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A NEKHOROSHEV-TYPE THEOREM FOR THE NONLINEAR SCHRÖDINGER EQUATION ON THE TORUS

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We prove a Nekhoroshev type theorem for the nonlinear Schrödinger equation

$$iu_t = -\Delta u + V \star u + \partial_{\bar{u}} g(u, \bar{u}), \quad x \in \mathbb{T}^d,$$

where V is a typical smooth Fourier multiplier and g is analytic in both variables. More precisely, we prove that if the initial datum is analytic in a strip of width $\rho > 0$ whose norm on this strip is equal to ε, then if ε is small enough, the solution of the nonlinear Schrödinger equation above remains analytic in a strip of width $\rho/2$, with norm bounded on this strip by $C\varepsilon$ over a very long time interval of order $\varepsilon^{-\sigma|\ln\varepsilon|^{\beta}}$, where $0 < \beta < 1$ is arbitrary and $C > 0$ and $\sigma > 0$ are positive constants depending on β and ρ.

1. Introduction and statements

We consider the nonlinear Schrödinger equation

$$iu_t = -\Delta u + V \star u + \partial_{\bar{u}} g(u, \bar{u}), \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \tag{1-1}$$

where V is a smooth convolution potential and g is an analytic function on a neighborhood of the origin in $\mathbb{C}^2$ which has a zero of order at least 3 at the origin and satisfies $g(z, \bar{z}) \in \mathbb{R}$. In more standard models, the convolution term is replaced by a multiplicative potential. The use of a convolution potential makes the analysis of the resonances easier.

For instance, when

$$g(u, \bar{u}) = \frac{a}{p+1}|u|^{2p+2}$$

with $a \in \mathbb{R}$ and $p \in \mathbb{N}$, we recover the standard NLS equation $iu_t = -\Delta u + V \star u + a|u|^{2p}u$. Equation (1-1) is a Hamiltonian system associated with the Hamiltonian function

$$H(u, \bar{u}) = \int_{\mathbb{T}^d} \big(|\nabla u|^2 + (V \star u)\bar{u} + g(u, \bar{u})\big)\,\mathrm{d}x$$

and the complex symplectic structure $i\,\mathrm{d}u \wedge \mathrm{d}\bar{u}$.

This equation has been considered with Hamiltonian tools in [Bambusi and Grébert 2003; Eliasson and Kuksin 2010]. The first of these papers (see also [Bambusi and Grébert 2006; Bourgain 1996] for related results) contains a Birkhoff normal form theorem adapted to this equation and discusses dynamical consequences on the long time behavior of the solutions with small initial Cauchy data in Sobolev spaces.

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More precisely, it is proved that for s sufficiently large, if the Sobolev norm of index s of the initial datum u_0 is sufficiently small (of order ε), then the Sobolev norm of index s of the solution is bounded by 2ε during a very long time (of order ε^{-r} with r arbitrary). In the second paper cited, Eliasson and Kuksin obtain a KAM theorem adapted to this equation. In particular, they prove that in a neighborhood of $u = 0$, many finite-dimensional invariant tori associated with the linear part of the equation are preserved by small Hamiltonian perturbations. In other words, (1-1) has many quasiperiodic solutions. In both cases, nonresonance conditions have to be imposed on the frequencies of the linear part, and thus on the potential V (these are not exactly the same in the two different cases).

Both results are related to the stability of the zero solution, which is an elliptic equilibrium of the linear equation. The first result establishes the stability for polynomials' times with respect to the size of the (small) initial datum, while the second proves the stability for all time of certain solutions. In the present work, we extend the technique of normal forms, establishing the stability of the solutions for times of order $\varepsilon^{-\sigma|\ln\varepsilon|^\beta}$ for some constants $\sigma > 0$ and $\beta < 1$, with ε being the size of the initial datum in an analytic space.

We now state our result more precisely. We assume that for $m > d/2$, $R > 0$, V belongs to the space

$$\mathcal{W}_m = \left\{ V(x) = \sum_{a\in\mathbb{Z}^d} w_a e^{ia\cdot x} \;\middle|\; v_a := \frac{w_a(1+|a|)^m}{R} \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \text{ for any } a \in \mathbb{Z}^d \right\}, \tag{1-2}$$

which we endow with the product probability measure. Here, for $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$, we set $|a|^2 = a_1^2 + \cdots + a_d^2$.

For $\rho > 0$, we denote by $\mathcal{A}_\rho \equiv \mathcal{A}_\rho(\mathbb{T}^d; \mathbb{C})$ the space of functions ϕ that are analytic on the complex neighborhood of a d-dimensional torus $\mathbb{T}^d$ given by $I_\rho = \{x + iy \mid x \in \mathbb{T}^d, y \in \mathbb{R}^d \text{ and } |y| < \rho\}$ and continuous on the closure of this strip. We then denote by $|\cdot|_\rho$ the usual norm on $\mathcal{A}_\rho$:

$$|\phi|_\rho = \sup_{z\in I_\rho} |\phi(z)|.$$

We note that $(\mathcal{A}_\rho, |\cdot|_\rho)$ is a Banach space.

Our main result is a Nekhoroshev type theorem:

Theorem 1.1. *There exists a subset $\mathcal{V} \subset \mathcal{W}_m$ of full measure, such that for $V \in \mathcal{V}$, $\beta < 1$ and $\rho > 0$, the following holds: there exist $C > 0$ and $\varepsilon_0 > 0$ such that if*

$$u_0 \in \mathcal{A}_{2\rho} \quad \textit{and} \quad |u_0|_{2\rho} = \varepsilon \le \varepsilon_0,$$

then the solution of (1-1) *with initial datum u_0 exists in $\mathcal{A}_{\rho/2}$ for times $|t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}$ and satisfies*

$$|u(t)|_{\rho/2} \le C\varepsilon \quad \textit{for } |t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}, \tag{1-3}$$

with $\sigma_\rho = \min\{\frac{1}{10}, \frac{1}{2}\rho\}$. Furthermore, writing $u(t) = \sum_{k\in\mathbb{Z}^d} \xi_k(t) e^{ik\cdot x}$, we have

$$\sum_{k\in\mathbb{Z}^d} e^{\rho|k|} \big||\xi_k(t)| - |\xi_k(0)|\big| \le \varepsilon^{3/2} \quad \textit{for } |t| \le \varepsilon^{-\sigma_\rho|\ln\varepsilon|^\beta}. \tag{1-4}$$

Estimate (1-4) asserts that there is almost no variation of the actions[1].

In finite dimension n, the standard Nekhoroshev result [1977] controls the dynamic over times of order $\exp(\sigma/\varepsilon^{1/(\tau+1)})$ for some $\sigma > 0$ and $\tau > n+1$ (see, for instance, [Benettin et al. 1985; Giorgilli and Galgani 1985; Pöschel 1993]), which is of course much better than $\varepsilon^{-\sigma|\ln\varepsilon|^{\beta}} = e^{\sigma|\ln\varepsilon|^{(1+\beta)}}$. Nevertheless, this standard result does not extend to the infinite-dimensional context. Actually, that the term $\varepsilon^{-1/(\tau+1)}$ in the exponential validity time can be replaced by $|\ln\varepsilon|^{(1+\beta)}$ at the limit $n \to \infty$ is good news!

To our knowledge, the only previous works in the direction of obtaining Nekhoroshev estimates for PDEs were obtained by Bambusi [1999a; 1999b]. However, the result in [Bambusi 1999a], which develops ideas expressed by Bourgain [1996], concerns a smaller set of functions made of entire analytic functions only, and nevertheless yields a weaker control on a large but finite number of modes.

The five main differences with the previous works on normal forms are:

- In the finite-dimensional case and in Bambusi's work, the central argument consists in optimizing the order of the Birkhoff normal form with respect to the size of the initial datum. Here we introduce a Fourier truncation and we optimize the order of the Birkhoff normal form *and* the order of the truncation.
- We prove in the Appendix that, generically with respect to V, the spectrum of $-\Delta + V\star$ satisfies a nonresonance condition much more efficient than the standard one (see Remark 2.7).
- We use ℓ^1-type norms to control the Fourier coefficients and the vector fields instead of the usual ℓ^2-type norms. Of course this choice does not allow us to work in Hilbert spaces and induces a slight loss of regularity each time the estimates are transposed from the Fourier space to the initial space of analytic functions. But it turns out that this choice simplifies the estimates on the vector fields (see Proposition 2.5 below and [Faou and Grébert 2011] for a similar framework in the context of numerical analysis).
- We use the zero momentum condition: in the Fourier space, the nonlinear term contains only monomials $z_{j_1} \dots z_{j_k}$ with $j_1 + \cdots + j_k = 0$ (see Definition 2.4). This property allows us to control the largest index by the others.
- We notice that the Hamiltonian vector field of a monomial $z_{j_1} \dots z_{j_k}$ containing at least three Fourier modes z_ℓ with large indices ℓ induces a flow whose dynamics is controlled during a very long time in the sense that the dynamic almost excludes exchanges between high Fourier modes and low Fourier modes (see Proposition 2.11). In [Bambusi 2003; Bambusi and Grébert 2006], such terms were neglected since the vector field of a monomial containing at least three Fourier modes with large indices is small in *Sobolev norm* (but not in analytic norm), and thus will almost keep all the modes invariant. This more subtle analysis was also used in [Faou et al. 2010].

Our method could be generalized by considering not only zero momentum monomials but also monomials with finite or exponentially decreasing momentum. This would certainly allow us to consider a nonlinear Schrödinger equation with a multiplicative potential V and nonlinearities depending periodically

[1] Here the actions are the square of the modulus of the Fourier coefficients, $I_k = |\xi_k|^2$.

on x:

$$iu_t = -\Delta u + Vu + \partial_{\bar u} g(x, u, \bar u), \quad x \in \mathbb{T}^d.$$

Nevertheless, this generalization would generate a lot of technicalities and we prefer to focus in the present article on the simplicity of the arguments.

2. Setting and hypothesis

2A. ***Hamiltonian formalism.*** Equation (1-1) is a semilinear PDE locally well posed in the Sobolev space $H^s(\mathbb{T}^d)$ with $s > d/2$ (see, for instance, [Cazenave 2003]). Let u be a (local) solution of (1-1) and consider $(\xi, \eta) = (\xi_a, \eta_a)_{a\in\mathbb{Z}^d}$ the Fourier coefficients of u, $\bar u$

$$u(x) = \sum_{a\in\mathbb{Z}^d} \xi_a e^{ia\cdot x} \quad \text{and} \quad \bar u(x) = \sum_{a\in\mathbb{Z}^d} \eta_a e^{-ia\cdot x}. \tag{2-1}$$

A standard calculation shows that u is a solution in $H^s(\mathbb{T}^d)$ of (1-1) if and only if (ξ, η) is a solution in[2] $\ell^2_s \times \ell^2_s$ of the system

$$\begin{cases} \dot\xi_a = -i\omega_a \xi_a - i\dfrac{\partial P}{\partial \eta_a}, & a \in \mathbb{Z}^d, \\ \dot\eta_a = i\omega_a \eta_a - i\dfrac{\partial P}{\partial \xi_a}, & a \in \mathbb{Z}^d, \end{cases} \tag{2-2}$$

where the linear frequencies are given by $\omega_a = |a|^2 + v_a$. As in (1-2), the notation is $V = \sum v_a e^{ia\cdot x}$. The nonlinear part is given by

$$P(\xi, \eta) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g\Big(\sum \xi_a e^{ia\cdot x}, \sum \eta_a e^{-ia\cdot x}\Big)\, \mathrm{d}x. \tag{2-3}$$

This system is Hamiltonian when endowing the set of pairs $(\xi_a, \eta_a) \in \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}$ with the symplectic structure

$$i \sum_{a\in\mathbb{Z}^d} \mathrm{d}\xi_a \wedge \mathrm{d}\eta_a. \tag{2-4}$$

We define the set $\mathcal{Z} = \mathbb{Z}^d \times \{\pm 1\}$. For $j = (a, \delta) \in \mathcal{Z}$, we define $|j| = |a|$ and we denote by $\bar j$ the index $(a, -\delta)$.

We identify a pair $(\xi, \eta) \in \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}$ with $(z_j)_{j\in\mathcal{Z}} \in \mathbb{C}^{\mathcal{Z}}$ via the formula

$$j = (a, \delta) \in \mathcal{Z} \implies \begin{cases} z_j = \xi_a & \text{if } \delta = 1, \\ z_j = \eta_a & \text{if } \delta = -1. \end{cases} \tag{2-5}$$

By a slight abuse of notation, we often write $z = (\xi, \eta)$ to denote such an element.

For a given $\rho > 0$, we consider the Banach space $\mathcal{L}_\rho$ made of elements $z \in \mathbb{C}^{\mathcal{Z}}$ such that

$$\|z\|_\rho := \sum_{j\in\mathcal{Z}} e^{\rho|j|} |z_j| < \infty,$$

[2] As usual, $\ell^2_s = \big\{(\xi_a)_{a\in\mathbb{Z}^d} \mid \sum (1 + |a|^{2s})|\xi_a|^2 < +\infty\big\}$.

using the symplectic form (2-4). We say that $z \in \mathcal{L}_\rho$ is *real* when $z_{\bar{j}} = \bar{z}_j$ for any $j \in \mathcal{Z}$. In this case, we write $z = (\xi, \bar{\xi})$ for some $\xi \in \mathbb{C}^{\mathbb{Z}^d}$. In this situation, we can associate with z the function u defined by (2-1).

The next lemma shows the relation with the space $\mathcal{A}_\rho$ defined above:

Lemma 2.1. *Let u be a complex valued function analytic on a neighborhood of $\mathbb{T}^d$, and let $(z_j)_{j\in\mathcal{Z}}$ be the sequence of its Fourier coefficients defined by* (2-1) *and* (2-5). *Then for all $\mu < \rho$, we have*

$$\text{if } u \in \mathcal{A}_\rho, \quad \text{then } z \in \mathcal{L}_\mu \quad \text{and} \quad \|z\|_\mu \le c_{\rho,\mu}|u|_\rho, \tag{2-6}$$

$$\text{if } z \in \mathcal{L}_\rho, \quad \text{then } u \in \mathcal{A}_\mu \quad \text{and} \quad |u|_\mu \le c_{\rho,\mu}\|z\|_\rho, \tag{2-7}$$

where $c_{\rho,\mu}$ is a constant depending on ρ and μ and the dimension d.

Proof. Assume that $u \in \mathcal{A}_\rho$. Then by using the Cauchy formula, we get $|z_j| \le |u|_\rho e^{-\rho|j|}$ for all $j \in \mathcal{Z}$. Hence, for $\mu < \rho$ we have

$$\|z\|_\mu \le |u|_\rho \sum_{j\in\mathcal{Z}} e^{(\mu-\rho)|j|} \le |u|_\rho \left(2\sum_{n\in\mathbb{Z}} e^{\frac{(\mu-\rho)}{\sqrt{d}}|n|}\right)^d \le \left(\frac{2}{1-e^{\frac{(\mu-\rho)}{\sqrt{d}}}}\right)^d |u|_\rho.$$

Conversely, assume that $z \in \mathcal{L}_\rho$. Then $|\xi_a| \le \|z\|_\rho e^{-\rho|a|}$ for all $a \in \mathbb{Z}^d$, and thus by (2-1), for all $x \in \mathbb{T}^d$ and $y \in \mathbb{R}^d$ with $|y| \le \mu$, we get

$$|u(x+iy)| \le \sum_{a\in\mathbb{Z}^d} |\xi_a| e^{|ay|} \le \|z\|_\rho \sum_{a\in\mathbb{Z}^d} e^{-(\rho-\mu)|a|} \le \left(\frac{2}{1-e^{\frac{(\mu-\rho)}{\sqrt{d}}}}\right)^d \|z\|_\rho.$$

Hence, u is bounded on the strip I_μ. □

For a function F of $\mathcal{C}^1(\mathcal{L}_\rho, \mathbb{C})$, we define its Hamiltonian vector field by $X_F = J\nabla F$, where J is the symplectic operator on $\mathcal{L}_\rho$ induced by the symplectic form (2-4), $\nabla F(z) = (\partial F/\partial z_j)_{j\in\mathcal{Z}}$, and where by definition, for $j = (a, \delta) \in \mathbb{Z}^d \times \{\pm 1\}$ we set

$$\frac{\partial F}{\partial z_j} = \begin{cases} \dfrac{\partial F}{\partial \xi_a} & \text{if } \delta = 1, \\[2ex] \dfrac{\partial F}{\partial \eta_a} & \text{if } \delta = -1. \end{cases}$$

For two functions F and G, the Poisson bracket is (formally) defined as

$$\{F, G\} = \nabla F^T J \nabla G = i \sum_{a\in\mathbb{Z}^d} \frac{\partial F}{\partial \eta_a}\frac{\partial G}{\partial \xi_a} - \frac{\partial F}{\partial \xi_a}\frac{\partial G}{\partial \eta_a}. \tag{2-8}$$

We say that a Hamiltonian function H is *real* if $H(z)$ is real for all real z.

Definition 2.2. For a given $\rho > 0$, we denote by $\mathcal{H}_\rho$ the space of real Hamiltonians P satisfying

$$P \in \mathcal{C}^1(\mathcal{L}_\rho, \mathbb{C}) \quad \text{and} \quad X_P \in \mathcal{C}^1(\mathcal{L}_\rho, \mathcal{L}_\rho).$$

For F and G in $\mathcal{H}_\rho$, the formula (2-8) is well defined. With a given Hamiltonian function $H \in \mathcal{H}_\rho$, we associate the Hamiltonian system

$$\dot{z} = X_H(z) = J\nabla H(z),$$

which also reads

$$\dot{\xi}_a = -i\frac{\partial H}{\partial \eta_a} \quad \text{and} \quad \dot{\eta}_a = i\frac{\partial H}{\partial \xi_a}, \quad a \in \mathbb{Z}^d. \tag{2-9}$$

We define the local flow $\Phi^t_H(z)$ associated with the previous system (for an interval of times $t \geq 0$ depending a priori on the initial condition z). If $z = (\xi, \bar{\xi})$ and if H is real, the flow $(\xi^t, \eta^t) = \Phi^t_H(z)$ is also real; $\xi^t = \bar{\eta}^t$ for all t. Choosing the Hamiltonian given by

$$H(\xi, \eta) = \sum_{a\in\mathbb{Z}^d} \omega_a \xi_a \eta_a + P(\xi, \eta),$$

P being given by (2-3), we recover the system (2-2), that is, the expression of the NLS equation (1-1) in Fourier modes.

Remark 2.3. The quadratic Hamiltonian $H_0 = \sum_{a\in\mathbb{Z}^d} \omega_a \xi_a \eta_a$ corresponding to the linear part of (1-1) does not belong to $\mathcal{H}_\rho$. Nevertheless, it generates a flow which maps $\mathcal{L}_\rho$ into $\mathcal{L}_\rho$ explicitly given for all time t and for all indices a by $\xi_a(t) = e^{-i\omega_a t}\xi_k(0)$, $\eta_a(t) = e^{i\omega_a t}\eta_k(0)$. On the other hand, we will see that, in our setting, the nonlinearity P belongs to $\mathcal{H}_\rho$.

2B. ***Space of polynomials.*** In this subsection we define a class of polynomials on $\mathbb{C}^{\mathcal{Z}}$.

We first need more notations concerning multi-indices: letting $\ell \geq 2$ and $\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell$ with $j_i = (a_i, \delta_i)$, we define

- the monomial associated with $\boldsymbol{j}$

$$z_{\boldsymbol{j}} = z_{j_1} \dots z_{j_\ell};$$

- the momentum of $\boldsymbol{j}$

$$\mathcal{M}(\boldsymbol{j}) = a_1\delta_1 + \dots + a_\ell\delta_\ell,; \tag{2-10}$$

- and the divisor associated with $\boldsymbol{j}$

$$\Omega(\boldsymbol{j}) = \delta_1\omega_{a_1} + \dots + \delta_\ell\omega_{a_\ell}, \tag{2-11}$$

  where for $a \in \mathbb{Z}^d$, $\omega_a = |a|^2 + v_a$ are the frequencies of the linear part of (1-1).

We then define the set of indices with *zero momentum* by

$$\mathcal{I}_\ell = \big\{\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell \;\big|\; \mathcal{M}(\boldsymbol{j}) = 0\big\}. \tag{2-12}$$

On the other hand, we say that $\boldsymbol{j} = (j_1, \dots, j_\ell) \in \mathcal{Z}^\ell$ is *resonant*, and we write $\boldsymbol{j} \in \mathcal{N}_\ell$, if ℓ is even and $\boldsymbol{j} = \boldsymbol{i} \cup \bar{\boldsymbol{i}}$ for some choice of $\boldsymbol{i} \in \mathcal{Z}^{\ell/2}$. In particular, if $\boldsymbol{j}$ is resonant, then its associated divisor vanishes, $\Omega(\boldsymbol{j}) = 0$, and its associated monomials depend only on the actions

$$z_{\boldsymbol{j}} = z_{j_1} \dots z_{j_\ell} = \xi_{a_1}\eta_{a_1} \dots \xi_{a_{\ell/2}}\eta_{a_{\ell/2}} = I_{a_1} \dots I_{a_{\ell/2}},$$

where $I_a(z) = \xi_a \eta_a$ denotes the action associated with the index a for all $a \in \mathbb{Z}^d$.

Finally, if z is real, then $I_a(z) = |\xi_a|^2$, and for odd r, the resonant set $\mathcal{N}_r$ is empty.

Definition 2.4. For $k \geq 2$, a (formal) polynomial $P(z) = \sum a_{\boldsymbol{j}} z_{\boldsymbol{j}}$ belongs to $\mathcal{P}_k$ if P is real, of degree k, has a zero of order at least 2 in $z = 0$, and satisfies the following conditions:

- P contains only monomials having zero momentum (i.e., such that $\mathcal{M}(\boldsymbol{j}) = 0$ when $a_{\boldsymbol{j}} \neq 0$), and thus P reads
$$P(z) = \sum_{\ell=2}^{k} \sum_{\boldsymbol{j} \in \mathcal{I}_\ell} a_{\boldsymbol{j}} z_{\boldsymbol{j}} \tag{2-13}$$
with the relation $a_{\bar{\boldsymbol{j}}} = \bar{a}_{\boldsymbol{j}}$.
- The coefficients $a_{\boldsymbol{j}}$ are bounded: $\sup\limits_{\boldsymbol{j} \in \mathcal{I}_\ell} |a_{\boldsymbol{j}}| < +\infty$ for all $\ell = 2, \ldots, k$.

We endow $\mathcal{P}_k$ with the norm
$$\|P\| = \sum_{\ell=2}^{k} \sup_{\boldsymbol{j} \in \mathcal{I}_\ell} |a_{\boldsymbol{j}}|. \tag{2-14}$$

The zero momentum assumption in Definition 2.4 is crucial to obtaining the following proposition:

Proposition 2.5. *Let $k \geq 2$ and $\rho > 0$. We have $\mathcal{P}_k \subset \mathcal{H}_\rho$, and for P a homogeneous polynomial of degree k in $\mathcal{P}_k$, we have the estimates*
$$|P(z)| \leq \|P\| \, \|z\|_\rho^k \tag{2-15}$$
and
$$\|X_P(z)\|_\rho \leq 2k \|P\| \, \|z\|_\rho^{k-1} \quad \textit{for all } z \in \mathcal{L}_\rho. \tag{2-16}$$
Furthermore, for $P \in \mathcal{P}_k$ and $Q \in \mathcal{P}_\ell$, we have $\{P, Q\} \in \mathcal{P}_{k+\ell-2}$ and the estimate
$$\|\{P, Q\}\| \leq 2k\ell \|P\| \|Q\|. \tag{2-17}$$

Proof. Let
$$P(z) = \sum_{\boldsymbol{j} \in \mathcal{I}_k} a_{\boldsymbol{j}} z_{\boldsymbol{j}};$$
we have
$$|P(z)| \leq \|P\| \sum_{\boldsymbol{j} \in \mathcal{Z}^k} |z_{j_1}| \ldots |z_{j_k}| \leq \|P\| \, \|z\|_{\ell^1}^k \leq \|P\| \|z\|_\rho^k,$$
and the first inequality (2-15) is proved.

To prove the second estimate, let $\ell \in \mathcal{Z}$; by using the zero momentum condition, we get
$$\left| \frac{\partial P}{\partial z_\ell} \right| \leq k \|P\| \sum_{\substack{\boldsymbol{j} \in \mathcal{Z}^{k-1} \\ \mathcal{M}(\boldsymbol{j}) = -\mathcal{M}(\ell)}} |z_{j_1} \ldots z_{j_{k-1}}|.$$
Therefore

$$\|X_{\boldsymbol{P}}(z)\|_\rho = \sum_{\ell\in\mathcal{Z}} e^{\rho|\ell|}\left|\frac{\partial P}{\partial z_\ell}\right| \leq k\|P\| \sum_{\ell\in\mathcal{Z}} \sum_{\substack{\boldsymbol{j}\in\mathcal{Z}^{k-1}\\ \mathcal{M}(\boldsymbol{j})=-\mathcal{M}(\ell)}} e^{\rho|\ell|}|z_{j_1}\dots z_{j_{k-1}}|.$$

But if $\mathcal{M}(\boldsymbol{j}) = -\mathcal{M}(\ell)$, then

$$e^{\rho|\ell|} \leq \exp\bigl(\rho(|j_1|+\cdots+|j_{k-1}|)\bigr) \leq \prod_{n=1,\dots,k-1} e^{\rho|j_n|}.$$

Hence, after summing in ℓ, we get[3]

$$\|X_{\boldsymbol{P}}(z)\|_\rho \leq 2k\|P\| \sum_{\boldsymbol{j}\in\mathcal{Z}^{k-1}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}| \leq 2k\|P\|\,\|z\|_\rho^{k-1},$$

which yields (2-16).

Assume now that P and Q are homogeneous polynomials of degrees k and ℓ respectively and with coefficients $a_{\boldsymbol{k}}$, $\boldsymbol{k}\in\mathcal{I}_k$ and $b_{\boldsymbol{\ell}}$, $\boldsymbol{\ell}\in\mathcal{I}_\ell$. It is clear that $\{P,Q\}$ is a monomial of degree $k+\ell-2$ satisfying the zero momentum condition. Furthermore, we can write

$$\{P,Q\}(z) = \sum_{\boldsymbol{j}\in\mathcal{I}_{k+\ell-2}} c_{\boldsymbol{j}}\, z_{\boldsymbol{j}},$$

where $c_{\boldsymbol{j}}$ is expressed as a sum of coefficients $a_{\boldsymbol{k}} b_{\boldsymbol{\ell}}$ for which there exists an $a\in\mathbb{Z}^d$ and $\epsilon\in\{\pm1\}$ such that

$$(a,\epsilon)\subset \boldsymbol{k}\in\mathcal{I}_k \quad\text{and}\quad (a,-\epsilon)\subset\boldsymbol{\ell}\in\mathcal{I}_\ell,$$

and such that if for instance $(a,\epsilon)=k_1$ and $(a,-\epsilon)=\ell_1$, we necessarily have $(k_2,\dots,k_k,\ell_2,\dots,\ell_\ell)=\boldsymbol{j}$. Hence, for a given $\boldsymbol{j}$, the zero momentum condition on $\boldsymbol{k}$ and on $\boldsymbol{\ell}$ determines the value of ϵa, which in turn determines two possible values of (ϵ,a).

This proves (2-17) for monomials. The extension to polynomials follows from the definition of the norm (2-14).

The last assertion and the fact that the Poisson bracket of two real Hamiltonian is real follow immediately from the definitions. □

2C. *Nonlinearity.* We assume that the nonlinearity g is analytic in a neighborhood of the origin in $\mathbb{C}^2$: There exist positive constants M and R_0 such that the Taylor expansion

$$g(v_1,v_2) = \sum_{k_1,k_2\geq 0} \frac{1}{k_1!k_2!}\partial_{k_1}\partial_{k_2} g(0,0) v_1^{k_1} v_2^{k_2}$$

is uniformly convergent and bounded by M on the ball $|v_1|+|v_2|\leq 2R_0$. Hence, formula (2-3) defines an analytic function P on the ball $\|z\|_\rho \leq R_0$ in $\mathcal{L}_\rho$, and we have

$$P(z) = \sum_{k\geq 0} P_k(z),$$

[3]Note that $\mathcal{M}(a,\delta) = \mathcal{M}(-a,-\delta)$, whence we get the coefficient 2.

where P_k for all $k \geq 0$ is a homogeneous polynomial given by

$$P_k = \sum_{k_1+k_2=k} \sum_{(\boldsymbol{a},\boldsymbol{b})\in(\mathbb{Z}^d)^{k_1}\times(\mathbb{Z}^d)^{k_2}} p_{\boldsymbol{a},\boldsymbol{b}} \xi_{a_1} \dots \xi_{a_{k_1}} \eta_{b_1} \dots \eta_{b_{k_2}},$$

with

$$p_{\boldsymbol{a},\boldsymbol{b}} = \frac{1}{k_1!k_2!} \partial_{k_1} \partial_{k_2} g(0,0) \int_{\mathbb{T}^d} e^{i\mathcal{M}(\boldsymbol{a},\boldsymbol{b})\cdot x}\, \mathrm{d}x$$

and $\mathcal{M}(\boldsymbol{a},\boldsymbol{b}) = a_1 + \dots + a_{k_1} - b_1 - \dots - b_{k_2}$ the moment of $\xi_{a_1} \dots \xi_{a_{k_1}} \eta_{b_1} \dots \eta_{b_{k_2}}$. Therefore, it is clear that P_k satisfies the zero momentum condition, and thus $P_k \in \mathcal{P}_k$ for all $k \geq 0$. Furthermore, we have the estimate $\|P_k\| \leq M R_0^{-k}$ for all $k \geq 0$.

2D. *Nonresonance condition.* In order to control the divisors (2-11), we need to impose a nonresonance condition on the linear frequencies ω_a, $a \in \mathbb{Z}^d$.

For $r \geq 3$ and $\boldsymbol{j} = (j_1, \dots, j_r) \in \mathcal{Z}^r$, we define $\mu(\boldsymbol{j})$ as the third largest integer amongst $|j_1|, \dots, |j_r|$. We recall that the resonant set $\mathcal{N}_r$ is the set of multi-indices $\boldsymbol{j} \in \mathcal{Z}^r$ such that $\boldsymbol{j} = \boldsymbol{i} \cup \bar{\boldsymbol{i}}$ for some $\boldsymbol{i} \in \mathcal{Z}^{r/2}$.

Hypothesis 2.6. There exist $\gamma > 0$, $\nu \geq 1$ and $c_0 > 0$ such that for all $r \geq 3$ and for all nonresonant $\boldsymbol{j} \in \mathcal{Z}^r \backslash \mathcal{N}_r$, we have

$$|\Omega(\boldsymbol{j})| \geq \frac{\gamma c_0^r}{\mu(\boldsymbol{j})^{\nu r}}. \tag{2-18}$$

Remark 2.7. Classically, a nonresonance condition reads (see, for instance, [Bambusi and Grébert 2006]): for all $r \geq 3$, there exist $\gamma(r) > 0$ and $\nu(r) > 0$ such that for all nonresonant $\boldsymbol{j} \in \mathcal{Z}^r$, we have

$$|\Omega(\boldsymbol{j})| \geq \frac{\gamma(r)}{\mu(\boldsymbol{j})^{\nu(r)}}.$$

In Hypothesis 2.6, we make precise the dependence of γ and ν with respect to r. In particular, we impose that ν be linear: $\nu(r) = \nu r$. This is crucial to optimizing the choice of r as a function of ε in Section 3B.

Recall that for $V = \sum_{a\in\mathbb{Z}^d} w_a e^{ia\cdot x}$ in the space $\mathcal{W}_m$ defined in (1-2), the frequencies are

$$\omega_a = |a|^2 + w_a = |a|^2 + \frac{R v_a}{(1+|a|)^m}, \quad a \in \mathbb{Z}^d,$$

with $v_a \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for all a. In the Appendix, we prove:

Proposition 2.8. *Fix $\gamma > 0$ small enough and $m > d/2$. There exist positive constants c_0 and ν depending only on m, R and d, and a set $F_\gamma \subset \mathcal{W}_m$ whose measure is larger than $1 - 4\gamma^{1/7}$, such that if $V \in F_\gamma$, then* (2-18) *holds true for all nonresonant $\boldsymbol{j} \in \mathcal{Z}^r$ and for all $r \geq 3$.*

Thus Hypothesis 2.6 is satisfied for all $V \in \mathcal{V}$, where

$$\mathcal{V} = \bigcup_{\gamma>0} F_\gamma \tag{2-19}$$

is a subset of full measure in $\mathcal{W}_m$.

2E. ***Normal forms.*** We fix an index $N \geq 1$. For a fixed integer $k \geq 3$, we set

$$\mathcal{J}_k(N) = \{\boldsymbol{j} \in \mathcal{I}_k \mid \mu(\boldsymbol{j}) > N\}.$$

Definition 2.9. Let N be an integer. We say that a polynomial $Z \in \mathcal{P}_k$ is in N-normal form if it can be written

$$Z = \sum_{\ell=3}^{k} \sum_{\boldsymbol{j} \in \mathcal{N}_\ell \cup \mathcal{J}_\ell(N)} a_{\boldsymbol{j}} z_{\boldsymbol{j}}.$$

In other words, Z contains either monomials depending only on the actions or monomials whose indices $\boldsymbol{j}$ satisfy $\mu(\boldsymbol{j}) > N$, that is, monomials involving at least three modes with index greater than N.

We now motivate the introduction of this definition. First, we recall:

Lemma 2.10. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function and $y : \mathbb{R} \to \mathbb{R}_+$ a differentiable function satisfying the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \leq 2 f(t) \sqrt{y(t)} \quad \textit{for all } t \in \mathbb{R}.$$

Then we have the estimate

$$\sqrt{y(t)} \leq \sqrt{y(0)} + \int_0^t f(s)\,\mathrm{d}s \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Let $\epsilon > 0$ and define $y_\epsilon = y + \epsilon$, a nonnegative function whose square root is differentiable. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \sqrt{y_\epsilon(t)} \leq 2 f(t) \frac{\sqrt{y(t)}}{\sqrt{y_\epsilon(t)}} \leq 2 f(t),$$

and thus

$$\sqrt{y_\varepsilon(t)} \leq \sqrt{y_\epsilon(0)} + \int_0^t f(s)\,\mathrm{d}s.$$

The claim is proved by taking $\epsilon \to 0$. □

For a given number N and for $z \in \mathcal{L}_\rho$, we define

$$\mathrm{R}_\rho^N(z) = \sum_{|j|>N} e^{\rho|j|} |z_j|.$$

Notice that if $z \in \mathcal{L}_{\rho+\mu}$, then

$$\mathrm{R}_\rho^N(z) \leq e^{-\mu N} \|z\|_{\rho+\mu}. \tag{2-20}$$

Proposition 2.11. *Let $N \in \mathbb{N}$ and $k \geq 3$. Suppose that Z is a homogeneous polynomial of degree k in N-normal form. Let $z(t)$ be a real solution of the flow generated by the Hamiltonian $H_0 + Z$. Then we have*

$$\mathrm{R}_\rho^N(z(t)) \leq \mathrm{R}_\rho^N(z(0)) + 4k^3 \|Z\| \int_0^t \mathrm{R}_\rho^N(z(s))^2 \|z(s)\|_\rho^{k-3}\,\mathrm{d}s \tag{2-21}$$

and

$$\|z(t)\|_\rho \leq \|z(0)\|_\rho + 4k^3 \|Z\| \int_0^t \mathrm{R}_\rho^N(z(s))^2 \|z(s)\|_\rho^{k-3}\,\mathrm{d}s. \tag{2-22}$$

Proof. Fix $a \in \mathbb{Z}^d$ and let $I_a(t) = \xi_a(t)\eta_a(t)$ be the actions associated with the solution of the Hamiltonian system generated by $H_0 + Z$. Let us recall that as $z(t) = (\xi(t), \eta(t))$ is a real solution, we have $\xi_a(t) = \bar{\eta}_a(t)$ for all times where the solution is defined. Using (2-17) and $H_0 = H_0(I)$, we have

$$|e^{2\rho|a|}\dot{I}_a| = |e^{2\rho|a|}\{I_a, Z\}| \le 2k\|Z\|\,|e^{\rho|a|}\sqrt{I_a}|\Bigg(\sum_{\substack{\mathcal{M}(\boldsymbol{j})=\pm a\\ 2\text{ indices}>N}} e^{\rho|a|}|z_{j_1}\dots z_{j_{k-1}}|\Bigg).$$

Then using Lemma 2.10, we get

$$e^{\rho|a|}\sqrt{I_a(t)} \le e^{\rho|a|}\sqrt{I_a(0)} + 2k\|Z\|\int_0^t\Bigg(\sum_{\substack{\mathcal{M}(\boldsymbol{j})=\pm a\\ 2\text{ indices}>N}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}|\Bigg)\,\mathrm{d}s. \tag{2-23}$$

Ordering the multi-indices such that $|j_1|$ and $|j_2|$ are the largest, and using the fact that $z(t)$ is real (and thus $|z_j| = \sqrt{I_a}$ for $j = (a, \pm 1) \in \mathcal{Z}$), we obtain, after summation in $|a| > N$,

$$\begin{aligned}\mathsf{R}^N_\rho(z(t)) &\le \mathsf{R}^N_\rho(z(0)) + 4k^3\|Z\|\int_0^t\Bigg(\sum_{\substack{|j_1|,|j_2|\ge N\\ j_3,\dots,j_{k-1}\in\mathcal{Z}}} e^{\rho|j_1|}|z_{j_1}|\dots e^{\rho|j_{k-1}|}|z_{j_{k-1}}|\Bigg)\,\mathrm{d}s\\ &\le \mathsf{R}^N_\rho(z(0)) + 4k^3\|Z\|\int_0^t \mathsf{R}^N_\rho(z(s))^2\|z(s)\|_\rho^{k-3}\,\mathrm{d}s.\end{aligned}$$

Inequality (2-22) is proved in the same way. □

Remark 2.12. These estimates will be central to the final bootstrap argument. Actually, as a consequence of Proposition 2.11, we have: if $z(t)$ is the solution of a Hamiltonian system in N-normal form with an initial datum z_0 satisfying $\|z_0\|_{2\rho} = \varepsilon$, then, as $\mathsf{R}^N_\rho(z_0) = \mathcal{O}(\varepsilon e^{-\rho N})$, Equations (2-21) and (2-22) guarantee that $\mathsf{R}^N_\rho(z(t))$ remains of order $\mathcal{O}(\varepsilon e^{-\rho N})$ and the norm of $z(t)$ remains of order ε over exponentially long time $t = \mathcal{O}(e^{\rho N})$.

The next result is an easy consequence of the nonresonance condition and of the definition of normal forms:

Proposition 2.13. *Assume that the nonresonance condition* (2-18) *is satisfied and let N be fixed. Let Q be a homogenous polynomial of degree k. Then the homological equation*

$$\{\chi, H_0\} - Z = Q \tag{2-24}$$

admits a polynomial solution (χ, Z) homogeneous of degree k, such that Z is in N-normal form, and such that

$$\|Z\| \le \|Q\| \quad \textit{and} \quad \|\chi\| \le \frac{N^{\nu k}}{\gamma c_0^k}\|Q\|. \tag{2-25}$$

Proof. Assume that $Q = \sum_{\boldsymbol{j}\in\mathcal{I}_k} Q_{\boldsymbol{j}} z_{\boldsymbol{j}}$ and seek $Z = \sum_{\boldsymbol{j}\in\mathcal{I}_k} Z_{\boldsymbol{j}} z_{\boldsymbol{j}}$ and $\chi = \sum_{\boldsymbol{j}\in\mathcal{I}_k} \chi_{\boldsymbol{j}} z_{\boldsymbol{j}}$ such that (2-24) is satisfied. Equation (2-24) can be written in terms of polynomial coefficients

$$i\Omega(\boldsymbol{j})\chi_{\boldsymbol{j}} - Z_{\boldsymbol{j}} = Q_{\boldsymbol{j}}, \quad \boldsymbol{j}\in\mathcal{I}_k,$$

where $\Omega(\boldsymbol{j})$ is given in (2-11). We then define

$$Z_{\boldsymbol{j}} = Q_{\boldsymbol{j}}, \quad \chi_{\boldsymbol{j}} = 0 \qquad \text{if } \boldsymbol{j} \in \mathcal{N}_k \text{ or } \mu(\boldsymbol{j}) > N,$$
$$Z_{\boldsymbol{j}} = 0, \qquad \chi_{\boldsymbol{j}} = \frac{Q_{\boldsymbol{j}}}{i\Omega(\boldsymbol{j})} \quad \text{if } \boldsymbol{j} \notin \mathcal{N}_k \text{ and } \mu(\boldsymbol{j}) \le N.$$

In view of (2-18), this leads to (2-25). □

3. Proof of the main theorem

3A. *Recursive equation.* We aim to construct a canonical transformation τ such that in the new variables, the Hamiltonian $H_0 + P$ is in normal form modulo a small remainder term. Using Lie transforms to generate τ, the problem can be written thus: Find a polynomial $\chi = \sum_{k=3}^{r} \chi_k$, a polynomial $Z = \sum_{k=3}^{r} Z_k$ in normal form, and a smooth Hamiltonian R satisfying $\partial^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^{\mathcal{Z}}$ with $|\alpha| \le r$, such that

$$(H_0 + P) \circ \Phi_\chi^1 = H_0 + Z + R. \tag{3-1}$$

Then the exponential estimate (1-3) will be obtained by optimizing the choice of r and N.

We recall that for χ and K two Hamiltonian functions, for all $k \ge 0$ we have

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}(K \circ \Phi_\chi^t) = \big\{\chi, \{\cdots\{\chi, K\}\cdot\}\big\}(\Phi_\chi^t) = (\mathrm{ad}_\chi^k K)(\Phi_\chi^t),$$

where $\mathrm{ad}_\chi K = \{\chi, K\}$. Also, if K, L are homogeneous polynomials of degrees k and ℓ, then $\{K, L\}$ is a homogeneous polynomial of degree $k + \ell - 2$. Therefore, by using Taylor's formula, we obtain

$$(H_0 + P) \circ \Phi_\chi^1 - (H_0 + P) = \sum_{k=0}^{r-3} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k(\{\chi, H_0 + P\}) + \mathbb{O}_r, \tag{3-2}$$

where $\mathbb{O}_r$ stands for a smooth function R satisfying $\partial^\alpha R(0) = 0$ for all $\alpha \in \mathbb{N}^{\mathcal{Z}}$ with $|\alpha| \le r$.

On the other hand, we know that for $\zeta \in \mathbb{C}$, the following relation holds:

$$\left(\sum_{k=0}^{r-3} \frac{B_k}{k!}\zeta^k\right)\left(\sum_{k=0}^{r-3} \frac{1}{(k+1)!}\zeta^k\right) = 1 + O(|\zeta|^{r-2}),$$

where B_k are the Bernoulli numbers defined by the expansion of the generating function $\frac{z}{e^z - 1}$. Therefore, defining the two differential operators

$$A_r = \sum_{k=0}^{r-3} \frac{1}{(k+1)!} \mathrm{ad}_\chi^k \quad \text{and} \quad B_r = \sum_{k=0}^{r-3} \frac{B_k}{k!} \mathrm{ad}_\chi^k,$$

we get

$$B_r A_r = \mathrm{Id} + C_r,$$

where C_r is a differential operator satisfying

$$C_r \mathbb{O}_3 = \mathbb{O}_r.$$

Applying B_r to the two sides of (3-2), we obtain

$$\{\chi, H_0 + P\} = B_r(Z - P) + \mathbb{O}_r.$$

Plugging the decompositions in homogeneous polynomials of χ, Z and P into this equation and equating the terms of same degree, we obtain after a straightforward calculation the recursive equations

$$\{\chi_m, H_0\} - Z_m = Q_m, \quad m = 3, \dots, r, \tag{3-3}$$

where

$$Q_m = -P_m + \sum_{k=3}^{m-1}\{P_{m+2-k}, \chi_k\} + \sum_{k=1}^{m-3} \frac{B_k}{k!} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k \\ 3\le \ell_i \le m-k}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} (Z_{\ell_{k+1}} - P_{\ell_{k+1}}). \tag{3-4}$$

In the last sum, $\ell_i \le m-k$ as a consequence of $3 \le \ell_i$ and $\ell_1 + \cdots + \ell_{k+1} = m+2k$.

Once these recursive equations are solved, we define the remainder term as $R = (H_0+P)\circ\Phi^1_\chi - H_0 - Z$. By construction, R is analytic on a neighborhood of the origin in $\mathscr{L}_\rho$ and $R = \mathbb{O}_r$. As a consequence, by Taylor's formula,

$$R = \sum_{m\ge r+1} \sum_{k=1}^{m-3} \frac{1}{k!} \sum_{\substack{\ell_1+\cdots+\ell_k=m+2k \\ 3\le \ell_i\le r}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} H_0 \\ + \sum_{m\ge r+1} \sum_{k=0}^{m-3} \frac{1}{k!} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k \\ 3\le \ell_1+\cdots+\ell_k\le r \\ 3\le \ell_{k+1}}} \mathrm{ad}_{\chi_{\ell_1}} \dots \mathrm{ad}_{\chi_{\ell_k}} P_{\ell_{k+1}}. \tag{3-5}$$

Lemma 3.1. *Assume that the nonresonance condition* (2-18) *is fulfilled for some constants γ, c_0, ν. Then there exists $C > 0$ such that for all r and N, and for $m = 3, \dots, r$, there exist homogeneous polynomials χ_m and Z_m of degree m, with Z_m in N-normal forms, which are solutions of the recursive equation* (3-3) *and satisfy*

$$\|\chi_m\| + \|Z_m\| \le (CmN^\nu)^{m^2}. \tag{3-6}$$

Proof. We define χ_m and Z_m by induction using Proposition 2.13. Note that (3-6) is clearly satisfied for $m = 3$, provided C is big enough. Estimate (2-25) yields

$$\gamma c_0^m N^{-\nu m}\|\chi_m\| + \|Z_m\| \le \|Q_m\|. \tag{3-7}$$

Using the definition (3-4) of the term Q_m and the estimate on the Bernoulli numbers, $|B_k| \le k!c^k$ for some $c > 0$, together with (2-17), which implies that for all $\ell \ge 3$, $\|\mathrm{ad}_{\chi_\ell} R\| \le 2m\ell\|R\|$ for any polynomial R of degree less than m, we have, for all $m \ge 3$,

$$\|Q_m\| \le \|P_m\| + 2\sum_{k=3}^{m-1} k(m+2-k)\|P_{m+2-k}\|\,\|\chi_k\| \\ + 2\sum_{k=1}^{m-3}(Cm)^k \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k \\ \le \ell_i \le m-k}} \ell_1\|\chi_{\ell_1}\| \dots \ell_k\|\chi_{\ell_k}\|\,\|Z_{\ell_{k+1}} - P_{\ell_{k+1}}\|. \tag{3-8}$$

for some constant C. Let us set $\beta_m = m(\|\chi_m\| + \|Z_m\|)$. Equation (3-7) implies that

$$\beta_m \le (CN^\nu)^m m\|Q_m\|,$$

for some constant C independent of m.

Using that $\|P_m\| \le MR_0^{-m}$ (see the end of Section 2D), we have that $\|P_m\|$ and $m\|P_m\|$ are uniformly bounded with respect to m. Hence, the previous inequality implies that

$$\beta_m \le \beta_m^{(1)} + \beta_m^{(2)},$$

where

$$\beta_m^{(1)} = (CN^\nu)^m m\Big(1 + \sum_{k=3}^{m-1} \beta_k\Big) \tag{3-9}$$

and

$$\beta_m^{(2)} = N^{\nu m}(Cm)^{m-2}\sum_{k=1}^{m-3} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k\\ \le\ell_i\le m-k}} \beta_{\ell_1}\dots\beta_{\ell_k}(\beta_{\ell_{k+1}}+1), \tag{3-10}$$

for some constant C depending on M, R_0, γ and c_0. It remains to prove that $\beta_m \le (CmN^\nu)^{\delta m^2}$ by induction, for some constant δ. Again, this is true for $m = 3$ by adapting C if necessary. Thus, assume that $\beta_j \le (CjN^\nu)^{j^2}$, $j = 3, \dots, m-1$. As soon as $C > 1$,

$$1 \le (CmN^\nu)^{m^2} \quad \text{for all } m \ge 3, \tag{3-11}$$

so we get

$$\beta_m^{(1)} \le (CN^\nu)^m m^{m+2}(CmN^\nu)^{(m-1)^2} \le \tfrac{1}{2}(CmN^\nu)^{m^2}$$

as soon as $m \ge 3$ and provided $C > 2$.

Using (3-11) again and the induction hypothesis, we get

$$\beta_m^{(2)} \le N^{\nu m}(Cm)^{m-2}\sum_{k=1}^{m-3} \sum_{\substack{\ell_1+\cdots+\ell_{k+1}=m+2k\\ 3\le\ell_i\le m-k}} \big(CN^\nu(m-k)\big)^{\ell_1^2+\cdots+\ell_{k+1}^2}.$$

The maximum of $\ell_1^2 + \cdots + \ell_{k+1}^2$ when $\ell_1 + \cdots + \ell_{k+1} = m+2k$ and $3 \le \ell_i \le m-k$ is obtained for $\ell_1 = \cdots = \ell_k = 3$ and $\ell_{k+1} = m-k$ and its value is $(m-k)^2 + 9k$. Furthermore, the cardinality of $\{\ell_1 + \cdots + \ell_{k+1} = m+2k,\ 3 \le \ell_i \le m-k\}$ is smaller than m^{k+1}, and hence we obtain, for $m \ge 4$ and after adapting C if necessary,

$$\beta_m^{(2)} \le \max_{k=1,\dots,m-3} N^{\nu m}(Cm)^{m-2}Cm^{k+2}\big(CN^\nu(m-k)\big)^{(m-k)^2+9k} \le \tfrac{1}{2}(CmN^\nu)^{m^2}. \qquad \square$$

3B. ***Normal form result.*** For any $R_0 > 0$, we set $B_\rho(R_0) = \{z \in \mathscr{L}_\rho \mid \|z\|_\rho < R_0\}$.

Theorem 3.2. *Assume that P is analytic on a ball $B_\rho(R_0)$ for some $R_0 > 0$ and $\rho > 0$. Assume that the nonresonance condition* (2-18) *is satisfied, and let $\beta < 1$ and $M > 1$ be fixed. Then there exist constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that for all $\varepsilon < \varepsilon_0$, there exist a polynomial χ, a polynomial Z in $N = |\ln \varepsilon|^{1+\beta}$ normal form, and a Hamiltonian R analytic on $B_\rho(M\varepsilon)$, such that*

$$(H_0 + P) \circ \Phi_\chi^1 = H_0 + Z + R. \tag{3-12}$$

Furthermore, for all $z \in B_\rho(M\varepsilon)$,

$$\|X_Z(z)\|_\rho + \|X_\chi(z)\|_\rho \le 2\varepsilon^{3/2} \quad \textit{and} \quad \|X_R(z)\|_\rho \le \varepsilon e^{-\frac{1}{4}|\ln \varepsilon|^{1+\beta}}. \tag{3-13}$$

Proof. Using Lemma 3.1, for all N and r, we can construct polynomial Hamiltonians

$$\chi(z) = \sum_{k=3}^{r} \chi_k(z) \quad \text{and} \quad Z(z) = \sum_{k=3}^{r} Z_k(z),$$

with Z in N-normal form, such that (3-12) holds with $R = \mathbb{O}_r$. Now for fixed $\varepsilon > 0$, we choose

$$N \equiv N(\varepsilon) = |\ln \varepsilon|^{1+\beta} \quad \text{and} \quad r \equiv r(\varepsilon) = |\ln \varepsilon|^{\beta}.$$

This choice is motivated by the necessity of a balance between Z and R in (3-12): The error induced by Z is controlled as in Remark 2.12, while the error induced by R is controlled by Lemma 3.1. By (3-6), we have

$$\begin{aligned} \|\chi_k\| &\le (CkN^\nu)^{k^2} \le \exp\bigl(k(\nu k(1+\beta)\ln|\ln\varepsilon| + k\ln Ck)\bigr) \\ &\le \exp\bigl(k(\nu r(1+\beta)\ln|\ln\varepsilon| + r\ln Cr)\bigr) \\ &\le \exp\bigl(k|\ln\varepsilon|\bigl(\nu|\ln\varepsilon|^{\beta-1}(1+\beta)\ln|\ln\varepsilon| + |\ln\varepsilon|^{\beta-1}\ln C|\ln\varepsilon|^{\beta}\bigr)\bigr) \le \varepsilon^{-k/8}, \end{aligned} \tag{3-14}$$

as $\beta < 1$, and for $\varepsilon \le \varepsilon_0$ sufficiently small. Therefore, using Proposition 2.5, for $z \in B_\rho(M\varepsilon)$ we obtain

$$|\chi_k(z)| \le \varepsilon^{-k/8}(M\varepsilon)^k \le M^k \varepsilon^{7k/8},$$

and thus

$$|\chi(z)| \le \sum_{k\ge 3} M^k \varepsilon^{7k/8} \le \varepsilon^{3/2},$$

for ε small enough. Similarly, for all $k \le r$, we have

$$\|X_{\chi_k}(z)\|_\rho \le 2k\varepsilon^{-k/8}(M\varepsilon)^{k-1} \le 2kM^{k-1}\varepsilon^{7k/8-1}$$

and

$$\|X_\chi(z)\|_\rho \le \sum_{k\ge 3} 2kM^{k-1}\varepsilon^{7k/8-1} \le C\varepsilon^{-1}\varepsilon^{21/8} \le \varepsilon^{3/2},$$

for ε small enough. Similar bounds clearly hold for $Z = \sum_{k=3}^{r} Z_k$, which shows the first estimate in (3-13).

On the other hand, using $\mathrm{ad}_{\chi_{\ell_k}} H_0 = Z_{\ell_k} + Q_{\ell_k}$ (see (3-3)) and then using Lemma 3.1 and the definition of Q_m (see (3-4)), we get $\|\mathrm{ad}_{\chi_{\ell_k}} H_0\| \le (CkN^\nu)^{\ell_k{}^2} \le \varepsilon^{-\ell_k/8}$, where the last inequality proceeds as in (3-14). Thus, using (3-5), (3-14) and $\|P_{\ell_{k+1}}\| \le MR_0^{-\ell_{k+1}}$, we obtain by Proposition 2.5 that for $z \in B_\rho(M\varepsilon)$,

$$\|X_R(z)\|_\rho \le \sum_{m\ge r+1} \sum_{k=0}^{m-3} m(Cr)^{3m} \varepsilon^{-\frac{m+2k}{8}} \varepsilon^{m-1} \le \sum_{m\ge r+1} m^2(Cr)^{3m}\varepsilon^{m/2} \le (Cr)^{3r}\varepsilon^{r/2}.$$

Therefore, since $r = |\ln\varepsilon|^\beta$, we get $\|X_R(z)\|_\rho \le \varepsilon\, e^{-\frac{1}{4}|\ln\varepsilon|^{1+\beta}}$ for $z \in B_\rho(M\varepsilon)$ and ε small enough. □

3C. ***Bootstrap argument.*** We are now in position to prove the main theorem of Section 1. It is a direct consequence of Theorem 3.2.

Let $u_0 \in \mathcal{A}_{2\rho}$ with $|u_0|_{2\rho} = \varepsilon$, and denote by $z(0)$ the corresponding sequence of its Fourier coefficients which belongs, by Lemma 2.1, to $\mathcal{L}_{(3/2)\rho}$ with $\|z(0)\|_{(3/2)\rho} \le (c_\rho/4)\varepsilon$ and

$$c_\rho = \frac{2^{d+2}}{(1-e^{-\rho/2\sqrt{d}})^d}.$$

Let $z(t)$ be the local solution in $\mathcal{L}_\rho$ of the Hamiltonian system associated with $H = H_0 + P$.

Let χ, Z and R be given by Theorem 3.2 with $M = c_\rho$ and let $y(t) = \Phi_\chi^1(z(t))$. We recall that since $\chi(z) = O(\|z\|^3)$, the transformation Φ_χ^1 is close to the identity: $\Phi_\chi^1(z) = z + O(\|z\|^2)$, and thus, for ε small enough, we have $\|y(0)\|_{(3/2)\rho} \le (c_\rho/2)\varepsilon$. In particular, as given in (2-20),

$$R_\rho^N(y(0)) \le \frac{c_\rho}{2}\varepsilon\, e^{-(\rho/2)N} \le \frac{c_\rho}{2}\varepsilon\, e^{-\sigma N},$$

where $\sigma = \sigma_\rho \le \rho/2$.

Let T_ε be the largest time T such that $R_\rho^N(y(t)) \le c_\rho\varepsilon\, e^{-\sigma N}$ and $\|y(t)\|_\rho \le c_\rho\varepsilon$ for all $|t| \le T$. By construction, we have

$$y(t) = y(0) + \int_0^t X_{H_0+Z}(y(s))\,ds + \int_0^t X_R(y(s))\,ds.$$

So using (2-21) for the first vector field and (3-13) for the second one, we get, for $|t| < T_\varepsilon$,

$$\begin{aligned} R_\rho^N(y(t)) &\le \tfrac{1}{2}c_\rho\varepsilon e^{-\sigma N} + 4|t|\sum_{k=3}^{r}\|Z_k\|k^3(c_\rho\varepsilon)^{k-1}e^{-2\sigma N} + |t|\varepsilon e^{-\frac{1}{4}|\ln\varepsilon|^{1+\beta}} \\ &\le \biggl(\tfrac{1}{2} + 4|t|\sum_{k=3}^{r}\|Z_k\|k^3(c_\rho\varepsilon)^{k-2}e^{-\sigma N} + |t|\varepsilon e^{-\frac{1}{8}|\ln\varepsilon|^{1+\beta}}\biggr)c_\rho\varepsilon e^{-\sigma N}, \end{aligned} \tag{3-15}$$

where in the last inequality we used $\sigma = \min\{\frac{1}{10}, \frac{1}{2}\rho\}$ and $N = |\ln\varepsilon|^{1+\beta}$.

Using Lemma 3.1, we then verify that

$$R_\rho^N(y(t)) \le \bigl(\tfrac{1}{2} + C|t|\varepsilon\, e^{-\sigma N}\bigr)c_\rho\varepsilon e^{-\sigma N},$$

and thus, for ε small enough,

$$R_\rho^N(y(t)) \le c_\rho\varepsilon\, e^{-\sigma N} \quad \text{for all } |t| \le \min\{T_\varepsilon, e^{\sigma N}\}. \tag{3-16}$$

Similarly, we obtain

$$\|y(t)\|_\rho \leq c_\rho \varepsilon \quad \text{for all } |t| \leq \min\{T_\varepsilon, e^{\sigma N}\}. \tag{3-17}$$

In view of the definition of T_ε, inequalities (3-16) and (3-17) imply $T_\varepsilon \geq e^{\sigma N}$. In particular, $\|z(t)\|_\rho \leq 2c_\rho\varepsilon$ for $|t| \leq e^{\sigma N} = \varepsilon^{-\sigma|\ln\varepsilon|^\beta}$, and using (2-7), we finally obtain (1-3) with

$$C = \frac{2^{2d+5}}{(1-e^{-\rho/2\sqrt{d}})^{2d}}.$$

Estimate (1-4) is another consequence of the normal form result and Proposition 2.11. Actually, we use that the Fourier coefficients of $u(t)$ are given by $z(t)$, which is ε^2-close to $y(t)$, which in turn is almost invariant: in view of (2-23) and as in (3-15), we have

$$\sum_{j\in\mathbb{Z}} e^{\rho|j|}\big||y_j(t)|-|y_j(0)|\big| \leq \left(4|t|\sum_{k=3}^{r}\|Z_k\|k^3(c_\rho\varepsilon)^{k-1}e^{-2\sigma N} + |t|\varepsilon e^{-\frac{1}{4}|\ln\varepsilon|^{1+\beta}}\right),$$

from which we deduce

$$\sum_{j\in\mathbb{Z}} e^{\rho|j|}\big||y_j(t)|-|y_j(0)|\big| \leq |t|e^{-\sigma N},$$

and then (1-4).

Appendix: Proof of the nonresonance hypothesis

Instead of proving Proposition 2.8, we prove a slightly more general result. For a multi-index $\boldsymbol{j} \in \mathcal{L}^r$, we define

$$N(\boldsymbol{j}) = \prod_{k=1}^{r}(1+|j_k|).$$

Proposition A.1. *Fix $\gamma > 0$ small enough and $m > d/2$. There exist positive constants C and ν depending only on m, R and d, and a set $F_\gamma \subset \mathcal{W}_m$ (see (1-2)) whose measure is larger than $1-4\gamma$, such that if $V \in F_\gamma$, then for any $r \geq 1$,*

$$\big|\Omega(\boldsymbol{j}) + \varepsilon_1\omega_{\ell_1} + \varepsilon_2\omega_{\ell_2}\big| \geq \frac{C^r\gamma^7}{N(\boldsymbol{j})^\nu} \tag{A-1}$$

for any $\boldsymbol{j} \in \mathcal{L}^r$, any indices $\ell_1, \ell_2 \in \mathbb{Z}^d$, and any $\varepsilon_1, \varepsilon_2 \in \{0, 1, -1\}$ such that $(\boldsymbol{j}, (\ell_1, \varepsilon_1), (\ell_2, \varepsilon_2))$ is nonresonant[4].

In order to prove Proposition A.1, we first prove that $\Omega(\boldsymbol{j})$ cannot accumulate on $\mathbb{Z}$. Precisely, we have:

Lemma A.2. *Fix $\gamma > 0$ and $m > d/2$. There exist $0 < C < 1$ depending only on m, R and d, and a set $F'_\gamma \subset \mathcal{W}_m$ whose measure is larger than $1-4\gamma$, such that if $V \in F'_\gamma$, then for any $r \geq 1$,*

$$|\Omega(\boldsymbol{j}) - b| \geq \frac{C^r\gamma}{N(\boldsymbol{j})^{m+d+3}} \tag{A-2}$$

[4]The resonant set $\mathcal{N}_r$, $r \geq 2$, is defined in Section 2D.

for any nonresonant $\boldsymbol{j} \in \mathbb{Z}^r$ *and for any* $b \in \mathbb{Z}$.

Proof. Let $(\alpha_1, \dots, \alpha_r) \neq 0$ in $\mathbb{Z}^r$, $M > 0$ and $c \in \mathbb{R}$. The set

$$\mathscr{E}(\eta) = \left\{ x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^r \;\middle|\; \left| \sum_{i=1}^r \alpha_i x_i + c \right| < \eta \right\}$$

is a slice of thickness 2η of the hypercube $[-M, M]^r$ guided by the hyperplane $\left\{\sum_{i=1}^r \alpha_i x_i + c = 0\right\}$, whose normal α has a norm larger than 1. Since the largest diagonal in the hypercube $\left[-\frac{1}{2}, \frac{1}{2}\right]^r$ has a length equal to $\sqrt{r}$, we get that the base of the slice $\mathscr{E}(\eta)$ is included in a hyperdisc of dimension $r-1$ and radius $\frac{1}{2}\sqrt{r}$. Recall that the volume of a ball in $\mathbb{R}^m$ of radius ρ equals $\pi^{m/2}\rho^m / \Gamma(m/2+1)$. So we deduce that the volume of $\mathscr{E}(\eta)$ is smaller than[5]

$$2\eta\pi^{(r-1)/2} \frac{\left(\frac{1}{2}\sqrt{r}\right)^{r-1}}{\Gamma\left(\frac{r-1}{2}+1\right)} \leq 2\eta \frac{\left(\frac{1}{2}\sqrt{\pi r}\right)^{r-1}}{\left(\frac{r-1}{2}\right)!} \leq C^r \eta$$

for a constant C independent of r. Hence, given $\boldsymbol{j} = (a_i, \delta_i)_{i=1}^r \in \mathscr{Z}^r$ and $b \in \mathbb{Z}$, the Lebesgue measure of

$$\mathscr{X}_\eta := \left\{ x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^r \;\middle|\; \left| \sum_{i=1}^r \delta_i (|a_i|^2 + x_i) - b \right| < \eta \right\}$$

is smaller than $2\eta r^{\frac{r-1}{2}}$. Now consider the set (using the notation (1-2))

$$\left\{ V \in \mathscr{W}_m \;\middle|\; |\Omega(\boldsymbol{j}) - b| < \eta \right\} = \left\{ V \in \mathscr{W}_m \;\middle|\; \left| \sum_{i=1}^r \delta_i \left(|a_i|^2 + \frac{v_{a_i} R}{(1+|a_i|)^m} \right) - b \right| < \eta \right\}. \tag{A-3}$$

It is contained in the set of the V's such that $(Rv_{a_i}/(1+|a_i|)^m)_{i=1}^r \in \mathscr{X}_\eta$. Hence the measure of (A-3) is smaller than $R^{-r} N(\boldsymbol{j})^m C^r \eta$. To conclude the proof, we have to sum over all the possible $\boldsymbol{j}$'s and all the possible b's. Now for a given $\boldsymbol{j}$, if $|\Omega(\boldsymbol{j}) - b| \geq \eta$ with $\eta \leq 1$, then $|b| \leq 2N(\boldsymbol{j})^2$. So to guarantee (A-2) for all possible choices of $\boldsymbol{j}$, b and r, it suffices to remove from $\mathscr{W}_m$ a set of measure

$$4\gamma \sum_{\boldsymbol{j} \in \mathscr{Z}^r} \frac{C^r}{R^r N(\boldsymbol{j})^{m+3+d}} N(\boldsymbol{j})^{m+2} \leq 4\gamma \left[\frac{2C}{R} \sum_{\ell \in \mathbb{Z}^d} \frac{1}{(1+|\ell|)^{d+1}} \right]^r.$$

Choosing $C \leq \frac{1}{2} R \left(\sum_{\ell \in \mathbb{Z}^d} \frac{1}{(1+|\ell|)^{d+1}} \right)^{-1}$ proves the result. □

Proof of Proposition A.1. First of all, for $\varepsilon_1 = \varepsilon_2 = 0$, (A-1) is a direct consequence of Lemma A.2, choosing $\nu \geq m+d+3$, $\gamma \leq 1$ and $F_\gamma = F'_\gamma$ (recall that $r \geq 1$).

When $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = 0$, we will prove that for some constants C and ν, we have

$$\left| \Omega(\boldsymbol{j}) \pm \omega_{\ell_1} \right| \geq \frac{C^r \gamma}{N(\boldsymbol{j})^\nu}, \tag{A-4}$$

[5] We use the formula of the gamma function valid for even integers, but the asymptotic is the same in the odd case.

which implies inequality (A-1) for $\gamma \le 1$. Notice that $|\Omega(\boldsymbol{j})| \le N(\boldsymbol{j})^2$ and thus, if $|\ell_1| \ge 2N(\boldsymbol{j})$, (A-4) is always true. When $|\ell_1| \le 2N(\boldsymbol{j})$, using that $N(\boldsymbol{j}, \ell) = N(\boldsymbol{j})(1+|\ell_1|)$, applying Lemma A.2 with $b = 0$ and $V \in F'_\gamma = F_\gamma$, we get

$$\left|\Omega(\boldsymbol{j}) + \varepsilon_1 \omega_{\ell_1}\right| = \left|\Omega(\boldsymbol{j}, (\ell_1, \varepsilon_1))\right| \ge \frac{C^{r+1}\gamma}{N(\boldsymbol{j})^{m+d+3}(3N(\boldsymbol{j}))^{m+d+3}} \ge \frac{\tilde{C}^r \gamma}{N(\boldsymbol{j})^{\nu}},$$

with $\nu = 2(m+d+3)$ and $\tilde{C} = 2C^2/3^{m+d+3}$.

When $\varepsilon_1 \varepsilon_2 = 1$, a similar argument yields an estimate of the form

$$\left|\Omega(\boldsymbol{j}) \pm (\omega_{\ell_1} + \omega_{\ell_2})\right| \ge \frac{C^r \gamma}{N(\boldsymbol{j})^{\nu}},$$

for some constants C, ν, and for $V \in F'_\gamma = F_\gamma$.

So it remains to establish an estimate of the form

$$\left|\Omega(\boldsymbol{j}) + \omega_{\ell_1} - \omega_{\ell_2}\right| \ge \frac{\tilde{C}^r \gamma^7}{N(\boldsymbol{j})^{\nu}}, \tag{A-5}$$

for some constant $\tilde{C}$ and $V \in F_\gamma$ to be defined. Assuming $|\ell_1| \le |\ell_2|$, we have

$$\left|\omega_{\ell_1} - \omega_{\ell_2} - \ell_1^2 + \ell_2^2\right| \le \left|\frac{R|v_{\ell_1}|}{(1+|\ell_1|)^m} - \frac{R|v_{\ell_2}|}{(1+|\ell_2|)^m}\right| \le \frac{R}{(1+|\ell_1|)^m},$$

for all v_{ℓ_1} and v_{ℓ_2} in $\left[-\frac{1}{2}, \frac{1}{2}\right]$; see (1-2). Therefore, if $(1+|\ell_1|)^m \ge (2R/C^r\gamma)N(\boldsymbol{j})^{m+d+3}$, we obtain (A-5) directly from Lemma A.2 applied with $b = \ell_1^2 - \ell_2^2$ and choosing $\nu = m+d+3$, $\tilde{C} = C/2$ and $F_\gamma = F'_\gamma$.

Finally, assume $(1+|\ell_1|)^m \le (2R/C^r\gamma)N(\boldsymbol{j})^{m+d+3}$. Then taking into account $|\Omega(\boldsymbol{j})| \le N(\boldsymbol{j})^2$, inequality (A-5) is satisfied when $\ell_2^2 - \ell_1^2 \ge 2N(\boldsymbol{j})^2$. It remains to consider the case when

$$1+|\ell_1| \le 1+|\ell_2| \le \left[2\left(\frac{2R}{C^r\gamma}N(\boldsymbol{j})^{m+d+3}\right)^{2/m} + 4N(\boldsymbol{j})^2\right]^{1/2} \le 2\left(\frac{3R}{C^r\gamma}\right)^{1/m} N(\boldsymbol{j})^{\frac{m+d+3}{m}}.$$

Again we use Lemma A.2 to conclude that

$$\begin{aligned}\left|\Omega(\boldsymbol{j}) + \omega_{\ell_1} - \omega_{\ell_2}\right| &\ge \frac{C^{r+2}\gamma}{\left[N(\boldsymbol{j})(1+|\ell_1|)(1+|\ell_2|)\right]^{m+d+3}} \\ &\ge \frac{C^{r+2}\gamma\left(\frac{C^r\gamma}{3.2^m R}\right)^{\frac{m+d+3}{m}}}{N(\boldsymbol{j})^{m+d+3}N(\boldsymbol{j})^{2\frac{(m+d+3)^2}{m}}} \ge \frac{\tilde{C}^r\gamma^{4+3/m}}{N(\boldsymbol{j})^{\nu}},\end{aligned}$$

as $m > \dfrac{d}{2}$, and with $\nu = m+d+3+\dfrac{(m+d+3)^2}{m}$ and $\tilde{C} = \dfrac{C^{\frac{4m+d+3}{m}}}{3.2^m R}$. This last estimate implies (A-1). □

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