\beta)\, d\beta,$$

which, together with the estimates about Ω in Section 5 and also about the operator $\mathcal{T}_1$ in the Appendix, yields

$$\|\mathrm{BR}(X, \omega)\|_{L^2} \leq P\big(\|X\|_4^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty}\big).$$

To estimate derivatives of order 3, we consider $\partial^3_{\alpha_i}(\mathrm{BR}(X, \omega))$, and observe that the most dangerous terms are given by

$$I_1 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{(\partial^3_{\alpha_i} X(\alpha) - \partial^3_{\alpha_i} X(\alpha-\beta)) \wedge \omega(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta,$$

$$I_2 = \frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} (X(\alpha) - X(\alpha-\beta)) \wedge \omega(\alpha-\beta) \frac{(X(\alpha) - X(\alpha-\beta)) \cdot (\partial^3_{\alpha_i} X(\alpha) - \partial^3_{\alpha_i} X(\alpha-\beta))}{|X(\alpha) - X(\alpha-\beta)|^5}\, d\beta,$$

$$I_3 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{(X(\alpha) - X(\alpha-\beta)) \wedge (\partial^3_{\alpha_i}\omega)(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta.$$

In the Appendix, we find all the ingredients needed to estimate these terms I_j, while the remainder in $\partial^3_{\alpha_i}(\mathrm{BR}(X, \omega))$ is easily bounded: in I_3 we can recognize an operator with the form of $\mathcal{T}_1$ in (A-5), so the estimate for ω in Section 5 gives the desired control for I_3. Regarding I_1, we may use the splitting $I_1 = J_1 + J_2$, where

$$J_1 = \frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{(\partial^3_{\alpha_i} X(\alpha) - \partial^3_{\alpha_i} X(\alpha-\beta)) \wedge (\omega(\alpha) - \omega(\alpha-\beta))}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta,$$

$$J_2 = \frac{\omega(\alpha)}{4\pi} \wedge \mathrm{PV}\int_{\mathbb{R}^2} \frac{(\partial^3_{\alpha_i} X(\alpha) - \partial^3_{\alpha_i} X(\alpha-\beta))}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta.$$

Then the identity $\partial^3_{\alpha_i} X(\alpha) - \partial^3_{\alpha_i} X(\alpha-\beta) = \beta \cdot \int_0^1 \nabla\partial^3_{\alpha_i} X(\alpha + (s-1)\beta)\, ds$ allows us to find in J_1 a kernel of degree -1 which we know how to handle (see the Appendix). One uses the estimate for $\mathcal{T}_3$ (A-7) to deal with J_2, and we proceed similarly to control I_2.

7. In search of the Rayleigh–Taylor condition

As was pointed out in Section 4 (outline of the proof), our approach is based on energy estimates, and a crucial step is to characterize those terms involving higher derivatives which are controlled because they have the appropriate sign. In our terminology, they constitute the Rayleigh–Taylor condition, which is supposed to hold at time $T = 0$, it being an important part of the proof to show that it prevails under the evolution.

Let us introduce the notation

$$|||X|||_k^2 = \|X\|_k^2 + \|F(X)\|_{L^\infty}^2 + \||N|^{-1}\|_{L^\infty},$$

where

$$\|X\|_k = \|X_1-\alpha_1\|_{L^3} + \|X_2-\alpha_2\|_{L^3} + \|X_3\|_{L^2} + \|\nabla(X-(\alpha,0))\|^2_{H^{k-1}} \tag{7-1}$$

and

$$\|\nabla(X-(\alpha,0))\|^2_{H^{k-1}} = \|\nabla(X-(\alpha,0))\|^2_{L^2} + \|\partial^k_{\alpha_1}(X-(\alpha,0))\|^2_{L^2} + \|\partial^k_{\alpha_2}(X-(\alpha,0))\|^2_{L^2}.$$

In order to justify the formula

$$\frac{d}{dt}\|X\|^2_k(t) \le -\sum_{i=1,2}\frac{2^{3/2}}{(\mu_1+\mu_2)}\int_{\mathbb{R}^2}\frac{\sigma(\alpha,t)}{|\nabla X(\alpha,t)|^3}\,\partial^k_{\alpha_i}X(\alpha,t)\cdot\Lambda(\partial^k_{\alpha_i}X)(\alpha,t)\,d\alpha + P(|||X|||_k(t)),$$

(here $k\ge 4$, although for the sake of simplicity we will present the explicit computations when $k=4$, leaving the other cases as an exercise for the interested reader), it will be convenient to make use of the following tools, which give us different kinds of cancellations, and which constitute our particular bestiary of formulas for this paper.

From the definition of the isothermal parametrization, we have the identities

$$|\partial_{\alpha_1}X|^2 = |\partial_{\alpha_2}X|^2, \tag{7-2}$$

$$\partial_{\alpha_1}X\cdot\partial_{\alpha_2}X = 0, \tag{7-3}$$

which yield

$$\tfrac12\Delta(|\partial_{\alpha_1}X|^2) = |\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X, \tag{7-4}$$

$$\partial^4_{\alpha_1}X\cdot\partial_{\alpha_1}X = -3\partial^3_{\alpha_1}X\cdot\partial^2_{\alpha_1}X + (\partial^2_{\alpha_1}\Delta^{-1}\partial_{\alpha_1})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big), \tag{7-5}$$

$$\partial^4_{\alpha_2}X\cdot\partial_{\alpha_2}X = -3\partial^3_{\alpha_2}X\cdot\partial^2_{\alpha_2}X + (\partial^2_{\alpha_2}\Delta^{-1}\partial_{\alpha_2})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big). \tag{7-6}$$

Using (7-3) and (7-4), we obtain

$$\partial^4_{\alpha_1}X\cdot\partial_{\alpha_2}X = -2\partial^3_{\alpha_1}X\cdot\partial_{\alpha_1}\partial_{\alpha_2}X - \partial^2_{\alpha_1}\partial_{\alpha_2}X\cdot\partial^2_{\alpha_1}X - (\partial_{\alpha_1}\partial_{\alpha_2}\Delta^{-1}\partial_{\alpha_1})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big), \tag{7-7}$$

$$\partial^4_{\alpha_2}X\cdot\partial_{\alpha_1}X = -2\partial^3_{\alpha_2}X\cdot\partial_{\alpha_1}\partial_{\alpha_2}X - \partial^2_{\alpha_2}\partial_{\alpha_1}X\cdot\partial^2_{\alpha_2}X - (\partial_{\alpha_1}\partial_{\alpha_2}\Delta^{-1}\partial_{\alpha_2})\big(|\partial_{\alpha_1}\partial_{\alpha_2}X|^2 - \partial^2_{\alpha_1}X\cdot\partial^2_{\alpha_2}X\big). \tag{7-8}$$

And Sobolev inequalities imply that if $\nabla(X-(\alpha,0))\in H^3$, then $\partial^4_{\alpha_i}X\cdot\partial_{\alpha_j}X\in H^3$ for $i,j=1,2$.

With the help of the estimates above, we may now determine σ. There is a part that may be considered a mere “algebraic” manipulation to detect the relevant characters and, in so doing, we disregard many terms because they are of lower order in the sense of Sobolev spaces. At the end, we shall present how to deal with those lower-order terms — if not for the whole collection of them, at least for the ones that we may consider to be the most “dangerous” characters. Here it is convenient to recommend to the reader our previous works [Córdoba and Gancedo 2007; Córdoba et al. 2011], where similar estimates were carried out.

Low-order norms. Since $X_i(\alpha)\to\alpha_i$ for $i=1,2$ at infinity, let us consider the evolution of the L^3 norm. That is,

$$\frac13\frac{d}{dt}\|X_1-\alpha_1\|^3_{L^3}(t) = \int_{\mathbb{R}^2}|X_1-\alpha_1|(X_1-\alpha_1)X_{1t}\,d\alpha = I_1+I_2+I_3,$$

where

$$I_1 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)\,\mathrm{BR}_1\,d\alpha,$$
$$I_2 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)C_1\partial_{\alpha_1}X_1 d\alpha,$$
$$I_3 = \int_{\mathbb{R}^2} |X_1 - \alpha_1|(X_1 - \alpha_1)C_2\partial_{\alpha_2}X_1 d\alpha.$$

Then we have

$$I_1 \le \|X_1 - \alpha_1\|_{L^3}^2 \|\,\mathrm{BR}\,\|_{L^3} \le C\big(\|X_1 - \alpha_1\|_{L^3}^3 + \|\,\mathrm{BR}\,\|_{L^\infty}\|\,\mathrm{BR}\,\|_{L^2}^2\big),$$

and Sobolev estimates, together with (6-1), yield the appropriate control in terms of $P(|||X|||_k)$.

Next, since $\partial_{\alpha_1}X_1 \to 1$ as $\alpha \to \infty$, we have

$$I_2 \le \|\partial_{\alpha_1}X_1\|_{L^\infty}\|X_1 - \alpha_1\|_{L^3}^2\|C_1\|_{L^3},$$

and it remains to get control of C_1. Using (3-1), we introduce the splitting $C_1 = \sum_{j=1}^4 C_1^j$, where

$$C_1^1(\alpha) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_2}\cdot\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\,d\beta, \qquad C_1^2(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_1}\cdot\frac{X_{\beta_1}}{|X_{\beta_2}|^2}\,d\beta,$$
$$C_1^3(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_2 - \beta_2}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_1}\cdot\frac{X_{\beta_2}}{|X_{\beta_1}|^2}\,d\beta, \qquad C_1^4(\alpha) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}_{\beta_2}\cdot\frac{X_{\beta_1}}{|X_{\beta_1}|^2}\,d\beta.$$

We shall show how to control C_1^1, because the estimates for the other terms follow by similar arguments. Integrating by parts, one obtains $C_1^1 = D_1 + D_2$, where

$$D_1 = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{\alpha_1 - \beta_1}{|\alpha-\beta|^2}\,\mathrm{BR}\cdot\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\,d\beta, \quad D_2 = -\frac{1}{\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)}{|\alpha-\beta|^4}\,\mathrm{BR}\cdot\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\,d\beta.$$

Regarding D_1, we write $D_1 = E_1 + E_2$, where

$$E_1 = \frac{-1}{2\pi}\int_{|\beta|<1} \frac{\beta_1}{|\beta|^2}\,\mathrm{BR}(\alpha-\beta)\cdot\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)(\alpha-\beta)\,d\beta,$$
$$E_2 = \frac{-1}{2\pi}\int_{|\beta|>1} \frac{\beta_1}{|\beta|^2}\,\mathrm{BR}(\alpha-\beta)\cdot\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)(\alpha-\beta)\,d\beta.$$

The Minkowski and Young inequalities yield, respectively,

$$\|E_1\|_{L^3} \le C\Big\|\mathrm{BR}\cdot\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^3} \le P(|||X|||_4),$$
$$\|E_2\|_{L^3} \le C\Big\|\mathrm{BR}\cdot\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^1} \le C\|\,\mathrm{BR}\,\|_{L^2}\Big\|\partial_{\beta_2}\Big(\frac{X_{\beta_2}}{|X_{\beta_2}|^2}\Big)\Big\|_{L^2} \le P(|||X|||_4),$$

and the desired control is achieved. In the term D_2, we have a double Riesz transform, and the standard

Calderón–Zygmund theory yields

$$\|D_2\|_{L^3} \le C \left\| \mathrm{BR} \cdot \frac{X_{\beta_2}}{|X_{\beta_2}|^2} \right\|_{L^3} \le C \| |X_{\beta_2}|^{-1} \|_{L^\infty} \| \mathrm{BR} \|_{L^3} \le P(\|\!|X|\!\|_4).$$

The estimate for I_3 follows on a similar path, and the case of the second coordinate is also identical:

$$\frac{1}{3}\frac{d}{dt}\|X_2 - \alpha_2\|^3_{L^3}(t) \le P(\|\!|X|\!\|_4).$$

Regarding the third coordinate, we have stronger decay because of the asymptotic flatness hypothesis:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|X_3\|^2_{L^2}(t) &= \int_{\mathbb{R}^2} X_3\,\mathrm{BR}_3\; d\alpha + \int_{\mathbb{R}^2} X_3 C_1 \partial_{\alpha_1} X_3\, d\alpha + \int_{\mathbb{R}^2} X_3 C_2 \partial_{\alpha_2} X_3\, d\alpha \\ &= \int_{\mathbb{R}^2} X_3\,\mathrm{BR}_3\, d\alpha - \frac{1}{2}\int_{\mathbb{R}^2} (\partial_{\alpha_1} C_1 + \partial_{\alpha_2} C_2)|X_3|^2\, d\alpha,\end{aligned}$$

and therefore the use of Sobolev's embedding in the formulas for C_1 (3-1) and C_2 (3-2), together with the estimates for BR (6-1), allows us to obtain:

$$\frac{1}{2}\frac{d}{dt}\|X_3\|^2_{L^2}(t) \le P(\|\!|X|\!\|_4).$$

Once we have control of higher-order derivatives, we can use the estimates of the Appendix to get

$$\frac{1}{2}\frac{d}{dt}\big\|\nabla(X - (\alpha, 0))\big\|^2_{L^2}(t) \le P(\|\!|X|\!\|_4).$$

Higher-order norms. Let us now consider

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\partial^4_{\alpha_1} X\|^2_{L^2}(t) \\ &= \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1}\,\mathrm{BR}(X,\omega)\, d\alpha + \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1}(C_1 \partial_{\alpha_1} X)\, d\alpha + \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1}(C_2 \partial_{\alpha_2} X)\, d\alpha \\ &= I_1 + I_2 + I_3. \qquad (7\text{-}9)\end{aligned}$$

The higher-order terms in I_2 and I_3 are given by

$$J_1 = \int_{\mathbb{R}^2} C_1 \partial^4_{\alpha_1} X \cdot \partial^5_{\alpha_1} X\, d\alpha, \qquad J_2 = \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X \partial^4_{\alpha_1} C_1\, d\alpha,$$

$$J_3 = \int_{\mathbb{R}^2} C_2 \partial^4_{\alpha_1} X \cdot \partial^4_{\alpha_1}\partial_{\alpha_2} X\, d\alpha, \qquad J_4 = \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X \partial^4_{\alpha_1} C_2\, d\alpha.$$

Integration by parts yields

$$J_1 + J_3 = -\frac{1}{2}\int_{\mathbb{R}^2} (\partial_{\alpha_1} C_1 + \partial_{\alpha_2} C_2)|\partial^4_{\alpha_1} X|^2 d\alpha,$$

and therefore

$$J_1 + J_3 \le \tfrac{1}{2}\big(\|\partial_{\alpha_1} C_1\|_{L^\infty} + \|\partial_{\alpha_2} C_2\|_{L^\infty}\big)\|\partial^4_{\alpha_1} X\|^2_{L^2} \le P(\|\!|X|\!\|_4).$$

Then in J_2 we use (7-5) to get

$$J_2 = -\int_{\mathbb{R}^2} \partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X)\partial^3_{\alpha_1} C_1\, d\alpha \le \|\partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_1} X)\|_{L^2}\|\partial^3_{\alpha_1} C_1\|_{L^2}.$$

In J_4, we use (7-7) to obtain

$$J_4 = -\int_{\mathbb{R}^2} \partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X)\partial^3_{\alpha_1} C_2 \, d\alpha \leq \|\partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_2} X)\|_{L^2} \|\partial^3_{\alpha_1} C_2\|_{L^2}.$$

From formulas (3-1), (3-2), one realizes that C_1 and C_2 are at the same level as Birkhoff–Rott (2-5), and therefore, we can use the estimates for BR (6-1) to control $\|\partial^3_{\alpha_1} C_i\|_{L^2}$, $i = 1, 2$. Then formulas (7-5) and (7-7) indicate how to estimate $\|\partial_{\alpha_1}(\partial^4_{\alpha_1} X \cdot \partial_{\alpha_i} X)\|_{L^2}$, $i = 1, 2$. That is, we have

$$J_2 + J_4 \leq P(|||X|||_4).$$

In I_1, the most singular terms are given by

$$\begin{aligned}
J_5 &= -\frac{1}{4\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \cdot \frac{(\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \wedge \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta, \\
J_6 &= \frac{3}{4\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta) \frac{(X(\alpha) - X(\beta)) \cdot (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta, \\
J_7 &= -\frac{1}{4\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \cdot \frac{(X(\alpha) - X(\beta)) \wedge (\partial^4_{\alpha_1} \omega)(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta.
\end{aligned} \tag{7-10}$$

Let us consider now the splitting $J_5 = K_1 + K_2$:

$$\begin{aligned}
K_1 &= -\frac{1}{8\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \wedge (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta, \\
K_2 &= \frac{1}{8\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \wedge (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \cdot \frac{\omega(\alpha) - \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,
\end{aligned}$$

Next we exchange α and β in K_1 to get

$$\begin{aligned}
K_1 &= \frac{1}{8\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\beta) \wedge (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta, \\
&= -\frac{1}{16\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \wedge (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta)) \cdot \frac{\omega(\beta) + \omega(\alpha)}{|X(\alpha) - X(\beta)|^3} \, d\alpha \, d\beta,
\end{aligned}$$

and therefore we can conclude that $K_1 = 0$. In K_2 we find a singular integral with a kernel of degree -2:

$$K_2 = -\frac{1}{8\pi} \mathrm{PV}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \cdot \int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\beta) \wedge \frac{\omega(\alpha) - \omega(\beta)}{|X(\alpha) - X(\beta)|^3} \, d\beta \, d\alpha,$$

and as is proved in the Appendix, we have

$$K_2 \leq P(|||X|||_4).$$

Let us now decompose $J_6 = K_3 + K_4^1 + K_4^2 + K_5^1 + K_5^2$, where

$$K_3 = \frac{3}{4\pi} \mathrm{PV}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha) \cdot (X(\alpha) - X(\beta)) \wedge \omega(\beta) \frac{A(\alpha, \beta) \cdot (\partial^4_{\alpha_1} X(\alpha) - \partial^4_{\alpha_1} X(\beta))}{|X(\alpha) - X(\beta)|^5} \, d\alpha \, d\beta,$$

with $A(\alpha,\beta)=X(\alpha)-X(\beta)-\nabla X(\alpha)(\alpha-\beta)$,

$$K_4^i=-\frac{3}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot(X(\alpha)-X(\beta))\wedge\omega(\beta)\frac{(\alpha_i-\beta_i)(\partial_{\alpha_i}X(\alpha)-\partial_{\alpha_i}X(\beta))\cdot\partial_{\alpha_1}^4X(\beta)}{|X(\alpha)-X(\beta)|^5}\,d\alpha\,d\beta$$

$$K_5^i=\frac{3}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot(X(\alpha)-X(\beta))\wedge\omega(\beta)\times\frac{(\alpha_i-\beta_i)(\partial_{\alpha_i}X(\alpha)\cdot\partial_{\alpha_1}^4X(\alpha)-\partial_{\alpha_i}X(\beta)\cdot\partial_{\alpha_1}^4X(\beta))}{|X(\alpha)-X(\beta)|^5}\,d\alpha\,d\beta.$$

In K_3 and K_4^i we find kernels of degree -2, and as shown in the Appendix, they behave as a Riesz transform acting on $\partial_{\alpha_1}^4X$. In K_5^i the kernels have degree -3 and act as a Λ operator on $\partial_{\alpha_i}X\cdot\partial_{\alpha_1}^4X$. Then using formulas (7-5) and (7-7), we get finally the desired estimate.

We will find the R-T condition in J_7. Let us take $J_7=K_6+K_7$, where

$$K_6=-\frac{1}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\int_{\mathbb{R}^2}\Big(\frac{(X(\alpha)-X(\beta))}{|X(\alpha)-X(\beta)|^3}-\frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\Big)\wedge(\partial_{\alpha_1}^4\omega)(\beta)\,d\beta\,d\alpha,$$

$$K_7=-\frac{1}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\int_{\mathbb{R}^2}\frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\wedge(\partial_{\alpha_1}^4\omega)(\beta)\,d\beta\,d\alpha.$$

The term K_6 is controlled by (A-8) in the Appendix. Using (7-2) and (7-3), we get

$$K_7=-\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}^4X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\cdot\big(\partial_{\alpha_1}X(\alpha)\wedge R_1(\partial_{\alpha_1}^4\omega)(\alpha)+\partial_{\alpha_2}X(\alpha)\wedge R_2(\partial_{\alpha_1}^4\omega)(\alpha)\big)\,d\alpha.$$

Formula (2-3) helps us to detect the most singular terms inside K_7, which will be denoted by L_i, $i=1,\dots,8$, and are the following:

$$L_1=-\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega\partial_{\alpha_1}X)(\alpha)\,d\alpha,$$

$$L_2=-\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_2}\Omega\partial_{\alpha_1}^5X)(\alpha)\,d\alpha,$$

$$L_3=\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}^5\Omega\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_4=\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge R_1(\partial_{\alpha_1}\Omega\partial_{\alpha_1}^4\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_5=-\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}^4\partial_{\alpha_2}\Omega\partial_{\alpha_1}X)(\alpha)\,d\alpha,$$

$$L_6=-\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_2}\Omega\partial_{\alpha_1}^5X)(\alpha)\,d\alpha,$$

$$L_7=\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}^5\Omega\partial_{\alpha_2}X)(\alpha)\,d\alpha,$$

$$L_8=\tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial_{\alpha_1}^4X(\alpha)\cdot\frac{\partial_{\alpha_2}X(\alpha)}{|\partial_{\alpha_2}X(\alpha)|^3}\wedge R_2(\partial_{\alpha_1}\Omega\partial_{\alpha_1}^4\partial_{\alpha_2}X)(\alpha)\,d\alpha.$$

In L_1 we get a kernel of degree -1 of the form

$$L_1 = \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2} \partial^4_{\alpha_1} X(\alpha)\cdot \int_{\mathbb{R}^2} \frac{\alpha_1-\beta_1}{|\alpha-\beta|^3}\frac{\partial_{\alpha_1}X(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\wedge(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\beta))\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\beta\,d\alpha,$$

which can be estimated integrating by parts throughout $\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega$; the term L_7 also follows in a similar manner. In order to estimate L_2, L_4, L_6 and L_8, we realize that they can be written like (A-3) in the Appendix plus commutators of the form (A-1). Next we have to deal with L_3 and L_5: with L_3, we proceed as follows:

$$L_3 \le \tilde{L}_3 + \|\partial_{\alpha_1}|X|^{-2}\|_{L^\infty}\|\partial^4_{\alpha_1}X\|_{L^2}\left\|R_1(\partial^5_{\alpha_1}\Omega\partial_{\alpha_2}X)-R_1(\partial^5_{\alpha_1}\Omega)\partial_{\alpha_2}X\right\|_{L^2},$$

where $\tilde{L}_3$ is given by

$$\tilde{L}_3 = \tfrac{1}{2}\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}\Omega)(\alpha)\,d\alpha, \tag{7-11}$$

and the commutator estimates (A-1) show that it only remains to control $\tilde{L}_3$. We now use formula (2-8) to get $\tilde{L}_3 = M_1 + M_2$, where

$$M_1 = -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)(\alpha)\,d\alpha$$

and

$$M_2 = -A_\mu\,\mathrm{PV}\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X(\alpha)\cdot\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}(R_1\partial_{\alpha_1})\big(\partial^3_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)\big)(\alpha)\,d\alpha.$$

Then we write $M_1 = O_1 + O_2 + O_3$, where

$$\begin{aligned}
O_1 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}(\partial_{\alpha_1}X_2\partial_{\alpha_2}X_3-\partial_{\alpha_1}X_3\partial_{\alpha_2}X_2)(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha,\\
O_2 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}(\partial_{\alpha_1}X_3\partial_{\alpha_2}X_1-\partial_{\alpha_1}X_1\partial_{\alpha_2}X_3)(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha,\\
O_3 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha.
\end{aligned} \tag{7-12}$$

Next we consider $O_1 = P_1 + P_2 + P_3$, with

$$\begin{aligned}
P_1 &= -A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_3\partial^4_{\alpha_1}X_3)\,d\alpha,\\
P_2 &= A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_2(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_3\partial^4_{\alpha_1}X_3)\,d\alpha,\\
P_3 &= A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_2\big[(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_3\partial^4_{\alpha_1}X_3)-\partial_{\alpha_2}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\big]\,d\alpha\\
&\quad + A_\rho\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X_1}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_2\big[\partial_{\alpha_1}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)-(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_3\partial^4_{\alpha_1}X_3)\big]\,d\alpha,
\end{aligned}$$

and the commutator estimate allows us to control the term P_3.

Now we use (7-7) to write $P_1 = Q_1 + Q_2 + Q_3$:

$$Q_1 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$Q_2 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$Q_3 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 (R_1 \partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

The term Q_3 is easily estimated. Regarding P_2, equality (7-5) allows us to write $P_2 = Q_4 + Q_5 + Q_6$, where

$$Q_4 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$Q_5 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$Q_6 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 (R_1 \partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

Let us recall the identity $P_1 + P_2 = (Q_4 + Q_1) + (Q_2 + Q_5) + (Q_3 + Q_6)$, where Q_3 and Q_6 are easily estimated. With respect to $Q_2 + Q_5$, we have

$$\begin{aligned} Q_2 + Q_5 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 \big[(R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2) - \partial_{\alpha_2} X_2 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_2)\big]\, d\alpha \\ + A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 \big[\partial_{\alpha_1} X_2 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_2) - (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\big]\, d\alpha, \end{aligned}$$

and again the commutator estimates yield the desired control.

Next we have

$$\begin{aligned} Q_4 + Q_1 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_2 \big[\partial_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) - (R_1 \partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\big]\, d\alpha \\ + A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_1}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_2 \big[(R_1 \partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1) - \partial_{\alpha_2} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\big]\, d\alpha \\ - A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha. \end{aligned}$$

The first two integrals above are easily handled, allowing us to get

$$O_1 = P_1 + P_2 + P_3 \le P(\|\!|X|\!\|_4) - A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1} X|^3}\, \partial^4_{\alpha_1} X_1 (R_1 \partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\, d\alpha. \tag{7-13}$$

For the term O_2, we proceed in a similar manner, first checking that $O_2 = P_4 + P_5 + P_6$:

$$P_4 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial^4_{\alpha_1} X_3)\, d\alpha,$$

$$P_5 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_3 \partial^4_{\alpha_1} X_3)\, d\alpha,$$

$$P_6 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 \big[\partial_{\alpha_2} X_3 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_3) - (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_3 \partial^4_{\alpha_1} X_3)\big]\, d\alpha$$
$$+ A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 \big[(R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_3 \partial^4_{\alpha_1} X_3) - \partial_{\alpha_1} X_3 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_3)\big]\, d\alpha.$$

We control P_6 as before. Regarding P_4, we use (7-7) to write it in the form $P_4 = S_1 + S_2 + S_3$, where

$$S_1 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$S_2 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$S_3 = -A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

The identity (7-5) allows us to write $P_5 = S_4 + S_5 + S_6$, where

$$S_4 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1)\, d\alpha,$$

$$S_5 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_2 \partial^4_{\alpha_1} X_2)\, d\alpha,$$

$$S_6 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\text{lower-order terms})\, d\alpha.$$

Next, we reorganize the sum in the form

$$P_4 + P_6 = (S_1 + S_4) + (S_2 + S_5) + (S_3 + S_6),$$

where the term $S_3 + S_6$ can be easily estimated. Regarding $S_1 + S_4$, we have

$$S_1 + S_4 = A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_1} X_1 \big[\partial_{\alpha_2} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1) - (R_1\partial_{\alpha_1})(\partial_{\alpha_2} X_1 \partial^4_{\alpha_1} X_1)\big]\, d\alpha$$
$$+ A_\rho \,\mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1} X_2}{|\partial_{\alpha_1} X|^3}\, \partial_{\alpha_2} X_1 \big[(R_1\partial_{\alpha_1})(\partial_{\alpha_1} X_1 \partial^4_{\alpha_1} X_1) - \partial_{\alpha_1} X_1 (R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\big]\, d\alpha,$$

and the commutator estimates give us precise control.

Let us consider now

$$\begin{aligned} S_2+S_5 = A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_1}X_1\big[\partial_{\alpha_2}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)-(R_1\partial_{\alpha_1})(\partial_{\alpha_2}X_2\partial^4_{\alpha_1}X_2)\big]\,d\alpha \\ + A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^4_{\alpha_1}X_2}{|\partial_{\alpha_1}X|^3}\,\partial_{\alpha_2}X_1\big[(R_1\partial_{\alpha_1})(\partial_{\alpha_1}X_2\partial^4_{\alpha_1}X_2)-\partial_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\big]\,d\alpha \\ - A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\,d\alpha. \end{aligned}$$

Here again the commutator estimates control the first two integrals above, allowing us to conclude that

$$O_2 = P_4+P_5+P_6 \le P(|||X|||_4) - A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\,d\alpha. \tag{7-14}$$

Furthermore, inequalities (7-13), (7-14) and (7-12) yield

$$M_1 = O_1+O_2+O_3 \le P(|||X|||_4) - A_\rho\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{N_3}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X)\,d\alpha, \tag{7-15}$$

and at this point we begin to recognize the Rayleigh–Taylor condition in the nonintegrable terms. Let us return now to the term M_2, which can be written in the form

$$M_2 = A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\,\partial^4_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X))\,d\alpha, \tag{7-16}$$

and whose most dangerous components are given by

$$O_4 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\wedge\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,$$

$$O_5 = \frac{3A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}B(\alpha,\beta)(X(\alpha)-X(\beta))\wedge\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,$$

with

$$B(\alpha,\beta) = \frac{(X(\alpha)-X(\beta))\cdot(\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_2}X(\beta))}{|X(\alpha)-X(\beta)|^5},$$

$$O_6 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\beta)}{|X(\alpha)-X(\beta)|^3}\wedge\partial^4_{\alpha_1}\omega(\beta)\cdot\partial_{\alpha_1}X(\alpha)\,d\alpha,$$

and

$$O_7 = A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\partial_{\alpha_1}(\mathrm{BR}(X,\omega)\cdot\partial^4_{\alpha_1}X)(\alpha)\,d\alpha.$$

The remainder terms are less singular and can be estimated with the same methods used before.

To deal with O_4, we decompose it further as $O_4 = P_7 + P_8$:

$$P_7 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot\omega(\beta)\wedge(\partial_{\alpha_1}X(\beta)-\partial_{\alpha_1}X(\alpha))\,d\beta\,d\alpha,$$

$$P_8 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{\partial^4_{\alpha_1}X(\alpha)-\partial^4_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\cdot N(\beta)\partial_{\alpha_1}\Omega(\beta)\,d\beta\,d\alpha,$$

where in P_8, we have used formula (2-3) to get

$$\omega\wedge\partial_{\alpha_1}X = N\partial_{\alpha_1}\Omega.$$

In the integral (with respect to β) of P_7, we have a kernel of degree -2 applied to 4 derivatives, which can be estimated easily. Next let us consider $P_8 = Q_7 + Q_8 + Q_9$, where

$$Q_7 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\partial^4_{\alpha_1}X(\alpha)\cdot\int_{\mathbb{R}^2}\frac{N(\alpha)\partial_{\alpha_1}\Omega(\alpha)-N(\beta)\partial_{\alpha_1}\Omega(\beta)}{|X(\alpha)-X(\beta)|^3}\,d\beta\,d\alpha,$$

$$Q_8 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\big((\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\alpha)-(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\beta)\big)C(\alpha,\beta)\,d\beta\,d\alpha,$$

and

$$C(\alpha,\beta) = \frac{1}{|X(\alpha)-X(\beta)|^3}-\frac{1}{|\nabla X(\alpha)(\alpha-\beta)|^3},$$

$$Q_9 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\frac{1}{|\partial_{\alpha_1}X(\alpha)|^3}\Lambda(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)(\alpha)\,d\alpha.$$

In Q_7, we have

$$Q_7 \le \Big\|R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big\|_{L^2}\|\partial^4_{\alpha_1}X\|_{L^2}\sup_\alpha\Big|\int_{\mathbb{R}^2}\frac{N(\alpha)\partial_{\alpha_1}\Omega(\alpha)-N(\beta)\partial_{\alpha_1}\Omega(\beta)}{|X(\alpha)-X(\beta)|^3}\,d\beta\Big|,$$

giving us the appropriate control, which can be also obtained in Q_8 because the corresponding kernel has degree -2. Regarding Q_9, we have the expression

$$Q_9 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big[\frac{1}{|\partial_{\alpha_1}X|^3}\Lambda(\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X)-\Lambda\Big(\frac{\partial_{\alpha_1}\Omega N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3}\Big)\Big]\,d\alpha$$
$$+\frac{A_\mu}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Lambda\Big(\partial_{\alpha_1}\Omega\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\,d\alpha.$$

Then we use (A-2) to control the first integral above, and since $\Lambda = R_1\partial_{\alpha_1}+R_2\partial_{\alpha_2}$ by (A-4), we can also take care of the second term.

With O_5, one proceeds as we did with J_6 (7-10) to get the desired estimate.

Next, we use (2-3) to catch the most singular terms in O_6, which are given by

$$S_7 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\beta)\cdot\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\,\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\alpha,$$

$$S_8 = -\frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\cdot\partial_{\alpha_2}\Omega(\beta)\partial^5_{\alpha_1}X(\beta)\,d\alpha,$$

$$S_9 = \frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_2}X(\beta)\cdot\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\,\partial^5_{\alpha_1}\Omega(\beta)\,d\alpha,$$

$$S_{10} = \frac{A_\mu}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge\partial_{\alpha_1}X(\alpha)}{|X(\alpha)-X(\beta)|^3}\cdot\partial_{\alpha_1}\Omega(\beta)\partial^4_{\alpha_1}\partial_{\alpha_2}X(\beta)\,d\alpha.$$

One may write

$$S_7 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2}\frac{(X(\alpha)-X(\beta))\wedge(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\beta))\cdot\partial_{\alpha_1}X(\beta)}{|X(\alpha)-X(\beta)|^3}\,\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega(\beta)\,d\alpha,$$

expressing the fact that we have a kernel of degree -1 applied to $\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega$, and therefore an integration by parts gives us the desired control, as before. To treat S_8, we further decompose $S_8 = T_1 + T_2$:

$$T_1 = -\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\int_{\mathbb{R}^2} D(\alpha,\beta)\cdot\partial_{\alpha_2}\Omega(\beta)\partial^5_{\alpha_1}X(\beta)\,d\alpha,$$

where

$$D(\alpha,\beta) = \Big(\frac{(X(\alpha)-X(\beta))}{|X(\alpha)-X(\beta)|^3} - \frac{\nabla X(\alpha)(\alpha-\beta)}{|\nabla X(\alpha)(\alpha-\beta)|^3}\Big)\wedge\partial_{\alpha_1}X(\alpha)$$

and

$$T_2 = \frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(\alpha)\frac{N(\alpha)}{|\partial_{\alpha_1}X(\alpha)|^3}\cdot R_2(\partial_{\alpha_2}\Omega\partial^5_{\alpha_1}X)(\alpha)\,d\alpha.$$

In T_1, we use the estimate for the operator (A-8). The term T_2 reads as follows:

$$\begin{aligned}T_2 = &-\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\frac{N}{|\partial_{\alpha_1}X|^3}\cdot R_2(\partial_{\alpha_2}\partial_{\alpha_1}\Omega\partial^4_{\alpha_1}X)\,d\alpha\\ &+\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)\Big[\frac{N}{|\partial_{\alpha_1}X|^3}\cdot(R_2\partial_{\alpha_1})(\partial_{\alpha_2}\Omega\partial^4_{\alpha_1}X)-(R_2\partial_{\alpha_1})(\partial_{\alpha_2}\Omega\frac{N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3})\Big]\,d\alpha\\ &-\frac{A_\mu}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1}X\cdot N}{|\partial_{\alpha_1}X|^3}\Big)(R_2\partial_{\alpha_1})\Big(\partial_{\alpha_2}\Omega\frac{N\cdot\partial^4_{\alpha_1}X}{|\partial_{\alpha_1}X|^3}\Big)\,d\alpha.\end{aligned}$$

The first integral above is easy to estimate, while for the second one we use (A-1), and (A-4) for the third.

For the next term, one has $S_9 = T_3 + T_4$, where

$$T_3 = \frac{A_\mu}{4\pi} \,\mathrm{PV}\!\int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1} X\cdot N}{|\partial_{\alpha_1} X|^3}\Big)(\alpha)\int_{\mathbb{R}^2} \frac{(X(\alpha)-X(\beta))\cdot \partial_{\alpha_2} X(\beta)\wedge(\partial_{\alpha_1} X(\alpha)-\partial_{\alpha_1} X(\beta))}{|X(\alpha)-X(\beta)|^3}\,\partial^5_{\alpha_1}\Omega(\beta)\,d\alpha,$$

$$T_4 = -A_\mu \int_{\mathbb{R}^2} R_1\Big(\frac{\partial^4_{\alpha_1} X\cdot N}{|\partial_{\alpha_1} X|^3}\Big)\mathfrak{D}(\partial^5_{\alpha_1}\Omega)\,d\alpha,$$

Proceeding as before, we get bounds for T_3, and the double-layer potential estimates help us to control T_4.

For S_{10}, one can adapt exactly the same approach used for S_8. Finally, we have to deal with O_7, which is given by

$$O_7 = -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \mathrm{BR}(X,\omega)\cdot\partial^4_{\alpha_1} X(R_1\partial_{\alpha_1})\Big(\frac{\partial^4_{\alpha_1} X\cdot N}{|\partial_{\alpha_1} X|^3}\Big)\,d\alpha,$$

after an integration by parts. Let us introduce the splitting $O_7 = \sum_{j,k=1}^3 U^k_j$, where

$$U^k_j = -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \mathrm{BR}_j(X,\omega)\partial^4_{\alpha_1} X_j(R_1\partial_{\alpha_1})\Big(\frac{\partial^4_{\alpha_1} X_k N_k}{|\partial_{\alpha_1} X|^3}\Big)\,d\alpha.$$

Then the commutator estimates allow us to write $U^k_j = V^k_j + \text{lower order terms}$, where

$$V^k_j = -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \mathrm{BR}_j(X,\omega)\partial^4_{\alpha_1} X_j \frac{N_k}{|\partial_{\alpha_1} X|^3}(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_k)\,d\alpha.$$

Using (7-5) and (7-7), one has

$$N_1\partial^4_{\alpha_1} X_2 = N_2\partial^4_{\alpha_1} X_1 + \text{lower-order terms},$$

so that V^1_2 becomes

$$V^1_2 = -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\mathrm{BR}_2(X,\omega)N_2}{|\partial_{\alpha_1} X|^3}\,\partial^4_{\alpha_1} X_1(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\,d\alpha - A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} f(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\,d\alpha,$$

where f is at the level of $\partial^3_{\alpha_i} X$. Integration by parts in the last integral allows us to conclude that

$$V^1_2 \le -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\mathrm{BR}_2(X,\omega)N_2}{|\partial_{\alpha_1} X|^3}\,\partial^4_{\alpha_1} X_1(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\,d\alpha + P(|||X|||_4).$$

With the help of (7-5) and (7-7), we also get

$$N_1\partial^4_{\alpha_1} X_3 = N_3\partial^4_{\alpha_1} X_1 + \text{lower-order terms},$$

and therefore

$$V^1_3 \le -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\mathrm{BR}_3(X,\omega)N_3}{|\partial_{\alpha_1} X|^3}\,\partial^4_{\alpha_1} X_1(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\,d\alpha + P(|||X|||_4).$$

Using the two inequalities above, we obtain

$$V^1_1 + V^1_2 + V^1_3 \le -A_\mu \,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1} X|^3}\,\partial^4_{\alpha_1} X_1(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1} X_1)\,d\alpha + P(|||X|||_4). \tag{7-17}$$

Next, let us observe that

$$N_2\partial^4_{\alpha_1}X_1 = N_1\partial^4_{\alpha_1}X_2 + \text{lower-order terms},\qquad N_2\partial^4_{\alpha_1}X_3 = N_3\partial^4_{\alpha_1}X_2 + \text{lower-order terms},$$

which implies the estimate

$$V_1^2+V_2^2+V_3^2 \le -A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_2(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_2)\,d\alpha + P(|||X|||_4). \tag{7-18}$$

Regarding V_1^3 and V_2^3, the identities

$$N_3\partial^4_{\alpha_1}X_1 = N_1\partial^4_{\alpha_1}X_3 + \text{lower-order terms},\qquad N_3\partial^4_{\alpha_1}X_3 = N_2\partial^4_{\alpha_1}X_3 + \text{lower-order terms}$$

yield

$$V_1^3+V_2^3+V_3^3 \le -A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X_3(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X_3)\,d\alpha + P(|||X|||_4). \tag{7-19}$$

Finally (7-17), (7-18) and (7-19) imply

$$\sum_{j,k=1}^{3}V_j^k \le -A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4).$$

Now we put together the estimates (7-16)–(7-19) to conclude that

$$M_2 \le -A_\mu\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\mathrm{BR}(X,\omega)\cdot N}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4),$$

and taking into account (7-15), we obtain

$$\tilde{L}_3 = M_1+M_2 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_1\partial_{\alpha_1})(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4). \tag{7-20}$$

Finally, we have to work with L_5, which can be written in the following manner:

$$L_5 = \tilde{L}_5 - \tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\frac{\partial_{\alpha_2}X}{|\partial_{\alpha_2}X|^3}\wedge\big[R_2(\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega\partial_{\alpha_1}X) - R_2(\partial^4_{\alpha_1}\partial_{\alpha_2}\Omega)\partial_{\alpha_1}X\big]\,d\alpha,$$

where

$$\tilde{L}_5 = \tfrac{1}{2}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\partial^4_{\alpha_1}X\cdot\frac{N}{|\partial_{\alpha_2}X|^3}(R_2\partial_{\alpha_2})(\partial^4_{\alpha_1}\Omega)\,d\alpha.$$

Using the commutator estimate, once more, it remains only to consider $\tilde{L}_5$, but let us point out that replacing the operator $R_1\partial_{\alpha_1}$ by $R_2\partial_{\alpha_2}$, the term $\tilde{L}_3$ (7-11) becomes $\tilde{L}_5$. Therefore, proceeding exactly as we did before, one obtains the inequality

$$\tilde{L}_5 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot(R_2\partial_{\alpha_2})(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4). \tag{7-21}$$

Introducing now the identity $\Lambda = (R_1\partial_{\alpha_1}) + (R_2\partial_{\alpha_2})$ in (7-20) and (7-21), we get

$$\tilde{L}_3+\tilde{L}_5 \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot\Lambda(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4).$$

Finally, all the estimates so far obtained, beginning with (7-9), allow us to write

$$\frac{1}{2}\frac{d}{dt}\|\partial^4_{\alpha_1}X\|^2_{L^2}(t) \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_1}X\cdot\Lambda(\partial^4_{\alpha_1}X)\,d\alpha + P(|||X|||_4). \tag{7-22}$$

In a similar manner, now using equations (2-9), (7-6) and (7-8) instead of (2-8), (7-5) and (7-7) respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^4_{\alpha_2}X\|^2_{L^2}(t) \le -\frac{1}{\mu_2+\mu_1}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{\sigma}{|\partial_{\alpha_1}X|^3}\,\partial^4_{\alpha_2}X\cdot\Lambda(\partial^4_{\alpha_2}X)\,d\alpha + P(|||X|||_4). \tag{7-23}$$

The two inequalities (7-22) and (7-23) are the main purpose of this section.

8. Estimates for the evolution of $\|F(X)\|_{L^\infty}$ and R-T

In this section we analyze the evolution of the no-self-intersection condition of the free surface as well as the Rayleigh–Taylor property, but in order to do that, we shall need precise bounds for both ∇X_t and Ω_t.

We shall estimate $\|\nabla X_t\|_{H^k}$ by means of equality (2-4) to get

$$\|\nabla X_t\|_{H^k} \le P\big(\|X\|^2_{k+2} + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big), \tag{8-1}$$

for $k \ge 2$. In fact

$$\|\nabla X_t\|_{H^k} \le \|\nabla\,\mathrm{BR}(X,\omega)\|_{H^k} + \|\nabla(C_1\partial_{\alpha_1}X + C_2\partial_{\alpha_2}X)\|_{H^k},$$

and with the help of (6-1), we can handle both terms on the right.

Next we shall consider the norms $\|\Omega_t\|_{H^k}$ to obtain the inequality

$$\|\Omega_t\|_{H^k} \le P\big(\|X\|^2_{k+1} + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big), \tag{8-2}$$

for $k \ge 3$. To do that, let us take a time derivative in the identity (2-6) to get

$$\Omega_t(\alpha,t) - A_\mu\mathcal{D}(\Omega_t)(\alpha,t) = A_\mu I_1(\alpha,t) - 2A_\rho\partial_t X_3(\alpha,t),$$

which yields

$$\|\Omega_t\|_{H^1} \le C\|(I-A_\mu\mathcal{D})^{-1}\|_{H^1}\big(\|I_1\|_{H^1} + \|\partial_t X_3\|_{H^1}\big),$$

and since we have control of $\|(I-A_\mu\mathcal{D})^{-1}\|_{H^1}$ and $\|\partial_t X_3\|_{H^1}$, it only remains to estimate $\|I_1\|_{H^1}$. For that purpose, let us consider the splitting $I_1 = J_1 + J_2 + J_3$, where

$$J_1 = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{X_t(\alpha)-X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_2 = -\frac{3}{4\pi}\int_{\mathbb{R}^2}\big(X(\alpha)-X(\alpha-\beta)\big)\cdot\big(X_t(\alpha)-X_t(\alpha-\beta)\big)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_3 = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N_t(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta.$$

Proceeding as we did with the operator $\mathcal{T}_2$ (A-6) (with X_t instead of $\partial_{\alpha_j}X_k$), one gets

$$\|J_1\|_{L^2} + \|J_2\|_{L^2} \le P\big(\|X\|_4 + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big).$$

Regarding J_3, we split further:

$$J_3 = \frac{1}{2\pi}\int_{|\beta|>1} d\beta + \frac{1}{2\pi}\int_{|\beta|<1} d\beta = K_1 + K_2.$$

Since

$$|K_1(\alpha)| \le \|F(X)\|^2_{L^\infty}\int_{|\beta|>1}\frac{|N_t(\alpha-\beta)||\Omega(\alpha-\beta)|}{2\pi|\beta|^2}\,d\beta,$$

Young's inequality yields

$$\|K_1\|_{L^2} \le \|F(X)\|^2_{L^\infty}\|N_t\Omega\|_{L^1} \le C\|F(X)\|^2_{L^\infty}\|N_t\|_{L^2}\|\Omega\|_{L^2},$$

and since we know that $\|N_t\|_{L^2} \le \|\nabla X\|_{L^\infty}\|\nabla X_t\|_{L^2}$, estimate (8-1) allows us to handle the terms K_1. The estimate for K_2 is similar to the one obtained for I_2 (A-13) in the Appendix.

Next we consider the most singular terms in $\partial_{\alpha_1} I_1$, which are given by

$$J_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_5 = -\frac{3}{4\pi}\int_{\mathbb{R}^2}\big(X(\alpha)-X(\alpha-\beta)\big) \cdot\big(\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)\big)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\cdot N(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

$$J_6 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot\partial_{\alpha_1}N_t(\alpha-\beta)\Omega(\alpha-\beta)\,d\beta,$$

because the remainder terms are easier to handle. Let us write $J_4 = K_3 + K_4$, where

$$K_3 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot\big(N(\alpha-\beta)\Omega(\alpha-\beta)-N(\alpha)\Omega(\alpha)\big)\,d\beta,$$

$$K_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\cdot N(\alpha)\Omega(\alpha)\,d\beta.$$

In K_3, the identity $\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta) = \int_0^1 \nabla\partial_{\alpha_1}X_t(\alpha+(s-1)\beta)\,ds\cdot\beta$ together with (8-1) gives us the desired control. Regarding K_4, we may observe its similarity with $\mathcal{T}_3$ (A-7), so that an application to (8-1) yields the appropriate bound; J_5 can be treated in a similar manner, and J_6 is analogous to J_3. By symmetry, one could get the same estimate for $\partial_{\alpha_2} I_1$, so that finally

$$\|\Omega_t\|_{H^1} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big). \tag{8-3}$$

Next, we will show how to deal with $\|\Omega_t\|_{H^2}$. Using Equation (2-8), one gets

$$\partial^2_{\alpha_1}\Omega_t = -2A_\mu\partial_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X) - 2A_\rho\partial^2_{\alpha_1}\partial_t X_3,$$

and with the help of (8-1), the last term above is properly controlled. To continue, we shall consider the most singular remainder terms. Namely, in $-\partial_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)$, we have

$$L_1 = -\mathrm{BR}(X,\omega)\cdot\partial^2_{\alpha_1}X_t,$$
$$L_2 = \frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$
$$L_3 = -\frac{3}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}A(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5}\wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

where $A(\alpha,\beta)=\big(X(\alpha)-X(\alpha-\beta)\big)\cdot\big(\partial_{\alpha_1}X_t(\alpha)-\partial_{\alpha_1}X_t(\alpha-\beta)\big)$,

$$L_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}^2}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\omega_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Let us observe that $\|L_1\|_{L^2}\le\|\mathrm{BR}(X,\omega)\|_{L^\infty}\|\partial^2_{\alpha_1}X_t\|_{L^2}$, where both quantities have been appropriately controlled before. In L_2 and L_3, we have kernels of degree -2, and therefore operators analogous to $\mathcal{T}_3$ (A-7) acting on $\partial_{\alpha_1}X_t$. Therefore, using (8-1), its control follows easily. In L_4, we use the decomposition

$$L_4 = \frac{1}{2\pi}\,\mathrm{PV}\int_{|\beta|>1}d\beta+\frac{1}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}d\beta = M_1+M_2.$$

Thus, an integration by parts yields

$$\|M_1\|_{L^2}\le C\|F(X)\|^3_{L^\infty}\|\nabla X\|^2_{L^\infty}\|w_t\|_{L^2}.$$

Formula (2-3), together with estimates (8-1) and (8-3), provides the appropriated bound.

Next, let us expand (2-3) to obtain the most singular terms in M_2, which are given by the integrals

$$O_1 = -\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_2}\Omega(\alpha-\beta)\partial^2_{\alpha_1}X_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$
$$O_2 = -\frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta)\partial_{\alpha_1}X(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$
$$O_3 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\Omega(\alpha-\beta)\partial_{\alpha_1}\partial_{\alpha_2}X_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$
$$O_4 = \frac{A_\mu}{2\pi}\,\mathrm{PV}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\partial_{\alpha_2}X(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Estimate (8-1) help us with the terms O_1 and O_3, which can be treated with the same approach used for I_2 (A-13) in the Appendix. Let us write O_2 as

$$O_2 = \frac{A_\mu}{2\pi}\int_{|\beta|<1}\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3}\wedge\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta)\big(\partial_{\alpha_1}X(\alpha)-\partial_{\alpha_1}X(\alpha-\beta)\big)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

which can be estimated integrating by parts in the variable β_1 using the identity

$$\partial_{\alpha_1}\partial_{\alpha_2}\Omega_t(\alpha-\beta) = -\partial_{\beta_1}(\partial_{\alpha_2}\Omega_t(\alpha-\beta)).$$

Let us point out that the kernel in the integral O_2 has degree -1, and therefore one can use (8-3) to control it. It remains to deal with O_4, which is decomposed in the form $O_4 = P_1 + P_2$, where

$$P_1 = \frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\big(\partial_{\alpha_2}X(\alpha-\beta)-\partial_{\alpha_2}X(\alpha)\big)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$P_2 = -\frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \partial^2_{\alpha_1}\Omega_t(\alpha-\beta)\,d\beta\cdot N(\alpha).$$

P_1 is estimated like O_2. We rewrite P_2 as follows:

$$P_2 = -\frac{A_\mu}{2\pi} \,\mathrm{PV}\!\int_{|\beta|<1} \left(\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{\nabla X(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\right)\partial^2_{\alpha_1}\Omega(\alpha-\beta)\,d\beta\cdot N(\alpha),$$

and this expression shows that the above integral can be estimated like $\mathcal{T}_4$ (A-8).

Using (8-3), we obtain

$$\|\partial^2_{\alpha_1}\Omega_t\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big),$$

and the identity

$$\partial^2_{\alpha_2}\Omega_t = -2A_\mu\partial_{\alpha_2}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_2}X) - 2A_\rho\partial^2_{\alpha_2}\partial_t X_3$$

yields

$$\|\partial^2_{\alpha_2}\Omega_t\|_{L^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big),$$

that is,

$$\|\Omega_t\|_{H^2} \le P\big(\|X\|_4^2 + \|F(X)\|^2_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big). \tag{8-4}$$

Next we consider third-order derivatives:

$$\partial^3_{\alpha_1}\Omega_t = -2A_\mu\partial^2_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X) - 2A_\rho\partial^3_{\alpha_1}\partial_t X_3.$$

Since (8-1) gives us control of the last term, we will concentrate on the other one, which is of a much more difficult character. In particular, for $-\partial^2_{\alpha_1}\partial_t(\mathrm{BR}(X,\omega)\cdot\partial_{\alpha_1}X)$, the most singular components are given by

$$L_5 = -\,\mathrm{BR}(X,\omega)\cdot\partial^3_{\alpha_1}X_t,$$

$$L_6 = \frac{1}{4\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{\partial^2_{\alpha_1}X_t(\alpha)-\partial^2_{\alpha_1}X_t(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

$$L_7 = -\frac{3}{8\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} B(\alpha,\beta)\frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} \wedge\omega(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha),$$

where $B(\alpha,\beta) = \big(X(\alpha)-X(\alpha-\beta)\big)\cdot\big(\partial^2_{\alpha_1}X_t(\alpha)-\partial^2_{\alpha_1}X_t(\alpha-\beta)\big)$,

$$L_8 = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}^2} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge\partial^2_{\alpha_1}\omega_t(\alpha-\beta)\,d\beta\cdot\partial_{\alpha_1}X(\alpha).$$

Inequalities (8-1) and (8-4) show how to handle L_i, $i=5,\dots,8$ as L_j, $j=1,\dots,4$ respectively, and then a similar approach for $\partial^3_{\alpha_2}\Omega_t$ allows us to get finally (8-2) for $k=3$. The cases $k>3$ are similar to deal with.

Our next goal is to obtain estimates for the evolution of $\|F(X)\|_{L^\infty}$ and R-T. Regarding the quantity $F(X)$, we have

$$\begin{aligned}\frac{d}{dt}F(X)(\alpha,\beta,t) &= -\frac{|\beta|\big(X(\alpha,t)-X(\alpha-\beta,t)\big)\cdot\big(X_t(\alpha,t)-X_t(\alpha-\beta,t)\big)}{|X(\alpha,t)-X(\alpha-\beta,t)|^3}\\ &\le (F(X)(\alpha,\beta,t))^2\|\nabla X_t\|_{L^\infty}(t).\end{aligned} \tag{8-5}$$

Then Sobolev inequalities in $\|\nabla X_t\|_{L^\infty}(t)$, together with (8-1), yield

$$\frac{d}{dt}F(X)(\alpha,\beta,t)\le F(X)(\alpha,\beta,t)P\big(\|X\|_4^2(t)+\|F(X)\|_{L^\infty}^2(t)+\||N|^{-1}\|_{L^\infty}(t)\big),$$

and an integration in time gives us

$$F(X)(\alpha,\beta,t+h)\le F(X)(\alpha,\beta,t)\exp\bigg(\int_t^{t+h}P(s)\,ds\bigg),$$

for $h>0$, where

$$P(s)=P\big(\|X\|_4^2(s)+\|F(X)\|_{L^\infty}^2(s)+\||N|^{-1}\|_{L^\infty}(s)\big).$$

Hence

$$\|F(X)\|_{L^\infty}(t+h)\le\|F(X)\|_{L^\infty}(t)\exp\bigg(\int_t^{t+h}P(s)\,ds\bigg).$$

This inequality, applied to the limit

$$\frac{d}{dt}\|F(X)\|_{L^\infty}(t)=\lim_{h\to0^+}\frac{\|F(X)\|_{L^\infty}(t+h)-\|F(X)\|_{L^\infty}(t)}{h},$$

allows us to get

$$\frac{d}{dt}\|F(X)\|_{L^\infty}(t)\le\|F(X)\|_{L^\infty}(t)P\big(\|X\|_4^2+\|F(X)\|_{L^\infty}^2+\||N|^{-1}\|_{L^\infty}\big).$$

Next we search for an a priori estimate for the evolution of the infimum of the difference of the gradients of the pressure in the normal direction to the interface. Let us recall the formula

$$\sigma(\alpha,t)=(\mu^2-\mu^1)\,\mathrm{BR}(X,\omega)(\alpha,t)\cdot N(\alpha,t)+(\rho^2-\rho^1)N_3(\alpha,t)$$

to obtain

$$\frac{d}{dt}\bigg(\frac{1}{\sigma(\alpha,t)}\bigg)=-\frac{\sigma_t(\alpha,t)}{\sigma^2(\alpha,t)},$$

with $\sigma_t(\alpha,t)=I_1+I_2$, where

$$\begin{aligned}I_1&=\big((\mu^2-\mu^1)\mathrm{BR}(X,\omega)(\alpha,t)+(\rho^2-\rho^1)(0,0,1)\big)\cdot N_t(\alpha,t),\\ I_2&=(\mu^2-\mu^1)\mathrm{BR}_t(X,\omega)(\alpha,t)\cdot N(\alpha,t).\end{aligned}$$

First we deal with $\|I_1\|_{L^\infty}$ using the estimates (8-1) for ∇X_t, and then we focus our attention on I_2 using the splitting $I_2 = J_1 + J_2 + J_3$, where

$$J_1 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta,$$

$$J_2 = \frac{3}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \big(X(\alpha) - X(\alpha-\beta)\big) \wedge \omega(\alpha-\beta) \frac{(X(\alpha) - X(\alpha-\beta))\cdot(X_t(\alpha) - X_t(\alpha-\beta))}{|X(\alpha) - X(\alpha-\beta)|^5}\, d\beta,$$

$$J_3 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}^2} \frac{X(\alpha) - X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega_t(\alpha-\beta)\, d\beta.$$

The terms J_1 and J_2 are similar and can be treated with the same method. Let us consider $J_1 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = -\frac{1}{4\pi}\int_{|\beta|>1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \omega(\alpha-\beta)\, d\beta,$$

$$K_2 = \frac{1}{4\pi}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3} \wedge \big(\omega(\alpha) - \omega(\alpha-\beta)\big)\, d\beta,$$

$$K_3 = -\frac{1}{4\pi}\int_{|\beta|<1} \left[\frac{1}{|X(\alpha) - X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\right] \big(X_t(\alpha) - X_t(\alpha-\beta)\big) \wedge \omega(\alpha)\, d\beta,$$

$$K_4 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta)}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha)\, d\beta.$$

First we have

$$\|K_1\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X_t\|_{L^\infty}\|\omega\|_{L^2}\left(\int_{|\beta|>1} |\beta|^{-4} d\beta\right)^{1/2},$$

giving us an appropriate control. Next, we get

$$\|K_2\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X_t\|_{L^\infty}\|\nabla\omega\|_{L^\infty}\int_{|\beta|<1} |\beta|^{-1} d\beta,$$

and an analogous estimate for K_3. Therefore, Sobolev's embedding helps us to obtain the desired control. Regarding K_4, we have

$$K_4 = -\frac{1}{4\pi}\int_{|\beta|<1} \frac{X_t(\alpha) - X_t(\alpha-\beta) - \nabla X_t(\alpha)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3} \wedge \omega(\alpha)\, d\beta.$$

Inequality (A-15) yields

$$\|K_4\|_{L^\infty} \le C\|\nabla X\|^3_{L^\infty}\||N|^{-1}\|^3_{L^\infty}\|\omega\|_{L^\infty}\|\nabla X_t\|_{C^\delta}\int_{|\beta|<1} |\beta|^{-2+\delta}\, d\beta,$$

and the control $\|\nabla X_t\|_{C^\delta}$ follows again by (8-1) and Sobolev's embedding. Next let us continue with $J_3 = K_5 + K_6$, where

$$K_5 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|>1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \big(\partial_{\beta_1}((\Omega\partial_{\alpha_2}X)_t(\alpha-\beta)) - \partial_{\beta_2}((\Omega\partial_{\alpha_1}X)_t(\alpha-\beta))\big)\,d\beta,$$

$$K_6 = -\frac{1}{4\pi}\,\mathrm{PV}\int_{|\beta|<1} \frac{X(\alpha)-X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^3} \wedge \omega_t(\alpha-\beta)\,d\beta.$$

Integration by parts yields

$$\|K_5\|_{L^\infty} \le C\|F(X)\|^3_{L^\infty}\|\nabla X\|_{L^\infty}\big(\|\Omega\|_{L^\infty}\|\nabla X_t\|_{L^\infty} + \|\Omega_t\|_{L^\infty}\|\nabla X\|_{L^\infty}\big),$$

where $4\pi C = \int_{|\beta|>1}|\beta|^{-3}d\beta + \int_{|\beta|=1} dl(\beta)$, and we may use (8-2) to estimate $\|\Omega_t\|_{L^\infty}$. With K_6, we introduce a similar splitting to obtain

$$\|K_6\|_{L^\infty} \le P\big(\|X-(\alpha,0)\|_{C^2} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|\omega_t\|_{C^\delta}.$$

Then it remains to estimate $\|\omega_t\|_{C^\delta}$, for which purpose we use formula (2-3) and inequalities (8-1), (8-2). Therefore, we have the estimate

$$\frac{d}{dt}\left(\frac{1}{\sigma(\alpha,t)}\right) \le \frac{1}{\sigma^2(\alpha,t)} P\big(\|X\|_4(t) + \|F(X)\|_{L^\infty}(t) + \||N|^{-1}\|_{L^\infty}(t)\big),$$

and proceeding similarly as we did for $F(X)$, we finally get

$$\frac{d}{dt}\|\sigma^{-1}\|_{L^\infty}(t) \le \|\sigma^{-1}\|^2_{L^\infty}(t)P\big(\|X\|_4(t) + \|F(X)\|_{L^\infty}(t) + \||N|^{-1}\|_{L^\infty}(t)\big).$$

Remark 8.1. Having obtained the a priori bounds of the preceding sections, we are in position to successfully implement the same approximation scheme developed in [Córdoba et al. 2011] to conclude local existence.

Appendix

Here we prove first some helpful inequalities regarding commutators of the Riesz transform (R_j, $j = 1, 2$) with several differential operators. Next we analyze the singular integral operators associated to the non-self-intersecting surface which appears throughout the paper. But the main goal of this section is to simplify the presentation of the main result.

Lemma A.1. *Consider $f \in L^2(\mathbb{R}^2)$ and $g \in C^{1,\delta}(\mathbb{R}^2)$, with $0 < \delta < 1$. Then for any $k, l = 1, 2$, we have the estimate*

$$\big\|(R_k\partial_{\alpha_l})(gf) - g(R_k\partial_{\alpha_l})(f)\big\|_{L^2} \le C\|g\|_{C^{1,\delta}}\|f\|_{L^2}. \tag{A-1}$$

An application of these inequalities to the operator $\Lambda = (R_1\partial_{\alpha_1}) + (R_2\partial_{\alpha_2})$ yields

$$\|\Lambda(gf) - g\Lambda(f)\|_{L^2} \le C\|g\|_{C^{1,\delta}}\|f\|_{L^2}. \tag{A-2}$$

For vector fields, we have:

Lemma A.2. *Consider* $f, g : \mathbb{R}^2 \to \mathbb{R}^3$ *vector fields, where* $f \in L^2(\mathbb{R}^2)$ *and* $g \in C^{1,\delta}(\mathbb{R}^2)$, *with* $0 < \delta < 1$. *Then for any* $k, l = 1, 2$, *the following inequality holds*:

$$\left| \int_{\mathbb{R}^2} (g \wedge f) \cdot (R_k \partial_{\alpha_l})(f) d\alpha \right| \le C \|g\|_{C^{1,\delta}} \|f\|_{L^2}^2. \tag{A-3}$$

Proof. Denoting by I the integral above, and since the operator $R_k \partial_{\alpha_l}$ is self-adjoint, we may write

$$I = \int_{\mathbb{R}^2} f_1 \big[(R_k \partial_{\alpha_l})(g_2 f_3) - g_2 (R_k \partial_{\alpha_l})(f_3) \big] \, d\alpha + \int_{\mathbb{R}^2} f_2 \big[(R_k \partial_{\alpha_l})(g_3 f_1) - g_3 (R_k \partial_{\alpha_l})(f_1) \big] \, d\alpha + \int_{\mathbb{R}^2} f_3 \big[(R_k \partial_{\alpha_l})(g_1 f_2) - g_1 (R_k \partial_{\alpha_l})(f_2) \big] \, d\alpha.$$

Then estimate (A-1) yields (A-3). □

Lemma A.3. *Consider* $f \in L^2(\mathbb{R}^2)$ *and* $g \in C^{1,\delta}(\mathbb{R}^2)$, *with* $0 < \delta < 1$. *Then for any* $j, k, l = 1, 2$, *the following inequality holds*:

$$\left| \int_{\mathbb{R}^2} R_j(f) (R_k \partial_{\alpha_l})(g f) \, d\alpha \right| \le C \|g\|_{C^{1,\delta}} \|f\|_{L^2}^2. \tag{A-4}$$

Proof. Let J be the integral to be bounded; then we have

$$J = \int_{\mathbb{R}^2} R_j(f) \big[(R_k \partial_{\alpha_l})(g f) - g (R_k \partial_{\alpha_l})(f) \big] \, d\alpha - \int_{\mathbb{R}^2} \big[R_j(f g) - g R_j(f) \big] (R_k \partial_{\alpha_l})(f) \, d\alpha + \int_{\mathbb{R}^2} R_j(f g) (R_k \partial_{\alpha_l})(f) \, d\alpha.$$

Since $R_j^* = -R_j$ and $R_k \partial_{\alpha_l}$ is self-adjoint, we get

$$J = \frac{1}{2} \int_{\mathbb{R}^2} R_j(f) \big[(R_k \partial_{\alpha_l})(g f) - g (R_k \partial_{\alpha_l})(f) \big] \, d\alpha - \frac{1}{2} \int_{\mathbb{R}^2} \big[R_j(f g) - g R_j(f) \big] (R_k \partial_{\alpha_l})(f) \, d\alpha.$$

An integration by parts in the second integral above yields

$$J = \frac{1}{2} \int_{\mathbb{R}^2} R_j(f) \big[(R_k \partial_{\alpha_l})(g f) - g (R_k \partial_{\alpha_l})(f) \big] \, d\alpha + \frac{1}{2} \int_{\mathbb{R}^2} \big[(R_j \partial_{\alpha_l})(f g) - g (R_j \partial_{\alpha_l})(f) \big] (R_k)(f) \, d\alpha - \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{\alpha_l} g) R_j(f) R_k(f) \, d\alpha,$$

allowing us to conclude the proof. □

Lemma A.4. *Let us define, for any $j = 1, 2$ and $k = 1, 2, 3$, the following operators*:

$$\mathcal{T}_1(\partial_{\alpha_j} f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{X_k(\alpha) - X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, \partial_{\alpha_j} f(\alpha-\beta)\, d\beta, \tag{A-5}$$

$$\mathcal{T}_2(f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{\partial_{\alpha_j} X_k(\alpha) - \partial_{\alpha_j} X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, f(\alpha-\beta)\, d\beta, \tag{A-6}$$

$$\mathcal{T}_3(f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \frac{f(\alpha) - f(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta, \tag{A-7}$$

$$\mathcal{T}_4(\partial_{\alpha_j} f)(\alpha) = \mathrm{PV}\int_{\mathbb{R}^2} \left(\frac{(X(\alpha) - X(\beta))}{|X(\alpha) - X(\beta)|^3} - \frac{\nabla X(\alpha)\cdot(\alpha-\beta)}{|\nabla X(\alpha)\cdot(\alpha-\beta)|^3} \right) \partial_{\alpha_j} f(\beta)\, d\beta\, d\alpha, \tag{A-8}$$

where $\nabla X(\alpha)\cdot\beta = \partial_{\alpha_1} X(\alpha)\beta_1 + \partial_{\alpha_2} X(\alpha)\beta_2$. Assume that $X(\alpha) - (\alpha, 0) \in C^{2,\delta}(\mathbb{R}^2)$, and that both $F(X)$ and $|N|^{-1}$ are in L^∞, where

$$F(X)(\alpha,\beta) = |\beta| \big/ |X(\alpha) - X(\alpha-\beta)| \quad \text{and} \quad N(\alpha) = \partial_{\alpha_1} X(\alpha) \wedge \partial_{\alpha_2} X(\alpha).$$

Then the following estimates hold:

$$\|\mathcal{T}_1(\partial_{\alpha_j} f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{1,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)(\|f\|_{L^2} + \|\partial_{\alpha_j} f\|_{L^2}), \tag{A-9}$$

$$\|\mathcal{T}_2(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{L^2}, \tag{A-10}$$

$$\|\mathcal{T}_3(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{H^1}, \tag{A-11}$$

$$\|\mathcal{T}_4(f)\|_{L^2} \le P\big(\|X - (\alpha,0)\|_{C^{2,\delta}} + \|F(X)\|_{L^\infty} + \||N|^{-1}\|_{L^\infty}\big)\|f\|_{L^2}, \tag{A-12}$$

with P a polynomial function.

Proof. To estimate the first set of operators, we first consider the splitting

$$\mathcal{T}_1(\partial_{\alpha_j} f) = \mathrm{PV}\int_{|\beta|>1} d\beta + \mathrm{PV}\int_{|\beta|<1} d\beta = I_1 + I_2, \tag{A-13}$$

and an integration by parts allows us to write $I_1 = J_1 + J_2 + J_3$, where

$$J_1 = \int_{|\beta|>1} -\frac{\partial_{\alpha_j} X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, f(\alpha-\beta)\, d\beta,$$

$$J_2 = 3\int_{|\beta|>1} \frac{\big(X_k(\alpha) - X_k(\alpha-\beta)\big)\big(X(\alpha) - X(\alpha-\beta)\big)\cdot\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^5}\, f(\alpha-\beta)\, d\beta,$$

$$J_3 = \int_{|\beta|=1} \frac{X_k(\alpha) - X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, f(\alpha-\beta)\, dl(\beta).$$

The above decomposition shows that

$$|I_1| \le C\|X - (\alpha,0)\|_{C^1}\|F(X)\|_{L^\infty}^3 \left(\int_{|\beta|>1} \frac{|f(\alpha-\beta)|}{|\beta|^3}\, d\beta + \int_{|\beta|=1} |f(\alpha-\beta)|\, dl(\beta), \right)$$

and then Minkowski's inequality gives the desired control.

Regarding I_2, we write $I_2 = J_4 + J_5 + J_6$, with

$$J_4 = \int_{|\beta|<1} \frac{X_k(\alpha) - X_k(\alpha-\beta) - \nabla X_k(\alpha)\cdot\beta}{|X(\alpha) - X(\alpha-\beta)|^3}\, \partial_{\alpha_j} f(\alpha-\beta)\, d\beta,$$

$$J_5 = \nabla X_k(\alpha)\cdot \int_{|\beta|<1} \beta\Big[\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3}\Big]\partial_{\alpha_j} f(\alpha-\beta)\, d\beta,$$

$$J_6 = \nabla X_k(\alpha)\cdot \mathrm{PV}\int_{|\beta|<1} \frac{\beta}{|\nabla X(\alpha)\cdot\beta|^3}\, \partial_{\alpha_j} f(\alpha-\beta)\, d\beta.$$

It is easy to see that

$$J_4 \le \|X - (\alpha, 0)\|_{C^{1,\delta}} \|F(X)\|^3_{L^\infty} \int_{|\beta|<1} \frac{|\partial_{\alpha_j} f(\alpha-\beta)|}{|\beta|^{2-\delta}}\, d\beta, \qquad \text{(A-14)}$$

and therefore that term can also be estimated with the use of Minkowski's inequality.

Some elementary algebraic manipulations allow us to get

$$J_5 \le C\|X - (\alpha, 0)\|^2_{C^{1,\delta}} \int_{|\beta|<1} \Big((F(X)(\alpha, \beta))^4 + \frac{|\beta|^4}{|\nabla X(\alpha)\cdot\beta|^4}\Big) \frac{|\partial_{\alpha_j} f(\alpha-\beta)|}{|\beta|^{2-\delta}}\, d\beta,$$

and then the inequality

$$\frac{|\beta|}{|\nabla X(\alpha)\cdot\beta|} \le 2\|\nabla X\|_{L^\infty} \||N|^{-1}\|_{L^\infty} \qquad \text{(A-15)}$$

yields for J_5 the same estimate (A-14).

The term J_6 can be written as

$$J_6 = \nabla X_k(\alpha)\cdot \mathrm{PV}\int_{|\beta|<1} \frac{\Sigma(\alpha, \beta)}{|\beta|^2}\, \partial_{\alpha_j} f(\alpha-\beta)\, d\beta,$$

where

(i) $\Sigma(\alpha, \lambda\beta) = \Sigma(\alpha, \beta)$ for all $\lambda > 0$,

(ii) $\Sigma(\alpha, -\beta) = -\Sigma(\alpha, \beta)$,

(iii) $\sup_\alpha |\Sigma(\alpha, \beta)| \le 8\|\nabla X\|^3_{L^\infty} \||N|^{-1}\|^3_{L^\infty}$,

as a consequence of (A-15).

Here we have a singular integral operator with odd kernel [Córdoba and Gancedo 2007; Stein 1993], and therefore a bounded linear map on $L^2(\mathbb{R}^2)$, giving us

$$\|J_6\|_{L^2} \le C\|\nabla X\|^4_{L^\infty} \||N|^{-1}\|^3_{L^\infty} \|\partial_{\alpha_j} f\|_{L^2}.$$

For the family of operators $\mathcal{T}_2(f)(\alpha)$, we use the splitting $\mathcal{T}_2(f) = I_3 + I_4$, where

$$I_3 = \int_{|\beta|>1} \frac{\partial_{\alpha_j} X_k(\alpha) - \partial_{\alpha_j} X_k(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, f(\alpha-\beta)\, d\beta.$$

We easily get

$$I_3 \le 2\|X - (\alpha, 0)\|_{C^1} \|F(X)\|^3_{L^\infty} \int_{|\beta|>1} \frac{|f(\alpha-\beta)|}{|\beta|^3}\, d\beta,$$

while for I_4, we proceed with the same method used with I_2, now replacing $X_k(\alpha)$ by $\partial_{\alpha_j} X_k(\alpha)$ and $\partial_{\alpha_j} f(\alpha-\beta)$ by $f(\alpha-\beta)$.

Next we shall show that the operator $\mathcal{T}_3$ behaves like $\Lambda = (-\Delta)^{1/2}$. To do that, we split it as $I_5 + I_6$, where

$$I_5 = \int_{|\beta|>1} \frac{f(\alpha) - f(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}\, d\beta$$

can be easily estimated by

$$I_5 \le \|F(X)\|^3_{L^\infty} \biggl(2\pi |f(\alpha)| + \int_{|\beta|>1} \frac{|f(\alpha-\beta)|}{|\beta|^3}\, d\beta \biggr).$$

The other term is written in the form $I_6 = J_7 + J_8$, where

$$J_7 = \int_{|\beta|<1} \biggl(\frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} - \frac{1}{|\nabla X(\alpha)\cdot\beta|^3} \biggr) (f(\alpha) - f(\alpha-\beta))\, d\beta.$$

The identity

$$f(\alpha) - f(\alpha-\beta) = \beta \cdot \int_0^1 \nabla f(\alpha + (s-1)\beta)\, ds$$

allows us to treat J_7 as we did with J_5. To estimate J_8, the equality

$$\frac{1}{|\nabla X(\alpha)\cdot\beta|^3} = -\partial_{\beta_1} \biggl(\frac{\beta_1}{|\nabla X(\alpha)\cdot\beta|^3} \biggr) - \partial_{\beta_2} \biggl(\frac{\beta_2}{|\nabla X(\alpha)\cdot\beta|^3} \biggr) \tag{A-16}$$

will be very useful. After a careful integration by parts, it yields

$$J_8 = \mathrm{PV}\int_{|\beta|<1} \frac{\nabla f(\alpha-\beta)\cdot\beta}{|\nabla X(\alpha)\cdot\beta|^3}\, d\beta - \int_{|\beta|=1} \frac{(f(\alpha) - f(\alpha-\beta))|\beta|}{|\nabla X(\alpha)\cdot\beta|^3}\, dl(\beta).$$

The principal value in J_8 is treated with the same method used for J_6, and since the integral on the circle is inoffensive, so long as $|N|^{-1}$ is in L^∞, the estimate for $\mathcal{T}_3$ follows.

For the remaining operator, one integrates by parts to get $\mathcal{T}_4 = I_7 + I_8$, where

$$I_7 = \mathrm{PV}\int_{\mathbb{R}^2} P_1(\alpha, \beta) f(\alpha-\beta)\, d\beta, \qquad I_8 = \mathrm{PV}\int_{\mathbb{R}^2} P_2(\alpha, \beta) f(\alpha-\beta)\, d\beta,$$

with

$$P_1(\alpha, \beta) = \frac{\partial_{\alpha_j} X(\alpha)}{|\nabla X(\alpha)\cdot\beta|^3} - \frac{\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}$$

and

$$P_2(\alpha, \beta) = 3\, \frac{\bigl(X(\alpha)-X(\alpha-\beta)\bigr)\bigl(X(\alpha)-X(\alpha-\beta)\bigr)\cdot\partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha)-X(\alpha-\beta)|^5} - 3\, \frac{\nabla X(\alpha)\cdot\beta\bigl((\nabla X(\alpha)\cdot\beta)\cdot\partial_{\alpha_j} X(\alpha)\bigr)}{|\nabla X(\alpha)\cdot\beta|^5}.$$

Next we will show how to treat I_7, because the estimate for I_8 follows similarly. For P_1 we introduce the decomposition $P_1 = Q_1 + Q_2$, where

$$Q_1 = \partial_{\alpha_j} X(\alpha) \biggl(\frac{1}{|\nabla X(\alpha)\cdot\beta|^3} - \frac{1}{|X(\alpha)-X(\alpha-\beta)|^3} \biggr), \quad Q_2 = \frac{\partial_{\alpha_j} X(\alpha) - \partial_{\alpha_j} X(\alpha-\beta)}{|X(\alpha) - X(\alpha-\beta)|^3}.$$

Since the kernel Q_2 has already appeared in the operator $\mathcal{T}_1$, it only remains to control J_9, which is given by

$$J_9 = \partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{\mathbb{R}^2} Q_1(\alpha,\beta) f(\alpha-\beta)\,d\beta.$$

The decomposition

$$J_9 = \partial_{\alpha_j} X(\alpha)\int_{|\beta|>1} d\beta + \partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} d\beta = K_1 + K_2$$

shows that the term K_1 trivializes. Regarding K_2, let us write

$$Q_1 = \frac{\big(|A|^4+|B|^2|A|^2+|B|^4\big)(A+B)\cdot(A-B)}{|A|^3|B|^3(|A|^3+|B|^3)},$$

where

$$A(\alpha,\beta) = X(\alpha) - X(\alpha-\beta), \quad B(\alpha,\beta) = \nabla X(\alpha)\cdot\beta.$$

This formula shows that inside Q_1 lies a kernel of degree -2. Then let us take $Q_1 = S_1 + S_2$, where

$$S_2 = \frac{3|B|^4 B\cdot(A-B)}{|B|^9} = \frac{3B\cdot(A-B)}{|B|^5}.$$

Next we check that the kernel S_1 has degree -1, and is therefore easy to handle. Finally, we have to consider the kernel S_2 appearing in the integral

$$L = 3\partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot\big(X(\alpha)-X(\alpha-\beta)-\nabla X(\alpha)\cdot\beta\big)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\,d\beta.$$

To do that, we introduce a further decomposition $L = M_1 + M_2$, with

$$M_1 = 3\partial_{\alpha_j} X(\alpha)\int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot\big(X(\alpha)-X(\alpha-\beta)-\nabla X(\alpha)\cdot\beta-\frac{1}{2}\beta\cdot\nabla^2 X(\alpha)\cdot\beta\big)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\,d\beta$$

and

$$M_2 = \tfrac{3}{2}\partial_{\alpha_j} X(\alpha)\,\mathrm{PV}\int_{|\beta|<1} \frac{(\nabla X(\alpha)\cdot\beta)\cdot(\beta\cdot\nabla^2 X(\alpha)\cdot\beta)}{|\nabla X(\alpha)\cdot\beta|^5} f(\alpha-\beta)\,d\beta,$$

where $\frac{1}{2}\beta\cdot\nabla^2 X(\alpha)\cdot\beta$ is the second-order term in the Taylor expansion of X. It is now easy to check that

$$M_1 \le C\|\nabla X\|^5_{L^\infty}\|X-(\alpha,0)\|_{C^{2,\delta}}\||N|^{-1}\|^4_{L^\infty}\int_{|\beta|<1}\frac{|f(\alpha-\beta)|}{|\beta|^{2-\delta}}\,d\beta.$$

Then we also check that M_2 is controlled like J_6 through the estimate

$$\|M_2\|_{L^2} \le C\|\nabla X\|^5_{L^\infty}\|\nabla^2 X\|_{L^\infty}\||N|^{-1}\|^4_{L^\infty}\|f\|_{L^2},$$

which allows us to finish the proof. □

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