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BILINEAR DISPERSIVE ESTIMATES VIA SPACE-TIME RESONANCES I: THE ONE-DIMENSIONAL CASE

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We prove new bilinear dispersive estimates. They are obtained and described via a bilinear time-frequency analysis following the space-time resonances method, introduced by Masmoudi, Shatah, and the second author. They allow us to understand the large time behavior of solutions of quadratic dispersive equations.

1. Introduction

Linear dispersive and Strichartz estimates. A linear, hyperbolic equation is called dispersive if the group velocity of a wave packet depends on its frequency. In order to remain concise, we discuss in this section only the Schrödinger equation

$$\begin{cases} \partial_t u - i\Delta u = 0, \\ u_{|t=0} = f, \end{cases}$$

whose solution we denote $u(t) = e^{it\Delta} f$. This is the prototype of a dispersive equation. A first way to quantify dispersion is provided by the "dispersive estimates", which, in the case of the linear Schrödinger equation, read

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim t^{d/p-d/2} \|f\|_{L^{p'}(\mathbb{R}^d)} \quad \text{if } 2 \le p \le \infty.$$

Another way of quantifying dispersion is provided by Strichartz estimates, which first appeared in [Strichartz 1977] (and were later extended by Ginibre and Velo [1992], with the endpoints due to Keel and Tao [1998]). They read

$$\|e^{it\Delta} f\|_{L^p L^q(\mathbb{R}^+ \times \mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}$$

for every admissible exponents (p, q), which means $2 \le p, q \le \infty$, $(p, q, d) \ne (2, \infty, 2)$ and

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

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Let us just point out the situation if the Euclidean space $\mathbb{R}^d$ is replaced by a compact Riemannian manifold. In that case, any constant function is a solution of the free Schrödinger equation and therefore the dispersive estimate fails for large t. It also fails locally in time. Then Strichartz estimates may only hold with a finite time scale and a loss of derivatives (the data f is controlled in a positive order Sobolev space), which were obtained for the torus by Bourgain [1993b; 1993a] and then extended to general manifolds by Burq, Gérard, and Tzvetkov [Burq et al. 2004].

Bilinear Strichartz estimates. Recently bilinear (and more generally multilinear) analogs of such inequalities have appeared. They correspond to controlling the size of the (pointwise) product of two linear solutions, for instance

$$\|vw\|_{L^pL^q(\mathbb{R}^+\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)} \quad \text{with} \quad \begin{cases} i\partial_t v + \Delta v = 0, & v(t=0) = f, \\ i\partial_t w + \Delta w = 0, & w(t=0) = g, \end{cases} \tag{1-1}$$

or the solution to the inhomogeneous linear problem, the right hand side being given by the product of two linear solutions:

$$\|u\|_{L^pL^q(\mathbb{R}^+\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)} \quad \text{with} \quad \begin{cases} i\partial_t v + \Delta v = 0, & v(t=0) = f, \\ i\partial_t w + \Delta w = 0, & w(t=0) = g, \\ i\partial_t u + \Delta u = vw, & u(t=0) = 0. \end{cases} \tag{1-2}$$

A first line of research, where $p = q = 2$, is related to the use of $X^{s,b}$ spaces in order to solve nonlinear dispersive equations; see, in particular, [Bourgain 1993b] and [Tao 2001]. If the Euclidean space is replaced by a manifold, we refer to [Burq et al. 2005] and [Hani 2010]. The case of the wave equation is treated by Klainerman, Machedon, Bourgain, and Tataru [Klainerman and Machedon 1996], and Foschi and Klainerman [2000]. In all these works, f and g are chosen with vastly different frequency supports, and the focus is on understanding the effect on the implicit constant.

Another line of research considers the case where p and q are not 2: see [Wolff 2001] for the case of the wave equation and [Tao 2003] for the Schrödinger equation. The problem then becomes related to deep harmonic analysis questions (the restriction conjecture), and the optimal estimates are not known in high dimension.

In this article our goal is different from the two directions mentioned: we aim at finding a decay rate in time (rather than integrability properties), and at understanding the effect of localized data.

The set up. From now on, the dimension d of the ambient space is set equal to 1. Let a, b, c be smooth real-valued functions on $\mathbb{R}$, and fix a smooth, compactly supported bilinear symbol m on the frequency plane $\mathbb{R}^2$. We denote by T_m the associated pseudoproduct operator. (a precise definition of T_m is given in Section 1; T_m can be thought of as a generalized product operator, and our setting of course includes classical products between functions that are compactly supported in Fourier space.) Consider then the equation

$$\begin{cases} i\partial_t u + a(D)u = T_m(v, w), \\ i\partial_t v + b(D)v = 0, \\ i\partial_t w + c(D)w = 0, \end{cases} \quad \text{with} \quad \begin{cases} u(t=0) = 0, \\ v(t=0) = f, \\ w(t=0) = g. \end{cases} \tag{1-3}$$

The unknown functions are complex-valued, and this system is set in the whole space: f and g map $\mathbb{R}$ to $\mathbb{C}$, whereas u, v, and w map $\mathbb{R}^2$ to $\mathbb{C}$. The above system is meant to help understand the nonlinear interaction of free waves, which is of course the first step towards understanding a nonlinear problem.

Most of the time, but not always, we assume

$$\text{The second derivatives } a'', b'', c'' \text{ are bounded away from zero.} \tag{H}$$

Under this hypothesis, it is well known that the groups $e^{ita(D)}$, $e^{itb(D)}$, $e^{itc(D)}$ satisfy the following estimates (we denote by $S(t)$ any of these groups):

- Dispersive estimates: $\|S(t)f\|_{L^{p'}} \lesssim |t|^{1/2-1/p}\|f\|_{L^p}$ for $p \in [1,2]$.
- Strichartz estimates: $\|S(t)f\|_{L^p_t L^q} \lesssim \|f\|_{L^2}$ if $2/p + 1/q = \frac{1}{2}$ and $2 \le p, q \le \infty$.

The question we want to answer is this: *Given f and g in L^2 (or weighted L^2 spaces), how does u grow or decay in L^p spaces, $2 \le p \le \infty$?*

The answer of course depends on a, b, c, and the crucial notion is that of space-time resonance.

Space-time resonances. Using Duhamel's formula, we see that $u(t,\cdot)$ is given by the bilinear operator T_t defined by

$$T_t(f,g)(x) = \int_0^t \iint e^{ix(\xi+\eta)} e^{ita(\xi)} e^{is\phi(\xi,\eta)} m(\xi,\eta) \hat{f}(\eta)\hat{g}(\xi-\eta)\, d\xi\, d\eta\, ds,$$

or, more concisely,

$$T_t(f,g) \stackrel{\text{def}}{=} -ie^{ita(D)} \int_0^t T_{me^{is\phi}}(f,g)\, ds,$$

where

$$\phi(\xi,\eta) \stackrel{\text{def}}{=} -a(\xi+\eta) + b(\xi) + c(\eta).$$

Thus the goal of this article is to understand the behavior for large time $t \gg 1$ and some exponent $q \in [2,\infty]$ of

$$\|T_t(f,g)\|_{L^q}, \quad f, g \in L^2.$$

We sometimes find it convenient to write $u(t)$ as

$$u(t) = \mathcal{F}^{-1} \int_0^t \int_{\mathbb{R}} e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta\, ds,$$

where

$$\Phi(\xi,\eta) \stackrel{\text{def}}{=} -a(\xi) + b(\xi-\eta) + c(\eta) = \phi(\xi-\eta,\eta) \quad \text{and} \quad \mu(\xi,\eta) \stackrel{\text{def}}{=} m(\xi-\eta,\eta).$$

Viewing this double integral as a stationary phase problem, it becomes clear that the sets where the phase is stationary in s or η,

$$\Gamma \stackrel{\text{def}}{=} \{(\xi,\eta) \text{ such that } \Phi(\xi,\eta) = 0\} \quad \text{and} \quad \Delta \stackrel{\text{def}}{=} \{(\xi,\eta) \text{ such that } \partial_\eta \Phi(\xi,\eta) = 0\},$$

play a crucial role. Even more important is their intersection $\Gamma \cap \Delta$.

The sets Γ and Δ are, respectively, the sets of time and space resonances; their intersection is the set of space-time resonant sets. A general presentation, stressing their relevance to PDE problems, can be found in [Germain 2010b]; for applications see [Germain et al. 2009; 2012a; 2012b; Germain 2010a; Germain and Masmoudi 2011].

In order to answer the question from the previous page, one has to distinguish between various possible geometries of Γ and Δ (which can be reduced to a discrete set, or curves, with vanishing curvature or not, etc....), possible orders of vanishing of Φ and $\partial_\eta \Phi$ on Γ and Δ, respectively, and different types of intersections of Γ and Δ (at a point or on a dimension 1 set, transverse or not, etc....). Considering all the possible configurations would be a daunting task. We therefore focus on a few relevant and "generic" examples.

- We study the influence of time resonances alone, ignoring space resonances: in other words, we study various configurations for Γ, without making any assumptions on Δ. This essentially amounts to considering the worst possible case as far as Δ is concerned.
- Similarly, we study the influence of space resonances alone, ignoring about time resonances.
- When putting space and time resonances together, we assume a "generic" configuration: Γ and Δ are smooth curves, and they intersect transversally at a point. Aside from being generic, this configuration is of key importance for many nonlinear PDE; this is explained in the next subsection.

Space-time resonant set reduced to a point. As was just mentioned, the case where Γ and Δ are curves which intersect transversally at a point will be examined carefully in this article. It is of course the generic situation, but it also occurs in a number of important models from physics; we give a few examples here. We restrict the discussion to one-dimensional models.

For simple equations of the form $i\partial_t u + \tau(D)u = Q(u, \bar{u})$, where u is scalar-valued, Q quadratic (that is, we retain only the quadratic part of the nonlinearity), and $\tau(\xi) = |\xi|^\alpha$ is homogeneous, the space-time resonant set of the various possible interactions between u and $\bar{u}$ is never reduced to a point. This is the case for standard equations such as NLS, KdV, and wave equations.

However, if τ is no longer supposed to be homogeneous, the space-time resonant set might be reduced to a point. In particular, this is the case for the water wave equation (ideal fluid with a free surface) in the following setting: close to the equilibrium given by a flat surface and zero velocity, including the effects of gravity g and capillarity c, with a constant depth d (perhaps infinite). The dispersion relation for the linearized problem is then

$$\tau(\xi) = \tanh(d|\xi|)\sqrt{g|\xi| + c|\xi|^3}.$$

For more complex models, u is vector-valued, and the system accounts for the interaction of waves with different dispersion relations. It is then often the case that the space-time resonance set is reduced to a point. We mention in particular the following.

- The Euler–Maxwell system, describing the interaction of a charged fluid with an electromagnetic field (see [Germain and Masmoudi 2011] for a mathematical treatment of this equation dealing with space-time resonances). Many other models of plasma physics could also be mentioned here.

- Systems where wave and (generalized) Schrödinger equations are coupled: for instance, the Davey–Stewartson, Ishimori, Maxwell–Schrödinger, and Zakharov systems.

A sample of our results. Among the many results proved in this paper, we record in Theorem 1.1 a few that are illustrative and interesting. We need a definition: a curve in (ξ, η) is *characteristic* if it has tangents parallel to one or more of the directions $\xi = 0$, $\eta = 0$, or $\xi + \eta = 0$, and *noncharacteristic* otherwise.

Theorem 1.1. *Recall that m is smooth and compactly supported. Assume that (H) holds.*

(i) *If Γ is a noncharacteristic curve along which Φ vanishes at order* 1,

$$\|u(t)\|_{L^q} \lesssim \langle \log t \rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \text{for } s > \tfrac{1}{4}.$$

(ii) *If $\Delta = \varnothing$, then, for any $\delta > 0$,*

$$\|u(t)\|_{L^q} \lesssim t^{1/q-1/2+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \text{for } s > 1 - \tfrac{1}{q}.$$

Furthermore, this rate of decay is optimal.

(iii) *If Γ and Δ intersect transversely at a single point in the support of m, then, for any $\delta > 0$*

$$\|u(t)\|_{L^q} \lesssim t^{-(1/4-1/(2q))+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}} \quad \text{for } s > 1.$$

Furthermore, this rate of decay is optimal (up to the loss δ as small as we want).

Organization of the article. In Section 2 we derive asymptotic equivalents for u when f and g smooth and localized. Three cases are considered: $\Gamma = \varnothing$, $\Delta = \varnothing$, and Γ and Δ are curves intersecting transversally at a point (in particular we prove the second part of Theorem 1.1). In Section 3, relying only on time resonances, we establish estimates for u when f and g belong to L^2. In Section 4, we establish estimates for u when f and g belong to weighted L^2 spaces. In particular we consider the case when the space-time resonant set is reduced to a point, and thereby prove the first part of Theorem 1.1. In Appendix A, we detail some results on boundedness of multilinear operators. Finally, in Appendix B, one-dimensional oscillatory integrals are studied.

Notations. We adopt the following notations.

- $A \lesssim B$ if $A \leq CB$ for some implicit constant C. The value of C may change from line to line.
- $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$.
- If f is a function over $\mathbb{R}^d$, its Fourier transform, denoted $\hat{f}$, or $\mathcal{F}(f)$, is given by

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\xi} f(x)\, dx, \quad \text{thus } f(x) = \frac{1}{(2\pi)^{d/2}} \int e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

  (In the text, we systematically drop constants such as $1/(2\pi)^{d/2}$ since they are not relevant.)
- The Fourier multiplier with symbol $m(\xi)$ is defined by

$$m(D) f = \mathcal{F}^{-1} [m \mathcal{F} f].$$

- The bilinear Fourier multiplier with symbol m is given by

$$T_m(f,g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)\,d\xi d\eta = \mathscr{F}^{-1}\int m(\xi-\eta,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta.$$

- The Japanese bracket $\langle\cdot\rangle$ stands for $\langle x\rangle = \sqrt{1+x^2}$.
- The weighted Fourier space $L^{p,s}$ is given by the norm $\|f\|_{L^{p,s}} = \|\langle x\rangle^s f\|_{L^p}$.
- If E is a set in $\mathbb{R}^d$, E_ϵ is the set of points of $\mathbb{R}^d$ that are within ϵ of a point of E.

2. Asymptotic equivalents

Preliminary discussion. Our aim in this section is to obtain asymptotic equivalents, as $t\to\infty$, for the solution u of (1-3), under the simplifying assumption that f and g are very smooth and localized. Hypotheses on a, b, c are needed, and the variety of possible situations is huge; we try to focus on the most representative, or generic situations. First, we assume in this whole section that (H) holds: this gives decay for the linear waves. For bilinear estimates, everything hinges on the vanishing properties of Φ and $\partial_\eta\Phi$, where

$$\Phi(\xi,\eta) = -a(\xi)+b(\xi-\eta)+c(\eta).$$

We distinguish three situations: Φ does not vanish (Theorem 2.2), Φ_η does not vanish (Theorem 2.4), $\{\Phi=0\}$ and $\{\Phi_\eta=0\}$ are curves intersecting transversally (Theorem 2.5). Additional assumptions will be specified as needed.

Asymptotics for the linear Cauchy problem. They are obtained easily by stationary phase; see for instance [Stein 1993].

Lemma 2.1. *Assume that $F\in\mathscr{S}$ is such that $\widehat{F}$ is compactly supported; and suppose that a'' does not vanish on* Supp F. *Then*

$$e^{ita(D)}F(x) = e^{it[a(\xi_0)+X\xi_0]}e^{i(\pi/4)\sigma}\frac{1}{\sqrt{|a''(\xi_0)|}}\frac{1}{\sqrt{t}}\widehat{F}(\xi_0)+O\Big(\frac{1}{t}\Big),$$

where

$$X \overset{\text{def}}{=} \frac{x}{t},\qquad a'(\xi_0)+X \overset{def}{=} 0,\qquad \sigma \overset{\text{def}}{=} \operatorname{sign} a''(\xi_0).$$

The point of view of stationary phase. The solution of (1-3) is

$$u(t,x) = -\frac{i}{\sqrt{2\pi}}\int_0^t\iint e^{ix\xi}e^{i[(t-s)a(\xi)+sb(\xi-\eta)+sc(\eta)]}\mu(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,ds.$$

Recalling that $X\overset{\text{def}}{=}x/t$ and $\mu(\xi,\eta)\overset{\text{def}}{=}m(\xi-\eta,\eta)$, this is equal to

$$u(t,x) = -\frac{i}{\sqrt{2\pi}}t\int_0^1\iint e^{it[(1-\sigma)a(\xi)+\sigma b(\xi-\eta)+\sigma c(\eta)+X\xi]}\mu(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,d\sigma. \tag{2-0}$$

This is now a (nonstandard) stationary phase problem, with phase

$$\psi(\xi,\eta,\sigma) \overset{\text{def}}{=} (1-\sigma)a(\xi)+\sigma b(\xi-\eta)+\sigma c(\eta)+X\xi = a(\xi)+\sigma\Phi(\xi,\eta)+X\xi.$$

The phase of the gradient is

$$\nabla_{\xi,\eta,\sigma}\psi = \begin{pmatrix} a' + \sigma\Phi_\xi + X \\ \sigma\Phi_\eta \\ \Phi \end{pmatrix},$$

which vanishes if either

$$\begin{cases} \sigma = 0, \\ \Phi = 0, \\ a' + X = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \Phi = 0, \\ \Phi_\eta = 0, \\ a' + \sigma\Phi_\xi + X = 0. \end{cases} \tag{2-1}$$

The Hessian of ψ is given by

$$\mathrm{Hess}_{\xi,\eta,\sigma}\,\psi = \begin{pmatrix} a'' + \sigma\Phi_{\xi\xi} & \sigma\Phi_{\xi\eta} & \Phi_\xi \\ \sigma\Phi_{\xi\eta} & \sigma\Phi_{\eta\eta} & \Phi_\eta \\ \Phi_\xi & \Phi_\eta & 0 \end{pmatrix}.$$

On stationary points of the first type in (2-1), the Hessian is degenerate if and only if (ξ, η) belongs to the space-time resonant set. On stationary points of the second type in (2-1), the Hessian is generically nondegenerate.

The main difficulty in the analysis is handling the stationary points on the boundary of the integration domain, namely those for which $\sigma = 0$ or 1; this is even more complicated when they are degenerate.

Theorem 2.2 (absence of time resonances). *Assume that* $\Phi(\xi, \eta)$ *does not vanish on* $\operatorname{Supp} m$ (*that is*, $\Gamma = \varnothing$), *and that* f *and* g *belong to* $\mathcal{S}$. *Then, as* $t \to \infty$,

$$u(t) = e^{ita(D)}F + O\Big(\frac{1}{t}\Big).$$

with

$$F = T_{m/\phi}(f, g).$$

Remark 2.3. The asymptotic behavior of $e^{ita(D)}F$ is given by Lemma 2.1.

Proof. The proof is very easy: u is given by

$$u(t) = -ie^{ita(D)}\int_0^t T_{me^{is\phi}}(f, g)\,ds,$$

or

$$u(t) = -T_{m/\phi}(e^{itb(D)}f, e^{itc(D)}g) + e^{ita(D)}T_{m/\phi}(f, g).$$

The theorem follows since the first term above is $O(1/t)$, by the linear decay estimates. □

Theorem 2.4 (absence of space resonances). *Assume that* ψ_η *does not vanish on* $\operatorname{Supp} m$ (*that is*, $\Delta = \varnothing$), *that* $\psi_{\xi\xi}(\xi, \eta, \sigma)$ *does not vanish on* $\operatorname{Supp} m \times [0, 1]$, *and that* f, g *belong to* $\mathcal{S}$. *Fix* $M > 0$ *and* $N \in \mathbb{N}$. *Then, as* $t \to \infty$,

$$u(t) = e^{ita(D)}F + O\Big(\frac{1}{M^N\sqrt{t}}\Big)$$

where

$$F = -i\int_0^M e^{i(t-s)a(D)}T_m(e^{isb(D)}f, e^{isc(D)}g)\,ds.$$

(*In other words*, $e^{ita(D)}F$ *is the solution of*

$$i\partial_t u + a(D)u = \begin{cases} T_m(v,w) & \text{if } 0 < t < M, \\ 0 & \text{if } t > M, \end{cases} \qquad i\partial_t v + b(D)v = 0, \qquad i\partial_t w + c(D)w = 0,$$

with the data $u(t=0) = 0$, $v(t=0) = f$, *and* $w(t=0) = g$.)

Proof. Starting from the stationary phase formulation (see page 692), it suffices to show that

$$\int_{M/t}^{1} \iint e^{it\psi(\xi,\eta,\sigma)} \mu(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta\, d\xi\, d\sigma \tag{2-2}$$

is $O(1/(M^N t^{3/2}))$.

First apply the stationary phase lemma in ξ in the above. The vanishing set of ψ_ξ depends on X. If X is such that ψ_ξ does not vanish, (2-2) is $O(1/t^N)$ for any N and we are done. Otherwise, ψ_ξ vanishes for some ξ, which we denote ξ_0, and which is a function of X, η, and σ. We can assume without loss of generality that ξ_0 is unique. Since $\psi_{\xi\xi}$ does not vanish by assumption, the stationary phase lemma gives

$$(2\text{-}2) = \int_{M/t}^{1} \int e^{it\psi(\xi_0,\eta,\sigma)} \left(\frac{\alpha(\xi,\eta,\sigma)}{\sqrt{t}} + \frac{\beta(\xi,\eta,\sigma)}{t} + \frac{\gamma(\xi,\eta,\sigma)}{t\sqrt{t}} + O\left(\frac{1}{t^2}\right) \right) d\eta\, d\sigma,$$

where α, β, and γ are smooth functions which we do not specify. The fourth summand in (2-2) is already small enough. We will now show how to deal with the first one, and this will conclude the proof since the second and third ones are easier (better decay). Thus we now want to show that

$$\int_{M/t}^{1} \int e^{it\psi(\xi_0,\eta,\sigma)} \frac{\alpha(\xi,\eta,\sigma)}{\sqrt{t}}\, d\eta\, d\sigma \tag{2-3}$$

is $O(1/(M^N t))$. In order to take advantage of oscillations in η, observe that

$$\partial_\eta[\psi(\xi_0(\eta,\sigma,X),\eta,\sigma)] = \partial_\eta \xi_0 [\partial_\xi \psi](\xi_0,\eta,\sigma) + [\partial_\eta \psi](\xi_0,\eta,\sigma) = [\partial_\eta \psi](\xi_0,\eta,\sigma) = \sigma[\partial_\eta \Phi](\xi_0,\eta).$$

By hypothesis, $\partial_\eta \Phi$ does not vanish, therefore

$$|\partial_\eta[\psi(\xi_0,\eta,\sigma)]| \gtrsim \sigma.$$

Integrating by parts $N+1$ times with the help of the identity

$$\frac{1}{t\partial_\eta[\psi(\xi_0,\eta,\sigma)]} \partial_\eta e^{it\psi(\xi_0,\eta,\sigma)} = i e^{it\psi(\xi_0,\eta,\sigma)},$$

we obtain

$$|(2\text{-}3)| \lesssim \int_{M/t}^{1} \frac{1}{(\sigma t)^{N+1}\sqrt{t}}\, d\sigma \lesssim \frac{1}{M^N t^{3/2}},$$

which concludes the proof. ☐

Theorem 2.5 (space-time resonance set reduced to a point). *Assume that f, g belong to $\mathcal{S}$, that there exists a unique (ξ_0, η_0) such that*

$$\Phi(\xi_0, \eta_0) = \Phi_\eta(\xi_0, \eta_0) = 0,$$

and that the following technical, generic hypotheses are satisfied:

- (*we are under the standing assumption* (H), *but only the fact that a'' is nonvanishing is used here*;)
- $\Phi_\xi(\xi_0, \eta_0) \neq 0$;
- $\Phi_{\eta\eta}(\xi_0, \eta_0) \neq 0$;

and that $\operatorname{Supp} m$ *is contained in a small enough neighborhood of* (ξ_0, η_0).

Recall that $X = x/t$, and set

$$\Sigma(X) \stackrel{\text{def}}{=} -\frac{1}{\Phi_\xi(\xi_0, \eta_0)}(a'(\xi_0) + X).$$

Let $\epsilon > 0$ be small enough. Assume without loss of generality that $\Phi_\xi(\xi_0, \eta_0) > 0$. Then:

- *If $X < -\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) - \epsilon$,*
$$u(t) = O\Big(\frac{1}{t^N}\Big)$$
for any N.
- *If $-\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) - \epsilon < X < -\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) + \epsilon$,*
$$u(t) = \frac{1}{\sqrt{t}} A_2(\Sigma)\mathcal{G}_1(\sqrt{t}[\Sigma - 1]) + O\Big(\frac{1}{t}\Big)$$
for a smooth function A_2.
- *If $-\Phi_\xi(\xi_0, \eta_0) - a'(\xi_0) + \epsilon < X < -a'(\xi_0) - \epsilon$,*
$$u(t, x) = \frac{1}{\sqrt{t}} \frac{A_1}{\sqrt{\Sigma(X)}} e^{it\psi(\xi_0, \eta_0, \Sigma)} + O\Big(\frac{1}{t}\Big)$$
for a constant A_1.
- *If $-a'(\xi_0) - \epsilon < X < -a'(\xi_0) + \epsilon$,*
$$u(t) = A_0(\Sigma)\frac{1}{t^{1/4}}\mathcal{G}_2(\sqrt{t}\Sigma) + \begin{cases} O(t^{-3/4}) & \text{if } |\sqrt{t}\Sigma| < 1, \\ O(|\log t|/\sqrt{t}) & \text{if } |\sqrt{t}\Sigma| > 1. \end{cases}$$
for a smooth function A_0;
- *if $-a'(\xi_0) + \epsilon < X$,*
$$u(t) = O\Big(\frac{1}{t^N}\Big)$$
for any N.

Remark 2.6. (1) Theorem 2.5 provides an efficient equivalent of $u(t)$ for large t in all the zones of the space-time plane (x, t), except where Σ is small, but larger than $1/|\log t|^2$ (because then $|\log t|/\sqrt{t} > (1/t^{1/4})|\mathcal{G}_2(\sqrt{t}\Sigma)|$). Dealing with this region would require fairly technical developments, from which we refrain.

(2) If Φ vanishes at order 1 on Γ and Δ, the conditions $\Phi_\xi(\xi_0, \eta_0) \neq 0$ and $\Phi_{\eta\eta}(\xi_0, \eta_0) \neq 0$ are equivalent to Γ and Δ intersecting transversally at (ξ_0, η_0). Indeed, a tangent vector to Γ (respectively, Δ) at (ξ_0, η_0) is given by

$$\begin{pmatrix} \partial_\eta \Phi(\xi_0, \eta_0) \\ -\partial_\xi \Phi(\xi_0, \eta_0) \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_\xi \Phi(\xi_0, \eta_0) \end{pmatrix} \qquad \left(\text{respectively} \begin{pmatrix} \partial_\eta^2 \Phi(\xi_0, \eta_0) \\ -\partial_\eta \partial_\xi \Phi(\xi_0, \eta_0) \end{pmatrix}\right).$$

These two vectors are not collinear if $\partial_\xi \Phi(\xi_0, \eta_0) \partial_\eta^2 \Phi(\xi_0, \eta_0) \neq 0$.

(3) The hypothesis that $\operatorname{Supp} m$ is restricted in a small enough neighborhood is not restrictive: away from (ξ_0, η_0), either Φ or Φ_η is nonzero, so either Theorem 2.2 or Theorem 2.4 applies.

The proof distinguishes three regions: σ away from 0 and 1, σ close to 0, and σ close to 1. Starting from Equation (2-0), we split the time integral as follows:

$$\begin{aligned} u(t, x) &= -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint e^{it\psi(\xi,\eta,\sigma)} \mu(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\xi \, d\eta \, d\sigma \\ &= -\frac{i}{\sqrt{2\pi}} t \int_0^t \iint \big(\chi_I(\sigma) + \chi_{II}(\sigma) + \chi_{III}(\sigma)\big) \dots d\xi \, d\eta \, d\sigma \overset{\text{def}}{=} I + II + III. \end{aligned} \tag{2-4}$$

Here χ_I, χ_{II}, and χ_{III} are three smooth, positive functions, adding up to 1 for each σ and such that

$$\chi_{II}(\sigma) = \begin{cases} 0 & \text{if } \sigma < \delta, \\ 1 & \text{if } \sigma > 2\delta, \end{cases} \qquad \chi_I(\sigma) = \begin{cases} 0 & \text{if } \sigma < \delta \text{ or } \sigma > 1 - \delta, \\ 1 & \text{if } 2\delta < \sigma < 1 - 2\delta, \end{cases} \qquad \chi_{III}(\sigma) = \begin{cases} 0 & \text{if } \sigma < 1 - 2\delta, \\ 1 & \text{if } \sigma > 1 - \delta. \end{cases}$$

Here $\delta > 0$ is a sufficiently small number.

The contribution of σ away from 0 *and* 1. This is the simplest case since it can be settled by resorting to elementary stationary phase considerations. Our aim is to estimate

$$I = -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint \chi_I(\sigma) e^{it\psi(\xi,\eta,\sigma)} \mu(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\xi \, d\eta \, d\sigma.$$

The phase $\psi(\xi, \eta, \sigma)$ is also a function of X, but from now on we consider X to be fixed.

Since σ does not vanish on $\operatorname{Supp} \chi_I$, the gradient

$$\nabla_{\xi,\eta,\sigma} \psi = \begin{pmatrix} a' + \sigma \Phi_\xi + X \\ \sigma \Phi_\eta \\ \Phi \end{pmatrix}$$

vanishes if

$$\Phi(\xi, \eta) = \Phi_\eta(\xi, \eta) = 0 \quad \text{and} \quad a'(\xi) + \sigma \Phi_\xi(\xi, \eta) + X = 0.$$

The first two conditions impose $(\xi, \eta) = (\xi_0, \eta_0)$ whereas the third one gives

$$\sigma = \Sigma(X) \stackrel{\text{def}}{=} -\frac{1}{\Phi_\xi(\xi_0, \eta_0)}(a'(\xi_0) + X).$$

(This makes sense under the assumption that $\Phi_\xi(\xi_0, \eta_0) \neq 0$.) We assume that X is such that σ given by the above line lies in $\operatorname{Supp} m$; if this is not the case, the contribution of I is negligible. The Hessian at (Σ, ξ_0, η_0) is

$$\operatorname{Hess}_{\xi,\eta,\sigma} \psi(\xi_0, \eta_0, \Sigma) = \begin{pmatrix} a'' + \Sigma\Phi_{\xi\xi}(\xi_0, \eta_0) & \Sigma\Phi_{\xi\eta}(\xi_0, \eta_0) & \Phi_\xi(\xi_0, \eta_0) \\ \Sigma\Phi_{\xi\eta}(\xi_0, \eta_0) & \Sigma\Phi_{\eta\eta}(\xi_0, \eta_0) & 0 \\ \Phi_\xi(\xi_0, \eta_0) & 0 & 0 \end{pmatrix}$$

with determinant

$$\det \operatorname{Hess}_{\xi,\eta,\sigma} \psi(\xi_0, \eta_0, \Sigma) = -\Sigma\Phi_\xi(\xi_0, \eta_0)^2 \Phi_{\eta\eta}(\xi_0, \eta_0).$$

Let us assume that $\Phi_{\eta\eta}(\xi_0, \eta_0)$ is not zero, which is generically satisfied. The stationary phase principle then gives [Stein 1993]

$$u(t, x) = \frac{1}{\sqrt{t}} \frac{\chi_I(\Sigma(X))}{\sqrt{\Sigma(X)}} e^{it\psi(\xi_0,\eta_0,\Sigma)} A_1 + O\Big(\frac{1}{t}\Big)$$

with

$$A_1 \stackrel{\text{def}}{=} \frac{(2\pi)^{3/2} e^{i(\pi/4)S}}{|\Phi_\xi(\xi_0, \eta_0)| \sqrt{|\Phi_{\eta\eta}(\xi_0, \eta_0)|}} \mu(\xi_0, \eta_0) \hat{f}(\xi_0 - \eta_0) \hat{g}(\eta_0)$$

where S is the signature of $\operatorname{Hess}_{\xi,\eta,\sigma} \psi(\xi_0, \eta_0, \Sigma)$.

The contribution of σ close to 0.

<u>Step 1</u>: splitting between small and large times. Our aim is to estimate

$$II = -\frac{i}{\sqrt{2\pi}} t \int_0^1 \iint \chi_{II}(\sigma) e^{it\psi(\xi,\eta,\sigma)} \mu(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\xi \, d\eta \, d\sigma,$$

which we split into

$$II = -\frac{i}{\sqrt{2\pi}} t \left[\int_0^{1/t} + \int_{1/t}^1 \dots d\sigma \right] \stackrel{\text{def}}{=} II_1 + II_2.$$

Rescaling II_1, we see that it can be written

$$u(t) = e^{ita(D)} F \quad \text{where } F = -\frac{i}{\sqrt{2\pi}} \int_0^1 \int e^{i(s-t)a(D)} T_m(e^{-isb(D)} f, e^{-isc(D)} g) \, ds,$$

so that it reduces to a linear solution for t sufficiently large. We now focus on II_2.

<u>Step 2</u>: stationary phase in ξ. We want to apply the stationary phase lemma in the variable ξ. Observe that

$$\partial_\xi \psi(\xi, \eta, \sigma) = a'(\xi) + \sigma\Phi_\xi + X.$$

Thus for η, σ, and X fixed, $\partial_\xi \psi(\xi, \eta, \sigma) = 0$ may or may not have a solution in $\operatorname{Supp} m$. If not, the contribution is negligible, so let us assume that this equation has a solution $\xi = \Xi(X, \eta, \sigma)$. Next,

$$\partial_\xi^2 \psi(\xi, \eta, \sigma) = a''(\xi) + \sigma \Phi_{\xi\xi}.$$

Since we are assuming that a'' does not vanish, taking δ small enough, we can ensure that $\partial_\xi^2 \psi(\xi, \eta, \sigma)$ does not vanish. Applying the stationary phase lemma then gives

$$II_2 = t \int_{1/t}^1 \int G(\Xi, \eta) e^{it\psi(\Xi,\eta,\sigma)} \left(\frac{\sqrt{2\pi}\, e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(\Xi, \eta, \sigma)}\sqrt{t}} + \frac{\alpha(\eta, \sigma)}{t} + \frac{\beta(\eta, \sigma)}{t\sqrt{t}} + O\left(\frac{1}{t^2}\right) \right) d\eta\, d\sigma, \quad (2\text{-}5)$$

where $S_0 = \operatorname{sign}(\psi_{\xi\xi}(\Xi, \eta, \sigma))$, α and β are smooth functions, and for simplicity we denoted

$$G(\xi, \eta, \sigma) = -\frac{i}{\sqrt{2\pi}} \chi_{II}(\sigma) \mu(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta).$$

The last term in (2-5), containing $O(1/t^2)$, contributes $O(1/t^2)$ to u; thus we can discard it and focus on

$$t \int_{1/t}^1 \int G(\Xi, \eta) e^{it\psi(\Xi,\eta,\sigma)} \left(\frac{\sqrt{2\pi}\, e^{iS_0 \frac{\pi}{4}}}{\sqrt{\psi_{\xi\xi}(\Xi, \eta, \sigma)}\sqrt{t}} + \frac{\alpha(\xi, \eta, \sigma)}{t} + \frac{\beta(\xi, \eta, \sigma)}{t\sqrt{t}} \right) d\eta\, d\sigma. \quad (2\text{-}6)$$

<u>Step 3</u>: stationary phase in η. Observe that

$$\partial_\eta[\psi(\Xi(\eta, \sigma), \eta, \sigma)] = \partial_\eta \Xi [\partial_\xi \psi](\Xi, \eta, \sigma) + [\partial_\eta \psi](\Xi, \eta, \sigma) = [\partial_\eta \psi](\Xi, \eta, \sigma) = \sigma [\partial_\eta \Phi](\Xi, \eta).$$

Just as for the stationary phase in ξ, we denote by $\eta = H(\sigma, X)$ the solution of $[\partial_\eta \Phi](\Xi, \eta) = 0$ (if no solution exist, the contribution is negligible). Next, set

$$\partial_\eta^2[\psi(\Xi(\eta, \sigma), \eta, \sigma)] = \sigma \partial_\eta \Xi [\partial_\xi \partial_\eta \Phi](\Xi, \eta) + \sigma [\partial_\eta^2 \Phi](\Xi, \eta) \overset{\text{def}}{=} \sigma Z(\eta, \sigma).$$

We need to assume that

$$Z(\eta, \sigma) \neq 0$$

if $(\sigma, \Xi, \eta) \in \operatorname{Supp} m \chi_{II}$. Since the support of m, as well as δ, is assumed to be small enough, it suffices that $Z(\eta_0, 0) \neq 0$; but a simple computation reveals that $Z(\eta_0, 0) = \phi_{\eta\eta}(\eta_0, \xi_0)$, which is nonzero by hypothesis. The stationary phase lemma in η applied to Theorem 2.5 then gives

$$\begin{aligned} (2\text{-}6) = t \int_{1/t}^1 G(\Xi, H) e^{it\psi(\Xi,H,\sigma)} & \frac{\sqrt{2\pi}\, e^{iS_1\pi/4}}{\sqrt{t}\sqrt{\sigma} Z(H, \sigma)} \\ & \times \left(\frac{\sqrt{2\pi}\, e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(\Xi, H, \sigma)}\sqrt{t}} + \frac{\alpha(H, \sigma)}{t} + \frac{\beta(H, \sigma)}{t\sqrt{t}} + O\left(\frac{1}{t\sqrt{t\sigma}}\right) \right) d\sigma, \end{aligned} \quad (2\text{-}7)$$

where $S_1 = \operatorname{sign}(Z(H, \sigma))$, $\tilde{\alpha}$ and $\tilde{\beta}$ are smooth functions. The last summand in (2-7) contributes

$$O\left(\frac{t}{t\sqrt{t}} \int_{1/t}^1 \frac{d\sigma}{\sigma} \right) = O\left(\frac{\log t}{\sqrt{t}} \right).$$

We discard this and focus on

$$t\int_{1/t}^{1} G(\Xi,H)e^{it\psi(\Xi,H,\sigma)}\frac{\sqrt{2\pi}e^{iS_1\pi/4}}{\sqrt{t}\sqrt{\sigma}Z(H,\sigma)}\left(\frac{\sqrt{2\pi}e^{iS_0\pi/4}}{\sqrt{\psi_{\xi\xi}(H,\sigma)}\sqrt{t}}+\frac{\alpha(H,\sigma)}{t}+\frac{\beta(\xi,H,\sigma)}{t\sqrt{t}}\right)d\sigma. \tag{2-8}$$

Step 4: stationary phase in σ. In this final step, we are not going to apply the standard stationary phase lemma, but rather its variant given in Proposition B.2. Differentiating in σ, the phase in (2-7) gives

$$\partial_\sigma\big[\psi\big(\Xi(H(\eta,\sigma),\sigma),H(\sigma),\sigma\big)\big]=[\partial_\sigma\psi](\Xi,H,\sigma)=\Phi(\Xi(H,\sigma),H(\sigma)),$$

since $\partial_\xi\psi=\partial_\eta\psi=0$ at the point (Ξ,H,σ). Thus $\partial_\sigma\phi=0$ if $\Phi(\Xi,H)=0$. On the other hand, since $\partial_\eta\Phi(\Xi,H)=0$ by definition of H,

$$\Phi(\Xi(H,\sigma),H(\sigma))=0 \quad \text{if and only if} \quad H(\sigma)=\eta_0 \quad \text{and} \quad \Xi(\eta_0,\sigma)=\xi_0.$$

But by definition of Ξ this implies

$$\sigma=\Sigma(X)=-\frac{X+a'(\xi_0)}{\psi_\xi(\xi_0,\eta_0)}.$$

In order to apply Proposition B.2, we need to check that

$$\partial_\sigma^2\big[\psi\big(\Xi(H(\eta,\sigma),\sigma),H(\sigma),\sigma\big)\big](\Sigma)=\partial_\sigma[\Phi(\Xi(H,\sigma),H(\sigma))](\Sigma)\neq 0.$$

Since δ is chosen small enough, it suffices to check that it holds for $\Sigma=0$ (that is, when X is such that $\Sigma(X)=0$). This follows from the following computation:

$$\begin{aligned}\partial_\sigma[\Phi(\Xi(H,\Sigma),H(\Sigma))](0)&=\partial_\xi\Phi(\xi_0,\eta_0)(\partial_\sigma\Xi(\eta_0,0)+\partial_\sigma H(0)\partial_\eta\Xi(\eta_0,0))+\partial_\eta\Phi(\xi_0,\eta_0)\partial_\sigma H(0)\\&=\partial_\xi\Phi(\xi_0,\eta_0)\partial_\sigma\Xi(\eta_0,0)=-\frac{\phi_\xi(\xi_0,\eta_0)^2}{a''(\xi_0)}\neq 0,\end{aligned}$$

where we used that $\partial_\eta\Phi(\xi_0,\eta_0)=\partial_\eta\Xi(\eta_0,0)=0$ and $\partial_\sigma\Xi(\eta_0,0)=-\phi_\xi(\xi_0,\eta_0)/a''(\xi_0)$. We now write

$$\text{(2-8)}=t\int_{1/t}^{1}\dots\,d\sigma=t\int_0^1 -t\int_0^{1/t}\dots\,d\sigma.$$

The second summand, $t\int_0^{1/t}\dots\,d\sigma$, is directly estimated to be $O(1/\sqrt{t})$. As for the first summand, $t\int_0^1\dots\,d\sigma$, apply Proposition B.2(iv) to obtain

$$\text{(2-8)}=A_0(\Sigma)\mathcal{G}_2(\sqrt{t}\Sigma)+\begin{cases}O(t^{-3/4}) & \text{if } |\sqrt{t}\Sigma|<1,\\ O(\sqrt{|\Sigma|/t}) & \text{if } |\sqrt{t}\Sigma|>1,\end{cases}$$

where A_0 is a smooth function which we do not detail here.

The contribution of σ close to 1. In order to estimate

$$III=-\frac{i}{\sqrt{2\pi}}t\int_0^1\iint\chi_{III}(\sigma)e^{it\psi(\xi,\eta,\sigma)}\mu(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\,d\xi\,d\sigma.$$

an approach similar to the one used for II can be followed, the details being simpler: first apply the stationary phase Lemma in the (ξ,η) variables, then Proposition B.2(i). We do not give details here.

Conclusion.

Space-time localization of the waves. As a conclusion of the asymptotic analysis of waves which has just been carried out, it is interesting to compare the space-time localizations of the emerging wave u, the solution of (1-3), in the three situations we examined. To simplify, suppose that f and g are localized in space close to 0, and in frequency close to $\tilde{\xi} - \tilde{\eta}$ and $\tilde{\eta}$, respectively. Then

- in the absence of space-time resonances, u will be localized where $X \sim -a'(\tilde{\xi})$, where it will have size $\sim 1/\sqrt{t}$;
- if the space-time resonant set is reduced to a point, then, under the assumptions of Theorem 2.5, u will have size $\sim 1/t^{1/4}$ if $-\Phi_\xi(\eta_0, \xi_0) - a'(\xi_0) < X < -a'(\xi_0)$, and size $\sim 1/\sqrt{t}$ if $X \sim -a'(\xi_0)$.

Lower bound. The asymptotic equivalents which have been computed also provide lower bounds for L^p norms of u. In the absence of space time resonances, we do not learn anything, since the equivalent for u is similar to a linear solution. However, in the case when Theorem 2.5 applies (that is, when Δ and Γ intersect transversally at a point), for t large we get

$$\|u(t)\|_{L^q} \gtrsim \begin{cases} \log t & \text{for } q = 2, \\ t^{1/(2q)-1/4} & \text{for } 2 < q \leq \infty, \end{cases} \tag{2-9}$$

which corresponds to the lower bound states in Theorem 1.1.

3. Nonlocalized data

In this section, the data are only supposed to belong to L^2, as opposed to in Section 4, where the data will belong to weighted L^2 spaces.

Main results.

Theorem 3.1. *Assume that m is smooth and compactly supported and a, b, c are real-valued. In the various possible situations that follow, for $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim \alpha(t) \|f\|_{L^2} \|g\|_{L^2}$$

with $\alpha(t)$ as follows:

$$\alpha(t) = \begin{cases} t & \textit{in general}, \\ 1 & \textit{if } \Gamma \textit{ is empty}, \\ t^{1/2+1/(2q)} & \textit{if } \Gamma \textit{ is a point where } \phi \textit{ vanishes at order two}, \\ t^{1/q} & \textit{if } 2 \leq q < \infty \textit{ and } \Gamma \textit{ is a noncharacteristic curve where } \phi \textit{ vanishes at order one}, \\ \langle \log t \rangle & \textit{if } q = \infty \textit{ and } \Gamma \textit{ is a noncharacteristic curve where } \phi \textit{ vanishes at order one}, \\ t^{1/4+1/(2q)} & \textit{when } \Gamma \textit{ is a curve with nonvanishing curvature where } \phi \textit{ vanishes at order one}, \\ t^{1/2} & \textit{if } \Gamma \textit{ is a general curve where } \phi \textit{ vanishes at order one}. \end{cases}$$

In the two first situations above, the bound can be improved if the unitary groups $e^{ita(D)}$, $e^{itb(D)}$, and $e^{itc(D)}$ give decay.

In this setting, using precisely the structure of the product of two linear solutions (which cannot be described by using only the set $\phi^{-1}(\{0\})$ as previously), we get the following improvement.

Theorem 3.2. *Assume that m is smooth and compactly supported, and that (H) holds. For all $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t^{1/2+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

If, moreover, we assume that $\Gamma = \varnothing$, then, for $p, q \in [2, \infty)$ with $\dfrac{1}{p} + \dfrac{1}{q} > \dfrac{1}{2}$, we get

$$\|u(t)\|_{L_t^p L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{3-1}$$

Remark 3.3. The last statement of the previous theorem gives decay in an integrated form (u belonging to some $L^p L^q$), as opposed to the pointwise in time rate of decay obtained earlier; of course, this has to do with the use of Strichartz estimates. Heuristically, (3-1) can be understood as giving the rate of decay $\|u(t)\|_{L^q} \lesssim t^{1/q-1/2} \|f\|_{L^2} \|g\|_{L^2}$.

Then, if the smooth symbol m does not have bounded support, we have the following result.

Corollary 3.4. *We want to track the dependence of the bounds in the above theorem on the size of the support of m. So assume that m is bounded by 1 along with sufficiently many of its derivatives, and that it is supported on $B(0, R)$. Then all the previous boundedness results hold with an extra factor R.*

Proof. In Theorems 3.1 or 3.2, we have obtained boundedness from $L^2 \times L^2$ to $\mathscr{B}$ (where $\mathscr{B} = L^q$ or $L_T^p L^q$) of the operator $T_t = T_\sigma$ with the symbol

$$\sigma(\xi, \eta) \stackrel{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta),$$

when m has a bounded support. So now, considering a smooth symbol m supported on $B(0, R)$, we split it (using a smooth partition of the unity) as

$$m = \sum_{k,l} m_{k,l}$$

with $m_{k,l}$ smooth symbols supported on $[k-1, k+1] \times [l-1, l+1]$. Applying the previous results (invariant by modulation), we get

$$\|T_\sigma\|_{\mathscr{B}} \le \sum_{k,l} \|T_{\sigma_{k,l}}\|_{\mathscr{B}} \le c_{\mathscr{B}} \sum_{k,l} \|\pi_k f\|_{L^2} \|\pi_l g\|_{L^2},$$

where $c_{\mathscr{B}}$ is the constant previously obtained for compactly supported symbols and $\pi_k f$ is a smooth truncation of f for frequencies around $[k-1, k+1]$. Using orthogonality, it follows that

$$\|T_\sigma\|_{\mathscr{B}} \le c_{\mathscr{B}} \left(\sum_{k,l} 1 \right)^{1/2} \|f\|_{L^2} \|g\|_{L^2},$$

which gives the desired results, since $k, l \in \{-R-1, \dots, R+1\}$. □

Proof of Theorem 3.1: the general case. Using no properties on Γ or a, b, c, we can get the following general bound.

Lemma 3.5. *Assume that m is compactly supported. For all $q \in [2, \infty]$, the solution u to (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. The solution $u(t)$ is given by

$$u(t) = T_t(f, g)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)\, d\xi\, d\eta = T_\sigma(f, g)(x),$$

with symbol

$$\sigma(\xi, \eta) \stackrel{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta).$$

In this general setting, we only know that σ is bounded by t and compactly supported. Lemma A.1 implies that

$$\|T_t(f, g)\|_{L^q} \lesssim t \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

The next several results improve on this bound under two different kinds of assumptions:

- using geometric properties of the resonance set Γ, or
- assuming linear Strichartz inequalities for the unitary groups $e^{ita(D)}$, $e^{itb(D)}$, and $e^{itc(D)}$, and using the structure of the product of two linear solutions.

Proof of Theorems 3.1 and 3.2: the case without resonances. We assume here that the phase function ϕ does not vanish.

Proposition 3.6. *Assume that $\Gamma = \varnothing$ and that m is compactly supported. For $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{3-2}$$

Proof. The solution $u(t)$ is given by

$$T_t(f, g)(x) = \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta = T_\sigma(f, g)(x),$$

with symbol

$$\sigma(\xi, \eta) \stackrel{\text{def}}{=} e^{ita(\xi+\eta)} \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} m(\xi, \eta).$$

Since a is real-valued and ϕ is nonvanishing, σ is bounded by a constant and compactly supported. Lemma A.1 yields

$$\|T_t(f, g)\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

Let us now deal with the improved bounds of Theorem 3.2 (using dispersive and Strichartz estimates on the linear evolution groups).

Proof of Theorem 3.2. Let us check the first claim. For every $s \in (0,t)$, we use the dispersive inequality (1-3) and the $L^2 \times L^2 \to L^1$ boundedness of T_m to get

$$\|e^{ita(D)} T_{me^{is\phi}}(f,g)\|_{L^\infty} = \|e^{i(t-s)a(D)} T_m(e^{isb(D)} f, e^{isc(D)} g)\|_{L^\infty} \lesssim \frac{1}{\sqrt{t-s}} \|f\|_{L^2} \|g\|_{L^2}.$$

Integrating for $s \in (0,t)$, it follows that

$$\|T_t(f,g)\|_{L^\infty} \lesssim t^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

Similarly, using the $L^2 \times L^\infty \to L^2$ boundedness of T_m, we have for all $s > 0$

$$\|e^{ita(D)} T_{me^{is\phi}}(f,g)\|_{L^2} = \|T_m(e^{isb(D)} f, e^{isc(D)} g)\|_{L^2} \lesssim \|f\|_{L^2} \|e^{isc(D)} g\|_{L^\infty},$$

which yields (using the Strichartz inequality)

$$\|T_t(f,g)\|_{L^2} \lesssim t^{3/4} \|f\|_{L^2} \|g\|_{L^2}.$$

The proof is concluded by interpolating between L^2 and L^∞.

Next, assume that $\Gamma = \varnothing$, which means that ϕ is nonvanishing on the support of m. Computing the integration over $s \in [0,t]$, we can split

$$iT_t(f,g)(x) = I_t(f,g) - II_t(f,g),$$

with

$$I_t(f,g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{it(b(\xi)+c(\eta))} \frac{1}{\phi(\xi,\eta)} m(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta$$

and

$$II_t(f,g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} e^{ita(\xi+\eta)} \frac{1}{\phi(\xi,\eta)} m(\xi,\eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$

In other words,

$$I_t(f,g) = T_{m/\phi}(e^{itb(D)} f, e^{itc(D)} g) \quad \text{and} \quad II_t = e^{ita(D)} T_{m/\phi}(f,g).$$

Since ϕ is assumed to be smooth and nonvanishing, m/ϕ is also smooth and compactly supported so that the bilinear operator $T_{m/\phi}$ is bounded from $L^P \times L^Q$ into L^R as soon as $1/P + 1/Q \geq 1/R$.

Choose now p and q as in the statement of the theorem. Using the dispersive estimates and Bernstein's inequality (indeed, since m has a compact support, it is possible to assume that $\hat{f}$ and $\hat{g}$ are compactly supported) gives

$$\|I_t(f,g)\|_{L^p L^q} \lesssim \|T_{m/\phi}(f,g)\|_{L^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}.$$

Therefore, $e^{itb(D)} f$ enjoys the usual Strichartz estimates, as well as, by Bernstein's inequality, the bounds $\|e^{itb(D)} f\|_{L^{2q}} \lesssim \|f\|_{L^q}$; the case of g is similar. This gives

$$\|II_t(f,g)\|_{L^p L^q} \lesssim \|e^{itb(D)} f\|_{L^{2p} L^q} \|e^{itc(D)} g\|_{L^{2p} L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \qquad \square$$

The case with resonance at only one point.

Proposition 3.7. *Assume that ϕ only vanishes at the point (ξ_0, η_0). Assume further that $\nabla\phi$ also vanishes at (ξ_0, η_0), but that* Hess ϕ *has a definite sign at that point. If $q \in [2, \infty]$, the solution u of (1-3) satisfies*

$$\|u(t)\|_{L^q} \lesssim t^{1/2+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. Assume for simplicity that ϕ vanishes at order 2 at 0. Take a smooth, compactly supported function χ, equal to 1 on $B(0, 1)$, and set $\psi = \chi(\cdot/2) - \chi$, so that

$$1 = \chi + \sum_{j\geq 1} \psi(2^{-j}\cdot).$$

Then decompose the symbol as

$$e^{ita(\xi)} m \frac{e^{it\phi}-1}{\phi} = \left(\chi(\sqrt{t}(\xi, \eta)) + \sum_{j\geq 1} \psi(2^{-j}\sqrt{t}(\xi, \eta))\right) e^{ita(\xi)} m \frac{e^{it\phi}-1}{\phi} \overset{\text{def}}{=} m_0(\xi, \eta) + \sum_{j\geq 1} m_j(\xi, \eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j\geq 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_j ($j \geq 0$) is supported on a ball of radius $\sim 2^j/\sqrt{t}$, and bounded by $2^{-2j} t$. It follows by Lemma A.1 that

$$\|T_{m_0}\|_{L^2\times L^2\to L^q} \lesssim t^{1/2+1/2q} \quad \text{and} \quad \|T_{m_j}\|_{L^2\times L^2\to L^q} \lesssim t^{1/2+1/(2q)} 2^{j(-1-1/q)}.$$

Therefore,

$$\|T_t\|_{L^2\times L^2\to L^q} \lesssim t^{1/2+1/(2q)} \left(1 + \sum_{j\geq 1} 2^{j(-1-1/q)}\right) \lesssim t^{1/2+1/(2q)},$$

which is the desired result. □

The case of resonances along a curve.

Proposition 3.8. *Assume that Γ is a smooth curve, where ϕ vanishes at order* 1. *If $q \in [2, \infty]$, the solution u of (1-3) satisfies the following.*

- *If Γ is noncharacteristic,*

$$\|u(t)\|_{L^q} \lesssim \|f\|_{L^2} \|g\|_{L^2} \begin{cases} t^{1/q} & \textit{if } 2 \leq q < \infty, \\ \langle \log t \rangle & \textit{if } q = \infty. \end{cases}$$

- *If Γ has nonvanishing curvature,*

$$\|u(t)\|_{L^q} \lesssim t^{1/4+1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$

- *Else,*

$$\|u(t)\|_{L^q} \lesssim t^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

As explained in Remark A.4, the estimate for a noncharacteristic curve Γ still holds if the only characteristic points are characteristic along the variable $\xi + \eta$.

Proof. We only treat the case where Γ is noncharacteristic and $2 \le q < \infty$; the other cases can be obtained by a similar argument. Similarly to Proposition 3.7, consider a smooth, compactly supported function χ, equal to 1 on [0, 1], and set $\psi = \chi(\cdot/2) - \chi$, so that

$$1 = \chi + \sum_{j \ge 1} \psi(2^{-j} \cdot).$$

We denote the distance function by $d_\Gamma(\xi,\eta) = d((\xi,\eta),\Gamma)$; since Γ is supposed to be a smooth curve and $\nabla\phi$ is nonvanishing near Γ, it follows that

$$d_\Gamma(\xi,\eta) \simeq |\phi(\xi,\eta)|.$$

Then decompose the symbol as

$$e^{ita(\xi)} m(\xi,\eta) \frac{e^{it\phi(\xi,\eta)}-1}{\phi(\xi,\eta)} = \left(\chi\,(t\phi(\xi,\eta)) + \sum_{j\ge 1} \psi(2^{-j} t\phi(\xi,\eta)) \right) e^{ita(\xi)} m(\xi,\eta) \frac{e^{it\phi(\xi,\eta)}-1}{\phi(\xi,\eta)}$$

$$\stackrel{\text{def}}{=} m_0(\xi,\eta) + \sum_{j\ge 1} m_j(\xi,\eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j\ge 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_0 $(j \ge 0)$ is supported on a neighborhood $\Gamma_{2^j/t}$ and bounded by $t 2^{-j}$, up to a numerical constant. If Γ is noncharacteristic, it follows by Lemma A.3 that $\|T_{m_0}\|_{L^2\times L^2\to L^q} \lesssim t^{1/q}$ and $\|T_{m_j}\|_{L^2\times L^2\to L^q} \lesssim t^{1/q} 2^{-j/q}$. Therefore,

$$\|T_t\|_{L^2\times L^2\to L^q} \lesssim t^{1/q}\left(1 + \sum_{j\ge 1} 2^{-j/q}\right) \lesssim t^{1/q},$$

which is the desired result. □

4. Localized data

We will now assume that the data belongs to a weighted Sobolev space, and study the decay of the solution of (1-3).

The role of time resonances.

Proposition 4.1. *Recall that m is smooth and compactly supported. Assume that ϕ only vanishes at (ξ_0,η_0), that $\nabla\phi$ also vanishes at that point, and that* Hess ϕ *at that point has a definite sign. If $q \in [2,\infty]$, the solution u of* (1-3) *satisfies the following.*

- *If $0 \le s < 1/2$, $\|u(t)\|_{L^q} \lesssim t^{1/2+1/(2q)-s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*
- *If $s > 1/2$ and $q < \infty$, $\|u(t)\|_{L^q} \lesssim t^{-1/(2q)} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*
- *If $s > 1/2$ and $q = \infty$, $\|u(t)\|_{L^\infty} \lesssim \langle \log t\rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*

Proof. As in the proof of Proposition 3.7, we decompose the symbol, giving the decomposition

$$u(t) = T_{m_0}(f, g) + \sum_{j \geq 1} T_{m_j}(f, g).$$

Again the symbol m_j is supported on a ball of radius $2^j t^{-1/2}$ and is bounded by $2^{-2j} t$. We conclude with Lemma A.1. □

Theorem 4.2. *Assume that ϕ vanishes at first order along a noncharacteristic curve Γ. Then for $2 \leq q < \infty$ and $s \geq 0$, the solution u of (1-3) satisfies the following estimates*:

- *If* $0 \leq s < 1/4$, $\|u(t)\|_{L^q} \lesssim t^{(1-4s)/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.
- *If* $s > 1/4$, $\|u(t)\|_{L^q} \lesssim \langle \log t \rangle \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.

If $q = \infty$, the solution u of (1-3) satisfies

$$\|u(t)\|_{L^\infty} \lesssim \langle \log t \rangle \|f\|_{L^2} \|g\|_{L^2}.$$

Remark 4.3. The L^∞ estimate of Proposition 3.8 does not improve if the data belong to weighted L^2 spaces. Also, notice that the L^2 estimate for $s > \frac{1}{4}$ is already as good as allowed by the lower bound Equation (2-9): any further assumption on space resonances will not improve the estimate.

Proof of Theorem 4.2. Just as in the proof of Proposition 3.8, split the symbol as

$$e^{ita(\xi)} m(\xi, \eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi, \eta)} = m_0(\xi, \eta) + \sum_{j \geq 1} m_j(\xi, \eta).$$

Obviously,

$$T_t = T_{m_0} + \sum_{j \geq 1} T_{m_j},$$

so it suffices to bound the summands above. The symbol m_j is supported on a neighborhood $\Gamma_{2^j/t}$ and bounded by $t 2^{-2j}$, up to a constant. Since Γ is noncharacteristic, it follows by Lemma A.5 that for $s < \frac{1}{4}$

$$\begin{aligned}
\|T_{m_j}\|_{L^{2,s} \times L^{2,s} \to L^q} &\lesssim (t^{-1} 2^j)^{1-1/q+4s/q} (t 2^{-2j}), \\
\|T_{m_0}\|_{L^{2,s} \times L^{2,s} \to L^q} &\lesssim t^{-1+1/q-4s/q} t,
\end{aligned}$$

with corresponding estimates if $s > \frac{1}{4}$. The proof of the proposition is concluded by summing the above bounds for the elementary operators T_{m_j}. □

Following the same reasoning and estimates as in [Bernicot and Germain 2012], it is possible to get similar results for a curve admitting characteristic points.

Theorem 4.4. *Assume that ϕ vanishes at first order along a curve Γ with nonvanishing curvature. Then, for $2 \leq q < \infty$, $s \geq 0$, and $\delta > 0$, the solution u of (1-3) satisfies the following estimates*:

- *If* $0 \leq s \leq 1/2$, $\|u(t)\|_{L^q} \lesssim t^{(1-2s)/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.
- *If* $s \geq 1/2$, $\|u(t)\|_{L^q} \lesssim \langle t \rangle^\delta \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.

If $q=\infty$, the solution u of (1-3) satisfies

$$\|u(t)\|_{L^\infty} \lesssim \langle \log t\rangle \|f\|_{L^2}\|g\|_{L^2}.$$

Proof. Use Lemma A.6 instead of Lemma A.5 and follow the proof of Theorem 4.2. □

The role of space resonances.

Theorem 4.5. *Assume that (H) holds and $\Delta=\varnothing$, or in other words that $(\partial_\xi-\partial_\eta)\phi$ never vanishes. Then the solution u of (1-3) satisfies the following bounds for any $\delta>0$.*

- *If $0\le s<1/q$, $\|u(t)\|_{L^q}\lesssim t^{1/(2q)+1/2-(3/2)s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*
- *If $1/q<s<1-1/q$, $\|u(t)\|_{L^q}\lesssim t^{1/2-s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*
- *If $s>1-1/q$, $\|u(t)\|_{L^q}\lesssim t^{1/q-1/2+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}}$.*

Proof. The proof proceeds by interpolating between the following L^2 and the L^∞ estimates. Indeed if $s<1/q$, then, for $\theta=2/q$, we have $L^q:=(L^2,L^\infty)_\theta$ and $L^{2,s}=(L^{2,qs/2},L^{2,0})_\theta$ with $qs/2\le 1/2$. We conclude similarly for the two other cases.

Recall that

$$\Phi(\xi,\eta)\stackrel{\text{def}}{=}\phi(\xi-\eta,\eta),$$

so that the hypothesis on ϕ translates into $\partial_\eta\Phi\neq 0$, and, in Fourier space, u reads

$$\hat{u}(t,\xi)=e^{ita(\xi)}\int_0^t\!\!\int e^{i\tau\Phi(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\,d\tau.$$

The L^2 estimate. We want to prove that, for every exponent $\delta>0$ (as small as we want),

$$\|u(t)\|_{L^2}\lesssim\begin{cases} t^{3/4-(3/2)s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } 0\le s\le \frac12,\\ \|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } s>\frac12.\end{cases}\tag{4-1}$$

The result for $s=0$ is given by Theorem 3.2. So let us study the case $s=\frac12$ so that (4-1) will follow by interpolation.

We first observe that the embedding $L^{2,1/2+\delta}\subset L^1$ and the dispersive estimates $L^1\to L^\infty$ give

$$\begin{aligned}\left\|\int e^{i\tau\phi(\xi-\eta,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\right\|_{L^2} &\lesssim \|T_m(e^{i\tau b(D)}f,e^{i\tau c(D)}g)\|_{L^2}\\ &\lesssim \|e^{i\tau b(D)}f\|_{L^\infty}\|g\|_{L^2}\lesssim \tau^{-1/2}\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^2}.\end{aligned}$$

Moreover, integrating by parts in η via the identity $is\partial_\eta\Phi e^{is\Phi}=\partial_\eta e^{is\Phi}$ gives

$$\begin{aligned}&\left\|\int e^{i\tau\Phi(\xi,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta\right\|_{L^2}\\ &\qquad\lesssim \tau^{-1}\left\|\int e^{i\tau\Phi(\xi,\eta)}\partial_\eta[\partial_\eta\Phi(\xi,\eta)^{-1}\hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)]\,d\eta\right\|_{L^2}\\ &\qquad\lesssim \tau^{-3/2}[\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^{2,1}}+\|f\|_{L^{2,1}}\|g\|_{L^{2,1/2+\delta}}],\end{aligned}\tag{4-2}$$

where we repeat the same arguments as previously.

So let us fix τ and consider the bilinear operator

$$U \stackrel{\text{def}}{=} (f,g) \to \int e^{i\tau\Phi(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta)m(\xi-\eta,\eta)\,d\eta. \tag{4-3}$$

We have obtained that

$$\|U\|_{L^{2,1/2+\delta}\times L^2\to L^2} + \|U\|_{L^2\times L^{2,1/2+\delta}\to L^2} \lesssim \tau^{-1/2} \tag{4-4}$$

and

$$\|U(f,g)\|_{L^2} \lesssim \tau^{-3/2}[\|f\|_{L^{2,1/2+\delta}}\|g\|_{L^{2,1}} + \|f\|_{L^{2,1}}\|g\|_{L^{2,1/2+\delta}}]. \tag{4-5}$$

We now explain how we can interpolate between these two estimates to obtain

$$\|U(f,g)\|_{L^2} \lesssim \tau^{-1+\delta}\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}}, \tag{4-6}$$

for any $\delta > 0$. We first consider the collection of dyadic intervals

$$I_0 \stackrel{\text{def}}{=} [-1,1] \qquad I_n \stackrel{\text{def}}{=} [-2^n, 2^{n-1}] \cup [2^{n-1}, 2^n] \quad \text{for } n \geq 1.$$

On each set I_n, the weight $\langle x\rangle$ is equivalent to 2^n, so for $n \leq m$, two integers, we know from (4-4) that

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-1/2}2^{n(1/2+\delta)}$$

and from (4-5) that

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-3/2}[2^{n(1/2+\delta)}2^m + 2^n 2^{m(1/2+\delta)}] \lesssim \tau^{-3/2}2^{n(1/2+\delta)}2^m.$$

Consequently, taking the geometric average with $\delta' > 2\delta$, we get

$$\|U\|_{L^2(I_n)\times L^2(I_m)\to L^2} \lesssim \tau^{-1+\delta'}2^{n(1/2+\delta)}2^{m(1/2-\delta')} \lesssim \tau^{-1+\delta}2^{(n+m)(1/2-\delta)}.$$

So we have

$$\begin{aligned}\|U(f,g)\|_{L^2} &\lesssim \tau^{-1+\delta'} \sum_{n,m\geq 0} 2^{(n+m)(1/2-\delta)}\|f\|_{L^2(I_n)}\|g\|_{L^2(I_m)} \\ &\lesssim \tau^{-1+\delta'}\Bigg(\sum_{n,m\geq 0} 2^{-(n+m)\delta}\Bigg)\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}} \\ &\lesssim \tau^{-1+\delta'}\|f\|_{L^{2,1/2}}\|g\|_{L^{2,1/2}}.\end{aligned}$$

Since δ, δ' can be chosen as small as we want with $\delta' > 2\delta > 0$, δ' can be chosen arbitrarily small, which concludes the proof of (4-6).

Finally from (4-6), we obtain (4-1) for $s = \frac{1}{2}$ by integrating in time for $\tau \in (0,t)$.

The L^∞ estimate. We want to prove that

$$\|u(t)\|_{L^\infty} \lesssim \begin{cases} t^{1/2-s+\delta}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } 0 \leq s \leq 1, \\ t^{-1/2}\|f\|_{L^{2,s}}\|g\|_{L^{2,s}} & \text{if } s > 1. \end{cases} \tag{4-7}$$

The case $s=0$ was stated in Theorem 3.2. Recall that, writing

$$u(t) \stackrel{\text{def}}{=} \int_0^t F(t,s)\,ds,$$

the $L^1 \to L^\infty$ dispersive estimate gives

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{\sqrt{t-s}} \|f\|_{L^2} \|g\|_{L^2}. \tag{4-8}$$

Next, integrating by parts via the formula $is\partial_\eta \Phi e^{is\Phi} = \partial_\eta e^{is\Phi}$ gives

$$\begin{aligned}
&\int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \hat{f}(\xi-\eta)\hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\
&\qquad = \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \frac{1}{is\partial_\eta \Phi(\xi,\eta)} \partial_\eta \hat{f}(\xi-\eta)\hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\
&\qquad\qquad + \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \frac{1}{is\partial_\eta \Phi(\xi,\eta)} \hat{f}(\xi-\eta)\partial_\eta \hat{g}(\eta) m(\xi-\eta,\eta)\,d\eta \\
&\qquad\qquad + \int e^{ita(\xi)} e^{is\Phi(\xi,\eta)} \partial_\eta \left(\frac{m(\xi-\eta,\eta)}{is\partial_\eta \Phi(\xi,\eta)} \right) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,
\end{aligned}$$

which becomes, in physical space,

$$F(t,s) = I + II + III, \tag{4-9}$$

with

$$\begin{aligned}
I &\stackrel{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} x f, e^{itc(D)} g), \\
II &\stackrel{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} f, e^{itc(D)} x g), \\
III &\stackrel{\text{def}}{=} \frac{1}{s} e^{i(t-s)a(D)} T_{\partial_\eta \frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} f, e^{itc(D)} g).
\end{aligned}$$

Using the $L^1 \to L^\infty$ dispersive estimate,

$$\|I\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \left\| T_{\frac{m}{i\partial_\eta \Phi}} (e^{itb(D)} x f,\ e^{itc(D)} g) \right\|_{L^1} \lesssim \frac{1}{s\sqrt{t-s}} \|xf\|_{L^2} \|g\|_{L^2} \lesssim \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

Similar estimates for II and III give

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}. \tag{4-10}$$

Repeating the argument, but integrating by parts twice via the identity $\dfrac{1}{is\partial_\eta \Phi} \partial_\eta e^{is\Phi} = e^{is\Phi}$, yields

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s^2\sqrt{t-s}} \|f\|_{L^{2,2}} \|g\|_{L^{2,2}}. \tag{4-11}$$

Finally, interpolating between (4-8), (4-10), and (4-11) gives

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s^\sigma \sqrt{t-s}} \|f\|_{L^{2,\sigma}} \|g\|_{L^{2,\sigma}} \quad \text{for } 0 \le \sigma \le 2.$$

Integrating this inequality in s (recall that $u(t) = \int_0^t F(t,s)\,ds$) gives the desired estimate. □

The role of space-time resonances. We want to consider here the case of a point which would be resonant both in space and in time; we need to combine the two approaches previously presented.

Theorem 4.6. *Assume as usual that m is smooth and compactly supported and that (H) holds. Assume further that the point*

$$p_0 \overset{\text{def}}{=} (\xi_0, \eta_0)$$

is the only point in the support of m that is resonant in space and time — in other words, the only point such that $\phi(p_0) = (\partial_\xi - \partial_\eta)\phi(p_0) = 0$. Moreover, assume that ϕ and $(\partial_\xi - \partial_\eta)\phi$ vanish at order one on their zero sets, and that the two smooth curves $\{\phi = 0\}$ and $\{(\partial_\xi - \partial_\eta)\phi = 0\}$ are non tangentially intersecting at p_0 with $\partial_\xi \phi(p_0) \neq 0$. Then the solution u of (1-3) satisfies the following bounds for $q \in [2, \infty]$ and every $\delta > 0$.

- *If $s \in \left[0, \frac{1}{2}\right]$,*
$$\|u(t)\|_{L^q} \lesssim t^{1/q - s(1/4 + 3/(2q)) + \delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}.$$
- *If $s \in \left(\frac{1}{2}, 1\right]$,*
$$\|u(t)\|_{L^q} \lesssim t^{-s(1/4 - 1/(2q)) + \delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}.$$

Remark 4.7.
- For $q = \infty$, the estimates follow from the ones with $q < \infty$ with the Bessel inequality (since δ can be as small as we want).
- The assumptions of the theorem imply that, if ϕ and $\nabla_\eta \phi$ vanish at order 1 on Γ and Δ, respectively, then, at the intersection point of Γ and Δ, Γ is characteristic along $\xi + \eta$. Fortunately, this turns out not be a problem in the estimates.
- The technical assumption $\partial_\xi \phi(p_0) \neq 0$ is exactly the same as that of Theorem 2.5: $\Phi_\xi(\xi_0, \eta_0) \neq 0$.
- In the previous results, for $s = 1$, we get that, for large $t \gg 1$,
$$\|u(t)\|_{L^q} \lesssim t^{1/(2q) - 1/4 + \delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$
for every $\delta > 0$. This estimate is optimal (up to δ which can be chosen as small as we want) due to the lower bound in (2-9).

Proof. The L^2 inequalities ($q = 2$) have already been proved in Theorems 4.2 and 4.4. Indeed, from Theorem 4.4 we know that $u(t)$ can be estimated in L^2 with a bound $t^{(1-s)/2+\delta}$ if $s \le \frac{1}{2}$ and t^δ for every $\delta > 0$ if $s \ge \frac{1}{2}$. Moreover, Theorem 4.2 yields that, for every $\delta > 0$,

$$\|u(t)\|_{L^\infty} \lesssim t^\delta \|f\|_{L^2} \|g\|_{L^2}.$$

So it suffices to check the only remaining extremal point, $q = \infty$ with $s = 1$. We now aim at proving that

$$\|u(t)\|_{BMO} \lesssim t^{-1/4+\delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}, \tag{4-12}$$

which implies the desired result by interpolation.

To prove (4-12), the main idea is to combine the two previous situations, so let us consider small parameters $\epsilon_1, \epsilon_2 \in (t^{-1/2}, 1)$ and a smooth partition of the unity with respect to the domains

$$\begin{aligned}
\Omega_1 &\stackrel{\text{def}}{=} \{(\xi,\eta),\ |\phi(\xi,\eta)| > \epsilon_1 + \tfrac{1}{2}|(\partial_\xi - \partial_\eta)\phi(\xi,\eta)|\},\\
\Omega_2 &\stackrel{\text{def}}{=} \{(\xi,\eta),\ |(\partial_\xi - \partial_\eta)\phi(\xi,\eta)| > \epsilon_2 + \tfrac{1}{2}|\phi(\xi,\eta)|\},\\
\Omega_3 &\stackrel{\text{def}}{=} \{(\xi,\eta),\ |\phi(\xi,\eta)| < 2\epsilon_1 \quad \text{and} \quad |(\partial_\xi - \partial_\eta)\phi(\xi,\eta)| < 2\epsilon_2\}.
\end{aligned}$$

More precisely, Ω_1 can be thought of as a truncated "cone" around the curve $|(\partial_\xi - \partial_\eta)\phi| = 0$ and of top p_0. Ω_2 can be thought of similarly, but around the other curve. This decomposition, from the smooth symbol m, gives rise to three symbols m_i, and we have

$$u(t) = u_1(t) + u_2(t) + u_3(t)$$

with

$$\hat{u_i}(t,\xi) := e^{ita(\xi)} \int_0^t \int e^{is\phi(\xi-\eta,\eta)} m_i(\xi-\eta,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta\, ds.$$

<u>Step 1</u>: estimate of u_1 in BMO with $s = 1$. We perform the same decomposition as was used in the proof of Theorem 3.2, so

$$u_1(t) = I_t(f,g) - II_t(f,g),$$

with

$$I_t(f,g) = T_{m_1/\phi}(e^{itb(D)}f, e^{itc(D)}g) \quad \text{and} \quad II_t = e^{ita(D)} T_{m_1/\phi}(f,g).$$

The symbol m_1 is of Coifman–Meyer type [Coifman and Meyer 1978] (up to a translation from p_0 to 0) and ϕ is smooth and lower-bounded by ϵ_1 so $T_{m_1/\phi}$ is bounded from $L^\infty \times L^\infty$ to a modulated BMO space [Meyer and Coifman 1991] with norm $\lesssim \epsilon_1^{-1}$. Using the dispersive inequalities for the linear evolution groups,

$$\|I_t(f,g)\|_{BMO} \lesssim \epsilon_1^{-1} \|e^{itb(D)}f\|_{L^\infty} \|e^{itc(D)}g\|_{L^\infty} \lesssim \epsilon_1^{-1} t^{-1} \|f\|_{L^1} \|g\|_{L^1} \lesssim \epsilon_1^{-1} t^{-1} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$

where we used $L^{2,1} \subset L^1$. Then we decompose the symbol m_1 around p_0 for scales 2^j from ϵ_1 to 1 as follows (here the scale means the distance in the frequency plane to the point p_0, which in Ω_1 is equivalent to $|\phi|$):

$$m_1 = \sum_{\epsilon_1 \le 2^j \lesssim 1} m_1 \chi(2^{-j}\phi),$$

where χ is a compactly supported and smooth function. The symbol $m_1\chi(2^{-j}\phi)/\phi$ is of Coifman–Meyer type (up to a translation) with a bound 2^{-j} so the operator $T_{m_1\chi(2^{-j}\phi)/\phi}$ is bounded from $L^2 \times L^2$ to L^1 with a bound 2^{-j}. Since when we evaluate $T_{m_1\chi(2^{-j}\phi)/\phi}(f,g)$, the functions f and g may be assumed supported in frequency on an interval of length 2^j, we deduce from Lemma A.2 that

$$\begin{aligned}
\|II_t(f,g)\|_{L^\infty} &\lesssim t^{-1/2} \|T_{m_1/\phi}(f,g)\|_{L^1}\\
&\lesssim t^{-1/2} \Bigg(\sum_{\epsilon_1 \le 2^j \lesssim 1} 2^j 2^{-j} \Bigg) \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim t^{-1/2} |\log \epsilon_1|\, \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.
\end{aligned}$$

So, since $\epsilon_1 \in [t^{-1/2}, 1]$, for every $\delta > 0$, we obtain

$$\|u_1(t)\|_{BMO} \lesssim t^{-1/2}\epsilon_1^{-\delta}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}. \tag{4-13}$$

<u>Step 2</u>: estimate of u_2 in L^∞ with $s = 1$. For u_2, we follow the proof of Theorem 4.5, with the symbol m_2 supported on a cone with $|(\xi,\eta) - p_0| \geq \epsilon_2$. In our current situation, the symbol m_2 satisfies the Hörmander regularity condition (which means $|\partial^\alpha m_2(\xi,\eta)| \lesssim |(\xi,\eta) - p_0|^{-|\alpha|}$) and is supported on Ω_2, which can be considered as a cone of top p_0. So Ω_2 can be split into different parts at distance 2^j from p_0 for $\epsilon_2 \leq 2^j \lesssim 1$:

$$m_2 = \sum_{\epsilon_2 \leq 2^j \lesssim 1} m_2 \chi(2^{-j}(\cdot - p_0)),$$

where χ is a smooth and compactly supported function. For each of these pieces, $\chi(2^{-j}(\cdot - p_0))$ restricts frequencies to a ball of radius $\sim 2^j$, so it is possible to add projections π_j on f and g, where π_j projects on intervals of length $\sim 2^j$ which we do not specify.

These considerations lead to the following modification of (4-10):

$$\|F(t,s)\|_{L^\infty} \lesssim \frac{1}{s\sqrt{t-s}} \sum_{\epsilon_2 \leq 2^j \lesssim 1} \big(I_j + II_j + III_j\big), \tag{4-14}$$

where

$$I_j \overset{\text{def}}{=} \Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|xf\|_{L^2}\|\pi_j g\|_{L^2},$$

$$II_j \overset{\text{def}}{=} \Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|\pi_j f\|_{L^2}\|x g\|_{L^2},$$

$$III_j \overset{\text{def}}{=} \Big\| T_{\partial_\eta \frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \|\pi_j f\|_{L^2}\|\pi_j g\|_{L^2}.$$

To bound I_j, observe that $2^j \dfrac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}$ is a Coifman–Meyer symbol; thus

$$\Big\| T_{\frac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}} \Big\|_{L^2\times L^2 \to L^1} \lesssim 2^j.$$

Furthermore, by Lemma A.2, $\|\pi_j g\|_{L^2} \lesssim 2^{j/2}\|g\|_{L^{2,1}}$. Therefore,

$$I_j \lesssim 2^{-j}\|f\|_{L^{2,1}}\|\pi_j g\|_{L^2} \lesssim 2^{-j/2}\|f\|_{L^{2,1}}\|\pi_j g\|_{L^{2,1}}.$$

Similarly,

$$II_j \lesssim 2^{-j}\|f\|_{L^2}\|g\|_{L^{2,1}} \lesssim 2^{-j/2}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}.$$

Finally, $2^{2j}\partial_\eta \dfrac{m_2\chi(2^{-j}(\cdot - p_0))}{\partial_\eta \Phi}$ is also a Coifman–Meyer symbol. Applying this and Lemma A.2 gives

$$III_j \lesssim 2^{-j}\|f\|_{L^{2,1}}\|g\|_{L^{2,1}}.$$

It follows that

$$\|F(t,s)\|_{L^\infty} \lesssim \Bigg(\sum_{\epsilon_2 \le 2^j \lesssim 1} 2^{-j/2} + 2^{-j}\Bigg) \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim \epsilon_2^{-1} \frac{1}{s\sqrt{t-s}} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}},$$

which means that in (4-10) we get a new extra factor ϵ_2^{-1}. Finally, applying similar arguments as for Theorem 4.5, we conclude that, for any $\delta > 0$, we have

$$\|u_2(t)\|_{L^\infty} \lesssim \epsilon_2^{-1-\delta} t^{-1/2+\delta} \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}. \tag{4-15}$$

<u>Step 3</u>: Estimate of u_3 in L^∞ with $s = 1$. For u_3, we know that the symbol m_3 is supported on a ball of radius $\epsilon := \max\{\epsilon_1, \epsilon_2\}$ around the space-time resonant point p_0.

We follow similar arguments as for Proposition 3.8, so we split the ball $B(p_0, \epsilon)$ into "strips" with scale ϕ from 0 to ϵ:

$$m_3 = \sum_{0 < 2^j \lesssim \epsilon} m_3 \chi(2^{-j}\phi),$$

which implies

$$u_3(t) = \sum_{0 < 2^k \lesssim \epsilon} T_{m_3^j}(f, g)$$

where $T_{m_3^j}$ is the bilinear Fourier multiplier associated to the symbol

$$m_3^j(\xi,\eta) = e^{ita(\xi+\eta)} m_3(\xi,\eta) \frac{e^{it\phi(\xi,\eta)} - 1}{\phi(\xi,\eta)} \chi(2^{-j}\phi(\xi,\eta)).$$

For each scale 2^j, the symbol m_3^j is bounded by $\max\{t, 2^{-j}\}$, so Lemmas A.3 and A.2 with Remark A.4 imply (the functions f, g may be supposed to be frequentially supported on an interval of length ϵ)

$$\|T_{m_3^j}(f,g)\|_{L^\infty} \lesssim \max\{t, 2^{-j}\} 2^j \|f\|_{L^2} \|g\|_{L^2} \lesssim \max\{t, 2^{-j}\} 2^j \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

By summing all these inequalities over the scale 2^j, we get

$$\begin{aligned} \|u_3(t)\|_{L^\infty} &\lesssim \Bigg(t \sum_{2^j \le t^{-1}} 2^j + \sum_{t^{-1} \le 2^j \le \epsilon} 1 \Bigg) \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \\ &\lesssim \langle \log(\epsilon t) \rangle \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \lesssim (\epsilon t)^\delta \epsilon \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}, \end{aligned} \tag{4-16}$$

for every $\delta > 0$, since $\epsilon t > 1$.

<u>Step 4</u>: End of the proof. Optimizing over ϵ_1 and ϵ_2 leads to

$$\epsilon_1 = \epsilon_2 = \epsilon_t := t^{-1/4+\delta}.$$

As required, we have $\epsilon_t \in [t^{-1/2}, 1]$. So by summing (4-13) and (4-16) with the estimate for u_2, we now have, for every small enough $\delta > 0$,

$$\|u(t)\|_{BMO} \lesssim [t^{-1/2}\epsilon_t^{-\delta} + (\epsilon t)^\delta \epsilon_t] \|f\|_{L^{2,1}} \|g\|_{L^{2,1}}.$$

Since $\epsilon_t \geq t^{-1/2}$, the main term in the previous inequality is the second one, so we deduce for every $\delta > 0$

$$\|u(t)\|_{BMO} \lesssim t^{-1/4+\delta},$$

which concludes the proof of (4-12). □

Appendix A: Multilinear estimates

Lemma A.1. *Suppose that the symbol $\sigma(\xi, \eta)$ is bounded (that is, $\|\sigma\|_{L^\infty} \lesssim 1$) and supported on a ball of radius ϵ, say $B(0, \epsilon)$. For $q \in [2, \infty]$ and $s < \frac{1}{2}$,*

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q+2s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$$

and

$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{2-1/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$$

if $s > \frac{1}{2}$.

Proof. Consider the first claim in the case $s = 0$. The lemma is obtained by interpolating between the endpoints $q = 2$ and $q = \infty$. If $q = 2$, it follows from an application of the Plancherel equality and the Cauchy–Schwarz inequality that

$$\begin{aligned} \|T_\sigma(f, g)\|_{L^2}^2 &= \int \left| \int \sigma(\xi - \eta, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta \right|^2 d\xi \\ &\leq \int \left(\int |\sigma(\xi - \eta, \eta)|^2 \, d\eta \right) \left(\int |\hat{f}(\xi - \eta) \hat{g}(\eta)|^2 d\eta \right) d\xi \\ &\lesssim \epsilon \|f\|_{L^2}^2 \|g\|_{L^2}^2. \end{aligned} \tag{A-1}$$

If $q = \infty$, use Cauchy–Schwarz again to get

$$\begin{aligned} \|T_\sigma(f, g)\|_{L^\infty} &= \left\| \iint e^{ix(\xi+\eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\eta \, d\xi \right\|_{L^\infty} \lesssim \iint_{B(0,\epsilon)} |\hat{f}(\xi) \hat{g}(\eta)| \, d\eta \, d\xi \\ &\lesssim \epsilon \left(\iint |\hat{f}(\xi) \hat{g}(\eta)|^2 \, d\eta \, d\xi \right)^{1/2} \lesssim \epsilon \|f\|_{L^2} \|g\|_{L^2}. \end{aligned} \tag{A-2}$$

Then, for $s > 0$, we use that the symbol is supported on a ball of radius ϵ, so f (respectively g) can be replaced with $\pi_I(f)$ (respectively $\pi_J(g)$), corresponding to the frequency-truncation of f on an interval I of length 2ϵ. We conclude by applying the previous reasoning with $\pi_I(f)$ and $\pi_J(g)$ and Lemma A.2. □

Lemma A.2. *Assume that I is an interval and consider π_I the Fourier multiplier, given by a smooth function supported on $2I$ and equal to 1 on I. For $q \in [2, \infty]$ and $s < \frac{1}{2}$,*

$$\|\pi_I(f)\|_{L^2} \lesssim |I|^s \|f\|_{L^{s,2}}.$$

Proof. The proof relies on the Sobolev embedding as follows:

$$\|\pi_I(f)\|_{L^2} \lesssim |I|^{1/2-1/\sigma} \|\hat{f}\|_{L^\sigma(2I)} \lesssim |I|^s \|\hat{f}\|_{L^\sigma} \lesssim |I|^s \|\hat{f}\|_{W^{s,2}} \lesssim |I|^s \|f\|_{L^{s,2}},$$

where the exponent σ is given by $1/\sigma = \frac{1}{2} - s$. □

Lemma A.3. *Consider a smooth curve Γ and a bounded symbol σ ($\|\sigma\|_\infty \lesssim 1$) supported on $\Gamma_\epsilon \cap B(0, M)$, for a positive constant M. Suppose $q \in [2, \infty]$.*

- *If the curve Γ is noncharacteristic,*
$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q} \|f\|_{L^2} \|g\|_{L^2}.$$
- *If the curve Γ has nonvanishing curvature,*
$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{3/4-1/(2q)} \|f\|_{L^2} \|g\|_{L^2}.$$
- *Otherwise,*
$$\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1/2} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. As for Lemma A.1, by interpolation it suffices to study the two extremal situations, $q = 2$ and $q = \infty$. First, for $q = 2$, employ the same reasoning as in Lemma A.1 (relying on the Plancherel equality). Since the support Γ_ϵ now has a measure bounded by ϵ (up to a constant), we get

$$\|T_\sigma(f, g)\|_{L^2} \lesssim \epsilon^{1/2} \|f\|_{L^2} \|g\|_{L^2}. \tag{A-3}$$

Let us point out that this estimate is the easiest situation (when the three exponents are equal to 2) described by Theorem 1.5 of [Bernicot and Germain 2012]. Moreover this estimate does not depend on geometric properties of the curve Γ.

Let us now study the case where $q = \infty$. If the curve Γ is noncharacteristic, then Proposition 6.2 of [Bernicot and Germain 2012] implies that

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim \epsilon \|f\|_{L^2} \|g\|_{L^2},$$

which, by interpolating with (A-3), proves the desired result. If the curve Γ has a nonvanishing curvature, the proposition just cited yields

$$\|T_\sigma(f, g)\|_{L^\infty} \lesssim \epsilon^{3/4} \|f\|_{L^2} \|g\|_{L^2},$$

and we similarly conclude by interpolation. □

Remark A.4. The estimate for a noncharacteristic curve Γ still holds if the curve admits some points that are characteristic only along the variable $\xi + \eta$, which means when the tangential vector of the curve at this point is parallel to $(-1, 1)$. Indeed the proof of Proposition 6.2 of [Bernicot and Germain 2012] only requires appropriate decompositions in the variables ξ and η for f and g and do not use specific properties on the third frequency variable $\xi + \eta$.

Lemma A.5. *Assume that Γ is a noncharacteristic curve. Consider a bounded symbol σ ($\|\sigma\|_\infty \lesssim 1$) supported on $\Gamma_\epsilon \cap B(0, M)$, for a positive constant M.*

- *If $0 \le s < 1/4$, $\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon^{1-1/q+4s/q} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*
- *If $s > 1/4$, $\|T_\sigma(f, g)\|_{L^q} \lesssim \epsilon \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}$.*

Proof. We follow the same steps as for the previous lemma. The $L^2 \times L^2$ to L^∞ estimate cannot be improved by replacing L^2 by $L^{2,s}$, so we simply focus on $L^2 \times L^2$ to L^2 estimates. Since the curve is assumed to be noncharacteristic, it follows that

$$|\langle T_\sigma(f,g),h\rangle| \lesssim \sum_i \epsilon^{1/2} \|\hat f\|_{L^2(I_i^1)} \|\hat g\|_{L^2(I_i^2)} \|\hat h\|_{L^2(I_i^3)}, \tag{A-4}$$

where the $(I_i^k)_i$ are collections of almost disjoint intervals of length ϵ for $k = 1, 2, 3$. As a consequence, from the Cauchy–Schwartz inequality it turns out

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2} (\sup_i \|\hat f\|_{L^2(I_i^1)}) \|g\|_{L^2}.$$

Using Sobolev embedding on the whole space $\mathbb{R}$, we get

$$\|\hat f\|_{L^2(I_i^1)} \lesssim \epsilon^{1/2-1/\sigma} \|\hat f\|_{L^\sigma(I_i^1)} \lesssim \epsilon^{2s} \|\hat f\|_{L^\sigma} \lesssim \epsilon^{2s} \|\hat f\|_{W^{2s,2}}$$

with the exponent σ given by $1/\sigma = \frac{1}{2} - 2s$ (we recall that $s \le \frac{1}{4}$). So finally we get

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2+2s} \|f\|_{L^{2,2s}} \|g\|_{L^2}.$$

By symmetry and then interpolation, we deduce

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon^{1/2+2s} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}. \qquad \square$$

Lemma A.6. *Assume that Γ has a nonvanishing curvature. Consider a bounded symbol σ ($\|\sigma\|_\infty \lesssim 1$) supported on $\Gamma_\epsilon \cap B(0, M)$, for a positive constant M. If $0 \le s \le \frac{1}{2}$, then, for every $\delta > 0$,*

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \begin{cases} \epsilon^{1/2+s+\delta} \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}, & \text{if } s < 1/2, \\ \epsilon\sqrt{|\log \epsilon|}\, \|f\|_{L^{2,s}} \|g\|_{L^{2,s}}, & \text{if } s > 1/2. \end{cases}$$

Proof. The case $s = 0$ is included in Lemma A.3, so by interpolation (with $L^{2,s} \subset L^1$ for $s > \frac{1}{2}$) it suffices to check that

$$\|T_\sigma(f,g)\|_{L^2} \lesssim \epsilon\sqrt{|\log \epsilon|}\, \|f\|_{L^1} \|g\|_{L^1}.$$

This estimate was already proved in [Bernicot and Germain 2012, Proposition 5.1]. For readability we quickly sketch the proof here. Assume that $(0, 0) \in \Gamma$ and let us work around this point. Then note that, for every L^2-function h,

$$\begin{aligned} \big|\langle T_\sigma(f,g),h\rangle\big| &= \left|\int \sigma(\xi,\eta) \hat f(\xi) \hat g(\eta) \hat h(\xi+\eta)\, d\xi\, d\eta\right| \lesssim \|\hat f\|_{L^\infty} \|\hat g\|_{L^\infty} \int \left|\sigma\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big)\right| |\hat h(u)|\, du\, dv \\ &\lesssim \|f\|_{L^1} \|g\|_{L^1} \int |\hat h(u)| \frac{\epsilon}{\sqrt{1+|u|}} \lesssim \|f\|_{L^1} \|g\|_{L^1} \|h\|_{L^2} \epsilon\sqrt{|\log \epsilon|}, \end{aligned}$$

where we have used (because of the nonvanishing curvature) that, uniformly with respect to λ_0,

$$\big|\{\xi - \eta,\ (\xi,\eta) \in \Gamma_\epsilon, \xi + \eta = \lambda_0\}\big| \lesssim \frac{\epsilon}{\sqrt{\epsilon + |\lambda_0|}}. \qquad \square$$

Appendix B: One-dimensional oscillatory integrals

Before stating the main proposition, we need to define the functions[1]

$$\mathcal{G}_1(x) \overset{\text{def}}{=} \int_x^\infty e^{i\sigma^2}\,d\sigma \quad \text{and} \quad \mathcal{G}_2(x) \overset{\text{def}}{=} \int_x^\infty e^{i\sigma^2}\frac{d\sigma}{\sqrt{\sigma - x}}.$$

Their qualitative behavior is given by the following lemma.

Lemma B.1. (i) *$\mathcal{G}_1$ is a smooth function such that*

$$\begin{cases} \mathcal{G}_1(x) = -e^{ix^2}/(2ix) + O(1/x^2) & \text{as } x \to \infty, \\ \mathcal{G}_1(x) = C_0 + O(1/x) & \text{as } x \to -\infty, \end{cases}$$

where C_0 is the constant $C_0 = \int_{-\infty}^{\infty} e^{i\sigma^2}\,d\sigma$.

(ii) *$\mathcal{G}_2$ is a smooth function such that*

$$\mathcal{G}_2(x) = \begin{cases} C_+ e^{ix^2}\sqrt{2/x} + O(1/|x|^{5/6}) & \text{as } x \to \infty, \\ C_- e^{ix^2}\sqrt{2/|x|} + \sqrt{\pi}\,e^{i\pi/4}e^{-ix^2}(1/\sqrt{|x|}) + O(1/|x|^{5/7}) & \text{as } x \to -\infty, \end{cases}$$

where $C_\pm = \int_0^\infty e^{\pm i\sigma^2}\,d\sigma$.

We now state the main result. Recall that $C_0 = \int_{-\infty}^{+\infty} e^{i\sigma^2}\,d\sigma$.

Proposition B.2. *Let χ be a smooth, compactly supported function, and let ζ be a smooth function.*

(i) *If $\zeta'' \geq c > 0$ and $\zeta'(\sigma_0) = 0$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\,d\sigma = \chi(\sigma_0)\sqrt{\frac{2}{\zeta''(\sigma_0)}}\frac{1}{\sqrt{t}}\mathcal{G}_1(\sqrt{t}\sigma_0) + O_c\Big(\frac{1}{t}\Big).$$

(ii) *If $|\zeta'| \geq c > 0$ does not vanish,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = \frac{\chi(0)}{\sqrt{\zeta'(0)}}e^{it\zeta(0)}\frac{C_0}{\sqrt{t}} + O_c\Big(\frac{1}{t}\Big).$$

(iii) *If $|\zeta''| \geq c > 0$, $\zeta'(\sigma_0) = 0$ with $\sigma_0 \geq c$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = \frac{\chi(0)}{\sqrt{\zeta'(0)}}e^{it\zeta(0)}\frac{C_0}{\sqrt{t}} + \sqrt{2\pi}\,e^{it\zeta(\sigma_0)}e^{i\,\mathrm{sign}(\zeta''(\sigma_0))\frac{\pi}{4}}\frac{\chi(\sigma_0)}{\sqrt{\sigma_0\zeta''(\sigma_0)}}\frac{1}{\sqrt{t}} + O_c\Big(\frac{1}{t}\Big).$$

(iv) *If $\zeta'' \geq c > 0$ and $\zeta'(\sigma_0) = 0$,*

$$\int_0^\infty e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} = C(\chi,\zeta)\mathcal{G}_2(\sqrt{t}\sigma_0) + \begin{cases} O_c(t^{-3/4}) & \text{if } |\sqrt{t}\sigma_0| < 1, \\ O_c(\sqrt{\sigma_0/t}) & \text{if } |\sqrt{t}\sigma_0| < 1, \end{cases}$$

where $C(\chi,\zeta)$ is a function of χ and ζ (and hence also of σ_0) which we do not make explicit here.

[1]The function $\mathcal{G}_1$ can obviously be obtained from the Fresnel integrals $S(x) = \int_0^x \sin t^2\,dt$ and $C(x) = \int_0^x \cos t^2\,dt$. In particular, the constants C_0 and $C_\pm$ appearing below can be computed via Fresnel integrals to yield $C_0 = (1+i)\sqrt{\pi/2}$ and $C_\pm = ((1 \pm i)/2)\sqrt{\pi/2}$. See [Abramowitz and Stegun 1964].

Remark B.3. Statements (ii) and (iii) on the one hand, and (iv) on the other, are complementary: (ii) and (iii) apply when ζ' vanishes away from zero, or not at all, whereas (iv) is meaningful if the point of vanishing of ζ' approaches zero.

Proof of Lemma B.1. Assertion (i) is proved by a simple integration by parts, so we skip it and focus on (ii). After the change of variable of integration to $\tau = \sqrt{\sigma - x}$, $\mathscr{G}_2$ becomes

$$\mathscr{G}_2(x) = 2e^{ix^2} \int_0^\infty e^{i\tau^2(\tau^2+2x)} d\tau \stackrel{\text{def}}{=} 2e^{ix^2} g(x).$$

The case $x \to \infty$. Split

$$g(x) = \int_0^R + \int_R^\infty \dots d\tau \stackrel{\text{def}}{=} I + II.$$

Start with

$$I = \int_0^R e^{i2x\tau^2} d\tau + \int_0^R [e^{i\tau^2(\tau^2+2x)} - e^{i2x\tau^2}] d\tau \stackrel{\text{def}}{=} I_1 + I_2.$$

The term I_1 can be written

$$I_1 = \int_0^\infty e^{i2x\tau^2} d\tau - \int_R^\infty e^{i2x\tau^2} d\tau = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O\left(\frac{1}{xR}\right),$$

where the inequality $\int_R^\infty e^{i2x\tau^2} d\tau = O(1/(\alpha R))$ follows by integration by parts.

As for I_2, estimate it brutally by

$$|I_2| \lesssim \int_0^R \tau^4 d\tau = O(R^5).$$

Finally, an integration by parts gives

$$II = \int_0^\infty e^{i\tau^2(\tau^2+2\tau x)} d\tau \lesssim \frac{1}{R^2 x}.$$

Gathering the above gives

$$g(x) = \sqrt{\frac{1}{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O(R^5) + O\left(\frac{1}{xR^2}\right);$$

finally, optimizing over R gives

$$g(x) = \sqrt{\frac{1}{2x}} \int_0^\infty e^{i\sigma^2} d\sigma + O\left(\frac{1}{x^{5/7}}\right),$$

which is the desired result.

The case $x \to -\infty$. Split

$$g(x) = \int_0^{\sqrt{-x/2}} + \int_{\sqrt{-x/2}}^\infty \dots d\tau \stackrel{\text{def}}{=} III + IV.$$

Start with III. Similarly to g in the case $x \to \infty$, we use the split

$$III = \int_0^R + \int_R^{\sqrt{-x/2}} \dots d\tau = III_1 + III_2,$$

and estimate

$$III_1 = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2}\, d\sigma + O\left(R^5 + \frac{1}{|x|R}\right) \quad \text{and} \quad III_2 = O\left(\frac{1}{R^2 x}\right).$$

Optimizing over R gives

$$III = \frac{1}{\sqrt{2x}} \int_0^\infty e^{i\sigma^2}\, d\sigma + O\left(\frac{1}{|x|^{5/7}}\right).$$

Turning now to IV, observe that the change of variable $\rho = -\tau^2/x$ gives

$$IV = \sqrt{-x} \int_{1/2}^\infty e^{ix^2\rho(\rho-2)} \frac{d\rho}{2\sqrt{\rho}} = \frac{\sqrt{\pi}}{2} e^{i\pi/4} e^{-ix^2} \frac{1}{\sqrt{|x|}} + O\left(\frac{1}{|x|}\right),$$

where the last equality follows by the stationary phase lemma. Putting together our estimates on III and IV gives the desired result. □

An intermediate result. The following proposition essentially corresponds to Proposition B.2, where ζ is replaced by either σ or $\sigma - \epsilon$ (in which case $\sigma_0 = \epsilon$).

Proposition B.4. *Let χ be a smooth function.*

(i) $\displaystyle\int_\epsilon^\infty e^{it\sigma^2} \chi(\sigma)\, d\sigma = \frac{\chi(0)}{\sqrt{t}} \mathscr{G}_1(\sqrt{t}\epsilon) + O\left(\frac{1}{t}\right).$

(ii) $\displaystyle\int_0^\infty e^{it\sigma} \chi(\sigma) \frac{d\sigma}{\sqrt{\sigma}} = \frac{C_0}{\sqrt{t}} \chi(0) + O\left(\frac{1}{t}\right)$ *(recall that $C_0 = \int_{-\infty}^{+\infty} e^{i\sigma^2}\, d\sigma$).*

(iii) $\displaystyle\int_\epsilon^\infty e^{it\sigma^2} \frac{1}{\sqrt{\sigma - \epsilon}} \chi(\sigma)\, d\sigma = \frac{\chi(0)}{t^{1/4}} \mathscr{G}_2(\sqrt{t}\epsilon) + \begin{cases} O(t^{-3/4}) & \text{if } |\sqrt{t}\epsilon| < 1 \\ O(\sqrt{\epsilon/t}) & \text{if } |\sqrt{t}\epsilon| > 1 \end{cases}$

Proof. We prove only (iii), since (i) and (ii) are simpler and can be proved using a similar procedure.

<u>First reduction for (iii).</u> The change of variable $\tau = \sqrt{t}\sigma$ gives

$$\int_\epsilon^\infty e^{it\sigma^2} \frac{1}{\sqrt{\sigma - \epsilon}} \chi(\sigma)\, d\sigma = t^{-1/4} \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \frac{1}{\sqrt{\tau - \sqrt{t}\epsilon}} \chi\left(\frac{\tau}{\sqrt{t}}\right) d\tau.$$

Thus the proposition is proved if we show that

$$\int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \frac{1}{\sqrt{\tau - \sqrt{t}\epsilon}} \left[\chi\left(\frac{\tau}{\sqrt{t}}\right) - \chi(0)\right] d\tau = \begin{cases} O(t^{-1/2}) & \text{if } |\sqrt{t}\epsilon| < 1, \\ O(\sqrt{\epsilon} t^{-1/4}) & \text{if } |\sqrt{t}\epsilon| > 1. \end{cases} \tag{B-1}$$

Define β a smooth, compactly supported function, equal to 1 on the support of χ. We can write

$$\text{(B-1)} = \chi(0) \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \left[\beta\left(\frac{\tau}{\sqrt{t}}\right) - 1\right] \frac{d\tau}{\sqrt{\tau - \sqrt{t}\epsilon}} + \int_{\sqrt{t}\epsilon}^\infty e^{i\tau^2} \beta\left(\frac{\tau}{\sqrt{t}}\right) \left[\chi\left(\frac{\tau}{\sqrt{t}}\right) - \chi(0)\right] \frac{d\tau}{\sqrt{\tau - \sqrt{t}\epsilon}}.$$

Since the first summand is easier to deal with, we focus on the second. Setting

$$Z(y) \overset{\text{def}}{=} \beta(y)[\chi(y) - \chi(0)],$$

reduces the question to proving that

$$\int_{\sqrt{t}\epsilon}^{\infty} e^{i\tau^2} Z\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \lesssim \begin{cases} t^{-1/2} & \text{if } |\sqrt{t}\epsilon|<1,\\ \sqrt{\epsilon}t^{-1/4} & \text{if } |\sqrt{t}\epsilon|>1,\end{cases} \tag{B-2}$$

where Z is a smooth function vanishing at 0.

Proof of (B-2). Split the left-hand side of (B-2) as

$$\int_{\sqrt{t}\epsilon}^{\sqrt{t}\epsilon+R} + \int_{\sqrt{t}\epsilon+R}^{\infty} \dots\, d\tau \overset{\text{def}}{=} I + II.$$

The term I is estimated directly, giving

$$I \le \int_{\sqrt{t}\epsilon}^{\sqrt{t}\epsilon+R}\left|Z\left(\frac{\tau}{\sqrt{t}}\right)\right|\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} \lesssim \begin{cases}\epsilon\sqrt{R} & \text{if } R<\sqrt{t}|\epsilon|,\\ t^{-1/2}R^{3/2} & \text{if } R>\sqrt{t}|\epsilon|.\end{cases}$$

The term II is submitted first to an integration by parts using the identity $\dfrac{1}{2\tau}\partial_\tau e^{i\tau^2}=e^{i\tau^2}$:

$$\begin{aligned} II &= \int_{\sqrt{t}\epsilon+R}^{\infty}\frac{1}{2\tau}\partial_\tau e^{i\tau^2} Z\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} = \frac{1}{\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty}\tfrac{1}{2}\partial_\tau e^{i\tau^2}\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}}\\ &= -\frac{1}{2\sqrt{t}\sqrt{R}}e^{i(\sqrt{t}\epsilon+R)^2}\widetilde{Z}\left(\frac{\sqrt{t}\epsilon+R}{\sqrt{t}}\right) - \frac{1}{2t}\int_{\sqrt{t}\epsilon+R}^{\infty} e^{i\tau^2}\widetilde{Z}'\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}}\\ &\qquad\qquad + \frac{1}{4\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty} e^{i\tau^2}\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\frac{d\tau}{(\tau-\sqrt{t}\epsilon)^{3/2}},\end{aligned}$$

where we set $\widetilde{Z}(y)\overset{\text{def}}{=}Z(y)/y$. The term II is then estimated directly:

$$\begin{aligned} II &\lesssim \frac{1}{2\sqrt{t}\sqrt{R}}\left|\widetilde{Z}\left(\frac{\sqrt{t}\epsilon+R}{\sqrt{t}}\right)\right| + \frac{1}{t}\int_{\sqrt{t}\epsilon+R}^{\infty}\left|\widetilde{Z}'\left(\frac{\tau}{\sqrt{t}}\right)\right|\frac{d\tau}{\sqrt{\tau-\sqrt{t}\epsilon}} + \frac{1}{\sqrt{t}}\int_{\sqrt{t}\epsilon+R}^{\infty}\left|\widetilde{Z}\left(\frac{\tau}{\sqrt{t}}\right)\right|\frac{d\tau}{(\tau-\sqrt{t}\epsilon)^{3/2}}\\ &\lesssim t^{-1/2}R^{-1/2}.\end{aligned}$$

Summing up, we have

$$I+II \lesssim \begin{cases}\epsilon\sqrt{R}+t^{-1/2}R^{-1/2} & \text{if } R<\sqrt{t}|\epsilon|,\\ t^{-1/2}R^{3/2}+t^{-1/2}R^{-1/2} & \text{if } R>\sqrt{t}|\epsilon|.\end{cases}$$

Optimizing over R (distinguishing between the cases $\sqrt{t}|\epsilon|>1$ and $\sqrt{t}|\epsilon|<1$) gives (B-2). □

Proof of Proposition B.2. We only prove (iv); the proofs of (i) and (ii) closely follow that of (iv), and (iii) simply requires an additional application of the stationary phase lemma. The idea is simply to perform a change of variable which reduces matters to Proposition B.4. We want to estimate

$$\int_0^{\infty} e^{it\zeta(\sigma)}\chi(\sigma)\frac{d\sigma}{\sqrt{\sigma}} \tag{B-3}$$

where $\zeta''\ge c>0$ and $\zeta'(\sigma_0)=0$. Now set

$$y=\Phi(\sigma)\overset{\text{def}}{=}\operatorname{sign}(\sigma-\sigma_0)\sqrt{\zeta(\sigma)}.$$

Notice that Φ is smooth, and that

$$\Phi^{-1}(0)=\sigma_0, \quad \Phi'(\sigma_0)=\sqrt{\frac{\zeta''(\sigma_0)}{2}}, \quad (\Phi^{-1})'(0)=\sqrt{\frac{2}{\zeta''(\sigma_0)}}.$$

Furthermore,

$$(\Phi^{-1})'(\Phi(0))=\operatorname{sign}(\Phi(0)-\sigma_0)\frac{2\sqrt{\zeta}(\Phi(0))}{\zeta'(\Phi(0))}\stackrel{\text{def}}{=}C(\zeta)^2,$$

which implies that $\sqrt{\Phi^{-1}(y)}$ can be written

$$\sqrt{\Phi^{-1}(y)}=C(\zeta)\sqrt{y}\gamma(y)$$

for some smooth, positive function γ. Performing the change of variable $y=\Phi(\sigma)$ gives

$$\text{(B-3)}=\int_{\Phi(0)}^{\infty}e^{ity^2}\chi\circ\Phi^{-1}(y)(\Phi^{-1})'(y)C(\zeta)^{-1}\frac{dy}{\sqrt{y}\sqrt{\gamma(y)}}.$$

Applying Proposition B.4 gives the desired result:

$$\text{(B-3)}=\chi(\sigma_0)\sqrt{\frac{2}{\zeta''(\sigma_0)}}\frac{1}{C(\zeta)}\frac{1}{\gamma(0)}\frac{1}{t^{1/4}}\mathscr{G}_2(\sqrt{t}\epsilon)+\begin{cases}O(t^{-3/4}) & \text{if } |\sqrt{t}\epsilon|<1,\\ O(\sqrt{\epsilon/t}) & \text{if } |\sqrt{t}\epsilon|>1.\end{cases} \qquad \square$$

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