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EMBEDDINGS OF INFINITELY CONNECTED PLANAR DOMAINS INTO $\mathbb{C}^2$

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We prove that every circled domain in the Riemann sphere admits a proper holomorphic embedding into the affine plane $\mathbb{C}^2$.

1. Introduction

It has been a longstanding open problem whether every open (noncompact) Riemann surface, in particular, every domain in the complex plane $\mathbb{C}$, admits a proper holomorphic embedding into $\mathbb{C}^2$. (By a *domain* we understand a connected open set.) Equivalently:

Is every open Riemann surface biholomorphic to a smoothly embedded, *topologically closed complex curve in* $\mathbb{C}^2$?

Every open Riemann surface properly embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$, but there is no constructive method of removing self-intersections of an immersed curve in $\mathbb{C}^2$. For a history of this subject see [Forstnerič and Wold 2009; Forstnerič 2011, §8.9–§8.10].

In this paper we prove the following general result in this direction.

Theorem 1.1. *Every domain in the Riemann sphere* $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ *with at most countably many boundary components*, *none of which are points*, *admits a proper holomorphic embedding into* $\mathbb{C}^2$.

By the uniformization theorem of He and Schramm [1993], every domain in Theorem 1.1 is conformally equivalent to a *circled domain*, that is, a domain whose complement is a union of pairwise disjoint closed round discs.

We prove the same embedding theorem also for generalized circled domains whose complementary components are discs and points (punctures), provided that all but finitely many of the punctures belong to the cluster set of the nonpoint boundary components (see Theorem 5.1). In particular, every domain in $\mathbb{C}$ or $\mathbb{P}^1$ with at most countably many boundary components, at most finitely many of which are isolated points, admits a proper holomorphic embedding into $\mathbb{C}^2$ (see Corollary 5.2 and Example 5.3).

For *finitely connected planar domains* without isolated boundary points, Theorem 1.1 was proved by Globevnik and Stensønes [1995]. More recently it was shown by the authors in [Forstnerič and Wold 2009] that for every embedded complex curve $\overline{C} \subset \mathbb{C}^2$, with smooth boundary bC consisting of finitely

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many Jordan curves, the interior $C = \overline{C} \setminus bC$ admits a proper holomorphic embedding into $\mathbb{C}^2$. This result was extended to some infinitely connected Riemann surfaces by I. Majcen [2009] under a nontrivial additional assumption on the accumulation set of the boundary curves. (These results can also be found in [Forstnerič 2011, Chapter 8].) Here we do not impose any restrictions whatsoever.

Our proofs of Theorems 1.1 and 5.1 are rather involved both from the analytic as well as the combinatorial point of view, something that seems inevitable in this notoriously difficult classical problem. Theorem 1.1 is proved in Section 4 after we develop the technical tools in Section 2 and Section 3. The main idea is to successively push the boundary components of an embedded complex curve in $\mathbb{C}^2$ to infinity by using holomorphic automorphisms of the ambient space, thereby ensuring that no self-intersections appear in the process, while at the same time controlling the convergence of the sequence of automorphisms in the interior of the curve. We employ the most advanced available analytic tools developed in recent years, sharpening further several of them. A novel part is our inductive scheme of dealing with an infinite sequence of boundary components, clustering them together into suitable subsets to which the analytic methods can be applied.

For simplicity of exposition we limit ourselves to domains in the Riemann sphere, although it seems likely that minor modifications yield similar results for domains in complex tori. Indeed, any punctured torus admits a proper holomorphic embedding in $\mathbb{C}^2$, and the uniformization theory of He and Schramm [1993] applies in this case as well. For infinitely connected domains in Riemann surfaces of genus > 1 the main problem is to find a suitable initial embedding of the uniformized surface into $\mathbb{C}^2$. One of the difficulties in working with nonuniformized boundary components is indicated in Remark 2.3; another one can be seen in the last part of proof of Lemma 3.1, which is a key ingredient in our construction.

Casting a view to the future, what is now needed to approach the general embedding problem is some progress on embedding *punctured* Riemann surfaces into $\mathbb{C}^2$. It is plausible that a method for answering the following question in the affirmative would lead to a complete solution to the embedding problem for planar domains with countably many boundary components.

Question 1.2. Assume that $f : \overline{\mathbb{D}} \to \mathbb{C}^2$ is a holomorphic embedding, $K \subset \mathbb{C}^2 \setminus f(b\mathbb{D})$ is a compact polynomially convex set, $C \subset \mathbb{D}$ is a compact set with $f^{-1}(K) \subset \mathring{C}$, and $a \in \mathbb{D} \setminus C$ is a point. Is f uniformly approximable on C by proper holomorphic embeddings $g : \overline{\mathbb{D}} \setminus \{a\} \hookrightarrow \mathbb{C}^2$ satisfying

$$g^{-1}(g(\overline{\mathbb{D}} \setminus \{a\}) \cap K) \subset \mathring{C}?$$

In another direction, one can ask to what extent does Theorem 1.1 hold for domains in $\mathbb{P}^1$ with uncountably many boundary components. A quintessential example of this type is a Cantor set, i.e., a compact, totally disconnected, perfect set. Recently Orevkov [2008] constructed an example of a Cantor set K in $\mathbb{C}$ whose complement $\mathbb{C} \setminus K$ embeds properly holomorphically in $\mathbb{C}^2$. (See also [Forstnerič 2011, Theorem 8.10.4]). His method, using compositions of rational shears of $\mathbb{C}^2$, does not seem to apply to a specific Cantor set. The methods explained in this paper offer some hope for future developments as indicated by Theorem 5.1 and Example 5.3 below.

The problem of embedding an open Riemann surface in $\mathbb{C}^2$ is purely complex analytic, and there are no topological obstructions. Indeed, Alarcón and López [2013] recently proved that every open Riemann surface X contains a domain $\Omega \subset X$, homeotopic to X, which embeds properly holomorphically in $\mathbb{C}^2$. In particular, every open orientable surface admits a smooth proper embedding in $\mathbb{C}^2$ whose image is a complex curve.

2. Preliminaries

In this and the following section we prepare the technical tools that will be used in the proof. The main result of this section, Theorem 2.8, gives holomorphic embeddings of bordered Riemann surfaces into $\mathbb{C}^2$ with exposed wedges at finitely many boundary points.

We begin by introducing the notation. Let $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc and by $\mathbb{D}_r = \{|z| < r\}$ the disc of radius r centered at the origin. Let (z_1, z_2) be complex coordinates on $\mathbb{C}^2$, and let $\pi_i : \mathbb{C}^2 \to \mathbb{C}$ denote the coordinate projection $\pi_i(z_1, z_2) = z_i$ for $i = 1, 2$. We denote by $\mathbb{B}_r$ and $\overline{\mathbb{B}}_r$ the open and the closed ball in $\mathbb{C}^2$, respectively, of radius r and centered at the origin. Let $\operatorname{Aut}\mathbb{C}^2$ denote the group of all holomorphic automorphisms of $\mathbb{C}^2$. By Id we denote the identity map; its domain will always be clear from the context. We denote by $\widehat{L}$ the polynomial hull of a compact set $L \subset \mathbb{C}^n$.

Definition 2.1. A domain $\Omega \subset \mathbb{P}^1$ is said to be a *circled domain* if the complement $\mathbb{P}^1 \setminus \Omega \neq \varnothing$ is a union of pairwise disjoint closed round discs $\overline{\triangle}_j \subset \mathbb{P}^1$ of positive radii.

Clearly a circled domain has at most countably many complementary discs. Mapping one of them onto $\mathbb{P}^1 \setminus \mathbb{D}$ by an automorphism of $\mathbb{P}^1$ (a fractional linear map) we see that a circled domain can be thought of as being contained in the unit disc $\mathbb{D}$.

The next lemma, and the remark following it, will serve to cluster together certain complementary components into finitely many discs; this will enable the use of holomorphic automorphisms for pushing these components towards infinity in the inductive process.

Lemma 2.2. *Let $\Omega \subset \mathbb{P}^1$ be a domain, let $K \subset \mathbb{P}^1 \setminus \Omega$ be a closed set that is a union of complementary connected components of Ω, and let $U \subset \mathbb{P}^1$ be an open set containing K. Then there exist finitely many pairwise disjoint, smoothly bounded discs $\overline{D}_j \subset U$ $(j = 1, \ldots, m)$ such that*

$$K \subset \bigcup_{j=1}^{m} D_j, \qquad bD_j \cap (\mathbb{P}^1 \setminus \Omega) = \varnothing \text{ for } j = 1, \ldots, m.$$

Proof. Let $K_j \subset \mathring{K}_{j+1} \subset K_{j+1}$ be an exhaustion of Ω by smoothly bounded connected compact sets K_j. Then $\mathbb{P}^1 \setminus K_j$ is the union of finitely many discs $\mathcal{U}_j = \{U_1^j, \ldots, U_{m(j)}^j\}$ for each j. Clearly $\mathcal{U}_j$ is a cover of K, and we claim that if j is large enough then $\mathcal{U}_j$ contains a subcover whose union is relatively compact in U. Otherwise there would exist a sequence of discs $U_{k(j)}^j \supset U_{k(j+1)}^{j+1}$ such that $U_{k(j)}^j \cap K \neq \varnothing$ and $U_{k(j)}^j \cap (\mathbb{P}^1 \setminus U) \neq \varnothing$ for each j; but then $\bigcap_{j=1}^{\infty} U_{k(j)}^j$ would be a connected complementary component of Ω that is contained in K and intersects $\mathbb{P}^1 \setminus U$, a contradiction. Hence for j large enough the discs $D_1, \ldots, D_m$ in $\mathcal{U}_j$ satisfy the stated properties. $\square$

Remark 2.3. When applying Lemma 2.2 to prove Theorem 1.1, it will be crucial that if $\Omega \subset \mathbb{P}^1$ is a circled domain with complementary discs $\overline{\triangle}_j$, and if $C \subset \mathbb{P}^1$ is any compact set, then the union of all discs $\overline{\triangle}_j$ intersecting C is a closed set that is a union of complementary connected components of Ω. The proof is elementary and is left to the reader. However, this fails in general if discs are replaced by more general connected closed sets. This is one of the reasons why our proof of Theorem 1.1 does not apply (at least not directly) to domains in compact Riemann surfaces of genus > 1. $\square$

Definition 2.4. Let $0 < \theta < 2\pi$. A domain $\Omega \subset \mathbb{C}$ is an (open) *θ-wedge with vertex $a \in b\Omega$* if there exist a $\mathscr{C}^2$ map of the form

$$\varphi(\zeta) = a + \lambda\zeta + O(|\zeta|^2), \quad \lambda \neq 0,$$

in a neighborhood of the origin $0 \in \mathbb{C}$, and for every sufficiently small $\epsilon > 0$ a neighborhood $U_\epsilon \subset \mathbb{C}$ of the point a such that

$$U_\epsilon \cap \Omega = \varphi\big(\{\zeta \in \mathbb{C}^* : 0 < \arg(\zeta) < \theta,\ 0 < |\zeta| < \epsilon\}\big).$$

The closure of an open wedge will be called a *closed wedge*.

If Ω is a domain in a Riemann surface Y, we apply the same definition of a θ-wedge in a local holomorphic coordinate near the point $a \in b\Omega \subset Y$. In particular, if $\Omega \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and $a = \infty \in b\Omega$, we apply the definition in the local chart $z \to 1/z$ on $\mathbb{P}^1$ mapping ∞ to 0.

Given a nonempty subset E of $\mathbb{C}^2$ and a linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, a point $p \in E$ is said to be *π-exposed*, and E is said to be *π-exposed at the point p*, if

$$E \cap \pi^{-1}(\pi(p)) = \{p\}. \tag{2-1}$$

Recall that a *bordered Riemann surface* is a compact one-dimensional complex manifold, $\overline{X}$, with boundary bX consisting of finitely many Jordan curves. The interior X of a bordered Riemann surface is biholomorphic to a relatively compact, smoothly bounded domain in a Riemann surface Y.

We shall use the following notion of an *exposed θ-wedge*.

Definition 2.5. Let X be a bordered Riemann surface, embedded as a smoothly bounded relatively compact domain in a Riemann surface Y. Pick a point $a \in bX$ and a number $\theta \in (0, 2\pi)$. An injective continuous map $f : \overline{X} \hookrightarrow \mathbb{C}^2$ is said to be a *holomorphic embedding with a π_1-exposed θ-wedge at $f(a)$* if f is holomorphic in X, and there exists an open neighborhood U of a in Y such that

(i) the domain $\Omega = (\pi_1 \circ f)(U \cap X) \subset \mathbb{C}$ is a θ-wedge with vertex $\pi_1(f(a))$ (see Definition 2.4),

(ii) $f(\overline{U \cap X})$ is a smooth graph over $\overline{\Omega}$ that is holomorphic over Ω, and

(iii) $\pi_1^{-1}(\overline{\Omega}) \cap f(\overline{X}) = f(\overline{U \cap X})$.

If the domain $\Omega \subset \mathbb{C}$ is instead smooth near the point $\pi_1(f(a)) \in b\Omega$, we say that f is a holomorphic embedding that is π_1-exposed at $f(a)$.

Remark 2.6 (on terminology). We shall consider embeddings $f : \overline{X} \hookrightarrow \mathbb{C}^2$ that are holomorphic in the interior X and smooth on $\overline{X}$, except at finitely many boundary points where $f(X)$ has (exposed) wedges in the sense of the above definition. Any such map will be called a *holomorphic embedding with corners*. We shall use embeddings with corners of a particular type: If X is a smoothly bounded, relatively compact domain in a Riemann surface Y, we will construct holomorphic embeddings $\widetilde{f}\colon Y \hookrightarrow \mathbb{C}^2$ and injective continuous maps $\varphi\colon \bar{X} \to Y$, holomorphic on X and smooth at all but finitely many boundary points $a_j \in bX$, such that

$$f := \widetilde{f} \circ \varphi\colon \bar{X} \hookrightarrow \mathbb{C}^2 \text{ is an embedding with corners at the points } a_j. \tag{2-2}$$

In the sequel we will refer to such maps simply as *being of the form* (2-2). The precise choice of the Riemann surface Y will not be important, and we will allow Y to shrink around $\bar{X}$ without saying it every time. □

The following lemma shows how to create wedges at smooth boundary points of a domain in a Riemann surface.

Lemma 2.7. *Let $X \Subset Y$ be Riemann surfaces, and assume that bX is smooth outside a finite set of points. Let $a_1, \dots, a_m \in bX$, $b_1, \dots, b_k \in \bar{X}$ be distinct points, with bX smooth near the points a_j, and let $\theta_j \in (0, 2\pi)$ for $j = 1, \dots, m$. Then there exists a sequence of injective continuous maps $\varphi_i\colon \bar{X} \to Y$, holomorphic on X and smooth on $\bar{X} \setminus \{a_1, \dots, a_m\}$, satisfying the following properties*:

(1) $\varphi_i \to \mathrm{Id}$ *uniformly on $\bar{X}$ as $i \to \infty$.*

(2) $\varphi_i(a_j) = a_j$ *and $\varphi_i(X)$ is a θ_j-wedge with vertex a_j $(j = 1, \dots, m)$.*

(3) $\varphi_i(x) = b_j + o(\mathrm{dist}(x, b_j)^2)$ *as $x \to b_j$ $(j = 1, \dots, k)$.*

Proof. The proof is similar to that of Lemma 8.8.3 in [Forstnerič 2011, p. 366], and it will help the reader to consult Figure 8.1 on p. 367 in that reference.

By enlarging the domain X slightly away from the points a_j we may assume that X is smoothly bounded. For simplicity of notation we explain the proof in the case when there is only one such point $a = a_1$; the same procedure can be performed simultaneously at finitely many points.

Choose a smoothly bounded disc D in Y such that $a \in bD$, $\bar{D}$ does not contain any of the points b_j, and $\bar{U} \cap \bar{X} \setminus \{a\} \subset D$ holds for some small open neighborhood U of the point a in Y. (The disc D is obtained by pushing the boundary of X slightly out near a and then rounding off.) We also choose a compact Cartan pair $(A, B) \subset Y$ with $\bar{X} \subset (A \cup B)^\circ$ and $C := A \cap B \subset D$. (For the notion of a Cartan pair see [Forstnerič 2011, Definition 5.7.1].) The set A is chosen such that it contains a neighborhood of a, and B contains $\bar{X} \setminus U'$ for a small neighborhood $U' \subset U$ of the point a.

The Riemann mapping theorem furnishes a sequence of injective continuous maps $\psi_i\colon \bar{D} \to Y$ that are holomorphic in D and smooth on $\bar{D} \setminus \{a\}$ such that $\psi_i(a) = a$, $\psi_i(D)$ is a θ_1-wedge with vertex a (see Definition 2.4), and the sets $\psi_i(\bar{D})$ converge to $\bar{D}$ as $i \to \infty$. We may assume that $\psi_i \to \mathrm{Id}$ uniformly on $\bar{D}$ (see [Goluzin 1969, Theorem 2, p. 59]). This implies that $\psi_i(C) \subset D$ for all sufficiently large $i \in \mathbb{N}$.

By Theorem 8.7.2 in [Forstnerič 2011, p. 359] there exist an integer $i_0 \in \mathbb{N}$ and sequences of injective holomorphic maps $f_i\colon A \to Y$ and $g_i\colon B \to Y$ $(i \geq i_0)$, both converging to the identity map and tangent to the identity to second order at those points a and b_j which are contained in their respective domains, such that

$$\psi_i \circ f_i = g_i \ \text{ holds on } C.$$

The sequence of maps $\varphi_i : \bar{X} \to Y$, defined by

$$\varphi_i = \psi_i \circ f_i \text{ on } A \cap \bar{X} \quad \text{and} \quad \varphi_i = g_i \text{ on } \bar{X} \cap B$$

then satisfies the conclusion of the lemma. Injectivity of φ_i on $\bar{X}$ for sufficiently large index i can be seen exactly as in the proof of [Forstnerič 2011, Lemma 8.8.3] (see bottom of p. 359 in the cited source). □

Using Lemma 2.7 we obtain the following version of the main tool introduced in [Forstnerič and Wold 2009] for exposing boundary points of bordered Riemann surfaces. (See also Theorem 8.9.10 and Figure 8.2 in [Forstnerič 2011, pp. 372–373].) The main novelty here is that we create *exposed points with wedges*.

Theorem 2.8. *Let $\bar{X}$ be a smoothly bounded domain in a Riemann surface Y, $f : \bar{X} \hookrightarrow \mathbb{C}^2$ a holomorphic embedding with corners of the form* (2-2), *and $a_1, \dots, a_m \in bX$, $b_1, \dots, b_k \in \bar{X}$ distinct points such that f is smooth near the points a_j. Let $\gamma_j : [0, 1] \to \mathbb{C}^2$ $(j = 1, \dots, m)$ be smooth embedded arcs with pairwise disjoint images satisfying the following properties*:

- *$\gamma_j([0, 1]) \cap f(\bar{X}) = \gamma_j(0) = f(a_j)$ for $j = 1, \dots, m$.*
- *The image $E := f(\bar{X}) \cup \bigcup_{j=1}^{m} \gamma_j([0, 1])$ is π_1-exposed at the point $\gamma_j(1)$ for $j = 1, \dots, m$ (see* (2-1)).

Given an open set $V \subset \mathbb{C}^2$ containing $\bigcup_{j=1}^{m} \gamma_j([0, 1])$, an open set $U \subset Y$ containing the points a_j and satisfying $f(\overline{U \cap X}) \subset V$, and numbers $0 < \theta_j < 2\pi$ $(j = 1, \dots, m)$ and $\epsilon > 0$, there exists a holomorphic embedding with corners $g : \bar{X} \hookrightarrow \mathbb{C}^2$ of the form (2-2) *satisfying the following properties*:

(1) $\|g - f\|_{\bar{X} \setminus U} < \epsilon$.

(2) $g(\overline{U \cap X}) \subset V$.

(3) $g(x) = f(x) + o(\operatorname{dist}(x, b_j)^2)$ *as* $x \to b_j$ $(j = 1, \dots, k)$.

(4) $g(a_j) = \gamma_j(1)$ *and* $g(\bar{X})$ *is* π_1*-exposed with a* θ_j*-wedge at* $g(a_j)$ *for every* $j = 1, \dots, m$.

(5) g *is smooth near all points* $x \in bX \setminus \{a_1, \dots, a_m\}$ *at which* f *is smooth.*

If for some $j \in \{1, \dots, k\}$ we have that $b_j \in bX$ and $f(X)$ is a wedge at the point $f(b_j)$, then property (3) ensures that $g(X)$ remains a wedge with the same angle at $f(b_j) = g(b_j)$. In addition, property (4) ensures that $g(X)$ is an exposed wedge at each of the points $g(a_j)$.

Proof. Since f is of the form (2-2), we write $f = \tilde{f} \circ \varphi$ where $\tilde{f} : Y \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding. Set $X' = \varphi(X) \Subset Y$. Lemma 2.7, applied to the domain X' and the points $a'_j = \varphi(a_j) \in bX'$, $b'_j = \varphi(b_j) \in \overline{X'}$, gives an injective continuous map $\psi : \overline{X'} \to Y$ close to the identity map, with ψ holomorphic on X' and smooth on $\bar{X}' \setminus \{a'_1, \dots, a'_m\}$, such that

(2′) $\psi(a'_j) = a'_j$ and $\psi(X')$ is a θ_j-wedge with vertex a'_j $(j = 1, \dots, m)$, and

(3′) $\psi(x) = b'_j + o(\operatorname{dist}(x, b'_j)^2)$ as $x \to b'_j$ $(j = 1, \dots, k)$.

(The map ψ is one of the maps φ_i in Lemma 2.7, and the properties (2′), (3′) correspond to (2), (3) in that lemma, respectively.)

Set $\widetilde{\varphi} = \psi \circ \varphi \colon \bar{X} \to Y$; this is an embedding with the analogous properties as φ, but with additional θ_j-wedges at the points $a'_j \in bX'$. The embedding with corners $\widetilde{f} \circ \widetilde{\varphi} \colon \bar{X} \hookrightarrow \mathbb{C}^2$ then satisfies properties (1)–(3) and (5) (for the map g) in Theorem 2.8.

In order to achieve also condition (4) we apply Theorem 8.9.10 in [Forstnerič 2011] and the proof thereof. (The original source for this result is [Forstnerič and Wold 2009, Theorem 4.2].) We recall the main idea and refer to the cited works for the details. By pushing the boundary bX' slightly outward away from the points a'_j we obtain a smoothly bounded domain $Z \Subset Y$ such that $\bar{X}' \subset Z \cup \{a'_1, \dots, a'_m\}$. We attach to $\bar{Z}$ short pairwise disjoint embedded arcs $\Gamma_j \subset Y$ intersecting $\bar{Z}$ only at the points a'_j. By Mergelyan's theorem we can change the embedding $\widetilde{f}$ so that it maps the arc Γ_j approximately onto the arc γ_j for each $j = 1, \dots, m$, taking the other endpoint c_j of Γ_j to the exposed endpoint $\gamma_j(1) \in \mathbb{C}^2$ of γ_j and remaining close to the initial embedding on $\bar{Z}$. At each point $a'_j \in bZ$ we choose a small smoothly bounded disc $D_j \subset Y$ with the same properties as in the proof of Lemma 2.7; in particular, $a'_j \in bD_j$ and D_j contains $\bar{Z} \setminus \{a'_j\}$ near the point a'_j. By the Riemann mapping theorem we find for each $j \in \{1, \dots, m\}$ a holomorphic map $h_j \colon \bar{D}_j \to Y$ stretching $\bar{D}_j$ to contain the arc Γ_j, mapping a'_j to the other endpoint c_j of Γ_j and remaining close to the identity except very near the point a'_j. We then glue the maps h_j to an approximation of the identity map on the rest of the domain $\bar{Z}$, using again Theorem 8.7.2 in [Forstnerič 2011, p. 359]. This gives an injective holomorphic map $h \colon \widetilde{Y} \hookrightarrow Y$ in an open neighborhood $\widetilde{Y}$ of $\bar{Z}$ such that $h|_{\bar{Z}}$ is close to the identity, except very near the points $a'_j \in bZ$. The holomorphic embedding $\widetilde{g} := \widetilde{f} \circ h \colon \widetilde{Y} \hookrightarrow \mathbb{C}^2$ is then close to $\widetilde{f}$ on $\bar{Z}$, except near the points a'_j. By the construction, $\widetilde{g}(a'_j)$ is a π_1-exposed point of $\widetilde{g}(\bar{Z})$ for $j = 1, \dots, m$. The embedding with corners $g = \widetilde{g} \circ \widetilde{\varphi} \colon \bar{X} \hookrightarrow \mathbb{C}^2$ is then of the form (2-2) and satisfies properties (1)–(5) in Theorem 2.8. □

3. The main lemma

In this section we prove the following key lemma that will be used in the proof of Theorem 1.1. It is similar in spirit to Lemma 1 in [Wold 2006, p. 4] (see also [Forstnerič 2011, Lemma 4.14.4, p. 150]), but with improvements needed to deal with the more complicated situation at hand.

Lemma 3.1. *Let $\Omega = \mathbb{P}^1 \setminus \bigcup_{j=0}^{\infty} \overline{\triangle}_j$ be a circled domain, and let $\Omega' = \mathbb{P}^1 \setminus \bigcup_{j=0}^{k} \overline{\triangle}_j$ for some $k \in \mathbb{N}$. Pick a point $c_j \in b\triangle_j$ for $j = 0, 1, \dots, k$. Assume that $f \colon \overline{\Omega'} \hookrightarrow \mathbb{C}^2$ is a holomorphic embedding with a π_1-exposed θ_j-wedge at each point $f(c_j)$ and $\theta_0 + \cdots + \theta_k < 2\pi$. Let g be a rational shear map of the form*

$$g(z_1, z_2) = \left(z_1, z_2 + \sum_{j=0}^{k} \frac{\beta_j}{z_1 - \pi_1(f(c_j))} \right).$$

Assume that there exist open neighborhoods $U_j \subset \mathbb{P}^1$ of the points c_j such that $(\pi_2 \circ g \circ f)(U_j) \subset \mathbb{P}^1$ are θ_j-wedges whose closures only intersect at their common vertex $\infty \in \mathbb{P}^1$. (This can be arranged by a suitable choice of the arguments of the numbers β_j, while at the same time keeping $|\beta_j| > 0$ arbitrarily *small.) Given a compact polynomially convex set* $K \subset \mathbb{C}^2$ *with*

$$K \cap (g \circ f)\biggl(b\Omega' \cup \biggl(\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i\biggr)\biggr) = \varnothing$$

and numbers $N \in \mathbb{N}$ *and* $\epsilon > 0$, *there exists a* $\psi \in \operatorname{Aut}\mathbb{C}^2$ *such that*

(1) $(\psi \circ g \circ f)(b\Omega' \cup (\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i)) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$, *and*

(2) $\|\psi - \mathrm{Id}\,\|_K < \epsilon$.

Proof. We may assume that $\triangle_0 = \mathbb{P}^1 \setminus \overline{\mathbb{D}}$, so $\Omega = \mathbb{D} \setminus \bigcup_{j=1}^{\infty} \overline{\triangle}_j$. By increasing the number $N \in \mathbb{N}$ we may also assume that $K \subset \mathbb{B}_N$.

Set $X = (g \circ f)(\Omega')$, $\gamma_j = (g \circ f)(b\triangle_j \setminus \{c_j\})$ $(j = 0, \dots, k)$, and $\gamma = \bigcup_{j=0}^{k} \gamma_j$. Then $\overline{X}$ is an embedded bordered Riemann surface in $\mathbb{C}^2$ whose boundary $bX = \gamma$ consists of pairwise disjoint properly embedded real curves γ_j diffeomorphic to $\mathbb{R}$, and the second coordinate projection $\pi_2 \colon \overline{X} \to \mathbb{C}$ is proper. Let $\triangle'_i = (g \circ f)(\triangle_i) \subset X$ for $i = k+1, k+2, \dots$; then

$$X \setminus \bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i = (g \circ f)(\Omega).$$

To prove the lemma we must find an automorphism $\psi \in \operatorname{Aut}\mathbb{C}^2$ sending the boundary curves $bX = \gamma$ and all the discs $\overline{\triangle}'_i$ for $i > k$ out of the ball $\overline{\mathbb{B}}_N$, while at the same time approximating the identity map on the compact set K. We seek ψ of the form

$$\psi = \phi_1 \circ \phi_2, \quad \text{where } \phi_1, \phi_2 \in \operatorname{Aut}\mathbb{C}^2.$$

We begin by constructing ϕ_1.

The conditions on f and g ensure that for any sufficiently large disc $D \subset \mathbb{C}$ centered at the origin the projection $\pi_2 \colon \overline{X} \setminus \pi_2^{-1}(D) \to \mathbb{C} \setminus D$ is injective and maps $\overline{X} \setminus \pi_2^{-1}(D)$ onto the union of $k+1$ pairwise disjoint wedges with the common vertex at ∞; furthermore, the closed set

$$\overline{D} \cup \pi_2\biggl(\gamma \cup \bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i\biggr) \subset \mathbb{C} \tag{3-1}$$

can be exhausted by polynomially convex compact sets. To see this, note that if $V'_j \subset V_j$ are small round discs in $\mathbb{C}$ centered at the point c_j such that $V_j \subset U_j$ for $j = 0, 1, \dots, k$, where the neighborhoods U_j satisfy the hypotheses of the lemma, then the sets

$$(bV_j \setminus \triangle_j) \cup \bigl(b\triangle_j \cap (\overline{V}_j \setminus V'_j)\bigr) \cup \biggl(\bigcup_{i=k+1}^{\infty} \overline{\triangle}_i \cap (\overline{V}_j \setminus V'_j)\biggr) \subset \mathbb{C}$$

are polynomially convex, and the map $\pi_2 \circ g \circ f \colon \bigcup_{j=0}^{k} \overline{V}_j \cap \Omega' \to \mathbb{C}$ is an injection onto a union of wedges such that the closures of any two of them intersect only at their common vertex at ∞. An exhaustion of the set in (3-1) by polynomially convex compact sets is constructed by letting the radii of the discs V'_j go to 0.

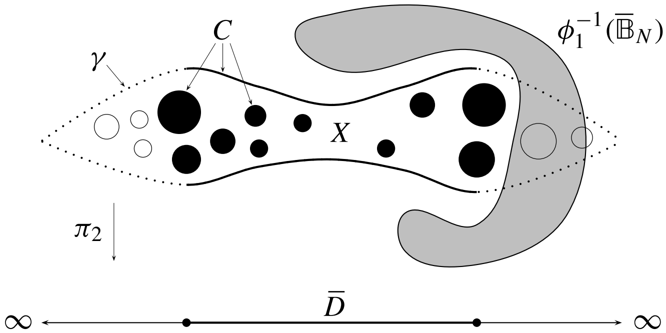

Figure 1. The set C.

Let $J = \{i \in \mathbb{N} : i \geq k+1,\ \pi_2(\overline{\triangle}_i') \cap \overline{D} \neq \varnothing\}$. Consider the compact set

$$C := \left[\gamma \cap \pi_2^{-1}(\overline{D})\right] \cup \Big[\bigcup_{i \in J} \overline{\triangle}_i'\Big] \subset \overline{X}.$$

(Figure 1 shows C with bold lines and black discs.) We claim that C is polynomially convex. Clearly C is holomorphically convex in $\overline{X}$ since its complement is connected. Furthermore, $\overline{X}$ can be exhausted by compact smoothly bounded subdomains $X_j \subset \overline{X}$ such that each boundary component of X_j intersects the boundary of X. (It suffices to take the intersection of $\overline{X}$ with a sufficiently large ball and smoothen the corners.) Then $\widehat{X}_j \setminus X_j$ is either empty or a pure one-dimensional complex subvariety of $\mathbb{C}^2 \setminus X_j$ (see [Stolzenberg 1966]), the latter being impossible since the variety would have to be unbounded. Hence every such set X_j is polynomially convex, and by choosing it large enough to contain C we see that C is polynomially convex.

We will construct ϕ_1 as a composition $\phi_1 = \sigma_2 \circ \sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on K and satisfies $\phi_1(C) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$; equivalently, $C \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing$.

By [Wold 2006, Lemma 1] (see also [Forstnerič 2011, Corollary 4.14.5]) there exists $\sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on K and satisfies $\sigma_1(\gamma) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$.

Let K' be the union of all discs $\overline{\triangle}_i$ $(i \in J)$ whose images $\overline{\triangle}_i'$ satisfy

$$\sigma_1(\overline{\triangle}_i') \cap \overline{\mathbb{B}}_N \neq \varnothing.$$

Since $\sigma_1(\gamma) \cap \overline{\mathbb{B}}_N = \varnothing$, the set $(\sigma_1 \circ g \circ f)^{-1}(\overline{\mathbb{B}}_N) \subset \Omega'$ is compact, and hence K' is also compact (see Remark 2.3). Lemma 2.2 gives pairwise disjoint smoothly bounded discs $D_1, \ldots, D_m$ in Ω' whose union $\bigcup_{j=1}^m D_j$ contains K' and whose closures $\overline{D}_j$ avoid $b\Omega' \cup (g \circ f)^{-1}(K)$. Set $D_j' = (g \circ f)(D_j) \subset X$ for $j = 1, \ldots, m$. The set

$$L := K \cup \left(C \setminus \bigcup_{j=1}^m \overline{D}_j'\right) \subset \mathbb{C}^2$$

is then polynomially convex (argue as above for the set C, using the fact that K is disjoint from C). The union of discs $E_0 := \bigcup_{j=1}^m \sigma_1(\overline{D}_j')$ is polynomially convex and disjoint from $\sigma_1(L)$, so it can be moved out of the ball $\overline{\mathbb{B}}_N$ by a holomorphic isotopy in the complement of the polynomially convex set

$\sigma_1(L)$. (It suffices to first contract each disc $\sigma_1(\overline{D}'_j)$ into a small ball around one of its points and then move these small balls out of the set $\sigma_1(L)$ along pairwise disjoint arcs.) Furthermore, letting $E_t \subset \mathbb{C}^2$ ($t \in [0, 1]$) denote the trace of E_0 under this isotopy, we can ensure that for every t the union $E_t \cup \sigma_1(L)$ is polynomially convex. The Andersén–Lempert theory (see [Forstnerič 2011, Theorem 4.12.1]) now furnishes an automorphism $\sigma_2 \in \operatorname{Aut}\mathbb{C}^2$ that is close to the identity on the set $\sigma_1(L)$ and satisfies

$$(\sigma_2 \circ \sigma_1)\biggl(\bigcup_{j=1}^{m} \overline{D}'_j\biggr) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N.$$

The automorphism $\phi_1 = \sigma_2 \circ \sigma_1 \in \operatorname{Aut}\mathbb{C}^2$ is then close to the identity map on K, and $\phi_1(C) \subset \mathbb{C}^2 \setminus \overline{\mathbb{B}}_N$.

Next we shall find a shear automorphism $\phi_2 \in \operatorname{Aut}\mathbb{C}^2$ of the form

$$\phi_2(z_1, z_2) = (z_1 + h(z_2), z_2) \tag{3-2}$$

that is close to the identity on $\mathbb{C} \times (\pi_2(C) \cup \overline{D})$ and satisfies

$$\phi_2\biggl(\gamma \cup \biggl(\bigcup_{i=k+1}^{\infty} \overline{\triangle}'_i\biggr)\biggr) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing.$$

The automorphism $\psi = \phi_1 \circ \phi_2 \in \operatorname{Aut}\mathbb{C}^2$ will then satisfy Lemma 3.1.

Choose a large number $R > 0$ such that

$$\pi_1(\phi_1^{-1}(\overline{\mathbb{B}}_N)) \subset \mathbb{D}_R \quad \text{and} \quad \pi_2(\phi_1^{-1}(\overline{\mathbb{B}}_N)) \cup \overline{D} \subset \mathbb{D}_R.$$

We shall find ϕ_2 as a composition $\phi_2 = \tau_2 \circ \tau_1$ of two shears of the same type (3-2). The values of the function $h \in \mathscr{O}(\mathbb{C})$ in (3-2) on $\mathbb{C} \setminus \mathbb{D}_R$ are unimportant since $\phi_1^{-1}(\overline{\mathbb{B}}_N)$ projects into $\mathbb{D}_R$.

Recall that the projection $\pi_2 \colon \overline{X} \setminus \pi_2^{-1}(D) \to \mathbb{C} \setminus D$ maps $\overline{X} \setminus \pi_2^{-1}(D)$ bijectively onto a union of pairwise disjoint closed wedges with the common vertex at ∞ (see Figure 2 below). Hence the geometry of subsets of $\overline{X} \setminus \pi_2^{-1}(D)$ is the same as the geometry of their π_2-projections in $\mathbb{C} \setminus D$, an observation that will be tacitly used in the sequel.

By [Wold 2006, Lemma 1] there is an entire function $h_1 \in \mathscr{O}(\mathbb{C})$ that is small on the set $\overline{D} \cup \pi_2(C)$ and takes suitable values on the projected curves $\pi_2(\gamma) \setminus D$ so that the shear $\tau_1(z_1, z_2) = (z_1 + h_1(z_2), z_2)$ satisfies

$$\tau_1(\gamma \cup C) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) = \varnothing.$$

Set $\widetilde{J} = \{i \in \mathbb{N} : i \geq k+1,\ \pi_2(\overline{\triangle}'_i) \cap \overline{\mathbb{D}}_R \neq \varnothing\}$. Consider the compact set

$$\widetilde{C} := \bigl[\gamma \cap \pi_2^{-1}(\overline{\mathbb{D}}_R)\bigr] \cup \biggl[\bigcup_{i \in \widetilde{J}} \overline{\triangle}'_i\biggr] \subset \overline{X}.$$

Let K'' be the union of all discs $\overline{\triangle}_i$ ($i \in \widetilde{J}$) whose images $\overline{\triangle}'_i = (g \circ f)(\overline{\triangle}_i)$ satisfy the condition

$$\tau_1(\overline{\triangle}'_i) \cap \phi_1^{-1}(\overline{\mathbb{B}}_N) \neq \varnothing.$$

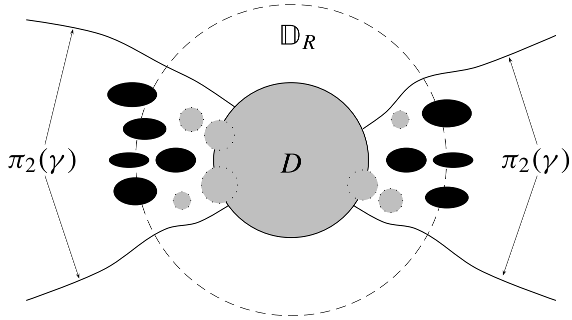

Figure 2. Geometry in the z_2-plane.

Our choices of ϕ_1 and τ_1 imply that for every disc $\overline{\triangle}_i \subset K''$ the projection $\pi_2(\overline{\triangle}'_i)$ intersects the disc $\overline{\mathbb{D}}_R$ and avoids the set $\pi_2(C) \cup \overline{D}$. Remark 2.3 shows that K'' is compact. Using Lemma 2.2 we find smoothly bounded discs $B_1, \dots, B_l \subset \Omega'$ with pairwise disjoint closures whose union $\bigcup_{j=1}^{l} B_j$ contains K'' and is disjoint from $b\Omega' \cup (g \circ f)^{-1}(C)$, and whose boundaries bB_j belong to Ω. (Hence every disc $\overline{\triangle}_i$ for $i > k$ is either completely contained in $\bigcup_{j=1}^{l} \overline{B}_j$ or else is disjoint from it.) It follows that the set

$$\widetilde{L} := \bigcup_{j=1}^{l} (\pi_2 \circ g \circ f)(\overline{B}_j) \subset \mathbb{C}$$

is a disjoint union of discs contained in $\mathbb{C} \setminus (\overline{D} \cup \pi_2(\gamma))$. Hence the sets $\widetilde{L}$ and $\pi_2(\widetilde{C}) \setminus \widetilde{L}$ are polynomially convex, and so is their union. (Figure 2 shows $\widetilde{L}$ as the union of black ellipses, while $\pi_2(\widetilde{C}) \setminus \widetilde{L}$ is shown in gray.)

Let $h_2 \in \mathbb{O}(\mathbb{C})$ be such that $|h_2| > R$ on $\widetilde{L}$ and $|h_2|$ is small on $\pi_2(\widetilde{C}) \setminus \widetilde{L}$. Let $\tau_2(z_1, z_2) = (z_1 + h_2(z_2), z_2)$ and $\phi_2 = \tau_2 \circ \tau_1$. The automorphism $\psi = \phi_1 \circ \phi_2 \in \operatorname{Aut} \mathbb{C}^2$ then clearly satisfies Lemma 3.1.

Note that $\phi_2(z_1, z_2) = (z_1 + h(z_2), z_2)$ with $h = h_1 + h_2$, so it is possible to boil down the construction of ϕ_2 to one step. $\square$

4. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. The construction is similar to the proof of Majcen's theorem [2009] as given in [Forstnerič 2011, §8.10], but the induction scheme is altered and improved at several key points.

Every holomorphic embedding with corners will be assumed to be of the form (2-2).

Let $\Omega \subset \mathbb{P}^1$ be a domain with countably many complementary components, none of which are points. (We assume that there are infinitely many components, for otherwise the result is due to Globevnik and Stensønes [1995]. Our proof also applies in the latter case, but it could be made much simpler.) By the uniformization theorem of He and Schramm [1993] we may assume that Ω is a circled domain. By mapping one of the complementary discs in $\mathbb{P}^1 \setminus \Omega$ onto the complement $\mathbb{P}^1 \setminus \mathbb{D}$ of the unit disc $\mathbb{D}$ we may further assume that $\Omega = \mathbb{D} \setminus \bigcup_{j=1}^{\infty} \overline{\triangle}_j$, where $\overline{\triangle}_j$ are pairwise disjoint closed discs in $\mathbb{D}$.

We construct a proper holomorphic embedding $\Omega \hookrightarrow \mathbb{C}^2$ by induction.

Choose an exhaustion $\varnothing = K_0 \subset K_1 \subset K_2 \subset \ldots \subset \bigcup_{j=1}^{\infty} K_j = \Omega$ of Ω by compact, connected, $\mathcal{O}(\Omega)$-convex sets with smooth boundaries, satisfying $K_j \subset \mathring{K}_{j+1}$ for $j = 0, 1, 2, \ldots$. These conditions imply that for each index $j \in \mathbb{N}$ the set $\widehat{K}_j \setminus K_j \subset \mathbb{D}$ is a union of finitely many open discs, i.e., sets homeomorphic to the standard disc.

We begin the induction at $n = 0$. Set $\Gamma_0 = b\,\mathbb{D}$, $m_0 = k_0 = 0$. Pick a point $c_0 \in \Gamma_0$ and a number $\epsilon_0 > 0$. At the n-th step of the construction we shall obtain the following data:

- Integers $m_n, k_n \in \mathbb{N}$.
- A number ϵ_n such that $0 < \epsilon_n < \frac{1}{2}\epsilon_{n-1}$ (the last inequality is void for $n = 0$).
- Circles $\Gamma_j = b\triangle_{i(j)}$ $(j = 1, \ldots, k_n)$ from the family $\{b\triangle_i\}_{i\in\mathbb{N}}$, at least one in each connected component of $\widehat{K}_{m_n} \setminus K_{m_n}$.
- The domain $\Omega_n = \mathbb{D}\setminus \bigcup_{j=1}^{k_n} \overline{\triangle}_{i(j)}$ with boundary $b\Omega_n = \bigcup_{j=0}^{k_n} \Gamma_j$.
- Points $c_j \in \Gamma_j$ for $j = 0, \ldots, k_n$.
- Numbers $\theta_j > 0$ $(j = 0, \ldots, k_n)$ with $\sum_{j=0}^{k_n} \theta_j < 2\pi$.
- A holomorphic embedding with corners $f_n \colon \overline{\Omega}_n \hookrightarrow \mathbb{C}^2$ such that the points $c_0, \ldots, c_{k_n}$ are π_1-exposed with θ_j-wedges (see Definition 2.5) and f_n is smooth near $b\Omega_n \setminus \{c_0, \ldots, c_{k_n}\}$.
- A rational shear with poles at the exposed points $f_n(c_j)$ of $f_n(b\Omega_n)$,
$$g_n(z_1, z_2) = \left(z_1, z_2 + \sum_{j=0}^{k_n} \frac{\beta_j}{z_1 - \pi_1(f_n(c_j))}\right),$$
such that $(\pi_2 \circ g_n \circ f_n)(\Omega_n) \subset \mathbb{C}$ is a union of θ_j-wedges whose closures intersect only at their common vertex $\infty \in \mathbb{P}^1$.
- An automorphism ϕ_n of $\mathbb{C}^2$.

In addition, setting
$$F_{n-1} = \Phi_{n-1} \circ g_n \circ f_n, \qquad \Phi_n = \phi_n \circ \Phi_{n-1} = \phi_n \circ \phi_{n-1} \ldots \circ \phi_1,$$
the following conditions hold:
$$|g_n \circ f_n(x) - g_{n-1} \circ f_{n-1}(x)| < \epsilon_n, \qquad x \in K_{m_n}. \tag{4-1}$$
$$|\Phi_{n-1} \circ g_n \circ f_n(x) - \Phi_{n-1} \circ g_{n-1} \circ f_{n-1}(x)| < \epsilon_n, \quad x \in K_{m_n}. \tag{4-2}$$
$$\overline{\mathbb{B}}_{n-1} \cap F_{n-1}(\Omega_n) \subset F_{n-1}(\mathring{K}_{m_n}). \tag{4-3}$$
$$|\phi_n(z) - z| < \epsilon_n, \qquad z \in \overline{\mathbb{B}}_{n-1} \cup F_{n-1}(K_{m_n}). \tag{4-4}$$
$$|\Phi_n \circ g_n \circ f_n(x)| > n, \qquad x \in b\Omega_n \cup (\Omega_n \setminus \Omega). \tag{4-5}$$

Remark 4.1. Setting $J_n = \mathbb{N} \setminus \{i(j) : j = 1, \ldots, k_n\}$, we have
$$\Omega_n = \Omega \cup \bigcup_{j\in J_n} \overline{\triangle}_j, \quad \Omega_n \setminus \Omega = \bigcup_{j\in J_n} \overline{\triangle}_j.$$

Clearly $\mathbb{D} \supset \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega$, but the intersection $\bigcap_{j=1}^{\infty} \Omega_j$ need not equal Ω. That is, the set of all circles Γ_j that get opened up in the course of the construction may be a proper subset of the family $\{b\triangle_i\}_{i\in\mathbb{Z}_+}$ of all boundary circles of Ω. The only reason for opening a boundary circle contained in $\widehat{K}_{m_n} \setminus K_{m_n}$ is to ensure that the image of K_{m_n} in $\mathbb{C}^2$ becomes polynomially convex; see (4-7) below. $\square$

We begin the induction at $n = 0$ by choosing an embedding $f_0(\zeta) = (\tau_0(\zeta), 0)$ of $\overline{\mathbb{D}}$ in $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ with a θ_0-wedge at the point $c_0 \in \Gamma_0 = b\mathbb{D}$ (see Theorem 2.8). We also choose a shear

$$g_0(z_1, z_2) = \left(z_1,\ z_2 + \frac{\beta_0}{z_1 - \pi_1 \circ f_0(c_0)}\right)$$

sending the exposed point $\pi_1 \circ f_0(c_0) = \tau_0(c_0)$ to infinity. Let $\phi_0 = \Phi_0 = \Phi_{-1} = \mathrm{Id}$. Conditions (4-1)–(4-4) are then vacuous for $n = 0$ (recall that $K_0 = \varnothing$), and (4-5) is satisfied after a small translation of the embedding $g_0 \circ f_0 \colon \overline{\mathbb{D}} \setminus \{c_0\} \hookrightarrow \mathbb{C}^2$ which removes the image off the origin.

We now explain the inductive step $n \to n+1$. By (4-5) there exists an integer $m_{n+1} > m_n$ such that

$$\overline{\mathbb{B}}_n \cap \big(\Phi_n \circ g_n \circ f_n(\Omega_n)\big) \subset \Phi_n \circ g_n \circ f_n(\mathring{K}_{m_{n+1}}). \tag{4-6}$$

By the inductive hypothesis the polynomial hull $\widehat{K}_{m_{n+1}}$ contains the boundary circles $\Gamma_j \subset b\Omega$ for $1 \le j \le k_n$. (This is vacuous if $n = 0$.) In each of the (finitely many) connected components of $\widehat{K}_{m_{n+1}} \setminus K_{m_{n+1}}$ that does not contain any of the above circles we pick another boundary circle of Ω (such exists since the set $K_{m_{n+1}}$ is $\mathscr{O}(\Omega)$-convex); we label these additional curves $\Gamma_{k_n+1}, \ldots, \Gamma_{k_{n+1}}$. As before, we have $\Gamma_j = b\triangle_{i(j)}$ for some index $i(j)$. Let

$$\Omega_{n+1} = \mathbb{D} \setminus \bigcup_{j=1}^{k_{n+1}} \overline{\triangle}_{i(j)}.$$

Setting $J_{n+1} = \mathbb{N} \setminus \{i(j) : j = 1, \ldots, k_{n+1}\}$, we have that

$$\Omega_{n+1} = \Omega \cup \bigcup_{j\in J_{n+1}} \overline{\triangle}_j.$$

Each of these additional curves will now be opened up. Pick a point $c_j \in \Gamma_j$ for each $j = k_n+1, \ldots, k_{n+1}$ and positive numbers $\theta_{k_n+1}, \ldots, \theta_{k_{n+1}}$ such that $\sum_{j=0}^{k_{n+1}} \theta_j < 2\pi$. Also choose a number $\epsilon_{n+1} \in (0, \epsilon_n/2)$ such that any holomorphic map $h \colon \Omega \to \mathbb{C}^2$ satisfying $\|h - g_n \circ f_n\|_{K_{m_{n+1}}} < 2\epsilon_{n+1}$ is an embedding on K_{m_n}. Theorem 2.8 furnishes a holomorphic embedding $f_{n+1} \colon \overline{\Omega}_{n+1} \hookrightarrow \mathbb{C}^2$ with corners such that f_{n+1} agrees with f_n to the second order at each of the points $c_0, \ldots, c_{k_n}$, it additionally makes the boundary points $c_{k_n+1}, \ldots, c_{k_{n+1}}$ π_1-exposed with θ_j-wedges, and it approximates f_n as closely as desired outside of small neighborhoods of these points. The image $f_{n+1}(\overline{\Omega}_{n+1})$ stays as close as desired to the union of $f_n(\overline{\Omega}_{n+1})$ with the family of arcs that were attached to this set in order to expose the points $c_{k_n+1}, \ldots, c_{k_{n+1}}$. In particular, we ensure that none of the complex lines $z_1 = \pi_1 \circ f_{n+1}(c_j)$ for $j = k_n+1, \ldots, k_{n+1}$ intersect the set $\Phi_n^{-1}(\overline{\mathbb{B}}_n)$. The rational shear

$$g_{n+1}(z_1, z_2) = g_n(z_1, z_2) + \left(0,\ \sum_{j=k_n+1}^{k_{n+1}} \frac{\beta_j}{z_1 - \pi_1(f_{n+1}(c_j))}\right)$$

sends the exposed points $f_{n+1}(c_0), \dots, f_{n+1}(c_{k_{n+1}})$ to infinity. A suitable choice of the arguments of $\beta_j \in \mathbb{C}^*$ for $j = k_n + 1, \dots, k_{n+1}$ ensures that, in a neighborhood of infinity, $(\pi_2 \circ g_{n+1} \circ f_{n+1})(\overline{\Omega}_{n+1})$ is a union of pairwise disjoint θ_j-wedges with the common vertex at $\infty \in \mathbb{P}^1$; at the same time the absolute values $|\beta_j| > 0$ can be chosen arbitrarily small in order to obtain good approximation of g_n by g_{n+1}.

Set $F_n = \Phi_n \circ g_{n+1} \circ f_{n+1}$. If the approximations of f_n, g_n by f_{n+1}, g_{n+1}, respectively, were close enough, then the conditions (4-1)–(4-3) hold with n replaced by $n+1$.

Since every connected component of $\widehat{K}_{m_{n+1}} \setminus K_{m_{n+1}}$ contains at least one of the points $c_1, \dots, c_{m_{n+1}}$ which F_n sends to infinity, the set $F_n(K_{m_{n+1}}) \subset \mathbb{C}^2$ is polynomially convex. (See [Wold 2006, Proposition 3.1] for the details of this argument.) From (4-6) we also infer that $\overline{\mathbb{B}}_n \cap F_n(\Omega_{n+1}) \subset F_n(\mathring{K}_{m_{n+1}})$ provided that the approximations were close enough. It follows that the set

$$L_n := \overline{\mathbb{B}}_n \cup F_n(K_{m_{n+1}}) \subset \mathbb{C}^2 \tag{4-7}$$

is polynomially convex.

Now comes the last, and the main step in the induction: We use Lemma 3.1 to find an automorphism $\phi_{n+1} \in \operatorname{Aut}\mathbb{C}^2$ which satisfies conditions (4-4) and (4-5) with n replaced by $n+1$. We look for ϕ_{n+1} of the form

$$\phi_{n+1} = \Phi_n \circ \psi \circ \Phi_n^{-1}, \quad \psi \in \operatorname{Aut}\mathbb{C}^2.$$

(Therefore $\Phi_{n+1} = \phi_{n+1} \circ \Phi_n = \Phi_n \circ \psi$.) Pick a small constant $\delta > 0$ such that for any pair of points $z, z' \in \mathbb{C}^2$, with $z \in \Phi_n^{-1}(L_n)$ and $|z - z'| < \delta$, we have $|\Phi_n(z) - \Phi_n(z')| < \epsilon_{n+1}$. (Such δ exists by continuity of Φ_n.) We also pick a large constant $R > 0$ such that $|\Phi_n(z)| > n+1$ for all $z \in \mathbb{C}^2$ with $|z| > R$. (Equivalently, $\Phi_n^{-1}(\overline{\mathbb{B}}_n) \subset \overline{\mathbb{B}}_R$.) Since the set $\Phi_n^{-1}(L_n)$ is polynomially convex, Lemma 3.1 furnishes an automorphism $\psi \in \operatorname{Aut}\mathbb{C}^2$ satisfying the following two conditions:

(4.4′) $|\psi(z) - z| < \delta$ for $z \in \Phi_n^{-1}(L_n)$.

(4.5′) $|\psi(z)| > R$ for $z \in g_{n+1} \circ f_{n+1}(b\Omega_{n+1} \cup \bigcup_{j \in J_{n+1}} \overline{\triangle}_j)$.

By (4-3) (applied with $n+1$) the two sets appearing in these conditions are disjoint. It is now immediate that ϕ_{n+1} satisfies conditions (4-4), (4-5).

This completes the induction step, so the induction may proceed.

We now conclude the proof. By (4-1) and the choice of the numbers $\epsilon_n > 0$ we see that the limit map $G = \lim_{n\to\infty} g_n \circ f_n \colon \Omega \to \mathbb{C}^2$ is a holomorphic embedding. Condition (4-4) implies that the sequence $\Phi_n \in \operatorname{Aut}\mathbb{C}^2$ converges on the domain $O = \bigcup_{n=2}^{\infty} \Phi_n^{-1}(\overline{\mathbb{B}}_{n-1}) \subset \mathbb{C}^2$ to a Fatou–Bieberbach map $\Phi = \lim_{n\to\infty} \Phi_n \colon O \to \mathbb{C}^2$, i.e., a biholomorphic map of O onto $\mathbb{C}^2$ (see [Forstnerič 2011, Corollary 4.4.2]). Conditions (4-2) and (4-4) show that the sequence Φ_n converges on $G(\Omega)$, so $G(\Omega) \subset O$. From (4-3) and (4-5) we see that G embeds Ω properly into O. Hence the map

$$F = \Phi \circ G = \lim_{n\to\infty} \Phi_n \circ g_n \circ f_n \colon \Omega \hookrightarrow \mathbb{C}^2$$

is a proper holomorphic embedding of Ω into $\mathbb{C}^2$. $\square$

Remark 4.2. If we choose an initial holomorphic embedding $f_0 \colon \overline{\mathbb{D}} \hookrightarrow \mathbb{C}^2$, a compact set $K = K_0 \subset \Omega$ and a number $\epsilon > 0$, then the above construction is easily modified to yield a proper holomorphic embedding $F \colon \Omega \hookrightarrow \mathbb{C}^2$ satisfying $\|F - f\|_K < \epsilon$. Furthermore, we can choose F to agree with f at finitely many points of Ω. All these additions are standard.

5. Domains with punctures

Theorem 1.1 can be extended to domains Ω in $\mathbb{P}^1$ with certain boundary punctures. By a *puncture* we mean a connected component of $\mathbb{P}^1 \setminus \Omega$ that is a point. We say that a domain $\Omega \subset \mathbb{P}^1$ is a *generalized circled domain* if each complementary component is either a round disc or a puncture. By [He and Schramm 1993], any domain in $\mathbb{P}^1$ with at most countably many boundary components is conformally equivalent to a generalized circled domain.

Our main result in this direction is the following.

Theorem 5.1. *Let Ω be a generalized circled domain in $\mathbb{P}^1$. If all but finitely many punctures in the complement $K := \mathbb{P}^1 \setminus \Omega$ are limit points of discs in K, then Ω embeds properly holomorphically into $\mathbb{C}^2$.*

Corollary 5.2. *If Ω is a circled domain in $\mathbb{C}$ or in $\mathbb{P}^1$ and $p_1, \dots, p_l \in \Omega$ is an arbitrary finite set of points in Ω, then the domain $\Omega \setminus \{p_1, \dots, p_l\}$ admits a proper holomorphic embedding into $\mathbb{C}^2$.*

By He and Schramm, Corollary 5.2 also holds for $\Omega \setminus \{p_1, \dots, p_l\}$, where $\Omega \subset \mathbb{P}^1$ is a domain as in Theorem 1.1.

Proof of Theorem 5.1. We make the following modifications to the proof of Theorem 1.1. We may assume as before that Ω is contained in the unit disc $\mathbb{D}$, with $\Gamma_0 = b\mathbb{D}$ being one of its boundary components. Let $f_0 \colon \Omega \hookrightarrow \mathbb{C}^2$ be the embedding $\zeta \mapsto (\zeta, 0)$. Assume that $p_1, \dots, p_l \in b\Omega$ are the finitely many punctures which do not belong to the cluster set of $\bigcup_i \overline{\Delta}_i$. A rational shear $g_0(z_1, z_2) = (z_1, z_2 + \sum_{j=1}^{l} \beta_j/(z_1 - p_j))$ sends the points $p_1, \dots, p_l$ to infinity. We then apply the rest of the proof exactly as before, ensuring at each step of the inductive construction that the embedding with corners $f_n \colon \overline{\Omega}_n \hookrightarrow \mathbb{C}^2$ agrees with f_0 at the points $p_1, \dots, p_l$ and leaves these points π_1-exposed, and the shear g_n has poles at these points. The coordinate projection $\pi_2 \colon \overline{X}_n = g_n \circ f_n(\overline{\Omega}_n) \to \mathbb{C}$ is no longer injective near infinity due to the poles of g_n at the points $p_1, \dots, p_l$. However, since the discs $\overline{\Delta}_i$ do not accumulate on any of the points $p_1, \dots, p_l$, the discs $(g_n \circ f_n)(\overline{\Delta}_i) \subset X_n$ which approach infinity are still mapped bijectively to a finite union of pairwise disjoint wedges at ∞, and the additional sheets of the projection $\pi_2 \colon \overline{X}_n \to \mathbb{C}$ are irrelevant for the construction of the automorphism, which removes the discs and the boundary curves of X_n out of a given ball in $\mathbb{C}^2$.

The remaining punctures p_λ in $b\Omega$ (a possibly uncountable set) can be treated in the same way as the complementary discs. Indeed, since each of these points is a limit point of the sequence of discs Δ_i, every connected component of the set $\widehat{K}_m \setminus K_m$ (where K_m is a sequence of compacts exhausting the domain Ω, see Section 4) that contains one of these punctures p_λ also contains a disc $\overline{\Delta}_i$. By exposing a boundary point of Δ_i and removing it to infinity by a rational shear we thus ensure that the image of p_λ does not belong to the polynomial hull of the image of K_m in $\mathbb{C}^2$. (See Remark 4.1.) The conclusion of Remark 2.3 is still valid, and hence the arguments in the proof of Theorem 1.1 concerning moving compact sets by automorphisms of $\mathbb{C}^2$ still apply without any changes. $\square$

Example 5.3. Assume that $E \subset \mathbb{P}^1$ is any compact totally disconnected set. (In particular, E could be a Cantor set). Then we may choose a sequence of pairwise disjoint closed round discs $\overline{\triangle}_j \subset \mathbb{P}^1 \setminus E$ such that each point of E is a cluster point of the sequence $\{\triangle_j\}$ and such that $\Omega := \mathbb{P}^1 \setminus (E \cup (\bigcup_j \overline{\triangle}_j))$ is a domain. Then Ω embeds properly in $\mathbb{C}^2$.

There exists a Cantor set in $\mathbb{P}^1$ whose complement embeds properly holomorphically into $\mathbb{C}^2$ [Orevkov 2008], but it is an open problem whether this holds for each Cantor set. $\square$

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FRANC FORSTNERIČ: franc.forstneric@fmf.uni-lj.si
Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska 19, 1000 Ljubljana, Slovenia
http://www.fmf.uni-lj.si/~forstneric/

ERLEND FORNÆSS WOLD: erlendfw@math.uio.no
Matematisk Institutt, Universitetet i Oslo, Postboks 1053, 0316 Oslo, Norway

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