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SEMICLASSICAL MEASURES FOR INHOMOGENEOUS SCHRÖDINGER EQUATIONS ON TORI

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The purpose of this note is to investigate the high-frequency behavior of solutions to linear Schrödinger equations. More precisely, Bourgain (1997) and Anantharaman and Macià (2011) proved that any weak-$*$ limit of the square density of solutions to the time-dependent homogeneous Schrödinger equation is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{T}^d$. The contribution of this article is that the same result automatically holds for nonhomogeneous Schrödinger equations, which allows for abstract potential type perturbations of the Laplace operator.

1. Introduction

In this note we are interested in understanding the high-frequency behavior of solutions of linear Schrödinger equations on tori, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Consider a sequence of initial data $(u_{0,n})$, bounded in $L^2(\mathbb{T}^d)$ and denote by (u_n) the sequence of solutions to the Schrödinger equation and by (ν_n) their concentration measures given by

$$u_n = e^{it\Delta} u_{0,n}, \quad \nu_n = |u_n|^2(t,x)\,dt\,dx.$$

The sequence ν_n on $\mathbb{R}_t \times \mathbb{T}^d$ is bounded (in mass) on any time interval $(0,T)$ by $T \sup_n \|u_{0,n}\|^2_{L^2(\mathbb{T}^d)}$. The following result was proved in [Bourgain 1997, Remark, page 108] and later, using a completely different approach that follows a more geometric path, in [Anantharaman and Macià 2011, Theorem 1]. (See also [Jakobson 1997; Macià 2011; Burq and Zworski 2004; 2005;Aïssiou et al. 2011] for related works.)

Theorem 1. *Any weak-$*$ limit of the sequence (ν_n) is absolutely continuous with respect to the Lebesgue measure $dt\,dx$ on $\mathbb{R}_t \times \mathbb{T}^d$.*

Remark 1.1. Actually, in [Anantharaman and Macià 2011] a more precise description of the possible limits is given and the result is proved in the case of Schrödinger operators $\Delta + V(t,x)$, if $V \in L^\infty(\mathbb{R}_t \times \mathbb{T}^2)$ is also continuous except possibly on a set of (spacetime) Lebesgue measure 0.

The purpose of this note is to show that the result in Theorem 1 extends to the case of solutions to the nonhomogeneous Schrödinger equation, and, consequently, to the case of Schrödinger operators $\Delta + V$ where $V \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(L^2(\mathbb{T}^d)))$ (we also give as an illustration an application to a simple nonlinear equation). Let us emphasize that our approach uses no particular property of the Laplace operator on tori

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other than selfadjointness (to get L^2 bounds for the time evolution) and the fact that Theorem 1 holds, which is used as a black box, and establishes an abstract link between the study of weak-$*$ limits of solutions of the homogeneous and inhomogeneous Schrödinger equations.

2. Inhomogeneous Schrödinger equations

Definition 2.1. Let $T > 0$. For any sequence (u_n) bounded in $L^2((0, T) \times \mathbb{T}^d)$, we say that the sequence (u_n) satisfies property (AC_T) if any weak-$*$ limit ν of (ν_n) is absolutely continuous with respect to the Lebesgue measure on $(0, T) \times \mathbb{T}^d$.

Theorem 2. *Let $(u_{n,0})$ and (f_n) be two sequences bounded in $L^2(\mathbb{T}^d)$ and $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{T}^d))$, respectively. Let u_n be the solution of*

$$(i\partial_t + \Delta)u_n = f_n, \quad u_n|_{t=0} = u_{n,0}, \quad u_n = e^{it\Delta}u_{n,0} + \frac{1}{i}\int_0^t e^{i(t-s)\Delta} f_n(s)\, ds.$$

Then, for any $T > 0$, the sequence (u_n), which is clearly bounded in $L^2((0, T) \times \mathbb{T}^2)$ by

$$T^{1/2} \sup_n (\|u_{n,0}\|_{L^2(\mathbb{T}^d)} + \|f_n\|_{L^1((0,T);L^2(\mathbb{T}^d))}),$$

satisfies property (AC_T).

Corollary 2.2. *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathscr{L}(L^2(\mathbb{T}^2)))$ (for example, V can be a potential in $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^\infty(\mathbb{T}^2))$ acting by pointwise multiplication). For any sequence $(u_{n,0})_{n\in\mathbb{N}}$ bounded in $L^2(\mathbb{T}^2)$, let (u_n) be the sequence of the unique solutions in $C^0(\mathbb{R}; L^2(\mathbb{T}^2))$ of*

$$(i\partial_t + \Delta + V(t))u_n = 0, \quad u_n|_{t=0} = u_{n,0}.$$

Then the sequence (u_n) satisfies the property (AC_T) for any $T > 0$.

Indeed, since

$$\frac{d}{dt}\|u_n\|^2_{L^2(\mathbb{T}^d)} = 2\Re(\partial_t u, u)_{L^2(\mathbb{T}^d)} = 2\Re(i\Delta u + iVu, u)_{L^2(\mathbb{T}^d)} = -2\Im(Vu, u)_{L^2(\mathbb{T}^d)},$$

by Gronwall's inequality, we obtain

$$\|u_n(t)\|^2_{L^2(\mathbb{T}^d)} \le \|u_{n,0}\|^2_{L^2(\mathbb{T}^d)} e^{\int_0^t \|V(s)\|_{\mathscr{L}(L^2(\mathbb{T}^d))}\, ds},$$

and, consequently, the sequence $(f_n) = (-V(t)u_n)$ is clearly bounded in $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{T}^d))$ and we can apply Theorem 2.

Remark 2.3. Any time independent $V \in \mathscr{L}(L^2(\mathbb{T}^d))$ satisfies the assumptions above, and, consequently, if (u_n) is a sequence of L^2 normalized eigenfunctions of $\Delta + V$, it follows from Corollary 2.2 that any weak-$*$ limit of $|u_n|^2(x)\, dx$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^d$. The proof we present below seems to be intrinsically time-dependent. However, it would be interesting to obtain a proof of this result avoiding the detour via the study of the time-dependent Schrödinger equation.

Proof of Theorem 2. If (u_n) satisfies property (AC_T), then the sequence (u_n+v_n) satisfies property (AC_T) if and only if the sequence (v_n) satisfies property (AC_T). This is because if $|u_n|^2\,dt\,dx$ and $|v_n|^2\,dt\,dx$ converge weakly to ν and μ, respectively, then, according to the Cauchy–Schwarz inequality, any weak-$*$ limit of $|u_n+v_n|^2\,dt\,dx$ is absolutely continuous with respect to $\nu+\mu$. The following result shows that the set of sequences satisfying property (AC_T) is closed in some weak-strong topology.

Lemma 2.4. *Consider (u_n) bounded in $L^2((0,T)\times\mathbb{T}^2)$. Assume that there exists for any $k\in\mathbb{N}$ a sequence $(u_n^{(k)})_{n\in\mathbb{N}}$ such that*

(1) *for any k, the sequence $(u_n^{(k)})_{n\in\mathbb{N}}$ satisfies property (AC_T);*

(2) *the sequences $(u_n^{(k)})_{n\in\mathbb{N}}$ are approximating the sequence (u_n) in the sense that*

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}\|u_n-u_n^{(k)}\|_{L^2((0,T)\times\mathbb{T}^2)}=0. \tag{2-1}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies property (AC_T).

Proof. Indeed, for any $\epsilon>0$, let k_0 be such that, for any $k\geq k_0$,

$$\limsup_n \|u_n-u_{n,k}\|_{L^2((0,T)\times\mathbb{T}^2)}<\epsilon.$$

Then, if ν and $\nu^{(k)}$ are weak-$*$ limits of the sequences $(u_n)_{n\in\mathbb{N}}$ and $(u_n^{(k)})_{n\in\mathbb{N}}$, respectively, associated to the same subsequence $n_p\to+\infty$, we have, for any $f\in C^0((0,T)\times\mathbb{T}^2)$ and large n,

$$\begin{aligned}\int_{(0,T)\times\mathbb{T}^2}|u_{n_p}|^2\chi\,dx\,dt &\leq \int_{(0,T)\times\mathbb{T}^2}2(|u_{n_p}-u_{n_p}^{(k)}|^2+|u_{n_p}^{(k)}|^2)\,dx\,dt\\ &\leq 2\epsilon^2+2\int_{(0,T)\times\mathbb{T}^2}2|u_{n_p}^{(k)}|^2)\chi\,dx\,dt.\end{aligned}\tag{2-2}$$

Passing to the limit $p\to+\infty$, we obtain

$$\langle\nu,\chi\rangle\leq 2\epsilon^2+2\langle\nu^{(k)},\chi\rangle.$$

On the other hand, according to the Riesz theorem (see, for example, [Rudin 1987, Theorem 2.14]), the measures ν, $\nu^{(k)}$ which are defined on the Borelian σ-algebra, $\mathcal{M}$, are *regular*, and, consequently,

$$\begin{aligned}\forall E\in\mathcal{M},\ \nu(E)&=\sup_{F\text{closed},\,F\subset E}\nu(U)=\inf_{U\text{open},\,E\subset U}\nu(U),\\ \forall E\in\mathcal{M},\ \nu^{(k)}(E)&=\sup_{F\text{closed},\,F\subset E}\nu^{(k)}(U)=\inf_{U\text{open},\,E\subset U}\nu^{(k)}(U).\end{aligned}\tag{2-3}$$

For any $E\in\mathcal{M}$, taking $F_p\subset E$ and $E\subset O_p$ such that

$$\lim_{p\to+\infty}\nu(F_p)=\nu(E),\qquad \lim_{p\to+\infty}\nu^{(k)}(O_p)=\nu^{(k)}(E)$$

and $\chi_p\in C_0((0,1)\times\mathbb{T}^d;[0,1])$ is equal to 1 on F_p and supported in O_p, we obtain, according to (2-2),

$$\nu(E)\leq 2\epsilon^2+2\nu^{(k)}(E).$$

Now consider E a subset of $(0, T) \times \mathbb{T}^d$-Lebesgue measure 0. Since by assumption $\nu^{(k)}$ is absolutely continuous with respect to the Lebesgue measure, we have $\nu^{(k)}(E) = 0$, and hence $\nu(E) \leq 2\epsilon^2$. Consequently, since $\epsilon > 0$ can be taken arbitrarily small, we have $\nu(E) = 0$, which proves that ν is also absolutely continuous with respect to the Lebesgue measure. □

We come back to the proof of Theorem 2 and fix $T > 0$. According to Duhamel's formula,

$$u_n = e^{it\Delta} u_{0,n} + \frac{1}{i} \int_0^t e^{i(t-s)\Delta} f_n(s)\, ds.$$

According to the remark above, since we know that the sequence $(e^{it\Delta} u_{0,n})$ satisfies property (AC_T), it is enough to prove that the sequence $(v_n) = (\int_0^t e^{i(t-s)} f_n(s)\, ds)$ satisfies property (AC_T). The key point of the analysis is that if instead of v_n we had

$$\tilde{v}_n = \int_0^T e^{i(t-s)\Delta} V u_n(s)\, ds = e^{it\Delta} g_n, \quad g_n = \int_0^T e^{-is\Delta} V e^{is(\Delta+V)} u_{n,0}(s)\, ds,$$

we could conclude using Theorem 1, because $\tilde{v}_n$ is a solution to the homogeneous Schrödinger equation with initial data the bounded sequence (g_n). To pass from $\tilde{v}_n$ to v_n, we adapt an idea borrowed from harmonic analysis (the Christ–Kiselev Lemma [2001]) in the simple form written in [Burq and Planchon 2006] (see also [Burq 2011]). Here the idea is to show that the sequence (v_n) can be approximated by other sequences $(v_n^{(k)})$ in the sense of (2-1) (actually, we get a stronger convergence, as we can replace the lim sup in (2-1) by a sup), where each $(v_n^{(k)})$ is a finite sum of solutions of the homogeneous Schrödinger equation, properly truncated in time, and hence satisfy property (AC_T). Let

$$\|f_n\|_{L^1((0,T);L^2(\mathbb{T}^2))} = c_n \leq C.$$

We decompose the interval $(0, T)$ into dyadic pieces on which the $L^1((0, T); L^2(\mathbb{T}^d))$-norm of f_n is equal to $2^{-q} c_n$. For this, we recursively construct (on the index $q \in \mathbb{N}$) certain sequences $(t_{p,q,n})_{\substack{q\in\mathbb{N}\\ p=1,\dots,2^q}}$ such that

- $0 = t_{0,q,n} < t_{1,q,n} < \cdots < t_{2^q,q,n} = T$,
- $\|f_n\|_{L^1((t_{p,q,n},t_{p+1,q,n});L^2(\mathbb{T}^2))} = 2^{-q} c_n$,
- $t_{2p,q,n} = t_{p,q-1,n}$ for any $p = 0, \dots, 2^{q-1}$.

Notice that if the function

$$G_n : t \in [0, T] \mapsto \|f_n\|_{L^1((0,t);L^2(\mathbb{T}^d))} \in [0, c_n]$$

is strictly increasing, the points $t_{p,q,n}$ are uniquely determined by the relation $G_n(t_{p,q,n}) = p2^{-q} c_n$, and the last condition above is automatic. In the general case, the function G_n (which is clearly nondecreasing) can have some flat parts, and, consequently, the points $t_{p,q,n}$ may not be unique anymore. The last condition above ensures that the choice made at step $q + 1$ is consistent with the choice made at step q. For $j = 0, \dots, 2^q - 1$, let

$$I_{j,q,n} = [t_{2j,q,n}, t_{2j+1,q,n}[, \quad J_{j,q,n} = [t_{2j+1,q,n}, t_{2j+2,q,n}[, \quad Q_{j,q,n} = J_{j,q,n} \times I_{j,q,n}.$$

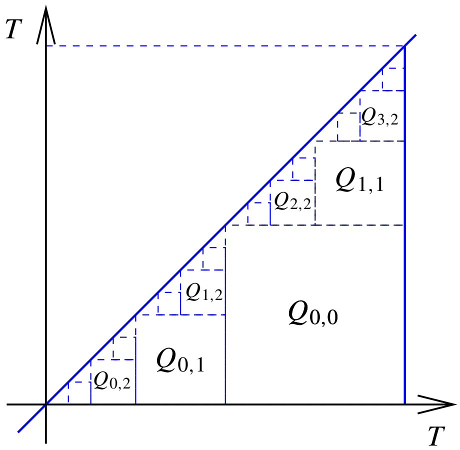

Figure 1. Decomposition of a triangle as a union of disjoint squares.

Notice that

$$\{((t,s)\in[0,T[^2;\, s\le t\}=\bigsqcup_{q=0}^{+\infty}\bigsqcup_{j=0}^{2^q-1}Q_{j,q,n}\Rightarrow 1_{s\le t}=\sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1}1_{Q_{j,q,n}}(t,s).$$

Now (if we are able to prove that the series in q converges) we have

$$\begin{aligned}v_n&=\int_0^t e^{i(t-s)\Delta}f_n(s)\,ds=\int_0^T 1_{s\le t}e^{i(t-s)\Delta}f_n(s)\,ds\\&=\sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1}1_{t\in J_{j,q,n}}\int_0^T e^{i(t-s)\Delta}1_{s\in I_{j,q,n}}f_n(s)\,ds=\sum_{q=0}^{+\infty}\sum_{j=0}^{2^q-1}1_{t\in J_{j,q,n}}e^{it\Delta}g_{j,q,n}\,ds,\end{aligned}\tag{2-4}$$

with

$$\begin{aligned}g_{j,q,n}(x)&=\int_0^T e^{-is\Delta}1_{s\in I_{j,q,n}}f_n(s)\,ds=\int_{t_{2j,q,n}}^{t_{2j+1,q,n}}e^{-is\Delta}f_n(s)\,ds,\\ \|g_{j,q,n}\|_{L^2(\mathbb{T}^d)}&\le\|f_n\|_{L^1((t_{2j,q,n},t_{2j+1,q,n}T);L^2(\mathbb{T}^d))}=2^{-q}c_n.\end{aligned}\tag{2-5}$$

Let

$$v_n^{(k)}=\sum_{q=0}^{k}\sum_{j=0}^{2^q-1}1_{t\in J_{j,q,n}}e^{it\Delta}g_{j,q,n}\,ds.$$

Noticing that if a sequence (w_n) satisfies property(AC_T), then, for any sequences $0\le t_{1,n}<t_{2,n}\le T$, the sequence $(1_{t\in(t_{1,n},t_{2,n})}w_n)$ satisfies property(AC_T), we see that for any $k\in\mathbb{N}$, the sequence $(v_n^{(k)})$ satisfies property (AC_T). On the other hand, since for $j\ne j'$, $1_{t\in J_{j,q,n}}$ and $1_{t\in J_{j',q,n}}$ have disjoint supports, we get, according to (2-5),

$$\begin{aligned}\Big\|\sum_{j=0}^{2^q-1}1_{t\in J_{j,q,n}}e^{it\Delta}g_{j,q,n}\Big\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}&\le\sup_{0\le j\le 2^q-1}\|1_{t\in J_{j,q,n}}e^{it\Delta}g_{j,q,n}\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}\\&\le\sup_{0\le j\le 2^q-1}\|g_{j,q,n}\|_{L^2(\mathbb{T}^d))}\le 2^{-q}c_n.\end{aligned}\tag{2-6}$$

As a consequence, we get that the series (2-4) is convergent and

$$\|v_n - v_n^{(k)}\|_{L^2((0,T)\times\mathbb{T}^d)} \le \sqrt{T} c_n 2^{-k} \le C 2^{-k},$$

which, according to Lemma 2.4, concludes the proof of Theorem 2. $\square$

3. An illustration

We consider here the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u + V(u,t)u = 0 \quad \text{on } \mathbb{T}^d, \quad u|_{t=0} = 0 \tag{3-1}$$

where the function $z \in \mathbb{C} \mapsto V(z,t)z \in \mathbb{C}$ is globally Lipschitz with respect to the z variable, with a time-integrable Lipschitz constant; that is, there exists $C \in L^1_{\text{loc}}(\mathbb{R})$ such that $C(t) > 0$ for all t and

$$|V(z,t)z - V(z',t)z'| \le C(t)|z - z'| \quad \text{for all } z, z' \in \mathbb{C}.$$

Notice, for example, that the choice $V(u,t) = |u|^2/(1+\epsilon|u|^2)$ satisfies these assumptions for any $\epsilon > 0$.

Proposition 3.1. *For any $u_0 \in L^2(\mathbb{T}^d)$, there exists a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^d))$ to* (3-1). *Furthermore, there exists a continuous increasing function, $F(t)$, such that, for any $u_0 \in L^2(\mathbb{T}^d)$, the solution u satisfies*

$$\|u\|_{L^2(\mathbb{T}^d)}(t) \le F(t)\|u_0\|_{L^2(\mathbb{T}^d)}. \tag{3-2}$$

Corollary 3.2. *For any sequence of initial data $(u_{0,n})$ bounded in $L^2(\mathbb{T}^d)$, the sequence (u_n) of solutions to* (3-1) *satisfies*

$$\|V(u_n,t)u_n\|_{L^2(\mathbb{T}^d)} \le C(t)\|u_n\|_{L^\infty((0,t);L^2(\mathbb{T}^d))} \le C(t)f(t)\|u_{0,n}\|_{L^2(\mathbb{T}^d)} \in L^1_{\text{loc}}(\mathbb{R}_t),$$

and, consequently, the sequence (u_n) satisfies property (AC_T) for any $T > 0$.

Proof of Proposition 3.1. Let

$$K : u \in L^\infty((0,T); L^2(\mathbb{T}^d)) \mapsto e^{it\Delta}u_0 + \frac{1}{i}\int_0^t e^{i(t-s)}(V(u(s),s)u(s))\,ds.$$

We have

$$\begin{aligned} \|K(u) - e^{it\Delta}u_0\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} &\le \int_0^T C(s)\,ds\,\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}, \\ \|K(u) - K(v)\|_{L^\infty((0,T);L^2(\mathbb{T}^d))} &\le \int_0^T C(s)\,ds\,\|u - v\|_{L^\infty((0,T);L^2(\mathbb{T}^d))}. \end{aligned} \tag{3-3}$$

We obtain that the map K has a unique fixed point on the ball centered on $e^{it\Delta}u_0$ with radius $\|u_0\|_{L^2(\mathbb{T}^d)}$ in $L^\infty((0,T); L^2(\mathbb{T}^d))$, as soon as $\int_0^T C(s)\,ds \le \frac{1}{2}$. This proves the local existence claim. To obtain existence on any time interval $[0, \widetilde{T}]$, we write $[0,\widetilde{T}] = \bigcup_{j=1}^N [t_j, t_{j+1}]$, where we choose t_j recursively such that $\int_{t_j}^{t_{j+1}} C(s)\,ds \le \frac{1}{2}$. Taking $\int_{t_j}^{t_{j+1}} C(s)\,ds = \frac{1}{2}$ for all $j < N-1$ gives the bound

$$N \le 1 + 2\int_0^{\widetilde{T}} C(s)\,ds. \tag{3-4}$$

Then applying the first step recursively gives a solution on $[0, \widetilde{T}]$ that, according to (3-4), satisfies

$$\|u\|_{L^2(\mathbb{T}^d)}(\widetilde{T}) \leq 2^N \|u_0\|_{L^2(\mathbb{T}^d)} \leq 2^{1+2\int_0^t C(s)\,ds} \|u_0\|_{L^2(\mathbb{T}^d)}.$$

The uniqueness claim in Proposition 3.1 follows now from standard methods. □

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