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A CODIMENSION-TWO STABLE MANIFOLD OF NEAR SOLITON EQUIVARIANT WAVE MAPS

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We consider finite-energy equivariant solutions for the wave map problem from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ which are close to the soliton family. We prove asymptotic orbital stability for a codimension-two class of initial data which is small with respect to a stronger topology than the energy.

1. Introduction

We consider wave maps $U : \mathbb{R}^{2+1} \to \mathbb{S}^2$ which are equivariant with corotation index 1. In particular, they satisfy $U(t, \omega x) = \omega U(t, x)$ for $\omega \in SO(2, \mathbb{R})$, where the latter group acts in standard fashion on $\mathbb{R}^2$, and the action on $\mathbb{S}^2$ is induced from that on $\mathbb{R}^2$ via stereographic projection. Wave maps are characterized by being critical with respect to the functional

$$U \to \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha U, \partial^\alpha U \rangle \, d\sigma, \quad \alpha = 0, 1, 2,$$

where Einstein's summation convention is in force, $\partial^\alpha = m^{\alpha\beta}\partial_\beta$, $m_{\alpha\beta} = (m^{\alpha\beta})^{-1}$ is the Minkowski metric on $\mathbb{R}^{2+1}$, and $d\sigma$ is the associated volume element. Also, $\langle \cdot\,, \cdot \rangle$ refers to the standard inner product on $\mathbb{R}^3$ if we use ambient coordinates to describe u, $\partial_\alpha u$, etc. Recall that the energy is preserved:

$$\mathscr{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \langle DU(\cdot\,, t), DU(\cdot\,, t) \rangle \, dx = \text{const.}$$

The problem at hand is *energy critical*, meaning that the conserved energy is invariant under the natural re-scaling $U \to U(\lambda t, \lambda x)$.

We focus on a particular subset of equivariant maps characterized by the additional property that $U(t, r, \theta) = (u(t, r), \theta)$ in spherical coordinates, where, on the right-hand side, u stands for the longitudinal angle and θ stands for the latitudinal angle, while on the left-hand side, r, θ are the polar coordinates on $\mathbb{R}^2$. Now $u(t, r)$, a scalar function, satisfies the equation

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2}. \tag{1-1}$$

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Then the energy has the form

$$\mathcal{E}(u) = \pi \int_{\mathbb{R}^2} \left(|u_t|^2 + |u_r|^2 + \frac{\sin^2 u}{r^2} \right) r\, dr. \tag{1-2}$$

We shall be interested in corotational maps that are topologically nontrivial, that is, with

$$u(t, 0) = 0, \quad u(t, \infty) = \pi.$$

A natural space adapted to the elliptic part of this energy is $\dot{H}_e^1$:

$$\|f\|_{\dot{H}_e^1}^2 = \|\partial_r f\|_{L^2}^2 + \left\| \frac{f}{r} \right\|_{L^2}^2.$$

This is the equivariant translation of the usual two-dimensional space $\dot{H}^1$. The size of the elliptic part of the energy of u in (1-2) and its $\dot{H}_e^1$ norm are comparable, provided that u is small pointwise. This is not true directly for u, but it is true after we subtract from u the "nearby" soliton that we describe below.

The solitons for (1-1) have the form

$$Q_\lambda(r) = Q(\lambda r), \quad Q(r) = 2 \arctan r, \quad \lambda \in \mathbb{R}_+ = (0, \infty),$$

and are global minimizers of the energy $\mathcal{E}$ within their homotopy class, $\mathcal{E}(Q_\lambda) = 4\pi$.

We consider solutions u which are close to the soliton in the sense that

$$\mathcal{E}(u) - \mathcal{E}(Q) \ll 1. \tag{1-3}$$

As it turns out, such solutions must stay close to the soliton family $\{Q_\lambda\}$, due to the bound

$$\inf_\lambda \|(u(t) - Q_\lambda)\|_{\dot{H}_e^1}^2 + \|u_t(t)\|_{L^2}^2 \sim \mathcal{E}(u) - \mathcal{E}(Q). \tag{1-4}$$

Indeed, this follows, for example, from [Cote 2005]. Thus at any given t, one can choose some $\lambda(t)$ such that

$$\|(u(t) - Q_\lambda)\|_{\dot{H}_e^1}^2 + \|u_t(t)\|_{L^2}^2 \sim \mathcal{E}(u) - \mathcal{E}(Q). \tag{1-5}$$

Such a parameter λ is uniquely determined up to an error of size $O((\mathcal{E}(u) - \mathcal{E}(Q))^1/2)$. One can, for instance, choose λ to be the minimizer in (1-4), though there are no obvious benefits to be derived from that. Another equivalent choice is more direct, namely by the relation

$$u(t, \lambda^{-1}(t)) = \frac{\pi}{2}, \tag{1-6}$$

and this still satisfies (1-5); see, for instance, [Bejenaru and Tataru 2014]. Since this problem is locally well-posed in the energy space, scaling considerations show that (for well-chosen $\lambda(t)$), we have

$$\left| \frac{d}{dt} \lambda(t) \right| \lesssim \lambda^{-2}, \tag{1-7}$$

so at least locally λ stays bounded. Then the main question to ask is as follows:

Open problem. *What is the behavior of the function $\lambda(t)$ for equivariant maps satisfying (1-3)?*

We can distinguish several interesting plausible scenarios:

- *Type 1:* $\lambda(t) \to \infty$ as $t \to t_0$ (finite time blow-up). By (1-7), this can only happen at rates $\lambda(t) \gtrsim |t - t_0|^{-1}$. The above extreme corresponds to self-similar concentration; this can also be thought of as a consequence of the finite speed of propagation. In effect, by the important work [Struwe 2003], it is known that such a blow-up can only occur with speed strictly faster than self-similar:
$$\lambda(t)|t - t_0| \to \infty.$$
- *Type 2:* $\lambda(t) \to \infty$ as $t \to \infty$ (infinite time focusing).
- *Type 3:* $\lambda(t) \to 0$ as $t \to \infty$ (infinite time relaxation). By (1-7), this can only happen at rates $\lambda(t) \gtrsim t^{-1}$, which corresponds to self-similar relaxation.
- *Type 4:* $\lambda(t)$ stays in a compact set globally in time. Then we have a global solution, and possibly a resolution into a soliton plus a dispersive part.

Blow-up solutions of Type 1 were constructed not long ago in two quite different papers, [Krieger et al. 2008] and [Rodnianski and Sterbenz 2010], and the result of the latter paper was significantly strengthened and generalized in [Raphaël and Rodnianski 2012]. The behavior of $\lambda(t)$ in [Krieger et al. 2008] as $t \to 0$ is given by
$$\lambda(t) = t^{-1-\nu}, \quad \nu \geq 1$$
(here the restriction $\nu \geq 1$ seems technical, and should really be $\nu > 0$), while that in [Raphaël and Rodnianski 2012] is
$$\lambda(t) \sim t^{-1} e^{c\sqrt{\log t}}.$$
The latter solutions were also proved to be stable with respect to a class of small smooth perturbations. It is not implausible that the set of all blow-up solutions is open in a suitable topology, although numerical evidence in [Bizoń et al. 2001] appears to suggest the existence of a codimension-one manifold of data leading to an unstable blow-up, which separates scattering solutions from a stable regime of finite time blow-up solutions.

Up to this point we are not aware of any examples of solutions of Type 2, 3 or 4 other than the Q_λ's in the wave maps context, although recent work [Gustafson et al. 2010] revealed unusual solutions of this type in the context of the Landau–Lifshitz equation. Earlier work [Krieger and Schlag 2007] showed the existence of Type 4 solutions for the critical focusing nonlinear wave equation on $\mathbb{R}^{3+1}$.

Understanding the general picture for data in the energy space seems out of reach for now. However, there is a simpler question one may ask, namely, what happens for data which is close to a soliton in a stronger topology which includes both extra regularity and extra decay at infinity. Neither the results of [Krieger et al. 2008] nor of [Raphaël and Rodnianski 2012] apply in this context. A good starting point for this investigation is the following:

Conjecture. *There exists a codimension-one set of (small) data leading to Type 4 solutions, which separates Type 1 and Type 3 solutions.*

One should take this only as a rough guide; some fine adjustments may be needed. Our main result is to construct a large class of Type 4 solutions:

Theorem 1.1. *There exists a codimension-two set of Type 4 equivariant wave maps satisfying (1-3).*

For a more precise formulation of the theorem, see page 834. Compared with the conjecture above, one can see that we are one dimension short. At this point it is not clear if this is a technical issue, or if something new happens. A plausible scenario might be that the missing dimension may include Type 2 solutions, as well as slowly relaxing Type 4 solutions.

One should also compare this result with the related problem for Schrödinger maps. Although the solitons are the same and the operator H arising below in the linearization is also the same for Schrödinger maps, in [Bejenaru and Tataru 2014] it is shown that the solitons are stable with respect to small localized perturbations. One way to explain this is that the linear growth in the resonant direction occurring in the H-wave equation has a stronger destabilizing effect than the corresponding lack of decay in the H-Schrödinger equation.

Notation. Here we introduce some notation which will be used throughout the paper. We slightly modify the use of $\langle \cdot \rangle$ in the following sense:

$$\langle x \rangle = \sqrt{4+x^2}, \quad x \in \mathbb{R}.$$

For a real number a, we define $a^+ = \max\{0, a\}$ and $a^- = \min\{0, a\}$.

We will use a dyadic partition of $\mathbb{R}_+$ into sets $\{A_m\}_{m\in\mathbb{Z}}$ given by

$$A_m = \{2^{m-1} < r < 2^{m+1}\}.$$

For given $M > 0$, we use smooth localization functions $\chi_{\lesssim M}, \chi_{\gtrsim M}$ forming a partition of unity for $\mathbb{R}_+$ and such that

$$|(r\partial_r)^\alpha \chi_{\lesssim M}| + |(r\partial_r)^\alpha \chi_{\gtrsim M}| \lesssim_\alpha 1.$$

2. The gauge derivative and linearizations

The linearized equation (1-1) around the soliton Q has the form

$$-v_{tt} - Hv = 0, \quad H = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\cos(2Q)}{r^2}. \tag{2-1}$$

The elliptic operator H admits the factorization

$$H = L^*L, \quad L = h_1 \partial_r h_1^{-1} = \partial_r + \frac{h_3}{r}, \quad L^* = -h_1^{-1}\partial_r h_1 - \frac{1}{r} = -\partial_r + \frac{h_3 - 1}{r}, \tag{2-2}$$

where[1] $h_1 = \sin Q = \frac{2r}{1+r^2}$, $h_3 = -\cos Q = \frac{r^2-1}{r^2+1}$. H is nonnegative and has a zero resonance

$$\phi_0 = h_1 = \frac{2r}{1+r^2}.$$

[1]Throughout this paper we use $\sin Q, \cos Q$ instead of h_1, h_3; however, the reader may need this correspondence in order to relate this work to [Bejenaru and Tataru 2014].

This resonance is the reason why (2-1) does not have good dispersive estimates. Since ϕ_0 fails to be an eigenvalue, we cannot project it away as is usually done in standard modulation theory. This suggests that working with the variable u and its equation (1-1) runs into problems due to the lack of good linear estimates needed to treat the nonlinearity. Therefore, instead of working with the solution u, we introduce a new variable

$$w = \partial_r u - \frac{1}{r}\sin u, \tag{2-3}$$

which has the nice property that

$$w = 0 \iff u = Q_\lambda$$

for some $\lambda \in \mathbb{R}_+$. Indeed, by rearranging (1-2) and using $u(0) = 0$, $u(\infty) = \pi$, we obtain

$$\mathscr{E}(u) = \pi\int_0^\infty (|u_t|^2 + |w|^2)r\,dr + \pi\int_0^\infty 2\sin u\cdot\partial_r u\,dr = \pi\int_0^\infty (|u_t|^2 + |w|^2)r\,dr + 4\pi,$$

from which the above observation follows. This type of change of variables originates at least with the work [Gustafson et al. 2008]. If $\lambda(t)$ is chosen such that (1-5) holds, then using (1-3), a direct computation shows that

$$\|u - Q_\lambda\|_{\dot{H}^1_e} \approx \|w\|^2_{L^2}. \tag{2-4}$$

Then a direct computation shows that w solves

$$w_{tt} - \Delta w + \frac{2(1+\cos u)}{r^2}w = \frac{1}{r}\sin u(u_t^2 - w^2). \tag{2-5}$$

The function u appears in this equation, but it can be recovered from w by solving the ODE (2-3) with Q-like "data" at $r = \infty$.

We remark that the linearized form of (2-3) near Q is

$$z = \Big(\partial_r - \frac{1}{r}\cos Q\Big)v = Lv, \tag{2-6}$$

where L was introduced above in (2-2).

On the other hand, the linearized equation for w near Q has the form

$$z_{tt} - \Delta z + \frac{2(1+\cos Q)}{r^2}z = 0. \tag{2-7}$$

This wave equation is governed by the operator

$$\tilde{H} = -\Delta + \frac{2(1+\cos Q)}{r^2} = -\Delta + \frac{4}{r^2(1+r^2)} = LL^*.$$

This operator is better behaved than H; in particular, its zero mode ψ_0 grows logarithmically at infinity.

The plan is to treat (2-5) in a perturbative manner for the most part. To fix things, we will rewrite it in the form

$$(\partial_t^2 + \tilde{H})w = \frac{2(\cos Q - \cos u)}{r^2}w + \frac{1}{r}\sin u(u_t^2 - w^2) := N(w, u) \tag{2-8}$$

and work with this from here on. Equation (2-8) for w is preferable due to the nice dispersive properties of its linear part. However, as u occurs in the w equation, one has to also keep track of it through the elliptic equation (2-3). In addition, u_t also appears in the above equation. This is related to w_t by differentiating (2-3):

$$w_t = \Big(\partial_r - \frac{1}{r}\cos u\Big)u_t. \tag{2-9}$$

In order to study this equation, we need to understand better the structure of its linear part, and, in particular, the spectral theory for the operator $\tilde{H}$. This is the subject of Section 3.

Setup of the problem. The starting point is to consider $\bar{w}$ to be an exact real solution to the linear homogeneous equation

$$(\partial_t^2 + \tilde{H})\bar{w} = 0, \quad w(0) = w_0, \quad w_t(0) = w_1, \tag{2-10}$$

where w_0 and w_1 are real Schwartz functions which are assumed to satisfy the nonresonance conditions

$$\langle w_0, \psi_0\rangle = 0, \quad \langle w_1, \psi_0\rangle = 0. \tag{2-11}$$

We denote by $\bar{u}$ the corresponding map, see (2-3) (this will be made precise in Proposition 5.2), obtained by solving the ODE

$$\partial_r \bar{u} - \frac{1}{r}\sin\bar{u} = \bar{w}, \quad \bar{u} \sim Q \text{ as } r \to \infty. \tag{2-12}$$

Now we seek a solution to the nonlinear equation u and its associated gauge derivative w close to $\bar{u}$, $\bar{w}$ respectively,

$$u = \bar{u} + \varepsilon, \quad w = \bar{w} + \gamma, \tag{2-13}$$

so that u and w match $\bar{u}$ and $\bar{w}$ asymptotically as $t \to \infty$.

By a slight abuse of notation, we use $\|\cdot\|_S$ to denote a norm obtained by adding sufficiently many seminorms of the Schwartz space S. We also use $\lesssim_S$ for inequalities where the implicit constant depends on $\|(w_0, w_1)\|_S$. Modulo defining the X and LX norms, we are now in a position to restate our main result in a more detailed fashion.

Theorem 2.1. *Let w_0, w_1 be Schwartz functions satisfying the nonresonance conditions (2-11). Let $\bar{u}$ and $\bar{w}$ be defined as above. Then there exist $T \lesssim_S 1$ and a unique wave map u in $[T, \infty)$ so that u and w match $\bar{u}$ and $\bar{w}$ as $t \to \infty$ in the following asymptotic fashion for $t \in [T, \infty)$:*

$$\|\gamma(t)\|_{LX} \lesssim_S t^{-3/2}, \quad \|\partial_t\gamma(t)\|_{LX} \lesssim_S t^{-5/2}, \quad \|\gamma(t)\|_{\dot{H}^1} \lesssim_S t^{-5/2}, \tag{2-14}$$

respectively

$$\|\varepsilon(t)\|_X \lesssim_S t^{-3/2}, \quad \|\partial_t\varepsilon(t)\|_{LX} \lesssim_S t^{-5/2}. \tag{2-15}$$

Furthermore, the map u and its corresponding gauge derivative w have a Lipschitz dependence on (w_0, w_1) with respect to the above norms.

One would expect the above result to be in terms of L^2 and $\dot{H}^1_e$ spaces. However, these spaces are very disconnected from the spectral structure of H and $\tilde{H}$, particularly at low frequencies, and this makes them unsuitable. The spaces $X \subset \dot{H}^1_e$ and $LX \subset L^2$ have been introduced in [Bejenaru and Tataru 2014] to address exactly this issue: they are low-frequency corrections of $\dot{H}^1_e$, respectively L^2. Their exact definition is provided in the next section.

In view of (2-8), the function γ solves

$$(\partial_t^2 + \tilde{H})\gamma = N(\bar{w} + \gamma, \bar{u} + \varepsilon) \tag{2-16}$$

with zero Cauchy data at infinity. By (2-3), (2-9), (2-13) and (2-12), the functions ε and ε_t are determined from the equations

$$\begin{aligned} \gamma &= \partial_r \varepsilon - \frac{\sin(\varepsilon + \bar{u}) - \sin \bar{u}}{r}, \\ \gamma_t &= \left(\partial_r - \frac{\cos(\varepsilon + \bar{u})}{r}\right)\varepsilon_t - \frac{\cos(\varepsilon + \bar{u}) - \cos \bar{u}}{r}\bar{u}_t. \end{aligned} \tag{2-17}$$

We proceed as follows. In the next section we recall from [Bejenaru and Tataru 2014] the spectral theory for H (which in fact originates in [Krieger et al. 2008]) and $\tilde{H}$ and the definitions and some properties of the spaces X and LX. Then, in Section 4, we provide linear estimates for the linear (inhomogeneous) wave equation corresponding to (2-10). In Section 5, we analyze the first approximations $\bar{w}$ and $\bar{u}$ using (2-12). Then, in Section 6, we continue with the study of the relation between ε and γ based on (2-17). All the analysis carried out in Sections 4–6 is done in the context of X and LX spaces. In the end, in Section 7, we study the solvability of Equation (2-16) using perturbative methods in LX based spaces.

3. The modified Fourier transform

In this section, we recall the spectral theory associated with the operators H, $\tilde{H}$. The spectral theory for H was developed in [Krieger et al. 2008], and the one for $\tilde{H}$ was derived from the one for H in [Bejenaru and Tataru 2014]. In this paper, we follow closely the exposition in [Bejenaru and Tataru 2014].

Generalized eigenfunctions. We consider H acting as an unbounded self-adjoint operator in $L^2(r\,dr)$. Then H is nonnegative, and its spectrum $[0, \infty)$ is absolutely continuous. H has a zero resonance, namely $\phi_0 = h_1$:

$$Hh_1 = 0.$$

For each $\xi > 0$, one can choose a normalized generalized eigenfunction ϕ_ξ,

$$H\phi_\xi = \xi^2 \phi_\xi.$$

These are unique up to a ξ dependent multiplicative factor, which is chosen as described below.

To these one associates a generalized Fourier transform $\mathcal{F}_H$ defined by

$$\mathcal{F}_H f(\xi) = \int_0^\infty \phi_\xi(r) f(r) r\,dr,$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \phi_\xi(r) \mathcal{F}_H f(\xi)\,d\xi.$$

The functions ϕ_ξ are smooth with respect to both r and ξ. To describe them, one considers two distinct regions, $r\xi \lesssim 1$ and $r\xi \gtrsim 1$.

In the first region, $r\xi \lesssim 1$, the functions ϕ_ξ admit a power series expansion of the form

$$\phi_\xi(r) = q(\xi)\biggl(\phi_0 + \frac{1}{r}\sum_{j=1}^{\infty}(r\xi)^{2j}\phi_j(r^2)\biggr), \quad r\xi \lesssim 1, \tag{3-1}$$

where $\phi_0 = h_1$ and the functions ϕ_j are analytic and satisfy

$$|(r\partial_r)^\alpha \phi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log(1+r). \tag{3-2}$$

This bound is not spelled out in [Krieger et al. 2008], but it follows directly from the integral recurrence formula for the f_j given on p. 578 of that paper. The smooth positive weight q satisfies

$$q(\xi) \approx \begin{cases} \dfrac{1}{\xi^{1/2}|\log\xi|} & \text{if } \xi \ll 1, \\ \xi^{3/2} & \text{if } \xi \gg 1, \end{cases} \qquad |(\xi\partial_\xi)^\alpha q| \lesssim_\alpha q. \tag{3-3}$$

Defining the weight

$$m_k^1(r) = \begin{cases} \min\Bigl\{1, r2^k\dfrac{\log(1+r^2)}{\langle k\rangle}\Bigr\} & \text{if } k < 0, \\ \min\{1, r^3 2^{3k}\} & \text{if } k \geq 0, \end{cases} \tag{3-4}$$

it follows that the nonresonant part of ϕ_ξ satisfies

$$\bigl|(\xi\partial_\xi)^\alpha (r\partial_r)^\beta \bigl(\phi_\xi(r) - q(\xi)\phi_0(r)\bigr)\bigr| \lesssim_{\alpha\beta} 2^{k/2} m_k^1(r), \quad \xi \approx 2^k, \ r\xi \lesssim 1. \tag{3-5}$$

In the other region, $r\xi \gtrsim 1$, we begin with the functions

$$\phi_\xi^+(r) = r^{-1/2} e^{ir\xi}\sigma(r\xi, r), \quad r\xi \gtrsim 1, \tag{3-6}$$

solving

$$H\phi_\xi^+ = \xi^2 \phi_\xi^+,$$

where for σ, we have the asymptotic expansion

$$\sigma(q, r) \approx \sum_{j=0}^{\infty} q^{-j}\phi_j^+(r), \quad \phi_0^+ = 1, \quad \phi_1^+ = \frac{3i}{8} + O\biggl(\frac{1}{1+r^2}\biggr),$$

with $\sup_{r>0}|(r\partial_r)^k \phi_j^+| < \infty$ in the following sense:

$$\sup_{r>0}\biggl|(r\partial r)^\alpha (q\partial_q)^\beta \biggl(\sigma(q, r) - \sum_{j=0}^{j_0} q^{-j}\phi_j^+(r)\biggr)\biggr| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}.$$

Then we have the representation

$$\phi_\xi(r) = a(\xi)\phi_\xi^+(r) + \overline{a(\xi)\phi_\xi^+(r)}, \tag{3-7}$$

where the complex-valued function a satisfies

$$|a(\xi)| = \sqrt{\frac{2}{\pi}}, \quad |(\xi\partial_\xi)^\alpha a(\xi)| \lesssim_\alpha 1. \tag{3-8}$$

The spectral theory for $\tilde{H}$ is derived from the spectral theory for H due to the conjugate representations

$$H = L^* L, \quad \tilde{H} = L L^*.$$

This allows us to define generalized eigenfunctions ψ_ξ for $\tilde{H}$ using the generalized eigenfunctions ϕ_ξ for H,

$$\psi_\xi = \xi^{-1} L\phi_\xi, \quad L^*\psi_\xi = \xi\phi_\xi. \tag{3-9}$$

It is easy to see that ψ_ξ are real and smooth, vanish at $r = 0$, and solve

$$\tilde{H}\psi_\xi = \xi^2\psi_\xi.$$

With respect to this frame, we can define the generalized Fourier transform adapted to $\tilde{H}$ by

$$\mathcal{F}_{\tilde{H}} f(\xi) = \int_0^\infty \psi_\xi(r) f(r) r\, dr,$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \psi_\xi(r)\mathcal{F}_{\tilde{H}} f(\xi) d\xi. \tag{3-10}$$

To see this, we compute, for a Schwartz function f,

$$\mathcal{F}_{\tilde{H}} Lf(\xi) = \int_0^\infty \psi_\xi(r) Lf(r) r\, dr = \int_0^\infty L^*\psi_\xi(r) f(r) r\, dr = \int_0^\infty \xi\phi_\xi(r) f(r) r\, dr = \xi\mathcal{F}_H f(\xi).$$

Hence

$$\|\mathcal{F}_{\tilde{H}} Lf\|_{L^2}^2 = \|\xi\mathcal{F}_H f(\xi)\|_{L^2}^2 = \langle Hf, f\rangle_{L^2(r\,dr)} = \|Lf\|_{L^2}^2,$$

which suffices, since Lf spans a dense subset of L^2.

The representation of ψ_ξ in the two regions $r\xi \lesssim 1$ and $r\xi \gtrsim 1$ is obtained from the similar representation of ϕ_ξ. In the first region, $r\xi \lesssim 1$, the functions ψ_ξ admit a power series expansion of the form

$$\psi_\xi = \xi q(\xi)\Big(\psi_0(r) + \sum_{j\geq 1}(r\xi)^{2j}\psi_j(r^2)\Big), \tag{3-11}$$

where

$$\psi_j(r) = (h_3 + 1 + 2j)\phi_{j+1}(r) + r\partial_r\phi_{j+1}(r).$$

From (3-2), it follows that

$$|(r\partial_r)^\alpha\psi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log(1+r^2).$$

In addition, ψ_0 solves $L^*\psi_0 = \phi_0$, and therefore a direct computation shows that

$$\psi_0 = \frac{1}{2}\left(\frac{(1+r^2)\log(1+r^2)}{r^2} - 1\right).$$

In particular, defining the weights

$$m_k(r) = \begin{cases} \min\left\{1, \dfrac{\log(1+r^2)}{\langle k\rangle}\right\} & \text{if } k < 0, \\ \min\{1, r^2 2^{2k}\} & \text{if } k \geq 0, \end{cases} \tag{3-12}$$

we have the pointwise bound for ψ_ξ

$$\left|(r\partial_r)^\alpha(\xi\partial_\xi)^\beta\psi_\xi(r)\right| \lesssim_{\alpha\beta} 2^{k/2} m_k(r), \quad \xi \approx 2^k,\ r\xi \lesssim 1. \tag{3-13}$$

On the other hand, in the regime $r\xi \gtrsim 1$, we define

$$\psi^+ = \xi^{-1}L\phi^+,$$

and we obtain the representation

$$\psi_\xi(r) = a(\xi)\psi_\xi^+(r) + \overline{a(\xi)\psi_\xi^+(r)}. \tag{3-14}$$

For ψ^+, we obtain the expression

$$\psi_\xi^+(r) = r^{-1/2}e^{ir\xi}\tilde{\sigma}(r\xi, r), \quad r\xi \gtrsim 1, \tag{3-15}$$

where $\tilde{\sigma}$ has the form

$$\tilde{\sigma}(q,r) = i\sigma(q,r) - \tfrac{1}{2}q^{-1}\sigma(q,r) + \frac{\partial}{\partial q}\sigma(q,r) + \xi^{-1}L\sigma(q,r),$$

and therefore it has exactly the same properties as σ. In particular, for fixed ξ, we obtain that

$$\tilde{\sigma}(r\xi, r) = i - \tfrac{7}{8}r^{-1}\xi^{-1} + O(r^{-2}). \tag{3-16}$$

We conclude our description of the generalized eigenfunctions and of the associated Fourier transforms with a bound on the $\tilde{H}$ Fourier transforms of Schwartz functions.

Lemma 3.1. *If f is a Schwartz function satisfying $\langle f, \psi_0\rangle = 0$, then*

$$\left|(\xi\partial_\xi)^\alpha \mathcal{F}_{\tilde{H}} f(\xi)\right| \lesssim_{\alpha,N} \begin{cases} \dfrac{\xi^{5/2}}{\langle\log\xi\rangle} & \text{if } \xi \lesssim 1, \\ \langle\xi\rangle^{-N} & \text{if } \xi \gtrsim 1. \end{cases} \tag{3-17}$$

Proof. We start from the definition of the modified Fourier transform and use that $\langle f, \psi_0\rangle = 0$:

$$\begin{aligned} |\mathcal{F}_{\tilde{H}} f(\xi)| &\lesssim \left(\left|\int_0^{\xi^{-1}} \psi_\xi(r)f(r)r\,dr\right| + \left|\int_{\xi^{-1}}^\infty \psi_\xi(r)f(r)r\,dr\right|\right) \\ &\lesssim \xi q(\xi)\left(\int_{\xi^{-1}}^\infty \left|\psi_0(r)f(r)\right|r\,dr + \int_0^{\xi^{-1}} \sum_{j\geq 1}(r\xi)^{2j}\psi_j(r^2)f(r)r\,dr\right) + \int_{\xi^{-1}}^\infty |f(r)|r^{1/2}\,dr \\ &\lesssim \xi^3 q(\xi). \end{aligned}$$

A similar argument takes care of the case $\alpha > 0$. □

The spaces X and LX. The operator L maps $\dot{H}^1_e$ into L^2. Conversely, one would like that, given some $f \in L^2$, we could solve $Lu = f$ and obtain a solution u which is in $\dot{H}^1_e$ and satisfies

$$\|u\|_{\dot{H}^1_e} \lesssim \|f\|_{L^2}.$$

However, this is not the case. The first observation is that the solution is only unique modulo a multiple of the resonance ϕ_0. Moreover, the inequality above is not expected to be true, even assuming that somehow we choose the "best" u from all candidates.

The spaces X and LX are in part introduced in order to remedy both the ambiguity in the inversion of L and the failing inequality.

Definition 3.2. (a) The space X is defined as the completion of the subspace of $L^2(r\,dr)$ for which the following norm is finite:

$$\|u\|_X = \Big(\sum_{k\geq 0} 2^{2k}\|P_k^H u\|_{L^2}^2\Big)^{1/2} + \sum_{k<0} \frac{1}{|k|}\|P_k^H u\|_{L^2},$$

where P_k^H is the Littlewood–Paley operator localizing at frequency $\xi \approx 2^k$ in the H calculus.

(b) LX is the space of functions of the form $f = Lu$ with $u \in X$, with norm $\|f\|_{LX} = \|u\|_X$. Expressed in the $\tilde{H}$ calculus, the LX norm is written as

$$\|f\|_{LX} = \Big(\sum_{k\geq 0} \|P_k^{\tilde{H}} f\|_{L^2}^2\Big)^{1/2} + \sum_{k<0} \frac{2^{-k}}{|k|}\|P_k^{\tilde{H}} f\|_{L^2}.$$

In this article we work with equivariant wave maps u for which $\|u - Q\|_X \ll 1$. This corresponds to functions w which satisfy $\|w\|_{LX} \ll 1$. The simplest properties of the space X are summarized as follows (see Proposition 4.2 in [Bejenaru and Tataru 2014]):

Proposition 3.3. *The following embeddings hold for the space X:*

$$H^1_e \subset X \subset \dot{H}^1_e. \tag{3-18}$$

In addition, for f in X, we have the bounds

$$\|\langle r\rangle^{1/2} f\|_{L^\infty} \lesssim \|f\|_X, \tag{3-19}$$

$$\Big\|\frac{f}{\log(1+r)}\Big\|_{L^2} \lesssim \|f\|_X, \tag{3-20}$$

$$\|\langle r\rangle^{1/2} f\|_{L^4} \lesssim \|f\|_X. \tag{3-21}$$

Now we turn our attention to the space LX. From [Bejenaru and Tataru 2014, Lemma 4.4 and Proposition 4.5], we have:

Lemma 3.4. *If $f \in L^2$ is localized at $\tilde{H}$-frequency 2^k, then*

$$|f(r)| \lesssim 2^k m_k(r)(1+2^k r)^{-1/2}\|f\|_{L^2}. \tag{3-22}$$

Proposition 3.5. *The following embeddings hold for LX:*

$$L^1 \cap L^2 \subset LX \subset L^2. \tag{3-23}$$

4. Linear estimates for the $\tilde{H}$ wave equation

In this section, we prove estimates for the linear equation

$$(\partial_t^2 + \tilde{H})\psi = f, \tag{4-1}$$

with zero Cauchy data at infinity. The solution is given by $\psi = Kf$, where

$$Kf(r,t) = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \frac{\sin(t-s)\xi}{\xi} \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds. \tag{4-2}$$

We also need its time derivative, which is given by

$$\partial_t Kf = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \cos(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

Finally, we need the following formula, which follows from (3-9):

$$L^* Kf = -\mathcal{F}_H^{-1} \int_t^\infty \sin(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

The following result is a modification of the standard energy estimate for the wave equation:

Lemma 4.1. *Assume that $f(s) \in LX$. Then for every $\alpha > 0$, the solution of (4-1) with zero data at ∞ satisfies*

$$t^\alpha \|\psi(t)\|_{LX} + t^{\alpha+1}\big(\|\partial_t \psi(t)\|_{LX} + \|\psi(t)\|_{\dot{H}^1_e}\big) \lesssim \sup_s s^{\alpha+2} \|f(s)\|_{LX}. \tag{4-3}$$

Proof. The solution of (4-1) with zero data at ∞ is given by $\psi = Kf$. The estimate for the first term follows from the bound $\big|(\sin(t-s)\xi)/\xi\big| \lesssim |t-s|$ and the representation of the spaces LX on the Fourier side. The estimate for the second term is similar.

The argument for the third term is more involved. We define g by

$$\mathcal{F}_{\tilde{H}} g(t, \xi) = -\int_t^\infty \sin((t-s)\xi) \mathcal{F}_{\tilde{H}} f(\xi, s)\, ds.$$

Then

$$\xi \mathcal{F}_{\tilde{H}} \psi(t, \xi) = \mathcal{F}_{\tilde{H}} g(t, \xi).$$

We estimate, as above,

$$\|g(t)\|_{LX} \lesssim \int_t^\infty \|f(s)\|_{LX}\, ds \lesssim t^{-\alpha-1} \sup_s s^{\alpha+2} \|f(s)\|_{LX}.$$

Hence it suffices to show that for ψ and g related as above, we have

$$\|\psi\|_{\dot{H}^1_e} \lesssim \|g\|_{LX}. \tag{4-4}$$

Here the time variable plays no role and is discarded. Recalling the form of L^* from (2-2), namely $L^* = -\partial_r + (h_3 - 1)/r$, it follows that

$$\|\psi\|_{\dot{H}^1_e} \lesssim \|L^*\psi\|_{L^2} + \left\| \frac{\psi}{r} \right\|_{L^2}.$$

For the first term, we use Plancherel to write

$$\|L^*\psi(t)\|^2_{L^2} = \langle \psi(t), \tilde{H}\psi(t)\rangle = \|\xi \mathcal{F}_{\tilde{H}}\psi(\xi)\|^2_{L^2} = \|g\|_{L^2} \lesssim \|g\|^2_{LX}.$$

For the second term, the L^2 bound for g no longer suffices, and we need to use the LX norm of g. We consider a Littlewood–Paley decomposition for both ψ and g, and denote their dyadic pieces by ψ_k, respectively g_k. Then

$$\|\psi_k\|_{L^2} \approx 2^{-k}\|g_k\|_{L^2}.$$

By using (3-13)–(3-14) and the Cauchy–Schwartz inequality, we obtain pointwise bounds for ψ_k, namely,

$$|\psi_k| \lesssim \frac{m_k(r)}{\langle 2^k r\rangle^{1/2}} 2^k \|\psi_k\|_{L^2} \lesssim \frac{m_k(r)}{\langle 2^k r\rangle^{1/2}} \|g_k\|_{L^2},$$

with m_k as in (3-12). For $k \geq 0$, the contributions are almost orthogonal, and we obtain

$$\left\| \frac{\psi_{\geq 0}}{r} \right\|_{L^2} \lesssim \|g_{\geq 0}\|_{L^2}.$$

However, if $k < 0$, then the weaker logarithmic decay for small r no longer suffices for such an argument. Instead, by direct computation, we obtain a weaker bound,

$$\left\| \frac{\psi_k}{r} \right\|_{L^2} \lesssim |k|^{1/2}\|g_k\|_{L^2} \lesssim |k|^{3/2} 2^k \|g\|_{LX}.$$

Then the k summation is easily accomplished. □

5. Analysis of the first approximations $\bar{w}$ and $\bar{u}$

Pointwise bounds for $\bar{w}$. We define f_0 and f_1 by $f_0 = \mathcal{F}_{\tilde{H}} w_0$ and $f_1 = \mathcal{F}_{\tilde{H}} w_1$. Then for $\bar{w}$, we have the representation

$$\bar{w}(t, r) = \int_0^\infty \psi_\xi(r) \Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi) \Big)\, d\xi.$$

Since w_0, w_1 are Schwartz functions satisfying (2-11), from (3-17) we obtain

$$\big|(\xi\partial_\xi)^\alpha f_0(\xi)\big| + \big|(\xi\partial_\xi)^\alpha f_1(\xi)\big| \lesssim_{\alpha,N} \|(w_0, w_1)\|_S \begin{cases} \dfrac{\xi^{5/2}}{\langle \log \xi\rangle} & \text{if } \xi \lesssim 1, \\ \langle \xi\rangle^{-N} & \text{if } \xi \gtrsim 1. \end{cases} \tag{5-1}$$

Here, by a slight abuse of notation, we use $\| \,.\, \|_S$ to denote a finite collection of the S seminorms. This will allow us to obtain pointwise bounds for $\bar{w}$:

Lemma 5.1. *If w_0, w_1 are Schwartz functions satisfying the moment conditions (2-11), then $\bar{w}$ satisfies*

$$|\bar{w}(r,t)| \lesssim \frac{\log(1+r^2)}{\log\langle r+t\rangle} \frac{1}{\langle t+r\rangle^{1/2}\langle t-r\rangle^{5/2}\log\langle r-t\rangle} \|(w_0,w_1)\|_S. \tag{5-2}$$

Proof. We fix k and consider

$$\bar{w}_k(t,r) = \int_0^\infty \psi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\chi_k(\xi)\,d\xi.$$

For $\psi_\xi(r)$, we use the representation (3-11) in the region $\{r\xi \lesssim 1\}$, respectively (3-14) in the region $\{r\xi \gtrsim 1\}$. Then via a standard stationary phase argument, we obtain

$$|w_k(r,t)| \lesssim_N \frac{2^{k/2}\langle 2^k r\rangle^{-1/2} m_k(r)}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{5k/2} 2^{-Nk^+}.$$

The desired estimate (5-2) follows by summing these bounds with respect to k. □

Bounds for $\bar{u}$, $\bar{u}_t$. Next we consider $\bar{u}$, which is recovered from $\bar{w}$ via (2-12). This equation contains a nonlinear part coming from the sine function. Consequently, we split $\bar{u}$ into a linear and a nonlinear part:

$$\bar{u} = Q + \bar{u}^l + \bar{u}^{nl},$$

where $\bar{u}^l$ solves the linear part of (2-12),

$$L\bar{u}^l = \bar{w},$$

and $\bar{u}^{nl}$ solves

$$L\bar{u}^{nl} = N(\bar{u}^l, \bar{u}^{nl}), \tag{5-3}$$

where

$$N(u,v) = \frac{1}{r}\big[\sin Q\cdot\big(\cos(u+v)-1\big) + \cos Q\cdot\big(\sin(u+v)-(u+v)\big)\big].$$

Both of the above ODE's are taken with zero Cauchy data at infinity or, equivalently, can be interpreted via the diffeomorphism $L : X \to LX$. The linear part, $\bar{u}^l$, is recovered from the explicit formula

$$\bar{u}^l := L^{-1}\bar{w} = \int_0^\infty \xi^{-1}\phi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\,d\xi,$$

and will be split into a resonant and a nonresonant part: $\bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr}$.

For the nonlinear part, we use an iterative argument based on the fact that there is enough decay on the right-hand side that we can recover it via

$$\bar{u}^{nl} = h_1(r)\int_r^\infty \frac{N(\bar{u}^l,\bar{u}^{nl})}{h_1(s)}\,ds. \tag{5-4}$$

At this stage, we also want to keep track of the differences of solutions. For this, we denote by δw_0, δw_1, $\delta\bar{w}$, $\delta\bar{u}$ the corresponding differences.

Proposition 5.2. (a) *Assume that w_0, w_1 are Schwartz functions satisfying (2-11). Then*

$$\bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr}, \tag{5-5}$$

where $\bar{u}^{l,r}$ and $\bar{u}^{l,nr}$ satisfy the bounds

$$|\bar{u}^{l,r}|+r|\partial_r\bar{u}^{l,r}|+\langle r+t\rangle|\partial_t\bar{u}^{l,r}|\lesssim \frac{h_1(r)}{\langle t+r\rangle\log^2\langle t+r\rangle}\|(w_0,w_1)\|_S,$$
$$|\bar{u}^{l,nr}|+\frac{r\langle r-t\rangle}{\langle t+r\rangle}|\partial_r\bar{u}^{l,nr}|+\langle r-t\rangle|\partial_t\bar{u}^{l,nr}|\lesssim \frac{r}{r+\langle t\rangle}\frac{1}{\langle t+r\rangle^{1/2}\langle t-r\rangle^{3/2}\log\langle t-r\rangle}\|(w_0,w_1)\|_S. \tag{5-6}$$

In addition,

$$\left|(\partial_r+\partial_t)\bar{u}^l+\frac{1}{2r}\bar{u}^l\right|\lesssim\frac{1}{t^{5/2}\langle r-t\rangle^{1/2}\log\langle t-r\rangle}\|(w_0,w_1)\|_S,\qquad r\sim t. \tag{5-7}$$

(b) *For $t\gtrsim_S 1$, the nonlinear part $\bar{u}^{nl}$ satisfies the bounds*

$$|\bar{u}^{nl}(r,t)|\lesssim_S\ h_1(r)t^{-1.5}\|(w_0,w_1)\|_S,\qquad \left|\partial_t\bar{u}^{nl}+\tfrac{1}{12}h_1(\bar{u}^l)^3\right|\lesssim_S\ h_1(r)t^{-2}\|(w_0,w_1)\|_S. \tag{5-8}$$

(c) *The above estimates hold true for $\delta\bar{u}^{nl}$ and $\delta\partial_t\bar{u}_l$:*

$$|\delta\bar{u}^{nl}(r,t)|\lesssim_S\ h_1(r)t^{-1.5}\|(\delta w_0,\delta w_1)\|_S,\qquad \left|\delta\partial_t\bar{u}^{nl}+\tfrac{1}{12}h_1\delta(\bar{u}^l)^3\right|\lesssim_S\ h_1(r)t^{-2}\|(\delta w_0,\delta w_1)\|_S. \tag{5-9}$$

Remark 5.3. By finite speed of propagation arguments, it is not difficult to show that $\bar{u}^l$ decays rapidly outside the cone. However, for our purposes, the decay established in the above proposition suffices.

Remark 5.4. The bound (5-7) shows that a double cancellation occurs on the light cone, as opposed to the expected single cancellation. This is a consequence of the exact decay properties at infinity for the potential in $\tilde{H}$.

Remark 5.5. The second estimate in part (b) is the outcome of a more subtle nonlinear cancellation, rather then a brute force computation.

Proof. (a) We first split $\bar{u}^l$ into two parts,

$$\bar{u}^l(r,t)=\sum_k\bar{u}^l_k(r,t)=\sum_{2^k\lesssim r^{-1}}\bar{u}^l_k(r,t)+\sum_{2^k\gtrsim r^{-1}}\bar{u}^l_k(r,t):=\bar{u}^l_{\mathrm{low}}(r,t)+\bar{u}^l_{\mathrm{hi}}(r,t),$$

where

$$\bar{u}^l_k=\int\xi^{-1}\phi_\xi(r)\chi_k(\xi)\left(\cos(t\xi)\cdot\widehat{f_0}(\xi)+\frac{\sin(t\xi)}{\xi}\widehat{f_1}(\xi)\right)d\xi.$$

The functions $\widehat{f_0}(\xi)$ and $\widehat{f_1}(\xi)$ belong to the same class, and for large ξ they are smooth and rapidly decaying. Hence the first term in the above formula is better than the second, and will be neglected in the sequel. Then using the power series (3-1), we can write

$$\bar{u}^l_k=\int\xi^{-2}q(\xi)\sin(t\xi)\left(\phi_0(r)+\frac{1}{r}\sum_{j\geq1}(r\xi)^{2j}\phi_j(r^2)\right)\widehat{f_1}(\xi)\chi_k(\xi)\,d\xi,\qquad 2^kr\lesssim1,$$

which leads to a corresponding decomposition

$$\bar{u}^l_{\mathrm{low}}=\bar{u}^{l,0}_{\mathrm{low}}+\sum_{j\geq1}\bar{u}^{l,j}_{\mathrm{low}}.$$

Then we set

$$\bar{u}^{l,r} = \bar{u}^{l,0}_{\text{low}}, \quad \bar{u}^{l,nr} = \bar{u}^{l}_{\text{hi}} + \sum_{j\geq 1} \bar{u}^{l,j}_{\text{low}}, \tag{5-10}$$

and proceed to estimate all of the above components of $\bar{u}^l$.

The terms in $\bar{u}^l_{\text{hi}}$ are estimated by stationary phase using (5-1) and the ϕ_ξ representation in (3-7). This yields

$$|\bar{u}^l_k| \lesssim \frac{r^{-1/2}2^{3k/2}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{-Nk^+}, \quad 2^k r \gtrsim 1, \tag{5-11}$$

which, after summation with respect to k, gives the bound

$$|\bar{u}^l_{\text{hi}}| \lesssim \sum_{2^k \gtrsim r^{-1}} |\bar{u}^l_k(r,t)| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{1/2}\langle r-t\rangle^{3/2}\log\langle r-t\rangle}.$$

The bounds for the time derivative are obtained from the explicit formula

$$\partial_t \bar{u}^l = \int_0^\infty \phi_\xi(r)\Big(-f_0(\xi)\sin(t\xi) + \frac{1}{\xi} f_1(\xi)\cos(t\xi)\Big)\,d\xi,$$

which shows that we produce an extra 2^k factor in (5-11). Similarly, an r derivative applied to ϕ_ξ yields an additional 2^k factor in the asymptotic expansion. Thus we obtain

$$|\partial_t \bar{u}^l_k| + |\partial_r \bar{u}^l_k| \lesssim \frac{r^{-1/2}2^{5k/2}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle} 2^{-Nk^+}, \quad 2^k r \gtrsim 1, \tag{5-12}$$

which leads to

$$|\partial_t \bar{u}^l_{hi}| + |\partial_r \bar{u}^l_{hi}| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{1/2}\langle r-t\rangle^{5/2}\log\langle r-t\rangle}.$$

We now consider the terms in $\bar{u}^{l,j}_{\text{low}}$. The main contribution comes from f_1, so we take $f_0 = 0$ for convenience. For $j = 0$, we have

$$\bar{u}^{l,0}_{\text{low}} = \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r) \int \xi^{-2} q(\xi)\sin(t\xi)\widehat{f_1}(\xi)\chi_k(\xi)\,d\xi := \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r) g^0_k(t) := \phi_0(r) g^0(r,t).$$

Using stationary phase and the properties of q, we have

$$|g^0_k(t)| + 2^{-k}|\partial_t g^0_k(t)| \lesssim \frac{2^k}{\langle k^-\rangle^2 \langle 2^k t\rangle^N} 2^{-Nk^+}.$$

By summing with respect to k, we obtain

$$|g^0(r,t)| + \langle t+r\rangle\big(|\partial_r g^0(r,t)| + |\partial_t g^0(r,t)|\big) \lesssim \frac{1}{\langle t+r\rangle \log^2\langle t+r\rangle}, \tag{5-13}$$

which yields the $\bar{u}^{l,r}$ bound in (5-6).

For $j \geq 1$, we have

$$u^{l,j}_{\text{low}} = \sum_k \chi_{\{r \lesssim 2^{-k}\}} \frac{1}{r} \int \xi^{-2} q(\xi) \sin(t\xi) \sum_{j\geq 1} (r\xi)^{2j} \phi_j(r^2) \widehat{f_1}(\xi) \chi_k(\xi)\, d\xi$$
$$:= r^{2j-1}\phi_j(r^2) \sum_k \chi_{\lesssim 2^{-k}}(r) g^j_k(t) := r^{2j-1}\phi_j(r^2) g^j(r,t).$$

By stationary phase and the properties of q and $\widehat{f_1}$, we have

$$|g^j_k(r,t)| + 2^{-k}\big(|\partial_t g^j_k(r,t)| + |\partial_r g^j_k(r,t)|\big) \lesssim \frac{2^{(2j+1)k}}{\langle k^-\rangle^2 \langle 2^k t\rangle^N} 2^{-Nk^+}.$$

Summing up over k, we obtain

$$|g^j(r,t)| + \langle t+r\rangle\big(|\partial_r g^j(r,t)| + |\partial_t g^j(r,t)|\big) \lesssim \frac{1}{\langle t+r\rangle^{2j+1} \log^2\langle t+r\rangle}. \tag{5-14}$$

Hence, using the bound (3-2) for ϕ_j, we obtain a bound for $\bar{u}^{l,j}_{\text{low}}$, namely

$$|\bar{u}^{l,j}_{\text{low}}(r,t)| + |r\partial_r \bar{u}^{l,j}_{\text{low}}(r,t)| + \langle t+r\rangle |\partial_t \bar{u}^{l,j}_{\text{low}}(r,t)| \lesssim \frac{C^j}{j!} \frac{r^{2j-1}\log(1+r^2)}{\langle t+r\rangle^{2j+1}\log^2\langle t+r\rangle}. \tag{5-15}$$

Thus these contributions satisfy the bounds required of $\bar{u}^{l,nr}$.

We now turn our attention to the estimate (5-7), which applies in the region where $r \approx t$. By (5-6) (for $\bar{u}^l$) and (5-15), the contributions of the term $\bar{u}^l_{\text{low}}$ are all below the required threshold, so it remains to consider $\bar{u}^l_{\text{hi}}$. We have

$$\bar{u}^l_{\text{hi}}(r,t) = \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi)\xi^{-1}\phi_\xi(r)\Big(f_0(\xi)\cos(t\xi) + \frac{1}{\xi} f_1(\xi)\sin(t\xi)\Big)\, d\xi.$$

For ϕ_ξ, we use the representation (3-7), with ϕ^+_ξ as in (3-6),

$$\phi_\xi = r^{-1/2}\big(a(\xi)\sigma(r\xi,r)e^{ir\xi} + \bar{a}(\xi)\bar{\sigma}(r\xi,r)e^{-ir\xi}\big), \quad r\xi \gtrsim 1.$$

We notice that the operator $\partial_r + \partial_t$ kills the resonant factors $e^{\pm i(r-t)\xi}$. Precisely, we have

$$\Big(\partial_r + \partial_t + \frac{1}{2r}\Big)\phi_\xi(r)\sin(t\xi) = 2r^{-1/2}\Re\big(e^{i\xi(r+t)}\xi a(\xi)\sigma(r\xi,r)\big) + 2r^{-1/2}\Re\big(e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\big)\sin(t\xi),$$

and a similar computation where $\sin(t\xi)$ is replaced by $\cos(t\xi)$. This leads to

$$\begin{aligned}\Big(\partial_r + \partial_t + \frac{1}{2r}\Big)\bar{u}^l_{\text{hi}} &= \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-1/2}\Re\big(2\xi e^{i(r+t)\xi}a(\xi)\sigma(r\xi,r) + 2e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\cos(t\xi)\big)\frac{f_0(\xi)}{\xi}\, d\xi \\ &\quad + \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-1/2}\Re\big(2\xi e^{i(r+t)\xi}a(\xi)\sigma(r\xi,r) + 2e^{ir\xi}a(\xi)\partial_r\sigma(r\xi,r)\sin(t\xi)\big)\frac{f_1(\xi)}{\xi^2}\, d\xi.\end{aligned}$$

The two integrals above are treated as before, using stationary phase. The first term in each of the last integrals has a nonresonant phase; therefore each integration by parts gains a factor of $(\xi t)^{-1}$. Thus, taking (5-1) into account, their contributions can be estimated by

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) t^{-1/2} \xi (t\xi)^{-N} \frac{\xi^{5/2}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^3 \log t}.$$

The second term contains the expression $\partial_r \sigma(r\xi, r)$, which (see the description of σ in Section 3) brings an additional factor of $r^{-1}(r\xi)^{-1} \approx t^{-2}\xi^{-1}$. The contribution of the part with phase $e^{i\xi(r+t)}$ is better than above, while the contribution of the part with phase $e^{i\xi(r-t)}$ is of the form

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) a(\xi) t^{-1/2} t^{-1} (t\xi)^{-1} e^{i\xi(t-r)} \frac{\xi^{5/2}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^{5/2} \langle t-r\rangle^{1/2} \log\langle t-r\rangle},$$

as desired.

(b) We find u^{nl} from (5-4) using a fixed point argument in the Banach space Z^{nl} with norm

$$\|f\|_{Z^{nl}} = \|h_1^{-1} t^{1.5} f\|_{L^\infty}.$$

Denoting by Z^l the Banach space of functions of the form $\bar{u}^{l,r} + \bar{u}^{l,nr}$ with norm as in (5-5)–(5-6), we will show that the map

$$T : (u, v) \to L^{-1} N(u, v) = h_1(r) \int_r^\infty \frac{N(u, v)}{h_1(s)} ds$$

is locally Lipschitz from $Z^l \times Z^{nl}$ into Z^{nl}, and that in addition, the Lipschitz constant with respect to the second variable v can be made small if either both arguments are small or if u and v are in a bounded set B and the time t is large enough, depending on the size of B. This would imply the existence and uniqueness of $\bar{u}^{nl}$, as well as its Lipschitz dependence on $\bar{u}^l$ and, implicitly, on (w_0, w_1). Recall that

$$N(u, v) = \frac{1}{r}\Big[\sin Q \cdot \big(\cos(u+v) - 1\big) + \cos Q \cdot \big(\sin(u+v) - (u+v)\big)\Big]$$

Then

$$|N(u, v)| \lesssim \frac{1}{r^2+1}(|u|^2 + |v|^2) + \frac{1}{r}(|u|^3 + |v|^3),$$
$$|\nabla N(u, v)| \lesssim \frac{1}{r^2+1}(|u| + |v|) + \frac{1}{r}(|u|^2 + |v|^2).$$

Hence, it remains to show that

$$\int_0^\infty \frac{1}{r}(|u|^2 + |v|^2) + \frac{r^2+1}{r^2}(|u|^3 + |v|^3) dr \lesssim t^{-1.5}\big(\|u\|_{Z^l}^2 + \|v\|_{Z^{nl}}^2 + \|u\|_{Z^l}^3 + \|v\|_{Z^{nl}}^3\big).$$

For u, we have two components u^r and u^{nr}, and therefore we need to consider the six integrals

$$\int_0^\infty \frac{1}{r}|u^r|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{h_1^2(r)}{(t \log^2 t)^2} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2 \log^4 t}\|u\|_{Z^l}^2,$$
$$\int_0^\infty \frac{1}{r}|u^{nr}|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{r^2}{(t+r)^2 t \langle t-r\rangle^3 \log^2\langle t-r\rangle} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2}\|u\|_{Z^l}^2,$$
$$\int_0^\infty \frac{1}{r}|v|^2 dr \lesssim \int_0^\infty \frac{1}{r} h_1^2(r) t^{-3} dr \cdot \|v\|_{Z^{nl}}^2 \approx \frac{1}{t^3}\|v\|_{Z^{nl}}^2,$$

$$\int_0^\infty \frac{r^2+1}{r^2}|u^r|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}\frac{h_1^3(r)}{(t\log^2 t)^3}dr\cdot \|u\|_{Z^l}^3 \approx \frac{1}{t^3\log^6 t}\|u\|_{Z^l}^3,$$
$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}\frac{r^3}{(t+r)^3 t^{3/2}\langle t-r\rangle^{9/2}\log^3\langle t-r\rangle}dr\cdot \|u\|_{Z^l}^3 \approx \frac{1}{t^{1.5}}\|u\|_{Z^l}^3,$$
$$\int_0^\infty \frac{r^2+1}{r^2}|v|^3 dr \lesssim \int_0^\infty \frac{r^2+1}{r^2}h_1^3(r)t^{-4.5}dr\cdot \|v\|_{Z^{nl}}^3 \approx \frac{1}{t^{4.5}}\|v\|_{Z^{nl}}^3.$$

We remark that the worst decay $t^{-1.5}$ comes from the penultimate integral above; all other terms are better. Furthermore, this term comes solely from the u dependence of $N(u, v)$. Thus, with our choice of norms, the Lipschitz constant for $L^{-1}N(u, v)$ with respect to u cannot be made small by taking t large; however, the Lipschitz constant with respect to v does have a negative power of t in it.

The argument for $\partial_t\bar{u}^{nl}$ is more involved. Differentiating (5-3), we obtain

$$\begin{aligned} L\left(\partial_t\bar{u}^{nl}+\frac{h_1}{12}(\bar{u}^l)^3\right) &= N_u(\bar{u}^l,\bar{u}^{nl})\partial_t\bar{u}^l + N_v(\bar{u}^l,\bar{u}^{nl})\partial_t\bar{u}^{nl} + \frac{h_1}{12}\partial_r(\bar{u}^l)^3 \\ &= N_v(\bar{u}^l,\bar{u}^{nl})\left(\partial_t\bar{u}^{nl}+\frac{h_1}{12}(\bar{u}^l)^3\right) + \left[N_u(\bar{u}^l,\bar{u}^{nl}) - \frac{h_1}{4}(\bar{u}^l)^2\right]\partial_t\bar{u}^l \\ &\quad - \tfrac{1}{12}N_v(\bar{u}^l,\bar{u}^{nl})h_1(\bar{u}^l)^3 + \frac{h_1}{12}(\partial_t+\partial_r)(\bar{u}^l)^3. \end{aligned} \tag{5-16}$$

We interpret this as a linear equation for $w = \partial_t\bar{u}^{nl} + (h_1/12)(\bar{u}^l)^3$, namely,

$$Lw = N_v(\bar{u}^l,\bar{u}^{nl})w + N_1(\bar{u}^l,\bar{u}^{nl}).$$

The approach is similar to what we have done before. We adjust the base space to

$$\|f\|_{\tilde{Z}^{nl}} = \|h_1^{-1}t^2 f\|_{L^\infty}$$

and prove that $w \to L^{-1}(N_v(\bar{u}^l,\bar{u}^{nl})w)$ is bounded from $\tilde{Z}^{nl}$ to $\tilde{Z}^{nl}$ with small norm, and also Lipschitz with respect to $(\bar{u}^l,\bar{u}^{nl}) \in Z^l \times Z^{nl}$ (but not necessarily with small Lipschitz constant), and also that $L^{-1}N_1$ is Lipschitz from $Z^l \times Z^{nl}$ to $\tilde{Z}^{nl}$ (no smallness needed).

The first bound above follows from the previous computation. The main cancellation occurs in the first term in N_1, where the $(\bar{u}^l)^2$ term disappears. Precisely, we have

$$N_u(u,v) - \tfrac{1}{4}h_1u^2 = -\frac{2}{1+r^2}\sin(u+v) - \frac{1-r^2}{r(1+r^2)}\big(1-\cos(u+v)\big) - \frac{r}{2(1+r^2)}u^2,$$

and therefore

$$\left|N_u(u,v) - \tfrac{1}{4}h_1u^2\right| \lesssim \frac{1}{1+r^2}(|u|+|v|) + \frac{1}{r}\big(|u|^3+|u||v|+|v|^2\big) + \frac{1}{r(1+r^2)}|u|^2.$$

For $\partial_t\bar{u}^l$, we use the same bounds as for $\bar{u}^l$. Then, compared with the previous computation, we need to reestimate the terms involving $|u|^3$, $|u||v|$ and $|u|^2$. The resonant part of u yields better bounds, so we only estimate terms involving u^{nr}:

$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^4 dr \lesssim \|u\|_{Z^l}^4 \cdot \int_0^\infty \frac{r^2+1}{r^2}\frac{r^4}{(t+r)^4 t^2 \langle t-r\rangle^6 \log^4\langle t-r\rangle} dr \approx \frac{1}{t^2}\|u\|_{Z^l}^4,$$

$$\int_0^\infty \frac{r^2+1}{r^2}|u^{nr}|^2|v| dr \lesssim \|u\|_{Z^l}^2\|v\|_{Z^{nl}} \int_0^\infty \frac{r^2+1}{r^2}\frac{r^2}{(t+r)^2 t^{2.5}\langle t-r\rangle^3 \log^2\langle t-r\rangle} dr \approx \frac{1}{t^{2.5}}\|u\|_{Z^l}^2\|v\|_{Z^{nl}},$$

$$\int_0^\infty \frac{1}{r^2}|u^{nr}|^3 dr \lesssim \|u\|_{Z^l}^3 \cdot \int_0^\infty \frac{1}{r^2}\frac{r^3}{(t+r)^3 t^{1.5}\langle t-r\rangle^{4.5} \log^3\langle t-r\rangle} dr \approx \frac{1}{t^{3.5}}\|u\|_{Z^l}^4.$$

The third term on the right in (5-16) is better behaved than the second. Finally, for the last term in (5-16), we invoke (5-7) so that we use the same bounds for $(\partial_t+\partial_r)(\bar{u}^l)$ as for $r^{-1}\bar{u}^l$. Then the integral to estimate is

$$\int_0^\infty \frac{1}{r}|u|^3 dr \lesssim \frac{1}{t^{2.5}}\|u\|_{Z^l}^3.$$

(c) In the case of $\bar{u}^l$, this part follows from the linearity. In the case of $\bar{u}^{nl}$, the Lipschitz dependence on $\bar{u}^l$ has already been discussed above. An additional argument is required for $\delta\partial_t\bar{u}^{nl}$. However, nothing new happens there, and the details are left for the reader. □

6. The transition between γ and ε

In this section, we study the transition from γ to ε, which were both introduced in (2-13). This transition is described by (2-17), which we recall for convenience:

$$\gamma = \partial_r\varepsilon - \frac{\sin(\varepsilon+\bar{u}) - \sin\bar{u}}{r}.$$

The main result of this section is the following:

Proposition 6.1. (a) *Assume that $\gamma \in LX$ is small and $\bar{u}$, $\bar{w}$ are as in Proposition 5.2. Then for t large enough, there exists a unique solution $\varepsilon \in X$ of (2-17) which satisfies*

$$\|\varepsilon\|_X \lesssim_S \|\gamma\|_{LX}. \tag{6-1}$$

Furthermore, ε has a Lipschitz dependence on both γ and the linear data (w_0, w_1) for $\bar{w}$:

$$\|\delta\varepsilon\|_X \lesssim_S \|\delta\gamma\|_{LX} + \frac{1}{t\log^2 t}\|(\delta w_0, \delta w_1)\|_S\|\gamma\|_{LX}. \tag{6-2}$$

(b) *Also, if γ is a function of t, then*

$$\|\partial_t\varepsilon\|_X \lesssim_S \|\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\gamma\|_{LX}, \tag{6-3}$$

with the corresponding Lipschitz dependence

$$\|\delta\partial_t\varepsilon\|_X \lesssim_S \|\delta\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\delta\gamma\|_{LX} + \|(\delta w_0, \delta w_1)\|_S\Big(\|\partial_t\gamma\|_{LX} + \frac{1}{t\log^2 t}\|\gamma\|_{LX}\Big). \tag{6-4}$$

(c) *Assume in addition that $\gamma \in L^\infty$. Then*

$$|\varepsilon(r)| \lesssim_S r \log r \|\gamma\|_{LX\cap L^\infty}, \qquad r \ll 1, \tag{6-5}$$

with a similar Lipschitz dependence.

Proof. (a) Equation (2-17) is rewritten as

$$L\varepsilon = \gamma + \frac{\sin(\varepsilon+\bar{u}) - \sin\bar{u} - \cos Q\cdot\varepsilon}{r} := \gamma + F(\varepsilon, \bar{u}-Q). \tag{6-6}$$

Hence, in order to prove both (6-1) and (6-2), it suffices to show that at fixed large enough time, the map F is Lipschitz:

$$F : X \times (Z^l + Z^{nl}) \to LX,$$

with a small Lipschitz constant in the second variable. For the X norm, we use the embeddings (3-18)–(3-21). For the LX norm, we use (3-23), which shows that it is enough to estimate $F(\bar{u}, \varepsilon)$ in $L^1 \cap L^2$. We expand F as follows:

$$\begin{aligned} F(\beta, v) &= \frac{\sin(\beta+Q+v) - \sin(Q+v) - \cos Q\cdot\beta}{r} \\ &= \frac{(\cos(Q+v)-\cos Q)\cdot\beta}{r} - \frac{\sin(Q+v)\cdot\beta^2}{2r} + \frac{O(\beta^3)}{r} \\ &= -\frac{\sin Q\cdot v\beta}{r} - \frac{\sin Q\cdot\beta^2}{2r} + \frac{O(v^2\beta)}{r} + \frac{O(\beta^3)}{r}. \end{aligned}$$

Hence

$$|F(\beta, v)| \lesssim \frac{|v||\beta|}{1+r^2} + \frac{|\beta|^2}{1+r^2} + \frac{|\beta|^3}{r} + \frac{|v|^2|\beta|}{r}. \tag{6-7}$$

By using (3-20), (3-18) and (5-6), we bound this first in L^2,

$$\begin{aligned} \|F(\beta, v)\|_{L^2} &\lesssim \left\|\frac{\beta}{\log(1+r)}\right\|_{L^2}\left(\|\beta\|_{L^\infty} + \|\beta\|_{L^\infty}^2 + \left\|\frac{v}{1+r}\right\|_{L^\infty} + \left\|\frac{v^2\log(1+r)}{r}\right\|_{L^\infty}\right) \\ &\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X\left(\frac{1}{t\log^2 t}\|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^2}\|v\|_{Z^l+Z^{nl}}^2\right), \end{aligned}$$

and then in L^1,

$$\begin{aligned} \|F(\beta, v)\|_{L^1} &\lesssim \left\|\frac{\beta}{\log(1+r)}\right\|_{L^2}^2 (1+\|\beta\|_{L^\infty}) \\ &\quad + \left\|\frac{\beta}{\log(1+r)}\right\|_{L^2}\left(\left\|\frac{v\log(1+r)}{1+r^2}\right\|_{L^2} + \left\|\frac{v^2\log(1+r)}{r}\right\|_{L^2}\right) \\ &\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X\left(\frac{1}{t\log^2 t}\|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{3/2}}\|v\|_{Z^l+Z^{nl}}^2\right). \end{aligned}$$

Hence we obtain

$$\|F(\beta, v)\|_{LX} \lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t \log^2 t} \|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{3/2}} \|v\|_{Z^l+Z^{nl}}^2 \right).$$

A similar analysis yields

$$\|\beta_1 F_\beta(\beta, v)\|_{LX} \lesssim \|\beta_1\|_X \left(\|\beta\|_X + \|\beta\|_X^2 + \frac{1}{t \log^2 t} \|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{3/2}} \|v\|_{Z^l+Z^{nl}}^2 \right),$$

$$\|v_1 F_v(\beta, v)\|_{LX} \lesssim \|v_1\|_{Z^l+Z^{nl}} \|\beta\|_X \left(\frac{1}{t \log^2 t} + \frac{\log t}{t^{3/2}} \|v\|_{Z^l+Z^{nl}} \right).$$

By the contraction principle, this proves both (6-1) and (6-2). The time decaying factors guarantee that for any size of $\bar{u} - Q$, the problem can be solved for large enough time.

(b) To prove (6-3), we differentiate with respect to t in (6-6):

$$L \partial_t \varepsilon = \partial_t \gamma + F_\varepsilon(\varepsilon, \bar{u}) \partial_t \varepsilon + F_{\bar{u}}(\varepsilon, \bar{u}) \partial_t \bar{u}.$$

Since $\partial_t \bar{u}$ satisfies the same pointwise bounds as $\bar{u}$, the last two estimates above show that the contraction principle still applies.

(c) Due to the embedding $X \subset \dot{H}^1_e \subset L^\infty$, we already have a small uniform bound for ε. We solve the ODE (6-6) in [0, 1] with Cauchy data at $r = 1$. Making the bootstrap assumption

$$|\varepsilon| \le M r \left| \log \frac{r}{2} \right|, \tag{6-8}$$

we rewrite (6-6) in the form

$$|L\varepsilon - \gamma| \le M^3 r^2 \left| \log^3 \frac{r}{2} \right| + C, \qquad C \approx_S \|\varepsilon\|_{L^\infty}.$$

Then solving the linear L evolution, we have

$$|\varepsilon| \lesssim r(|\gamma(1)| + M^3) + Cr \left| \log \frac{r}{2} \right| \lesssim_S M^3 r + r \left| \log \frac{r}{2} \right| \|\varepsilon\|_{L^\infty}.$$

If $\|\varepsilon\|_{L^\infty}$ is sufficiently small, then we can choose M small enough that the above bound is stronger than our bootstrap assumption (6-8). The proof of (6-5) is concluded. □

7. Perturbative analysis in the γ equation

Our main goal is to solve (2-16) for γ with zero Cauchy data at $t = \infty$. Using the backward linear parametrix K introduced in (4-2), Equation (2-16) is rewritten in the form

$$\gamma = K N(\bar{u} + \varepsilon, \bar{w} + \gamma), \tag{7-1}$$

where the auxiliary function ε and its time derivative ε_t are uniquely determined by γ and γ_t via Proposition 6.1.

Our strategy is to solve (7-1) using the contraction principle in the space $\boldsymbol{E}$ with norm

$$\|\gamma\|_{\boldsymbol{E}} = \sup_{t > t_0} t^{1.5} \|\gamma\|_{LX} + t^{2.5} \big(\|\partial_t \gamma\|_{LX} + \|\gamma\|_{\dot{H}^1_e} \big),$$

for a suitably chosen t_0. By Proposition 6.1, this yields control for ε in the space $\boldsymbol{G}$ with norm

$$\|\varepsilon\|_{\boldsymbol{G}} = \sup_{t>t_0} t^{1.5}\big(\|\varepsilon\|_X + \|r^{-1/2}\varepsilon\|_{L^\infty}\big) + t^{2.5}\|\partial_t\varepsilon\|_{LX}.$$

For the linear $\tilde{H}$ wave equation, we use the LX bounds in Lemma 4.1 with $\alpha = 1.5$. Thus we need to estimate the nonlinearity $N(\bar{u}+\varepsilon, \bar{w}+\gamma)$ in the space $\boldsymbol{N}$ with norm

$$\|N\|_{\boldsymbol{N}} = \sup_{t>t_0} t^{3.5}\|N(t)\|_{LX}.$$

Finally, all the implicit constants in our estimates depend on $\|(w_0, w_1)\|_S$ and need not be small. Thus we need a different source of smallness, which is an additional time decay factor, incorporated in the stronger norm $\boldsymbol{N}^\sharp$ defined by

$$\|N\|_{\boldsymbol{N}^\sharp} = \sup_{t>t_0} t^{3.5}(\log t)^2\|N(t)\|_{LX}.$$

With this notation, our main estimates for the nonlinearity $N(\bar{u}+\varepsilon, \bar{w}+\gamma)$ are as follows:

Proposition 7.1. *Assume that the Schwartz functions (w_0, w_1) satisfy the nonresonance conditions (2-11). Then:*

(a) *The map $(w_0, w_1) \to N(\bar{u}, \bar{w})$ is locally Lipschitz from S to $\boldsymbol{N}$.*

(b) *The map $(w_0, w_1, \gamma, \varepsilon) \to N(\bar{u}+\varepsilon, \bar{w}+\gamma) - N(\bar{u}, \bar{w})$ is locally Lipschitz from $S \times \boldsymbol{E} \times \boldsymbol{G}$ to $\boldsymbol{N}^\sharp$.*

In view of Lemma 4.1 and Proposition 6.1, the above result allows us to solve (7-1) for γ in the ball

$$B = \{\|\gamma - KN(\bar{u}, \bar{w})\|_{\boldsymbol{E}}\},$$

for $t > t_0$, via the contraction principle, provided that t_0 is chosen to be sufficiently large. This concludes the proof of Theorem 2.1.

We note that in terms of time decay we gain only logarithms, whereas the implicit constants in our estimates are all polynomial in $\|(w_0, w_1)\|_S$. This implies that for large Schwartz data (w_0, w_1) in the linear equation, our solutions are only defined for $t > T$, with T exponentially large.

Proof of Proposition 7.1. We recall that N is given by

$$N(w, u) = \frac{2(\cos Q - \cos u)}{r^2} w + \frac{1}{r}\sin u(u_t^2 - w^2).$$

We split the difference $N(\bar{w}+\gamma, \bar{u}+\varepsilon) - N(\bar{w}, \bar{u})$ as

$$N(w, u) - N(\bar{w}, \bar{u}) = N^l(\bar{w}, \bar{u}, \gamma, \varepsilon) + N^n(\bar{w}, \bar{u}, \gamma, \varepsilon).$$

The term N^l contains the linear contributions in ε, γ in the difference $N(w, u) - N(\bar{w}, \bar{u})$:

$$N^l = \frac{2(\cos Q - \cos \bar{u})}{r^2}\gamma + \frac{2\sin\bar{u}\cdot\varepsilon}{r^2}\bar{w} + \frac{\sin\bar{u}(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma) + \cos\bar{u}\cdot\varepsilon(\bar{u}_t^2 - \bar{w}^2)}{r}.$$

The remaining term N^n contains the genuinely nonlinear contributions in ε, γ in the difference $N(w,u)-N(\bar{w},\bar{u})$:

$$N^n = \frac{2(\cos\bar{u}-\cos u-\sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w}+\frac{2(\cos\bar{u}-\cos(\bar{u}+\varepsilon))}{r^2}\gamma+\frac{\sin\bar{u}(\varepsilon_t^2-\gamma^2)}{r} + \frac{(\sin u-\sin\bar{u})(2\bar{u}_t\varepsilon_t-2\bar{w}\gamma+\bar{u}_t^2-\bar{w}^2)}{r}+\frac{(\sin u-\sin\bar{u}-\cos\bar{u}\cdot\varepsilon)(\bar{u}_t^2-\bar{w}^2)}{r}.$$

We will consider separately the expressions $N(\bar{u},\bar{w})$, N^l and N^n.

The term $N(\bar{w},\bar{u})$. Our main goal here is to prove the estimate

$$\|N(\bar{w},\bar{u})\|_{LX}\lesssim_S t^{-3.5}. \tag{7-2}$$

We also need to show that $N(\bar{w},\bar{u})$ has a Lipschitz dependence on (w_0,w_1). However, as the leading order part of $N(\bar{w},\bar{u})$ is multilinear, the proof of that follows the same lines as below and is omitted.

To establish (7-2), we split

$$N(\bar{w},\bar{u})=\chi_{r\ll t}N(\bar{w},\bar{u})+\chi_{r\gg t}N(\bar{w},\bar{u})+\chi_{r\approx t}N(\bar{w},\bar{u})=N_1+N_2+N_3.$$

For the first two terms, it suffices to use a direct estimate:

$$|N(\bar{w},\bar{u})|\lesssim\frac{\sin Q}{r^2}|\bar{u}-Q||\bar{w}|+\frac{1}{r^2}|\bar{u}-Q|^2|\bar{w}|+\frac{1}{r}(\sin Q+|\bar{u}|)(|\bar{u}_t|^2+|\bar{w}|^2).$$

Using the bounds (5-6) and (5-8) for $\bar{u}-Q$, as well as the bound (5-2) for $\bar{w}$, this gives

$$|N_1(\bar{w},\bar{u})|\lesssim_S\chi_{r\ll t}\frac{1}{\langle r\rangle^4 t^4},$$

where the leading contribution comes from $u^{l,r}$. This implies that

$$\|N_1\|_{L^1\cap L^2}\lesssim_S t^{-4},$$

which suffices for (7-2) in view of the embedding (3-23). Similarly,

$$|N_2|\lesssim_S\chi_{r\gg t}\frac{1}{\langle r\rangle^8},$$

which also gives

$$\|N_2\|_{L^1\cap L^2}\lesssim_S t^{-4}.$$

However, a similar direct computation for N_3 only gives

$$|N_3(\bar{w},\bar{u})|\lesssim_S\chi_{r\sim t}\frac{1}{t^{2.5}\langle t-r\rangle^{5.5}},$$

which fails by two units,

$$\|N_3\|_{L^1\cap L^2}\lesssim_S t^{-1.5}.$$

Hence, in order to conclude the proof of (7-2), we need to better exploit the structure of N and capture a double cancellation on the null cone. In the computations below (through the end of the subsection), we work in the regime $r \approx t$. We expand $N(\bar{w}, \bar{u})$ as

$$N(\bar{w}, \bar{u}) = 2\frac{\sin Q}{r^2}(\bar{u}-Q)\bar{w} + \frac{\cos Q}{r^2}(\bar{u}-Q)^2\bar{w} + \frac{\sin Q}{r}(\bar{u}_t^2 - \bar{w}^2) + \frac{\cos Q}{r}(\bar{u}_t^2 - \bar{w}^2)(\bar{u}-Q)$$
$$+ \frac{\sin Q}{r^2} w\, O((\bar{u}-Q)^3) + \frac{\cos Q}{r^2} w\, O((\bar{u}-Q)^4)$$
$$+ \frac{\sin Q}{r}(\bar{u}_t^2 - \bar{w}^2) O((\bar{u}-Q)^2) + \frac{\cos Q}{r}(\bar{u}_t^2 - \bar{w}^2) O((\bar{u}-Q)^3).$$

The terms on the second line are already acceptable; i.e., it can be estimated by $t^{-4.5}\langle t-r\rangle^{-3.5}$. For further progress, we observe that by (5-8) we have

$$\bar{u}^{nl} = O_S(t^{-2.5}), \quad \partial_t \bar{u}^{nl} = O_S\big(t^{-2.5}\langle t-r\rangle^{-0.5}\big),$$

and that by (5-7), we can write

$$\partial_t \bar{u} + \bar{w} = \partial_t \bar{u}^{nl} + \partial_t \bar{u}^l + \partial_r \bar{u}^l + \frac{\cos Q}{r}\bar{u}^l = O_S\big(t^{-1.5}\langle t-r\rangle^{-1.5}\big). \tag{7-3}$$

The first relation above allows us to dispense with $\bar{u}^{nl}$ everywhere and replace $\bar{u} - Q$ by $\bar{u}^l$, and the second allows us to estimate the third line in $N(\bar{w}, \bar{u})$. We are left with

$$N(\bar{w}, \bar{u}) = 2\frac{\sin Q}{r^2}\bar{u}^l\bar{w} + \frac{\cos Q}{r^2}(\bar{u}^l)^2\bar{w} + \frac{\sin Q}{r}\big((\bar{u}_t^l)^2 - \bar{w}^2\big) + \frac{\cos Q}{r}\big((\bar{u}_t^l)^2 - \bar{w}^2\big)\bar{u}^l + O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big).$$

To advance further, we substitute $\bar{w} = \partial_r \bar{u}^l - (\cos Q/r)\bar{u}^l$ everywhere. The $(\cos Q/r)\bar{u}^l$ is acceptable in the first two terms of N, that is, it gives contributions of $O_S(t^{-4.5}\langle t-r\rangle^{-3.5})$, and we discard it. For the last two terms, we use the better approximation from (5-7):

$$\bar{u}_t^l = -\partial_r \bar{u}^l - \frac{1}{2r}\bar{u}^l + O\big(t^{-2.5}\langle t-r\rangle^{-0.5}\big).$$

Then we can write

$$(\bar{u}_t^l)^2 - \bar{w}^2 = \Big(\partial_r \bar{u}^l + \frac{1}{2r}\bar{u}^l\Big)^2 - \Big(\partial_r \bar{u}^l - \frac{\cos Q}{r}\bar{u}^l\Big)^2 + O_S\big(t^{-3}\langle t-r\rangle^{-3}\big)$$
$$= -\frac{1}{r}\bar{u}^l\partial_r \bar{u}^l + O_S\big(t^{-3}\langle t-r\rangle^{-3}\big).$$

It is also harmless to replace $\sin Q$ by r^{-1} and $\cos Q$ by -1 everywhere. Returning to N, we obtain

$$N(\bar{w}, \bar{u}) = \frac{2}{r^3}\bar{u}^l\partial_r \bar{u}^l - \frac{1}{r^2}(\bar{u}^l)^2\partial_r \bar{u}^l - \frac{1}{r^3}u^l\partial_r u^l + \frac{1}{r^2}(\bar{u}^l)^2\partial_r \bar{u}^l + O_S\big(\langle t\rangle^{-4.5}\langle t-r\rangle^{3.5}\big)$$
$$= \frac{1}{2r^3}\partial_r(\bar{u}^l)^2 + O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big)$$

in the region $r \approx t$, which we rewrite as

$$N_3 = Lg + \chi_{r\approx t} O_S\big(t^{-4.5}\langle t-r\rangle^{-3.5}\big), \quad g = \chi_{r\approx t}\frac{1}{2r^3}(\bar{u}^l)^2.$$

The last term can be directly estimated in $L^1 \cap L^2$. For the leading term Lg, we estimate g in H^1_e and use the embedding (3-18). We have

$$|g| \lesssim_S \frac{1}{t^4 \langle t-r\rangle^3}, \qquad |\partial_r g| \lesssim_S \frac{1}{t^4 \langle t-r\rangle^4},$$

and therefore

$$\|g\|_{H^1_e} \lesssim_S \frac{1}{t^{3.5}}.$$

This concludes the proof of (7-2).

The bound for N^l. Our goal here is to establish the bound

$$\|N^l(t)\|_{LX} \lesssim_S \frac{1}{t^{3.5}\log^2 t}\big(\|\gamma\|_{\boldsymbol{G}} + \|\varepsilon\|_{\boldsymbol{E}}\big). \tag{7-4}$$

The proof of the Lipschitz dependence on (w_0, w_1) is again similar and therefore omitted.

We recall that

$$N^l = \frac{2(\cos Q - \cos \bar u)}{r^2}\gamma + \frac{2\sin \bar u \cdot \varepsilon}{r^2}\bar w + \frac{\sin \bar u(2\bar u_t \varepsilon_t - 2\bar w \gamma) + \cos \bar u \cdot \varepsilon(\bar u_t^2 - \bar w^2)}{r}.$$

The pointwise estimate

$$\left|\frac{2(\cos Q - \cos \bar u)}{r}\right| \lesssim \frac{1}{r^2+1}|\bar u - Q| + \frac{1}{r}|\bar u - Q|^2,$$

combined with the pointwise bounds for $\bar u$ from (5-6), leads to

$$\left\|\frac{2(\cos Q - \cos \bar u)}{r}\right\|_{L^\infty \cap L^2} \lesssim_S \frac{1}{t\log^2 t},$$

with the worst contribution arising from the resonant part of $\bar u$. From (3-23), it follows that

$$\left\|\frac{2(\cos Q - \cos \bar u)}{r^2}\gamma\right\|_{LX} \lesssim \left\|\frac{2(\cos Q - \cos \bar u)}{r}\right\|_{L^\infty \cap L^2} \cdot \left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim_S \frac{1}{t^{3.5}\log^2 t}\|\gamma\|_{\boldsymbol{G}}.$$

Next, from (5-6) and (5-2), it follows that

$$\left\|\frac{\bar u \cdot \bar w}{r^2}\log(2+r)\right\|_{L^\infty \cap L^2} \lesssim_S \frac{\log t}{t^{2.5}},$$

which, combined with

$$\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2} \lesssim \|\varepsilon\|_X \lesssim t^{-1.5}\|\varepsilon\|_{\boldsymbol{E}}$$

(recall (3-20)), gives

$$\left\|\frac{2\sin \bar u \cdot \varepsilon}{r^2}\bar w\right\|_{LX} \lesssim_S \frac{\log t}{t^4}\|\varepsilon\|_{\boldsymbol{E}}.$$

Using (5-6), we obtain

$$\left\|\frac{\bar u \bar u_t}{r}\log(2+r)\right\|_{L^\infty \cap L^2} \lesssim_S \frac{\log t}{t^{1.5}},$$

and therefore, by invoking (3-23) and (3-20), it follows that

$$\left\|\frac{\sin(\bar u)\cdot\bar u_t\varepsilon_t}{r}\right\|_{LX}\lesssim\left\|\frac{\bar u\bar u_t}{r}\log(2+r)\right\|_{L^\infty\cap L^2}\left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2}\lesssim_S\frac{\log t}{t^4}\|\varepsilon\|_{\boldsymbol{E}}.$$

The following term in N^l requires some extra work. Using (5-6) and (5-2), we note that away from the cone, we have $|\sin(\bar u)|\lesssim\sin Q$, and continue with

$$\left\|\chi_{r\not\approx t}\frac{\bar w\sin\bar u}{r}\right\|_{L^1\cap L^2}\lesssim_S t^{-2},$$

followed by

$$\left\|\chi_{r\not\approx t}\frac{\sin(\bar u)\cdot\bar w\gamma}{r}\right\|_{LX}\lesssim\left\|\chi_{r\not\approx t}\frac{\bar u\bar w}{r}\right\|_{L^1\cap L^2}\|\gamma\|_{L^\infty}\lesssim_S t^{-4.5}\|\gamma\|_{\boldsymbol{G}}.$$

Near the cone, we write

$$\begin{aligned}\chi_{r\approx t}\frac{\bar w\sin\bar u}{r}&=\chi_{r\approx t}\biggl(\frac{2\bar w}{1+r^2}-\frac{\bar w(\bar u-Q)}{r}\cos Q+\frac{\bar w\,O((\bar u-Q)^2)}{1+r^2}+\frac{\bar w\,O((\bar u-Q)^3)}{r}\biggr)\\&=\chi_{r\approx t}\frac{\bar w(\bar u-Q)}{r}+O_S\bigl(t^{-2.5}\langle t-r\rangle^{-2.5}\bigr)\\&=L\bigl(\chi_{r\approx t}r^{-1}(\bar u^l)^2\bigr)+O_S\bigl(t^{-2.5}\langle t-r\rangle^{-2.5}\bigr).\end{aligned}$$

The output of the second term is estimated as above in $L^1\cap L^2$, and yields a contribution of $t^{-4}\|\varepsilon\|_{\boldsymbol{E}}$ to the $\|N^l\|_{LX}$ bound. For the first term, we write its contribution to N^l in the form

$$L\bigl(\chi_{r\approx t}r^{-1}(\bar u^l)^2\bigr)\gamma=L\bigl(\chi_{r\approx t}r^{-1}(\bar u^l)^2\gamma\bigr)+\chi_{r\approx t}r^{-1}(\bar u^l)^2\partial_r\gamma.$$

Then, using (3-18) for the first term and (3-23) for the second term, we have

$$\begin{aligned}\bigl\|L\bigl(\chi_{r\approx t}r^{-1}(\bar u^l)^2\bigr)\gamma\bigr\|_{LX}&\lesssim\bigl\|\chi_{r\approx t}r^{-1}(\bar u^l)^2\gamma\bigr\|_{H^1_e}+\bigl\|\chi_{r\approx t}r^{-1}(\bar u^l)^2\partial_r\gamma\bigr\|_{L^1\cap L^2}\\&\lesssim\bigl\|\chi_{r\approx t}r^{-1}(\bar u^l)^2\bigr\|_{H^1_e}\|\gamma\|_{\dot H^1_e}+\bigl\|\chi_{r\approx t}r^{-1}(\bar u^l)^2\bigr\|_{L^2\cap L^\infty}\|\partial_r\gamma\|_{L^2}\\&\lesssim_S t^{-1.5}\|\gamma\|_{\dot H^1_e}\lesssim_S t^{-4}\|\gamma\|_{\boldsymbol{G}}.\end{aligned}$$

It remains to bound the last term in N^l. For this, we take advantage of the first-order cancellation on the cone in the expression $\bar u_t-\bar w$ (see (7-3)), which, combined with (5-6) and (5-2), gives

$$\left\|\frac{\cos\bar u(\bar u_t^2-\bar w^2)\log(2+r)}{r}\right\|_{L^2\cap L^\infty}\lesssim_S\frac{\log t}{t^{2.5}}.$$

This leads to

$$\left\|\varepsilon\frac{\cos\bar u(\bar u_t^2-\bar w^2)}{r}\right\|_{L^1\cap L^2}\lesssim_S\frac{\log t}{t^{2.5}}\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2}\lesssim_S\frac{\log t}{t^{2.5}}\|\varepsilon\|_X\lesssim_S\frac{\log t}{t^4}\|\varepsilon\|_{\boldsymbol{E}}.$$

This concludes the proof of the N^l bound (7-4).

The bound for N^n. Our goal here will be to prove the bound

$$\|N^n\|_{LX} \lesssim_S \frac{\log t}{t^4}(M^2+M^3), \qquad M = \|\gamma\|_{\boldsymbol{G}} + \|\varepsilon\|_{\boldsymbol{E}}, \tag{7-5}$$

which is almost $t^{-.5}$ better than what we need. The corresponding Lipschitz dependence argument is similar and thus omitted. We recall the expression of N^n:

$$N^n = \frac{2(\cos\bar{u}-\cos u-\sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w} + \frac{2(\cos\bar{u}-\cos(\bar{u}+\varepsilon))}{r^2}\gamma + \frac{\sin u(\varepsilon_t^2-\gamma^2)}{r} + \frac{(\sin u-\sin\bar{u})(2\bar{u}_t\varepsilon_t-2\bar{w}\gamma)}{r} + \frac{(\sin u-\sin\bar{u}-\cos\bar{u}\cdot\varepsilon)(\bar{u}_t^2-\bar{w}^2)}{r}.$$

We successively consider the terms on the right. For the first one, we start with

$$\left|\frac{2(\cos\bar{u}-\cos u-\sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w}\right| \lesssim \frac{\varepsilon^2|\bar{w}|}{r^2}.$$

Then, using (5-2) and (3-20), we obtain

$$\left\|\frac{\varepsilon^2\bar{w}}{r^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^\infty\cap L^2}\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2}\left\|\frac{\bar{w}}{r^2}\log^2(2+r)\right\|_{L^\infty} \lesssim_S \frac{\log^2 t}{t^{5.5}}M^2.$$

The second term in N^n is estimated by

$$\left|\frac{\cos\bar{u}-\cos(\bar{u}+\varepsilon)}{r^2}\gamma\right| \lesssim \frac{|\sin\bar{u}\cdot\varepsilon\gamma|}{r^2} + \frac{|\varepsilon^2\gamma|}{r^2} \lesssim \frac{|\varepsilon\gamma|}{r\langle r\rangle^2} + \frac{|(\bar{u}-Q)\varepsilon\gamma|}{r^2} + \frac{|\varepsilon^2\gamma|}{r^2}.$$

The first two terms can be estimated in $L^1\cap L^2$ as before:

$$\left\|\frac{\varepsilon\gamma}{r\langle r\rangle^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\gamma}{r}\right\|_{L^2}\left\|\frac{\varepsilon}{\langle r\rangle^2}\right\|_{L^\infty\cap L^2} \lesssim_S t^{-4}M^2,$$

$$\left\|\frac{(\bar{u}-Q)\varepsilon\gamma}{r^2}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\gamma}{r}\right\|_{L^2}\left\|\frac{\bar{u}-Q}{r}\right\|_{L^2\cap L^\infty}\|\varepsilon\|_{L^\infty} \lesssim_S t^{-5}M^2.$$

For the last term, we first get the L^1 bound

$$\left\|\frac{\varepsilon^2\gamma}{r^2}\right\|_{L^1} \lesssim \|\varepsilon\|_{L^\infty}\left\|\frac{\varepsilon}{r}\right\|_{L^2}\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3.$$

However, getting the L^2 bound is more delicate:

$$\left\|\frac{\varepsilon^2\gamma}{r^2}\right\|_{L^2} \lesssim \left\|\frac{\varepsilon}{\sqrt{r}}\right\|_{L^\infty}^2\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3,$$

where the pointwise bound for $\varepsilon/\sqrt{r}$ near $r=0$ comes from (6-5).

The third term in N is estimated by using (5-6):

$$\left|\frac{\sin u(\varepsilon_t^2-\gamma^2)}{r}\right| \lesssim \frac{|\varepsilon_t|^2}{1+r} + \frac{|\gamma^2|}{1+r}.$$

We successively consider all terms:

$$\begin{aligned}\left\|\frac{|\varepsilon_t|^2}{1+r}\right\|_{L^1\cap L^2} &\lesssim \left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2\cap L^\infty}\left\|\frac{\varepsilon_t}{\log(2+r)}\right\|_{L^2} \lesssim \frac{1}{t^5}M^2,\\ \left\|\frac{|\gamma|^2}{1+r}\right\|_{L^1\cap L^2} &\lesssim \|\gamma\|_{L^2\cap L^\infty}\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim \frac{1}{t^4}M^2.\end{aligned}$$

Next we estimate the fourth term in N^n:

$$\left|\frac{(\sin u-\sin\bar u)(2\bar u_t\varepsilon_t-2\bar w\gamma)}{r}\right| \lesssim \frac{|\varepsilon|\big(|\bar u_t\varepsilon_t|+|\bar w\gamma|\big)}{r}.$$

On behalf of (5-2), (5-6) and (3-20), we have

$$\begin{aligned}\left\|\frac{\varepsilon\bar u_t\varepsilon_t}{r}\right\|_{L^1\cap L^2} &\lesssim \|\varepsilon_t\|_{L^\infty}\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2}\left\|\frac{\bar u_t}{r}\log(2+r)\right\|_{L^\infty\cap L^2} \lesssim_S \frac{\log t}{t^4}M^2,\\ \left\|\frac{\varepsilon\bar w\gamma}{r}\right\|_{L^1\cap L^2} &\lesssim \|\varepsilon\|_{L^\infty}\|\bar w\|_{L^2\cap L^\infty}\left\|\frac{\gamma}{r}\right\|_{L^2} \lesssim_S t^{-4}M^2.\end{aligned}$$

Finally we consider the last term in N^n,

$$\left|\frac{(\sin u-\sin\bar u-\cos\bar u\cdot\varepsilon)(\bar u_t^2-\bar w^2)}{r}\right| \lesssim \frac{\varepsilon^2(\bar u_t^2+\bar w^2)}{r},$$

which, by using (5-2), (5-6) and (3-20), we further bound as follows:

$$\left\|\frac{\varepsilon^2(\bar u_t^2+\bar w^2)}{r}\right\|_{L^1\cap L^2} \lesssim \left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2}\left\|\frac{\varepsilon}{\log(2+r)}\right\|_{L^2\cap L^\infty}\left\|\frac{\bar u_t^2+\bar w^2}{r}\log^2(2+r)\right\|_{L^\infty} \lesssim_S \frac{\log^2 t}{t^5}M^2. \qquad \square$$

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