

# ANALYSIS & PDE

Volume 6

No. 7

2013

MARÍA DEL MAR GONZÁLEZ AND JIE QING

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## FRACTIONAL CONFORMAL LAPLACIANS AND FRACTIONAL YAMABE PROBLEMS

MARÍA DEL MAR GONZÁLEZ AND JIE QING

Based on the relations between scattering operators of asymptotically hyperbolic metrics and Dirichlet-to-Neumann operators of uniformly degenerate elliptic boundary value problems observed by Chang and González, we formulate fractional Yamabe problems that include the boundary Yamabe problem studied by Escobar. We observe an interesting Hopf-type maximum principle together with interplay between analysis of weighted trace Sobolev inequalities and conformal structure of the underlying manifolds, which extends the phenomena displayed in the classic Yamabe problem and boundary Yamabe problem.

### 1. Introduction

In this paper, based on the relations between scattering operators of asymptotically hyperbolic metrics and Dirichlet-to-Neumann operators of uniformly degenerate elliptic boundary value problems observed in [Chang and González 2011], we formulate and solve fractional order Yamabe problems that include the boundary Yamabe problem studied in [Escobar 1992].

Suppose that  $X^{n+1}$  is a smooth manifold with smooth boundary  $M^n$  for  $n \geq 3$ . A function  $\rho$  is a defining function of the boundary  $M^n$  in  $X^{n+1}$  if

$$\rho > 0 \text{ in } X^{n+1} \quad \rho = 0 \text{ on } M^n, \quad d\rho \neq 0 \text{ on } M^n.$$

We say that  $g^+$  is conformally compact if, for some defining function  $\rho$ , the metric  $\bar{g} = \rho^2 g^+$  extends to  $\bar{X}^{n+1}$  so that  $(\bar{X}^{n+1}, \bar{g})$  is a compact Riemannian manifold. This induces a conformal class of metrics  $\hat{h} = \bar{g}|_{TM^n}$  on  $M^n$  when defining functions vary. The conformal manifold  $(M^n, [\hat{h}])$  is called the conformal infinity of  $(X^{n+1}, g^+)$ . A metric  $g^+$  is said to be asymptotically hyperbolic if it is conformally compact and the sectional curvature approaches  $-1$  at infinity.

Graham and Zworski [2003] introduced the meromorphic family of scattering operators  $S(s)$ , which is a family of pseudodifferential operators, for a given asymptotically hyperbolic manifold  $(X^{n+1}, g^+)$  and a choice of the representative  $\hat{h}$  of the conformal infinity  $(M^n, [\hat{h}])$ . Instead one often considers the normalized scattering operators

$$P_\gamma[g^+, \hat{h}] = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right).$$

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González is supported by Spain Government project MTM2008-06349-C03-01 and GenCat 2009SGR345. Qing is partially supported by NSF 0700535 and CNSF 10728103.

MSC2010: 35J70, 53A30, 35R11.

Keywords: fractional Laplacian, conformal geometry, Yamabe problem.

The normalized scattering operators  $P_\gamma[g^+, \hat{h}]$  are conformally covariant,

$$P_\gamma[g^+, w^{4/(n-2\gamma)}\hat{h}]\phi = w^{-(n+2\gamma)/(n-2\gamma)} P_\gamma[g^+, \hat{h}](w\phi),$$

with principal symbol

$$\sigma(P_\gamma[g^+, \hat{h}]) = \sigma((-\Delta_{\hat{h}})^\gamma).$$

Hence they may be considered to be conformal fractional Laplacians for  $\gamma \in (0, 1)$  for a given asymptotically hyperbolic metric  $g^+$ . As proven in [Graham and Zworski 2003; Fefferman and Graham 2012], when  $g^+$  is Poincaré–Einstein,  $P_1$  is the conformal Laplacian,  $P_2$  is the Paneitz operator, and, in general,  $P_k$  for  $k \in \mathbb{N}$  are the conformal powers of the Laplacian discovered in [Graham et al. 1992].

When  $g^+$  is a fixed asymptotically hyperbolic metric, we may simply denote

$$P_\gamma^{\hat{h}} := P_\gamma[g^+, \hat{h}].$$

We will consider the associated “fractional order curvature”

$$Q_\gamma^{\hat{h}} = P_\gamma^{\hat{h}}(1)$$

and the normalized total curvature

$$I_\gamma[\hat{h}] = \frac{\int_{M^n} Q_\gamma^{\hat{h}} dv_{\hat{h}}}{\left(\int_{M^n} dv_{\hat{h}}\right)^{(n-2\gamma)/n}}.$$

When a background metric  $\hat{h}$  is fixed, we may write

$$I_\gamma[w, \hat{h}] = I_\gamma[w^4/(n-2\gamma)\hat{h}] = \frac{\int_{M^n} w P_\gamma^{\hat{h}} w dv_{\hat{h}}}{\left(\int_{M^n} w^{2n/(n-2\gamma)} dv_{\hat{h}}\right)^{(n-2\gamma)/n}}.$$

This functional  $I_\gamma[\hat{h}]$  is clearly an analogue to the Yamabe functional. Hence one may ask if there is a metric which is the minimizer of  $I_\gamma$  among metrics in the class  $[\hat{h}]$  and whose curvature  $Q_\gamma$  is a constant. We will refer to that problem as a fractional Yamabe problem when  $\gamma \in (0, 1)$ . For the original Yamabe problem readers are referred to [Lee and Parker 1987; Schoen and Yau 1994]. A similar question was studied in [Qing and Raske 2006] for  $\gamma > 1$  and  $g^+$  being a Poincaré–Einstein metric. Because of the lack of a maximum principle, these generalized Yamabe problems are, in general, difficult to solve. Yet this new window to the analytic aspects of conformal geometry remains fascinating. For example, it was proven [Guillarmou and Qing 2010] that the location of the first scattering pole is dictated by the sign of the Yamabe constant and the Green’s function of  $P_\gamma^{\hat{h}}$  is positive for  $\gamma \in (0, 1)$  when the Yamabe constant is positive, at least in the case where  $g^+$  is conformally compact Einstein.

On the other hand, see [González 2012] for an interpretation of the fractional curvature  $Q_\gamma$  in relation to the first variation of some weighted volume. The singular version of the fractional Yamabe problem has been considered in [González et al. 2012], but there are still many open questions in this field.

It turns out that one may use the relations of scattering operators and the Dirichlet-to-Neumann operators to reformulate the above fractional Yamabe problems as degenerate elliptic boundary value problems. The correspondence between pseudodifferential equations and degenerate elliptic boundary value problems is inspired by [Caffarelli and Silvestre 2007]. Interestingly, the corresponding degenerate

elliptic boundary value problem is a natural extension of the boundary Yamabe problem raised and studied in [Escobar 1992].

Recall from [Chang and González 2011] that, given an asymptotically hyperbolic manifold  $(X^{n+1}, g^+)$  and a representative  $\hat{h}$  of the conformal infinity  $(M^n, [\hat{h}])$ , one can find a geodesic defining function  $\rho$  such that the compactified metric can be written as

$$\bar{g} := \rho^2 g^+ = d\rho^2 + h_\rho = d\rho^2 + \hat{h} + h^{(1)}\rho + h^{(2)}\rho^2 + o(\rho^2)$$

near infinity. One may consider the degenerate elliptic boundary value problem of  $\bar{g}$  as follows:

$$\begin{cases} -\operatorname{div}(\rho^a \nabla U) + E(\rho)U = 0 & \text{in } (X^{n+1}, \bar{g}), \\ U|_{\rho=0} = f & \text{on } M^n, \end{cases} \tag{1-1}$$

where

$$E(\rho) = \rho^{-1-s}(-\Delta_{g^+} - s(n-s))\rho^{n-s},$$

$s = n/2 + \gamma$ , and  $a = 1 - 2\gamma$ .

**Lemma 1.1** [Chang and González 2011]. *Let  $(X^{n+1}, g^+)$  be an asymptotically hyperbolic manifold. Suppose that  $U$  is the solution to the boundary value problem (1-1). Then*

(1) *for  $\gamma \in (0, 1/2)$  and  $-n^2/4 + \gamma^2$  not an  $L^2$ -eigenvalue for the Laplacian of  $g^+$ ,*

$$P_\gamma[g^+, \hat{h}]f = -d_\gamma^* \lim_{\rho \rightarrow 0} \rho^a \partial_\rho U, \tag{1-2}$$

where

$$d_\gamma^* = -\frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}; \tag{1-3}$$

(2) *for  $\gamma = 1/2$ ,*

$$P_{1/2}[g^+, \hat{h}]f = -\lim_{\rho \rightarrow 0} \partial_\rho U + \frac{n-1}{2} Hf,$$

where  $H := (1/(2n)) \operatorname{Tr}_{\hat{h}}(h^{(1)})$  is the mean curvature of  $M$ ;

(3) *for  $\gamma \in (1/2, 1)$ , (1-2) still holds if  $H = 0$ .*

In light of Lemma 1.1, consider, for  $\gamma \in (0, 1)$ ,

$$I_\gamma^*[U, \bar{g}] = \frac{d_\gamma^* \int_{X^{n+1}} (\rho^a |\nabla U|^2 + E(\rho)U^2) dv_{\bar{g}}}{\int_{M^n} U^{2n/(n-2\gamma)} dv_{\hat{h}}}.$$

It is then a very natural variational problem for  $I_\gamma^*$ . For instance, right away one sees that a minimizer of  $I_\gamma^*$  is automatically nonnegative, which was a huge issue for the functional  $I_\gamma$ .

One key ingredient in our work here is the following Hopf-type maximum principle. We drew inspiration from some version of Hopf’s lemma for the Euclidean half space case [Cabré and Sire 2010, Proposition 4.11].

**Proposition 1.2.** *Let  $\gamma \in (0, 1)$ . Suppose  $U$  is a nonnegative solution to (1-1) in  $X^{n+1}$ . Let  $p_0 \in M^n = \partial X^{n+1}$  and  $B_r$  be a geodesic ball of radius  $r$  centered at  $p_0$  in  $M^n$ . Then, for sufficiently small  $r_0$ , if*

$U(q_0) = 0$  for  $q_0 \in B_{r_0} \setminus \overline{B_{1/2r_0}}$  and  $U > 0$  on  $\partial B_{1/2r_0}$ ,

$$y^a \partial_y U|_{q_0} > 0. \tag{1-4}$$

It seems weaker than the original one, but it suffices for our purposes. A nice and immediate consequence of the above maximum principle is that the first eigenfunction of the fractional conformal Laplacian  $P_\gamma^{\hat{h}}$  is always positive, which has been a rather challenging question in general for the pseudodifferential operators  $P_\gamma^{\hat{h}}$ ; see [Guillarmou and Qing 2010]. Hence one can produce a metric in the class  $[\hat{h}]$  that has positive, negative, or zero  $Q_\gamma$  curvature when the first eigenvalue is positive, negative, or zero.

Our approach to solving the  $\gamma$ -Yamabe problem is very similar to that taken in [Escobar 1992], where one of the crucial steps is the understanding of a trace inequality. In our case, the relevant sharp weighted trace Sobolev inequality appears in [Lieb 1983; Cotioliis and Tavoularis 2004; Nekvinda 1993].

**Proposition 1.3.** *Let  $\gamma \in (0, 1)$  and  $a = 1 - 2\gamma$ . Suppose that  $U \in W^{1,2}(\mathbb{R}_+^{n+1}, y^a)$  with trace  $TU = w$ . Then, for some constant  $\bar{S}(n, \gamma)$ ,*

$$\|w\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq \bar{S}(n, \gamma) \int_{\mathbb{R}_+^{n+1}} y^a |\nabla U|^2 dx dy, \tag{1-5}$$

where  $2^* = 2n/(n - 2\gamma)$ . Moreover the equality holds if and only if

$$w(x) = c \left( \frac{\mu}{|x - x_0|^2 + \mu^2} \right)^{(n-2\gamma)/2}, \quad x \in \mathbb{R}^n,$$

for  $c \in \mathbb{R}$ ,  $\mu > 0$  and  $x_0 \in \mathbb{R}^n$  fixed, and  $U$  is its Poisson extension of  $w$  as given in (2-13).

As in the case of the original Yamabe problem, one can define the  $\gamma$ -Yamabe constant

$$\Lambda_\gamma(M^n, [\hat{h}]) = \inf_{h \in [\hat{h}]} I_\gamma[h].$$

It is then easily seen that

$$\Lambda_\gamma(S^n, [g_c]) = \frac{d_\gamma^*}{\bar{S}(n, \gamma)}$$

where  $[g_c]$  is the canonical conformal class of metrics on the sphere  $S^n$ . Analogous to the cases of the original Yamabe problem, we obtain the following.

**Theorem 1.4.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold. Suppose, in addition, that  $H = 0$  when  $\gamma \in (1/2, 1)$ . Then, if*

$$-\infty < \Lambda_\gamma(M, [\hat{h}]) < \Lambda_\gamma(S^n, [g_c]), \tag{1-6}$$

the  $\gamma$ -Yamabe problem is solvable for  $\gamma \in (0, 1)$ .

**Remark.** It is easily seen that  $\Lambda_\gamma(M, [\hat{h}]) > -\infty$  in light of (1.4) in Theorems 1.1 and 1.2 of [Jin and Xiong 2013] when  $\gamma \in (0, 1/2]$  or if some additional assumptions in Theorem 1.2 of [Jin and Xiong 2013] hold.

Based on computations similar to ones in [Escobar 1992], we have the following.

**Theorem 1.5.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold and that*

$$\rho^{-2}(R[g^+] - \text{Ric}[g^+](\rho\partial_\rho) + n^2) \rightarrow 0 \quad \text{as } \rho \rightarrow 0. \tag{1-7}$$

*If  $X^{n+1}$  has a nonumbilic point on  $\partial X^{n+1}$  and*

$$\frac{n+a-3}{1-a} 2^{2\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} + \frac{n-1+a}{a+1} < 0, \tag{1-8}$$

*then*

$$\Lambda_\gamma(M, [\hat{h}]) < \Lambda_\gamma(S^n, [g_c])$$

*and hence the  $\gamma$ -Yamabe problem is solvable for  $\gamma \in (0, 1)$ .*

We remark now that the 1/2-Yamabe problem introduced here reduces back to the boundary Yamabe problem considered in [Escobar 1992] in this way. Notice that, in this case, we have

$$I_{1/2}^*[U, \phi^{4/n-1}\bar{g}] = I_{1/2}^*[U\phi, \bar{g}] \tag{1-9}$$

for any positive function  $\phi$  on  $\bar{X}^{n+1}$ , and therefore (1-7) is no longer needed. Also notice that the condition (1-8) becomes  $n > 5$  when  $\gamma = 1/2$ , which agrees with the conclusion in [Escobar 1992].

Suppose we start with a compact Riemannian manifold  $(X^{n+1}, \bar{g})$  and its boundary  $(M^n, \hat{h})$ . Then one can construct an asymptotically hyperbolic manifold  $(X^{n+1}, g^+)$  which is conformal to  $(X^{n+1}, \bar{g})$ . For example, as observed in [Chang and González 2011], one may, according to [Mazzeo 1991; Andersson et al. 1992], require that

$$R[g^+] = -n(n+1). \tag{1-10}$$

Then the induced degenerate equation becomes

$$-\text{div}(\rho^a \nabla U) + \frac{n-1+a}{4n} R[\bar{g}] \rho^a U = 0 \quad \text{in } (X^{n+1}, \bar{g}), \tag{1-11}$$

whose associated variational functional becomes

$$F[U] = \int_X \rho^a |\nabla U|_{\bar{g}}^2 dv_{\bar{g}} + \frac{n-1+a}{4n} \int_X R[\bar{g}] \rho^a |U|^2 dv_{\bar{g}}. \tag{1-12}$$

In Section 2 we recall [Chang and González 2011] to make possible the passage from pseudodifferential equations to second order elliptic boundary value problems as in [Caffarelli and Silvestre 2007]. In Section 3 we study regularity ( $L^\infty$  and Schauder estimates) for degenerate elliptic boundary value problems, and, more importantly, we establish the Hopf-type maximum principle. In Section 4 we formulate the fractional Yamabe problem and obtain some properties for the fractional case that are analogous to the original Yamabe problem with the help of the Hopf-type maximum principle. In Section 5 we analyze sharp weighted Sobolev trace inequalities. We define, on any conformal manifold, the fractional Yamabe constant associated with an asymptotically hyperbolic metric, and show that one of the standard round spheres associated to the standard hyperbolic metric is the largest. In Section 6 we take a subcritical approximation and prove Theorem 1.4. In Section 7 we adopt the calculation from [Escobar 1992] and prove Theorem 1.5 by choosing a suitable test function.

We finally mention the two related works [Barrios et al. 2012; Servadei 2013] on nonlinearities with critical exponents for the fractional Laplacian.

## 2. Conformal fractional Laplacians

In this section we introduce [Chang and González 2011] to relate two equivalent definitions of conformal fractional Laplacians. Conformal fractional Laplacians are defined via scattering theory on asymptotically hyperbolic manifolds [Graham and Zworski 2003; Fefferman and Graham 2012]. We have also seen fractional Laplacians defined as Dirichlet-to-Neumann operators for degenerate equations on compact manifolds with boundary [Caffarelli and Silvestre 2007]. It turns out that, in some way, these two fractional Laplacians are the same.

Let  $X^{n+1}$  be a smooth manifold of dimension  $n + 1$  with compact boundary  $\partial X = M^n$ . A function  $\rho$  is a *defining function* of  $\partial X$  in  $X$  if

$$\rho > 0 \text{ in } X, \quad \rho = 0 \text{ on } \partial X, \quad d\rho \neq 0 \text{ on } \partial X.$$

We say that  $g^+$  is *conformally compact* if the metric  $\bar{g} = \rho^2 g^+$  extends to  $\bar{X}^{n+1}$  for a defining function  $\rho$  so that  $(\bar{X}^{n+1}, \bar{g})$  is a compact Riemannian manifold. This induces a conformal class of metrics  $\hat{h} = \bar{g}|_{TM^n}$  on  $M^n$  when the defining function varies, which is called the *conformal infinity* of  $(X^{n+1}, g^+)$ . A metric  $g^+$  is said to be *asymptotically hyperbolic* if it is conformally compact and the sectional curvature approaches  $-1$  at infinity.

Given an asymptotically hyperbolic manifold  $(X^{n+1}, g^+)$  and a representative  $\hat{h}$  of the conformal infinity  $(M^n, [\hat{h}])$ , there exists a uniquely geodesic defining function  $\rho$  such that, on a neighborhood  $M \times (0, \delta)$  in  $X$ ,  $g^+$  has the normal form

$$g^+ = \rho^{-2}(d\rho^2 + h_\rho) \tag{2-1}$$

where  $h_\rho$  is a one parameter family of metrics on  $M$  such that

$$h_\rho = \hat{h} + h^{(1)}\rho + O(\rho^2). \tag{2-2}$$

From [Mazzeo and Melrose 1987; Graham and Zworski 2003] it follows that, given  $f \in \mathcal{C}^\infty(M)$ ,  $\operatorname{Re}(s) > n/2$  and  $s(n-s)$  is not an  $L^2$ -eigenvalue for  $-\Delta_{g^+}$ , the generalized eigenvalue problem

$$-\Delta_{g^+}u - s(n-s)u = 0 \quad \text{in } X \tag{2-3}$$

has a solution of the form

$$u = F\rho^{n-s} + G\rho^s, \quad F, G \in \mathcal{C}^\infty(\bar{X}), \quad F|_{\rho=0} = f. \tag{2-4}$$

The *scattering operator* on  $M$  is then defined as

$$S(s)f = G|_M.$$

It is shown in [Graham and Zworski 2003] that, by a meromorphic continuation,  $S(s)$  is a meromorphic



family of pseudodifferential operators in the whole complex plane. Instead, it is often useful to consider the normalized scattering operators  $P_\gamma[g^+, \hat{h}]$  defined as

$$P_\gamma[g^+, \hat{h}] := d_\gamma S\left(\frac{n}{2} + \gamma\right), \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2-5}$$

Note that  $s = n/2 + \gamma$ . With this regularization the principal symbol of  $P_\gamma[g^*, \hat{h}]$  is exactly the principal symbol of the fractional Laplacian  $(-\Delta_{\hat{h}})^\gamma$ . Hence we will call (assuming implicitly the dependence on the extension metric  $g^+$ )

$$P_\gamma^{\hat{h}} := P_\gamma[g^+, \hat{h}]$$

a conformal fractional Laplacian for each  $\gamma \in (0, 1)$  which is not a pole of the scattering operator, that is,  $n^2/4 - \gamma^2$  is not an  $L^2$ -eigenvalue for  $-\Delta_{g^+}$ . It is a conformally covariant operator, in the sense that it behaves like

$$P_\gamma^{\hat{h}_w} \varphi = w^{-(n+2\gamma)/(n-2\gamma)} P_\gamma^{\hat{h}}(w\varphi) \tag{2-6}$$

for a conformal change of metric  $\hat{h}_w = w^{4/(n-2\gamma)} \hat{h}$ . We will call

$$Q_\gamma^{\hat{h}} = P_\gamma^{\hat{h}}(1)$$

the fractional scalar curvature associated to the conformal fractional Laplacian  $P_\gamma^{\hat{h}}$ . From (2-6) we have

$$P_\gamma^{\hat{h}}(w) = Q_\gamma^{\hat{h}_w} w^{(n+2\gamma)/(n-2\gamma)}. \tag{2-7}$$

The familiar case is  $\gamma = 1$ , where

$$P_1^{\hat{h}} = -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)} R[\hat{h}]$$

becomes the conformal Laplacian and the associated curvature is the scalar curvature

$$Q_1^{\hat{h}} = (n-2)/(4(n-1))R[\hat{h}]$$

of the metric  $\hat{h}$  which undergoes the change

$$P_1^{\hat{h}_w} w = \frac{n-2}{4(n-1)} R[\hat{h}_w] w^{(n+2)/(n-2)}$$

when taking conformal change of metrics, provided that  $(X^{n+1}, g^+)$  is a Poincaré–Einstein as established in [Graham and Zworski 2003; Fefferman and Graham 2012]. The conformal fractional Laplacians and fractional scalar curvatures should also be compared to the higher order generalization of the conformal Laplacian and scalar curvature, the Paneitz operator  $P_2^{\hat{h}}$  and its associated  $Q$ -curvature; see [Paneitz 2008; Branson 1995; Qing and Raske 2006].

It was observed in [Chang and González 2011] that the generalized eigenvalue problem (2-3) on a noncompact manifold  $(X^{n+1}, g^+)$  is equivalent to a linear degenerate elliptic problem on the compact manifold  $(\bar{X}^{n+1}, \bar{g})$ , for  $\bar{g} = \rho^2 g^+$ . Hence Chang and González reconciled the definition of the fractional Laplacians given above as normalized scattering operators and the one given in the spirit of the Dirichlet-

to-Neumann operators by Caffarelli and Silvestre [2007]. This observation plays a fundamental role in this paper and provides an alternative way to study the fractional partial differential equation (2-7). First, we know by the conformal covariance that

$$P_1^{g^+} u = \rho^{(n+3)/2} P_1^{\bar{g}}(\rho^{-(n-1)/2} u).$$

Let  $a = 1 - 2\gamma \in (-1, 1)$ ,  $s = n/2 + \gamma$ , and  $U = \rho^{s-n} u$ . Then we may write (2-3) as

$$-\operatorname{div}(\rho^a \nabla_{\bar{g}} U) + E(\rho)U = 0 \quad \text{in } (X^{n+1}, \bar{g}),$$

where

$$E(\rho) := \rho^{a/2} P_1^{\bar{g}} \rho^{a/2} - (s(n-s) + \frac{n-1}{4n} R[g^+]) \rho^{a-2}, \tag{2-8}$$

or, writing everything back in the metric  $g^+$ ,

$$E(\rho) = \rho^{-1-s} (-\Delta_{g^+} - s(n-s)) \rho^{n-s}. \tag{2-9}$$

Notice that, in a neighborhood  $M \times (0, \delta)$  where the metric  $g^+$  is in the normal form

$$E(\rho) = \frac{n-1+a}{4n} (R[\bar{g}] - (n(n+1) + R[g^+]) \rho^{-2}) \rho^a \quad \text{in } M \times (0, \delta). \tag{2-10}$$

**Proposition 2.1** [Chang and González 2011]. *Let  $(X^{n+1}, g^+)$  be an asymptotically hyperbolic manifold. Then, given  $f \in \mathcal{C}^\infty(M)$ , the generalized eigenvalue problem (2-3) and (2-4) is equivalent to*

$$\begin{cases} -\operatorname{div}(\rho^a \nabla U) + E(\rho)U = 0 & \text{in } (X, \bar{g}), \\ U|_{\rho=0} = f & \text{on } M, \end{cases} \tag{2-11}$$

where  $U = \rho^{n-s} u$  and  $U$  is the unique minimizer of the energy

$$F[V] = \int_X \rho^a |\nabla V|_{\bar{g}}^2 dv_{\bar{g}} + \int_X E(\rho) |V|^2 dv_{\bar{g}}$$

among all the functions  $V \in W^{1,2}(X, \rho^a)$  with fixed trace  $V|_{\rho=0} = f$ . Moreover,

(1) for  $\gamma \in (0, 1/2)$ ,

$$P_\gamma^{\hat{h}} f = -d_\gamma^* \lim_{\rho \rightarrow 0} \rho^a \partial_\rho U, \tag{2-12}$$

where the constant  $d_\gamma^*$  is given in (1-3);

(2) for  $\gamma = 1/2$ , we have an extra term

$$P_{1/2}^{\hat{h}} f = -\lim_{\rho \rightarrow 0} \partial_\rho U + \frac{n-1}{2} Hf,$$

where  $H := (1/(2n)) \operatorname{Tr}_{\hat{h}}(h^{(1)})$  is the mean curvature of  $M$ ;

(3) for  $\gamma \in (1/2, 1)$ , (2-12) still holds if and only if  $H = 0$ .

**Remark.** It should be noted here that there are many asymptotically hyperbolic manifolds  $(X^{n+1}, g^+)$  whose conformal infinity is prescribed as  $(M^n, [\hat{h}])$ . If one insists on  $(X^{n+1}, g^+)$  being Poincaré–Einstein, then the normalized scattering operators  $P_\gamma^{\hat{h}}$  are a bit more intrinsic, at least at positive integers as

observed in [Graham and Zworski 2003; Fefferman and Graham 2012]. It should also be noted that one can simply start with a compact Riemannian manifold  $(\bar{X}^{n+1}, \bar{g})$  with boundary  $(M^n, \hat{h})$  and easily build an asymptotically hyperbolic manifold whose conformal infinity is given by  $(M^n, [\hat{h}])$ . Please see the details of this observation in [Chang and González 2011].

The simplest example of a conformally compact Einstein manifold is the hyperbolic space  $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$ . It can be characterized as the upper half-space (with coordinates  $x \in \mathbb{R}^n, y \in \mathbb{R}_+$ ), endowed with the metric

$$g^+ = \frac{dy^2 + |dx|^2}{y^2}.$$

Then (2-11) with Dirichlet condition  $w$  reduces to

$$\begin{cases} -\operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ U|_{y=0} = w & \text{on } \mathbb{R}^n, \end{cases}$$

and the fractional Laplacian at the boundary  $\mathbb{R}^n$  is just

$$P_\gamma^{|dx|^2} w = (-\Delta_{|dx|^2})^\gamma w = -d_\gamma^* \lim_{y \rightarrow 0} (y^a \partial_y U).$$

This is precisely the Caffarelli–Silvestre extension [2007]. Note that this extension  $U$  can be written in terms of the Poisson kernel  $K_\gamma$  as follows:

$$U(x, y) = K_\gamma *_{x,y} w = C_{n,\gamma} \int_{\mathbb{R}^n} \frac{y^{1-a}}{(|x - \xi|^2 + |y|^2)^{(n+1-a)/2}} w(\xi) d\xi, \tag{2-13}$$

for some constant  $C_{n,\gamma}$ . Moreover, given  $w \in H^\gamma(\mathbb{R}^n)$ ,  $U$  is the minimizer of the functional

$$F[V] = \int_{\mathbb{R}_+^{n+1}} y^a |\nabla V|^2 dx dy$$

among all the possible extensions in the set

$$\left\{ V : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R} : \int_{\mathbb{R}_+^{n+1}} y^a |\nabla V|^2 dx dy < \infty, V(\cdot, 0) = w \right\}.$$

Based on (2-9), it is observed in [Chang and González 2011] that one may use

$$\rho^* = v^{1/(n-s)}$$

as a defining function, where  $v$  solves

$$-\Delta_{g^+} v - s(n-s)v = 0$$

and  $\rho^{s-n} v = 1$  on  $M$ , to eliminate  $E(\rho^*)$  from (2-11). It suffices to show that  $v$  is strictly positive in the interior. But this is true because, away from the boundary, it is the solution of a uniformly elliptic equation in divergence form. Thus it cannot have a nonpositive minimum. Hence we arrive at an improvement of Proposition 2.1 as follows.

**Proposition 2.2.** *The function  $\rho^*$  is a defining function of  $M$  in  $X$  such that  $E(\rho^*) \equiv 0$ . Hence  $U = (\rho^*)^{s-n}u$  solves*

$$\begin{cases} -\operatorname{div}((\rho^*)^a \nabla U) = 0 & \text{in } (X, \bar{g}^*), \\ U = w & \text{on } M, \end{cases} \tag{2-14}$$

with respect to the metric  $\bar{g}^* = (\rho^*)^2 g^+$  and  $U$  is the unique minimizer of the energy

$$F[V] = \int_X (\rho^*)^a |\nabla V|_{\bar{g}^*}^2 dv_{\bar{g}^*} \tag{2-15}$$

among all the extensions  $V \in W^{1,2}(X, (\rho^*)^a)$  satisfying  $V|_M = w$ . Moreover,

$$\rho^*(\rho) = \rho \left[ 1 + \frac{Q_\gamma^{\hat{h}}}{(n-s)(-d_\gamma^*/(2\gamma))} \rho^{2\gamma} + O(\rho^2) \right]$$

near infinity and

$$P_\gamma^{\hat{h}} w = -d_\gamma^* \lim_{\rho^* \rightarrow 0} (\rho^*)^a \partial_{\rho^*} U + w Q_\gamma^{\hat{h}}, \tag{2-16}$$

provided that  $H = 0$  when  $\gamma \in (1/2, 1)$ .

We will sometimes use the defining function  $\rho^*$ , denoted by  $y$  unless explicitly stated otherwise, because it allows us to work with a pure divergence equation with no lower order terms.

We end this section by discussing the assumption that  $H = 0$  for an asymptotically hyperbolic metric  $g^+$ . It turns out that this indeed is an intrinsic condition.

**Lemma 2.3.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold and that  $\rho$  and  $\tilde{\rho}$  are the geodesic defining functions of  $M$  in  $X$  associated with representatives  $\hat{h}$  and  $\tilde{h}$  of the conformal infinity  $(M^n, [\hat{h}])$ , respectively. Hence*

$$g^+ = \rho^{-2}(d\rho^2 + h_\rho) = \tilde{\rho}^{-2}(d\tilde{\rho}^2 + \tilde{h}_{\tilde{\rho}})$$

where

$$h_\rho = \hat{h} + \rho h^{(1)} + O(\rho^2) \quad \text{and} \quad \tilde{h}_{\tilde{\rho}} = \tilde{h} + \tilde{\rho} \tilde{h}^{(1)} + O(\tilde{\rho}^2)$$

near infinity. Then  $\tilde{h}^{(1)} = h^{(1)}$  on  $M$ . In particular

$$H = \frac{\tilde{\rho}}{\rho} \Big|_{\rho=0} \tilde{H} \quad \text{on } M.$$

*Proof.* This simply follows from the equations that define the geodesic defining functions. Let

$$\tilde{\rho} = e^w \rho$$

near infinity. Then

$$1 = |d(e^w \rho)|_{e^{2w} \rho^2 g^+}^2 = |d\rho|_{\rho^2 g^+}^2 + 2\rho \langle dw, d\rho \rangle_{\rho^2 g^+} + \rho^2 |dw|_{\rho^2 g^+}^2,$$

which implies

$$2 \frac{\partial w}{\partial \rho} + \rho \left[ \left( \frac{\partial w}{\partial \rho} \right)^2 + |\nabla w|_{h_\rho}^2 \right] = 0.$$



Hence it is rather obvious that  $\partial w / \partial \rho = 0$  at  $\rho = 0$ . Therefore the proof is complete, since

$$\tilde{g} = \tilde{\rho}^2 g^+ = e^{2w} \rho^2 g^+ = e^{2w} \bar{g}. \quad \square$$

### 3. Uniformly degenerate elliptic equations

Considering the fractional powers of the Laplacian as Dirichlet-to-Neumann operators in Proposition 2.2 allows us to relate the properties of nonlocal operators to those of uniformly degenerate elliptic equations in one more dimension. The same strategy has been used, for instance, in [Cabré and Sire 2010].

Fix  $\gamma \in (0, 1)$ . Let  $y = \rho^*$  be the special defining function given in Proposition 2.2 and set  $\bar{g}^* = y^2 g^+$ . We are concerned with the uniformly degenerate elliptic equation

$$\begin{cases} -\operatorname{div}(y^a \nabla U) = 0 & \text{in } (X, \bar{g}^*), \\ U = w & \text{on } M. \end{cases} \quad (3-1)$$

For our purpose we concentrate on the local behaviors of the solutions to (3-1) near the boundary. First, we write our equation in local coordinates near a fixed boundary point  $(p_0, 0)$ . More precisely, for some  $R > 0$ , we set

$$\begin{aligned} B_R^+ &= \{(x, y) \in \mathbb{R}^{n+1} : y > 0, |(x, y)| < R\}, \\ \Gamma_R^0 &= \{(x, 0) \in \partial \mathbb{R}_+^{n+1} : |x| < R\}, \\ \Gamma_R^+ &= \{(x, y) \in \mathbb{R}^{n+1} : y \geq 0, |(x, y)| = R\}. \end{aligned}$$

In local coordinates on  $\Gamma_R^0$  the metric  $\hat{h}$  is of the form  $|dx|^2(1 + O(|x|^2))$ , where  $x(p_0) = 0$ . Consider the matrix

$$A(x, y) = \sqrt{|\det \bar{g}^*|} y^a (\bar{g}^*)^{-1}.$$

Then (3-1) is equivalent to

$$\sum_{i,j=1}^{n+1} \partial_i (A_{ij} \partial_j U) = 0. \quad (3-2)$$

Moreover, we know that

$$\frac{1}{c} y^a I \leq A \leq c y^a I. \quad (3-3)$$

This shows that (3-2) is a uniformly degenerate elliptic equation. For instance, the weight  $\psi(y) = y^a$  is an  $\mathcal{A}_2$  weight in the sense of [Muckenhoupt 1972]. Equation (3-2) has been well understood in a series of papers by Fabes, Jerison, Kenig, and Serapioni [Fabes et al. 1982b; Fabes et al. 1982a]. Let us state a regularity result that is relevant to us. We will concentrate on problems of the form

$$\begin{cases} \operatorname{Div}(A(DU)) = 0 & \text{in } B_R^+, \\ -y^a \partial_y U = F, & \text{on } \Gamma_R^0, \end{cases} \quad (3-4)$$

where, for the rest of the section,  $A$  satisfies the ellipticity condition (3-3) for  $a \in (-1, 1)$ , the derivatives

are Euclidean, that is,  $D := (\partial_{x_1}, \dots, \partial_{x_n}, y)$ , and

$$\operatorname{Div}(A(DU)) := \sum_{i,j=1}^{n+1} \partial_i (A_{ij} \partial_j U).$$

**Definition 3.1.** Given  $R > 0$  and a function  $F \in L^1(\Gamma_R^0)$ , we call  $U$  a weak solution of (3-4) if  $U$  satisfies

$$(DU)^t A(DU) \in L^1(B_R^+)$$

and

$$\int_{B_R^+} (D\phi)^t A(DU) \, dx \, dy - \int_{\Gamma_R^0} F \phi \, dx = 0$$

for all  $\phi \in \mathcal{C}^1(\overline{B_R^+})$  such that  $\phi \equiv 0$  on  $\Gamma_R^+$  and  $(D\xi)^t A(D\phi) \in L^1(B_R^+)$ .

Hölder regularity for weak solutions was shown in [Fabes et al. 1982b, Lemma 2.3.12] for any  $A$  satisfying (3-3). Using this main result, regularity of weak solutions up to the boundary was carefully shown in [Cabré and Sire 2010, Lemma 4.3], at least when  $A = y^a I$ . However, their proof only depends on the divergence structure of the equation and the behavior of the weight. Hence we have the following.

**Proposition 3.2.** Let  $\gamma \in (0, 1)$ ,  $\gamma = (1 - a)/2$  and  $\beta \in (0, \min\{1, 1 - a\})$ . Let  $R > 0$  and

$$U \in L^\infty(B_{2R^+}) \cap W^{1,2}(B_{2R^+}, y^a)$$

be a weak solution of

$$\begin{cases} \operatorname{Div}(A(DU)) = 0 & \text{in } B_{2R}^+, \\ -y^a \partial_y U = F(U) & \text{on } \Gamma_{2R}^0, \end{cases} \tag{3-5}$$

for  $A$  satisfying (3-3). If  $F \in \mathcal{C}^{1,\beta}$ ,  $U \in \mathcal{C}^{0,\tilde{\beta}}(\overline{B_R^+})$  and  $\partial_{x_i} U \in \mathcal{C}^{0,\tilde{\beta}}(\overline{B_R^+})$ ,  $i = 1, \dots, n$ , for some  $\tilde{\beta} \in (0, 1)$ .

Particularly, when  $F(x, t) = \alpha(x)t + \beta(x)t^{(n+2\gamma)/(n-2\gamma)}$ , to get smoothness it is necessary to know the local boundedness of weak solutions  $U$  on  $\overline{B_R^+}$ . To get this local boundedness for weak solutions, we employ the usual Moser iteration scheme adapted to boundary valued problems (see Theorem 3.4). However, a new idea is required: we will perform two coupled iterations, one in the interior and one at the boundary, that need to be handled simultaneously. Note that in the linear case when  $F \equiv 0$ , local boundedness was shown in [Fabes et al. 1982b, Corollary 2.3.4], using the weighted Sobolev embeddings in the interior described in Proposition 3.3. However, when a nonlinearity  $F(U)$  is present at the boundary term, instead we need to use weighted trace Sobolev embeddings. (For the half-Laplacian with some particular nonlinearities,  $L^\infty$  estimates were shown in [González and Monneau 2012].)

First, we recall a weighted Sobolev embedding theorem in the interior (compare [Fabes et al. 1982b, Theorem 1.3]; see also [Chiarenza and Frasca 1985]).

**Proposition 3.3.** Let  $\Omega$  be an open bounded set in  $\mathbb{R}^{n+1}$ . Take  $1 < p < \infty$ . There exist positive constants  $C_\Omega$  and  $\delta$  such that for all  $u \in \mathcal{C}_0^\infty(\Omega)$  and all  $k$  satisfying  $1 \leq k \leq (n + 1)/n + \delta$ ,

$$\|u\|_{L^{kp}(\Omega, y^a)} \leq C_\Omega \|\nabla u\|_{L^p(\Omega, y^a)}.$$

$C_\Omega$  may be taken to depend only on  $n, p, a$ , and the diameter of  $\Omega$ .

Now we can state the theorem. Note that we actually prove it in the flat case but it is straightforward to generalize it to the manifold setting.

**Theorem 3.4.** *Let  $U$  be a weak solution of the problem*

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } B_{2R}^+, \\ -y^a \partial_y U = F(U) & \text{on } \Gamma_{2R}^0, \end{cases} \tag{3-6}$$

where  $F(z)$  satisfies

$$F(z) = O(|z|^{\beta-1}),$$

when  $|z| \rightarrow \infty$  for some  $2 < \beta < 2^*$ . Assume, in addition, that  $\int_{\Gamma_{2r_0}^0} |U|^{2^*} dx =: V < \infty$ . Then, for each  $\bar{p} > 1$ , there exists a constant  $C_{\bar{p}} = C(\bar{p}, V) > 0$  such that

$$\sup_{B_R^+} |U| + \sup_{\Gamma_R^0} |U| \leq C_{\bar{p}} \left[ \left( \frac{1}{R^{n+1+a}} \right)^{1/\bar{p}} \|U\|_{L^{\bar{p}}(B_{2R}, y^a)} + \left( \frac{1}{R^n} \right)^{1/\bar{p}} \|U\|_{L^{\bar{p}}(\Gamma_{2R}^0)} \right].$$

*Proof.* Let  $p \in \partial X$ . Note that we can work with normal coordinates  $x_1, \dots, x_n \in \mathbb{R}^n, y > 0$  near  $p$ . Without loss of generality, assume that  $R = 1$ . Then the general case is obtained by rescaling. Let  $\eta = \eta(r), r = (|x|^2 + y^2)^{1/2}$ , be a smooth cutoff function such that  $\eta = 1$  if  $r < 1, \eta = 0$  if  $r \geq 2, 0 \leq \eta \leq 1$  if  $r \in (1, 2)$ . Next, by working with  $U^+ := \max\{U, 0\}, U^- := \max\{-U, 0\}$  separately, we can assume that  $U$  is positive.

A good reference for Moser iteration arguments in divergence structure equations is [Gilbarg and Trudinger 1983, Chapter 8]. We generalize this method, considering a double iteration: one at the boundary, using Sobolev trace inequalities to handle the nonlinear term  $F(U)$ , the other in the interior domain.

The first step is to use that  $U$  is a weak solution of (3-6) by finding a good test function. Formally we can write the following: multiply (3-6) by  $\eta^2 U^\alpha$  and integrate by parts:

$$0 = 2 \int_{B_2^+} y^a \eta U^\alpha \nabla \eta \nabla U \, dx \, dy + \alpha \int_{B_2^+} y^a \eta^2 U^{\alpha-1} |\nabla U|^2 \, dx \, dy + \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) \, dx. \tag{3-7}$$

This implies, using Hölder estimates to handle the crossed term,

$$\int_{B_2^+} y^a \eta^2 U^{\alpha-1} |\nabla U|^2 \, dx \, dy \leq \frac{2}{\alpha} \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) \, dx + \frac{4}{\alpha^2} \int_{B_2^+} y^a |\nabla \eta|^2 U^{\alpha+1} \, dx \, dy. \tag{3-8}$$

On the other hand, again using Hölder’s inequality, we have

$$\int_{B_2^+} y^a |\nabla(\eta U^\delta)|^2 \, dx \, dy \leq 2\delta^2 \int_{B_2^+} y^a \eta^2 U^{2(\delta-1)} |\nabla U|^2 \, dx \, dy + 2 \int_{B_2^+} y^a U^{2\delta} |\nabla \eta|^2 \, dx \, dy.$$

If we insert formula (3-8) into the inequality above, for the choice  $\alpha = 2\delta - 1$ , we obtain

$$\begin{aligned} J &:= \int_{B_2^+} y^\alpha |\nabla(\eta U^\delta)|^2 dx dy \\ &\leq 2\left(1 + \left(\frac{\alpha+1}{\alpha}\right)^2\right) \int_{B_2^+} y^\alpha |\nabla\eta|^2 U^{2\delta} dx dy + \frac{(\alpha+1)^2}{\alpha} \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) dx \\ &=: I_1 + I_2. \end{aligned} \tag{3-9}$$

For the left hand side above, recall the trace Sobolev embedding (Corollary 5.3)

$$J = \int_{B_2^+} y^\alpha |\nabla(\eta U^\delta)|^2 dx dy \gtrsim \left( \int_{\Gamma_2^0} (\eta U^\delta)^{2^*} dx \right)^{2/2^*}, \tag{3-10}$$

and the standard weighted Sobolev embedding from Proposition 3.3.

$$J = \int_{B_2^+} y^\alpha |\nabla(\eta U^\delta)|^2 dx dy \gtrsim \left( \int_{B_2^+} y^\alpha (\eta U^\delta)^k \right)^{2/k} \tag{3-11}$$

for some  $1 < k < 2(n + 1)/n$ .

Next, we estimate from above the terms  $I_1, I_2$  in (3-9).  $I_1$  can be easily handled since  $|\nabla\eta| \leq C$ :

$$I_1 = \int_{B_2^+} y^\alpha |\nabla\eta|^2 U^{2\delta} dx dy \lesssim \int_{B_2^+} y^\alpha U^{2\delta} dx dy. \tag{3-12}$$

Now we consider the second term. To estimate  $I_2$ , if we write  $U^{2\delta-2+\beta} = U^{\beta-2}U^{2\delta}$ , then, using Hölder’s inequality with  $p = 2^*/(\beta - 2)$ ,  $1/p + 1/q = 1$ , we obtain

$$\int_{\Gamma_2^0} \eta^2 U^{2\delta-1} F(U) dx \leq \left[ \int_{\Gamma_2^0} U^{2^*} dx \right]^{1/p} \left[ \int_{\Gamma_2^0} \eta^{2q} U^{2\delta q} dx \right]^{1/q} \leq V^{1/p} \left[ \int_{\Gamma_2^0} \eta^{2q} U^{2\delta q} dx \right]^{1/q}. \tag{3-13}$$

This last integral can be handled as follows. Call  $\chi = 2^*/2$ , for simplicity. Because our hypothesis on  $\beta$ , we know that  $q \in (1, \chi)$ . Then there exists  $\lambda \in (0, 1)$  such that  $q = \lambda + (1 - \lambda)\chi$ , and an interpolation inequality gives

$$\left[ \int f^q \right]^{1/q} \leq \left[ \int f \right]^{\lambda/q} \left[ \int f^\chi \right]^{(1-\lambda)/q} = \left[ \int f^\chi \right]^{1/\chi} \left( \left[ \int f \right] \left[ \int f^\chi \right]^{-1/\chi} \right)^{\lambda/q}. \tag{3-14}$$

Since  $\lambda/q < 1$ , Young’s inequality reads

$$z^{\lambda/q} \leq C_\epsilon z + \epsilon,$$

for  $\epsilon$  small. If we substitute  $z = \left[ \int f \right] \left[ \int f^\chi \right]^{-1/\chi}$  above, together with (3-14), we arrive at

$$\left[ \int f^q \right]^{1/q} \leq \epsilon \left[ \int f^\chi \right]^{1/\chi} + C_\epsilon \int f.$$



Then from (3-13) it follows that

$$I_2 \leq V^{1/p} \left\{ \epsilon \left( \int_{\Gamma_2^0} (\eta U^\delta)^{2^*} dx \right)^{2/2^*} + C_\epsilon \int_{\Gamma_2^0} \eta^2 U^{2\delta} dx \right\}, \tag{3-15}$$

where  $\epsilon$  will be chosen later and will depend on the value of  $\alpha, \delta$ .

We go back now to the main iteration formula (3-9). It is clear from (3-10) that the first integral of the right hand side of the formula for  $I_2$  (3-15) can be absorbed into the left hand side of (3-9), and, using (3-11) and (3-10), we get that

$$\left( \int_{\Gamma_1^0} U^{\delta 2^*} dx \right)^{2/2^*} + \left( \int_{B_1^+} U^{2k\delta} dx dy \right)^{1/k} \leq C(\delta) \left[ \int_{\Gamma_2^0} U^{2\delta} dx + \int_{B_2^+} U^{2\delta} dx dy \right],$$

for some suitable choice of  $\epsilon$ . Or, switching notation from  $2\delta$  to  $\delta$ ,

$$\left( \int_{\Gamma_1^0} U^{\delta\chi} dx \right)^{1/\chi} + \left( \int_{B_1^0} U^{k\delta} dx dy \right)^{1/k} \leq C(\delta) \left[ \int_{\Gamma_2^0} U^\delta dx + \int_{B_2^0} U^\delta dx dy \right]. \tag{3-16}$$

Next, because we will always have  $\delta > 1$ , we can use that

$$C_1(a^{1/\delta} + b^{1/\delta}) \leq (a + b)^{1/\delta} \leq C_2(a^{1/\delta} + b^{1/\delta}),$$

so from (3-16) we get that

$$\|U\|_{L^{\chi\delta}(\Gamma_1^0)} + \|U\|_{L^{k\delta}(B_1^+, y^a)} \leq \|U\|_{L^\delta(\Gamma_2^0)} + \|U\|_{L^\delta(B_2^+, y^a)}.$$

For simplicity, we set

$$\theta := \min\{\chi, k\} > 1,$$

and

$$\Phi(\delta, R) := \left( \frac{1}{R^n} \right)^{1/\delta} \|U\|_{L^\delta(\Gamma_1^0)} + \left( \frac{1}{R^{n+1+a}} \right)^{1/\delta} \|U\|_{L^\delta(B_1^+, y^a)}.$$

Then, after explicitly writing all the constants involved, formula (3-16) simply reduces to

$$\Phi(\theta\delta, 1) \leq [C(1 + \delta)^\sigma]^{2/\delta} \Phi(\delta, 2),$$

for some positive number  $\sigma$ . It is clear that the same proof works if we replace  $B_1, B_2$  by  $B_{R_1}, B_{R_2}$ . The only difference is in (3-12), where we need to estimate  $|\nabla\eta| \leq C(R_2 - R_1)^{-1}$ . Thus we would obtain

$$\Phi(\theta\delta, R_1) \leq \left[ \frac{C(1 + \delta)^\sigma}{R_2 - R_1} \right]^{2/\delta} \Phi(\delta, R_2). \tag{3-17}$$

Now we iterate (3-17): set  $R_m = 1 + 1/2^m$  and  $\theta_m = \theta^m \bar{\rho}$ . Then

$$\Phi(\theta_m, 1) \leq \Phi(\theta_m, R_m) \leq (c_1\theta)^{c_2 \sum_{i=0}^{m-1} i/\theta^i} \Phi(\bar{\rho}, 2) \leq C\Phi(\bar{\rho}, 2), \tag{3-18}$$

for some constant  $C$ , because the series  $\sum_{i=0}^\infty i/\theta^i$  is convergent.

Finally, note that

$$\sup_{\Gamma_1^0} U = \lim_{\delta \rightarrow \infty} \|U\|_{L^\delta(\Gamma_1^0)}, \quad \sup_{B_1^+} U = \lim_{\delta \rightarrow \infty} \|U\|_{L^\delta(B_1^+, y^a)},$$

so that (3-18) is telling us that

$$\sup_{B_1^+} U + \sup_{\Gamma_1^0} U \leq C[\|U\|_{L^{\bar{p}}(B_2, y^a)} + \|U\|_{L^{\bar{p}}(\Gamma_2^0)}].$$

Rescaling to a ball of radius  $R$  concludes the proof of the theorem. □

The next main ingredient is the proof of the positivity of a solution to (3-5). We observed that a Hopf lemma, some version of which was known for the Euclidean half space case [Cabré and Sire 2010, Proposition 4.10], can be obtained for the uniformly degenerate elliptic equation (3-1). This nice Hopf lemma turns out to be one of the keys for us in this paper. It is interesting to observe a different behavior between the cases  $\gamma \in (0, 1/2)$  and  $\gamma \in [1/2, 1)$  in our proof — this dichotomy does not seem to appear in the flat case in [Cabré and Sire 2010].

We continue to use the setting as in Proposition 2.2. Let  $p_0 \in \partial X$  and  $(x, y)$  be the local coordinate at  $p_0$  for  $\bar{X}$  with  $x(p_0) = 0$ , where  $x$  is the normal coordinate at  $p_0$  with respect to the metric  $\hat{h}$  on the boundary  $M^n$ .

**Theorem 3.5.** *Suppose that  $U$  is a nonnegative solution to (3-1) in  $X^{n+1}$ . Then, for sufficiently small  $r_0$ , if  $U(q_0) = 0$  for  $q_0 \in \Gamma_{r_0}^0 \setminus \Gamma_{1/2r_0}^0$  and  $U > 0$  on  $\partial\Gamma_{1/2r_0}^0$  on the boundary  $M^n$ , then*

$$y^a \partial_y U|_{q_0} > 0. \tag{3-19}$$

*Proof.* First we assume that  $\gamma \in [1/2, 1)$ , that is,  $a \in (-1, 0]$ . We consider a positive function

$$W = y^{-a}(y + Ay^2)(e^{-B|x|} - e^{-Br_0}). \tag{3-20}$$

To calculate  $\text{div}(y^a \nabla W)$  in the metric  $\bar{g}^*$ , we first calculate from Proposition 2.2 that

$$\bar{g}^* = (1 + \alpha_1 y)dy^2 + (1 + \alpha_2 y)\hat{h} + o(y)$$

for some constants  $\alpha_1, \alpha_2$  and

$$\det \bar{g}^* = \det \hat{h}(1 + \alpha_3 y) + o(y),$$

for some constant  $\alpha_3$ . Then

$$\text{div}(y^a \nabla W) = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{\det \bar{g}^*}} \partial_y (\sqrt{\det \bar{g}^*} (\bar{g}^*)^{yy} ((1-a) + (2-a)yA)(e^{-B|x|} - e^{-Br_0})) \\ &= (\alpha_4 + (2-a)A + o(1))(e^{-B|x|} - e^{-Br_0}), \\ I_2 &= \frac{1}{\sqrt{\det \bar{g}^*}} \partial_{x^k} (\sqrt{\det \bar{g}^*} (\bar{g}^*)^{ky} ((1-a) + (2-a)yA)(e^{-B|x|} - e^{-Br_0})) \\ &= o(1)(e^{-B|x|} - e^{-Br_0}) + o(y)Be^{-Br}, \end{aligned}$$

for some constant  $\alpha_4$ ,

$$I_3 = \frac{1}{\sqrt{\det \bar{g}^*}} \partial_y (\sqrt{\det \bar{g}^*} (\bar{g}^*)^{yk} (y + y^2 A) \partial_{x^k} (e^{-B|x|} - e^{-Br_0})) = o(y)Be^{-Br},$$

and

$$\begin{aligned} I_4 &= \frac{y + y^2 A}{\sqrt{\det \bar{g}^*}} \partial_{x^k} (\sqrt{\det \bar{g}^*} (\bar{g}^*)^{kj} \partial_{x^j} (e^{-B|x|} - e^{-Br_0})) \\ &= \frac{y + y^2 A}{\sqrt{\det \bar{g}^*}} \partial_{x^k} (\sqrt{\det \bar{g}^*} (\bar{g}^*)^{kj} (-\frac{x_j}{r} B e^{-Br})) \\ &= y B^2 e^{-Br} + o(y) B^2 e^{-Br} + y B^2 o(r^2) e^{-Br} + o(y) B e^{-Br}. \end{aligned}$$

Thus

$$\operatorname{div}(y^a \nabla W) = (\alpha_4 + (2 - a)A + o(1))(e^{-B|x|} - e^{-Br_0}) + (B^2 + o(1)B)y e^{-Br}.$$

We remark here that all  $\alpha$ 's can be explicit, but it would not be any more use. Take  $r_0$  sufficiently small and  $A$  and  $B$  sufficiently large so that

$$\operatorname{div}(y^a \nabla W) \geq 0$$

provided that  $a \leq 0$ . Now we know

$$\operatorname{div}(y^a \nabla (U - \epsilon W)) \leq 0$$

in  $(\Gamma_{r_0}^0 \setminus \overline{\Gamma_{1/2r_0}^0}) \times (0, r_0)$  for all  $\epsilon > 0$ , and, moreover,

$$U - \epsilon W \geq 0$$

on  $\partial\{(\Gamma_{r_0}^0 \setminus \overline{\Gamma_{1/2r_0}^0}) \times (0, r_0)\}$ , provided we choose  $\epsilon$  appropriately small. Therefore, due to the maximum principle, we know that

$$U - \epsilon W > 0$$

in  $(\Gamma_{r_0}^0 \setminus \overline{\Gamma_{1/2r_0}^0}) \times (0, r_0)$ . Thus, when  $U(x(q_0), 0) = 0$ , we have

$$y^a \partial_y (U - \epsilon W)|_{(x(q_0), 0)} \geq 0,$$

which implies

$$y^a \partial_y U|_{(x(q_0), 0)} \geq \epsilon y^a \partial_y W|_{(x(q_0), 0)} = \epsilon(1 - a)(e^{-B|x(q_0)|} - e^{-Br_0}) > 0,$$

as desired.

When  $a \in (0, 1)$ , or equivalently,  $\gamma \in (0, 1/2)$ , we instead use the function

$$W = y^{-a} (y + Ay^{2-a})(e^{-B|x|} - e^{-Br_0}).$$

Then a similar calculation will prove that the conclusion still holds. □

Positivity of solutions for (3-1) is now clear:

**Corollary 3.6.** *Suppose that  $U \in \mathcal{C}^2(X) \cap \mathcal{C}(\bar{X})$  is a nonnegative solution to the equation*

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } (X, \bar{g}^*), \\ y^a \partial_y U = F(U) & \text{on } M, \end{cases}$$

where  $F(0) = 0$ . Then  $U > 0$  on  $\bar{X}$  unless  $U \equiv 0$ .

*Proof.* First,  $U > 0$  in  $X$ , and  $U$  is not identically zero on the boundary if it is not identically zero on  $\bar{X}$ . Then, on the boundary, the set where  $U$  is positive is nonempty and open. Hence, if the set where  $U$  vanishes is not empty, then, for any small number  $r_0$ , there always exist points  $p_0$  and  $q_0$  as given in the assumptions of Theorem 3.5. Thus we would arrive at the contradiction from Theorem 3.5.  $\square$

#### 4. The $\gamma$ -Yamabe problem

Now we are ready to set up the fractional Yamabe problem for  $\gamma \in (0, 1)$ . On the conformal infinity  $(M^n, [\hat{h}])$  of an asymptotically hyperbolic manifold  $(X^{n+1}, g^+)$ , we consider a scale-free functional on metrics in the class  $[\hat{h}]$  given by

$$I_\gamma[\hat{h}] = \frac{\int_M Q_\gamma^{\hat{h}} dv_{\hat{h}}}{(\int_M dv_{\hat{h}})^{(n-2\gamma)/n}}. \tag{4-1}$$

Or, if we set a base metric  $\hat{h}$  and write a conformal metric

$$\hat{h}_w = w^{4/(n-2\gamma)} \hat{h},$$

then

$$I_\gamma[w, \hat{h}] = \frac{\int_M w P_\gamma^{\hat{h}}(w) dv_{\hat{h}}}{(\int_M w^{2^*} dv_{\hat{h}})^{2/2^*}} \tag{4-2}$$

where  $2^* = 2n/(n - 2\gamma)$ . We call  $I_\gamma$  the  $\gamma$ -Yamabe functional.

The  $\gamma$ -Yamabe problem is to find a metric in the conformal class  $[\hat{h}]$  that minimizes the  $\gamma$ -Yamabe functional  $I_\gamma$ . It is clear that a metric  $\hat{h}_w$ , where  $w$  is a minimizer of  $I_\gamma[w, \hat{h}]$ , has a constant fractional scalar curvature  $Q_\gamma^{\hat{h}_w}$ , that is,

$$P_\gamma^{\hat{h}}(w) = cw^{(n+2\gamma)/(n-2\gamma)}, \quad w > 0, \tag{4-3}$$

for some constant  $c$  on  $M$ .

This suggests that we define the  $\gamma$ -Yamabe constant

$$\Lambda_\gamma(M, [\hat{h}]) = \inf\{I_\gamma[h] : h \in [\hat{h}]\}. \tag{4-4}$$

It is then apparent that  $\Lambda_\gamma(M, [\hat{h}])$  is an invariant on the conformal class  $[\hat{h}]$  when  $g^+$  is fixed.

In the mean time, based on Proposition 2.1, we set

$$I_\gamma^*[U, \bar{g}] = \frac{d_\gamma^* \int_X \rho^a |\nabla U|_{\bar{g}}^2 dv_{\bar{g}} + \int_X E(\rho) |U|^2 dv_{\bar{g}}}{(\int_M |U|^{2^*} dv_{\hat{h}})^{2/2^*}}, \tag{4-5}$$

or similarly, using Proposition 2.2, we may set

$$I_\gamma^*[U, \bar{g}^*] = \frac{d_\gamma^* \int_X y^a |\nabla U|_{\bar{g}^*}^2 dv_{\bar{g}^*} + \int_M Q_\gamma^{\hat{h}} |U|^2 dv_{\hat{h}}}{(\int_M |U|^{2^*} dv_{\hat{h}})^{2/2^*}}. \tag{4-6}$$

It is obvious that it is equivalent to solve the minimizing problems for  $I_\gamma$  and  $I_\gamma^*$ . But a very pleasant surprise is that this immediately tells us that

$$\Lambda_\gamma(X, [\hat{h}]) = \inf\{I_\gamma^*[U, \bar{g}] : U \in W^{1,2}(X, y^a)\}. \tag{4-7}$$



(Please see the definitions and discussions of the weighted Sobolev spaces in Section 5.) Note that one has that  $I_\gamma^*[|U|] \leq I_\gamma^*[U]$  to handle positivity issues. Therefore we have the following.

**Lemma 4.1.** *Suppose that  $U$  is a minimizer of the functional  $I_\gamma^*[\cdot, \bar{g}]$  in the weighted Sobolev space  $W^{1,2}(X, y^a)$  with  $\int_M |TU|^{2^*} dv_{\hat{h}} = 1$ . Then its trace  $w = TU \in H^\gamma(M)$  solves the equation*

$$P_\gamma^{\hat{h}}(w) = \Lambda_\gamma(X, [\hat{h}])w^{(n+2\gamma)/(n-2\gamma)}.$$

To resolve the  $\gamma$ -Yamabe problem is to verify  $I_\gamma$  has a minimizer  $w$ , which is positive and smooth. But before launching our resolution to the  $\gamma$ -Yamabe problem we are first due to discuss the sign of the  $\gamma$ -Yamabe constant. These statements are familiar and easy ones for the Yamabe problem but not so easy at all for the  $\gamma$ -Yamabe problem, where the conformal fractional Laplacians are just pseudodifferential operators. One knows that eigenvalues and eigenfunctions of the conformal fractional Laplacians are even more difficult to study than the differential operators. There are some affirmative results analogous to the conformal Laplacian proven in [Guillarmou and Qing 2010] when the Yamabe constant of the conformal infinity is assumed to be positive. Here we will take advantage of our Hopf lemma and the interpretation of conformal fractional Laplacians through extensions provided in Proposition 2.2.

For each  $\gamma \in (0, 1)$  we know that each conformal fractional Laplacian is selfadjoint; see [Graham and Zworski 2003; Fefferman and Graham 2002]. Hence we may look for the first eigenvalue  $\lambda_1$  by minimizing the quotient

$$\frac{\int_M w P_\gamma^{\hat{h}} w dv_{\hat{h}}}{\int_M w^2 dv_{\hat{h}}}. \tag{4-8}$$

Moreover, again in light of Proposition 2.2, it is equivalent to minimizing

$$\frac{d_\gamma^* \int_X y^a |\nabla U|_{\bar{g}^*}^2 dv_{\bar{g}^*} + \int_M Q_\gamma^{\hat{h}} |U|^2 dv_{\hat{h}}}{\int_M |U|^2 dv_{\hat{h}}}. \tag{4-9}$$

We arrive at the eigenvalue equation

$$P_\gamma^{\hat{h}} w = \lambda_1 w \quad \text{on } M.$$

Or, equivalently,

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } (X, \bar{g}^*), \\ -d_\gamma^* \lim_{y \rightarrow 0} y^a \partial_y U + Q_\gamma^{\hat{h}} U = \lambda_1 U & \text{on } M, \end{cases} \tag{4-10}$$

As a consequence of Proposition 2.2 and Theorem 3.5 we have the following.

**Theorem 4.2.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold. For each  $\gamma \in (0, 1)$  there is a smooth, positive first eigenfunction for  $P_\gamma^{\hat{h}}$  and the first eigenspace is of dimension one, provided  $H = 0$  when  $\gamma \in (1/2, 1)$ .*

*Proof.* We use the variational characterization (4-9) of the first eigenvalue. We first observe that one may always assume there is a nonnegative minimizer for (4-9). Then regularity and the maximum principle in

Section 3 insure that such a first eigenfunction is smooth and positive. To show that the first eigenspace is of dimension one, we suppose that  $\phi$  and  $\psi$  are positive first eigenfunctions for  $P_\gamma^{\hat{h}}$ . Then

$$\begin{aligned} P_\gamma^{\hat{h}_\phi} \frac{\psi}{\phi} &= \phi^{-(n+2\gamma)/(n-2\gamma)} P_\gamma^{\hat{h}} \psi = \lambda_1 \phi^{-(n+2\gamma)/(n-2\gamma)} \psi \\ &= (\phi^{-(n+2\gamma)/(n-2\gamma)} P_\gamma^{\hat{h}} \phi) \frac{\psi}{\phi} \\ &= Q_\gamma^{\hat{h}_\phi} \frac{\psi}{\phi}, \end{aligned}$$

where  $\hat{h}_\phi = \phi^{4/(n-2\gamma)} \hat{h}$ . That is, there is a function  $U$  satisfying

$$\begin{cases} \operatorname{div}(y_\phi^a \nabla U) = 0 & \text{in } (X, \bar{g}_\phi^*), \\ \lim_{y_\phi \rightarrow 0} y_\phi^a \frac{\partial U}{\partial y_\phi} U = 0 & \text{on } M, \end{cases}$$

and  $U = \psi/\phi$  on  $M$ , where  $y_\phi$  and  $\bar{g}_\phi^*$  are associated with  $\hat{h}_\phi$  as  $y$  and  $\bar{g}^*$  are associated with  $\hat{h}$  in Proposition 2.2, respectively. Replace  $U$  by  $U - U_m$  for  $U_m = \min_{\bar{X}} U$  and apply Theorem 3.5 and Corollary 3.6 to conclude that  $U$  has to be a constant. □

Consequently, we get the following.

**Corollary 4.3.** *Suppose  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold. Assume that  $\gamma \in (0, 1)$  and that  $H = 0$  when  $\gamma \in (1/2, 1)$ . Then there are three mutually exclusive possibilities for the conformal infinity  $(M^n, [\hat{h}])$ .*

- (1) *The first eigenvalue of  $P_\gamma^{\hat{h}}$  is positive, the  $\gamma$ -Yamabe constant is positive, and  $M$  admits a metric in  $[\hat{h}]$  that has pointwise positive fractional scalar curvature.*
- (2) *The first eigenvalue of  $P_\gamma^{\hat{h}}$  is negative, the  $\gamma$ -Yamabe constant is negative, and  $M$  admits a metric in  $[\hat{h}]$  that has pointwise negative fractional scalar curvature.*
- (3) *The first eigenvalue of  $P_\gamma^{\hat{h}}$  is zero, the  $\gamma$ -Yamabe constant is zero, and  $M$  admits a metric in  $[\hat{h}]$  that has vanishing fractional scalar curvature.*

*Proof.* First, it is obvious that the sign of the first eigenvalue of the conformal fractional Laplacian  $P_\gamma^{\hat{h}}$  does not change within the conformal class due to the conformal covariance property of the conformal fractional Laplacian. The three possibilities are distinguished by the sign of the first eigenvalue  $\lambda_1$  of the conformal fractional Laplacian  $P_\gamma^{\hat{h}}$ . Because, if  $\phi$  is the positive first eigenfunction of  $P_\gamma^{\hat{h}}$ , we have

$$Q_\gamma^{\hat{h}_\phi} = \lambda_1^{\hat{h}} \phi^{-4\gamma/(n-2\gamma)},$$

where  $\hat{h}_\phi = \phi^{4/(n-2\gamma)} \hat{h}$ . □

### 5. Weighted Sobolev trace inequalities

Let us continue in the setting provided by Proposition 2.2. On the compact manifold  $M^n$ , for  $\gamma \in (0, 1)$ , we recall the fractional order Sobolev space  $H^\gamma(M)$ , with its usual norm

$$\|w\|_{H^\gamma(M)}^2 := \|w\|_{L^2(M)}^2 + \int_M w(-\Delta_{\hat{h}})^\gamma w \, dv_{\hat{h}}.$$

An equivalent norm on this space is

$$\|w\|_{H^\gamma(M)}^2 := A\|w\|_{L^2(M)}^2 + \int_M w P_\gamma^{\hat{h}} w \, dv_{\hat{h}},$$

for some appropriately large number  $A$ , since  $P_\gamma^{\hat{h}}$  is an elliptic pseudodifferential operator of order  $2\gamma$  with its principal symbol being the same as that of  $(-\Delta_{\hat{h}})^\gamma$ .

Note that in  $\mathbb{R}^n$ , this Sobolev norm can be easily written in terms of the Fourier transform as

$$\|w\|_{H^\gamma(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^\gamma \hat{w}^2(\xi) \, d\xi. \tag{5-1}$$

We would also like to recall the definition of the weighted Sobolev spaces. For  $\gamma \in (0, 1)$  and  $a = 1 - 2\gamma$ , consider the norm

$$\|U\|_{W^{1,2}(X, y^a)}^2 = \int_X y^a |\nabla U|_{g^*}^2 \, dv_{g^*} + \int_X y^a U^2 \, dv_{g^*}.$$

The following is then known.

**Lemma 5.1.** *There exists a unique linear bounded operator*

$$T : W^{1,2}(X, y^a) \rightarrow H^\gamma(M)$$

*such that  $TU = U|_M$  for all  $U \in \mathcal{C}^\infty(\bar{X})$ , which is called the trace operator.*

Lemma 5.1 was explored by Nekvinda [1993] in the case when  $X$  is a subset of  $\mathbb{R}^{n+1}$  and  $M^n$  a piece of its boundary; see also [Maz’ja 1985]. It then takes some standard argument to derive Lemma 5.1 from, for instance, [Nekvinda 1993].

The classical Sobolev trace inequality on Euclidean space is well known (see, for instance, [Escobar 1988]) and reads

$$\left( \int_{\mathbb{R}^n} |Tu|^{2n/(n-1)} \, dx \right)^{(n-1)/(2n)} \leq C(n) \left( \int_{\mathbb{R}^{n+1}} |\nabla u|^2 \, dx \, dy \right)^{1/2} \tag{5-2}$$

where the constant  $C(n)$  is sharp and the equality case is completely characterized. This corresponds to  $a = 0$  for our cases. The same result is true for any other real  $a \in (-1, 1)$ . Indeed there are general weighted Sobolev trace inequalities. Let us first recall the well known fractional Sobolev inequalities. They were first considered in a remarkable paper by Lieb [1983]; see also [Frank and Lieb 2012; Cotioliis and Tavoularis 2004] or the survey [Di Nezza et al. 2012].

**Lemma 5.2.** *Let  $0 < \gamma < n/2$ ,  $2^* = 2n/(n - 2\gamma)$ . Then, for all  $w \in H^\gamma(\mathbb{R}^n)$ , we have*

$$\|w\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq S(n, \gamma) \|(-\Delta)^{\gamma/2} w\|_{H^\gamma(\mathbb{R}^n)}^2 = S(n, \gamma) \int_{\mathbb{R}^n} w(-\Delta)^\gamma w \, dx, \tag{5-3}$$

where

$$S(n, \gamma) = 2^{-2\gamma} \pi^{-\gamma} \frac{\Gamma((n - 2\gamma)/2)}{\Gamma((n + 2\gamma)/2)} \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{2\gamma/n} = \frac{\Gamma((n - 2\gamma)/2)}{\Gamma((n + 2\gamma)/2)} |\text{vol}(S^n)|^{-2\gamma/n}.$$

We have equality in (5-3) if and only if

$$w(x) = c \left( \frac{\mu}{|x - x_0|^2 + \mu^2} \right)^{(n-2\gamma)/2}, \quad x \in \mathbb{R}^n,$$

for  $c \in \mathbb{R}$ ,  $\mu > 0$  and  $x_0 \in \mathbb{R}^n$  fixed.

Note that we may interpret the above inequality as a calculation of the best  $\gamma$ -Yamabe constant on the standard sphere as the conformal infinity of the Hyperbolic space. Namely, if  $g_c$  is the standard round metric on the unit sphere,

$$\|w\|_{L^{2^*}(S^n)}^2 \leq S(n, \gamma) \int_{S^n} w P_\gamma^{g_c} w \, dv_{g_c}. \tag{5-4}$$

Such an inequality for the sphere case was also considered independently by Beckner [1993], Branson [1995], and Morpurgo [2002], in the setting of intertwining operators. Indeed, we have the following explicit expression for  $P_\gamma^{S^n}$ :

$$P_\gamma^{S^n} = \frac{\Gamma(B + \gamma + 1/2)}{\Gamma(B - \gamma + 1/2)}, \quad \text{where } B := \sqrt{-\Delta_{S^n} + \left(\frac{n-1}{2}\right)^2}.$$

It is clear from (5-4) that

$$\Lambda_\gamma(S^n, [g_c]) = \frac{1}{S(n, \gamma)}. \tag{5-5}$$

Sobolev trace inequalities can be obtained by the composition of the trace theorem and the Sobolev embedding theorem above. There have been some related works that deal with these types of energy inequalities, for instance, Nekvinda [1993], González [2009], and Cabré and Cinti [2012]. In particular, in light of the work of Caffarelli and Silvestre [2007] and Lemma 5.2, we easily see the more general form of (5-2) as follows.

**Corollary 5.3.** *Let  $w \in H^\gamma(\mathbb{R}^n)$ ,  $\gamma \in (0, 1)$ ,  $a = 1 - 2\gamma$ , and  $U \in W^{1,2}(\mathbb{R}_+^{n+1}, y^a)$  with trace  $TU = w$ . Then*

$$\|w\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq \bar{S}(n, \gamma) \int_{\mathbb{R}_+^{n+1}} y^a |\nabla U|^2 \, dx \, dy, \tag{5-6}$$

where

$$\bar{S}(n, \gamma) := d_\gamma^* S(n, \gamma). \tag{5-7}$$

Equality holds if and only if

$$w(x) = c \left( \frac{\mu}{|x - x_0|^2 + \mu^2} \right)^{(n-2\gamma)/2}, \quad x \in \mathbb{R}^n,$$

for  $c \in \mathbb{R}$ ,  $\mu > 0$  and  $x_0 \in \mathbb{R}^n$  fixed, and  $U$  is its Poisson extension of  $w$  as given in (2-13).

In the following lines we take a closer look at the extremal functions that attain the best constant in the inequality above. On  $\mathbb{R}^n$  we fix

$$w_\mu(x) := \left( \frac{\mu}{|x|^2 + \mu^2} \right)^{(n-2\gamma)/2}. \tag{5-8}$$

These correspond to the conformal diffeomorphisms of the sphere. We set

$$U_\mu = K_\gamma *_x w_\mu \tag{5-9}$$

as given in (2-13). Then we have the equality

$$\|w_\mu\|_{L^{2^*}(\mathbb{R}^n)}^2 = \bar{S}(n, \gamma) \int_{\mathbb{R}_+^{n+1}} y^a |\nabla U_\mu|^2 dx dy.$$

It is clear that

$$w_\mu(x) = \frac{1}{\mu^{(n-2\gamma)/2}} w_1\left(\frac{x}{\mu}\right) \quad \text{and} \quad U_\mu(x, y) = \frac{1}{\mu^{(n-2\gamma)/2}} U_1\left(\frac{x}{\mu}, \frac{y}{\mu}\right). \tag{5-10}$$

Moreover,  $U_\mu$  is the (unique) solution of the problem

$$\begin{cases} \operatorname{div}(y^a \nabla U_\mu) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\lim_{y \rightarrow 0} y^a \partial_y U_\mu = c_{n,\gamma} (w_\mu)^{(n+2\gamma)/(n-2\gamma)} & \text{on } \mathbb{R}^n. \end{cases} \tag{5-11}$$

On the other hand, if we multiply (5-11) by  $U_\mu$  and integrate by parts,

$$\int_{\mathbb{R}_+^{n+1}} y^a |\nabla U_\mu|^2 dx dy = c_{n,\gamma} \int_{\mathbb{R}^n} (w_\mu)^{2^*} dx. \tag{5-12}$$

Now we compare (5-12) with (5-6). Using (5-5), we arrive at

$$\Lambda(S^n, [g_c]) = c_{n,\gamma} d_\gamma^* \left[ \int_{\mathbb{R}^n} (w_\mu)^{2^*} dx \right]^{2\gamma/n}. \tag{5-13}$$

Before the end of this section we calculate the general upper bound of the  $\gamma$ -Yamabe constants. Indeed there is a complete analogue to the case of the usual Yamabe problem (compare [Aubin 1982; Lee and Parker 1987]). Namely:

**Proposition 5.4.** *Let  $\gamma \in (0, 1)$ . Then*

$$\Lambda_\gamma(M, [\hat{h}]) \leq \Lambda_\gamma(S^n, [g_c]).$$

*Proof.* First, we instead use the functional (4-6) to estimate the  $\gamma$ -Yamabe constant for a good reason. The approach is rather the standard method of gluing a “bubble” (5-8) to the manifold  $M$ ; see, for instance, Lemma 3.4 of [Lee and Parker 1987].

For any fixed  $\epsilon > 0$ , let  $B_\epsilon$  be the ball of radius  $\epsilon$  centered at the origin in  $\mathbb{R}^{n+1}$  and  $B_\epsilon^+$  be the half ball of radius  $\epsilon$  in  $\mathbb{R}_+^{n+1}$ . Choose a smooth radial cutoff function  $\eta$ ,  $0 \leq \eta \leq 1$  supported on  $B_{2\epsilon}$ , and satisfying  $\eta \equiv 1$  on  $B_\epsilon$ . Then consider the function  $V = \eta U_\mu$  with its trace  $v = \eta w_\mu$  on  $\mathbb{R}^n$ . We have

$$\int_{\mathbb{R}_+^{n+1}} y^a |\nabla V|^2 dx dy \leq (1 + \epsilon) \int_{\mathbb{R}_+^{n+1}} y^a |\nabla U_\mu|^2 dx dy + C(\epsilon) \int_{B_{2\epsilon}^+ \setminus B_\epsilon^+} U_\mu^2 dx dy. \tag{5-14}$$

Note that  $w_\mu = O(\mu^{(n-2\gamma)/2}|x|^{2\gamma-n})$  in the annulus  $\epsilon \leq |x| \leq 2\epsilon$  and  $U_\mu$  is  $O(\mu^{(n-2\gamma)/2})$  in the annulus  $B_{2\epsilon}^+ \setminus B_\epsilon^+$ . This allows us to estimate the second term in the right hand side of (5-14) by  $O(\mu^{n-2\gamma})$  as  $\mu \rightarrow 0$ , for  $\epsilon$  fixed. For the first term in the right hand side of (5-14) we first use the fact that  $w_\mu$  attains the best constant in the Sobolev inequality, so

$$\bar{S}(n, \gamma) \int_{\mathbb{R}_+^{n+1}} y^\alpha |\nabla U_\mu|^2 dx dy = \left( \int_{\mathbb{R}^n} w_\mu^{2^*} dx \right)^{2/2^*} \leq \left( \int_{\mathbb{R}^n} v^{2^*} dx \right)^{2/2^*} + O(\mu^n). \tag{5-15}$$

Now we need to transplant the function  $V$  to the manifold  $(\bar{X}, \bar{g}^*)$ . Fix a point on the boundary  $M$  and use normal coordinates  $\{x_1, \dots, x_n, y\}$  around it, in a half ball  $B_{2\epsilon}^+$  where  $V$  is supported. Two things must be modified: when  $\epsilon \rightarrow 0$ ,

$$|\nabla V|_{\bar{g}^*}^2 = |\nabla V|^2(1 + O(\epsilon)),$$

and

$$dv_{\bar{g}^*} = (1 + O(\epsilon)) dx dy,$$

so that

$$\begin{aligned} I_{\epsilon, \mu} &:= d_\gamma^* \int_{B_{2\epsilon}^+} y^\alpha |\nabla V|_{\bar{g}^*}^2 dv_{\bar{g}^*} + \int_{|x| \leq 2\epsilon} Q_\gamma^{\hat{h}} v^2 dv_{\hat{h}} \\ &\leq (1 + O(\epsilon)) \left( \int_{B_{2\epsilon}^+} y^\alpha |\nabla V|^2 dx dy + C \int_{|x| < 2\epsilon} v^2 dx \right). \end{aligned}$$

It is easily seen that

$$\int_{|x| < 2\epsilon} w_\mu^2 dx = o(1).$$

This is a small computation that can be found in Lemma 3.5 of [Lee and Parker 1987]. Then, from (5-15), fixing  $\epsilon$  small and then  $\mu$  small, we can get that

$$I_{\epsilon, \mu} \leq (1 + C\epsilon) \left( \frac{1}{S(n, \gamma)} \|v\|_{L^{2^*}(M)}^2 + C\mu \right),$$

which implies

$$\Lambda_\gamma(M, [\hat{h}]) \leq \frac{1}{S(n, \gamma)} = \Lambda_\gamma(S^n, [g_c]). \quad \square$$

We end this section by remarking that, although most of the results mentioned here were already known in different contexts, it is certainly very interesting to put all the analysis and geometry together in the context of conformal fractional Laplacians and the associated  $\gamma$ -Yamabe problems in a way that is analogous to what has been done on the subject of the Yamabe problem, which becomes fundamental to the development of geometric analysis.

### 6. Subcritical approximations

In this section we take a well-known subcritical approximation method to solve the  $\gamma$ -Yamabe problem and prove Theorem 1.4. There does not seem to be any more difficulty than usual after our discussions in previous sections. But, for the convenience of the readers, we present a brief sketch of the proof. Similar

to the case of the usual Yamabe problem we consider the following subcritical approximations to the functionals  $I_\gamma$  and  $I_\gamma^*$ , respectively. Set

$$I_\beta[w] = \frac{\int_M w P_\gamma^{\hat{h}} w dv_{\hat{h}}}{(\int_M w^\beta dv_{\hat{h}})^{2/\beta}}$$

and

$$I_\beta^*[U] = \frac{d_\gamma^* \int_X y^a |\nabla U|_{\bar{g}^*}^2 dv_{\bar{g}^*} + \int_M Q_\gamma^{\hat{h}} U^2 dv_{\hat{h}}}{(\int_M U^\beta dv_{\hat{h}})^{2/\beta}}$$

for  $\beta \in [2, 2^*)$ , where  $2^* = 2n/(n - 2\gamma)$  and  $\gamma \in (0, 1)$ . These are subcritical problems and can be solved through standard variational methods. For clarity we state the following.

**Proposition 6.1.** *For each  $2 \leq \beta < 2^*$ , there exists a smooth positive minimizer  $U_\beta$  for  $I_\beta^*[U]$  in  $W^{1,2}(X, y^a)$ , which satisfies the equations*

$$\begin{cases} \operatorname{div}(y^a \nabla U_\beta) = 0 & \text{in } (X, \bar{g}^*), \\ -d_\gamma^* \lim_{y \rightarrow 0} y^a U_\beta + Q_\gamma^{\hat{h}} U_\beta = c_\beta U_\beta^{\beta-1} & \text{on } M, \end{cases}$$

where the derivatives are taken with respect to the metric  $\bar{g}^*$  in  $X$  and  $c_\beta = I_\beta^*[U_\beta] = \min I_\beta^*$ . And the boundary value  $w_\beta$  of  $U_\beta$ , which is a positive smooth minimizer for  $I_\beta[w]$  in  $H^\gamma(M)$ , satisfies

$$P_\gamma^{\hat{h}} w_\beta = c_\beta w_\beta^{\beta-1}.$$

Using a similar argument as in the proof of Lemma 4.3 in [Lee and Parker 1987] (see also [Aubin 1982]), we have the following.

**Lemma 6.2.** *If  $\operatorname{vol}(M, \hat{h}) = 1$ ,  $|c_\beta|$  is nonincreasing as a function of  $\beta \in [2, 2^*]$ ; and if  $\Lambda_\gamma(M, [\hat{h}]) \geq 0$ ,  $c_\beta$  is continuous from the left at  $\beta = 2^*$ .*

Readers are referred to [Escobar 1992; Lee and Parker 1987; Schoen and Yau 1994] for more details.

*Proof of Theorem 1.4.* Instead of applying the standard Sobolev embedding in the Yamabe problem, we apply the weighted trace ones discussed in the previous section. To ensure that  $U_\beta$  as  $\beta \rightarrow 2^*$  produces a minimizer for the  $\gamma$ -Yamabe problem, we want to establish the a priori estimates for  $U_\beta$ . In light of the discussions in Section 3, we only need to have a uniform  $L^\infty$  bound for  $w_\beta$ . We establish the  $L^\infty$  bound for  $w_\beta$  by the so-called blow-up method.

Otherwise, assume there exist sequences  $\beta_k \rightarrow 2^*$ ,  $w_k := w_{\beta_k}$  and  $U_k := U_{\beta_k}$ ,  $x_k \in M$  such that  $w_k(x_k) = \max_M \{w_k\} = m_k \rightarrow \infty$  and  $x_k \rightarrow x_0 \in M$  as  $k \rightarrow \infty$ . Take a normal coordinate system centered at  $x_0$  and rescale

$$V_k(x, y) = m_k^{-1} U_k(\delta_k x + x_k, \delta_k y),$$

with the boundary value

$$v_k(x) = m_k^{-1} w_k(\delta_k x + x_k),$$

where  $\delta_k = m_k^{(1-\beta_k)/2\gamma}$ . Then  $V_k$  is defined in a half ball of radius  $R_k = (1 - |x_k|)/\delta_k$  and is a solution of

$$\begin{cases} \operatorname{div}(\rho^a \nabla V_k) = 0 & \text{in } B_{R_k}^+, \\ -d_\gamma^* \lim_{y \rightarrow 0} y^a \partial_y V_k + (Q_\gamma^{\hat{h}})_k v_k = c_k v_k^{\beta-1} & \text{on } B_{R_k}, \end{cases} \tag{6-1}$$

with respect to the metric  $\bar{g}^*(\delta_k x + x_k, \delta_k y)$ , where

$$(Q_\gamma^{\hat{h}})_k = \delta_k^{1-a} Q_\gamma^{\hat{h}}(\delta_k x + x_k) \rightarrow 0.$$

Due to, for example,  $\mathcal{C}^{2,\alpha}$  a priori estimates for the rescaled solutions  $V_k$ , to extract a subsequence, if necessary, we have  $V_k \rightarrow V_0$  in  $C_{\text{loc}}^{2,\alpha}$ . Moreover the metrics  $\bar{g}^*(\delta_k x + x_k, \delta_k y)$  converge to the Euclidean metric. Hence  $V_0$  is a nontrivial, nonnegative solution of

$$\begin{cases} -\operatorname{div}(y^a \nabla V_0) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -d_\gamma^* \lim_{y \rightarrow 0} y^a \partial_y V_0 = c_0 V_0^{(n+2\gamma)/(n-2\gamma)} & \text{on } \mathbb{R}^n. \end{cases} \tag{6-2}$$

Let  $v_0 = T V_0$ . It is easily seen that

$$\int_{\mathbb{R}^n} v_0^{2^*}(x) dx \leq 1. \tag{6-3}$$

Theorem 3.5 and Corollary 3.6 then assure that  $V_0 > 0$  on  $\overline{\mathbb{R}_+^{n+1}}$ . Therefore we can obtain

$$\int_{\mathbb{R}_+^{n+1}} y^a |\nabla V_0|^2 dx dy = c_0 d_\gamma^* \int_{\mathbb{R}^n} v_0^{2^*}(x) dx. \tag{6-4}$$

It is then obvious that  $c_0 > 0$ , that is,  $c_0 = \Lambda_\gamma(M, [\hat{h}])$  in light of Lemma 6.2. Moreover, by the trace inequalities from Corollary 5.3, we have

$$\left( \int_{\mathbb{R}^n} v_0^{2^*}(x) dx \right)^{2/2^*} \leq \bar{S}(n, \gamma) \int_{\mathbb{R}_+^{n+1}} y^a |\nabla V_0|^2 dx dy. \tag{6-5}$$

Then (6-3), (6-4), and (6-5), together with the definition of  $\Lambda_\gamma(S^n, [g_c])$  in (5-5), contradict the initial hypothesis (1-6).

Once we have a uniform  $L^\infty$  estimate, by the regularity theorems in Section 3 we may extract a subsequence if necessary and pass to a limit  $U_0$ , whose boundary value  $w_0$  satisfies

$$P_\gamma^{\hat{h}} w_0 = \Lambda w_0^{2^*-1}, \quad I_\gamma[w_0] = \Lambda, \quad \Lambda = \lim c_\beta. \tag{6-6}$$

Theorem 3.5 and Corollary 3.6 also ensure that  $w_0 > 0$  on  $M$ . It remains to check that  $\Lambda = \Lambda_\gamma(M, [\hat{h}])$ . However, this is a direct consequence of Lemma 6.2 when  $\Lambda_\gamma(M, [\hat{h}]) \geq 0$ . Meanwhile it is easily seen that by the definition of the  $\gamma$ -Yamabe constants and (6-6) that  $\Lambda$  can not be less than  $\Lambda_\gamma(M, [\hat{h}])$ . Hence it is also implied that  $\Lambda = \Lambda_\gamma(M, [\hat{h}])$  by Lemma 6.2 when  $\Lambda_\gamma(M, [\hat{h}]) < 0$ . Thus, in any case,  $w_0$  is a minimizer of  $I_\gamma$ , as desired.  $\square$



**7. A sufficient condition**

In this section we give the proof of Theorem 1.5, which provides a sufficient condition for the resolution of the  $\gamma$ -Yamabe problem. Here the precise structure of the metric plays a crucial role, since a careful computation of the asymptotics is required, following the calculation in [Escobar 1992]. The section is divided into two parts: the first contains the necessary estimates on the Euclidean case, while in the second we go back to the geometry setting and finish the proof of the theorem.

*Some preliminary results on  $\mathbb{R}_+^{n+1}$ .* Here we consider the divergence equation (2-11) on  $\mathbb{R}_+^{n+1}$ , as understood in [Caffarelli and Silvestre 2007; González 2009]. The main point is that by using the Fourier transform, a solution to this problem can be written in terms on its trace value on  $\mathbb{R}^n$  and the well-known Bessel functions. Indeed, let  $U$  be a solution of

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ U(x, 0) = w & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{7-1}$$

or equivalently,  $U = K_\gamma *_x w$ , where  $K_\gamma$  is the Poisson kernel as given in (2-13).

The main idea is to reduce (7-1) to an ODE by taking the Fourier transform in  $x$ . We obtain

$$\begin{cases} -|\xi|^2 \hat{u}(\xi, y) + \frac{a}{y} \hat{u}_y(\xi, y) + \hat{u}_{yy}(\xi, y) = 0, \\ \hat{U}(\xi, 0) = \hat{w}(\xi), \end{cases}$$

that is, an ODE for each fixed value of  $\xi$ .

On the other hand, consider the solution  $\varphi : [0, +\infty) \rightarrow \mathbb{R}$  of the problem

$$-\varphi(y) + \frac{a}{y} \varphi_y(y) + \varphi_{yy}(y) = 0, \tag{7-2}$$

subject to the conditions  $\varphi(0) = 1$  and  $\lim_{t \rightarrow +\infty} \varphi(t) = 0$ . This is a Bessel function and its properties are summarized in Lemma 7.1. Then we have that

$$\hat{U}(\xi, y) = \hat{w}(\xi) \varphi(|\xi|y). \tag{7-3}$$

For a review of Bessel functions, see, for instance, Lemma 5.1 of [González 2009] or Section 9.6.1 of [Abramowitz and Stegun 1964].

**Lemma 7.1.** *Consider the following ODE in the variable  $y > 0$ :*

$$-\varphi(y) + \frac{a}{y} \varphi_y(y) + \varphi_{yy}(y) = 0,$$

*with boundary conditions  $\varphi(0) = 1, \varphi(\infty) = 0$ . Its solution can be written in terms of Bessel functions:*

$$\varphi(y) = c_1 y^\gamma \mathfrak{K}_\gamma(y),$$

where  $\mathcal{K}_\gamma$  is the modified Bessel function of the second kind that has asymptotic behavior

$$\begin{aligned} \mathcal{K}_\gamma(y) &\sim \frac{\Gamma(\gamma)}{2} \left(\frac{2}{y}\right)^\gamma && \text{when } y \rightarrow 0^+, \\ \mathcal{K}_\gamma(y) &\sim \sqrt{\frac{\pi}{2y}} e^{-y} && \text{when } y \rightarrow +\infty, \end{aligned}$$

for a constant

$$c_1 = \frac{2^{1-\gamma}}{\Gamma(\gamma)}.$$

Now we are ready to prove the main technical lemmas in the proof of Theorem 1.5. More precisely, we will explicitly compute several energy terms through Fourier transforms, thanks to expression (7-3). Such precise computation is needed in order to obtain the exact value of the constant (1-8). For the rest of the section, we set

$$|\nabla U|^2 = (\partial_{x_1} U)^2 + \dots + (\partial_{x_n} U)^2 + (\partial_y U)^2, \quad |\nabla_x U|^2 = (\partial_{x_1} U)^2 + \dots + (\partial_{x_n} U)^2.$$

**Lemma 7.2.** *Given  $w \in H^\gamma(\mathbb{R}^n)$ , let  $U = K_\gamma * w$  defined on  $\mathbb{R}_+^{n+1}$ . Then*

$$\mathcal{A}_1(w) := \int_{\mathbb{R}_+^{n+1}} y^{a+2} |\nabla U|^2 dx dy = d_1 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{2(\gamma-1)} d\xi, \tag{7-4}$$

$$\mathcal{A}_2(w) := \int_{\mathbb{R}_+^{n+1}} y^{a+2} |\nabla_x U|^2 dx dy = d_2 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{2(\gamma-1)} d\xi, \tag{7-5}$$

$$\mathcal{A}_3(w) := \int_{\mathbb{R}_+^{n+1}} y^a U^2 dx dy = d_3 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{2(\gamma-1)} d\xi, \tag{7-6}$$

where

$$d_2 = \frac{-a+3}{6} d_1, \quad d_3 = \frac{1}{a+1} d_1.$$

*Proof.* We write  $\mathcal{A}_i := \mathcal{A}_i(w)$ ,  $i = 1, 2, 3$ , for simplicity. Note that the integrals in the right hand side of (7-4), (7-5), (7-6) are finite because  $w \in H^\gamma(\mathbb{R}^n) \hookrightarrow H^{\gamma-1}(\mathbb{R}^n)$  and because of the definition of the Sobolev norm (5-1).

Thanks to (7-3), we can easily compute, using the properties of the Fourier transform,

$$\begin{aligned} \mathcal{A}_1 &:= \int_{\mathbb{R}_+^{n+1}} y^{a+2} |\nabla U|^2 dx dy = \int_{\mathbb{R}_+^{n+1}} y^{a+2} (|\nabla_x U|^2 + |\partial_y U|^2) dx dy \\ &= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} (|\xi|^2 |\hat{U}|^2 + |\partial_y \hat{U}|^2) dy d\xi \\ &= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} |\hat{w}(\xi)|^2 |\xi|^2 (|\varphi(|\xi|y)|^2 + |\varphi'(|\xi|y)|^2) dy d\xi \\ &= \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{-1-a} \int_0^\infty t^{a+2} (|\varphi(t)|^2 + |\varphi'(t)|^2) dt d\xi \\ &= d_1 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{-1-a} d\xi \end{aligned} \tag{7-7}$$

for a constant

$$d_1 := \int_0^\infty t^{a+2} (|\varphi(t)|^2 + |\varphi'(t)|^2) dt. \tag{7-8}$$

Similarly,

$$\begin{aligned} \mathcal{A}_2 &:= \int_{\mathbb{R}_+^n} y^{a+2} |\nabla_x U|^2 dx dy = \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} |\xi|^2 |\widehat{U}|^2 dy d\xi \\ &= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} |\widehat{w}(\xi)|^2 |\xi|^2 |\varphi(|\xi|y)|^2 dy d\xi \\ &= \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a} \int_0^\infty t^{a+2} |\varphi(t)|^2 dt d\xi \\ &= d_2 \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a} d\xi \end{aligned}$$

for

$$d_2 := \int_0^\infty t^{a+2} |\varphi(t)|^2 dt. \tag{7-9}$$

And finally,

$$\begin{aligned} \mathcal{A}_3 &:= \int_{\mathbb{R}_+^{n+1}} y^a U^2 dx dy = \int_{\mathbb{R}^n} \int_0^\infty y^a |\widehat{U}|^2 dy d\xi = \int_{\mathbb{R}^n} \int_0^\infty y^a |\widehat{w}(\xi)|^2 |\varphi(|\xi|y)|^2 dy d\xi \\ &= \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a} \int_0^\infty t^a |\varphi(t)|^2 dt d\xi = d_3 \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a} d\xi, \end{aligned} \tag{7-10}$$

for

$$d_3 = \int_0^\infty t^a |\varphi(t)|^2 dt.$$

In the next step, we find the relation between the constants  $d_1, d_2, d_3$ . All the integrals are evaluated between zero and infinity in the following. Multiply (7-2) by  $\varphi_t t^{a+3}$  and integrate by parts:

$$- \int \varphi \varphi_t t^{a+3} + a \int \varphi_t^2 t^{a+2} + \int \varphi_{tt} \varphi_t t^{a+3} = 0. \tag{7-11}$$

In the above formula, we estimate the first term by

$$\int t^{a+3} \varphi \varphi_t = \frac{1}{2} \int t^{a+3} \partial_t (\varphi^2) = -\frac{a+3}{2} \int t^{a+2} \varphi^2,$$

and the last one by

$$\int t^{a+3} \varphi_{tt} \varphi_t = \frac{1}{2} \int t^{a+3} \partial_t (\varphi_t^2) = -\frac{a+3}{2} \int t^{a+2} \varphi_t^2,$$

so from (7-11) we obtain

$$(a+3) \int t^{a+2} \varphi^2 = (-a+3) \int t^{a+2} \varphi_t^2.$$

Together with (7-8) and (7-9) this gives

$$d_1 = \frac{6}{-a+3} d_2,$$

as desired.

Now, multiply (7-2) by  $\varphi t^{a+2}$  and integrate:

$$-\int t^{a+2}\varphi\varphi_t + a\int t^{a+1}\varphi_t^2 + \int t^{a+2}\varphi_{tt}\varphi = 0. \tag{7-12}$$

The third term above is computed as

$$\int t^{a+2}\varphi_{tt}\varphi = -\int t^{a+2}\varphi_t^2 - (a+2)\int t^{a+1}\varphi_t\varphi,$$

so (7-12) becomes

$$d_1 = -2\int t^{a+1}\varphi_t\varphi = (a+1)\int t^a\varphi^2 = (a+1)d_3. \tag{7-13}$$

This completes the proof of the lemma. □

In the following, we continue the estimates of the different error terms, although now we only need the asymptotic behavior and not the precise constant.

**Lemma 7.3.** *Let  $w$  be defined on  $\mathbb{R}^n$  and  $U = K_\gamma *_x w$ . Then*

(1) *for each  $k \in \mathbb{N}$ , if  $w \in H^{\nu-k/2}(\mathbb{R}^n)$ ,*

$$\mathcal{E}_k := \int_{\mathbb{R}_+^{n+1}} y^{a+k} |\nabla U|^2 dx dy < \infty; \tag{7-14}$$

(2) *if  $w \in H^{\nu-3/2}(\mathbb{R}^n)$  and  $(|x|w) \in H^{-1/2+\nu}(\mathbb{R}^n)$ ,*

$$\tilde{\mathcal{E}}_3 := \int_{\mathbb{R}_+^{n+1}} y^a |(x, y)|^3 |\nabla U|^2 dx dy < \infty. \tag{7-15}$$

*Proof.* Taking into account (7-3), we can proceed as in the calculation for  $\mathcal{A}_1$  in (7-7), easily arriving at

$$\mathcal{E}_k = c_k \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{1-k-a} d\xi,$$

where

$$c_k := \int_0^\infty t^{a+k} (\varphi^2(t) + \varphi_t^2(t)) dt < \infty,$$

and this last integral is finite for all  $k \in \mathbb{N}$  because of the asymptotics of the Bessel functions from Lemma 7.1. The second conclusion of the lemma is a little more involved. To show that the integral (7-15) is finite, first note that (7-14) with  $k = 3$  gives

$$\int_{\mathbb{R}_+^{n+1}} y^{a+3} |\nabla U|^2 dx dy < \infty.$$

It is clear that it only remains to prove

$$\int_{\mathbb{R}_+^{n+1}} y^a |x|^3 |\nabla U|^2 dx dy < \infty.$$

Since the computation of the previous integral can be made component by component, it is clear that it is enough to restrict to the case  $n = 1$ . Then we just need to show that

$$J := \int_0^\infty \int_{\mathbb{R}} y^a |x|^3 (\partial_x U)^2 dx dy < \infty. \tag{7-16}$$

This is an easy but tedious calculation using the Fourier transform. Without loss of generality, we drop all the constants  $2\pi$  appearing in the Fourier transform. First notice that

$$\begin{aligned} \int_{\mathbb{R}} |x|^3 (\partial_x U)^2 dx &= \|\{|x|^{3/2} \partial_x U\}\|_{L^2(\mathbb{R})}^2 = \|D_\xi^{3/2} \widehat{\partial_x U}\|_{L^2(\mathbb{R})}^2 = \|D^{3/2}(|\xi| \widehat{U})\|_{L^2(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} |\xi| \widehat{U} D_\xi^3(|\xi| \widehat{U}) d\xi. \end{aligned} \tag{7-17}$$

At this point we go back to (7-3) to substitute the explicit expression for  $\widehat{U}$ . We need to compute

$$\begin{aligned} D_\xi^3(|\xi| \widehat{w}(\xi) \varphi(|\xi| y)) &= \widehat{w}'''[|\xi| \varphi] + \widehat{w}''[3\varphi + 3|\xi| \varphi' y] + \widehat{w}'[6\varphi' y + 3|\xi| \varphi'' y^2] + \widehat{w}[|\xi| \varphi''' y^3 + 3\varphi'' y^2] \\ &= \widehat{w}'''[|\xi| \varphi] + \widehat{w}''[3\varphi + 3t \varphi'] + \widehat{w}'[6|\xi|^{-1} t \varphi' + 3|\xi|^{-1} t^2 \varphi''] + \widehat{w}[|\xi|^{-2} \varphi''' t^3 + 3|\xi|^{-2} t^2 \varphi''], \end{aligned}$$

after the change  $|\xi| y = t$ . When we substitute the above expression into (7-17) and then back into (7-16), taking into account the change of variables, we obtain

$$\begin{aligned} J &= \int_0^\infty t^a \varphi^2 dt \int_{\mathbb{R}} \widehat{w}''' \widehat{w} |\xi|^{1-a} d\xi + \int_0^\infty t^a [\varphi^2 + 3t \varphi \varphi'] dt \int_{\mathbb{R}} \widehat{w}'' \widehat{w} |\xi|^{-a} d\xi \\ &\quad + \int_0^\infty t^a [6t \varphi' \varphi + 3t^2 \varphi'' \varphi] dt \int_{\mathbb{R}} \widehat{w}' \widehat{w} |\xi|^{-a-1} d\xi + \int_0^\infty t^a [t^3 \varphi''' \varphi + 3t^2 \varphi'' \varphi] dt \int_{\mathbb{R}} \widehat{w}^2 |\xi|^{-a-2} d\xi \\ &=: c_1 J_1 + c_2 J_2 + c_3 J_3 + c_4 J_4. \end{aligned}$$

It is clear, looking at the asymptotic behavior of  $\varphi$  from Lemma 7.1 that the constants  $c_i, i = 1, 2, 3, 4$ , are finite. On the other hand, by a straightforward integration by parts argument, we can write each of the terms  $J_i, i = 1, 2, 3, 4$ , as a linear combination of just

$$\int_{\mathbb{R}} \widehat{w}^2(\xi) |\xi|^{-a-2} d\xi \quad \text{and} \quad \int_{\mathbb{R}} \widehat{w}'(\xi)^2 |\xi|^{-a} d\xi. \tag{7-18}$$

Finally, the proof is completed because the initial hypotheses show that both integrals in (7-18) are finite. In particular, these hypotheses show that all the derivations are rigorous.  $\square$

**Lemma 7.4.** *Let  $w$  be defined on  $\mathbb{R}^n$  and  $U = K_\gamma *_x w$ .*

(1) *For each  $k \in \mathbb{N}$ , if  $w \in H^{\gamma-k/2-1}(\mathbb{R}^n)$ ,*

$$\mathcal{F}_k := \int_{\mathbb{R}_+^{n+1}} y^{a+k} U^2 dx dy < \infty. \tag{7-19}$$

(2) *If  $w \in H^{\gamma-5/2}(\mathbb{R}^n)$  and  $(|x|w) \in H^{\gamma-3/2}(\mathbb{R}^n)$ ,*

$$\tilde{\mathcal{F}}_3 := \int_{\mathbb{R}_+^{n+1}} y^a |x|^3 U^2 dx dy < \infty. \tag{7-20}$$

*Proof.* The first assertion (7-19) follows as in (7-10):

$$\begin{aligned} \mathcal{F}_k &:= \int_{\mathbb{R}_+^{n+1}} y^{a+k} U^2 \, dx \, dy = \int_{\mathbb{R}^n} \int_0^\infty y^{a+k} |\widehat{U}|^2 \, dy \, d\xi = \int_{\mathbb{R}^n} \int_0^\infty y^{a+k} |\widehat{w}(\xi)|^2 |\varphi(|\xi|y)|^2 \, dy \, d\xi \\ &= \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a-k} \int_0^\infty |\varphi(t)|^2 t^{a+k} \, dt \, d\xi = c_k \int_{\mathbb{R}^n} |\widehat{w}(\xi)|^2 |\xi|^{-1-a-k} \, d\xi, \end{aligned}$$

for

$$c_k := \int_0^\infty |\varphi(t)|^2 t^{a+k} \, dt < \infty.$$

For the second assertion, in light of our previous discussions, it is enough to show that, in the one-dimensional case,

$$\int_{\mathbb{R}} |x|^3 U^2 \, dx = \| \{|x|^{3/2} U\} \|_{L^2(\mathbb{R})}^2 = \| D^{3/2} \widehat{U} \|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \widehat{U} D_\xi^3(\widehat{U}) \, d\xi.$$

Substitute the expression for  $\widehat{U}$  from (7-3). Then

$$\int_{\mathbb{R}} |x|^3 U^2 \, dx = \int \widehat{w}''' \widehat{w} \varphi^2 \, d\xi + 3 \int \widehat{w}'' \widehat{w}' \varphi' \varphi y \, d\xi + 3 \int \widehat{w}' \widehat{w}' \varphi' \varphi y^2 \, d\xi + \int \widehat{w}^2 \varphi''' \varphi y^3 \, d\xi,$$

so when we change variables  $t = |\xi|y$ ,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} y^a |x|^3 U^2 \, dx \, dy &= \int_0^\infty t^a \varphi^2 \, dt \int_{\mathbb{R}} \widehat{w}''' \widehat{w} |\xi|^{-1-a} \, d\xi + 3 \int_0^\infty t^{1+a} \varphi' \varphi \, dt \int_{\mathbb{R}} \widehat{w}'' \widehat{w} |\xi|^{-2-a} \, d\xi \\ &\quad + 3 \int_0^\infty t^{2+a} \varphi'' \varphi \, dt \int_{\mathbb{R}} \widehat{w}' \widehat{w} |\xi|^{-3-a} \, d\xi + \int_0^\infty t^{3+a} \varphi''' \varphi \, dt \int_{\mathbb{R}} \widehat{w}^2 |\xi|^{-4-a} \, d\xi \\ &= \tilde{c}_1 \tilde{J}_1 + \tilde{c}_2 \tilde{J}_2 + \tilde{c}_3 \tilde{J}_3 + \tilde{c}_4 \tilde{J}_4. \end{aligned}$$

Clearly, from the asymptotics of the Bessel functions from Lemma 7.1, the constants  $\tilde{c}_i$ ,  $i = 1, 2, 3, 4$ , are finite. At the same time, each of the integrals  $\tilde{J}_i$ ,  $i = 1, 2, 3, 4$ , can be written as a linear combination of

$$\int (\widehat{w}')^2 |\xi|^{-2-a} \, d\xi \quad \text{and} \quad \int (\widehat{w})^2 |\xi|^{-4-a} \, d\xi,$$

which are finite because of the hypothesis on  $w$ . □

Next, we check what happens with the previous two lemmas under rescaling. Here  $f = o(1)$  means

$$\lim_{\epsilon/\mu \rightarrow 0} f = 0.$$

Given any function  $w$  defined on  $\mathbb{R}^n$ , we consider its extension to  $\mathbb{R}_+^{n+1}$  as  $U = K_\gamma *_x w$ , and the rescaling, for each  $\mu > 0$ ,

$$U_\mu(x, y) := \frac{1}{\mu^{(n-2\gamma)/2}} U\left(\frac{x}{\mu}, \frac{y}{\mu}\right). \tag{7-21}$$

**Corollary 7.5.** Fix  $\epsilon, \mu > 0$  and let the hypotheses be the as in Lemma 7.3 (in each of the two cases).

(1) For each  $k \in \mathbb{N}$ ,

$$\int_{B_\epsilon^+} y^{a+k} |\nabla U_\mu|^2 dx dy = \mu^k \int_{B_{\epsilon/\mu}^+} y^{a+k} |\nabla U|^2 dx dy = \mu^k [\mathcal{E}_k + o(1)]. \tag{7-22}$$

(2) Moreover,

$$\int_{B_\epsilon^+} y^a |(x, y)|^3 |\nabla U_\mu|^2 dx dy = \mu^3 \int_{B_{\epsilon/\mu}^+} y^{a+k} |\nabla U|^2 dx dy = \mu^3 [\tilde{\mathcal{E}}_3 + o(1)], \tag{7-23}$$

where  $U_\mu$  is the rescaling (7-21), and  $\mathcal{E}_k, \tilde{\mathcal{E}}_3 < \infty$  are defined as in Lemma 7.3.

**Corollary 7.6.** Fix  $\epsilon, \mu > 0$  and let the hypotheses be as in Lemma 7.4 (in each of the two cases).

(1) For each  $k \in \mathbb{N}$ ,

$$\int_{B_\epsilon^+} y^{a+k} (U_\mu)^2 dx dy = \mu^{k+2} \int_{B_{\epsilon/\mu}^+} y^{a+k} U^2 dx dy = \mu^{k+2} [\mathcal{F}_k + o(1)]. \tag{7-24}$$

(2) Moreover,

$$\int_{B_\epsilon^+} y^a |(x, y)|^3 (U_\mu)^2 dx dy = \mu^5 \int_{B_{\epsilon/\mu}^+} y^a |x|^3 U^2 dx dy = \mu^5 [\tilde{\mathcal{F}}_3 + o(1)], \tag{7-25}$$

where  $U_\mu$  is the rescaling (7-21), and  $\mathcal{F}_k, \tilde{\mathcal{F}}_3 < \infty$  are defined as in Lemma 7.4.

**Proof of Theorem 1.5.** We first need to choose a very particular background metric for  $X$  near a nonumbilic point on  $M$ . We follow the same steps as in Lemmas 3.1–3.3 of [Escobar 1992]. But our situation is a little different. Our freedom of choice of metrics is restricted to the boundary. Hence we make some assumptions on the behavior of the asymptotically hyperbolic manifolds in order to allow us to see clearly what we can get for a good choice of representative from the conformal infinity.

**Lemma 7.7.** Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold and  $\rho$  is a geodesic defining function associated with a representative  $\hat{h}$  of the conformal infinity  $(M^n, [\hat{h}])$ . Assume that

$$\rho^{-2}(R[g^+] - \text{Ric}[g^+](\rho\partial_\rho) + n^2) \rightarrow 0 \quad \text{as } \rho \rightarrow 0. \tag{7-26}$$

Then, at  $\rho = 0$ ,

$$H := \text{Tr}_{\hat{h}} h^{(1)} = 0 \tag{7-27}$$

and

$$\text{Tr}_{\hat{h}} h^{(2)} = \frac{1}{2} \left( \|h^{(1)}\|_{\hat{h}}^2 + \frac{1}{2(n-1)} R[\hat{h}] \right), \tag{7-28}$$

where

$$g^+ = \frac{d\rho^2 + h_\rho}{\rho^2}, \quad h_\rho = \hat{h} + h^{(1)}\rho + h^{(2)}\rho^2 + o(\rho^2).$$

*Proof.* This simply follows from the calculations in [Graham 2000]. Recall (2.5) from [Graham 2000]:

$$\rho h''_{ij} + (1-n)h'_{ij} - h^{kl}h'_{kl}h_{ij} - \rho h^{kl}h'_{ik}h'_{jl} + \frac{1}{2}\rho h^{kl}h'_{kl}h'_{ij} - 2\rho R_{ij}[\hat{h}] = \rho(R_{ij}[g^+] + ng_{ij}^+), \tag{7-29}$$

where we use  $h$  to stand for  $h_\rho$  for simplicity. Taking its trace with respect to the metrics  $h$ , we have

$$\rho \operatorname{Tr}_h h'' + (1 - 2n) \operatorname{Tr}_h h' - \rho \|h'\|_h^2 + \frac{1}{2} \rho (\operatorname{Tr}_h h')^2 - 2\rho R[\hat{h}] = \rho^{-1} (R[g^+] - \operatorname{Ric}[g^+](x \partial_x) + n^2). \tag{7-30}$$

Immediately from (7-26) we see that

$$\operatorname{Tr}_h h' = 0 \quad \text{at } \rho = 0.$$

Then, dividing  $\rho$  in both sides of (7-30) and taking  $\rho \rightarrow 0$ , we have (7-28), under the assumption (7-26), because

$$(\operatorname{Tr}_h h')' = \operatorname{Tr}_{\hat{h}} h'' - \|h'\|_{\hat{h}}^2 \quad \text{at } \rho = 0. \quad \square$$

Notice that (7-26) is an intrinsic curvature condition of an asymptotically hyperbolic manifold, which is independent of the choice of geodesic defining functions. Consequently we have the following.

**Lemma 7.8.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold and (7-26) holds. Then, given a point  $p$  on the boundary  $M$ , there exists a representative  $\hat{h}$  of the conformal infinity such that*

- (i)  $H =: \operatorname{Tr}_{\hat{h}} h^{(1)} = 0$  on  $M$ ,
- (ii)  $\operatorname{Ric}[\hat{h}](p) = 0$  on  $M$ ,
- (iii)  $\operatorname{Ric}[\bar{g}](\partial_\rho)(p) = 0$  on  $M$ ,
- (iv)  $R[\bar{g}](p) = \|h^{(1)}\|_{\hat{h}}^2$  on  $M$ .

*Proof.* The proof, like that of Lemma 3.3 in [Escobar 1992], uses Theorem 5.2 of [Lee and Parker 1987]. Therefore we may choose a representative of the conformal infinity whose Ricci curvature vanishes at any given point  $p \in M$ . In light of Lemma 7.7 we get *i.* and *ii.* right away. We then calculate

$$\operatorname{Ric}[\bar{g}](\partial_x) = -\frac{1}{2} \operatorname{Tr}_{\hat{h}} h^{(2)} + \frac{1}{4} \|h^{(1)}\|_{\hat{h}}^2 = 0$$

at  $p \in M$  from (7-28). Finally we recall that

$$R[\bar{g}] = 2 \operatorname{Ric}[\bar{g}](\partial_\rho) + R[\hat{h}] + \|h^{(1)}\|_{\hat{h}}^2 - (\operatorname{Tr}_{\hat{h}} h^{(1)})^2 = \|h^{(1)}\|_{\hat{h}}^2.$$

The proof is complete. □

Assume that  $0 \in M = \partial \bar{X}$  is a nonumbilic point. Choose normal coordinates  $x_1, \dots, x_n$  around 0 on  $M$  and let  $(x_1, \dots, x_n, \rho)$  be the Fermi coordinates on  $X$  around 0. In particular, we can write

$$g^+ = \rho^{-2} (d\rho^2 + h_{ij}(x, \rho) dx_i dx_j), \quad \bar{g} = d\rho^2 + h_{ij}(x, \rho) dx_i dx_j.$$

In order to simplify the later notation, we denote the coordinate  $\rho$  by  $y$ . The only risk of confusion comes from the fact that we have previously used  $y$  for the special defining function  $\rho^*$  from Proposition 2.2, but we will not need it any longer. In the new notation we have

$$\bar{g} = dy^2 + h_{ij}(x, y) dx_i dx_j$$

for some functions  $h_{ij}(x, y)$ ,  $i, j = 1, \dots, n$ . From what we have in the above two lemmas, we get from Lemmas 3.1 and 3.2 of [Escobar 1992] the following.



**Lemma 7.9.** *Suppose that  $(X^{n+1}, g^+)$  is an asymptotically hyperbolic manifold satisfying (7-26). Given a nonumbilic point  $p$  on the boundary  $M$ , that is, one such  $\|h^{(1)}\|_{\hat{h}}(p) \neq 0$  for  $p \in M$ , where  $\hat{h}$  is chosen as in Lemma 7.8, we have*

- (1)  $\sqrt{|\bar{g}|} = 1 - \frac{1}{2}\|\pi\|^2 y^2 + O(|(x, y)|^3)$  and
- (2)  $\bar{g}^{ij} = \delta_{ij} + 2\pi^{ij} y - \frac{1}{3}R_{kl}^{ij} [\hat{h}] x_k x_l + \bar{g}^{ij}_{,ym} y x_m + (3\pi^{im} \pi_m^j + R^i_{jy} [\bar{g}]) y^2 + O(|(x, y)|^3)$ ,

where, for simplicity, we set  $\pi = h^{(1)}$ .

As in Proposition 5.4, we try to find a good test function for the Sobolev quotient given by

$$I_{\gamma}^*[U, \bar{g}] = \frac{d_{\gamma}^* \int_X y^a |\nabla U|_{\bar{g}}^2 dv_{\bar{g}} + \int_X E(y) U^2 dv_{\bar{g}}}{(\int_M |U|^{2^*} dv_{\hat{h}})^{2/2^*}},$$

where  $E(y)$  is given by (2-10), with respect to the metric  $\bar{g}$ :

$$E(y) = \frac{n-1+a}{4n} (R[\bar{g}] - (n(n+1) + R[g^+]) y^{-2}) y^a. \tag{7-31}$$

We need to perform a careful computation of the lower order terms in order to find an estimate for  $\Lambda_{\gamma}(M, [\hat{h}])$ . For simplicity, we introduce the following notation: for a subset  $\Omega \subset \mathbb{R}_+^{n+1}$ , we consider the energy functional restricted to  $\Omega$  given by

$$\mathcal{H}(U, \Omega) := d_{\gamma}^* \int_{\Omega} y^a |\nabla U|_{\bar{g}}^2 dv_{\bar{g}} + \int_{\Omega} E(y) U^2 dv_{\bar{g}}.$$

Given any  $\epsilon > 0$ , let  $B_{\epsilon}$  be the ball of radius  $\epsilon$  centered at the origin in  $\mathbb{R}^{n+1}$  and  $B_{\epsilon}^+$  be the half ball of radius  $\epsilon$  in  $\mathbb{R}_+^{n+1}$ . Choose a smooth radial cutoff function  $\eta$ ,  $0 \leq \eta \leq 1$ , supported on  $B_{2\epsilon}$ , and satisfying  $\eta = 1$  on  $B_{\epsilon}$ . We recall here the conformal diffeomorphisms of the sphere  $w_{\mu}$  given in (5-8) and their extension  $U_{\mu}$  as in (5-9). Our test function is simply

$$V_{\mu} := \eta U_{\mu}.$$

*Step 1: Computation of the energy in  $B_{\epsilon}^+$ .* It is clear that in the half ball  $B_{\epsilon}^+$ ,  $V_{\mu} = U_{\mu}$ , so that  $\mathcal{H}(V_{\mu}, B_{\epsilon}^+) = \mathcal{H}(U_{\mu}, B_{\epsilon}^+)$ . We compute the first term in the energy  $\mathcal{H}(U_{\mu}, B_{\epsilon}^+)$ . Using the asymptotics for  $\bar{g}$  from Lemma 7.9, we have (here the indexes  $i, j$  run from 1 to  $n$ )

$$\begin{aligned} \int_{B_{\epsilon}^+} y^a |\nabla U_{\mu}|_{\bar{g}}^2 dv_{\bar{g}} &= \int_{B_{\epsilon}^+} y^a [\bar{g}^{ij} (\partial_i U_{\mu})(\partial_j U_{\mu}) + (\partial_y U_{\mu})^2] dv_{\bar{g}} \\ &= J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \end{aligned} \tag{7-32}$$

where

$$\begin{aligned} J_1 &:= \int_{B_{\epsilon}^+} y^a |\nabla U_{\mu}|^2 dv_{\bar{g}}, \\ J_2 &:= 2\pi^{ij} \int_{B_{\epsilon}^+} y^{a+1} (\partial_i U_{\mu})(\partial_j U_{\mu}) dv_{\bar{g}}, \\ J_3 &:= \int_{B_{\epsilon}^+} y^{a+2} (3\pi^{im} \pi_m^j + R^i_{jy} [\bar{g}]) (\partial_i U_{\mu})(\partial_j U_{\mu}) dv_{\bar{g}}, \end{aligned}$$

$$\begin{aligned}
 J_4 &:= \int_{B_\epsilon^+} y^{a+1} \bar{g}^{ij}{}_{,tk} x_k (\partial_i U_\mu) (\partial_j U_\mu) dv_{\bar{g}} \\
 J_5 &:= -\frac{1}{3} \int_{B_\epsilon^+} y^a R^i{}_{kl}{}^j[\bar{g}] x_k x_l (\partial_i U_\mu) (\partial_j U_\mu) dv_{\bar{g}} \\
 J_6 &:= c \int_{B_\epsilon^+} y^a |(x, y)|^3 |\nabla U_\mu|^2 dv_{\bar{g}}.
 \end{aligned}$$

We estimate  $J_1$  using the estimate for the volume element  $\sqrt{|\bar{g}|}$  from Lemma 7.9:

$$\begin{aligned}
 J_1 &= \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dv_{\bar{g}} \\
 &\leq \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy - \frac{1}{2} \|\pi\|^2 \int_{B_\epsilon^+} y^{2+a} |\nabla U_\mu|^2 dx dy + c \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 |(x, y)|^3 dx dy \\
 &\leq \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy - \frac{1}{2} \|\pi\|^2 \mu^2 \mathcal{A}_1 + \mu^2 o(1) + c\mu^3 [\tilde{\mathcal{E}}_3 + o(1)],
 \end{aligned} \tag{7-33}$$

if we take into account the notation from (7-4) and Corollary 7.5.

Now we look closely at the equation for  $U_\mu$ . Multiply expression (5-11) by  $U_\mu$  and integrate by parts:

$$\int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy = c_{n,\gamma} \int_{\Gamma_\epsilon^0} w_\mu^{2*} dx + \int_{\Gamma_\epsilon^+} U_\mu (\partial_\nu U_\mu) d\sigma \leq c_{n,\gamma} \int_{\Gamma_\epsilon^0} w_\mu^{2*} dx, \tag{7-34}$$

where  $\nu$  is the exterior normal to  $B_\epsilon^+$ . Here we have used the properties of the convolution with a radially symmetric, nonincreasing kernel  $K_\gamma$ . More precisely, since  $w_\mu$  is radially symmetric and nonincreasing,  $U_\mu = K_\gamma * w_\mu$  also satisfies  $\partial_\nu U_\mu \leq 0$  on  $\Gamma_\epsilon^+$ ; see [Cabré and Roquejoffre 2013, Lemma 2.3], for instance.

From (7-34), using (5-13), we arrive at

$$\int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy \leq \Lambda(S^m, [g_c]) (d_\gamma^*)^{-1} \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2*} dx \right]^{(n-2\gamma)/n}. \tag{7-35}$$

For simplicity, we set  $\Lambda_1 := \Lambda(S^m, [g_c]) (d_\gamma^*)^{-1}$ . Equations (7-33) and (7-35) tell us that

$$J_1 = \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dv_{\bar{g}} \leq \Lambda_1 \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2*} dx \right]^{2/2^*} - \frac{1}{2} \|\pi\|^2 \mu^2 \mathcal{A}_1 + \mu^2 o(1) + c\mu^3. \tag{7-36}$$

On the other hand, the asymptotics for the metric  $\hat{h} = \bar{g}|_{y=0}$  near the origin are explicit. Indeed, from Lemma 7.8 we know that

$$\sqrt{|\hat{h}|} = 1 + O(|x|^3). \tag{7-37}$$

Moreover, we can compute from (5-10)

$$\int_{\Gamma_\epsilon^0} (w_\mu)^{2*} |x|^3 dx = \mu^3 \int_{\Gamma_{\epsilon/\mu}^0} (w_1)^{2*} |x|^3 dx \leq c\mu^3.$$

Consequently, from (7-37) we are able to relate the integrals in  $dv_{\hat{h}}$  and  $dx$ :

$$\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dx \leq \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} + c\mu^3.$$

And substituting the above expression into (7-36) we get

$$J_1 = \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dv_{\bar{g}} \leq \Lambda_1 \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*} - \frac{1}{2} \|\pi\|^2 \mu^2 \mathcal{A}_1 + \mu^2 o(1) + c\mu^3.$$

Now we go back to (7-32) and try to estimate the second term  $J_2$  in the right hand side. If we again use the asymptotics of the metric  $\bar{g}$  given in Lemma 7.9,

$$\int_{B_\epsilon^+} y^{a+1} (\partial_i U_\mu) (\partial_j U_\mu) dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+1} (\partial_i U_\mu) (\partial_j U_\mu) dx dy + \mathcal{B}, \tag{7-38}$$

for

$$\mathcal{B} \leq c \int_{B_\epsilon^+} y^{a+3} |\nabla U_\mu|^2 dx dy + c \int_{B_\epsilon^+} y^{a+1} |\nabla U_\mu|^2 |(x, y)|^3 dx dy.$$

We notice here that  $\mathcal{B}$  can be easily estimated from Corollary 7.5:

$$\mathcal{B} \leq c\mu^3 (\mathcal{E}_3 + o(1)) + c\mu^3 \epsilon (\tilde{\mathcal{E}}_3 + o(1)) \leq c\mu^3 + \mu^3 o(1). \tag{7-39}$$

Let us look at the cross terms  $(\partial_i U_\mu) (\partial_j U_\mu)$ ,  $1 \leq i, j \leq n$  in (7-38). We note that  $\partial_i U_\mu = K_\gamma *_x (\partial_i w_\mu)$ , just by taking the derivatives in the convolution. This last derivative can be explicitly written, and in particular,  $\partial_i w_\mu$  is an odd function in the variable  $x_i$ . By the properties of the convolution, we know that  $\partial_i U_\mu$  is also an odd function in the variable  $x_i$ . Then, using the symmetries of the half ball, the integral

$$\int_{B_\epsilon^+} y^{a+1} (\partial_i U_\mu) (\partial_j U_\mu) dx dy$$

is zero if  $i \neq j$ . If  $i = j$ , we use that the mean curvature at the point vanishes, that is,  $\pi_i^i = 0$  by Lemma 7.8. Then, when we substitute formula (7-38) in the expression for  $J_2$ , only the error term remains, and by (7-39) we conclude that

$$J_2 = 2\pi^{ij} \int_{B_\epsilon^+} y^{a+1} (\partial_i U_\mu) (\partial_j U_\mu) dv_{\bar{g}} \leq \mathcal{B} \leq \mu^3 (c + o(1)). \tag{7-40}$$

Now we estimate the next term in (7-32),  $J_3$ . Again using the asymptotics for the volume element  $dv_{\bar{g}}$  from Lemma 7.9, we have that

$$\int_{B_\epsilon^+} y^{a+2} (\partial_i U_\mu) (\partial_j U_\mu) dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+2} (\partial_i U_\mu) (\partial_j U_\mu) dx dy + \mathcal{B}', \tag{7-41}$$

for

$$\begin{aligned} \mathcal{B}' &\leq c \int_{B_\epsilon^+} y^{a+4} |\nabla U_\mu|^2 dx dy + c \int_{B_\epsilon^+} y^{a+2} |(x, y)|^3 |\nabla U_\mu|^2 dx dy \\ &\leq \mu^4 (\mathcal{E}_4 + o(1)) + \mu^3 \epsilon^2 (\tilde{\mathcal{E}}_3 + o(1)) \leq c\mu^3, \end{aligned}$$

where the last estimate follows again thanks to Corollary 7.5.

Notice that, again for  $i \neq j$ , the first integral in the right hand side of (7-41) vanishes — thanks to the symmetries of the half ball and the discussion above on the oddness of the derivatives of  $U_\mu$ . Then we recall the definition of  $\mathcal{A}_2$  from (7-5) and the estimate (7-22). When we put all these ingredients together, we get

$$\begin{aligned} J_3 &= (3\pi^{im}\pi_m^j + R^i{}^j{}_y[\bar{g}]) \int_{B_\epsilon^+} y^{a+2}(\partial_i U_\mu)(\partial_j U_\mu) dv_{\bar{g}} \\ &= \frac{1}{n}[3\|\pi\|^2 + \text{Ric}(v)]\mu^2\mathcal{A}_2 + c\mu^3 \\ &= \frac{3}{n}\|\pi\|^2\mu^2\mathcal{A}_2 + \mu^2 o(1) + c\mu^3, \end{aligned}$$

if we take into account that  $\text{Ric}(v)(0)[\hat{h}] = 0$  because of Lemma 7.8.

Next, the calculation for  $J_4$  is very similar to the previous one. Indeed,

$$\int_{B_\epsilon^+} y^{a+1}x_k(\partial_i U_\mu)(\partial_j U_\mu) dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+1}x_k(\partial_i U_\mu)(\partial_j U_\mu) dx dy + \mathcal{B}'',$$

and because of symmetries on the unit ball, the first integral in the right hand side above vanishes for all  $i, j, k$ , while  $\mathcal{B}'' \leq c\mu^3$ . Thus

$$J_4 = \bar{g}^{ij,tk} \int_{B_\epsilon^+} y^{a+1}x_k(\partial_i V_\mu)(\partial_j V_\mu) dv_{\bar{g}} \leq c\mu^3.$$

And finally  $J_5, J_6$  can be estimated in a similar manner.

Putting all the estimates together for the  $J_j, j = 1, \dots, 6$ , we have shown that (7-32) reduces to

$$\int_{B_\epsilon^+} y^a |\nabla U_\mu|_{\bar{g}}^2 dv_{\bar{g}} \leq \Lambda_1 \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*} + \left[ -\frac{1}{2}\mathcal{A}_1 + \frac{3}{n}\mathcal{A}_2 \right] \|\pi\|^2 \mu^2 + \mu^2 o(1) + c\mu^3. \tag{7-42}$$

Finally, we are able to complete the computation of the energy  $\mathcal{H}(U_\mu, B_\epsilon^+)$ . Note that in the half ball  $B_\epsilon^+$ , we have a very precise behavior for the lower order term (7-31). In particular, Lemma 7.8 gives that  $R[\bar{g}](p) = \|\pi\|^2$ , so

$$E(y) = \frac{n-1+a}{4n} \|\pi\|^2 y^a + O(y^{1+a}). \tag{7-43}$$

Then, again using the asymptotics for the volume element  $dv_{\bar{g}}$ ,

$$\int_{B_\epsilon^+} E(y)(U_\mu)^2 dv_{\bar{g}} = \frac{n-1+a}{4n} \|\pi\|^2 \int_{B_\epsilon^+} y^a (U_\mu)^2 dx dy + \mathcal{B}''', \tag{7-44}$$

where

$$\mathcal{B}''' \leq c \int_{B_\epsilon^+} y^{a+1} (U_\mu)^2 dx dy + c \int_{B_\epsilon^+} y^a |x|^3 (U_\mu)^3 dx dy$$

can be estimated from Corollary 7.6 as

$$\mathcal{B}''' \leq c\mu^3 + o(1). \tag{7-45}$$

Summarizing, from (7-44) and (7-45), and using the scaling properties of  $U_\mu$  as given in (5-10), we have

$$\begin{aligned} \int_{B_\epsilon^+} E(y)(U_\mu)^2 dv_{\bar{g}} &\leq \frac{n-1+a}{4n} \|\pi\|^2 \mu^2 \int_{B_{\epsilon/\mu}^+} y^a (U_1)^2 dx dy + c\mu^3 \\ &= \frac{n-1+a}{4n} \|\pi\|^2 \mu^2 \mathcal{A}_3 + c\mu^2 o(1) + c\mu^3, \end{aligned} \tag{7-46}$$

where for the last inequality we have used Corollary 7.6 and the definition of  $\mathcal{A}_3$  from (7-6).

The energy of  $V_\mu$  in the half ball  $B_\epsilon^+$  is computed from (7-42) and (7-46), noting the relation between  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$  from Lemma 7.2 and that  $\Lambda_1 = \Lambda(S^n, [g_c])d_\gamma^*$ :

$$\begin{aligned} \mathcal{H}(V_\mu, B_\epsilon^+) &= d_\gamma^* \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2 dv_{\bar{g}} + \int_{B_\epsilon^+} E(y)(U_\mu)^2 dv_{\bar{g}} \\ &\leq \Lambda(S^n, [g_c]) \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*} + \left[ d_\gamma^* \left(-\frac{1}{2}\mathcal{A}_1 + \frac{3}{n}\mathcal{A}_2\right) + \frac{n-1+a}{4n}\mathcal{A}_3 \right] \|\pi\|^2 \mu^2 + \mu^2 o(1) + c\mu^3 \\ &\leq \Lambda(S^n, [g_c]) \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*} + \theta_{n,\gamma} \|\pi\|^2 \mu^2 \int_{\mathbb{R}^n} |\xi|^{2(\gamma-1)} |\hat{w}_1(\xi)|^2 d\xi + \mu^2 o(1) + c\mu^3 \end{aligned}$$

for

$$\theta_{n,\gamma} = \frac{1}{4n} \left[ \frac{n+a-3}{1-a} 2^{2\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} + \frac{n-1+a}{a+1} \right] d_1. \tag{7-47}$$

Finally, we note that  $w_1 \in H^\gamma(\mathbb{R}^n)$  and  $(|x|w) \in H^\gamma(\mathbb{R}^n)$ , so that all our computations are well justified.

*Step 2: Computation of the energy in the half-annulus  $B_{2\epsilon}^+ \setminus B_\epsilon^+$ .* To compute  $\mathcal{H}(V_\mu, B_{2\epsilon}^+ \setminus B_\epsilon^+)$ , note that

$$|\nabla V_\mu|_{\bar{g}}^2 \leq c|\nabla V_\mu|^2 \leq c(\eta^2 |\nabla U_\mu|^2 + (U_\mu)^2 |\nabla \eta|^2)$$

so that, because of the structure of the cutoff function  $\eta$ ,

$$|\nabla V_\mu|_{\bar{g}}^2 \leq c|\nabla U_\mu|^2 + \frac{c}{\epsilon}(U_\mu)^2. \tag{7-48}$$

Moreover,

$$\int_{B_{2\epsilon}^+ \setminus B_\epsilon^+} y^a (U_\mu)^2 dx dy \leq \mu^2 \int_{B_{2\epsilon/\mu}^+ \setminus B_{\epsilon/\mu}^+} y^a (U_1)^2 dx dy = \mu^2 o(1), \tag{7-49}$$

because the integral  $\int_{\mathbb{R}^n} y^a (U_1)^2 dx dy$  is finite and  $\epsilon/\mu \rightarrow \infty$ . On the other hand, we know that

$$\left(\frac{\epsilon}{\mu}\right)^3 \int_{B_{2\epsilon/\mu}^+ \setminus B_{\epsilon/\mu}^+} y^a |\nabla U_1|^2 dx dy \leq \int_{B_{2\epsilon/\mu}^+ \setminus B_{\epsilon/\mu}^+} y^a |(x, y)|^3 |\nabla U_1|^2 dx dy \leq \tilde{\mathcal{E}}_3 < \infty$$

because of Lemma 7.4. As a consequence,

$$\int_{B_{2\epsilon}^+ \setminus B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy = \int_{B_{2\epsilon/\mu}^+ \setminus B_{\epsilon/\mu}^+} y^a |\nabla U_1|^2 dx dy \leq \left(\frac{\mu}{\epsilon}\right)^3 \tilde{\mathcal{E}}_3. \tag{7-50}$$

If we put together formulas (7-48), (7-49), and (7-50), we arrive at

$$\mathcal{H}(V_\mu, B_{2\epsilon}^+ \setminus B_\epsilon^+) = \int_{B_{2\epsilon}^+ \setminus B_\epsilon^+} y^a |\nabla U_\mu|^2 dx dy + \int_{B_{2\epsilon}^+ \setminus B_\epsilon^+} E(y)(U_\mu)^2 dx dy \leq \mu^2 o(1)$$

when  $\mu/\epsilon \rightarrow 0$ .

*Step 3: Completion of the proof.* We have very carefully computed

$$\begin{aligned} \mathcal{H}(V_\mu, X) &= d_\gamma^* \int_X y^a |\nabla V_\mu|^2 d\text{vol}_{\bar{g}} + \int_X E(y)(V_\mu)^2 d\text{vol}_{\bar{g}} \\ &\leq \Lambda(S^n, [g_c]) \left[ \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*} + \theta_{n,\gamma} \|\pi\|^2 \mu^2 \int_{\mathbb{R}^n} |\hat{w}_1(\xi)|^2 |\xi|^{2(\gamma-1)} d\xi + \mu^2 o(1) + c\mu^3, \end{aligned}$$

where  $\theta_{n,\gamma}$  is given in (7-47).

If there is a nonumbilic point,  $\|\pi\|^2 \neq 0$  at that point. In the case that  $\theta_{n,\gamma} < 0$ , we are done, because fixing  $\epsilon$  small and then choosing  $\mu$  much smaller,

$$\mathcal{H}(V_\mu, X) < \Lambda(S^n, [g_c]) \left[ \int_M (w_\mu)^{2^*} dv_{\hat{h}} \right]^{2/2^*},$$

as desired. □

### Acknowledgements

The authors acknowledge the hospitality of USTC (Hefei, China), UC Santa Cruz (USA) and the Beijing International Center for Mathematical Research (China).

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Received 18 Sep 2011. Revised 5 Sep 2012. Accepted 18 Oct 2012.

MARÍA DEL MAR GONZÁLEZ: mar.gonzalez@upc.edu

ETSEIB, Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Av. Diagonal, 647, 08028 Barcelona, Spain

JIE QING: qing@ucsc.edu

Department of Mathematics, University of California, Santa Cruz, 4111 McHenry, Santa Cruz, CA 95064, United States



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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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APDE peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

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# ANALYSIS & PDE

Volume 6 No. 7 2013

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|  |      |
|--|------|
| Fractional conformal Laplacians and fractional Yamabe problems<br>MARÍA DEL MAR GONZÁLEZ and JIE QING  | 1535 |
| $L^p$ estimates for the Hilbert transforms along a one-variable vector field<br>MICHAEL BATEMAN and CHRISTOPH THIELE                                       | 1577 |
| Carleman estimates for anisotropic elliptic operators with jumps at an interface<br>JÉRÔME LE ROUSSEAU and NICOLAS LERNER                                  | 1601 |
| The semiclassical limit of the time dependent Hartree–Fock equation: The Weyl symbol of the solution<br>LAURENT AMOUR, MOHAMED KHODJA and JEAN NOURRIGAT   | 1649 |
| The classification of four-end solutions to the Allen–Cahn equation on the plane<br>MICHAŁ KOWALCZYK, YONG LIU and FRANK PACARD                            | 1675 |
| Pseudoparabolic regularization of forward-backward parabolic equations: A logarithmic non-linearity<br>MICHIEL BERTSCH, FLAVIA SMARRAZZO and ALBERTO TESEI | 1719 |
| The heat kernel on an asymptotically conic manifold<br>DAVID A. SHER   | 1755 |



2157-5045(2013)6:7;1-B