ANALYSIS & PDE

Volume 6 No. 5 2013

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ANALYSIS AND PDE
Vol. 6, No. 5, 2013
dx.doi.org/10.2140/apde.2013.6.1213

NODAL COUNT OF GRAPH EIGENFUNCTIONS VIA MAGNETIC PERTURBATION

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We establish a connection between the stability of an eigenvalue under a magnetic perturbation and the number of zeros of the corresponding eigenfunction. Namely, we consider an eigenfunction of discrete Laplacian on a graph and count the number of edges where the eigenfunction changes sign (has a "zero"). It is known that the n-th eigenfunction has $n-1+s$ such zeros, where the "nodal surplus" s is an integer between 0 and the first Betti number of the graph.

We then perturb the Laplacian with a weak magnetic field and view the n-th eigenvalue as a function of the perturbation. It is shown that this function has a critical point at the zero field and that the Morse index of the critical point is equal to the nodal surplus s of the n-th eigenfunction of the unperturbed graph.

1. Introduction

Studying zeros of eigenfunctions is a question with rich history. While experimental observations have been mentioned by Leonardo da Vinci [MacCurdy 1938], Galileo [1638] and Hooke [Birch 1756], and greatly systematized by Chladni [1787], the first mathematical result is probably due to Sturm [1836]. The Oscillation Theorem of Sturm states that the number of internal zeros of the n-th eigenfunction of a Sturm–Liouville operator on an interval is equal to $n-1$. Equivalently, the zeros of the n-th eigenfunction divide the interval into n parts. In higher dimensions, the latter equality becomes a one-sided inequality: Courant [1923] (see also [Courant and Hilbert 1953]) proved that the zero curves (surfaces) of the n-th eigenfunction of the Laplacian divide the domain into at most n parts (called the "nodal domains").

Recently, there has been a resurgence of interest in counting the nodal domains of eigenfunctions, with many exciting conjectures and rigorous results. The nodal count seems to have universal features [Blum et al. 2002; Bogomolny and Schmit 2002; Nazarov and Sodin 2009], is conjectured to resolve isospectrality [Gnutzmann et al. 2006], and has connections to minimal partitions of the domain [Helffer et al. 2009; Berkolaiko et al. 2012a], to name but a few. For a selection of research articles and historical reviews, see [Smilansky and Stöckmann 2007].

On graphs, the question can be formulated regarding the signs of the eigenfunctions of the operator

$$H : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}, \qquad H = Q - C, \tag{1}$$

where V is the set of the vertices of the graph, Q is an arbitrary real diagonal matrix, and C is the adjacency matrix of the graph. The operator H is a discrete analogue of the Schrödinger operator with

The author is supported by the NSF grant DMS-0907968.
MSC2010: 05C50, 58J50, 81Q10, 81Q35.
Keywords: discrete Laplace operator, nodal count, discrete magnetic Schrödinger operator.

electric potential. In this discrete setting, by a "zero" we understand an edge on which the eigenfunction changes sign, and not the exceptional (with respect to perturbation of Q) situation of an eigenfunction having a zero entry.

The subject of sign changes and nodal domains (connected components of the graph left after cutting the above edges) was addressed by, among others, Fiedler [1975], who showed the analogue of Sturm equality for *tree graphs* (see also [Bıyıkoğlu 2003]); Davies, Gladwell, Leydold and Stadler [Davies et al. 2001], who proved an analogue of the Courant (upper) bound for the number of nodal domains; Berkolaiko [2008], who proved a lower bound for graphs with cycles; and Oren [2007], who found a bound for the nodal domains in terms of the chromatic number of the graph. A number of predictions regarding the nodal count in regular graphs (assuming an adaptation of the random wave model) is put forward in [Elon 2008]. For more information, the interested reader is referred to [Bıyıkoğlu et al. 2007; Band et al. 2008].

The study of the magnetic Schrödinger operator on graphs has a similarly rich history. To give a sample, Harper [1955] used the tight-binding model (discrete Laplacian) to describe the effect of the magnetic field on conduction (see also [Hofstadter 1976]). In mathematical literature, the discrete magnetic Schrödinger operator was introduced by Lieb and Loss [1993] and Sunada [1993; 1994], and studied in [Shubin 1994; Colin de Verdière 1998; Colin de Verdière et al. 2011], among other sources (see also [Sunada 2008] for a review).

In this paper, we present a surprising connection between the two topics, namely, the number of sign changes of the n-th eigenfunction and the behavior of the eigenvalue λ_n under the perturbation of the operator H by a magnetic field. To make a precise statement, we need to introduce some notation.

The eigenvalues of the operator H on a connected graph are ordered in increasing fashion,

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{|V|}.$$

We will only consider the eigenvalue–eigenfunction pairs $(\lambda_n, f^{(n)})$ such that the eigenvalue is simple and the eigenfunction is nonzero at the vertices of the graph. This situation is generic with respect to perturbations of the potential Q, and thus we will refer to the members of such a pair as a *generic eigenvalue* and a *generic eigenfunction* correspondingly. We denote by ϕ_n the number of *sign changes* (also called *sign flips*, hence the notation ϕ) which are defined as the edges of the graph at whose endpoints the eigenfunction $f^{(n)}$ has different signs. The combined results of [Fiedler 1975; Berkolaiko 2008; Berkolaiko et al. 2012b] bound the number ϕ_n by

$$n - 1 \leq \phi_n \leq n - 1 + \beta, \tag{2}$$

where $\beta := |E| - |V| + 1$ is the first Betti number (the number of independent cycles) of the graph. Here and throughout the manuscript, we assume that the graph is connected. We will call the quantity

$$\sigma_n = \phi_n - (n - 1), \qquad 0 \leq \sigma_n \leq \beta \tag{3}$$

the *nodal surplus*. This is the extra number of sign changes that an eigenfunction has due to the graph's nontrivial topology.

A magnetic field on discrete graphs has been introduced in, among other sources, [Lieb and Loss 1993; Sunada 1994; Colin de Verdière 1998]. Up to unitary equivalence, it can be specified using β phases $\vec{\alpha} = (\alpha_j)_{j=1}^{\beta} \in (-\pi, \pi]^{\beta}$ that will be described in Section 2. We consider the eigenvalues of the graph as functions of the parameters $\vec{\alpha}$. The zero phases, $\vec{\alpha} = 0$, correspond to the graph Γ without the magnetic field. We are now ready to formulate our main result, which connects the behavior of the eigenvalue $\lambda_n(\vec{\alpha})$ as a function of the magnetic phases to the number of zeros of the eigenfunction at $\vec{\alpha} = 0$.

Theorem 1.1. *The point $\vec{\alpha} = 0$ is the critical point of the function $\lambda_n(\vec{\alpha})$. If $\lambda_n(0)$, the n-th eigenvalue of the nonmagnetic operator, is generic, then this critical point is nondegenerate and its Morse index — the number of negative eigenvalues of the Hessian — is equal to the nodal surplus σ_n of the eigenfunction $f^{(n)}$ of the nonmagnetic operator.*

An immediate consequence of this theorem is the following.

Corollary 1.2. *The generic n-th eigenvalue of the discrete Schrödinger operator is stable with respect to magnetic perturbation of the operator if and only if the corresponding eigenfunction has exactly $n-1$ sign changes.* (By "stability" we mean that the eigenvalue has a local minimum at zero magnetic field.)

Other possible consequences of our result and links to several other questions are discussed in Section 6. The rest of the paper is structured as follows. In Section 2 we provide detailed definitions. Section 3 is devoted to a duality between the magnetic perturbation and a certain perturbation to the potential, coupled with removal of edges. This leads to an alternative proof of the result in the case $\beta = 1$ (Subsection 3.3), which, although unnecessary for the general proof, provides us with some important insights. Section 4 collects the tools necessary for the proof of Theorem 1.1, while Section 5 contains the proof itself, which is done by extending the magnetic phases into the complex plane and relating the purely imaginary phases to the edge-removal perturbation.

2. The magnetic Hamiltonian on discrete graphs

Let $\Gamma = (V, E)$ be a simple finite connected graph with vertex set V and edge set E. We define the Schrödinger operator with potential $q : V \to \mathbb{R}$ by

$$H : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}, \qquad (H\psi)_u = -\sum_{v \sim u} \psi_v + q_u \psi_u, \tag{4}$$

that is, the matrix H is

$$H = Q - C, \tag{5}$$

where Q is the diagonal matrix of site potentials q_u and C is the adjacency matrix of the graph. It is perhaps more usual (and physically motivated) to represent the Hamiltonian as $H = Q + L$, where the Laplacian L is given by $L = D - C$ with D being the diagonal matrix of vertex degrees. But since we will not be imposing any restrictions on the potential Q, we absorb the matrix D into Q.

The operator H has $|V|$ eigenvalues, which we number in increasing order:

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_{|V|}.$$

We define the magnetic Hamiltonian (magnetic Schrödinger operator) on discrete graphs as

$$(H\psi)_u = -\sum_{v\sim u} e^{iA_{v,u}}\psi_v + q_u\psi_u, \tag{6}$$

with the convention that $A_{v,u} = -A_{u,v}$, which makes H self-adjoint. For further details, the reader should consult [Lieb and Loss 1993; Sunada 1994; Colin de Verdière 1998; Colin de Verdière et al. 2011].

A sequence of directed edges $C = [u_1, u_2, \dots, u_n]$ is called a cycle if the terminus of edge u_j coincides with the origin of the edge u_{j+1} for all j (u_{n+1} is understood as u_1). The flux through the cycle C is defined as

$$\Phi_C = \left(A_{u_1,u_2} + \cdots + A_{u_{n-1},u_n} + A_{u_n,u_1}\right) \mod 2\pi. \tag{7}$$

Two operators which have the same flux through every cycle C are unitarily equivalent (by a gauge transformation). Therefore, the effect of the magnetic field on the spectrum is fully determined by β fluxes through a chosen set of basis cycles of the cycle space. We denote them by $\alpha_1, \dots, \alpha_\beta$ and consider the n-th eigenvalue of the graph as a function of $\vec{\alpha}$.

More precisely, fix an arbitrary spanning tree of the graph and let S be the set of edges that do not belong to the chosen tree. Obviously, S contains exactly β edges.

Lemma 2.1. *Any magnetic Schrödinger operator on the graph Γ is unitarily equivalent to one of the operators of the type*

$$H_{u,v} = \begin{cases} q_u, & u = v, \\ -1, & (u, v) \in \mathscr{E} \setminus S, \\ -e^{\pm i\alpha_s}, & (u, v) = s \in S, \end{cases} \tag{8}$$

where the sign in the exponent is plus if $u < v$ and minus if $u > v$.

Example 2.2. Consider the triangle graph — a graph with three vertices and three edges connecting them. One of the equivalent forms of the magnetic Hamiltonian for this graph is

$$H(\Gamma^{\alpha}_{\mathrm{mag}}) = \begin{pmatrix} q_1 & -e^{i\alpha} & -1 \\ -e^{-i\alpha} & q_2 & -1 \\ -1 & -1 & q_3 \end{pmatrix}.$$

The spectrum of $H(\Gamma^{\alpha}_{\mathrm{mag}})$ as a function of $\alpha \in (-\pi, \pi]$ is shown in Figure 1. The eigenfunctions of $H(\Gamma) = H(\Gamma^{\alpha=0}_{\mathrm{mag}})$ have $\phi_1 = 0$, $\phi_2 = 2$ and $\phi_3 = 2$ sign changes correspondingly (these are the only choices consistent with (2) and the topology of the graph). The nodal surpluses are $\sigma_1 = 0$, $\sigma_2 = 1$ and $\sigma_3 = 0$, which agrees with $\alpha = 0$ being the point of minimum, maximum and minimum of the corresponding curves.

3. A duality between a magnetic phase and a cut

In this section, we explore a simple result which shows a connection between two types of perturbations of the operator H that will be used to prove the main theorem. It illustrates the duality between the perturbation of a discrete Schrödinger operator by a magnetic phase on a cycle and the operation of

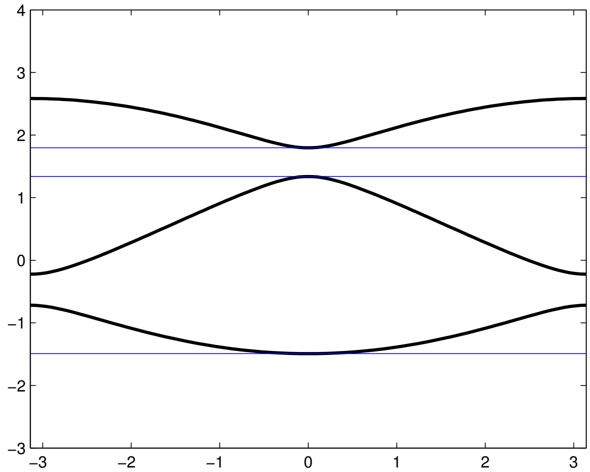

Figure 1. The eigenvalues of the triangle graph as functions of a magnetic phase α (bold curves) and the eigenvalues of the unperturbed graph (horizontal lines).

removing ("cutting") an edge that lies on the cycle. The latter operation was used to prove the lower bound on the number of nodal domains in [Berkolaiko 2008] and to study partitions on discrete graphs in [Berkolaiko et al. 2012b].

Tools used. The result of this section (Theorem 3.3 below) is based on the following version of Weyl's inequality of linear algebra that can be obtained using the variational characterization of the eigenvalues (see [Horn and Johnson 1985, Chapter 4] for similar results).

Theorem 3.1. *Let A be a self-adjoint matrix and B be a rank-one positive semidefinite self-adjoint matrix. Then*

$$\lambda_n(A-B) \le \lambda_n(A) \le \lambda_{n+1}(A-B), \tag{9}$$

where λ_n is the n-th eigenvalue, numbered in increasing order, of the corresponding matrix. Moreover, the inequalities are strict if and only if $\lambda_n(A)$ is simple and its eigenvector is not in the null-space of B.

Similarly, when B is negative definite, we have

$$\lambda_{n-1}(A-B) \le \lambda_n(A) \le \lambda_n(A-B), \tag{10}$$

with an analogous condition for strict inequalities.

Another useful result is the first term in the perturbation expansion of a parameter-dependent eigenvalue. Let $A(x)$ be a Hermitian matrix-valued analytic function of x. Let $\lambda(x)$ be an eigenvalue of the matrix A that is simple in a neighborhood of a point x_0. We know from standard perturbation theory [Kato 1976] that $\lambda(x)$ is an analytic function. Denote by $u(x)$ the normalized eigenvector corresponding to the eigenvalue λ. Then we have the following formula for the derivative of λ evaluated at the point $x = x_0$:

$$\frac{\partial}{\partial x}\lambda = \left\langle u, \frac{\partial A}{\partial x} u \right\rangle. \tag{11}$$

Two operations on a graph. Let λ_n be a simple eigenvalue and let the corresponding eigenfunction f be nonzero on vertices. Let (u_1, u_2) be an edge that belongs to one of the cycles of the graph. We allow the graph to have magnetic phases on some edges, but assume that there is no phase on the edge (u_1, u_2). Then the operator $H = Q - C$ has the following subblock corresponding to vertices u_1 and u_2:

$$H(\Gamma)_{[u_1,u_2]} = \begin{pmatrix} q_{u_1} & -1 \\ -1 & q_{u_2} \end{pmatrix}. \tag{12}$$

We consider two modifications of the original graph. The first modification of the graph is a cut: we remove the edge (u_1, u_2) and change the potential at sites u_1 and u_2. Namely, we change the $[u_1, u_2]$ subblock to

$$H(\Gamma_\gamma^{\text{cut}})_{[u_1,u_2]} = \begin{pmatrix} q_{u_1} - \gamma & 0 \\ 0 & q_{u_2} - 1/\gamma \end{pmatrix}, \tag{13}$$

and leave the rest of the matrix H intact. We denote this modification by $H(\Gamma_\gamma^{\text{cut}})$. Note that this modification is a rank-one perturbation of the original operator $H(\Gamma)$. Namely, $H(\Gamma_\gamma^{\text{cut}}) = H(\Gamma) - B^c$, where the matrix B^c has the $[u_1, u_2]$ subblock

$$B^c_{[u_1,u_2]} = \begin{pmatrix} \gamma & -1 \\ -1 & 1/\gamma \end{pmatrix}, \tag{14}$$

and the rest of the elements are zero. Then B^c is positive definite if $\gamma > 0$ and negative definite if $\gamma < 0$. Note that the cases $\gamma = \infty$ and $\gamma = 0$ can also be given the meaning of removing (or imposing the Dirichlet condition at) the vertex u_1 or the vertex u_2 correspondingly. However, we will not dwell on this issue, and exclude these cases from our consideration.

Notably, if f is an eigenfunction of $H(\Gamma)$ and $\gamma = f_{u_2}/f_{u_1} \in \mathbb{R}$, where f_u is the value of f at the vertex u, then f is also an eigenfunction of $H(\Gamma_\gamma^{\text{cut}})$. Equivalently, f is in the null-space of the perturbation B^c.

The second modification of the original graph is the introduction of a magnetic phase on the edge (u_1, u_2). The $[u_1, u_2]$ subblock of the new operator $H(\Gamma_{\text{mag}}^\alpha)$ is

$$H(\Gamma_{\text{mag}}^\alpha)_{[u_1,u_2]} = \begin{pmatrix} q_{u_1} & -e^{i\alpha} \\ -e^{-i\alpha} & q_{u_2} \end{pmatrix}, \tag{15}$$

while other entries coincide with those of $H(\Gamma)$. Note that $H(\Gamma_{\text{mag}}^\alpha)$ is *not* a rank-one perturbation of $H(\Gamma)$. However, it is a rank-one perturbation of the cut graph $H(\Gamma_\gamma^{\text{cut}})$ for any values of α and γ. Namely, $H(\Gamma_\gamma^{\text{cut}}) = H(\Gamma_{\text{mag}}^\alpha) - B^{mc}$, where

$$B^{mc}_{[u_1,u_2]} = \begin{pmatrix} \gamma & -e^{i\alpha} \\ -e^{-i\alpha} & 1/\gamma \end{pmatrix}, \tag{16}$$

and all other entries of B^{mc} are zero. Also, the spectra of $H(\Gamma_{\text{mag}}^\alpha)$ and $H(\Gamma)$ coincide when $\alpha = 0$ since the operators coincide.

A duality between the two operations. We now want to apply Theorem 3.1 to the spectra of Γ, $\Gamma_\gamma^{\mathrm{cut}}$ and $\Gamma_{\mathrm{mag}}^\alpha$. However, we must take care to distinguish the two cases that correspond to equations (9) and (10) ($\gamma > 0$ and $\gamma < 0$ correspondingly).

Definition 3.2. The eigenvalues of Γ, $\Gamma_\gamma^{\mathrm{cut}}$ and $\Gamma_{\mathrm{mag}}^\alpha$ will be numbered in increasing order starting from 1. We will also use the convention

$$\lambda_j(\Gamma) = \begin{cases} -\infty, & j < 1, \\ \infty, & j > n, \end{cases}$$

for the cases when the index of λ happens to be out of bounds.

Theorem 3.3. *Let $p(\gamma)$ be 1 if $\gamma < 0$ and 0 otherwise. Then the following inequalities hold*:

$$\lambda_{n-p(\gamma)}(\Gamma_\gamma^{\mathrm{cut}}) \le \lambda_n(\Gamma_{\mathrm{mag}}^\alpha) \le \lambda_{n-p(\gamma)+1}(\Gamma_\gamma^{\mathrm{cut}}), \tag{17}$$

for all values of α and γ. Furthermore, for any fixed n,

$$\max_\gamma \lambda_{n-p(\gamma)}(\Gamma_\gamma^{\mathrm{cut}}) = \min_\alpha \lambda_n(\Gamma_{\mathrm{mag}}^\alpha) =: M_1 \tag{18}$$

and

$$\max_\alpha \lambda_n(\Gamma_{\mathrm{mag}}^\alpha) = \min_\gamma \lambda_{n-p(\gamma)+1}(\Gamma_\gamma^{\mathrm{cut}}) =: M_2. \tag{19}$$

Finally, if there are no magnetic phases on the graph Γ (that is, all entries of $H(\Gamma)$ are real), then one of the extremal values M_1 or M_2 is equal to $\lambda_n(\Gamma) = \lambda_n(\Gamma_{\mathrm{mag}}^{\alpha=0})$, while the other is equal to $\lambda_n(\hat{\Gamma}) := \lambda_n(\Gamma_{\mathrm{mag}}^{\alpha=\pi})$.

Remark 3.4. Note that at this point we don't know which extremum, M_1 or M_2, is equal to $\lambda_n(\Gamma)$. In other words, $\alpha = 0$ may be either a maximum or a minimum of $\lambda_n(\Gamma_{\mathrm{mag}}^\alpha)$; see Figure 1. This information is related to the nodal surplus. The point $\alpha = \pi$ will then be a minimum or a maximum, correspondingly. Also, if the graph Γ had some magnetic phases on it before we added a phase α on the edge (u_1, u_2), the extrema with respect to α do not have to occur at 0 and π.

Note that we have also defined yet another modification of the graph Γ, the graph $\hat{\Gamma}$ whose adjacency matrix has -1 in place of 1 for the entries C_{u_1,u_2} and C_{u_1,u_2}.

Remark 3.5. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real line and $\widehat{\mathbb{R}} = \overline{\mathbb{R}}/[-\infty = \infty]$ be its projective ("wrapped") version. The eigenvalue $\lambda_{n-p(\gamma)}(\Gamma_\gamma^{\mathrm{cut}})$ is then a continuous function of γ, considered as a function from $\widehat{\mathbb{R}}$ to $\overline{\mathbb{R}}$; see Figure 2 for an example. Note that by our definitions, $\lambda_{n-p(\gamma)}(\Gamma_\gamma^{\mathrm{cut}}) = -\infty$ for $n = 1$ and $\gamma < 0$.

Proof of Theorem 3.3. The inequalities follow directly from Theorem 3.1, since for any α, the graph $\Gamma_{\mathrm{mag}}^\alpha$ is a rank-one perturbation of $\Gamma_\gamma^{\mathrm{cut}}$. Whether it is positive or negative definite depends on the sign of γ, and results in the shift by p.

We get the properties of the extrema as follows. Observe that if $\max \lambda_{n-p}(\Gamma_\gamma^{\mathrm{cut}}) = \min \lambda_{n-p+1}(\Gamma_\gamma^{\mathrm{cut}})$, then $\lambda_n(\Gamma_{\mathrm{mag}}^\alpha)$ is constant and equal to the common value of $\lambda_{n-p}(\Gamma_\gamma^{\mathrm{cut}})$ and $\lambda_{n-p+1}(\Gamma_\gamma^{\mathrm{cut}})$.

Let now $\max \lambda_{n-p}(\Gamma_\gamma^{\mathrm{cut}}) < \min \lambda_{n-p+1}(\Gamma_\gamma^{\mathrm{cut}})$. The eigenvalues of a one-parameter family can always be represented as a set of analytic functions (that can intersect). Let $\lambda'(\Gamma_\gamma^{\mathrm{cut}})$ be the analytic function that

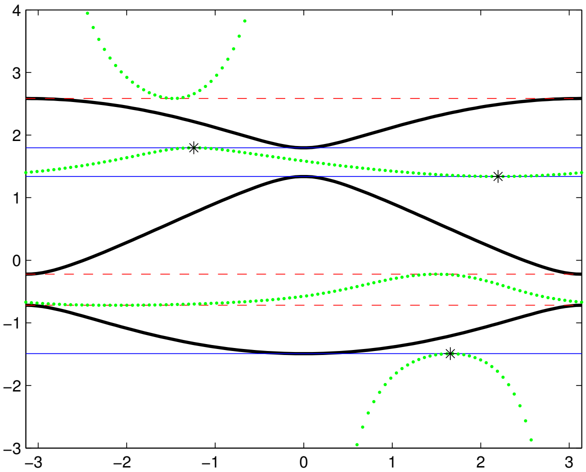

Figure 2. The duality between a magnetic field on one side and cut edge with added potential on the other. The graph is a triangle. The bold curves correspond to the eigenvalues as functions of the magnetic phase. The dotted curves correspond to varying the potential parameter γ after cutting the edge. The x-axis ranges from $-\pi/2$ to $\pi/2$ with the magnetic phase taken as $\alpha = 2x$ and the potential parameter $\gamma = \tan(x)$. The horizontal solid lines are the eigenvalues of the original graph, while the horizontal dashed lines are the eigenvalues of the graph with the magnetic phase π.

achieves the maximum $\max \lambda_{n-p}(\Gamma_\gamma^{\text{cut}})$ and f be the corresponding eigenfunction. We will differentiate $\lambda'(\Gamma_\gamma^{\text{cut}})$ using (11). At the maximum point $\gamma = \gamma^\circ$, we have, by (13),

$$0 = \frac{d\lambda'}{d\gamma} = \left\langle f, \frac{dB^c}{d\gamma} f \right\rangle = -|f_{u_1}|^2 + \frac{|f_{u_2}|^2}{(\gamma^\circ)^2}. \tag{20}$$

From here it follows that

$$\gamma^\circ = \pm \frac{|f_{u_2}|}{|f_{u_1}|} \qquad \text{or, equivalently,} \qquad \left| \frac{\gamma^\circ f_{u_1}}{f_{u_2}} \right| = 1. \tag{21}$$

Let $\widetilde{\alpha}$ be the solution of $e^{i\alpha} = \gamma^\circ f_{u_1}/f_{u_2}$. Direct calculation shows that the eigenfunction f is in the null-space of the perturbation B^{mc} of (16) with $\alpha = \widetilde{\alpha}$, and therefore f is both in the spectrum of $\Gamma_{\gamma^\circ}^{\text{cut}}$ and in the spectrum of $\Gamma_{\text{mag}}^{\widetilde{\alpha}}$, so (18) follows. The proof of (19) is completely analogous.

Note that we could instead differentiate the eigenvalue of $\Gamma_{\text{mag}}^{\alpha}$, leading to the condition

$$f_{u_2} \overline{f_{u_1}} e^{i\widetilde{\alpha}} \in \mathbb{R}, \tag{22}$$

instead of (20). One then sets $\gamma^\circ = e^{i\widetilde{\alpha}} f_{u_2}/f_{u_1} \in \mathbb{R}$, to the same effect.

Finally, when the matrix $H(\Gamma)$ is real, the eigenfunctions of $\Gamma_\gamma^{\text{cut}}$, $\Gamma_{\text{mag}}^{\alpha=0}$ and $\Gamma_{\text{mag}}^{\alpha=\pi}$ are real-valued. When $\alpha = 0$, we can verify directly that the eigenfunction f of $\Gamma_{\text{mag}}^{\alpha=0}$ is also an eigenfunction of $\Gamma_{\gamma^\circ}^{\text{cut}}$ by setting $\gamma^\circ = f_{u_2}/f_{u_1}$. When $\alpha = \pi$, we also set $\gamma = -f_{u_2}/f_{u_1}$ and do the same. $\square$

Theorem 3.3 highlights a sort of duality between the two modifications of the graph Γ. The spectra of the graphs with a magnetic phase form bands (as the phase is varied), while the spectra of the graphs with the cut fill the gaps between these bands. Minima of one correspond to maxima of the other, and in half of the cases correspond to eigenvalues of the original graph.

We now explain how the $\beta = 1$ case of Theorem 1.1 follows from Theorem 3.3. While for general β, the proof is significantly different (it bypasses the interlacing inequalities and goes straight to the quadratic form), some key features are the same as in this simple case.

Starting with the eigenvalue λ_n of Γ and the corresponding eigenfunction f, we cut an edge on the only cycle of Γ to obtain a family of trees $\Gamma_\gamma^{\mathrm{cut}}$. For $\gamma = \gamma^\circ := f_{u_2}/f_{u_1}$, we have either

$$\max_\gamma \lambda_{n-p(\gamma)}(\Gamma_\gamma^{\mathrm{cut}}) = \lambda_{n-p(\gamma^\circ)}(\Gamma_{\gamma^\circ}^{\mathrm{cut}}) = \lambda_n(\Gamma) = \min_\alpha \lambda_n(\Gamma_{\mathrm{mag}}^\alpha)$$

or

$$\max_\alpha \lambda_n(\Gamma_{\mathrm{mag}}^\alpha) = \lambda_n(\Gamma) = \lambda_{n-p(\gamma^\circ)+1}(\Gamma_{\gamma^\circ}^{\mathrm{cut}}) = \min_\gamma \lambda_{n-p(\gamma)+1}(\Gamma_\gamma^{\mathrm{cut}}).$$

In the first case, according to Fiedler's theorem (Equation (2) with $\beta = 0$), the function f has $n - p(\gamma^\circ) - 1$ sign changes *with respect to* the tree $\Gamma_\gamma^{\mathrm{cut}}$. Adding back the removed edge (u_1, u_2) adds another sign change if $\gamma^\circ < 0$, and doesn't change the number of sign changes otherwise. In other words, it adds $p(\gamma^\circ)$ sign changes. Thus, with respect to Γ, the function f has $n - 1$ sign changes and $\sigma_n = 0$. In the second case, we similarly conclude that f has $n - p(\gamma^\circ)$ sign changes with respect to $\Gamma_\gamma^{\mathrm{cut}}$, and n sign changes with respect to Γ. The nodal surplus is $\sigma_n = 1$.

On the other hand, in the first case, $\lambda_n(\Gamma)$ is a minimum of $\lambda_n(\Gamma_{\mathrm{mag}}^\alpha)$ (Morse index 0), while in the second, it is a maximum of $\lambda_n(\Gamma_{\mathrm{mag}}^\alpha)$ (Morse index 1), which shows that the Morse index coincides with σ_n in the case $\beta = 1$.

Remark 3.6. In the $\beta = 1$ case, the spectrum of the cut graph $\Gamma_\gamma^{\mathrm{cut}}$ completely fills the gaps in the magnetic spectrum (see Theorem 3.3 and Figure 2). This is not the case for $\beta > 1$, although an interesting relationship persists, as will become apparent in Section 5.

4. Tools of the main proof

In this section, we collect some basic facts that will be repeatedly used in the proof of Theorem 1.1.

Critical points of the quadratic form.

Definition 4.1. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function. If c is a critical point (that is, $\nabla F(c) = 0$), the *inertia* of c is the triple (n_-, n_0, n_+) that counts the number of negative, zero and positive eigenvalues correspondingly of the Hessian (the matrix of second derivatives) at the point c. The number n_- is called the *Morse index* (or simply *index*).

The next lemma is a reminder that the eigenvectors of a symmetric matrix are critical points of the quadratic form on the unit sphere.

Lemma 4.2. *Let A be a $d \times d$ real symmetric matrix and let $h(x) = \langle x, Ax\rangle$, $x \in \mathbb{R}^d$, be the associated quadratic form. Then the* (*real*) *eigenvectors of the matrix A are critical points of the function $h(x)$ on the unit sphere $\|x\| = 1$.*

Let λ_n be the n-th eigenvalue of A and let $f^{(n)}$ be the corresponding normalized eigenfunction. Define

$$n_- = \#\{\lambda_m < \lambda_n\}, \qquad n_0 = \#\{\lambda_m = \lambda_n,\ m \neq n\}, \qquad n_+ = \#\{\lambda_m > \lambda_n\}, \tag{23}$$

with $n_- + n_0 + n_+ = d - 1$. Then the inertia of the critical point $x = f^{(n)}$ is (n_-, n_0, n_+). In particular, if λ_n is a simple eigenvalue, the inertia is $(n-1, 0, d-n)$.

Remark 4.3. The value of the quadratic form h at the critical point $f^{(n)}$ is λ_n.

Proof. The idea is intuitively clear: n_- — which is the Morse index — counts the number of directions in which the quadratic form decreases relative to the value at $x = f^{(n)}$. These directions are the eigenvectors corresponding to the eigenvalues that are less than λ_n. Similar characterizations are valid for n_0 and n_+.

We note that by Sylvester's law of inertia, the inertia is invariant under the change of variables. Making the orthogonal change of coordinates to the eigenbasis of the matrix A, the quadratic form $h(a)$ becomes

$$h(a) = \lambda_1 a_1^2 + \lambda_2 a_2^2 + \cdots + \lambda_d a_d^2,$$

while the sphere is given by the equations

$$a_1^2 + a_2^2 + \cdots + a_d^2 = 1.$$

Thus, on the sphere, the quadratic form in terms of variables $a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_d$ is given by

$$h = \lambda_n + \sum_{j \neq n} (\lambda_j - \lambda_n) a_j^2,$$

and the Hessian is a diagonal matrix with $\lambda_j - \lambda_n$, $j = 1, \dots, d$, $j \neq n$. The statement of the lemma follows immediately. □

Reduction to the critical manifold. The tool introduced in this section is a simple idea already used in [Band et al. 2012; Berkolaiko et al. 2012a; 2012b]. If we have a function $f(x_1, \dots, x_n)$ with a critical point c, then under some general conditions, there is an $(n-1)$-dimensional manifold around the point c on which the local minimum of f is achieved when we vary the variable x_1 and keep the others fixed. Then the Morse index of f restricted to this manifold is the same as the Morse index of the unrestricted function. On the other hand, if the manifold is the locus of local maxima with respect to the variable x_1, the Morse index on the manifold is one less than the unrestricted Morse index. The following lemma is a simple generalization of this idea. The proof is a simplified finite-dimensional adaptation of the proof in [Berkolaiko et al. 2012a].

Lemma 4.4 (reduction lemma). *Let $X = Y \oplus Y'$ be a direct decomposition of a finite-dimensional vector space. Let $f : X \to \mathbb{R}$ be a smooth functional such that $(0, 0) \in X$ is its critical point with inertia $\mathcal{I}_X$. Further, for every $y \in Y$ locally around 0, let the functional $f(y, y')$ considered as a function of y' have a* *critical point at* $y' = 0$ *with inertia* $\mathcal{I}_{Y'}$, *that* (*locally*) *does not depend on* y. *Then the Hessian of* f *is reduced by the decomposition* $X = Y \oplus Y'$, *and the inertia of* f *with respect to the space* Y *is*

$$\mathcal{I}_Y = \mathcal{I}_X - \mathcal{I}_{Y'}. \tag{24}$$

Proof. We calculate the mixed derivative of f with respect to one variable from Y and the other from Y'. In a slight abuse of notation, we denote these variables simply by y and y'. We have

$$\frac{\partial^2 f}{\partial y\, \partial y'}(0, 0) = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y'}(y, 0)\right]\bigg|_{y=0} = 0, \tag{25}$$

since $y' = 0$ is the critical point of $f(y, y')$ as a function of y' for every y. Thus the Hessian of f has a block-diagonal form with two blocks that correspond to Y and Y'. The spectrum of the Hessian is the union of the spectra of the blocks and the inertia is the sum of the inertias of the blocks,

$$\mathcal{I}_X = \mathcal{I}_Y + \mathcal{I}_{Y'}.$$

Equation (24) follows immediately. □

Remark 4.5. Lemma 4.4 can be simply extended to the case when, for every fixed y, the critical point with respect to y' is located at $y' = q(y)$ (rather than $y' = 0$). The function $q(y)$ defines the critical manifold $\mathfrak{Q} = (y, q(y))$. If $q(y)$ is a smooth function of y and $q(0) = 0$, the change of variables

$$y \mapsto y, \qquad y' \mapsto y' - q(y)$$

is nondegenerate (its Jacobian is a triangular matrix with 1s on the diagonal) and makes f satisfy the assumptions of Lemma 4.4. By Sylvester's law of inertia, the conclusion of the lemma is invariant under the change of variables. Therefore, the inertia of $f\big|_{\mathfrak{Q}}$ at point 0 is

$$\mathcal{I}_{\mathfrak{Q}} = \mathcal{I}_X - \mathcal{I}_{Y'}.$$

5. Proof of the main theorem

We prove the main result in three steps. First we show by an explicit computation that the point 0 is a critical point of the function $\lambda_n(\vec{\alpha})$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\beta) \in (-\pi, \pi]^\beta$ are the magnetic phases.

Then we fix an eigenpair $\lambda = \lambda_n(\Gamma)$ and f. We cut β edges of the graph, turning it into a tree T, but modifying the potentials so that the eigenfunction f is also an eigenfunction of the tree T. It now corresponds to an eigenvalue number m, that is, $\lambda_m(T) = \lambda$. Considering the eigenvalue $\lambda_m(T)$ as a function of the potentials, we find its inertia by two applications of the reduction lemma to the corresponding quadratic form. The result of this step is related to the results on critical equipartitions [Berkolaiko et al. 2012b].

Finally, we relate the inertia of the function $\lambda_m(T)$ to the inertia of the function $\lambda_n(\vec{\alpha})$ at the corresponding critical points. This is done by complexifying $\vec{\alpha}$ and relating the function $\lambda_n(\vec{\alpha})$ on the imaginary axis to the function $\lambda_m(T)$ by a change of variables.

We recall that S is a set of β edges whose removal turns the graph Γ into a tree. By $\Gamma_{\mathrm{mag}}^{\vec{\alpha}}$, we denote the graph obtained from Γ by introducing magnetic phases $\vec{\alpha} = (\alpha_1, \dots, \alpha_\beta)$ on the edges from the set S. Similarly, by $\Gamma_{\vec{\gamma}}^{\mathrm{cut}}$ we denote the tree graph obtained by cutting every edge from S in the manner already described (see Equation (13) and around it). For future reference, we list the quadratic forms of the original graph and the graph $\Gamma_{\vec{\gamma}}^{\mathrm{cut}}$, grouping the terms to highlight the differences between the two forms:

$$h(\vec{x}) = \sum_u q_u x_u^2 - \sum_{(u,v)\in E\setminus S} 2x_u x_v - \sum_{(u,v)\in S} 2x_u x_v, \tag{26}$$

$$h_{\vec{\gamma}}^{\mathrm{cut}}(\vec{x}) = \sum_u q_u x_u^2 - \sum_{(u,v)\in E\setminus S} 2x_u x_v - \sum_{e_j=(u,v)\in S} \left(\gamma_j x_u^2 + \frac{x_v^2}{\gamma_j}\right). \tag{27}$$

Critical points. Let f be an eigenfunction of the graph Γ. We have seen in Theorem 3.3 and its proof that the points $\alpha = 0$ and $\gamma = \gamma^\circ$ (see Equation (21)) are special: at these points, f is an eigenfunction of the graphs $\Gamma_{\mathrm{mag}}^{\alpha}$ and $\Gamma_{\gamma}^{\mathrm{cut}}$. Moreover, they are critical points of the corresponding eigenvalues considered as functions of the parameters α and γ, respectively. The result of this section generalizes this observation.

Theorem 5.1. *Let f be an eigenfunction of $H(\Gamma)$ that corresponds to a simple eigenvalue $\lambda = \lambda_n(\Gamma)$. Assume f is nonzero on vertices of the graph Γ. For every edge $(u_j, v_j) \in S$, $j = 1, \dots, \beta$, let*

$$\gamma_j^\circ = \frac{f_{v_j}}{f_{u_j}}. \tag{28}$$

Let p denote the number of negatives among the values γ_j°:

$$p = \#\{\gamma_j^\circ < 0,\ j = 1, \dots, \beta\}.$$

Then

$$\lambda_n(\Gamma) = \lambda_{\phi_n - p+1}(\Gamma_{\vec{\gamma}^\circ}^{\mathrm{cut}}), \tag{29}$$

where ϕ_n is the number of sign changes of f with respect to the graph Γ. The eigenvalue λ_{ϕ_n-p+1} of the tree $\Gamma_{\vec{\gamma}^\circ}^{\mathrm{cut}}$ is simple. Moreover, the point $\vec{\gamma}^\circ = (\gamma_1^\circ, \dots, \gamma_\beta^\circ)$ is a critical point of the function $\lambda_{\phi_n-p+1}(\Gamma_{\vec{\gamma}}^{\mathrm{cut}})$.

Similarly for $\Gamma_{\mathrm{mag}}^{\vec{\alpha}}$,

$$\lambda_n(\Gamma) = \lambda_n(\Gamma_{\mathrm{mag}}^{0,\dots,0}) \tag{30}$$

and $(0, \dots, 0)$ is a critical point of the function $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$.

Proof. It can be verified directly that f is an eigenfunction of the graph $\Gamma_{\vec{\gamma}^\circ}^{\mathrm{cut}}$. The nodal bound (2) with $\beta = 0$ (proved by Fiedler [1975]; see also [Berkolaiko 2008]) shows that the eigenvalue corresponding to the function f has number $\mu' + 1$ in the spectrum of the tree $\Gamma_{\vec{\gamma}^\circ}^{\mathrm{cut}}$, where μ' is the number of sign changes of f with respect to the tree. In general, this number is different from ϕ_n because we might have cut some of the edges on which f was changing sign. However, according to (28), these edges gave rise to *negative* values of γ_j°, and therefore $\mu' = \phi_n - p$, proving (29). The eigenvalue that corresponds to a nonzero eigenvector on a tree is simple [Fiedler 1975], establishing simplicity of $\lambda_{\phi_n-p+1}(\Gamma_{\vec{\gamma}^\circ}^{\mathrm{cut}})$. Equation (30) is trivial since $\Gamma_{\mathrm{mag}}^{0,\dots,0} = \Gamma$.

To prove criticality of the points, we calculate the derivatives. Because the eigenvalues in question are simple, they are analytic functions of the parameters and can be differentiated.

The derivative of $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})$ with respect to γ_j has been calculated in (20), resulting in

$$\frac{\partial}{\partial \gamma_j}\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})\big|_{(\gamma_1^\circ,\dots,\gamma_\beta^\circ)} = -|f_{u_j}|^2 + \frac{|f_{v_j}|^2}{\gamma_j^{\circ 2}} = 0, \tag{31}$$

where we used the definition of γ_j° from (28).

The derivative of $\lambda_n(\Gamma^{\vec{\alpha}}_{\mathrm{mag}})$ can be evaluated similarly using (11), leading to

$$\frac{\partial}{\partial \alpha_j}\lambda_n(\Gamma^{\vec{\alpha}}_{\mathrm{mag}})\big|_{(0,\dots,0)} = -i\,\overline{f_{u_j}}\,f_{v_j} + i f_{u_j}\,\overline{f_{v_j}} = \mathrm{Im}(\overline{f_{u_j}}\,f_{v_j}) = 0, \tag{32}$$

since the eigenfunction f is real-valued. Alternatively, we can observe that $\lambda_n(\Gamma^{\vec{\alpha}}_{\mathrm{mag}})$ is invariant with respect to reflection $\alpha \mapsto -\alpha$. □

Index of the eigenvalue on the tree. In this section, we elaborate on the first part of the result of Theorem 5.1, namely that $(\gamma_1^\circ, \dots, \gamma_\beta^\circ)$ is a critical point of the function $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})$.

Theorem 5.2. *Let f be an eigenfunction of $H(\Gamma)$ that corresponds to a simple eigenvalue $\lambda = \lambda_n(\Gamma)$. Assume f is nonzero on vertices of the graph Γ and has ϕ_n sign changes. For every edge $(u_j, v_j) \in S$, $j = 1, \dots, \beta$, let*

$$\gamma_j^\circ = \frac{f_v}{f_u}. \tag{33}$$

As before, p denotes the number of negatives among the values γ_j°. Then the point $(\gamma_1^\circ, \dots, \gamma_\beta^\circ)$ as a critical point of the function $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})$ is nondegenerate and has inertia

$$\big(n-1+\beta-\phi_n,\ 0,\ \phi_n-n+1\big).$$

Proof. Denote by d the number of vertices of the graph Γ. Consider $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$, which is the quadratic form on the Hamiltonian of $\Gamma^{\mathrm{cut}}_{\vec{\gamma}}$, as a function of $d+\beta$ real variables $(x_1, \dots, x_d, \gamma_1, \dots, \gamma_\beta)$ on the manifold $x_1^2 + \cdots + x_d^2 = 1$. We note that the point $(f_1, \dots, f_d, \gamma_1^\circ, \dots, \gamma_\beta^\circ)$ is a critical point of $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$, as can be easily shown by explicit computation. Indeed, the value of the Lagrange multiplier is the eigenvalue λ_n and the gradient of

$$F(x_1, \dots, x_d, \gamma_1, \dots \gamma_\beta) = h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x}) - \lambda_n(x_1^2 + \cdots + x_d^2)$$

is zero: the first d equations become the eigenvalue condition $Hf = \lambda_n f$ and the last β are the same as (31).

We now describe the outline of the proof. Denote the inertia of the point $(f_1, \dots, f_d, \gamma_1^\circ, \dots, \gamma_\beta^\circ)$ by $\mathcal{I}$. To calculate it, we will look for critical points of $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$ as a function of $\gamma_1, \dots, \gamma_\beta$. These points will define a critical manifold to which we will apply Lemma 4.4 via Remark 4.5 (this reduction corresponds to the left arrow in Figure 3). On the critical manifold, the function $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$ will coincide with $h(x)$, the quadratic form of the original graph, whose inertia we know by Lemma 4.2. Having found the inertia of the critical point at the top of Figure 3, we will apply minimax with respect to variables $x_1, \dots, x_d$ to

$$\begin{tikzcd}
& h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x}) \arrow[dl, "{\min_{\gamma_j<0}\max_{\gamma_j>0},\ \beta - p}"'] \arrow[d, "{\text{eigenvector minimax},\ \phi_n - p}"] \\
h(\vec{x}) & \lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})
\end{tikzcd}$$

Figure 3. Schematic diagram of the proof of Theorem 5.2. The reductions are indicated by arrows, with the description of the parameters that are being reduced and the index of the reduction. Since we know the index of the critical point of $h(\vec{x})$, we can follow the diagram, applying the reduction lemma, to calculate the index of $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})$.

follow the vertical arrow of Figure 3. This will take us to the eigenvalue $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec{\gamma}})$, and we will be able to calculate its inertia applying Lemma 4.4 again.

Consider $\vec{x}$ varying locally around the point f, so that the elements of $\vec{x}$ remain bounded away from zero. For each fixed $\vec{x}$, we look for a critical point with respect to the variables $(\gamma_1, \dots, \gamma_\beta)$. The terms of $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$ that depend on a given γ have the form

$$T(\gamma) = -\gamma x_u^2 - \frac{x_v^2}{\gamma}. \tag{34}$$

The critical point is $\gamma = g(\vec{x}) = x_v/x_u$, which is a smooth function of $\vec{x}$. The points $(x_1, \dots, x_d, g_1, \dots g_\beta)$ define the critical manifold to which we apply Lemma 4.4 (via Remark 4.5). Note that the critical manifold includes the point $(f_1, \dots, f_d, \gamma_1^\circ, \dots, \gamma_\beta^\circ)$. Moreover, the critical point with respect to a given γ is a maximum if $g(\vec{x}) > 0$ and a minimum if $g(\vec{x}) < 0$. Each point is nondegenerate and, moreover, the sign of g_j is locally the same as the sign of γ_j° for all j. Different variables γ_j are not coupled, and thus the Hessian is diagonal. Therefore, the inertia of the points on the critical manifold is $(\beta - p, 0, p)$ — it is a minimum with respect to p variables and maximum with respect to $\beta - p$. We remind the reader that p is the number of negatives among $\{\gamma_j^\circ\}$.

Consider now the function $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$ on the critical manifold. When $\gamma = g$, the term (34) evaluates to

$$T(g) = -2x_u x_v,$$

and we find that, on the critical manifold, the function $h^{\mathrm{cut}}_{\vec{\gamma}}(\vec{x})$ coincides with the quadratic form of the original graph, $h(\vec{x})$. The point $\vec{x} = f$, being the n-th eigenfunction of the graph, is a nondegenerate critical point of $h(\vec{x})$ and has inertia $(n-1, 0, d-n)$. Applying Lemma 4.4, we obtain

$$\mathcal{I} = \big(n - 1 + (\beta - p),\ 0,\ d - n + p\big).$$

In particular, we conclude that the point $(f_1, \dots, f_d, \gamma_1^\circ, \dots, \gamma_\beta^\circ)$ is a nondegenerate critical point.

For every value of $(\gamma_1, \dots, \gamma_\beta)$ locally around the point $(\gamma_1^\circ, \dots, \gamma_\beta^\circ)$, consider the $(\phi_n - p + 1)$-th eigenvector $f_{\vec{\gamma}}^{\mathrm{cut}}$ of $\Gamma_{\vec{\gamma}}^{\mathrm{cut}}$. According to Lemma 4.2, it is a nondegenerate critical point of $h_{\vec{\gamma}}^{\mathrm{cut}}(\vec{x})$ as a function of $\vec{x}$ with inertia $(\phi_n - p, 0, d + p - \phi_n - 1)$. At the critical point, the value of the $h_{\vec{\gamma}}^{\mathrm{cut}}$ is

$$h_{\vec{\gamma}}^{\mathrm{cut}}(f_{\vec{\gamma}}^{\mathrm{cut}}) = \lambda_{\phi_n - p+1}(\Gamma_{\vec{\gamma}}^{\mathrm{cut}}),$$

which is the function whose inertia we strive to evaluate.

According to standard perturbation theory (see [Kato 1976], for example), the eigenvector $f_{\vec{\gamma}}^{\mathrm{cut}}$ is a smooth (indeed, analytic) function of $(\gamma_1, \dots, \gamma_\beta)$. This allows us to use Lemma 4.4 again, concluding that the critical point $(\gamma_1^\circ, \dots, \gamma_\beta^\circ)$ of $h_{\vec{\gamma}}^{\mathrm{cut}}(f_{\vec{\gamma}}^{\mathrm{cut}})$ has inertia

$$\mathcal{I} - (\phi_n - p, 0, d + p - \phi_n - 1) = (n - 1 + \beta - \phi_n, 0, \phi_n - n + 1). \qquad \square$$

Remark 5.3. In [Berkolaiko et al. 2012b], the eigenvalue of the tree graph $\Gamma_{\vec{\gamma}}^{\mathrm{cut}}$ was interpreted as the energy of the "partition" with the given number of domains. Theorem 5.2 gives another route for the proof of the results of that paper.

Index of the eigenvalue as a function of the magnetic field. Now we move from the critical point on the tree to the critical point of the eigenvalue of the graph with magnetic phases. We can apply the same method, retracing our steps, but now considering the quadratic forms $h_{\vec{\gamma}}^{\mathrm{cut}}(z)$ and

$$h_{\mathrm{mag}}^{\vec{\alpha}}(\vec{z}) = \sum_u q_u |z_u|^2 - \sum_{(u,v)\in E\setminus S} 2\,\mathrm{Re}(\bar{z}_u z_v) - \sum_{e_j=(u,v)\in S} 2\,\mathrm{Re}(\bar{z}_u e^{i\alpha_j} z_v) \tag{35}$$

as functions of complex variables z. Considering the complex space as a real space of double dimension leads to the inertia in the Hermitian analogue of Lemma 4.2 being $(2n_-, 2n_0 + 1, 2n_+)$. Finding extrema of $h_{\mathrm{mag}}^{\vec{\alpha}}(z)$ with respect to $\vec{\alpha}$ and of $h_{\vec{\gamma}}^{\mathrm{cut}}(z)$ with respect to $\vec{\gamma}$ results in the same values, and thus we relate the indices of $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ and $\lambda_{\phi_n - p+1}(\Gamma_{\vec{\gamma}}^{\mathrm{cut}})$ through a chain of four applications of the reduction lemma (Lemma 4.4), illustrated in Figure 4.

However, instead of following the above plan, we present a simpler yet more insightful proof which can be summarized as follows: after a change of variables, the function $\lambda_{\phi_n - p+1}(\Gamma_{\vec{\gamma}}^{\mathrm{cut}})$ coincides with the function $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ with purely imaginary values of the magnetic phases $\vec{\alpha}$. This will give us full understanding of the quadratic term (the Hessian) of the analytic function $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$.

Theorem 5.4. *Let f be an eigenfunction of $H(\Gamma)$ that corresponds to a simple eigenvalue $\lambda = \lambda_n(\Gamma)$. Assume f is nonzero on vertices of the graph Γ and has ϕ_n sign changes. Let $\Gamma_{\mathrm{mag}}^{\vec{\alpha}}$ be the graph with the magnetic phases $\vec{\alpha} = (\alpha_1, \dots, \alpha_\beta)$ introduced on the edges from the set S. Then the index of* $(0, \dots, 0)$ *as a critical point of the function $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ is the nodal surplus $\sigma_n := \phi_n - (n - 1)$. The critical point is nondegenerate.*

Proof. First we remark that analyticity of $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ is a consequence of standard perturbation theory applied to the simple eigenvalue $\lambda_n(0)$. Moreover, when $\alpha_j = i\xi_j$, with real ξ_j, the Hamiltonian $H(\Gamma_{\mathrm{mag}}^{i\vec{\xi}})$

$$\begin{tikzcd}
h^{\vec\alpha}_{\mathrm{mag}}(z) \arrow[d, "\substack{\text{eigenvector}\\ \text{minimax,}\\ 2(n-1)}"'] \arrow[dr, "{\min\limits_{\alpha_j}\max\limits_{\alpha_k},\ p}"] & & h^{\mathrm{cut}}_{\vec\gamma}(z) \arrow[dl, "{\min\limits_{\gamma_j<0}\max\limits_{\gamma_j>0},\ \beta-p}"'] \arrow[d, "\substack{\text{eigenvector}\\ \text{minimax,}\\ 2(\phi_n-p)}"] \\
\lambda_n(\Gamma^{\vec\alpha}_{\mathrm{mag}}) & \hat h(z) & \lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec\gamma})
\end{tikzcd}$$

Figure 4. Schematic diagram of a possible proof of Theorem 5.4. From Theorem 5.2, we know the index of the critical point of $\lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec\gamma})$ (bottom right corner). We then apply the reduction lemma four times to get the index of $\lambda_n(\Gamma^{\vec\alpha}_{\mathrm{mag}})$ (bottom left corner).

is a matrix with real entries. It is no longer Hermitian, but its complex eigenvalues must come in conjugate pairs; therefore a simple eigenvalue $\lambda_n(0)$ remains real for $\vec\xi$ in a small neighborhood of 0.

Let $\psi = \psi(\vec\xi)$ be the corresponding real eigenfunction. It is a perturbation of f; therefore it is nonzero locally around $\vec\xi = 0$. For every edge $(u_j, v_j) \in S$, we let

$$\gamma_j = \frac{e^{-\xi_j}\psi_v(\vec\xi)}{\psi_u(\vec\xi)}. \tag{36}$$

This defines a mapping

$$R : (\xi_1, \dots, \xi_\beta) \mapsto (\gamma_1, \dots \gamma_\beta), \tag{37}$$

which is smooth in a neighborhood of zero. We also have $R(0, \dots, 0) = (\gamma_1^\circ, \dots, \gamma_\beta^\circ) = \vec\gamma^\circ$, where the γ_j° are given by (33). The inverse of R, which can be directly calculated from (36), is also a smooth function in a neighborhood of the point $\vec\gamma^\circ$. Therefore R is a diffeomorphism.

Moreover, ψ is an eigenfunction of both $\Gamma^{i\vec\xi}_{\mathrm{mag}}$ (by construction) and $\Gamma^{\mathrm{cut}}_{\vec\gamma}$ with $\vec\gamma = R(\vec\xi)$ (since ψ is in the null-space of the perturbation B^{mc} of (16)), and their eigenvalues coincide, with the appropriate shift in numbering (see (29)). Namely, we have

$$\lambda_n(\Gamma^{i\vec\xi}_{\mathrm{mag}}) = \lambda_{\phi_n-p+1}(\Gamma^{\mathrm{cut}}_{\vec\gamma}), \qquad \vec\gamma = R(\vec\xi).$$

By Sylvester's law of inertia, the index is not affected by the diffeomorphism R, and we get from Theorem 5.2 that $\vec\xi = 0$ is a nondegenerate critical point of $\lambda_n(\Gamma^{i\vec\xi}_{\mathrm{mag}})$ of inertia

$$\mathcal{I}_\xi = (n-1+\beta-\phi_n,\ 0,\ \phi_n-n+1).$$

Finally, since $\vec\alpha = i\vec\xi$, the Hessian of λ_n with respect to $\vec\alpha$ is the Hessian with respect to $\vec\xi$ multiplied by $i^2 = -1$. The entries n_- and n_+ of the inertia get swapped; therefore the inertia of $\lambda_n(\Gamma^{\vec\alpha}_{\mathrm{mag}})$ is

$$\mathcal{I}_\alpha = (\phi_n-n+1,\ 0,\ n-1+\beta-\phi_n). \qquad \square$$

6. Discussion

Some simple extensions. As already mentioned, the criticality of the point $(0, \dots, 0)$ can be easily obtained from the fact that $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ is symmetric with respect to each variable α_j. In fact, there are 2^β points of symmetry of the function $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$, namely, the points where each α_j is equal to either 0 or π. Taking $\alpha_j = \pi$ makes the corresponding edge have the weight -1 (rather than 1) in the connectivity matrix. A statement similar to Theorem 1.1 can be proved about every point of symmetry, with the appropriate modification of the notion of a sign change: ϕ_n counts the number of edges (u, v) such that $H_{u,v} f_u f_v > 0$.

One can also easily extend the results to generalized Schrödinger operators on the graph, i.e., symmetric matrices H with the property that $H_{u,v} \neq 0$ if and only if the vertices u and v are connected. The magnetic field is introduced by multiplying off-diagonal matrix elements by phases. If $H_{u,v}$ is allowed to be positive, the notion of a "sign change" has to be modified to refer to the edges (u, v) with $H_{u,v} f_u f_v > 0$, as above. With this modification, the statement of Theorem 1.1 remains valid as stated.

The necessary modifications to the proofs are limited to having $H_{u_1,u_2}^2/\gamma$ in place of $1/\gamma$ in the definition of the "cut" Hamiltonian, Equation (13), and letting the critical value of γ_j be $\gamma_j^\circ = -H_{u_j,v_j} f_{v_j}/f_{u_j}$. All other considerations remain unchanged (in particular, Fiedler's theorem on tree eigenfunctions is already formulated in terms of "generalized sign changes").

Further consequences. Perhaps the most important feature of Theorem 1.1 is that it allows us to access some of the features of the eigenfunction via the behavior of the corresponding eigenvalue under perturbation. It is known that the eigenvalues of the Laplacian are connected to the statistics of the closed paths on the graph. The connection is given through the so-called "trace formulae", which can be obtained from a graph analogue of the Selberg zeta function, the Ihara zeta function [Ihara 1966; Bass 1992; Stark and Terras 1996]. An extension by Bartholdi [1999] (see also [Mizuno and Sato 2005]) was used in [Oren et al. 2009] to obtain a family of trace formulae including the ones for the magnetic Laplacian. Thus, the closed paths on the graph determine the spectrum of the magnetic Laplacian, which, in turn, determines the nodal count. This, in principle, establishes the existence of a general connection between the nodal count and the closed paths. However, we are not aware of any concrete general formulas. We note that such a connection has been earlier conjectured by Smilansky, with special cases reported in [Gnutzmann et al. 2006; Aronovitch and Smilansky 2010].

We would also like to mention that the result of this paper has already been used in an elegant proof by Band [2012] of the converse of Fiedler's theorem: if for all n, the n-th graph eigenfunction is generic and has $n - 1$ sign changes, the graph is a tree.

There is an interesting connection between the magnetic spectrum of a compact graph and the continuous spectrum of a periodic graph. Namely, the eigenvalue $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ featured in this paper is the dispersion relation for the maximal Abelian cover of the graph Γ, a well studied object. One of the interesting questions regarding this object is the "full spectrum property" [Higuchi and Shirai 2004; Higuchi and Nomura 2009; Sunada 2008]: whether the continuous spectrum of the cover graph of a regular graph — in our terms, the union of ranges of the functions $\lambda_n(\Gamma_{\mathrm{mag}}^{\vec{\alpha}})$ — contains no gaps. This question can be reformulated in terms of eigenfunctions of graphs $\Gamma^{\vec{\alpha}}_{\text{mag}}$ with all $\alpha_j = 0$ or π that have minimal and maximal number of sign changes.

This, in turn, is related to the question of whether the extrema of the dispersion relation are always achieved at the symmetry points described above. Examples to the contrary have been put forward in [Harrison et al. 2007; Exner et al. 2010]. However, an important question remains: how can one characterize the extremal points that are not points of symmetry? In this direction, the duality with the cut graphs (Section 3) might provide some answers. One can speculate that critical points of the dispersion relation correspond to critical points of the eigenvalues of the cut graph $\Gamma^{\text{cut}}_{\vec{\gamma}}$ that *do not* give rise to the eigenfunction of the graph Γ. Further, we conjecture that these "unclaimed" critical points correspond to eigenfunctions of Γ modified by enforcing Dirichlet conditions at some vertices.

The results of the present paper are derived under the assumption that the eigenvalue is nondegenerate. While this is the generic situation with respect to the change in the potential Q, it is also interesting to consider what happens in the degenerate case. The linear Zeeman effect (the magnetic perturbation splitting eigenvalues) suggests that the singularities of $\lambda_n(\Gamma^{\vec{\alpha}}_{\text{mag}})$ are conical. It should be possible to define the index of the singularity point that does not rely on differentiability.

Finally, it would be most interesting to generalize the results of the present paper to manifolds. However, we immediately encounter a conceptual problem — the "number" of zeros is infinite. Still, some measure of instability of the eigenvalue under magnetic perturbation should be related to some measure of the zero set of the corresponding eigenfunction. This can be intuitively visualized by approximating the domain eigenfunction as eigenfunctions of a discrete mesh. Moreover, the method of proof used in Section 5 might be appropriate for the manifolds as well: it is based on a connection between the magnetic spectrum and the energy of the equipartitions (see Remark 5.3), and on manifolds the equipartitions are well understood [Berkolaiko et al. 2012a].

After this manuscript had been submitted, the author was notified by Y. Colin de Verdière that he found an alternative proof Theorem 1.1, which appears in this issue [Colin de Verdiére 2013]. The proof is based on a direct application of the eigenvalue perturbation formulas and a clever choice of gauge that significantly simplifies the calculations. Colin de Verdière also succeeded in proving an analogue of Theorem 1.1 for continuous Schrödinger operators on a circle (also called Hill operators). An extension of Theorem 1.1 to general quantum graphs has been subsequently obtained in [Berkolaiko and Weyand 2012].

Acknowledgment

The impetus for considering magnetic perturbation was indirectly given to the author by two people: B. Helffer and P. Kuchment. B. Helffer, in a talk at "Selected topics in spectral theory" at the Erwin Schrödinger Institute (Vienna), January 13–15, 2011, described the conjecture that the minimal spectral partition of a domain can be obtained by finding the minimal energy of the Laplacian with a number of Aharonov–Bohm flux lines (the conjecture has since been proven by B. Helffer and T. Hoffmann-Ostenhof [2013]). P. Kuchment gave the author a preprint by Rueckriemen [2011], which discussed objects similar to the ones introduced in a study of minimal spectral partitions on graphs [Band et al. 2012]. We are also grateful to P. Kuchment for many subsequent enlightening discussions, in particular regarding the topics touched upon in Section 6. Interesting discussions with and corrections by R. Band, Y. Colin de Verdière, U. Smilansky and T. Weyand are gratefully acknowledged. The visit to the ESI in Vienna was possible through the generosity of the Institute.

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Received 10 Dec 2012. Accepted 19 Jan 2013.

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ANALYSIS & PDE

Volume 6 No. 5 2013

2157-5045(2013)6:5;1-D