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SOME MINIMIZATION PROBLEMS IN THE CLASS OF CONVEX FUNCTIONS WITH PRESCRIBED DETERMINANT

NAM Q. LE AND OVIDIU SAVIN

We consider minimizers of linear functionals of the type

$$L(u) = \int_{\partial\Omega} u \, d\sigma - \int_{\Omega} u \, dx$$

in the class of convex functions u with prescribed determinant $\det D^2 u = f$.

We obtain compactness properties for such minimizers and discuss their regularity in two dimensions.

1. Introduction

In this paper, we consider minimizers of certain linear functionals in the class of convex functions with prescribed determinant. We are motivated by the study of convex minimizers u for convex energies E of the type

$$E(u) = \int_{\Omega} F(\det D^2 u) \, dx + L(u), \quad \text{with } L \text{ a linear functional,}$$

which appear in the work of Donaldson [2002; 2009] in the context of existence of Kähler metrics of constant scalar curvature for toric varieties. The minimizer u solves a fourth-order elliptic equation with two nonstandard boundary conditions involving the second- and third-order derivatives of u (see (1-4) below). In this paper, we consider minimizers of L (or E) in the case when the determinant $\det D^2 u$ is prescribed. This allows us to understand better the type of boundary conditions that appear in such problems and to obtain estimates also for unconstrained minimizers of E.

The simplest minimization problem with prescribed determinant which is interesting in its own right is

$$\text{minimize } \int_{\partial\Omega} u \, d\sigma, \quad \text{with } u \in \mathscr{A}_0,$$

where Ω is a bounded convex set, $d\sigma$ is the surface measure of $\partial\Omega$, and $\mathscr{A}_0$ is the class of nonnegative solutions to the Monge–Ampère equation $\det D^2 u = 1$:

$$\mathscr{A}_0 := \left\{ u : \bar{\Omega} \to [0, \infty) \mid u \text{ convex, } \det D^2 u = 1 \right\}.$$

Question. *Is the minimizer u smooth up to the boundary $\partial\Omega$ if Ω is a smooth, say uniformly convex, domain?*

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In the present paper, we answer this question affirmatively in dimensions $n = 2$. First, we remark that the minimizer must vanish at x_0, the center of mass of $\partial\Omega$:

$$x_0 = \fint_{\partial\Omega} x \, d\sigma.$$

This follows easily since

$$u(x) - u(x_0) - \nabla u(x_0)(x - x_0) \in \mathcal{A}_0$$

and

$$\int_{\partial\Omega} \big[u(x) - u(x_0) - \nabla u(x_0)(x - x_0)\big] \, d\sigma = \int_{\partial\Omega} [u - u(x_0)] \, d\sigma \leq \int_{\partial\Omega} u \, d\sigma,$$

with strict inequality if $u(x_0) > 0$. Thus we can reformulate the problem above as minimizing

$$\int_{\partial\Omega} u \, d\sigma - \mathcal{H}^{n-1}(\partial\Omega) u(x_0)$$

in the set of all solutions to the Monge–Ampère equation $\det D^2 u = 1$ which are not necessarily nonnegative. This formulation is more convenient since we can now perturb functions in all directions.

More generally, we consider linear functionals of the type

$$L(u) = \int_{\partial\Omega} u \, d\sigma - \int_{\Omega} u \, dA,$$

with $d\sigma$, dA nonnegative Radon measures supported on $\partial\Omega$ and Ω respectively. In this paper, we study the existence, uniqueness and regularity properties for minimizers of L, that is,

$$\text{minimize } L(u) \text{ for all } u \in \mathcal{A} \tag{P}$$

in the class $\mathcal{A}$ of subsolutions (solutions) to a Monge–Ampère equation $\det D^2 u \geq f$:

$$\mathcal{A} := \big\{u : \overline{\Omega} \to \mathbb{R} \;\big|\; u \text{ convex}, \ \det D^2 u \geq f\big\}.$$

Notice that we are minimizing a linear functional L over a convex set $\mathcal{A}$ in the cone of convex functions.

Clearly, the minimizer of the problem (P) satisfies $\det D^2 u = f$ in Ω. Otherwise we can find $v \in \mathcal{A}$ such that $v = u$ in a neighborhood of $\partial\Omega$, and $v \geq u$ in Ω with strict inequality in some open subset, and thus $L(v) < L(u)$.

We assume throughout that the following 5 conditions are satisfied:

(1) Ω is a bounded, uniformly convex, $C^{1,1}$ domain.

(2) f is bounded away from 0 and ∞.

(3) $d\sigma = \sigma(x) \, d\mathcal{H}^{n-1} \lfloor \partial\Omega$, with the density $\sigma(x)$ bounded away from 0 and ∞.

(4) $dA = A(x) \, dx$ in a small neighborhood of $\partial\Omega$, with the density $A(x)$ bounded from above.

(5) $L(u) > 0$ for all u convex but not linear.

The last condition is known as the *stability* of L (see [Donaldson 2002]), and in two dimensions, is equivalent to saying that for all linear functions l, we have

$$L(l) = 0 \quad \text{and} \quad L(l^+) > 0 \quad \text{if } l^+ \not\equiv 0 \text{ in } \Omega,$$

where $l^+ = \max(l, 0)$ (see Proposition 2.4).

Notice that the stability of L implies that $L(l) = 0$ for any linear function l, and hence $d\sigma$ and dA must have the same mass and the same center of mass.

A minimizer u of the functional L is determined up to linear functions, since both L and $\mathcal{A}$ are invariant under addition with linear functions. We "normalize" u by subtracting its tangent plane at, say, the center of mass of Ω. In Section 2, we shall prove in Proposition 2.5 that there exists a unique normalized minimizer to the problem (P).

We also prove a compactness theorem for minimizers.

Theorem 1.1 (compactness). *Let u_k be the normalized minimizers of the functionals L_k with data $(f_k, d\sigma_k, dA_k, \Omega)$ that has uniform bounds in k. Precisely, the inequalities* (2-1) *and* (2-4) *below are satisfied uniformly in k and $\rho \le f_k \le \rho^{-1}$. If*

$$f_k \rightharpoonup f,\ d\sigma_k \rightharpoonup d\sigma,\ dA_k \rightharpoonup dA,$$

then $u_k \to u$ uniformly on compact sets of Ω, where u is the normalized minimizer of the functional L with data $(f, d\sigma, dA, \Omega)$.

If u is a minimizer, then the Euler–Lagrange equation reads (see Proposition 3.6)

$$\text{if } \varphi : \Omega \to \mathbb{R} \text{ solves } U^{ij}\varphi_{ij} = 0, \text{ then } L(\varphi) = 0,$$

where U^{ij} are the entries of the cofactor matrix U of the Hessian $D^2 u$. Since the linearized Monge–Ampère equation is also an equation in divergence form, we can always express the Ω-integral of a function φ in terms of a boundary integral. For this, we consider the solution v to the Dirichlet problem

$$U^{ij} v_{ij} = -dA \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

Integrating by parts twice and using $\partial_i(U^{ij}) = \partial_j(U^{ij}) = 0$, we can compute

$$\begin{aligned} \int_\Omega \varphi\, dA &= -\int_\Omega \varphi\, U^{ij} v_{ij} = \int_\Omega \varphi_i\, U^{ij} v_j - \int_{\partial\Omega} \varphi U^{ij} v_j \nu_i \\ &= -\int_\Omega (U^{ij}\varphi_{ij}) v + \int_{\partial\Omega} \varphi_i U^{ij} v \nu_j - \int_{\partial\Omega} \varphi U^{ij} v_j \nu_i = -\int_{\partial\Omega} \varphi U^{ij} v_i \nu_j. \end{aligned} \tag{1-1}$$

From the Euler–Lagrange equation, we obtain

$$U^{ij} v_i \nu_j = -\sigma \quad \text{on } \partial\Omega.$$

Since $v = 0$ on $\partial\Omega$, we have $v_i = v_\nu \nu_i$, and hence

$$U^{ij} v_i \nu_j = U^{ij} \nu_i \nu_j v_\nu = U^{\nu\nu} v_\nu = \left(\det D^2_{x'} u\right) v_\nu,$$

with $x' \perp \nu$ denoting the tangential directions along $\partial\Omega$. In conclusion, if u is a smooth minimizer, then there exists a function v such that (u, v) solves the system

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ U^{ij} v_{ij} = -dA & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ U^{\nu\nu} v_\nu = -\sigma & \text{on } \partial\Omega. \end{cases} \tag{1-2}$$

This system is interesting since the function v above satisfies two boundary conditions, Dirichlet and Neumann, while u has no boundary conditions. Heuristically, the boundary values for u can be recovered from the term $U^{\nu\nu} = \det D^2_{x'} u$, which appears in the Neumann boundary condition for v.

Our main regularity results for the minimizers u are in two dimensions.

Theorem 1.2. *Assume that $n = 2$, and the conditions (1)–(5) hold. If $\sigma \in C^\alpha(\partial\Omega)$, $f \in C^\alpha(\overline{\Omega})$, and $\partial\Omega \in C^{2,\alpha}$, then the minimizer $u \in C^{2,\alpha}(\overline{\Omega})$ and the system* (1-2) *holds.*

We obtain Theorem 1.2 by showing that u separates quadratically on $\partial\Omega$ from its tangent planes, and then we apply the boundary Hölder gradient estimates for v which were obtained in [Le and Savin 2013].

As a consequence of Theorem 1.2, we obtain higher regularity if the data $(f, d\sigma, dA, \Omega)$ is more regular.

Theorem 1.3. *Assume that $n = 2$ and the conditions (1)–(5) hold. If $\sigma \in C^\infty(\partial\Omega)$, $f \in C^\infty(\overline{\Omega})$, $A \in C^\infty(\overline{\Omega})$, and $\partial\Omega \in C^\infty$, then $u \in C^\infty(\overline{\Omega})$.*

In Section 6, we provide an example of Pogorelov type for a minimizer in dimensions $n \geq 3$ that shows that Theorem 1.3 does not hold in this generality in higher dimensions.

We explain briefly how Theorem 1.3 follows from Theorem 1.2. If $u \in C^{2,\alpha}(\overline{\Omega})$, then $U^{ij} \in C^\alpha(\overline{\Omega})$, and Schauder estimates give $v \in C^{2,\alpha}(\overline{\Omega})$, and thus $v_\nu \in C^{1,\alpha}(\partial\Omega)$. From the last equation in (1-2) we obtain $U^{\nu\nu} = \det D^2_{x'} u \in C^{1,\alpha}(\partial\Omega)$. This implies $u \in C^{3,\alpha}(\partial\Omega)$, and from the first equation in (1-2), we find $u \in C^{3,\alpha}(\overline{\Omega})$. We can repeat the same argument and obtain that $u \in C^{k,\alpha}$ for any $k \geq 2$.

As we mentioned above, our constraint minimization problem is motivated by the minimization of the Mabuchi energy functional from complex geometry in the case of toric varieties

$$M(u) = \int_\Omega -\log\det D^2 u + \int_{\partial\Omega} u \, d\sigma - \int_\Omega u \, dA.$$

In this case, $d\sigma$ and dA are canonical measures on $\partial\Omega$ and Ω. Minimizers of M satisfy the following fourth-order equation, called Abreu's equation [1998]:

$$u^{ij}_{ij} := \sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A,$$

where u^{ij} are the entries of the inverse matrix of $D^2 u$. This equation and the functional M have been studied extensively by Donaldson [2002; 2005; 2008; 2009]; see also [Zhou and Zhu 2008]. In Donaldson's papers, the domain Ω was taken to be a polytope $P \subset \mathbb{R}^n$ and A was taken to be a positive constant. The existence of smooth solutions with suitable boundary conditions has important implications in complex geometry. It says that we can find Kähler metrics of constant scalar curvature for toric varieties.

More generally, one can consider minimizers of the convex functional

$$E(u)=\int_\Omega F(\det D^2u)+\int_{\partial\Omega} u\,d\sigma-\int_\Omega u\,dA, \tag{1-3}$$

where $F(t^n)$ is a convex and decreasing function of $t\ge 0$. The Mabuchi energy functional corresponds to $F(t)=-\log t$, whereas in our minimization problem (P) (with $f\equiv 1$),

$$F(t)=\begin{cases}\infty & \text{if } t<1,\\ 0 & \text{if } t\ge 1.\end{cases}$$

Minimizers of E satisfy a system similar to (1-2):

$$\begin{cases}-F'(\det D^2u)=v & \text{in } \Omega,\\ U^{ij}v_{ij}=-dA & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\\ U^{\nu\nu}v_\nu=-\sigma & \text{on } \partial\Omega.\end{cases} \tag{1-4}$$

A similar system but with different boundary conditions was investigated by Trudinger and Wang [2008a]. If the function F is strictly decreasing, then we see from the first and third equations above that $\det D^2u=\infty$ on $\partial\Omega$, and therefore we cannot expect minimizers to be smooth up to the boundary (as is the case with the Mabuchi functional $M(u)$).

If F is constant for large values of t (as in the case we considered), then $\det D^2u$ becomes finite on the boundary and smoothness up to the boundary is expected. More precisely, assume that

$$F\in C^{1,1}((0,\infty)),\quad G(t):=F(t^n)\text{ is convex in } t,\quad\text{and}\quad G'(0^+)=-\infty,$$

and there exists $t_0>0$ such that

$$F(t)=0 \text{ on } [t_0,\infty),\qquad F''(t)>0 \text{ on } (0,t_0].$$

Theorem 1.4. *Assume $n=2$ and the conditions (1)–(5) and the above hypotheses on F are satisfied. If $\sigma\in C^\alpha(\partial\Omega)$, $A\in C^\alpha(\overline{\Omega})$, and $\partial\Omega\in C^{2,\alpha}$, then the normalized minimizer u of the functional E defined in* (1-3) *satisfies $u\in C^{2,\alpha}(\overline{\Omega})$, and the system* (1-4) *holds in the classical sense.*

The paper is organized as follows. In Section 2, we discuss the notion of stability for the functional L and prove existence, uniqueness and compactness of minimizers of the problem (P). In Section 3, we state a quantitative version of Theorem 1.2, Proposition 3.1, and we also obtain the Euler–Lagrange equation. Proposition 3.1 is proved in Sections 4 and 5, first under the assumption that the density A is bounded from below and then in the general case. In Section 6, we give an example of a singular minimizer in dimension $n\ge 3$. Finally, in Section 7, we prove Theorem 1.4.

2. Stability inequality and existence of minimizers

Let Ω be a bounded convex set and define

$$L(u) = \int_{\partial\Omega} u \, d\sigma - \int_{\Omega} u \, dA$$

for all convex functions $u : \overline{\Omega} \to \mathbb{R}$ with $u \in L^1(\partial\Omega, d\sigma)$. We assume that

$$\sigma \geq \rho \text{ on } \partial\Omega \text{ and } A(x) \leq \rho^{-1} \text{ in a neighborhood of } \partial\Omega \tag{2-1}$$

for some small $\rho > 0$ and that L is stable, that is,

$$L(u) > 0 \text{ for all } u \text{ convex but not linear.} \tag{2-2}$$

Assume for simplicity that 0 is the center of mass of Ω. We notice that (2-2) implies $L(l) = 0$ for any l linear, since l can be approximated by both convex and concave functions. We "normalize" a convex function by subtracting its tangent plane at 0, and this does not change the value of L. First we prove some lower semicontinuity properties of L with respect to normalized solutions.

Lemma 2.1 (lower semicontinuity). *Assume that* (2-1) *holds and* (u_k) *is a normalized sequence that satisfies*

$$\int_{\partial\Omega} u_k \, d\sigma \leq C, \qquad u_k \to u \textit{ uniformly on compact sets of } \Omega, \tag{2-3}$$

for some function $u : \Omega \to \mathbb{R}$. *Let* $\bar{u}$ *be the minimal convex extension of* u *to* $\overline{\Omega}$, *that is,*

$$\bar{u} = u \textit{ in } \Omega, \qquad \bar{u}(x) = \lim_{t \to 1^-} u(tx) \textit{ if } x \in \partial\Omega.$$

Then

$$\int_{\Omega} u \, dA = \lim \int_{\Omega} u_k \, dA, \qquad \int_{\partial\Omega} \bar{u} \, d\sigma \leq \liminf \int_{\partial\Omega} u_k \, d\sigma,$$

and thus

$$L(\bar{u}) \leq \liminf L(u_k).$$

Remark. The upper graph of the function $\bar{u}$ is the closure of the upper graph of u in $\mathbb{R}^{n+1}$.

Proof. Since u_k are normalized, they are increasing on each ray out of the origin. For each $\eta > 0$ small, we consider the set $\Omega_\eta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \eta\}$, and from (2-1) we obtain

$$\int_{\Omega_\eta} u_k \, dA \leq C\rho^{-1}\eta \int_{\partial\Omega} u_k \, d\sigma \leq C\eta.$$

Since this inequality holds for all small $\eta \to 0$, we easily obtain

$$\int_{\Omega} u \, dA = \lim \int_{\Omega} u_k \, dA.$$

For each $z \in \partial\Omega$ and $t < 1$ we have $u_k(tz) \leq u_k(z)$. We let $k \to \infty$ in the inequality

$$\int_{\partial\Omega} u_k(tz) \, d\sigma \leq \int_{\partial\Omega} u_k(z) \, d\sigma$$

and obtain

$$\int_{\partial\Omega} u(tz)\,d\sigma \leq \liminf \int_{\partial\Omega} u_k(z)\,d\sigma,$$

and then we let $t \to 1^-$:

$$\int_{\partial\Omega} \bar{u}\,d\sigma \leq \liminf \int_{\partial\Omega} u_k\,d\sigma. \qquad \square$$

Remark 2.2. From the proof we see that if we are given functionals L_k with measures σ_k, A_k that satisfy (2-1) uniformly in k and

$$\sigma_k \rightharpoonup \sigma, \quad A_k \rightharpoonup A,$$

and if (2-3) holds for a sequence u_k, then the statement still holds; that is,

$$L(\bar{u}) \leq \liminf L_k(u_k).$$

By compactness, one can obtain a quantitative version of (2-2) known as *stability inequality*. This was done by Donaldson [2002, Proposition 5.2.2]. For completeness, we sketch its proof here.

Proposition 2.3. *Assume that* (2-1) *and* (2-2) *hold. Then we can find* $\mu > 0$ *such that*

$$L(u) := \int_{\partial\Omega} u d\sigma - \int_{\Omega} u dA \geq \mu \int_{\partial\Omega} u d\sigma \tag{2-4}$$

for all convex functions u *normalized at* 0.

Proof. Assume the conclusion does not hold; then there is a sequence of normalized convex functions (u_k) with

$$\int_{\partial\Omega} u_k d\sigma = 1, \quad \lim L(u_k) = 0,$$

and thus

$$\lim \int_{\Omega} u_k dA = 1.$$

Using convexity, we may assume that u_k converges uniformly on compact subsets of Ω to a limiting function $u \geq 0$. Let $\bar{u}$ be the minimal convex extension of u to $\overline{\Omega}$. Then, from Lemma 2.1, we obtain

$$L(\bar{u}) = 0, \quad \int_{\Omega} \bar{u}\,dA = 1,$$

and thus $\bar{u} \geq 0$ is not linear and we contradict (2-2). $\square$

Donaldson [2002, Proposition 5.3.1] showed that when $n = 2$, the stability condition can be checked easily.

Proposition 2.4. *Assume that* $n = 2$, *that* (2-1) *holds, and that for all linear functions* l *we have*

$$L(l) = 0 \quad \textit{and} \quad L(l^+) > 0 \quad \textit{if } l^+ \not\equiv 0 \textit{ in } \Omega, \tag{2-5}$$

where $l^+ = \max(l, 0)$. *Then* L *is stable; that is, condition* (2-2) *is satisfied.*

Proof. For completeness, we sketch the proof. Assume by contradiction that $L(u) \le 0$ for some convex function u which is not linear in Ω. Let u^* be the convex envelope generated by the boundary values of $\bar{u}$ — the minimal convex extension of u to $\overline{\Omega}$. Notice that $u^* = \bar{u}$ on $\partial\Omega$. Since $L(u^*) \le L(\bar{u}) \le L(u)$, we find $L(u^*) \le 0$. Notice that u^* is not linear, since otherwise $0 = L(u^*) < L(\bar{u}) \le 0$ (we used that $\bar{u}$ is not linear). After subtracting a linear function, we may assume that u^* is normalized and u^* is not identically 0.

We obtain a contradiction by showing that u^* satisfies the stability inequality. By our hypotheses, there exists $\mu > 0$ small such that

$$L(l^+) \ge \mu \int_{\partial\Omega} l^+ \, d\sigma$$

for any l^+. Indeed, by (2-1), this inequality is valid if the "crease" $\{l = 0\}$ is near $\partial\Omega$, and for all other l's, it follows by compactness from (2-5). We approximate from below u^* by u_k^*, which is defined as the maximum of the tangent planes of u^* at some points $y_i \in \Omega$, $i = 1, \dots, k$. Since u^* is a convex envelope in two dimensions, u_k^* is a discrete sum of l^+'s, and hence it satisfies the stability inequality. Now we let $k \to \infty$; since $u_k^* \le u^*$, using Lemma 2.1, we obtain that u^* also satisfies the stability inequality. □

Proposition 2.5. *Assume that* (2-1) *and* (2-2) *hold. Then there exists a unique* (*up to linear functions*) *minimizer u of L subject to the constraint*

$$u \in \mathcal{A} := \{ v : \overline{\Omega} \to \mathbb{R} \mid v \text{ convex}, \ \det D^2 v \ge f \},$$

where $\rho \le f \le \rho^{-1}$ for some $\rho > 0$. The minimizer satisfies $\det D^2 u = f$, *and if $n = 2$, it is unique* (*up to linear functions*).

Proof. Let (u_k) be a sequence of normalized solutions such that $L(u_k) \to \inf_{\mathcal{A}} L$. By the stability inequality, we see that $\int_{\partial\Omega} u_k \, d\sigma$ are uniformly bounded, and after passing to a subsequence, we may assume that u_k converges uniformly on compact subsets of Ω to a function u. Then $u \in \mathcal{A}$, and from the lower semicontinuity we see that $L(u) = \inf_{\mathcal{A}} L$, that is, u is a minimizer. Notice that $\det D^2 u = f$. Indeed, if a quadratic polynomial P with $\det D^2 P > f$ touches u strictly from below at some point $x_0 \in \Omega$, in a neighborhood of x_0, then we can replace u in this neighborhood by $\max\{P + \epsilon, u\} \in \mathcal{A}$, and the energy decreases.

Next we assume w is another minimizer. We use the strict concavity of $M \mapsto \log(\det D^2 M)$ in the space of positive symmetric matrices M, and obtain that for almost every x where u, w are twice differentiable,

$$\log\det D^2\left(\frac{u+w}{2}\right)(x) \ge \tfrac{1}{2}\log\det D^2 u(x) + \tfrac{1}{2}\log\det D^2 w(x) \ge \log f(x).$$

This implies $(u + w)/2 \in \mathcal{A}$ is also a minimizer and $D^2 u = D^2 w$ almost everywhere in Ω. Since f is bounded above and below, we know that $u, w \in W^{2,1}_{\mathrm{loc}}$ (see [De Philippis and Figalli 2013]) in the open set Ω' where both u, w are strictly convex. This gives that $u - w$ is linear on each connected component of Ω'. If $n = 2$, then $\Omega' = \Omega$, and hence $u - w$ is linear. □

Remark. Uniqueness is expected to hold in any dimension. For this one needs to show that the set of strict convexity of a solution to the Monge–Ampère equation is always connected.

Remark. The arguments above show that the stability condition is also necessary for the existence of a minimizer. Indeed, if u is a minimizer and $L(u_0) = 0$ for some convex function u_0 that is not linear, then $u + u_0$ is also a minimizer and we contradict the uniqueness.

Proof of Theorem 1.1. We assume that the data $(f_k, d\sigma_k, dA_k, \Omega)$ satisfies (2-1), (2-4) uniformly in k and $\rho \leq f_k \leq \rho^{-1}$. For each k, let w_k be the convex solution to $\det D^2 w_k = f_k$ in Ω with $w_k = 0$ on $\partial\Omega$. Since f_k are bounded from above, we find $w_k \geq -C$, and so by the minimality of u_k,

$$L_k(u_k) \leq L_k(w_k) \leq C.$$

It follows from the stability inequality that

$$\int_{\partial\Omega} u_k \, d\sigma_k \leq C,$$

and we may assume, after passing to a subsequence, that $u_k \to u$ uniformly on compact subsets of Ω.

We need to show that u is a minimizer for L with data $(f, d\sigma, dA, \Omega)$. For this it suffices to prove that for any continuous $v : \overline{\Omega} \to \mathbb{R}$ which solves $\det D^2 v = f$ in Ω, we have $L(u) \leq L(v)$.

Let v_k be the solution to $\det D^2 v_k = f_k$ with boundary data $v_k = v$ on $\partial\Omega$. Using appropriate barriers, it is standard to check that $f_k \rightharpoonup f$, $f_k \leq \rho^{-1}$ implies $v_k \to v$ uniformly in $\overline{\Omega}$. Then we let $k \to \infty$ in $L_k(u_k) \leq L_k(v_k)$, use Remark 2.2, and obtain

$$L(u) \leq \liminf L_k(u_k) \leq \lim L_k(v_k) = L(v),$$

which finishes the proof. □

3. Preliminaries and the Euler–Lagrange equation

We rewrite our main hypotheses in a quantitative way. We assume that for some small $\rho > 0$, we have:

(H1) The curvatures of $\partial\Omega$ are bounded from below by ρ and from above by ρ^{-1}.

(H2) $\rho \leq f \leq \rho^{-1}$.

(H3) $d\sigma = \sigma(x)\, d\mathcal{H}^{n-1} \lfloor \partial\Omega$, with $\rho \leq \sigma(x) \leq \rho^{-1}$.

(H4) $dA = A(x)\, dx$ in a small neighborhood

$$\Omega_\rho := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \rho\}$$

of $\partial\Omega$ with $A(x) \leq \rho^{-1}$.

(H5) For any convex function u normalized at the center of mass of Ω, we have

$$L(u) := \int_{\partial\Omega} u \, d\sigma - \int_{\Omega} u \, dA \geq \rho \int_{\partial\Omega} u \, d\sigma.$$

We denote by c, C positive constants depending on ρ, and their values may change from line to line whenever there is no possibility of confusion. We refer to such constants as *universal constants*.

Our main theorem, Theorem 1.2, follows from the next proposition, which deals with less regular data.

Proposition 3.1. *Assume that $n = 2$ and that conditions* (H1)–(H5) *hold.*

(i) *Then the minimizer u obtained in Proposition 2.5 satisfies $u \in C^{1,\beta}(\overline{\Omega}) \cap C^{1,1}(\partial\Omega)$ for some universal $\beta \in (0, 1)$ and u separates quadratically from its tangent planes on $\partial\Omega$, that is,*

$$C^{-1}|x-y|^2 \le u(y) - u(x) - \nabla u(x)(y-x) \le C|x-y|^2 \quad \textit{for all } x, y \in \partial\Omega,$$

for some $C > 0$ universal.

(ii) *If in addition $\sigma \in C^\alpha(\partial\Omega)$, then $u\,|_{\partial\Omega} \in C^{2,\gamma}(\partial\Omega)$ with $\gamma := \min\{\alpha, \beta\}$, and*

$$\|u\|_{C^{2,\gamma}(\partial\Omega)} \le C\|\sigma\|_{C^\gamma(\partial\Omega)}.$$

We remark that in part (ii), we obtain $u \in C^{2,\gamma}(\partial\Omega)$ even though f and A are assumed to be only L^∞.

Proof that Proposition 3.1 implies Theorem 1.2. Theorem 7.3 of [Savin 2013] states that a solution to the Monge–Ampère equation which separates quadratically from its tangent planes on the boundary satisfies the classical C^α-Schauder estimates. Thus, if the assumptions of Proposition 3.1(ii) are satisfied and $f \in C^\alpha(\overline{\Omega})$, then $u \in C^{2,\gamma}(\overline{\Omega})$ with its $C^{2,\gamma}$ norm bounded by a constant C depending on ρ, α, $\|\sigma\|_{C^\alpha(\partial\Omega)}$, $\|\partial\Omega\|_{C^{2,\alpha}}$, and $\|f\|_{C^\alpha(\overline{\Omega})}$. This implies that the system (1-2) holds. If $\alpha \le \beta$, then we are done. If $\alpha > \beta$, then we use $v_\nu \in C^\alpha(\partial\Omega)$ in the last equation of the system and obtain $u \in C^{2,\alpha}(\partial\Omega)$, which gives $u \in C^{2,\alpha}(\overline{\Omega})$. □

We prove Proposition 3.1 in the next two sections. Part (ii) follows from part (i) and the boundary Harnack inequality for the linearized Monge–Ampère equation, which was obtained in [Le and Savin 2013, Theorem 2.4]. This theorem states that if a solution to the Monge–Ampère equation with bounded right-hand side separates quadratically from its tangent planes on the boundary, then the classical boundary estimate of Krylov holds for solutions of the associated linearized equation.

In order to simplify the ideas, we prove the proposition in the case when the hypotheses (H1), (H2), (H4) are replaced by

(H1′) $\Omega = B_1$.

(H2′) $f \in C^\infty(\overline{\Omega})$, $\rho \le f \le \rho^{-1}$.

(H4′) $dA = A(x)\,dx$ with $\rho \le A(x) \le \rho^{-1}$ in Ω and $A \in C^\infty(\Omega)$.

We use (H1′) only for simplicity of notation. We will see from the proofs that the same arguments carry to the general case. We use (H2′) so that D^2u is continuous in Ω and the linearized Monge–Ampère equation is well defined. Our estimates do not depend on the smoothness of f, and thus the general case follows by approximation from Theorem 1.1. Later, in Section 5, we show that (H4′) can be replaced by (H4), that is, the bound for A from below is not needed.

First, we establish a result on uniform modulus of convexity for minimizers of L in two dimensions.

Proposition 3.2. *Let u be a minimizer of L that satisfies the hypotheses above. Then, for any $\delta < 1$, there exist $c(\delta) > 0$ depending on ρ, δ such that*

$$x \in B_{1-\delta} \quad \Longrightarrow \quad S_h(x) \Subset B_1 \ \textit{ if } h \le c(\delta),$$

where $S_h(x)$ denotes the section of u centered at x at height h:

$$S_h(x) = \{ y \in \bar{B}_1 : u(y) < u(x) + \nabla u(x)(y-x) + h \}.$$

Although this result is well known (see [Trudinger and Wang 2008b, Remark 3.2] for example), we include its proof here for completeness.

Proof. Without loss of generality, assume u is normalized in B_1, that is, $u \geq 0$, $u(0) = 0$. From the stability inequality (2-4), we obtain

$$\int_{\partial B_1} u \, dx \leq C.$$

This integral bound and the convexity of u imply

$$|u|, |Du| \leq C(\delta) \text{ in } B_{1-\delta/2},$$

for any $\delta < 1$. We show that our statement follows from these bounds. Assume by contradiction that the conclusion is not true. Then we can find a sequence of convex functions u_k satisfying the bounds above such that

$$u_k(y_k) \leq u_k(x_k) + \nabla u_k(x_k)(y_k - x_k) + h_k \tag{3-1}$$

for sequences $x_k \in B_{1-\delta}$, $y_k \in \partial B_{1-\delta/2}$ and $h_k \to 0$. Because Du_k is uniformly bounded, after passing to a subsequence if necessary, we may assume

$$u_k \to u_* \text{ uniformly on } \overline{B}_{1-\delta/2}, \quad x_k \to x_*, \quad y_k \to y_*.$$

Moreover, u_* satisfies $\rho \leq \det D^2 u_* \leq \rho^{-1}$, and

$$u_*(y_*) = u_*(x_*) + \nabla u_*(x_*)(y_* - x_*),$$

that is, the graph of u_* contains a straight line in the interior. However, any subsolution v to $\det D^2 v \geq \rho$ in two dimensions does not have this property and we reach a contradiction. □

Since $f \in C^\alpha$, we obtain that $u \in C^{2,\alpha}(B_1)$, and thus the linearized Monge–Ampère equation is well defined in B_1. The next lemma deals with general linear elliptic equations in B_1 which may become degenerate as we approach ∂B_1.

Lemma 3.3. *Let $\mathscr{L}v := a^{ij}(x)v_{ij}$ be a linear elliptic operator with continuous coefficients $a^{ij} \in C^\alpha(B_1)$ that satisfy the ellipticity condition $(a^{ij}(x))_{ij} > 0$ in B_1. Given a continuous boundary data φ, there exists a unique solution $v \in C(\overline{B}_1) \cap C^2(\Omega)$ to the Dirichlet problem*

$$\mathscr{L}v = 0 \text{ in } B_1, \quad v = \varphi \text{ on } \partial B_1.$$

Proof. For each small δ, we consider the standard Dirichlet problem for uniformly elliptic equations $\mathscr{L}v_\delta = 0$ in $B_{1-\delta}$, $v_\delta = \varphi$ on $\partial B_{1-\delta}$. Since v_δ satisfies the comparison principle with linear functions, it follows that the modulus of continuity of v_δ at points on the boundary $\partial B_{1-\delta}$ depends only on the modulus of continuity of φ. Thus, from the maximum principle, we see that v_δ converges uniformly to a solution v of the Dirichlet problem above. The uniqueness of v follows from the standard comparison principle. $\square$

Remark 3.4. The modulus of continuity of v at points on ∂B_1 depends only on the modulus of continuity of φ.

Remark 3.5. If $\mathscr{L}_m$ is a sequence of operators satisfying the hypotheses of Lemma 3.3 with $a_m^{ij} \to a^{ij}$ uniformly on compact subsets of B_1 and $\mathscr{L}_m v_m = 0$ in B_1, $v_m = \varphi$ on ∂B_1, then $v_m \to v$ uniformly in $\overline{B}_1$.

Indeed, since v_m have a uniform modulus of continuity on ∂B_1 and, for all large m, a uniform modulus of continuity in any ball $B_{1-\delta}$, we see that we can always extract a uniform convergent subsequence in $\overline{B}_1$. Now it is straightforward to check that the limiting function v satisfies $\mathscr{L}v = 0$ in the viscosity sense.

Next, we establish an integral form of the Euler–Lagrange equations for the minimizers of L.

Proposition 3.6. *Assume that u is the normalized minimizer of L in the class $\mathscr{A}$. If $\varphi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to the linearized Monge–Ampère equation*

$$U^{ij}\varphi_{ij} = 0 \textit{ in } \Omega,$$

then

$$L(\varphi) := \int_{\partial\Omega} \varphi \, d\sigma - \int_\Omega \varphi \, dA = 0.$$

Proof. Consider the solution $u_\epsilon = u + \epsilon\varphi_\epsilon$ to

$$\begin{cases} \det D^2 u_\epsilon = f & \text{in } B_1, \\ u_\epsilon = u + \epsilon\varphi & \text{on } \partial B_1. \end{cases}$$

Since φ_ϵ satisfies the comparison principle and comparison with planes, its existence follows as in Lemma 3.3 by solving the Dirichlet problems in $B_{1-\delta}$ and then letting $\delta \to 0$.

In B_1, φ_ϵ satisfies

$$0 = \frac{1}{\epsilon}\big(\det D^2 u_\epsilon - \det D^2 u\big) = \frac{1}{\epsilon}\int_0^1 \frac{d}{dt} \det D^2(u + t\epsilon\varphi_\epsilon)\, dt = a_\epsilon^{ij}\partial_{ij}\varphi_\epsilon,$$

where (a_ϵ^{ij}) is the integral from 0 to 1 of the cofactor matrix of $D^2(u + t\epsilon\varphi_\epsilon)$, that is,

$$(a_\epsilon^{ij})_{ij} = \int_0^1 \det D^2(u + t\epsilon\varphi_\epsilon)\big(D^2(u + t\epsilon\varphi_\epsilon)\big)^{-1}\, dt.$$

Because u is strictly convex in two dimensions and $u_\epsilon \to u$ uniformly on $\overline{B}_1$, $D^2 u_\epsilon \to D^2 u$ uniformly on compact sets of B_1. Thus, as $\epsilon \to 0$, $a_\epsilon^{ij} \to U^{ij}$ uniformly on compact sets of B_1 and by Remark 3.5, we find $\varphi_\epsilon \to \varphi$ uniformly in $\overline{B}_1$. By the minimality of u, we find

$$0 \le \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\big(L(u_\epsilon) - L(u)\big) = \int_{\partial B_1} \varphi \, d\sigma - \int_{B_1} \varphi \, dA.$$

By replacing φ with $-\varphi$, we obtain the opposite inequality. □

4. Proof of Proposition 3.1

In this section, we prove Proposition 3.1 where (H1′), (H2′) and (H4′) are satisfied. Given a convex function $u \in C^\infty(B_1)$ (not necessarily a minimizer of L) with $\rho \le \det D^2u \le \rho^{-1}$, we let v be the solution to the Dirichlet problem

$$U^{ij} v_{ij} = -A \text{ in } B_1 \quad v = 0 \text{ on } \partial B_1. \tag{4-1}$$

Notice that $\Psi := C(1-|x|^2)$ is an upper barrier for v if C is large enough, since

$$U^{ij}\Psi_{ij} \le -C \operatorname{tr} U \le -C(\det D^2U)^{1/n} = -C(\det D^2u)^{(n-1)/n} \le -C\rho^{(n-1)/n} \le -A,$$

and hence

$$0 \le v(x) \le C(1-|x|^2) \sim \operatorname{dist}(x, \partial B_1). \tag{4-2}$$

As in Lemma 3.3, the function v is the uniform limit of the corresponding v_δ that solve the Dirichlet problem in $B_{1-\delta}$. Indeed, since v_δ also satisfies (4-2), we see that

$$|v_{\delta_1} - v_{\delta_2}|_{L^\infty} \le C \max\{\delta_1, \delta_2\}.$$

Let φ be the solution of the homogeneous problem

$$U^{ij}\varphi_{ij} = 0 \text{ in } B_1, \quad \varphi = l^+ \text{ on } \partial B_1,$$

where $l^+ = \max\{0, l\}$ for some linear function $l = b + \nu \cdot x$ of slope $|\nu| = 1$. Denote by $\mathcal{S} := \overline{B}_1 \cap \{l = 0\}$ the segment of intersection of the crease of l with $\overline{B}_1$. Then:

Lemma 4.1. $$\int_{B_1} \varphi \, dA = \int_{B_1} l^+ \, dA + \int_{\mathcal{S}} u_{\tau\tau} v \, d\mathcal{H}^1,$$

where τ is the unit vector in the direction of $\mathcal{S}$, and hence $\tau \perp \nu$.

Proof. It suffices to show the equality in the case when $u \in C^\infty(\overline{B}_1)$. The general case follows by writing the identity in $B_{1-\delta}$ with v_δ (which increases as δ decreases), and then letting $\delta \to 0$.

Let $\tilde{l}_\epsilon$ be a smooth approximation of l^+ with

$$D^2 l_\epsilon \rightharpoonup \nu \otimes \nu \, dH^1 \lfloor \mathcal{S} \text{ as } \epsilon \to 0,$$

and let φ_ϵ solve the corresponding Dirichlet problem with boundary $\tilde{l}_\epsilon$. Then we integrate by parts and use $\partial_i U^{ij} = 0$:

$$\int_{B_1} (\varphi_\epsilon - \tilde{l}_\epsilon)\, dA = -\int_{B_1} (\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v_{ij}\, dx = \int_{B_1} \partial_i (\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v_j\, dx$$
$$= -\int_{B_1} \partial_{ij} (\varphi_\epsilon - \tilde{l}_\epsilon) U^{ij} v\, dx = \int_{B_1} U^{ij} \partial_{ij} \tilde{l}_\epsilon v\, dx.$$

We let $\epsilon \to 0$ and obtain

$$\int_{B_1} (\varphi - l^+)\, dA = \int_{\mathcal{S}} U^{\nu\nu} v\, d\mathcal{H}^1,$$

which is the desired conclusion, since $U^{\nu\nu} = u_{\tau\tau}$. □

From Lemma 4.1 and Proposition 3.6, we obtain:

Corollary 4.2. *If u is a minimizer of L in the class $\mathcal{A}$, then*

$$\int_{\mathcal{S}} u_{\tau\tau} v\, d\mathcal{H}^1 = \int_{\partial B_1} l^+\, d\sigma - \int_{B_1} l^+\, dA.$$

The hypotheses on σ and A imply that if the segment $\mathcal{S}$ has length $2h$ with $h \le h_0$ small, universal then

$$ch^3 \le \int_{\mathcal{S}} u_{\tau\tau} v\, d\mathcal{H}^1 \le Ch^3,$$

for some c, C universal.

Lemma 4.3. *Let X_1 and X_2 be the endpoints of the segment $\mathcal{S}$ defined as above. Then*

$$\int_{\mathcal{S}} u_{\tau\tau}(1 - |x|^2)\, d\mathcal{H}^1 = 4h\left(\frac{u(X_1) + u(X_2)}{2} - ⨍_{\mathcal{S}} u\, d\mathcal{H}^1\right), \tag{4-3}$$

where $2h$ denotes the length of $\mathcal{S}$.

Proof. Again we may assume that $u \in C^2(\overline{B}_1)$, since the general case follows by approximating B_1 by $B_{1-\delta}$. Assume for simplicity that $\tau = e_1$. Then

$$\int_{\mathcal{S}} u_{\tau\tau}(1 - |x|^2)\, d\mathcal{H}^1 = \int_{-h}^{h} \partial_t^2 u(t, a)(h^2 - t^2)\, dt$$

for some fixed a, and integrating by parts twice, we obtain (4-3). □

We remark that the right-hand side in (4-3) represents twice the area between the segment with end points $(X_1, u(X_1))$, $(X_2, u(X_2))$ and the graph of u above $\mathcal{S}$.

Definition 4.4. We say that u admits a tangent plane at a point $z \in \partial B_1$ if there exists a linear function l_z such that

$$x_{n+1} = l_z(x)$$

is a supporting hyperplane for the graph of u at $(z, u(z))$ but for any $\epsilon > 0$,

$$x_{n+1} = l_z(x) - \epsilon z \cdot (x - z)$$

is not a supporting hyperplane. We call l_z a tangent plane for u at z.

Remark 4.5. Notice that if $\det D^2u \le C$, then the set of points where u admits a tangent plane is dense in ∂B_1. Indeed, using standard barriers, it is not difficult to check that any point on ∂B_1 where the boundary data $u|_{\partial B_1}$ admits a quadratic polynomial from below satisfies the definition above. In the definition above, we assumed $u = \bar{u}$ on ∂B_1 with $\bar{u}$ defined as in the Lemma 2.1; therefore $u|_{\partial B_1}$ is lower semicontinuous.

Assume that u admits a tangent plane at z, and define

$$\tilde{u} = u - l_z.$$

Lemma 4.6. *There exists $\eta > 0$ small, universal such that the section*

$$\tilde{S}_z := \{x \in \bar{B}_1 \mid \tilde{u} < \eta(x - z) \cdot (-z)\}$$

satisfies

$$\tilde{S}_z \subset B_1 \setminus B_{1-\rho}, \quad |\tilde{S}_z| \ge c,$$

for some small c universal.

Proof. We notice that (4-3) is invariant under additions with linear functions. We apply it to $\tilde{u}$ with $X_1 = z$, $X_2 = x$ and use $\tilde{u} \ge 0$, $\tilde{u}(z) = 0$ together with (4-2) and Corollary 4.2 to obtain

$$\tilde{u}(x) \ge c|x - z|^2, \qquad x \in \partial B_1 \cap B_{h_0}(z).$$

From the uniform strict convexity of $\tilde{u}$, which was obtained in Proposition 3.2, we find that the inequality above holds for all $x \in \partial B_1$ for possibly a different value of c. Thus, by choosing η sufficiently small, we obtain

$$\tilde{S}_z \subset B_1, \qquad \tilde{S}_z \cap B_{1-\rho} = \varnothing,$$

where the second statement follows also from Proposition 3.2.

Next we show that $|\tilde{S}_z|$ cannot be arbitrarily small. Otherwise, by the uniform strict convexity of $\tilde{u}$, we obtain that $\tilde{S}_z \subset B_{\epsilon^4}(z)$ for some small $\epsilon > 0$. Assume for simplicity of notation that $z = -e_2$. Then the function

$$w := \eta(x_2 + 1) + \frac{\epsilon}{2}x_1^2 + \frac{1}{2\rho\epsilon}(x_2 + 1)^2 - 2\epsilon(x_2 + 1)$$

is a lower barrier for $\tilde{u}$ in $B_1 \cap B_{\epsilon^4}(z)$. Indeed, notice that if ϵ is sufficiently small, then

$$w \le \eta(x_2 + 1) \le \tilde{u} \text{ on } \partial(B_1 \cap B_{\epsilon^4}(z)), \qquad \det D^2 w = \rho^{-1} \ge \det D^2 \tilde{u}.$$

In conclusion, $\tilde{u} \ge w \ge (\eta/2)(x_2 + 1)$ and we contradict that $x_{n+1} = 0$ is a tangent plane for $\tilde{u}$ at z. □

Lemma 4.7. *Let u be the normalized minimizer of L. Then $\|u\|_{C^{0,1}(\bar{B}_1)} \le C$, and u admits tangent planes at all points of ∂B_1. Also, u separates at least quadratically from its tangent planes, that is,*

$$u(x) \ge l_z(x) + c|x - z|^2 \quad \textit{for all } x, z \in \partial B_1.$$

Proof. Let z be a point on ∂B_1 where u admits a tangent plane l_z. From the previous lemma, we know that u satisfies the quadratic separation inequality at z and also that $\tilde{u} = u - l_z$ is bounded from above and below in $\tilde{S}_z$, that is,

$$|u - l_z| \le C \text{ in } \tilde{S}_z.$$

We obtain

$$\int_{\tilde{S}_z} |l_z|\, dx - C \le \int_{\tilde{S}_z} u\, dx \le \int_{B_1} u\, dx \le C \int_{\partial B_1} u\, d\sigma \le C,$$

and since $\tilde{S}_z \subset B_1$ has measure bounded from below, we find

$$l_z(z),\ |\nabla l_z| \le C.$$

By Remark 4.5, this holds for almost every $z \in \partial B_1$ and, by approximation, we find that any point in ∂B_1 admits a tangent plane that satisfies the bounds above. This also shows that u is Lipschitz and the lemma is proved. □

Lemma 4.8. *The function v satisfies the lower bound*

$$v(x) \ge c \ \mathrm{dist}(x, \partial B_1),$$

for some small c universal.

Proof. Let $z \in \partial B_1$ and let l be a linear functional with

$$l(x) = l_z(x) - b\, z \cdot (x - z), \quad \text{for some } 0 \le b \le \eta,$$

where l_z denotes a tangent plane at z. We consider all sections

$$S = \{x \in \overline{B}_1 \mid u < l\}$$

which satisfy

$$\inf_S (u - l) \le -c_0,$$

for some appropriate c_0 small, universal. We denote the collection of such sections $\mathcal{M}_z$. From Lemma 4.6, we see that $\mathcal{M}_z \ne \varnothing$ since $\tilde{S}_z$ (or $b = \eta$) satisfies the property above. Notice also that $S \subset \tilde{S}_z \subset B_1$ and $z \in \partial S$. For any section $S \in \mathcal{M}_z$, we consider its center of mass z^S, and from the property above we see that $z^S \in B_{1-c}$ for some small $c > 0$ universal.

First, we show that the lower bound for v holds on the segment $[z, z^S]$. Indeed, since

$$U^{ij}[c(l-u)]_{ij} = -2c \det D^2 u \ge -2c\rho^{-1} \ge -A = U^{ij} v_{ij}$$

and $c(l-u) \le 0 = v$ on ∂B_1, we conclude that

$$c(l-u)^+ \le v \text{ in } B_1. \tag{4-4}$$

Now we use the convexity of u and the fact that the property of S implies $(u-l)(z^S) < -c$, and conclude that

$$v(x) \ge c(l-u)(x) \ge c|x - z| \ge c\, \mathrm{dist}(x, \partial B_1) \quad \text{for all } x \in [z, z^S].$$

Now it remains to prove that the collection of segments $[z, z^S]$, $z \in \partial B_1$, $S \in \mathcal{M}_z$ cover a fixed neighborhood of ∂B_1. To this aim, we show that the multivalued map

$$z \in \partial B_1 \mapsto F(z) := \{z^S \mid S \in \mathcal{M}_z\}$$

has the following properties:

(1) the map F is *closed* in the sense that

$$z_n \to z_* \text{ and } z_n^{S_n} \to y_* \Rightarrow y^* \in F(z_*);$$

(2) $F(z)$ is a connected set for any z.

The first property follows easily from the following facts: z^S varies continuously with the linear map l that defines $S = \{u < l\}$; and if $l_{z_n} \to l_*$, then $l_* \le l_{z_*}$ for some tangent plane l_{z_*}.

To prove the second property, we notice that if we increase continuously the value of the parameter b (which defines l) up to η, then all the corresponding sections also belong to $\mathcal{M}_z$. This means that in $F(z)$ we can continuously connect z^S with $z^{\tilde{S}_z}$ for some section $\tilde{S}_z$. On the other hand, the set of all possible $z^{\tilde{S}_z}$ is connected, since the set l_z of all tangent planes at z is connected in the space of linear functions.

Since $F(z) \subset B_{1-c}$, it follows that for all $\delta < c$, the intersection map

$$z \mapsto G_\delta(z) = \{[z, y] \cap \partial B_{1-\delta} \mid y \in F(z)\}$$

also has properties (1) and (2) above. Now it is easy to check that the image of G_δ covers the whole $\partial B_{1-\delta}$, and hence the collection of segments $[z, z^S]$ covers $B_1 \setminus B_{1-c}$ and the lemma is proved. □

Now we are ready to prove the first part of Proposition 3.1.

Proof of Proposition 3.1(i). In Lemma 4.7, we obtained the quadratic separation from below for $\tilde{u} = u - l_z$. Next we show that $\tilde{u}$ separates at most quadratically on ∂B_1 in a neighborhood of z.

Assume for simplicity of notation that $z = -e_2$. We apply (4-3) to $\tilde{u}$ with $X_1 = (-h, a)$, $X_2 = (h, a)$, and then use Corollary 4.2 and Lemma 4.8 to obtain

$$\frac{\tilde{u}(X_1) + \tilde{u}(X_2)}{2} - \fint_{\mathcal{S}} \tilde{u} \le Ch^2.$$

On the other hand, for small h, the segment $[z, z^{\tilde{S}_z}]$ intersects $[X_1, X_2]$ at a point $y = (t, a)$ with $|t| \le Ch^2 \le h/2$. Moreover, since $y \in \tilde{S}_z$, we have $\tilde{u}(y) \le \eta(a+1) \le Ch^2$. On the segment $[X_1, X_2]$, $\tilde{u}$ satisfies the conditions of Lemma 4.9 which we prove below, and hence

$$\tilde{u}(X_1), \tilde{u}(X_2) \le Ch^2.$$

In conclusion, u separates quadratically on ∂B_1 from its tangent planes and therefore satisfies the hypotheses of the Localization Theorem in [Savin 2013; Le and Savin 2013]. From [Le and Savin 2013, Theorem 2.4 and Proposition 2.6], we conclude that

$$\|u\|_{C^{1,\beta}(\bar{B}_1)}, \ \|v\|_{C^\beta(\bar{B}_1)}, \ \|v_\nu\|_{C^\beta(\partial B_1)} \le C, \tag{4-5}$$

for some $\beta < 1$, C universal. □

Lemma 4.9. *Let $f : [-h, h] \to \mathbb{R}^+$ be a nonnegative convex function such that*

$$\frac{f(-h)+f(h)}{2} - \frac{1}{2h}\int_{-h}^{h} f(x)\,dx \le Mh^2, \qquad f(t) \le Mh^2,$$

for some $t \in [-h/2, h/2]$. Then

$$f(\pm h) \le Ch^2$$

for some C depending on M.

Proof. The inequality above states that the area between the line segment with end points $(-h, f(-h))$, $(h, f(h))$ and the graph of f is bounded by $2Mh^3$. By convexity, this area is greater than the area of the triangle with vertices $(-h, f(-h))$, $(t, f(t))$, $(h, f(h))$. Now the inequality of the heights $f(\pm h)$ follows from elementary euclidean geometry. □

Finally, we are ready to prove the second part of Proposition 3.1.

Proof of Proposition 3.1 (*ii*). Let φ be such that

$$U^{ij}\varphi_{ij} = 0 \text{ in } B_1, \quad \varphi \in C^{1,1}(\partial B_1) \cap C^0(\overline{B}_1).$$

Since u satisfies the quadratic separation assumption and f is smooth up to the boundary, we obtain from [Le and Savin 2013, Theorem 2.5 and Proposition 2.6]

$$\|v\|_{C^{1,\beta}(\overline{B}_1)},\ \|\varphi\|_{C^{1,\beta}(\overline{B}_1)} \le K, \quad \text{and} \quad |U^{ij}| \le K|\log\delta|^2 \text{ on } B_{1-\delta},$$

for some constant K depending on ρ, $\|f\|_{C^\beta(\overline{B}_1)}$, and $\|\varphi\|_{C^{1,1}(\partial B_1)}$.

We will use the following identity in two dimensions:

$$U^{ij}v_j\nu_i = U^{\tau\nu}v_\tau + U^{\nu\nu}v_\nu.$$

Integrating by parts twice, we obtain, as in (1-1),

$$\int_{B_{1-\delta}} \varphi\,dA = -\int_{B_{1-\delta}} \varphi\,U^{ij}v_{ij}\,dx = \int_{\partial B_{1-\delta}} \varphi_i U^{ij} v\nu_j - \int_{\partial B_{1-\delta}} \varphi U^{ij}v_j\nu_i = -\int_{\partial B_{1-\delta}} \varphi U^{\nu\nu}v_\nu + o(1),$$

where in the last equality we used the estimates

$$|v| \le C\delta, \quad |v_\tau| \le K\delta^\beta, \quad |\varphi|, |\nabla\varphi| \le K, \quad U^{ij} \le K|\log\delta|^2 \quad \text{on } \partial B_{1-\delta}.$$

Since on ∂B_r

$$U^{\nu\nu} = u_{\tau\tau} = r^{-2}u_{\theta\theta} + r^{-1}u_\nu,$$

$u \in C^{1,\beta}(\overline{B}_1)$ and $u(re^{i\theta})$ converges uniformly as $r \to 1$, and $u_{\theta\theta}$ is uniformly bounded from below, we obtain

$$U^{\nu\nu}\,d\mathcal{H}^1\lfloor_{\partial B_r} \rightharpoonup (u_{\theta\theta}+u_\nu)\,d\mathcal{H}^1\lfloor_{\partial B_1} \quad \text{as } r \to 1.$$

We let $\delta \to 0$ in the equality above and find

$$\int_{B_1} \varphi\,dA = -\int_{\partial B_1} \varphi\,(u_{\theta\theta}+u_\nu)v_\nu\,d\mathcal{H}^1.$$

Now the Euler–Lagrange equation, Proposition 3.6, gives

$$(u_{\theta\theta} + u_\nu)v_\nu = -\sigma \quad \text{on } \partial B_1.$$

We use that $\|v_\nu\|_{C^\beta(\partial B_1)} \le C$ and, from Lemma 4.8, $v_\nu \le -c$ on ∂B_1 and obtain

$$\|u\|_{C^{2,\gamma}(\partial B_1)} \le C\|\sigma\|_{C^\gamma(\partial B_1)}. \qquad \square$$

5. The general case for A

In this section, we remove the assumptions that A is bounded from below by ρ in B_1 and we also assume that A is bounded from above only in a neighborhood of the boundary. Precisely, we assume that $A \ge 0$ in B_1 and $A \le \rho^{-1}$ in $B_1 \setminus \overline{B}_{1-\rho}$. We may also assume A is smooth in B_1, since the general case follows by approximation. Notice that $\int_{B_1} A\,dx$ is bounded from above and below since it equals $\int_{\partial B_1} d\sigma$.

Let v be the solution of the Dirichlet problem

$$U^{ij} v_{ij} = -A, \qquad v = 0 \text{ on } \partial B_1. \tag{5-1}$$

In Section 4, we used that A is bounded from above when we obtained $v \le C(1-|x|^2)$, and we used that A is bounded from below in Lemma 4.8 (see (4-4)). We need to show that these bounds for v also hold in a neighborhood of ∂B_1 under the weaker hypotheses above. First, we show:

Lemma 5.1. $\quad v \le C$ *on* $\partial B_{1-\rho/2}$ *and* $v \ge c(\delta)$ *on* $B_{1-\delta}$,

with C universal and $c(\delta) > 0$ depending also on δ.

Proof. As before, we may assume that $u \in C^\infty(\overline{B}_1)$, since the general case follows by approximating B_1 by $B_{1-\epsilon}$.

We multiply the equation in (5-1) by $(1-|x|^2)$, integrate by parts twice, and obtain

$$\int_{B_1} 2v \operatorname{tr} U\,dx = \int_{B_1} A(x)(1-|x|^2)\,dx \le C,$$

and since $\operatorname{tr} U \ge c$, we obtain

$$\int_{B_1} v\,dx \le C.$$

We know this:

(1) $v \ge 0$ solves a linearized Monge–Ampère equation with bounded right-hand side in $B_1 \setminus B_{1-\rho}$.

(2) u has a uniform modulus of convexity on compact sets of B_1.

Now we use the Harnack inequality of Caffarelli and Gutierrez [1997] and conclude that

$$\sup_{\mathcal{V}} v \le C(\inf_{\mathcal{V}} v + 1), \qquad \mathcal{V} := B_{1-\rho/4} \setminus \overline{B}_{1-3\rho/4},$$

and the integral inequality above gives $\sup_{\mathcal{V}} v \le C$.

Next we prove the lower bound. We multiply the equation in (5-1) by $\varphi \in C_0^\infty(B_1)$ with

$$\varphi = 0 \text{ if } |x| \ge 1-\delta/2, \quad \varphi = 1 \text{ in } B_{1-\delta}, \quad \|D^2\varphi\| \le C/\delta^2,$$

integrate by parts twice, and obtain

$$C(\delta)\int_{\mathcal{U}} v\,\mathrm{tr}\,U \geq -\int_{B_1} v\,U^{ij}\varphi_{ij} = \int_{B_1} A\varphi \geq c, \qquad \mathcal{U} := B_{1-\delta/2}\setminus \overline{B}_{1-\delta},$$

where the last inequality holds provided that δ is sufficiently small. Since u is normalized, we obtain (see Proposition 3.2) $|\nabla u| \leq C(\delta)$ in $\mathcal{U}$, and thus

$$\int_{\mathcal{U}} \mathrm{tr}\; U = \int_{\mathcal{U}} \triangle u = \int_{\partial\mathcal{U}} u_\nu \leq C(\delta).$$

The last two inequalities imply $\sup_{\mathcal{U}} v \geq c(\delta)$, and hence there exists $x_0 \in \mathcal{U}$ such that $v(x_0) \geq c(\delta)$. We use (1), (2) above and the Harnack inequality and find $v \geq c(\delta)$ in $B_{\bar{\delta}}(x_0)$ for some small $\bar{\delta}$ depending on ρ and δ. Since v is a supersolution, that is, $U^{ij}v_{ij} \leq 0$, we can apply the weak Harnack inequality of Caffarelli and Gutierrez [1997, Theorem 4]. From property (2) above, we see that we can extend the lower bound of v from $B_{\bar{\delta}}(x_0)$ all the way to $\mathcal{U}$, and by the maximum principle, this bound holds also in $B_{1-\delta/2}$. □

The upper bound in Lemma 5.1 gives as in (4-2) the upper bound for v in a neighborhood of ∂B_1, that is,

$$v(x) \leq C(1-|x|^2) \quad \text{on} \quad B_1 \setminus B_{1-\rho/2}.$$

This implies, as in Section 4, that Lemma 4.7 holds, that is, u separates at least quadratically from its tangent planes on ∂B_1. It remains to show that also Lemma 4.8 holds. Since A is not strictly positive, $c(l-u)$ is no longer a subsolution for the equation (5-1) and we cannot bound v below as we did in (4-4). In the next lemma, we construct another barrier which allows us to bound v from below on the segment $[z, z^S]$.

Lemma 5.2. *Let $\tilde{u} : B_1 \to \mathbb{R}$ be a convex function with $\tilde{u} \in C(\overline{B}_1)\cap C^2(B_1)$, and*

$$\rho \leq \det D^2\tilde{u} \leq \rho^{-1}.$$

Assume that the section $S := \{\tilde{u} < 0\}$ is included in B_1 and is tangent to ∂B_1 at a point $z \in \partial B_1$, and also that

$$\inf_S \tilde{u} \leq -\mu,$$

for some $\mu > 0$. If

$$\tilde{U}^{ij} v_{ij} \leq 0 \text{ in } B_1, \quad v \geq 0 \text{ on } \partial B_1,$$

then

$$v(x) \geq c(\mu,\rho)|x-z| \inf_{S'} v \quad \text{for all } x \in [z, z^S], \qquad S' := \left\{\tilde{u} \leq \tfrac{1}{2}\inf_S \tilde{u}\right\},$$

where z^S denotes the center of mass of S and $c(\mu,\rho)$ is a positive constant depending on μ and ρ.

The functions $\tilde{u} = u - l$ and v in the proof of Lemma 4.8 satisfy the lemma above, if η in Lemma 4.6 is small, universal. Using also the lower bound on v from Lemma 5.1, we find

$$v \geq c|x-z| \text{ on } [z, z^S],$$

for some c universal, and the rest of the proof of Lemma 4.8 follows as before. This shows that Proposition 3.1 holds also with our assumptions on the measure A.

Proof of Lemma 5.2. We construct a lower barrier for v of the type

$$w := e^{k\bar{w}} - 1, \quad \bar{w} := -\tilde{u} + \frac{\epsilon}{2}(|x|^2 - 1),$$

for appropriate constants k large and $\epsilon \ll \mu$ small. Notice that $w \le 0$ on ∂B_1, since $\bar{w} \le 0$ on ∂B_1. Also

$$\bar{w} \ge c|x - z| \text{ on } [z, z^S],$$

since, by convexity, $-\tilde{u} \ge c|x - z|$ on $[z, z^S]$ for some c depending on μ and ρ. It suffices to check that

$$\tilde{U}^{ij} w_{ij} \ge 0 \text{ on } B_1 \setminus S',$$

since then we obtain $v \ge (\inf_{S'} v)cw$ in $B_1 \setminus S'$, which easily implies the conclusion. In $B_1 \setminus S'$ we have $|\nabla \bar{w}| \ge c(\mu) > 0$, provided that ϵ is sufficiently small, and thus

$$\tilde{U}^{ij} \bar{w}_i \bar{w}_j = (\det D^2\tilde{u})(\nabla\bar{w})^T (D^2\tilde{u})^{-1} \nabla\bar{w} \ge c\Lambda^{-1},$$

where Λ is the largest eigenvalue of $D^2\tilde{u}$. Then we use that $\operatorname{tr}\tilde{U} \ge c\lambda^{-1} \ge c\Lambda^{1/(n-1)}$, where λ is the smallest eigenvalue of $D^2\tilde{u}$, and obtain

$$\begin{aligned} \tilde{U}^{ij} w_{ij} &= ke^{k\bar{w}}\big(\tilde{U}^{ij}\bar{w}_{ij} + k\tilde{U}^{ij}\bar{w}_i\bar{w}_j\big) \ge ke^{k\bar{w}}\big(-n\rho^{-1} + \epsilon \operatorname{tr}\tilde{U} + kc\Lambda^{-1}\big) \\ &\ge ke^{k\bar{w}}\big(-n\rho^{-1} + c(\epsilon\Lambda^{1/(n-1)} + k\Lambda^{-1})\big) \ge 0, \end{aligned}$$

if k is chosen large depending on ϵ, ρ, μ and n. □

6. Singular minimizers in dimension $n \ge 3$.

Let

$$u(x) := |x'|^{2-2/n} h(x_n)$$

be the singular solution to $\det D^2 u = 1$ constructed by Pogorelov, with h a smooth even function, defined in a neighborhood of 0 and $h(0) = 1$, satisfying an ODE

$$\left(\left(1 - \frac{2}{n}\right)hh'' - \left(2 - \frac{2}{n}\right)h'^2\right)h^{n-2} = c.$$

We let

$$v(x) := |x'|^{2-2/n} q(x_n)$$

be obtained as the infinitesimal difference between u and a rescaling of u,

$$v(x', x_n) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big[u(x', x_n) - (1+\epsilon)^{-\gamma} u(x', (1+\epsilon)x_n)\big],$$

for some small $\gamma < \dfrac{2}{n}$. Notice that

$$q(t) = \gamma h(t) - h'(t)t$$

and $q > 0$ in a small interval $(-a, a)$ and q vanishes at its end points. Also,

$$U^{ij} v_{ij} = n\gamma - 2 < 0 \quad \text{in } \Omega := \mathbb{R}^{n-1} \times [-a, a],$$
$$v = 0, \quad U^{\nu\nu} v_\nu = U^{nn} v_n = -\sigma_0 \quad \text{on } \partial\Omega,$$

for some constant $\sigma_0 > 0$. The last equality follows since U^{nn} is homogeneous of degree $-(n-1)(2/n)$ in $|x'|$ and v_n is homogeneous of degree $2 - 2/n$ in $|x'|$.

Notice that u, v are solutions of the system (1-2) in the infinite cylinder Ω for uniform measures A and σ. In order to obtain a solution in a finite domain Ω_0, we modify v outside a neighborhood of the line $|x'| = 0$ by subtracting a smooth convex function ψ which vanishes in B_1 and increases rapidly outside B_1. Precisely, we let

$$\tilde{v} := v - \psi, \quad \Omega_0 := \{\tilde{v} > 0\},$$

and then we notice that u, $\tilde{v}$, solve the system (1-2) in the smooth bounded domain Ω_0 for smooth measures A and σ.

Since

$$|U^{ij}| \le C r^{(2/n)-2} \quad \text{if } |x'| \ge r,$$

we integrate by parts in the domain $\Omega_0 \setminus \{|x'| \le \epsilon\}$ and then let $\epsilon \to 0$ and find

$$\int_{\Omega_0} \varphi \, dA = -\int_{\Omega_0} U^{ij} \varphi_{ij} v + \int_{\partial\Omega_0} \varphi \, d\sigma, \quad \text{for all } \varphi \in C^2(\overline{\Omega}_0),$$

or

$$L(\varphi) = \int_{\Omega_0} U^{ij} \varphi_{ij} v.$$

This implies that L is stable, that is, $L(\varphi) > 0$ for any convex φ which is not linear. Also, if $w \in C^2(\overline{\Omega}_0)$ satisfies $\det D^2 w = 1$, then $U^{ij}(w - u)_{ij} \ge 0$, and we obtain

$$L(w) - L(u) = \int_{\Omega_0} U^{ij} (w - u)_{ij} v \ge 0,$$

that is, u is a minimizer of L.

We remark that the domain Ω_0 has flat boundary in a neighborhood of the line $\{|x'| = 0\}$, and therefore is not uniformly convex. However, this is not essential in our example. One can construct, for example, a function $\bar{v}$ in a uniformly convex domain by modifying v as

$$\bar{v} := |x'|^{2-2/n} q(x_n(1 + \delta |x'|^2)),$$

for some small $\delta > 0$.

7. Proof of Theorem 1.4

We assume for simplicity that $\Omega = B_1$. The existence of a minimizer u for the convex functional E follows as in Section 2. First, we show that

$$t_1 \le \det D^2 u \le t_0 \tag{7-1}$$

for some t_1 depending on F and ρ. The upper bound follows easily. If $\det D^2u > t_0$ in a set of positive measure, then the function w defined as

$$\det D^2 w = \min\{t_0, \det D^2 u\}, \quad w = u \text{ on } \partial B_1,$$

satisfies $E(w) < E(u)$, since $F(\det D^2 w) = F(\det D^2 u)$ and $L(w) < L(u)$.

In order to obtain the lower bound in (7-1), we need the following lemma.

Lemma 7.1. *Let w be a convex function in B_1 with*

$$(\det D^2 w)^{1/n} = g \in L^n(B_1).$$

Let $w+\varphi$ be another convex function in B_1 with the same boundary values as w such that

$$(\det D^2(w+\varphi))^{1/n} = g - h, \quad \textit{for some } h \geq 0.$$

Then

$$\int_{B_1} \varphi g^{n-1} \leq C(n) \int_{B_1} h g^{n-1}.$$

Proof. By approximation, we may assume that w, φ are smooth in $\overline{B}_1$. Using the concavity of the map $M \mapsto (\det M)^{1/n}$ in the space of symmetric matrices $M \geq 0$, we obtain

$$(\det D^2(w+\varphi))^{1/n} \leq (\det D^2 w)^{1/n} + \frac{1}{n}(\det D^2 w)^{(1/n)-1} W^{ij}\varphi_{ij},$$

and hence

$$-nhg^{n-1} \leq W^{ij}\varphi_{ij}.$$

We multiply both sides by $\Phi := \frac{1}{2}(1-|x|^2)$ and integrate. Since both φ and Φ vanish on ∂B_1 we integrate by parts twice and obtain

$$-C(n)\int_{B_1} hg^{n-1} \leq \int_{B_1} W^{ij}\Phi_{ij}\varphi = -\int_{B_1} (\operatorname{tr} W)\varphi.$$

Using

$$\operatorname{tr} W \geq c(n)(\det W)^{1/n} = c(n)(\det D^2 w)^{(n-1)/n} = c(n) g^{n-1},$$

we obtain the desired conclusion. □

Now we prove the lower bound in (7-1). Define w such that $w = u$ on ∂B_1 and

$$\det D^2 w = \max\{t_1, \det D^2 u\}$$

for some small t_1. Since $G(t) = F(t^n)$ is convex and $\det D^2 w \geq t_1$, we have

$$G\big((\det D^2 w)^{1/n}\big) \leq G\big((\det D^2 u)^{1/n}\big) + G'(t_1^{1/n})\big((\det D^2 w)^{1/n} - (\det D^2 u)^{1/n}\big).$$

We write

$$u - w = \varphi, \quad (\det D^2 w)^{1/n} = g, \quad (\det D^2 u)^{1/n} = g - h,$$

and we rewrite the inequality above as

$$F(\det D^2 w) \le F(\det D^2 u) + G'(t_1^{1/n})h.$$

From Lemma 7.1, we obtain

$$\int_{B_1} h\, g^{n-1} \ge c(n) \int_{B_1} \varphi\, g^{n-1},$$

and since h is supported on the set where the value of $g = t_1^{1/n}$ is minimal, we find that

$$\int_{B_1} h \ge c(n) \int_{B_1} \varphi.$$

This gives

$$\int_{B_1} F(\det D^2 w) - F(\det D^2 u) \le c(n) G'(t_1^{1/n}) \int_{B_1} \varphi,$$

and thus, using the minimality of u and $G'(0^+) = -\infty$,

$$0 \le E(w) - E(u) \le \int_{B_1} \varphi dA + c(n) G'(t_1^{1/n}) \int_{B_1} \varphi \le 0,$$

if t_1 is small enough. In conclusion, $\varphi = 0$ and $u = w$ and (7-1) is proved.

We write

$$\det D^2 u = f, \quad t_1 \le f \le t_0.$$

Any minimizer for L in the class of functions whose determinant equals f is a minimizer for E as well. In order to apply Theorem 1.2, we need f to be Holder continuous. However, we can approximate f by smooth functions f_n and find smooth minimizers u_n for approximate linear functionals L_n with the constraint

$$\det D^2 u_n = f_n.$$

By Proposition 3.1 (see (4-5)),

$$\|u_n\|_{C^{1,\beta}(\overline{B}_1)},\ \|v_n\|_{C^{\beta}(\overline{B}_1)} \le C,$$

and hence we may assume (see Theorem 1.1) that, after passing to a subsequence, $u_n \to u$ and $v_n \to v$ uniformly for some function $v \in C^{\beta}(\overline{B}_1)$. We show that

$$v = -F'(f). \tag{7-2}$$

Then by the hypotheses on F, we obtain $\det D^2 u = f \in C^{\beta}(\overline{B}_1)$, and from Theorem 1.2, we easily obtain

$$\|u\|_{C^{2,\alpha}(\overline{B}_1)},\ \|v\|_{C^{2,\alpha}(\overline{B}_1)} \le C$$

for some C depending on ρ, α, $\|\sigma\|_{C^{\alpha}(\overline{B}_1)}$, $\|A\|_{C^{\alpha}(\overline{B}_1)}$, and F.

In order to prove (7-2), we need a uniform integral bound (in two dimensions) between solutions to the Monge–Ampère equation and solutions of the corresponding linearized equation.

The proof of the following lemma will be given at the end of the section.

Lemma 7.2. *Assume $n = 2$ and let w be a smooth convex function in $\overline{B}_1$ with*

$$\lambda \leq \det D^2 w := g \leq \Lambda$$

for some positive constants λ, Λ. Let $w + \epsilon\varphi$ be a convex function with

$$\det D^2(w + \epsilon\varphi) = g + \epsilon h, \quad \varphi = 0 \text{ on } \partial B_1$$

for some smooth function h with $\|h\|_{L^\infty} \leq 1$. If $\epsilon \leq \epsilon_0$, then

$$\int_{B_1} |h - W^{ij}\varphi_{ij}| \leq C\epsilon$$

for some C, ϵ_0 depending only on λ, Λ.

Now let h be a smooth function, $\|h\|_{L^\infty} \leq 1$, and we solve the equations

$$\det D^2(u_n + \epsilon\varphi_n) = f_n + \epsilon h, \quad \varphi_n = 0 \text{ on } \partial B_1,$$

with u_n, f_n as above. From (1-1) we see that

$$L_n(\varphi_n) = \int_{B_1} (U_n^{ij}\partial_{ij}\varphi_n)v_n,$$

and hence, by the lemma above,

$$\left| L_n(\varphi_n) - \int_{B_1} hv_n \right| \leq C\epsilon$$

with C universal. We let $n \to \infty$ and obtain

$$\left| L(\varphi) - \int_{B_1} hv \right| \leq C\epsilon,$$

with φ the solution of

$$\det D^2(u + \epsilon\varphi) = f + \epsilon h, \quad \varphi = 0 \text{ on } \partial B_1.$$

The inequality $E(u + \epsilon\varphi) \geq E(u)$ implies

$$\int_{B_1} \big(F(f + \epsilon h) - F(f) + \epsilon hv\big) \geq -C\epsilon^2,$$

and hence, as $\epsilon \to 0$,

$$\int_{B_1} (F'(f) + v)h \geq 0 \quad \text{for any smooth } h,$$

which gives (7-2). □

Proof of Lemma 7.2. Using the concavity of $(\det D^2 w)^{1/n}$, we obtain

$$(g + \epsilon h)^{1/n} \leq g^{1/n} + \frac{\epsilon}{n} g^{1/n-1} W^{ij}\varphi_{ij},$$

and thus, for $\epsilon \leq \epsilon_0$,

$$h - C\epsilon \leq W^{ij}\varphi_{ij}. \tag{7-3}$$

Since $n = 2$, we have

$$\det D^2(w + \epsilon\varphi) = \det D^2 w + \epsilon W^{ij}\varphi_{ij} + \epsilon^2 \det D^2\varphi,$$

and hence

$$h - W^{ij}\varphi_{ij} = \epsilon \det D^2\varphi.$$

From the pointwise inequality (7-3), we see that in order to prove the lemma, it suffices to show that

$$\int_{B_1} \det D^2\varphi \geq -C.$$

Let $\Phi = (\Phi^{ij})$ be the cofactor matrix of $D^2\phi$. Integrating by parts and using $\varphi = 0$ on ∂B_1, we find

$$\int_{B_1} 2 \det D^2\varphi = \int_{B_1} \Phi^{ij}\varphi_{ij} = \int_{\partial B_1} \Phi^{ij}\varphi_i \nu_j = \int_{\partial B_1} \Phi^{\nu\nu}\varphi_\nu = \int_{\partial B_1} \varphi_\nu^2 \geq 0,$$

where we used $\Phi^{\nu\nu} = \varphi_{\tau\tau} = \varphi_\nu$. □

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