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CARLEMAN ESTIMATES FOR ANISOTROPIC ELLIPTIC OPERATORS WITH JUMPS AT AN INTERFACE

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We consider a second-order self-adjoint elliptic operator with an anisotropic diffusion matrix having a jump across a smooth hypersurface. We prove the existence of a weight function such that a Carleman estimate holds true. We also prove that the conditions imposed on the weight function are sharp.

1. Introduction

1A. *Carleman estimates.* Let $P(x, D_x)$ be a differential operator defined on some open subset of $\mathbb{R}^n$. A *Carleman estimate* for this operator is the weighted a priori inequality

$$\|e^{\tau\varphi} P w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi} w\|_{L^2(\mathbb{R}^n)}, \tag{1-1}$$

where the weight function φ is real-valued with a nonvanishing gradient, τ is a large positive parameter, and w is any smooth compactly supported function. This type of estimate was used for the first time by T. Carleman [1939] to handle uniqueness properties for the Cauchy problem for nonhyperbolic operators. To this day, it remains essentially the only method to prove unique continuation properties for ill-posed problems,[1] and in particular to handle uniqueness of the Cauchy problem for elliptic operators with nonanalytic coefficients.[2] This tool has been refined, polished and generalized by manifold authors.

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[1] F. John [1960] showed that, although the Hadamard well-posedness property is a privilege of hyperbolic operators, a weaker type of continuous dependence, which he called *Hölder continuous well-behavior*, could occur. Strong connections between the well-behavior property and Carleman estimates can be found in an article by H. Bahouri [1987].

[2] For analytic operators, Holmgren's theorem provides uniqueness for the noncharacteristic Cauchy problem, but that analytical result falls short of giving a control of the solution from the data.

A. P. Calderón [1958] gave a very important development of the Carleman method with a proof of an estimate of the form (1-1) using a pseudodifferential factorization of the operator, giving a new start to singular-integral methods in local analysis. L. Hörmander [1958; 1963, Chapter VIII] showed that local methods could provide the same estimates, with weaker assumptions on the regularity of the coefficients of the operator.

For instance, for second-order elliptic operators with real coefficients[3] in the principal part, Lipschitz continuity of the coefficients suffices for a Carleman estimate to hold and thus for unique continuation across a $\mathscr{C}^1$ hypersurface. Naturally, pseudodifferential methods require more derivatives, at least tangentially, that is, essentially on each level surface of the weight function φ. Chapters 17 and 28 in [Hörmander 1985b] contain more references and results.

Furthermore, it was shown by A. Pliś [1963] that Hölder continuity is not enough to get unique continuation: he constructed a real homogeneous linear differential equation of second order and of elliptic type on $\mathbb{R}^3$ without the unique continuation property, although the coefficients are Hölder-continuous with any exponent less than one. The constructions by K. Miller [1974] and later by N. Mandache [1998] and N. Filonov [2001] showed that Hölder continuity is not sufficient to obtain unique continuation for second-order elliptic operators, even in divergence form (see also [Buonocore and Manselli 2000; Schulz 1998] for the particular two-dimensional case where boundedness is essentially enough to get unique continuation for elliptic equations in the case of $W^{1,2}$ solutions).

The results cited above are related to the regularity of the principal part of the second-order operator. For strong unique continuation properties for second-order operators with Lipschitz-continuous coefficients, many results are also available for differential inequalities with singular potentials, originating with the seminal work of D. Jerison and C. Kenig [1985]. The reader is also referred to the work of C. Sogge [1989] and some of the most recent and general results of H. Koch and D. Tataru [2001; 2002].

In more recent years, the field of applications of Carleman estimates has gone beyond the original domain. They are also used in the study of inverse problems (see, for example, [Bukhgeim and Klibanov 1981; Isakov 1998; Imanuvilov et al. 2003; Kenig et al. 2007]) and control theory for PDEs. Through unique continuation properties, they are used for the exact controllability of hyperbolic equations [Bardos et al. 1992]. They also yield the null controllability of linear parabolic equations [Lebeau and Robbiano 1995] and the null controllability of classes of semilinear parabolic equations [Fursikov and Imanuvilov 1996; Barbu 2000; Fernández-Cara and Zuazua 2000].

1B. ***Jump discontinuities.*** Although the situation seems to be almost completely clarified by the previous results, with a minimal and somewhat necessary condition on Lipschitz continuity, we are interested in the following second-order elliptic operator $\mathcal{L}$:

$$\mathcal{L}w = -\operatorname{div}(A(x)\nabla w), \quad A(x) = (a_{jk}(x))_{1\le j,k\le n} = A^T(x), \quad \inf_{\|\xi\|_{\mathbb{R}^n}=1} \langle A(x)\xi, \xi\rangle > 0, \tag{1-2}$$

[3] S. Alinhac [1980] showed the nonunique continuation property for second-order elliptic operators with nonconjugate roots; of course, if the coefficients of the principal part are real, this is excluded.

in which the matrix A has a jump discontinuity across a smooth hypersurface. However, we shall impose some stringent — yet natural — restrictions on the domain of functions w, which will be required to satisfy some homogeneous *transmission conditions*, detailed in the next sections. Roughly speaking, this means that w must belong to the domain of the operator, with continuity at the interface, so that ∇w remains bounded, and continuity of the flux across the interface, so that $\operatorname{div}(A\nabla w)$ remains bounded, avoiding in particular the occurrence of a simple or multiple layer at the interface.[4]

A. Doubova, A. Osses, and J.-P. Puel [Doubova et al. 2002] tackled that problem in the isotropic case (the matrix A is $c\,\mathrm{Id}$ for scalar c) with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the "lowest". (The work of Doubova et al. [2002] concerns the case of a parabolic operator, but an adaptation to an elliptic operator is straightforward.) In the one-dimensional case, the monotonicity assumption was relaxed for general piecewise $\mathscr{C}^1$ coefficients by A. Benabdallah, Y. Dermenjian, and J. Le Rousseau [Benabdallah et al. 2007] and for coefficients with bounded variations [Le Rousseau 2007]. The case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved by J. Le Rousseau and L. Robbiano [2010]: there the isotropic case is treated, as well as a particular case of anisotropic medium. An extension of their approach to the case of parabolic operators can be found in [Le Rousseau and Robbiano 2011]. A. Benabdallah, Y. Dermenjian, and J. Le Rousseau [Benabdallah et al. 2011] also tackled the situation in which the interface meets the boundary, a case that is typical of stratified media. They treat particular forms of anisotropic coefficients.

The purpose of the present article is to show that a Carleman estimate can be proven for any operator of type (1-2) without an isotropy assumption: $A(x)$ is a symmetric positive-definite matrix with a jump discontinuity across a smooth hypersurface. We also provide conditions on the Carleman weight function that are rather simple to handle, and we prove that they are sharp.

The approach we follow differs from that of [Le Rousseau and Robbiano 2010], where the authors base their analysis on the usual Carleman method for certain microlocal regions and on Calderón projectors for others. The regions they introduce are determined by the ellipticity or nonellipticity of the conjugated operator. The method in [Benabdallah et al. 2011] exploits a particular structure of the anisotropy that allows one to use Fourier series. The analysis is then close to that of [Le Rousseau and Robbiano 2010; 2011] in the sense that second-order operators are inverted in some frequency ranges. Here, our approach is somewhat closer to A. Calderón's original work [1958] on unique continuation: the conjugated operator is factored out in first-order (pseudodifferential) operators, for which estimates are derived. Naturally, the quality of these estimates depends on their elliptic or nonelliptic nature; we thus recover microlocal regions that correspond to those of [Le Rousseau and Robbiano 2010]. Such a factorization is also used in [Imanuvilov and Puel 2003] to address nonhomogeneous boundary conditions.

1C. ***Notation and statement of the main result.*** Let Ω be an open subset of $\mathbb{R}^n$ and let Σ be a $\mathscr{C}^\infty$ oriented hypersurface of Ω; we have the partition

$$\Omega = \Omega_+ \cup \Sigma \cup \Omega_-, \quad \overline{\Omega_\pm} = \Omega_\pm \cup \Sigma, \quad \Omega_\pm \text{ open subsets of } \mathbb{R}^n, \tag{1-3}$$

[4] In the sections below, we shall also consider nonhomogeneous boundary conditions.

and we introduce the Heaviside-type functions

$$H_\pm = \mathbf{1}_{\Omega_\pm}. \tag{1-4}$$

We consider the elliptic second-order operator

$$\mathcal{L} = D \cdot AD = -\operatorname{div}(A(x)\nabla) \quad (D = -i\nabla), \tag{1-5}$$

where $A(x)$ is a symmetric positive-definite $n \times n$ matrix such that

$$A = H_- A_- + H_+ A_+, \quad A_\pm \in \mathscr{C}^\infty(\Omega). \tag{1-6}$$

We shall consider functions w of the type

$$w = H_- w_- + H_+ w_+, \quad w_\pm \in \mathscr{C}^\infty(\Omega). \tag{1-7}$$

We have $dw = H_- dw_- + H_+ dw_+ + (w_+ - w_-)\delta_\Sigma \nu$, where δ_Σ is the Euclidean hypersurface measure on Σ and ν is the unit conormal vector field to Σ pointing into Ω_+. To remove the singular term, we assume

$$w_+ = w_- \quad \text{at } \Sigma, \tag{1-8}$$

so that $A\,dw = H_- A_- dw_- + H_+ A_+ dw_+$ and

$$\operatorname{div}(A\,dw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+) + \langle A_+ dw_+ - A_- dw_-, \nu\rangle \delta_\Sigma.$$

Also, we shall assume that

$$\langle A_+ dw_+ - A_- dw_-, \nu\rangle = 0 \quad \text{at } \Sigma, \text{ that is, } \langle dw_+, A_+\nu\rangle = \langle dw_-, A_-\nu\rangle, \tag{1-9}$$

so that

$$\operatorname{div}(A\,dw) = H_- \operatorname{div}(A_- dw_-) + H_+ \operatorname{div}(A_+ dw_+). \tag{1-10}$$

Conditions (1-8)–(1-9) will be called *transmission conditions* on the function w, and we define the vector space

$$\mathcal{W} = \{H_- w_- + H_+ w_+\}_{w_\pm \in \mathscr{C}^\infty(\Omega)}, \quad w_\pm \text{ satisfying (1-8)–(1-9)}. \tag{1-11}$$

Note that (1-8) is a continuity condition of w across Σ and (1-9) is concerned with the continuity of $\langle A\,dw, \nu\rangle$ across Σ, that is, the continuity of the flux of the vector field $A\,dw$ across Σ. A weight function *suitable for observation from* Ω_+ is defined as a Lipschitz continuous function φ on Ω such that

$$\varphi = H_- \varphi_- + H_+ \varphi_+, \quad \varphi_\pm \in \mathscr{C}^\infty(\Omega), \quad \varphi_+ = \varphi_-, \quad \langle d\varphi_\pm, X\rangle > 0 \quad \text{at } \Sigma, \tag{1-12}$$

for any positively transverse vector field X to Σ (that is, $\langle \nu, X\rangle > 0$).

Theorem 1.1. *Let* $\Omega, \Sigma, \mathcal{L}, \mathcal{W}$ *be as in* (1-3), (1-5), *and* (1-11). *Then for any compact subset* K *of* Ω, *there exist a weight function* φ *satisfying* (1-12) *and positive constants* C, τ_1 *such that for all* $\tau \geq \tau_1$ *and* *all* $w \in \mathcal{W}$ *with* $\operatorname{supp} w \subset K$,

$$
\begin{aligned}
C\|e^{\tau\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)}
&\geq \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big\|H_+e^{\tau\varphi}\nabla w_+\big\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big\|H_-e^{\tau\varphi}\nabla w_-\big\|_{L^2(\mathbb{R}^n)} \\
&\quad + \tau^{3/2}\big|(e^{\tau\varphi}w)_{|\Sigma}\big|_{L^2(\Sigma)} + \tau^{1/2}\big|(e^{\tau\varphi}\nabla w_+)_{|\Sigma}\big|_{L^2(\Sigma)} + \tau^{1/2}\big|(e^{\tau\varphi}\nabla w_-)_{|\Sigma}\big|_{L^2(\Sigma)}. \quad (1\text{-}13)
\end{aligned}
$$

Remark 1.2. The proof of Theorem 1.1 provides an explicit construction of the weight function φ. The precise properties of φ are given in Section 2D, specifically (2-22), (2-24), and (2-26). The weight function is at first constructed only depending on x_n. Dependency upon the other variables, that is, convexification with respect to $\{x_n = 0\}$, is introduced in Section 4E.

Remark 1.3. It is important to notice that whenever a true discontinuity occurs for the vector field $A\nu$, the space $\mathcal{W}$ does *not* contain $\mathscr{C}^\infty(\Omega)$: the inclusion $\mathscr{C}^\infty(\Omega) \subset \mathcal{W}$ implies by (1-9) that for all $w \in \mathscr{C}^\infty(\Omega)$, $\langle dw, A_+\nu - A_-\nu\rangle = 0$ at Σ, so that $A_+\nu = A_-\nu$ at Σ, which is continuity for $A\nu$. The Carleman estimate which is proven in the present paper naturally takes into account these transmission conditions on the function w, and it is important to keep in mind that the occurrence of a jump excludes many smooth functions from the space $\mathcal{W}$. On the other hand, we have $\mathcal{W} \subset \operatorname{Lip}(\Omega)$.

Remark 1.4. We also point out the geometric content of our assumptions, which do not depend on the choice of a coordinate system. For each $x \in \Omega$, the matrix $A(x)$ is a positive-definite symmetric mapping from $T_x(\Omega)^*$ onto $T_x(\Omega)$, so that $A(x)dw(x)$ belongs indeed to $T_x(\Omega)$ and $A\,dw$ is a vector field with an L^2 divergence (inequality (1-13) yields the L^2 bound by density).

1D. ***Examples of applications.*** We mention some applications of the Carleman estimate of Theorem 1.1, namely, controllability for parabolic equations and stabilization for hyperbolic equations.

Following [Lebeau and Robbiano 1995; Lebeau and Zuazua 1998] (see also [Le Rousseau and Robbiano 2010]), we first deduce the following interpolation inequality. With $\alpha \in (0, X_0/2)$, we set $X = (0, X_0)\times\Omega$, $Y = (\alpha, X_0 - \alpha)\times\Omega$.

Theorem 1.5. *There exist* $C \geq 0$ *and* $\delta \in (0,1)$ *such that for* $u \in H^1(X)$ *that satisfies* $u_\pm = u|_{(0,X_0)\times\Omega_\pm} \in H^2((0,X_0)\times\Omega_\pm)$,

$$
u_+ = u_- \quad \textit{and} \quad \langle du_+, A_+\nu\rangle = \langle du_-, A_-\nu\rangle \quad \textit{at } (0,X_0)\times\Sigma,
$$

and

$$
u(x_0,x)|_{x\in\partial\Omega} = 0, \quad x_0 \in (0,X_0), \quad \textit{and} \quad u(0,x) = 0,\ x \in \Omega,
$$

we have

$$
\|u\|_{H^1(Y)} \leq C\|u\|^{\delta}_{H^1(X)}\Big(\big\|(D^2_{x_0} + \mathcal{L})u\big\|_{L^2(X)} + \big\|\partial_{x_0}u(0,x)\big\|_{L^2(\omega)}\Big)^{1-\delta}.
$$

This interpolation inequality was first proven in [Lebeau and Robbiano 1995; Lebeau and Zuazua 1998] for second-order elliptic operators with smooth coefficients and in [Le Rousseau and Robbiano 2010] in the case of an isotropic diffusion coefficient with a jump at an interface. Here, a jump for the whole diffusion matrix is permitted.

Remark 1.6. In fact, the interpolation inequality of Theorem 1.5 rather follows from the nonhomogeneous version of Theorem 1.1 stated in Theorem 2.2 below.

From Theorem 1.5 we can prove an estimation of the loss of orthogonality for the eigenfunctions $\phi_j(x)$, $j \in \mathbb{N}$, of the operator $\mathcal{L}$, with Dirichlet boundary conditions, when these eigenfunctions are restricted to some subset ω of Ω (see [Lebeau and Zuazua 1998; Jerison and Lebeau 1999] and also [Le Rousseau and Lebeau 2012]). We denote by μ_j, $j \in \mathbb{N}$, the associated eigenvalues, sorted in an increasing sequence.

Theorem 1.7. *There exists $C > 0$ such that for any $(a_j)_{j\in\mathbb{N}} \subset \mathbb{C}$, we have*

$$\Big(\sum_{\mu_j\le\mu}|a_j|^2\Big)^{1/2} = \Big\|\sum_{\mu_j\le\mu} a_j\phi_j\Big\|_{L^2(\Omega)} \le Ce^{C\sqrt{\mu}}\Big\|\sum_{\mu_j\le\mu} a_j\phi_j\Big\|_{L^2(\omega)}, \quad \mu > 0. \tag{1-14}$$

In turn, this yields the following null-controllability result for the associated anisotropic parabolic equation with jumps in the coefficients across Σ (see [Lebeau and Robbiano 1995; Lebeau and Zuazua 1998; Le Rousseau and Robbiano 2010] and also [Le Rousseau and Lebeau 2012]).

Theorem 1.8. *For an arbitrary time $T > 0$, an arbitrary nonempty open subset $\omega \subset \Omega$, and an initial condition $y_0 \in L^2(\Omega)$, there exists $v \in L^2((0,T)\times\Omega)$ such that the solution y of*

$$\begin{cases} \partial_t y + \mathcal{L}y = 1_\omega u & \text{in } (0,T)\times\Omega,\\ y(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,\\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases} \tag{1-15}$$

satisfies $y(T) = 0$ almost everywhere in Ω.

The interpolation inequality of Theorem 1.5 also yields the stabilization of the hyperbolic equation

$$\begin{cases} \partial_{tt} y + \mathcal{L}y + a(x)\partial_t y = 0 & \text{in } (0,T)\times\Omega,\\ y(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1-16}$$

where a is a nonvanishing nonnegative smooth function. From [Lebeau 1996; Lebeau and Robbiano 1997], we can obtain a resolvent estimate which in turn yields the following energy decay estimate.

Theorem 1.9 [Burq 1998, Theorem 3]. *For all $k \in \mathbb{N}$, there exists $C > 0$ such that*

$$\|\partial_t y(t)\|_{L^2(\Omega)} + \|y(t)\|_{H^1(\Omega)} \le \frac{C}{[\log(2+t)]^k}\big(\|\partial_t y_{|t=0}\|_{D(\mathcal{L}^{k/2})} + \|y_{|t=0}\|_{D(\mathcal{L}^{(k+1)/2})}\big), \quad t > 0,$$

for y a solution to (1-16).

The same decay can also be obtained in the case of a boundary damping (see [Lebeau and Robbiano 1997]).

Remark 1.10. Exponential decay cannot be achieved if the set $\mathcal{O} = \{a > 0\}$ does not satisfy the geometrical control condition of [Rauch and Taylor 1974; Bardos et al. 1992]. Because of the jump in the matrix coefficient $A(x)$ here, some bicharacteristics of the hyperbolic operators $\partial_{tt} + \mathcal{L}$ can be trapped in Ω_+ or Ω_- and may remain away from the stabilization region $\mathcal{O}$.

1E. ***Sketch of the proof.*** We provide in this subsection an outline of the main arguments used in our proof. To avoid technicalities, we somewhat simplify the geometric data and the weight function, keeping of course the anisotropy. We consider the operator

$$\mathcal{L}_0 = \sum_{1\le j\le n} D_j c_j D_j, \quad c_j(x) = H_+ c_j^+ + H_- c_j^-, \quad c_j^\pm > 0 \text{ constants}, \quad H_\pm = \mathbf{1}_{\{\pm x_n > 0\}}, \tag{1-17}$$

with $D_j = \dfrac{\partial}{i\partial x_j}$, and the vector space $\mathcal{W}_0$ of functions $H_+ w_+ + H_- w_-$, $w_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that

$$\text{at } x_n = 0, \quad w_+ = w_-, \quad c_n^+ \partial_n w_+ = c_n^- \partial_n w_- \text{ (transmission conditions across } x_n = 0). \tag{1-18}$$

As a result, for $w \in \mathcal{W}_0$, we have $D_n w = H_+ D_n w_+ + H_- D_n w_-$ and

$$\mathcal{L}_0 w = \sum_j \bigl(H_+ c_j^+ D_j^2 w_+ + H_- c_j^- D_j^2 w_-\bigr). \tag{1-19}$$

We also consider a weight function[5]

$$\varphi = \underbrace{\left(\alpha_+ x_n + \frac{\beta x_n^2}{2}\right)}_{\varphi_+} H_+ + \underbrace{\left(\alpha_- x_n + \frac{\beta x_n^2}{2}\right)}_{\varphi_-} H_-, \quad \alpha_\pm > 0, \quad \beta > 0, \tag{1-20}$$

a positive parameter τ, and the vector space $\mathcal{W}_\tau$ of functions $H_+ v_+ + H_- v_-$, $v_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, such that at $x_n = 0$,

$$v_+ = v_-, \tag{1-21}$$

$$c_n^+ (D_n v_+ + i\tau\alpha_+ v_+) = c_n^- (D_n v_- + i\tau\alpha_- v_-). \tag{1-22}$$

Observe that $w \in \mathcal{W}_0$ is equivalent to $v = e^{\tau\varphi} w \in W_\tau$. We have

$$e^{\tau\varphi}\mathcal{L}_0 w = \underbrace{e^{\tau\varphi}\mathcal{L}_0 e^{-\tau\varphi}}_{\mathcal{L}_\tau}(e^{\tau\varphi} w),$$

so that proving a weighted a priori estimate $\|e^{\tau\varphi}\mathcal{L}_0 w\|_{L^2(\mathbb{R}^n)} \gtrsim \|e^{\tau\varphi} w\|_{L^2(\mathbb{R}^n)}$ for $w \in \mathcal{W}_0$ amounts to getting $\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^n)} \gtrsim \|v\|_{L^2(\mathbb{R}^n)}$ for $v \in \mathcal{W}_\tau$.

Step 1 (pseudodifferential factorization). We have, using the Einstein convention on repeated indices $j \in \{1, \dots, n-1\}$,

$$\mathcal{L}_\tau = (D_n + i\tau\varphi')c_n(D_n + i\tau\varphi') + D_j c_j D_j,$$

and for $v \in \mathcal{W}_\tau$, by (1-19), with $m_\pm = m_\pm(D') = (c_n^\pm)^{-1/2}(c_j^\pm D_j^2)^{1/2}$,

$$\mathcal{L}_\tau v = H_+ c_n^+ \bigl((D_n + i\tau\varphi_+')^2 + m_+^2\bigr) v_+ + H_- c_n^- \bigl((D_n + i\tau\varphi_-')^2 + m_-^2\bigr) v_-,$$

[5] In the main text, we shall introduce some minimal requirements on the weight function and suggest other possible choices.

so that

$$\mathcal{L}_\tau v = H_+ c_n^+ \big(D_n + i(\overbrace{\tau\varphi'_+ + m_+}^{e_+})\big)\big(D_n + i(\overbrace{\tau\varphi'_+ - m_+}^{f_+})\big) v_+ \\ + H_- c_n^- \big(D_n + i(\underbrace{\tau\varphi'_- - m_-}_{f_-})\big)\big(D_n + i(\underbrace{\tau\varphi'_- + m_-}_{e_-})\big) v_-. \quad (1\text{-}23)$$

Note that $e_\pm$ are elliptic positive in the sense that $e_\pm = \tau\alpha_\pm + m_\pm \gtrsim \tau + |D'|$. At this point, we want to use certain natural estimates for first-order factors on the half-lines $\mathbb{R}_\pm$. Let us, for instance, check on $t > 0$ for $\omega \in \mathscr{C}_c^\infty(\mathbb{R})$, λ, γ positive:

$$\begin{aligned} \big\| D_t\omega + i(\lambda + \gamma t)\omega \big\|^2_{L^2(\mathbb{R}_+)} &= \| D_t\omega\|^2_{L^2(\mathbb{R}_+)} + \|(\lambda+\gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + 2\operatorname{Re}\big\langle D_t\omega, iH(t)(\lambda+\gamma t)\omega\big\rangle \\ &\geq \int_0^{+\infty} \big((\lambda+\gamma t)^2 + \gamma\big)|\omega(t)|^2\,dt + \lambda|\omega(0)|^2 \geq (\lambda^2+\gamma)\|\omega\|^2_{L^2(\mathbb{R}_+)} + \lambda|\omega(0)|^2, \quad (1\text{-}24) \end{aligned}$$

which is in a sense a perfect estimate of elliptic type, suggesting that the first-order factor containing e_+ should be easy to handle. Changing λ in $-\lambda$ gives

$$\big\| D_t\omega + i(-\lambda + \gamma t)\omega \big\|^2_{L^2(\mathbb{R}_+)} \geq 2\operatorname{Re}\big\langle D_t\omega, iH(t)(-\lambda+\gamma t)\omega\big\rangle = \int_0^{+\infty} \gamma|\omega(t)|^2\,dt - \lambda|\omega(0)|^2,$$

so that $\| D_t\omega + i(-\lambda+\gamma t)\omega\|^2_{L^2(\mathbb{R}_+)} + \lambda|\omega(0)|^2 \geq \gamma\|\omega\|^2_{L^2(\mathbb{R}_+)}$, an estimate of lesser quality, because we need to secure a control of $\omega(0)$ to handle this type of factor.

Step 2 (case $f_+ \geq 0$). Looking at formula (1-23), since the factor containing e_+ is elliptic in the sense given above, we have to discuss the sign of f_+. Identifying the operator with its symbol, we have $f_+ = \tau(\alpha_+ + \beta x_n) - m_+(\xi')$, and thus $\tau\alpha_+ \geq m_+(\xi')$, yielding a nonnegative f_+. Iterating the method outlined above on the half-line $\mathbb{R}_+$, we get a nice estimate of the form of (1-24) on $\mathbb{R}_+$; in particular, we obtain a control[6] of $v_+(0)$ and $D_n v_+(0)$. From the transmission condition, we have $v_+(0) = v_-(0)$, and hence this amounts to also controlling $v_-(0)$. That control, along with the natural estimates on $\mathbb{R}_-$, is enough to prove an inequality of the form of the Carleman estimate we seek.

Step 3 (case $f_+ < 0$). Here we assume that $\tau\alpha_+ < m_+(\xi')$. On $\mathbb{R}_+$ we can still use the factor containing e_+, and by (1-23) and (1-24) we can control the quantity

$$c_n^+(D_n + if_+)v_+(0) = \overbrace{c_n^+(D_n v_+ + i\tau\alpha_+)v_+(0)}^{=\mathcal{V}_+} - c_n^+ i m_+ v_+(0). \quad (1\text{-}25)$$

[6] In the case $f_+(0) = 0$, one needs to consider the estimation of

$$\big\|(D_n + ie_+)(D_n + if_+)v_+\big\|_{L^2(\mathbb{R}^+)} + \big\|(D_n + if_+)(D_n + ie_+)v_+\big\|_{L^2(\mathbb{R}^+)}$$

from below to obtain a control of $v_+(0)$ and $D_n v_+(0)$ with the previous estimates used in cascade. Indeed, the first term will give an estimate of $D_n v_+(0)$, and the second term one of $v_+(0)$.

Our key assumption is

$$f_+(0) < 0 \implies f_-(0) \le 0. \tag{1-26}$$

Under that hypothesis, we can use the negative factor f_- on $\mathbb{R}_-$ (note that f_- is increasing with x_n, so that $f_-(0) \le 0 \implies f_-(x_n) < 0$ for $x_n < 0$). We then control

$$c_n^-(D_n + ie_-)v_-(0) = \underbrace{c_n^-(D_n v_- + i\tau\alpha_-)v_-(0)}_{=\mathcal{V}_-} + c_n^- i m_- v_-(0). \tag{1-27}$$

Nothing more can be achieved with inequalities on each side of the interface. At this point, however, we notice that the second transmission condition in (1-22) implies $\mathcal{V}_- = \mathcal{V}_+$, yielding the control of the difference of (1-27) and (1-25), that is, of

$$c_n^- i m_- v_-(0) + c_n^+ i m_+ v_+(0) = i\big(c_n^- m_- + c_n^+ m_+\big)v(0).$$

Now, as $c_n^- m_- + c_n^+ m_+$ is elliptic positive, this gives a control of $v(0)$ in (tangential) H^1-norm, which is enough to then get an estimate on both sides that leads to the Carleman estimates we seek.

Step 4 (patching estimates together). The analysis we have sketched here relies on a separation into two zones in the (τ, ξ') space. Patching the estimates of the form of (1-13) in each zone together allows us to conclude the proof of the Carleman estimate.

1F. ***Explaining the key assumption.*** Our key assumption, condition (1-26), can be reformulated as

$$\text{for all } \xi' \in \mathbb{S}^{n-2}, \quad \frac{\alpha_+}{\alpha_-} \ge \frac{m_+(\xi')}{m_-(\xi')}. \tag{1-28}$$

In fact,[7] (1-26) means $\tau\alpha_+ < m_+(\xi') \implies \tau\alpha_- \le m_-(\xi')$, and since $\alpha_\pm, m_\pm$ are all positive, this is equivalent to having $m_+(\xi')/\alpha_+ \le m_-(\xi')/\alpha_-$, which is (1-28). An analogy with an estimate for a first-order factor may shed some light on this condition. With

$$f(t) = H(t)(\tau\alpha_+ + \beta t - m_+) + H(-t)(\tau\alpha_- + \beta t - m_-), \quad \tau, \alpha_\pm, \beta, m_\pm \text{ positive constants,}$$

we want to prove an injectivity estimate of the type $\|D_t v + if(t)v\|_{L^2(\mathbb{R})} \gtrsim \|v\|_{L^2(\mathbb{R})}$, say for $v \in \mathscr{C}_c^\infty(\mathbb{R})$. It is a classical fact (see, for example, Lemma 3.1.1 in [Lerner 2010]) that such an estimate (for a smooth f) is equivalent to the condition that $t \mapsto f(t)$ does not change sign from $+$ to $-$ while t increases: it means that the adjoint operator $D_t - if(t)$ satisfies the so-called condition (Ψ). Looking at the function f, we see that it increases on each half-line $\mathbb{R}_\pm$, so that the only place to get a "forbidden" change of sign from

[7] For the main theorem, we shall in fact require the stronger strict inequality

$$\frac{\alpha_+}{\alpha_-} > \frac{m_+(\xi')}{m_-(\xi')}. \tag{1-29}$$

This condition is then stable under perturbations, whereas (1-28) is not. This gives us the freedom to introduce microlocal cutoff in the analysis below.

However, we shall see in Section 5 that in the particular case presented here, where the matrix A is piecewise constant and the weight function φ depends solely on x_n, the inequality (1-28) is actually a *necessary and sufficient* condition to obtain a Carleman estimate with weight φ.

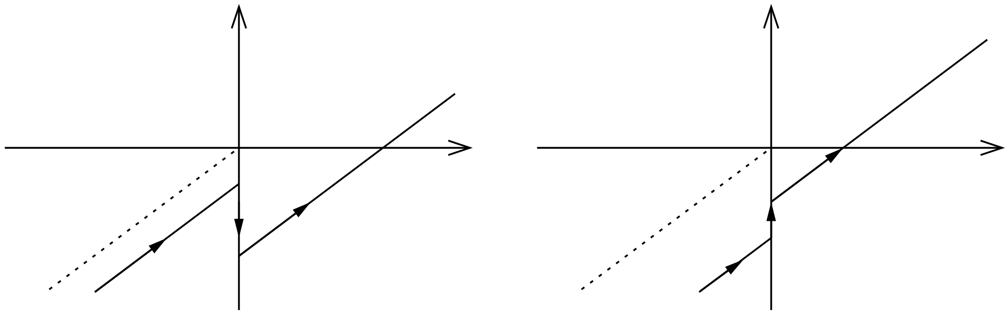

Figure 1. $f(0^-) \le 0$; $f(0^+) < 0$.

$+$ to $-$ is at $t = 0$: to get an injectivity estimate, we have to avoid the situation where $f(0^+) < 0$ and $f(0^-) > 0$, that is, we have to make sure that $f(0^+) < 0 \implies f(0^-) \le 0$, which is indeed the condition (1-28). The function f is increasing affine on $\mathbb{R}_\pm$ with the same slope β on both sides, with a possible discontinuity at 0; see Figure 1.

In Figure 1, when $f(0^+) < 0$, we should have $f(0^-) \le 0$, and the line on the left cannot go above the dotted line, in such a way that the discontinuous zigzag curve with the arrows has only a change of sign from $-$ to $+$.

When $f(0^+) \ge 0$, there is no other constraint on $f(0^-)$: even with a discontinuity, the change of sign can only occur from $-$ to $+$; see Figure 2.

We prove below (Section 5) that condition (1-28) is relevant to our problem in the sense that it is indeed necessary to have a Carleman estimate with this weight: if (1-28) is violated, we are able, for this model, to construct a quasimode for $\mathcal{L}_\tau$, that is, a τ-family of functions v with L^2-norm 1 such that $\|\mathcal{L}_\tau v\|_{L^2} \ll \|v\|_{L^2}$, as τ goes to ∞, ruining any hope of proving a Carleman estimate. As usual for this type of construction, it uses a certain complex geometrical optics method, which is easy in this case to implement directly, due to the simplicity of the expression of the operator.

Remark 1.11. A very particular case of anisotropic medium was tackled in [Le Rousseau and Robbiano 2010] for the purpose of proving a controllability result for linear parabolic equations. The condition

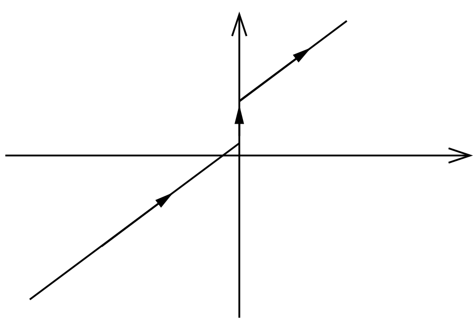

Figure 2. $f(0^-) \gtrless 0$; $f(0^+) \ge 0$.

imposed on the weight function in [Le Rousseau and Robbiano 2010, Assumption 2.1] is much more demanding than what we impose here. In the isotropic case, $c_j^{\pm} = c_{\pm}$ for all $j \in \{1, \dots, n\}$, we have $m_+ = m_- = |\xi'|$ and our condition (1-29) reads $\alpha_+ > \alpha_-$. Note also that the isotropic case $c_- \geq c_+$ was already considered in [Doubova et al. 2002].

In [Le Rousseau and Robbiano 2010], the controllability result concerns an isotropic parabolic equation. The Carleman estimate we derive here extends this result to an anisotropic parabolic equation.

2. Framework

2A. ***Presentation.*** Let Ω, Σ be as in (1-3). With

$$\Xi = \{\text{positive-definite } n \times n \text{ matrices}\},$$

we consider $A_{\pm} \in \mathscr{C}^{\infty}(\Omega; \Xi)$ and let $\mathcal{L}, \varphi$ be as in (1-5) and (1-12). We set

$$\mathcal{L}_{\pm} = D \cdot A_{\pm} D = -\operatorname{div}(A_{\pm} \nabla).$$

Here, we generalize our analysis to nonhomogeneous transmission conditions: for θ and Θ smooth functions of the interface Σ, we set

$$w_+ - w_- = \theta \quad \text{and} \quad \langle A_+ dw_+ - A_- dw_-, \nu \rangle = \Theta \quad \text{at } \Sigma \tag{2-1}$$

(compare with (1-8)-(1-9)) and introduce

$$\mathcal{W}_0^{\theta,\Theta} = \{H_- w_- + H_+ w_+\}_{w_{\pm} \in \mathscr{C}_c^{\infty}(\Omega)}, \quad w_{\pm} \text{ satisfying (2-1).} \tag{2-2}$$

For $\tau \geq 0$, we define the affine space

$$\mathcal{W}_\tau^{\theta,\Theta} = \{e^{\tau\varphi} w\}_{w \in \mathcal{W}_0^{\theta,\Theta}}. \tag{2-3}$$

For $v \in \mathcal{W}_\tau^{\theta,\Theta}$, we have $v = e^{\tau\varphi} w$ with $w \in \mathcal{W}_0^{\theta,\Theta}$, so that using the notation introduced in (1-4), (1-7), with $v_{\pm} = e^{\tau\varphi_{\pm}} w_{\pm}$, we have

$$v = H_- v_- + H_+ v_+, \tag{2-4}$$

and we see that the transmission conditions (2-1) on w read for v as

$$v_+ - v_- = \theta_\varphi, \quad \big\langle dv_+ - \tau v_+ d\varphi_+, A_+ \nu \big\rangle - \big\langle dv_- - \tau v_- d\varphi_-, A_- \nu \big\rangle = \Theta_\varphi \quad \text{at } \Sigma, \tag{2-5}$$

with

$$\theta_\varphi = e^{\tau\varphi_{|\Sigma}} \theta, \quad \Theta_\varphi = e^{\tau\varphi_{|\Sigma}} \Theta. \tag{2-6}$$

Observing that $e^{\tau\varphi_{\pm}} D e^{-\tau\varphi_{\pm}} = D + i\tau d\varphi_{\pm}$ for $w \in \mathcal{W}^{\theta,\Theta}$, we obtain

$$e^{\tau\varphi_{\pm}} \mathcal{L}_{\pm} w_{\pm} = e^{\tau\varphi_{\pm}} D \cdot A_{\pm} D e^{-\tau\varphi_{\pm}} v_{\pm} = (D + i\tau d\varphi_{\pm}) \cdot A_{\pm}(D + i\tau d\varphi_{\pm}) v_{\pm}.$$

We define

$$\mathcal{P}_{\pm} = (D + i\tau d\varphi_{\pm}) \cdot A_{\pm}(D + i\tau d\varphi_{\pm}). \tag{2-7}$$

Proposition 2.1. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}_\tau^{\theta,\Theta}$ be as in* (1-3), (1-5), *and* (2-3). *Then for any compact subset K of Ω, there exist a weight function φ satisfying* (1-12) *and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $v \in \mathcal{W}_\tau$ with* $\operatorname{supp} v \subset K$,

$$C\big(\|H_-\mathcal{P}_-v_-\|_{L^2(\mathbb{R}^n)} + \|H_+\mathcal{P}_+v_+\|_{L^2(\mathbb{R}^n)} + \mathcal{T}_{\theta,\Theta}\big)$$
$$\geq \tau^{3/2}|v_\pm|_{L^2(\Sigma)} + \tau^{1/2}|(\nabla v_\pm)|_{L^2(\Sigma)} + \tau^{3/2}\|v\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_+\nabla v_+\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\|H_-\nabla v_-\|_{L^2(\mathbb{R}^n)},$$

where $\mathcal{T}_{\theta,\Theta} = \tau^{3/2}|\theta_\varphi|_{L^2(\Sigma)} + \tau^{1/2}|\nabla_\Sigma\theta_\varphi|_{L^2(\Sigma)} + \tau^{1/2}|\Theta_\varphi|_{L^2(\Sigma)}$.

Here, ∇_Σ denotes the tangential gradient to Σ. The proof of this proposition will occupy a large part of the remainder of the article (Sections 3 and 4), as it implies the result of the following theorem, a nonhomogeneous version of Theorem 1.1.

Theorem 2.2. *Let $\Omega, \Sigma, \mathcal{L}, \mathcal{W}_0^{\theta,\Theta}$ be as in* (1-3), (1-5), *and* (2-2). *Then for any compact subset K of Ω, there exist a weight function φ satisfying* (1-12) *and positive constants C, τ_1 such that for all $\tau \geq \tau_1$ and all $w \in \mathcal{W}$ with* $\operatorname{supp} w \subset K$,

$$\begin{aligned}C\big(\big\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\big\|_{L^2(\mathbb{R}^n)} &+ \big\|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\big\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta}\big)\\ &\geq \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big(\big\|H_+e^{\tau\varphi}\nabla w_+\big\|_{L^2(\mathbb{R}^n)} + \big\|H_-e^{\tau\varphi}\nabla w_-\big\|_{L^2(\mathbb{R}^n)}\big)\\ &\qquad + \tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi}\nabla w_\pm|_{L^2(\Sigma)}, \end{aligned}\tag{2-8}$$

where $T_{\theta,\Theta} = \tau^{3/2}|e^{\tau\varphi_{|\Sigma}}\theta|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi_{|\Sigma}}\nabla_\Sigma\theta|_{L^2(\Sigma)} + \tau^{1/2}|e^{\tau\varphi_{|\Sigma}}\Theta|_{L^2(\Sigma)}$.

Theorem 1.1 corresponds to the case $\theta = \Theta = 0$, since by (1-10), we then have

$$\|e^{\tau\varphi}\mathcal{L}w\|_{L^2(\mathbb{R}^n)} = \big\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\big\|_{L^2(\mathbb{R}^n)} + \big\|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\big\|_{L^2(\mathbb{R}^n)}.$$

Remark 2.3. It is often useful to have such a Carleman estimate at hand for the case of nonhomogeneous transmission conditions, for example when one tries to patch such local estimates together in the neighborhood of the interface.

Here we derive local Carleman estimates. We can in fact consider a similar geometrical situation on a Riemannian manifold (with or without boundary) with a metric exhibiting jump discontinuities across interfaces. For the associated Laplace–Beltrami operator, the local estimates we derive can be patched together to yield a global estimate. We refer to [Le Rousseau and Robbiano 2011, Section 5] for such questions.

Proof that Proposition 2.1 implies Theorem 2.2. Replacing v by $e^{\tau\varphi}w$, we get

$$\begin{aligned}\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)} &+ \|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)} + T_{\theta,\Theta}\\ &\gtrsim \tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)} + \tau^{1/2}\big(\|H_+\nabla e^{\tau\varphi}w_+\|_{L^2(\mathbb{R}^n)} + \|H_-\nabla e^{\tau\varphi}w_-\|_{L^2(\mathbb{R}^n)}\big)\\ &\qquad + \tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)} + \tau^{1/2}|\nabla e^{\tau\varphi}w_\pm|_{L^2(\Sigma)}. \end{aligned}\tag{2-9}$$

Commuting ∇ with $e^{\tau\varphi}$ produces

$$\begin{aligned}C\big(\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)}+\|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)}+T_{\theta,\Theta}\big)\\ +C_1\tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)}+C_2\tau^{3/2}\big(|e^{\tau\varphi}w_{\pm|\Sigma}|_{L^2(\Sigma)}\big)\\ \ge\tau^{1/2}\|H_-e^{\tau\varphi}Dw_-\|_{L^2(\mathbb{R}^n)}+\tau^{1/2}\|H_+e^{\tau\varphi}Dw_+\|_{L^2(\mathbb{R}^n)}+\tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(\mathbb{R}^n)}\\ +\tau^{1/2}|e^{\tau\varphi}Dw_\pm|_{L^2(\Sigma)}+\tau^{3/2}|e^{\tau\varphi}w_\pm|_{L^2(\Sigma)},\end{aligned}$$

but by (2-9), we have

$$\begin{aligned}C_1\tau^{3/2}\|e^{\tau\varphi}w\|+C_2\tau^{3/2}|e^{\tau\varphi}w|\\ \le C\max(C_1,C_2)\big(\|H_-e^{\tau\varphi_-}\mathcal{L}_-w_-\|_{L^2(\mathbb{R}^n)}+\|H_+e^{\tau\varphi_+}\mathcal{L}_+w_+\|_{L^2(\mathbb{R}^n)}+T_{\theta,\Theta}\big),\end{aligned}$$

proving the implication. $\square$

2B. *Description in local coordinates.* Carleman estimates of types (1-13) and (2-8) can be handled locally, as they can be patched together. Assuming, as we may, that the hypersurface Σ is given locally by the equation $\{x_n=0\}$, we have, using the Einstein convention on repeated indices $j\in\{1,\dots,n-1\}$, and noting from the ellipticity condition that $a_{nn}>0$ (the matrix $A(x)=(a_{jk}(x))_{1\le j,k\le n}$),

$$\begin{aligned}\mathcal{L}&=D_na_{nn}D_n+D_na_{nj}D_j+D_ja_{jn}D_n+D_ja_{jk}D_k\\ &=D_na_{nn}\big(D_n+a_{nn}^{-1}a_{nj}D_j\big)+D_ja_{jn}D_n+D_ja_{jk}D_k.\end{aligned}$$

With $T=a_{nn}^{-1}a_{nj}D_j$, we have

$$\mathcal{L}=(D_n+T^*)a_{nn}(D_n+T)-T^*a_{nn}D_n-T^*a_{nn}T+D_ja_{jn}D_n+D_ja_{jk}D_k;$$

and since $T^*=D_ja_{nn}^{-1}a_{nj}$, we have $T^*a_{nn}D_n=D_ja_{nj}D_n=D_ja_{jn}D_n$ and

$$\mathcal{L}=(D_n+T^*)a_{nn}(D_n+T)+D_jb_{jk}D_k, \tag{2-10}$$

where the $(n-1)\times(n-1)$ matrix (b_{jk}) is positive-definite, since with $\xi'=(\xi_1,\dots,\xi_{n-1})$ and $\xi=(\xi',\xi_n)$,

$$\langle B\xi',\xi'\rangle=\sum_{1\le j,k\le n-1}b_{jk}\xi_j\xi_k=\langle A\xi,\xi\rangle,$$

where $a_{nn}\xi_n=-\sum_{1\le j\le n-1}a_{nj}\xi_j$. Note also that $b_{jk}=a_{jk}-(a_{nj}a_{nk}/a_{nn})$.

Remark 2.4. The positive-definite quadratic form B is the restriction of $\langle A\xi,\xi\rangle$ to the hyperplane $\mathcal{H}$ defined by $\{\langle A\xi,\xi\rangle,x_n\}=\partial_{\xi_n}\big(\langle A\xi,\xi\rangle\big)=0$, where $\{\cdot,\cdot\}$ stands for the Poisson bracket. In fact, the principal symbol of $\mathcal{L}$ is $\langle A(x)\xi,\xi\rangle$, and if Σ is defined by the equation $\psi(x)=0$ with $d\psi\ne0$ at Σ, we have

$$\tfrac12\big\{\langle A(x)\xi,\xi\rangle,\psi\big\}=\langle A(x)\xi,d\psi(x)\rangle,$$

so that $\mathcal{H}_x=(A(x)d\psi(x))^\perp=\big\{\xi\in T_x^*(\Omega),\langle\xi,A(x)d\psi(x)\rangle_{T_x^*(\Omega),T_x(\Omega)}=0\big\}$. When $x\in\Sigma$, that set does not depend on the choice of the defining function ψ of Σ, and we simply have

$$\mathcal{H}_x=(A(x)\nu(x))^\perp=\big\{\xi\in T_x^*(\Omega),\langle\xi,A(x)\nu(x)\rangle_{T_x^*(\Omega),T_x(\Omega)}=0\big\},$$

where $\nu(x)$ is the conormal vector to Σ at x (recall that from Remark 1.4, $\nu(x)$ is a cotangent vector at x, and $A(x)\nu(x)$ is a tangent vector at x). Now, for $x \in \Sigma$, we can restrict the quadratic form $A(x)$ to $\mathcal{H}_x$: this is the positive-definite quadratic form $B(x)$, providing a coordinate-free definition.

For $w \in \mathcal{W}_0^{\theta,\Theta}$, we have

$$\mathcal{L}_\pm w_\pm = (D_n + T_\pm^*) a_{nn}^\pm (D_n + T_\pm) w_\pm + D_j b_{jk}^\pm D_k w_\pm, \tag{2-11}$$

and the nonhomogeneous transmission conditions (2-1) read

$$w_+ - w_- = \theta, \quad a_{nn}^+ (D_n + T_+) w_+ - a_{nn}^- (D_n + T_-) w_- = \Theta \quad \text{at } \Sigma. \tag{2-12}$$

2C. *Pseudodifferential factorization on each side.* At first, we consider the weight function $\varphi = H_+\varphi_+ + H_-\varphi_-$, with $\varphi_\pm$ that solely depend on x_n. Later on, we shall allow for some dependency upon the tangential variables x' (see Section 4E). We define, for $m \in \mathbb{R}$, the class of tangential standard symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$\sup_{(x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1}} \langle \xi' \rangle^{-m+|\beta|} \big| \big(\partial_x^\alpha \partial_{\xi'}^\beta a\big)(x, \xi')\big| < \infty, \tag{2-13}$$

with $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$. Some basic properties of standard pseudodifferential operators are recalled in Section AA. Section 2B and formulae (2-7), (2-11) give

$$\mathcal{P}_\pm = \big(D_n + i\tau\varphi_\pm' + T_\pm^*\big) a_{nn}^\pm \big(D_n + i\tau\varphi_\pm' + T_\pm\big) + D_j b_{jk}^\pm D_k. \tag{2-14}$$

We define $m_\pm \in \mathcal{S}^1$ such that

$$\text{for } |\xi'| \ge 1, \quad m_\pm = \left(\frac{b_{jk}^\pm}{a_{nn}^\pm} \xi_j \xi_k\right)^{1/2}, \quad m_\pm \ge C\langle \xi' \rangle, \quad M_\pm = \operatorname{op}^w(m_\pm). \tag{2-15}$$

We then have $M_\pm^2 \equiv D_j b_{jk}^\pm D_k \mod \operatorname{op}(\mathcal{S}^1)$.

We define

$$\Psi^1 = \operatorname{op}(\mathcal{S}^1) + \tau \operatorname{op}(\mathcal{S}^0) + \operatorname{op}(\mathcal{S}^0) D_n. \tag{2-16}$$

Modulo the operator class Ψ^1, we may write

$$\mathcal{P}_+ \equiv \mathcal{P}_{E+} a_{nn}^+ \mathcal{P}_{F+}, \quad \mathcal{P}_- \equiv \mathcal{P}_{F-} a_{nn}^- \mathcal{P}_{E-}, \tag{2-17}$$

where

$$\mathcal{P}_{E\pm} = D_n + S_\pm + i\underbrace{(\tau\varphi_\pm' + M_\pm)}_{E_\pm}, \quad \mathcal{P}_{F\pm} = D_n + S_\pm + i\underbrace{(\tau\varphi_\pm' - M_\pm)}_{F_\pm}, \tag{2-18}$$

with

$$S_\pm = s^w(x, D'), \quad s_\pm = \sum_{1 \le j \le n-1} \frac{a_{nj}^\pm}{a_{nn}^\pm} \xi_j, \quad \text{so that } S_\pm^* = S_\pm, \quad S_\pm = T_\pm + \tfrac{1}{2} \operatorname{div} T_\pm, \tag{2-19}$$

where

$$T_\pm \text{ is the vector field } \sum_{1\le j\le n-1} \frac{a^\pm_{nj}}{i a^\pm_{nn}} \partial_j. \tag{2-20}$$

We denote by $f_\pm$ and $e_\pm$ the homogeneous principal symbols of $F_\pm$ and $E_\pm$, respectively, determined modulo the symbol class $\mathcal{S}^1 + \tau\mathcal{S}^0$. The transmission conditions (2-12) with our choice of coordinates read, at $x_n = 0$,

$$\begin{cases} v_+ - v_- = \theta_\varphi = e^{\tau\varphi_{|x_n=0}}\theta, \\ a^+_{nn}(D_n + T_+ + i\tau\varphi'_+)v_+ - a^-_{nn}(D_n + T_- + i\tau\varphi'_-)v_- = \Theta_\varphi = e^{\tau\varphi_{|x_n=0}}\Theta. \end{cases} \tag{2-21}$$

Remark 2.5. The Carleman estimate we shall prove is insensitive to terms in Ψ^1 in the conjugated operator $\mathcal{P}$. Formulae (2-17) and (2-18) for $\mathcal{P}_+$ and $\mathcal{P}_-$ will thus be the base of our analysis.

Remark 2.6. In [Le Rousseau and Robbiano 2010; 2011], the zero crossing of the roots of the symbol of $\mathcal{P}_\pm$, as seen as a polynomial in ξ_n, is analyzed. Here the factorization into first-order operators isolates each root. In fact, $f_\pm$ changes sign, and we shall impose a condition on the weight function at the interface to obtain a certain scheme for this change of sign; see Section 4.

2D. *Choice of weight function.* The weight function can be taken of the form

$$\varphi_\pm(x_n) = \alpha_\pm x_n + \frac{\beta x_n^2}{2}, \quad \alpha_\pm > 0, \quad \beta > 0. \tag{2-22}$$

The choice of the parameters $\alpha_\pm$ and β will be done below and will take into account the geometric data of our problem: $\alpha_\pm$ will be chosen to fulfill a geometric condition at the interface, and $\beta > 0$ will be chosen large. Here, we shall require $\varphi' \ge 0$, that is, we choose an "observation" region on the right-hand side of Σ. As we shall need β large, this amounts to working in a small neighborhood of the interface, that is, $|x_n|$ small. Also, we shall see below (Section 4E) that this weight can be perturbed by any smooth function with a small gradient.

Other choices for the weight functions are possible. In fact, two sufficient conditions can be put forward. We shall describe them now.

The operators $M_\pm$ have a principal symbol $m_\pm(x,\xi')$ in $\mathcal{S}^1$, which is positively homogeneous[8] of degree 1 and elliptic, that is, there exist $\lambda_0^\pm, \lambda_1^\pm$ positive such that for $|\xi'| \ge 1, x \in \mathbb{R}^n$,

$$\lambda_0^\pm|\xi'| \le m_\pm(x,\xi') \le \lambda_1^\pm|\xi'|. \tag{2-23}$$

We choose $\varphi'_{|x_n=0^\pm} = \alpha_\pm$ such that

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x',\xi' \\ |\xi'|\ge 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}. \tag{2-24}$$

[8] The homogeneity property means, as usual, $m_\pm(x,\rho\xi') = \rho m_\pm(x,\xi')$ for $\rho \ge 1$, $|\xi'| \ge 1$.

The consequence of this condition will be made clear in Section 4. We shall also prove that this condition is sharp in Section 5: a strong violation of this condition, namely, $\alpha_+/\alpha_- < \sup(m_+/m_-)_{|x_n=0}$, ruins any possibility of deriving a Carleman estimate of the form of Theorem 1.1.

Condition (2-24) concerns the behavior of the weight function at the interface. Conditions away from the interface are also needed. These conditions are more classical. From (2-14), the symbols of $\mathcal{P}_\pm$, modulo the symbol class $\mathcal{S}^1 + \tau\mathcal{S}^0 + \mathcal{S}^0\xi_n$, are given by $p_\pm(x,\xi,\tau) = a^\pm_{nn}\big(q_2^\pm + 2iq_1^\pm\big)$, with

$$q_2^\pm = (\xi_n + s_\pm)^2 + \frac{b^\pm_{jk}}{a^\pm_{nn}}\xi_j\xi_k - \tau^2(\varphi'_\pm)^2, \quad q_1^\pm = \tau\varphi'_\pm(\xi_n + s_\pm),$$

for φ solely depending on x_n, and from the construction of $m_\pm$, for $|\xi'| \geq 1$, we have

$$q_2^\pm = (\xi_n + s_\pm)^2 + m_\pm^2 - (\tau\varphi'_\pm)^2 = (\xi_n + s_\pm)^2 - f_\pm e_\pm. \tag{2-25}$$

We can then formulate the usual *subellipticity* condition, with *loss of a half-derivative*:

$$q_2^\pm = 0 \text{ and } q_1^\pm = 0 \implies \{q_2^\pm, q_1^\pm\} > 0, \tag{2-26}$$

which can be achieved by choosing β sufficiently large. It is important to note that this property is coordinate-free. For second-order elliptic operators with real smooth coefficients, this property is necessary and sufficient for a Carleman estimate such as that of Theorem 1.1 to hold (see [Hörmander 1963], or, for example, [Le Rousseau and Lebeau 2012]).

With the weight functions provided in (2-22), we choose $\alpha_\pm$ according to condition (2-24) and $\beta > 0$ large enough, and we restrict ourselves to a small neighborhood of Σ, that is, $|x_n|$ small, to have $\varphi' > 0$ and so that (2-26) is fulfilled.

Remark 2.7. Other "classical" forms for the weight function φ are also possible. For instance, one may use $\varphi(x_n) = e^{\beta\phi(x_n)}$ with the function ϕ depending solely on x_n of the form

$$\phi = H_-\phi_- + H_+\phi_+, \quad \phi_\pm \in \mathscr{C}_c^\infty(\mathbb{R}),$$

such that ϕ is *continuous* and $|\phi'_\pm| \geq C > 0$. In this case, property (2-24) can be fulfilled by properly choosing $\phi'_{|x_n=0^\pm}$, and (2-26) by choosing β sufficiently large.

Property (2-26) concerns the conjugated second-order operator. We show now that this condition concerns, in fact, only one of the first-order terms in the pseudodifferential factorization that we put forward above, namely, $\mathcal{P}_{F\pm}$.

Lemma 2.8. *There exist $C > 0$, $\tau_1 > 1$, and $\delta > 0$ such that for $\tau \geq \tau_1$,*

$$|f_\pm| \leq \delta\lambda \implies C^{-1}\tau \leq |\xi'| \leq C\tau \text{ and } \{\xi_n + s_\pm, f_\pm\} \geq C'\lambda,$$

with $\lambda^2 = \tau^2 + |\xi'|^2$.

See Appendix AB.1 for a proof. This is the form of the subellipticity condition, with loss of a half-derivative, that we shall use. This will be further highlighted by the estimates we derive in Section 3 and by the proof of the main theorem.

3. Estimates for first-order factors

Unless otherwise specified, the notation $\|\cdot\|$ will stand for the $L^2(\mathbb{R}^n)$-norm and $|\cdot|$ for the $L^2(\mathbb{R}^{n-1})$-norm. The $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{n-1})$ dot-products will be both denoted by $\langle\cdot,\cdot\rangle$.

In this section, we shall use the function space

$$\mathscr{S}_c(\mathbb{R}^n) = \{u \in \mathscr{S}(\mathbb{R}^n) : \operatorname{supp}(u) \subset \mathbb{R}^{n-1} \times (-L, L) \text{ for some } L > 0\}.$$

3A. ***Preliminary estimates.*** Most of our pseudodifferential arguments concern a calculus with large parameter $\tau \geq 1$: with

$$\lambda^2 = \tau^2 + |\xi'|^2, \tag{3-1}$$

we define for $m \in \mathbb{R}$ the class of tangential symbols $\mathcal{S}_\tau^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$, depending on the parameter $\tau \geq 1$, such that, for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$\sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} \lambda^{-m+|\beta|} \big|(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi', \tau)\big| < \infty. \tag{3-2}$$

Some basic properties of the calculus of the associated pseudodifferential operators are recalled in Section AA.2. We shall refer to this calculus as the semiclassical calculus (with a large parameter). In particular, we introduce the Sobolev norms

$$\|u\|_{\mathcal{H}^s} := \|\Lambda^s u\|_{L^2(\mathbb{R}^{n-1})}, \quad \text{with } \Lambda^s := \operatorname{op}(\lambda^s). \tag{3-3}$$

For $s \geq 0$, note that we have $\|u\|_{\mathcal{H}^s} \sim \tau^s \|u\|_{L^2(\mathbb{R}^{n-1})} + \|\langle D'\rangle^s u\|_{L^2(\mathbb{R}^{n-1})}$. Observe also that we have

$$\|u\|_{\mathcal{H}^s} \leq C\tau^{s-s'} \|u\|_{\mathcal{H}^{s'}}, \quad s \leq s'.$$

In what follows, we shall often refer implicitly to this inequality when invoking a large value for the parameter τ.

The operator $M_\pm$ is of pseudodifferential nature in the standard calculus. Observe, however, that in any region where $\tau \gtrsim |\xi'|$ the symbol, $m_\pm$ does not satisfy the estimates of $\mathcal{S}_\tau^1$. We shall circumvent this technical point by introducing a cut-off procedure.

Let $C_0, C_1 > 0$ be such that $\varphi' \geq C_0$ and

$$(M_\pm u, H^+ u) \leq C_1 \|H^+ u\|^2_{L^2(\mathbb{R}; H^{1/2}(\mathbb{R}^{n-1}))}. \tag{3-4}$$

We choose $\psi \in \mathscr{C}^\infty(\mathbb{R}^+)$ nonnegative such that $\psi = 0$ in $[0, 1]$ and $\psi = 1$ in $[2, +\infty)$. We introduce the Fourier multiplier

$$\psi_\epsilon(\tau, \xi') = \psi\left(\frac{\epsilon\tau}{\langle\xi\rangle}\right) \in \mathcal{S}_\tau^0, \quad \text{with } 0 < \epsilon \leq \epsilon_0, \tag{3-5}$$

such that $\tau \gtrsim \langle\xi'\rangle/\epsilon$ in its support. We choose ϵ_0 sufficiently small that $\operatorname{supp}(\psi_\epsilon)$ is disjoint from a conic neighborhood (for $|\xi'| \geq 1$) of the sets $\{f_\pm = 0\}$ (see Figure 3).

The following lemma states that we can obtain very natural estimates on both sides of the interface in the region $|\xi'| \ll \tau$, that is, for ϵ small. We refer to Section AB.2 for a proof.

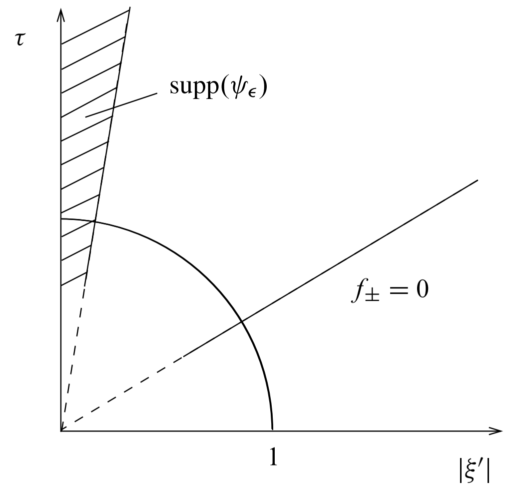

Figure 3. Relative positions of supp(ψ_ϵ) and the sets $\{f_\pm = 0\}$.

Lemma 3.1. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $0 < \epsilon_1 \leq \epsilon_0$ and $C > 0$ such that*

$$C \left\| H_+ \mathcal{A}_+ \mathrm{op}(\psi_\epsilon)\omega \right\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \geq \left| \mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^+} \right|_{\mathcal{H}^{\ell+1/2}} + \left\| H_+ \mathrm{op}(\psi_\epsilon)\omega \right\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$
$$C \left(\left\| H_- \mathcal{A}_- \mathrm{op}(\psi_\epsilon)\omega \right\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \left| \mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^-} \right|_{\mathcal{H}^{\ell+1/2}} \right) \geq \left\| H_- \mathrm{op}(\psi_\epsilon)\omega \right\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}$$

for $0 < \epsilon \leq \epsilon_1$, with $A_+ = \mathcal{P}_{E+}$ or $\mathcal{P}_{F+}$, $\mathcal{A}_- = \mathcal{P}_{E-}$ or $\mathcal{P}_{F-}$, for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

3B. ***Positive imaginary part on a half-line.*** We have the following estimates for the operators $\mathcal{P}_{E+}$ and $\mathcal{P}_{E-}$.

Lemma 3.2. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C \| H_+ \mathcal{P}_{E_+} \omega \|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \geq |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+1/2}} + \| H_+ \omega \|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \| H_+ D_n \omega \|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \tag{3-6}$$

and

$$C \left(\| H_- \mathcal{P}_{E_-} \omega \|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^-}|_{\mathcal{H}^{\ell+1/2}} \right) \geq \| H_- \omega \|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \| H_+ D_n \omega \|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \tag{3-7}$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

The first estimate, in $\mathbb{R}_+$, is of very good quality, as both the trace and the volume norms are dominated: we have a perfect elliptic estimate. In $\mathbb{R}_-$, we obtain an estimate of lesser quality. Observe also that no assumption on the weight function, apart from the positivity of φ', is used in the proof below.

Proof. Let ψ_ϵ be defined as in Section 3A. We let $\tilde{\psi} \in \mathscr{C}^\infty(\mathbb{R}^+)$ be nonnegative and such that $\tilde{\psi} = 1$ in $[4, +\infty)$ and $\tilde{\psi} = 0$ in $[0, 3]$. We then define $\tilde{\psi}_\epsilon$ according to (3-5), and we have $\tau \lesssim \langle \xi' \rangle$ in supp$(1 - \tilde{\psi}_\epsilon)$ and supp$(1 - \psi_\epsilon) \cap$ supp$(\tilde{\psi}_\epsilon) = \varnothing$. We set $\tilde{m}_\pm = m_\pm(1 - \tilde{\psi}_\epsilon)$ and observe that $\tilde{m}_\pm \in \mathcal{S}^1_\tau$. We define

$$\tilde{e}_\pm = \tau\varphi' + \tilde{m}_\pm \in \mathcal{S}^1_\tau, \quad \tilde{E}_\pm = \mathrm{op}^w(\tilde{e}_\pm).$$

From the definition of $\tilde{\psi}_\epsilon$, we have

$$\tilde{e}_\pm \geq C\lambda. \tag{3-8}$$

Next,

$$M_\pm \mathrm{op}(1-\psi_\epsilon)\omega = \mathrm{op}^w(\tilde{m}_\pm)\mathrm{op}(1-\psi_\epsilon)\omega + \mathrm{op}^w(m_\pm\tilde{\psi}_\epsilon)\mathrm{op}(1-\psi_\epsilon)\omega,$$

and since $m_\pm\tilde{\psi}_\epsilon \in \mathcal{S}^1$ and $1-\psi_\epsilon \in \mathcal{S}^0_\tau$, with the latter vanishing in a region $\langle\xi'\rangle \leq C\tau$, Lemma A.4 yields

$$M_\pm \mathrm{op}(1-\psi_\epsilon)\omega = \mathrm{op}^w(\tilde{m}_\pm)\mathrm{op}(1-\psi_\epsilon)\omega + R_1\omega, \quad \text{with } R_1 \in \mathrm{op}(\mathcal{S}^{-\infty}_\tau). \tag{3-9}$$

We set $u = \mathrm{op}(1-\psi_\epsilon)\omega$. For $s = 2\ell+1$, we compute

$$\begin{aligned} 2\,\mathrm{Re}\langle \mathcal{P}_{E_+}u, iH_+\Lambda^s u\rangle &= \langle i[D_n, H_+]u, \Lambda^s u\rangle + \langle i[S_+, \Lambda^s]u, H_+u\rangle + 2\,\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle \\ &\geq |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+1/2}} + 2\,\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle - C\|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})}. \end{aligned} \tag{3-10}$$

By (3-9), we have $E_+u = \tilde{E}_+u + R_1\omega$. This yields

$$\mathrm{Re}\langle E_+u, H_+\Lambda^s u\rangle + \|H_+\omega\|^2 \gtrsim \mathrm{Re}\langle \tilde{E}_+u, H_+\Lambda^s u\rangle \gtrsim \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

for τ sufficiently large, by (3-8) and Lemma A.2. We thus obtain

$$\mathrm{Re}\langle \mathcal{P}_{E_+}u, iH_+\Lambda^s u\rangle + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})} + \|H_+\omega\|^2 \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+1/2}} + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

With the Young inequality and taking τ sufficiently large, we then find

$$\|H_+\mathcal{P}_{E_+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| \gtrsim |u_{|x_n=0^+}|_{\mathcal{H}^{\ell+1/2}} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

We now invoke the corresponding estimate provided by Lemma 3.1,

$$\big\|H_+\mathcal{P}_{E+}\mathrm{op}(\psi_\epsilon)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim \big|\mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^+}\big|_{\mathcal{H}^{\ell+1/2}} + \|H_+\mathrm{op}(\psi_\epsilon)\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Adding the two estimates, with the triangle inequality we obtain

$$\begin{aligned} \big\|H_+\mathcal{P}_{E_+}\mathrm{op}(1-\psi_\epsilon)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\mathcal{P}_{E+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| \\ \gtrsim |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+1/2}} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}. \end{aligned}$$

Lemma A.4 gives $\big[\mathcal{P}_{E_+}, \mathrm{op}(1-\psi_\epsilon)\big] \in \mathrm{op}(\mathcal{S}^0_\tau)$. We thus have

$$\begin{aligned} \big\|H_+\mathcal{P}_{E_+}\mathrm{op}(1-\psi_\epsilon)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} &\lesssim \big\|H_+\mathrm{op}(1-\psi_\epsilon)\mathcal{P}_{E_+}\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \\ &\lesssim \|H_+\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}. \end{aligned}$$

By taking τ sufficiently large, we thus obtain

$$\|H_+\mathcal{P}_{E_+}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim |\omega_{|x_n=0^+}|_{\mathcal{H}^{\ell+1/2}} + \|H_+\omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}. \tag{3-11}$$

The term $\|H_+D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}$ can simply be introduced on the right-hand side of this estimate to yield (3-6), thanks to the form of the first-order operator $\mathcal{P}_{E_+}$. To obtain estimate (3-7), we compute $2\,\mathrm{Re}\langle \mathcal{P}_{E_-}\omega, iH_-\omega\rangle$. The argument is similar, but the trace term comes out with the opposite sign. □

For the operator $\mathcal{P}_{F+}$, we can also obtain a microlocal estimate. We place ourselves in a microlocal region where $f_+ = \tau\varphi^+ - m_+$ is positive. More precisely, let $\chi(x,\tau,\xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \le C\tau$ and $f_+ \ge C_1\lambda$ in $\operatorname{supp}(\chi)$, $C_1 > 0$, and $|\xi'| \ge C'\tau$ in $\operatorname{supp}(1-\chi)$.

Lemma 3.3. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \ge 1$ and $C > 0$ such that*

$$C\big(\big\| H_+\mathcal{P}_{F+}\operatorname{op}^w(\chi)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\|\big) \\ \ge \big|\operatorname{op}^w(\chi)\omega_{|x_n=0^+}\big|_{\mathcal{H}^{\ell+1/2}} + \big\|H_+\operatorname{op}^w(\chi)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \big\|H_+D_n\operatorname{op}^w(\chi)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)},$$

for $\tau \ge \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

As for (3-6) of Lemma 3.2, up to a harmless remainder term, we obtain an elliptic estimate in this microlocal region.

Proof. Let ψ_ϵ be as defined in Section 3A, and let $\tilde{\psi}_\epsilon$ be as in the proof of Lemma 3.2. We set

$$\tilde{f}_\pm = \tau\varphi' - \tilde{m}_\pm \in \mathcal{S}^1_\tau, \quad \tilde{F}_\pm = \operatorname{op}^w(\tilde{f}_\pm). \tag{3-12}$$

We have

$$\tilde{f}_\pm = \tau\varphi' - \tilde{m}_\pm = \tau\varphi' - m_\pm(1-\tilde{\psi}_\epsilon) = f_\pm + \tilde{\psi}_\epsilon m_\pm \ge f_\pm.$$

This gives $\tilde{f}_+ \ge C\lambda$ in $\operatorname{supp}(\chi)$.

We set $u = \operatorname{op}(1-\psi_\epsilon)\operatorname{op}^w(\chi)\omega$. Following the proof of Lemma 3.2, for $s = 2\ell+1$, we obtain

$$\operatorname{Re}\langle \mathcal{P}_{F+}u, iH_+\Lambda^s u\rangle + \|H_+\omega\|^2 + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})} \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+1/2}} + \operatorname{Re}\langle \tilde{F}_+u, H_+\Lambda^s u\rangle.$$

Let now $\tilde{\chi} \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with $\tilde{\chi} = 1$ on a neighborhood of $\operatorname{supp}(\chi)$. We then write

$$\tilde{f}_+ = \check{f}_+ + r, \quad \text{with } \check{f}_+ = \tilde{f}_+\tilde{\chi} + \lambda(1-\tilde{\chi}) \in \mathcal{S}^1_\tau, \quad r = (\tilde{f}_+ - \lambda)(1-\tilde{\chi}) \in \mathcal{S}^1_\tau.$$

As $\operatorname{supp}(1-\tilde{\chi}) \cap \operatorname{supp}(\chi) = \varnothing$, we find $r\sharp(1-\psi_\epsilon)\sharp\chi \in \mathcal{S}^{-\infty}_\tau$. Since $\check{f}_+ \ge C\lambda$ by construction, with Lemma A.2 we obtain

$$\operatorname{Re}\langle \mathcal{P}_{F+}u, iH_+\Lambda^s u\rangle + \|H_+\omega\|^2 + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})} \gtrsim |u_{|x_n=0^+}|^2_{\mathcal{H}^{\ell+1/2}} + \|H_+u\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

With the Young inequality, taking τ sufficiently large, we obtain

$$\|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| \gtrsim |u_{|x_n=0^+}|_{\mathcal{H}^{\ell+1/2}} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Invoking the corresponding estimate provided by Lemma 3.1 for $\operatorname{op}^w(\chi)\omega$,

$$\big\|H_+\mathcal{P}_{F+}\operatorname{op}(\psi_\epsilon)\operatorname{op}^w(\chi)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \\ \gtrsim \big|\operatorname{op}(\psi_\epsilon)\operatorname{op}^w(\chi)\omega_{|x_n=0^+}\big|_{\mathcal{H}^{\ell+1/2}} + \big\|H_+\operatorname{op}(\psi_\epsilon)\operatorname{op}^w(\chi)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

and arguing as in the end of the proof of Lemma 3.2, we obtain the result. □

For the operator $\mathcal{P}_{F-}$ we can also obtain a microlocal estimate. We place ourselves in a microlocal region where $f_- = \tau\varphi^- - m_-$ is positive. More precisely, let $\chi(x,\tau,\xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \le C\tau$ and $f_- \ge C_1\lambda$ in $\operatorname{supp}(\chi)$, $C_1 > 0$, and $|\xi'| \ge C'\tau$ in $\operatorname{supp}(1-\chi)$.

Lemma 3.4. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\big(\|H_-\mathcal{P}_{F_-}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_-\omega\| + \|H_-D_n\omega\| + |u_{|x_n=0^-}|_{\mathcal{H}^{\ell+1/2}}\big) \geq \|H_-u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}, \tag{3-13}$$

for $\tau \geq \tau_1$ and $u = a_{nn}^-\mathcal{P}_{E-}\mathrm{op}^w(\chi)\omega$ with $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. Let ψ_ϵ be defined as in Section 3A. We define $\tilde{f}_-$ and $\tilde{F}_-$ as in (3-12). We have $\tilde{f}_- \geq f_- \geq C\lambda$ in $\mathrm{supp}(\chi)$. We set $z = \mathrm{op}(1-\psi_\epsilon)u$ and for $s = 2\ell+1$, we compute

$$\begin{aligned} 2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle &= \langle i[D_n, H_-]z, \Lambda^s z\rangle + i\langle [S_-, \Lambda^s]z, H_-z\rangle + 2\,\mathrm{Re}\langle F_-z, H_-\Lambda^s z\rangle \\ &\geq -|z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+1/2}} + 2\,\mathrm{Re}\langle F_-z, H_-\Lambda^s z\rangle - C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})}. \end{aligned}$$

Arguing as in the proof of Lemma 3.2 (see (3-9) and (3-10)), we obtain

$$2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle + C\|H_-u\|^2 + |z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+1/2}} + C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})} \geq 2\,\mathrm{Re}\langle \tilde{F}_-z, H_-\Lambda^s z\rangle.$$

Let now $\tilde{\chi} \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with $\tilde{\chi} = 1$ on a neighborhood of $\mathrm{supp}(\chi)$. We then write

$$\tilde{f}_- = \check{f}_- + r, \quad \text{with } \check{f}_- = \tilde{f}_-\tilde{\chi} + \lambda(1-\tilde{\chi}) \in \mathcal{S}^1_\tau, \quad r = (\tilde{f}_- - \lambda)(1-\tilde{\chi}) \in \mathcal{S}^1_\tau.$$

As $\check{f}_- \geq C\lambda$ and $\mathrm{supp}(1-\tilde{\chi}) \cap \mathrm{supp}(\chi) = \varnothing$, with Lemma A.2 we obtain, for τ large,

$$\begin{aligned} 2\,\mathrm{Re}\langle \mathcal{P}_{F_-}z, iH_-\Lambda^s z\rangle + C\|H_-u\|^2 + |z_{|x_n=0^-}|^2_{\mathcal{H}^{\ell+1/2}} + C\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1/2})} + \|H_-\omega\|^2 + \|H_-D_n\omega\|^2 \\ \geq C'\|H_-z\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}. \end{aligned}$$

With the Young inequality and taking τ sufficiently large, we then find

$$\|H_-\mathcal{P}_{F_-}z\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_-u\| + |z_{|x_n=0^-}|_{\mathcal{H}^{\ell+1/2}} + \|H_-\omega\| + \|H_-D_n\omega\| \gtrsim \|H_-z\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

Invoking the corresponding estimate provided by Lemma 3.1 for u yields

$$\big\|H_-\mathcal{P}_{F_-}\mathrm{op}(\psi_\epsilon)u\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \big|\mathrm{op}(\psi_\epsilon)u_{|x_n=0^-}\big|_{\mathcal{H}^{\ell+1/2}} \gtrsim \big\|H_-\mathrm{op}(\psi_\epsilon)u\big\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

and arguing as in the end of Lemma 3.2, we obtain the result. □

3C. *Negative imaginary part on the negative half-line.* Here we place ourselves in a microlocal region where $f_- = \tau\varphi^- - m_-$ is negative. More precisely, let $\chi(x, \tau, \xi') \in \mathcal{S}^0_\tau$ be such that $|\xi'| \geq C\tau$ and $f_- \leq -C_1\lambda$ in $\mathrm{supp}(\chi)$, $C_1 > 0$. We have the following lemma, whose form is adapted to our needs in the next section. Up to harmless remainder terms, this can also be considered as a good elliptic estimate.

Lemma 3.5. *There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\big(\|H_-\mathcal{P}_{F_-}u\| + \|H_-\omega\| + \|H_-D_n\omega\|\big) \geq |u_{|x_n=0^-}|_{\mathcal{H}^{1/2}} + \|H_-u\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{3-14}$$

for $\tau \geq \tau_1$ and $u = a_{nn}^-\mathcal{P}_{E-}\mathrm{op}^w(\chi)\omega$ with $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. We compute

$$\begin{aligned}2\operatorname{Re}\langle \mathcal{P}_{F_-}u, -iH_-\Lambda^1 u\rangle &= \langle i[D_n, -H_-]u, \Lambda^1 u\rangle - i\langle [S_-, \Lambda^1]u, H_-u\rangle + 2\operatorname{Re}\langle -F_-u, H_-\Lambda^1 u\rangle \\ &\geq |u_{|x_n=0^-}|^2_{\mathcal{H}^{1/2}} + 2\operatorname{Re}\langle -F_-u, H_-\Lambda^1 u\rangle - C\|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^{1/2})}.\end{aligned}$$

Let now $\tilde{\chi} \in \mathcal{S}^0_\tau$ satisfy the same properties as χ, with $\tilde{\chi} = 1$ on a neighborhood of $\operatorname{supp}(\chi)$. We then write

$$f_- = \check{f}_- + r, \quad \text{with } \check{f}_- = f_-\tilde{\chi} - \lambda(1-\tilde{\chi}), \quad r = (f_- + \lambda)(1-\tilde{\chi}).$$

Observe that $f_-\tilde{\chi} \in \mathcal{S}^1_\tau$ because of the support of $\tilde{\chi}$. Hence $\check{f}_- \in \mathcal{S}^1_\tau$. As $-\check{f}_- \geq C\lambda$, with Lemma A.2 we obtain, for τ large, $\operatorname{Re}\langle -\operatorname{op}^w(\check{f}_-)u, H_-\Lambda^1 u\rangle \gtrsim \|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^1)}$. Note that r does not satisfy the estimates of the semiclassical calculus because of the term $m_-(1-\tilde{\chi})$. However, we have

$$\operatorname{op}^w(r)u = \operatorname{op}^w(r)a^-_{nn}\operatorname{op}^w(\chi)D_n\omega + \operatorname{op}^w(r)a^-_{nn}S_-\operatorname{op}^w(\chi)\omega + i\operatorname{op}^w(r)a^-_{nn}E_-\operatorname{op}^w(\chi)\omega.$$

Applying Lemma A.4 and using that $1-\tilde{\chi} \in \mathcal{S}^0_\tau \subset \mathcal{S}^0$ yields

$$\operatorname{op}^w(r)u = R\omega \qquad \text{with } R \in \operatorname{op}(\mathcal{S}^1_\tau)D_n + \operatorname{op}(\mathcal{S}^2_\tau).$$

As $\operatorname{supp}(1-\tilde{\chi}) \cap \operatorname{supp}(\chi) = \varnothing$, the composition formula (A-7) (which is valid in this case — see Lemma A.4) yields $R \in \operatorname{op}(\mathcal{S}^{-\infty}_\tau)D_n + \operatorname{op}(\mathcal{S}^{-\infty}_\tau)$. We thus find, for τ sufficiently large,

$$\operatorname{Re}\langle \mathcal{P}_{F_-}u, -iH_-\Lambda^1 u\rangle + \|H_-\omega\|^2 + \|H_-D_n\omega\|^2 \gtrsim |u_{|x_n=0^-}|^2_{\mathcal{H}^{1/2}} + \|H_-u\|^2_{L^2(\mathbb{R};\mathcal{H}^1)},$$

and we conclude with the Young inequality. □

3D. ***Increasing imaginary part on a half-line.*** Here we allow the symbols $f_\pm$ to change sign. For the first-order factor $\mathcal{P}_{F_\pm}$, this will lead to an estimate that exhibits a loss of a half-derivative, as can be expected.

Let ψ_ϵ be as defined in Section 3A, and let $\tilde{\psi}_\epsilon$ be as in the proof of Lemma 3.2. We define $\tilde{f}_\pm$ and $\tilde{F}_\pm$ as in (3-12), and set $\tilde{\mathcal{P}}_{F_\pm} = D_n + S_\pm + i\tilde{F}_\pm$.

As $\operatorname{supp}(\tilde{\psi}_\epsilon)$ remains away from the sets $\{f_\pm = 0\}$, the subellipticity property of Lemma 2.8 is preserved for $\tilde{f}_\pm$ in place of $f_\pm$. We shall use the following inequality.

Lemma 3.6. *There exist $C > 0$ such that for $\mu > 0$ sufficiently large, we have*

$$\rho_\pm = \mu\tilde{f}^2_\pm + \tau\{\xi_n + s_\pm, \tilde{f}_\pm\} \geq C\lambda^2,$$

with $\lambda^2 = \tau^2 + |\xi'|^2$.

Proof. If $|\tilde{f}_\pm| \leq \delta\lambda$, for δ small, then $\tilde{f}_\pm = f_\pm$ and $\tau\{\xi_n + s_\pm, \tilde{f}_\pm\} \geq C\lambda^2$, by Lemma 2.8.

If $|\tilde{f}_\pm| \geq \delta\lambda$, observing that $\tau\{\xi_n + s_\pm, \tilde{f}_\pm\} \in \tau\mathcal{S}^1_\tau \subset \mathcal{S}^2_\tau$, we obtain $\rho_\pm \geq C\lambda^2$, by choosing μ sufficiently large. □

We now prove the following estimate for $\mathcal{P}_{F_\pm}$.

Lemma 3.7. *Let $\ell \in \mathbb{R}$. There exist $\tau_1 \geq 1$ and $C > 0$ such that*

$$C\big(\|H_\pm \mathcal{P}_{F_\pm}\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + |\omega_{|x_n=0^\pm}|_{\mathcal{H}^{\ell+1/2}}\big) \geq \tau^{-1/2}\big(\|H_\pm \omega\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})} + \|H_\pm D_n\omega\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}\big),$$

for $\tau \geq \tau_1$ and $\omega \in \mathscr{S}_c(\mathbb{R}^n)$.

Proof. We set $u = \mathrm{op}(1-\psi_\epsilon)\omega$. We start by invoking (3-9), and the fact that $[\tilde{\mathcal{P}}_{F+}, \Lambda^\ell] \in \mathrm{op}(\mathcal{S}_\tau^\ell)$, and write

$$\begin{aligned} \|H_+\tilde{\mathcal{P}}_{F+}\Lambda^\ell u\| &\lesssim \|H_+\Lambda^\ell \tilde{\mathcal{P}}_{F+}u\| + \|H_+[\tilde{\mathcal{P}}_{F+}, \Lambda^\ell]u\| \\ &\lesssim \|H_+\tilde{\mathcal{P}}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \\ &\lesssim \|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| + \|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)}. \end{aligned} \tag{3-15}$$

We set $u_\ell = \Lambda^\ell u$. We then have

$$\begin{aligned} \|H_+\tilde{\mathcal{P}}_{F+}u_\ell\|^2 &= \big\|H_+(D_n+S_+)u_\ell\big\|^2 + \|H_+\tilde{F}_+u_\ell\|^2 + 2\,\mathrm{Re}\big\langle (D_n+S_+)u_\ell, iH_+\tilde{F}_+u_\ell\big\rangle \\ &\geq \tau^{-1}\,\mathrm{Re}\big\langle\big(\mu\tilde{F}_+^2 + i\tau\big[D_n+S_+, \tilde{F}_+\big]\big)u_\ell, H_+u_\ell\big\rangle + \langle i[D_n, H_+]u_\ell, \tilde{F}_+u_\ell\rangle, \end{aligned}$$

if $\mu\tau^{-1} \leq 1$. As the principal symbol (in the semiclassical calculus) of $\mu\tilde{F}_+^2 + i\tau\big[D_n+S_+, \tilde{F}_+\big]$ is $\rho_+ = \mu\tilde{f}_+^2 + \tau\{\xi_n + s_+, \tilde{f}_+\}$, Lemmata 3.6 and A.2 yield

$$\|H_+\tilde{\mathcal{P}}_{F+}u_\ell\|^2 + |u_\ell|^2_{\mathcal{H}^{1/2}} \gtrsim \tau^{-1}\|H_+u_\ell\|^2_{L^2(\mathbb{R};\mathcal{H}^1)},$$

for μ large, that is, τ large. With (3-15) we obtain, for τ sufficiently large,

$$\|H_+\mathcal{P}_{F+}u\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} + \|H_+\omega\| + |u|_{\mathcal{H}^{\ell+1/2}} \gtrsim \tau^{-1/2}\|H_+u\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.$$

We now invoke the corresponding estimate provided by Lemma 3.1,

$$\big\|H_+\mathcal{P}_{F+}\mathrm{op}(\psi_\epsilon)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^\ell)} \gtrsim \big|\mathrm{op}(\psi_\epsilon)\omega_{|x_n=0^+}\big|_{\mathcal{H}^{\ell+1/2}} + \big\|H_+\mathrm{op}(\psi_\epsilon)\omega\big\|_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})},$$

and we proceed as in the end of the proof of Lemma 3.2 to obtain the result for $\mathcal{P}_{F+}$. The same computation and arguments, *mutatis mutandis*, give the result for $\mathcal{P}_{F-}$. ☐

4. Proof of the Carleman estimate

With the estimates for the first-order factors obtained in Section 3, we shall now prove Proposition 2.1, which gives the result of Theorems 1.1 and 2.2 (see the end of Section 2A).

The Carleman estimates we prove are well known away from the interface $\{x_n = 0\}$. Since local Carleman estimates can be patched together, we may thus assume that the compact set K in the statements of Theorems 1.1 and 2.2 is such that $|x_n|$ is sufficiently small for the arguments below to be carried out. Hence, we shall assume the functions $w_\pm$ in Theorem 2.2 (resp. $v_\pm$ in Proposition 2.1) have small supports near 0 in the x_n-direction.

4A. ***The geometric hypothesis.*** In Section 2D, we chose a weight function φ that satisfies the condition

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}, \quad \alpha_\pm = \partial_{x_n}\varphi_{\pm|x_n=0^\pm}. \tag{4-1}$$

Let us explain the immediate consequences of that assumption. First of all, we can reformulate it by saying that

$$\frac{\alpha_+}{\alpha_-} = \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}} \quad \text{for some } \sigma > 1. \tag{4-2}$$

Let $1 < \sigma_0 < \sigma$.

Consider $(x',\xi',\tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{+,*}$, $|\xi'| \geq 1$, such that

$$\tau\alpha_+ \geq \sigma_0 m_+(x',\xi')_{|x_n=0^+}. \tag{4-3}$$

We then have

$$\tau\alpha_+ - m_+(x',\xi')_{|x_n=0^+} \geq \tau\alpha_+(1-\sigma_0^{-1}) \geq \frac{\sigma_0-1}{2\sigma_0}\tau\alpha_+ + \frac{\sigma_0-1}{2} m_+(x',\xi')_{|x_n=0^+} \geq C\lambda. \tag{4-4}$$

We choose τ sufficiently large, say $\tau \geq \tau_2 > 0$, that this inequality remains true for $0 \leq |\xi'| \leq 2$. It also remains true for $x_n > 0$ small. As $f_+ = \tau(\varphi' - \alpha_+) + \tau\alpha_+ - m_+(x,\xi')$, for $|x_n|$ small, we obtain $f_+ \geq C\lambda$, which means that f_+ is elliptic positive in that region.

Second, if we now have $|\xi'| \geq 1$ and

$$\tau\alpha_+ \leq \sigma m_+(x',\xi')_{|x_n=0^+}, \tag{4-5}$$

we get that $\tau\alpha_- \leq \sigma^{-1} m_-(x',\xi')_{|x_n=0^-}$: otherwise we would have $\tau\alpha_- > \sigma^{-1}m_-(x',\xi')_{|x_n=0^-}$ and thus

$$\frac{m_-(x',\xi')_{|x_n=0^-}}{\sigma\alpha_-} < \tau \leq \frac{\sigma m_+(x',\xi')_{|x_n=0^+}}{\alpha_+},$$

implying

$$\frac{\alpha_+}{\alpha_-} < \sigma^2 \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}} \leq \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\geq 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}} = \frac{\alpha_+}{\alpha_-}, \quad \text{which is impossible.}$$

As a consequence, we have

$$\begin{aligned} \tau\alpha_- - m_-(x',\xi')_{|x_n=0^-} &\leq -m_-(x',\xi')_{|x_n=0^-}\frac{(\sigma-1)}{\sigma} \\ &\leq -m_-(x',\xi')_{|x_n=0^-}\frac{(\sigma-1)}{2\sigma} - \frac{(\sigma-1)}{2}\tau\alpha_- \leq -C\lambda. \end{aligned} \tag{4-6}$$

With $f_- = \tau(\varphi' - \alpha_-) + \tau\alpha_- - m_-(x,\xi')$, for $|x_n|$ sufficiently small, we obtain $f_- \leq -C\lambda$, which means that f_- is elliptic negative in that region.

We have thus proven the following result.

Lemma 4.1. *Let $\sigma > \sigma_0 > 1$ and $\alpha_\pm$ be positive numbers such that* (4-2) *holds. For $s > 0$, we define the following "cones" in $\mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^*_+$:*

$$\Gamma_s = \big\{(x', \tau, \xi') : |\xi'| < 2 \text{ or } \tau\alpha_+ > s m_+(x', \xi')_{|x_n=0^+}\big\},$$
$$\widetilde{\Gamma}_s = \big\{(x', \tau, \xi') : |\xi'| > 1 \text{ and } \tau\alpha_+ < s m_+(x', \xi')_{|x_n=0^+}\big\}.$$

For $|x_n|$ sufficiently small and τ sufficiently large, we have $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+ = \Gamma_{\sigma_0} \cup \widetilde{\Gamma}_\sigma$ and

$$\Gamma_{\sigma_0} \subset \big\{(x', \xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+ :\ f_+(x, \xi') \geq C\lambda,\ \text{if } 0 \leq x_n \text{ small}\big\},$$
$$\widetilde{\Gamma}_\sigma \subset \big\{(x', \xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^*_+ :\ f_-(x, \xi') \leq -C\lambda,\ \text{if } |x_n| \text{ small},\ x_n \leq 0\big\}.$$

N.B. The key result for the sequel is that property (4-1) is securing the fact that the overlapping open regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$ are such that on Γ_{σ_0}, f_+ is elliptic positive and on $\widetilde{\Gamma}_\sigma$, f_- is elliptic negative. Using a partition of unity and symbolic calculus, we shall be able to assume that either F_+ is elliptic positive, or F_- is elliptic negative.

N.B. Note that we can keep the preliminary cut-off region of Section 3A away from the overlap of Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$ by choosing ϵ sufficiently small (see (3-5) and Lemma 3.1). This is illustrated in Figure 4.

With the two overlapping "cones", for $\tau \geq \tau_2$, we introduce a homogeneous partition of unity

$$1 = \chi_0(x', \xi', \tau) + \chi_1(x', \xi', \tau), \qquad \underbrace{\operatorname{supp}(\chi_0) \subset \Gamma_{\sigma_0}}_{|\xi'| \lesssim \tau,\ f_+ \text{ elliptic} > 0}, \qquad \underbrace{\operatorname{supp}(\chi_1) \subset \widetilde{\Gamma}_\sigma}_{|\xi'| \gtrsim \tau,\ f_- \text{ elliptic} < 0}. \tag{4-7}$$

Note that χ_j', $j = 0, 1$, are supported at the overlap of the regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$, where $\tau \lesssim |\xi'|$. Hence, χ_0 and χ_1 satisfy the estimates of the semiclassical calculus and we have $\chi_0, \chi_1 \in \mathcal{S}^0_\tau$. With these symbols

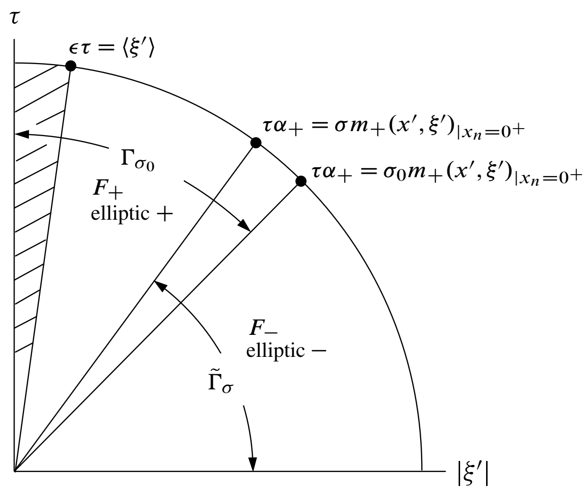

Figure 4. The overlapping microlocal regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$ in the $\tau, |\xi'|$ plane above a point x'. Dashed is the region used in Section 3A, which is kept away from the overlap of Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$.

we associate the operators

$$\Xi_j = \mathrm{op}^w(\chi_j), \ j = 0, 1, \quad \text{and we have } \Xi_0 + \Xi_1 = \mathrm{Id}. \tag{4-8}$$

Remark 4.2. Here we have chosen to let χ_0 and χ_1 (resp. Ξ_0 and Ξ_1) be independent of x_n. As the functions $v_\pm$ have supports in which $|x_n|$ is small (see the introductory paragraph of this section), we can further introduce a cut-off in the x_n direction. The lemmata of Section 3 can then be applied directly.

By the transmission conditions (2-21), we find

$$\Xi_j v_{+|x_n=0^+} - \Xi_j v_{-|x_n=0^-} = \Xi_j \theta_\varphi \tag{4-9}$$

and

$$\begin{aligned} a_{nn}^+ \big(D_n + T_+ + i\tau\varphi_+'\big)\Xi_j v_{+|x_n=0^+} - a_{nn}^- \big(D_n + T_- + i\tau\varphi_-'\big)\Xi_j v_{-|x_n=0^-} \\ = \Xi_j \Theta_\varphi + \mathrm{op}^w(\kappa_0) v_{|x_n=0^+} + \mathrm{op}^w(\tilde\kappa_0)\theta_\varphi, \quad j = 0, 1, \end{aligned}$$

with $\kappa_0, \tilde\kappa_0 \in \mathcal{S}_\tau^0$ that originate from commutators and (4-9). Defining

$$\mathcal{V}_{j,\pm} = a_{nn}^\pm \big(D_n + S_\pm + i\tau\varphi_\pm'\big)\Xi_j v_{\pm|x_n=0^\pm} \tag{4-10}$$

and recalling (2-19), we find

$$\mathcal{V}_{j,+} - \mathcal{V}_{j,-} = \Xi_j \Theta_\varphi + \mathrm{op}^w(\kappa_1) v_{|x_n=0^+} + \mathrm{op}^w(\tilde\kappa_1)\theta_\varphi, \quad \kappa_1, \tilde\kappa_1 \in \mathcal{S}_\tau^0. \tag{4-11}$$

We shall now prove microlocal Carleman estimates in the regions Γ_{σ_0} and $\widetilde{\Gamma}_\sigma$.

4B. ***Region $\boldsymbol{\Gamma_{\sigma_0}}$: both roots are positive on the positive half-line.*** On the one hand, by Lemma 3.2, we have

$$\big\| H_+ \mathcal{P}_+ \Xi_0 v_+ \big\| \gtrsim \big| \mathcal{V}_{0,+} - i a_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{1/2}} + \big\| H_+ \mathcal{P}_{F+} \Xi_0 v_+ \big\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \tag{4-12}$$

where the operator $\mathcal{P}_+$ is defined in (2-7) (see also (2-17)). The positive ellipticity of F_+ on the $\operatorname{supp}\chi_0 \cap \operatorname{supp}(v_+)$ allows us to reiterate the estimate by Lemma 3.3 to obtain

$$\begin{aligned} \big\| H_+ \mathcal{P}_+ \Xi_0 v_+ \big\| + \| H_+ v_+ \| \gtrsim \big| \mathcal{V}_{0,+} - i a_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{1/2}} + \big| \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{3/2}} \\ + \big\| H_+ \Xi_0 v_+ \big\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \big\| H_+ D_n \Xi_0 v_+ \big\|_{L^2(\mathbb{R};\mathcal{H}^1)}. \end{aligned}$$

Since we also have

$$|\mathcal{V}_{0,+}|_{\mathcal{H}^{1/2}} \lesssim \big| \mathcal{V}_{0,+} - i a_{nn}^+ M_+ \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{1/2}} + \big| \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{3/2}}, \tag{4-13}$$

writing the $\mathcal{H}^{1/2}$ norm as $|\,.\,|_{\mathcal{H}^{1/2}} \sim \tau^{1/2} |\,.\,|_{L^2} + |\,.\,|_{H^{1/2}}$ and using the regularity of $M_+ \in \mathrm{op}(\mathcal{S}^1)$ in the standard calculus, we obtain

$$\begin{aligned} \big\| H_+ \mathcal{P}_+ \Xi_0 v_+ \big\| + \| H_+ v_+ \| \gtrsim |\mathcal{V}_{0,+}|_{\mathcal{H}^{1/2}} + \big| \Xi_0 v_{+|x_n=0^+} \big|_{\mathcal{H}^{3/2}} \\ + \big\| H_+ \Xi_0 v_+ \big\|_{L^2(\mathbb{R};\mathcal{H}^2)} + \big\| H_+ \Xi_0 D_n v_+ \big\|_{L^2(\mathbb{R};\mathcal{H}^1)}. \end{aligned} \tag{4-14}$$

On the other hand, with Lemma 3.7, we have, for $k=0$ or $k=\frac{1}{2}$,

$$\left\| H_-\mathcal{P}_-\Xi_0 v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{-k})}+\left|\mathcal{V}_{0,-}+ia_{nn}^- M_-\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{1/2-k}}\gtrsim\tau^{-1/2}\left\| H_-\mathcal{P}_{E-}\Xi_0 v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}.$$

This gives

$$\left\| H_-\mathcal{P}_-\Xi_0 v_-\right\|+\tau^k\left|\mathcal{V}_{0,-}+ia_{nn}^- M_-\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{1/2-k}}\gtrsim\tau^{k-1/2}\left\| H_-\mathcal{P}_{E-}\Xi_0 v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})},$$

which with Lemma 3.2 yields

$$\begin{aligned}\left\| H_-\mathcal{P}_-\Xi_0 v_-\right\|+\tau^k\left|\mathcal{V}_{0,-}+ia_{nn}^- M_-\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{1/2-k}}+\tau^{k-1/2}\left|\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{3/2-k}}&\\ \gtrsim\tau^{k-1/2}\big(\left\| H_-\Xi_0 v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})}+\left\| H_-\Xi_0 D_n v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\big).&\end{aligned}$$

Arguing as for (4-13), we find

$$\begin{aligned}\left\| H_-\mathcal{P}_-\Xi_0 v_-\right\|+\tau^k|\mathcal{V}_{0,-}|_{\mathcal{H}^{1/2-k}}+\tau^k\left|\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{3/2-k}}&\\ \gtrsim\tau^{k-1/2}\big(\left\| H_-\Xi_0 v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})}+\left\| H_-\Xi_0 D_n v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\big).&\end{aligned}\tag{4-15}$$

Now, from the transmission conditions (4-9)–(4-11), by adding ε(4-15) + (4-14), we obtain

$$\begin{aligned}&\left\| H_-\mathcal{P}_-\Xi_0 v_-\right\|+\left\| H_+\mathcal{P}_+\Xi_0 v_+\right\|+\tau^k\big(|\theta_\varphi|_{\mathcal{H}^{3/2-k}}+|\Theta_\varphi|_{\mathcal{H}^{1/2-k}}+|v_{|x_n=0^+}|_{\mathcal{H}^{1/2-k}}\big)+\|H_+v_+\|\\ &\quad\gtrsim\tau^k\big(|\mathcal{V}_{0,-}|_{\mathcal{H}^{1/2-k}}+|\mathcal{V}_{0,+}|_{\mathcal{H}^{1/2-k}}+\left|\Xi_0 v_{-|x_n=0^-}\right|_{\mathcal{H}^{3/2-k}}+\left|\Xi_0 v_{+|x_n=0^+}\right|_{\mathcal{H}^{3/2-k}}\big)\\ &\qquad+\tau^{k-1/2}\big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})}+\left\| H_-\Xi_0 D_n v_-\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}+\left\| H_+\Xi_0 D_n v_+\right\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\big),\end{aligned}$$

by choosing $\varepsilon>0$ sufficiently small and τ sufficiently large. Finally, recalling the form of $\mathcal{V}_{0,\pm}$ and arguing as for (4-13), we obtain

$$\begin{aligned}&\|H_-\mathcal{P}_-\Xi_0 v_-\|+\|H_+\mathcal{P}_+\Xi_0 v_+\|+\tau^k\big(|\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}}+|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}}+|v_{|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\big)+\|H_+v_+\|\\ &\quad\gtrsim\tau^k\Big(|\Xi_0 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}}+|\Xi_0 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}+|\Xi_0 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}}+|\Xi_0 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}}\Big)\\ &\qquad+\tau^{k-\frac{1}{2}}\Big(\|\Xi_0 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})}+\|H_-\Xi_0 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}+\|H_+\Xi_0 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big),\end{aligned}\tag{4-16}$$

for $k=0$ or $k=\frac{1}{2}$.

Remark 4.3. In the case $k=0$, recalling the form of the second-order operators $\mathcal{P}_\pm$, we can estimate the additional terms $\tau^{-1/2}\left\| H_\pm\Xi_0 D_n^2 v_\pm\right\|$.

4C. ***Region $\tilde{\Gamma}_\sigma$: only one root is positive on the positive half-line.*** This case is more difficult a priori, since we cannot expect to control $v_{|x_n=0^+}$ directly from the estimates of the first-order factors. Nevertheless, when the positive ellipticity of F_+ is violated, F_- is elliptic negative: this is the result of our main geometric assumption in Lemma 4.1.

As in (4-12), we have

$$\left\| H_+\mathcal{P}_+\Xi_1 v_+\right\|\gtrsim\left|\mathcal{V}_{1,+}-ia_{nn}^+ M_+\Xi_1 v_{+|x_n=0^+}\right|_{\mathcal{H}^{1/2}}+\left\| H_+\mathcal{P}_{F+}\Xi_1 v_+\right\|_{L^2(\mathbb{R};\mathcal{H}^1)},$$

and using Lemma 3.5 for the negative half-line, we have

$$\begin{aligned}&\|H_-\mathcal{P}_-\Xi_1 v_-\| + \|H_- v_-\| + \|H_- D_n v_-\| \\ &\qquad\gtrsim \big|\mathcal{V}_{1,-} + i a_{nn}^- M_- \Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2}} + \big\|H_-\mathcal{P}_{E-}\Xi_1 v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^1)}.\end{aligned}$$

A quick glance at the above estimates shows that none could be iterated in a favorable manner, since F_+ could be negative on the positive half-line and E_- is indeed positive on the negative half-line. We have to use the additional information given by the transmission conditions. From the above inequalities, we control

$$\tau^k\Big(\big|\mathcal{V}_{1,-} + i a_{nn}^- M_- \Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|-\mathcal{V}_{1,+} + i a_{nn}^+ M_+ \Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}\Big)$$

for $k = 0$ or $\frac{1}{2}$, which, by the transmission conditions (4-9)–(4-11), implies the control of

$$\begin{aligned}&\tau^k\Big|\mathcal{V}_{1,-} - \mathcal{V}_{1,+} + i a_{nn}^- M_- \Xi_1 v_{-|x_n=0^-} + i a_{nn}^+ M_+ \Xi_1 v_{+|x_n=0^+}\Big|_{\mathcal{H}^{1/2-k}} \\ &\geq \tau^k\big|(a_{nn}^- M_- + a_{nn}^+ M_+)\Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} - C\tau^k\big(|\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}\big).\end{aligned}$$

Let now $\tilde{\chi}_1 \in \mathcal{S}_\tau^0$ satisfy the same properties as χ_1, with $\tilde{\chi}_1 = 1$ on a neighborhood of $\operatorname{supp}(\chi_1)$. We then write

$$m_\pm = \check{m}_\pm + r, \quad \text{with } \check{m}_\pm = m_\pm\tilde{\chi}_1 + \lambda(1-\tilde{\chi}_1), \quad r = (m_\pm + \lambda)(1-\tilde{\chi}_1).$$

We have $\check{m}_\pm \geq C\lambda$ and $\check{m}_\pm \in \mathcal{S}_\tau^1$ because of the support of $\tilde{\chi}_1$. Because of the supports of $1-\tilde{\chi}_1$ and χ_1, in particular $\tau \lesssim |\xi'|$ in $\operatorname{supp}(\chi_1)$, Lemma A.4 yields $r\sharp\chi_1 \in \mathcal{S}_\tau^{-\infty}$. With Lemma A.2 and (4-9), we thus obtain

$$\begin{aligned}&\big|\mathcal{V}_{1,-} + i a_{nn}^- M_- \Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|-\mathcal{V}_{1,+} + i a_{nn}^+ M_+ \Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} \\ &\quad + |\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} \gtrsim \big|\Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} + \big|\Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}}.\end{aligned}$$

From the form of $\mathcal{V}_{1,+}$ we obtain

$$\begin{aligned}&\big|\mathcal{V}_{1,-} + i a_{nn}^- M_- \Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|-\mathcal{V}_{1,+} + i a_{nn}^+ M_+ \Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} \\ &\qquad\qquad + |\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} \\ &\gtrsim \big|\Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} + \big|\Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}} + \big|\Xi_1 D_n v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|\Xi_1 D_n v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}.\end{aligned}$$

We thus have

$$\begin{aligned}&\|H_-\mathcal{P}_-\Xi_1 v_-\| + \|H_+\mathcal{P}_+\Xi_1 v_+\| \\ &\quad + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}\Big) + \|H_- v_-\| + \|H_- D_n v_-\| \\ &\qquad \gtrsim \tau^k\Big(\big|\Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} + \big|\Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}} + \big|\Xi_1 D_n v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} \\ &\qquad\qquad + \big|\Xi_1 D_n v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}} + \big\|H_-\mathcal{P}_{E-}\Xi_1 v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \big\|H_+\mathcal{P}_{F+}\Xi_1 v_+\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big),\end{aligned}$$

for $k=0$ or $\frac{1}{2}$. The remaining part of the discussion is very similar to the last part of the argument in the previous subsection. By Lemmata 3.2 and 3.7, we have

$$\big\| H_-\mathcal{P}_{E-}\,\Xi_1 v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \big|\Xi_1 v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} \gtrsim \big\| H_-\,\Xi_1 v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \big\| H_-\,\Xi_1 D_n v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}$$

and

$$\big\| H_+\mathcal{P}_{F_+}\,\Xi_1 v_+\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \big|\Xi_1 v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}} \gtrsim \tau^{-1/2}\Big(\big\| H_+\,\Xi_1 v_+\big\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \big\| H_+\,\Xi_1 D_n v_+\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).$$

Since $\big|\Xi_1 v_{\pm|x_n=0^\pm}\big|_{\mathcal{H}^{3/2-k}}$ are already controlled, we also control the right-hand side of the above inequalities and have

$$\begin{aligned}
&\|H_-\mathcal{P}_-\,\Xi_1 v_-\| + \|H_+\mathcal{P}_+\,\Xi_1 v_+\| + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big)\\
&\quad + \|H_- v_-\| + \|H_- D_n v_-\|\\
&\quad\gtrsim \tau^k\Big(|\Xi_1 v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Xi_1 v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |\Xi_1 D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |\Xi_1 D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big)\\
&\qquad + \tau^{k-\frac{1}{2}}\Big(\|\Xi_1 v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-\,\Xi_1 D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+\,\Xi_1 D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big). \qquad (4\text{-}17)
\end{aligned}$$

Remark 4.4. In the case $k=0$, recalling the form of the second-order operators $\mathcal{P}_\pm$, we can estimate the additional terms $\tau^{-1/2}\big\|H_\pm\,\Xi_1 D_n^2 v_\pm\big\|$.

4D. ***Patching together microlocal estimates.*** We now sum estimates (4-16) and (4-17) together. By the triangle inequality, this gives, for $k=0$ or $\frac{1}{2}$,

$$\begin{aligned}
&\sum_{j=0,1}\big(\|H_-\mathcal{P}_-\,\Xi_j v_-\| + \|H_+\mathcal{P}_+\,\Xi_j v_+\|\big) + \tau^k\Big(|\Theta_\varphi|_{\mathcal{H}^{\frac{1}{2}-k}} + |\theta_\varphi|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big)\\
&\quad + \|H_+ v_+\| + \|H_- v_-\| + \|H_- D_n v_-\|\\
&\qquad\gtrsim \tau^k\Big(|v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{3}{2}-k}} + |v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{3}{2}-k}} + |D_n v_{-|x_n=0^-}|_{\mathcal{H}^{\frac{1}{2}-k}} + |D_n v_{+|x_n=0^+}|_{\mathcal{H}^{\frac{1}{2}-k}}\Big)\\
&\qquad\quad + \tau^{k-\frac{1}{2}}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_- D_n v_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+ D_n v_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).
\end{aligned}$$

For τ sufficiently large, we now obtain

$$\begin{aligned}
&\sum_{j=0,1}\Big(\big\|H_-\mathcal{P}_-\,\Xi_j v_-\big\| + \big\|H_+\mathcal{P}_+\,\Xi_j v_+\big\|\Big) + \tau^k\big(|\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}}\big)\\
&\quad\gtrsim \tau^k\Big(\big|v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}} + \big|D_n v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|D_n v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}\Big)\\
&\qquad + \tau^{k-1/2}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \big\|H_- D_n v_-\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \big\|H_+ D_n v_+\big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).
\end{aligned}$$

Arguing with commutators, as in the end of Lemma 3.2, noting here that the second-order operators $\mathcal{P}_\pm$ belong to the semiclassical calculus, that is, $\mathcal{P}_\pm \in \mathcal{S}_\tau^2$, we obtain, for τ sufficiently large,

$$\begin{aligned}\|H_-\mathcal{P}_-v_-\| + \|H_+\mathcal{P}_+v_+\| + \tau^k\big(|\Theta_\varphi|_{\mathcal{H}^{1/2-k}} + |\theta_\varphi|_{\mathcal{H}^{3/2-k}}\big)\\ \gtrsim \tau^k\Big(\big|v_{-|x_n=0^-}\big|_{\mathcal{H}^{3/2-k}} + \big|v_{+|x_n=0^+}\big|_{\mathcal{H}^{3/2-k}} + \big|D_n v_{-|x_n=0^-}\big|_{\mathcal{H}^{1/2-k}} + \big|D_n v_{+|x_n=0^+}\big|_{\mathcal{H}^{1/2-k}}\Big)\\ + \tau^{k-1/2}\Big(\|v\|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \|H_-D_nv_-\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \|H_+D_nv_+\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}\Big).\end{aligned}$$

In particular, this estimate allows us to absorb the perturbation in Ψ^1 as defined by (2-16) by taking τ large enough. For $k = \frac{1}{2}$, we obtain the result of Proposition 2.1, which concludes the proof of the Carleman estimate.

N.B. The case $k = 0$ gives higher Sobolev norm estimates of the trace terms $v_{\pm|x_n=0^\pm}$ and $D_n v_{\pm|x_n=0^\pm}$. It also allows one to estimate $\tau^{-1/2}\|H_\pm D_n^2 v_\pm\|$, as noted in Remarks 4.3 and 4.4. These estimates are obtained at the price of higher requirements (one additional tangential half-derivative) on the nonhomogeneous transmission condition functions θ and Θ.

4E. ***Convexification.*** We want now to slightly modify the weight function φ, for instance to allow some convexification. We started with $\varphi = H_+\varphi_+ + H_-\varphi_-$, where $\varphi_\pm$ were given by (2-22) and our proof relied heavily on a smooth factorization in first-order factors. We modify $\varphi_\pm$ into

$$\Phi_\pm(x', x_n) = \underbrace{\alpha_\pm x_n + \tfrac{1}{2}\beta x_n^2}_{\varphi_\pm(x_n)} + \kappa(x', x_n), \quad \kappa \in \mathscr{C}^\infty(\Omega;\mathbb{R}), \quad |d\kappa| \text{ bounded on } \Omega.$$

We shall prove below that the Carleman estimates of Theorems 1.1 and 2.2 also hold in this case if we choose $\|\kappa'\|_{L^\infty}$ sufficiently small.

We start by inspecting what survives in our factorization argument. We have from (2-7) $\mathcal{P}_\pm = (D + i\tau d\Phi_\pm)\cdot A_\pm(D + i\tau d\Phi_\pm)$, so that, modulo Ψ^1,

$$\begin{aligned}\mathcal{P}_\pm \equiv a_{nn}^\pm\Big(\big[D_n + S_\pm(x, D') + i\tau\big(\partial_n\Phi_\pm + S_\pm(x, \partial_{x'}\Phi_\pm)\big)\big]^2\\ + \frac{b_{jk}^\pm}{a_{nn}^\pm}\big(D_j + i\tau\partial_j\Phi_\pm\big)\big(D_k + i\tau\partial_k\Phi_\pm\big)\Big).\end{aligned} \tag{4-18}$$

(See also (2-10).) The new difficulty comes from the fact that the roots in the variable D_n are not necessarily smooth: when Φ does not depend on x', the symbol of the term $b_{jk}^\pm(D_j + i\tau\partial_j\Phi_\pm)(D_k + i\tau\partial_k\Phi_\pm)$ equals $b_{jk}^\pm\xi_j\xi_k$ and thus is positive elliptic with a smooth positive square root. It is no longer the case when we have an actual dependence of Φ upon the variable x'; nevertheless, we have, as $\partial_{x'}\Phi_\pm = \partial_{x'}\kappa$,

$$\begin{aligned}\operatorname{Re}\Big(\frac{b_{jk}^\pm}{a_{nn}^\pm}(\xi_j + i\tau\partial_j\kappa)(\xi_k + i\tau\partial_k\kappa)\Big) = \frac{b_{jk}^\pm}{a_{nn}^\pm}\xi_j\xi_k - \tau^2\frac{b_{jk}^\pm}{a_{nn}^\pm}\partial_j\kappa\partial_k\kappa \ge (\lambda_0^\pm)^2|\xi'|^2 - \tau^2(\lambda_1^\pm)^2|\partial_{x'}\kappa|^2\\ \ge \frac{3}{4}(\lambda_0^\pm)^2|\xi'|^2 \quad \text{if } \tau\|\partial_{x'}\kappa\|_{L^\infty} \le \frac{\lambda_0^\pm}{2\lambda_1^\pm}|\xi'|,\end{aligned}$$

where

$$\lambda_0^{\pm} = \inf_{\substack{x',\xi \\ |\xi'|=1}} \left(\frac{b_{jk}^{\pm}}{a_{nn}^{\pm}} \xi_j \xi_k \right)^{1/2}_{|x_n=0^{\pm}}, \quad \lambda_1^{\pm} = \sup_{\substack{x',\xi \\ |\xi'|=1}} \left(\frac{b_{jk}^{\pm}}{a_{nn}^{\pm}} \xi_j \xi_k \right)^{1/2}_{|x_n=0^{\pm}}.$$

As a result, the roots are smooth when $\tau \|\partial_{x'}\kappa\|_{L^\infty} \le \dfrac{\lambda_0^{\pm}}{2\lambda_1^{\pm}} |\xi'|$.

In this case, we define $\mathfrak{m}_{\pm} \in \mathcal{S}^1$ such that

$$\text{for } |\xi'| \ge 1, \quad \mathfrak{m}_{\pm}(x,\xi') = \left(\frac{b_{jk}^{\pm}}{a_{nn}^{\pm}} (\xi_j + i\tau\partial_j\kappa)(\xi_k + i\tau\partial_k\kappa) \right)^{1/2}, \quad \mathfrak{m}_{\pm}(x,\xi') \ge C\langle \xi' \rangle.$$

Here we use the principal value of the square root function for complex numbers.

Introducing

$$\mathfrak{e}_{\pm} = \tau\big(\partial_n \Phi_{\pm} + S_{\pm}(x,\partial_{x'}\kappa)\big) + \operatorname{Re}\mathfrak{m}_{\pm}(x,\xi'), \mathfrak{f}_{\pm} = \tau\big(\partial_n \Phi_{\pm} + S_{\pm}(x,\partial_{x'}\kappa)\big) - \operatorname{Re}\mathfrak{m}_{\pm}(x,\xi'),$$

we set $\mathfrak{E}_{\pm} = \operatorname{op}(\mathfrak{e}_{\pm})$ and $\mathfrak{F}_{\pm} = \operatorname{op}(\mathfrak{f}_{\pm})$ and

$$\mathcal{P}_{\mathfrak{E}\pm} = D_n + S_{\pm}(x,D') - \operatorname{op}^w(\operatorname{Im}\mathfrak{m}_{\pm}) + i\,\mathfrak{E}_{\pm},$$
$$\mathcal{P}_{\mathfrak{F}\pm} = D_n + S_{\pm}(x,D') + \operatorname{op}^w(\operatorname{Im}\mathfrak{m}_{\pm}) + i\,\mathfrak{F}_{\pm}.$$

Modulo the operator class Ψ^1, as in Section 2C, we may write

$$\mathcal{P}_{+} \equiv \mathcal{P}_{\mathfrak{E}+} a_{nn}^{+} \mathcal{P}_{\mathfrak{F}+}, \quad \mathcal{P}_{-} \equiv \mathcal{P}_{\mathfrak{F}-} a_{nn}^{-} \mathcal{P}_{\mathfrak{E}-}.$$

We keep the notation $m_{\pm}$ for the symbols that correspond to the previous sections, that is, if κ vanishes:

$$m_{\pm}(x,\xi') = \left(\frac{b_{jk}^{\pm}}{a_{nn}^{\pm}} \xi_j \xi_k \right)^{1/2}, \quad |\xi'| \ge 1.$$

As above, see (4-1), we choose the weight function such that the following property is fulfilled:

$$\frac{\alpha_+}{\alpha_-} > \sup_{\substack{x',\xi' \\ |\xi'|\ge 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}, \quad \alpha_{\pm} = \partial_{x_n}\varphi_{\pm|x_n=0^{\pm}};$$

and we let $\sigma > 1$ be such that

$$\frac{\alpha_+}{\alpha_-} = \sigma^2 \sup_{\substack{x',\xi' \\ |\xi'|\ge 1}} \frac{m_+(x',\xi')_{|x_n=0^+}}{m_-(x',\xi')_{|x_n=0^-}}.$$

We also introduce $1 < \sigma_0 < \sigma$. As in Section 2C, we set $f_{\pm} = \tau\varphi'_{\pm} - m_{\pm}$ (compare with $\mathfrak{f}_{\pm}$ above).

We can choose $\alpha_+/\|\partial_{x'}\kappa\|_{L^\infty}$ large enough that

$$\frac{\sigma m^+_{|x_n=0^+}}{\alpha_+} < \frac{\lambda_0^+ |\xi'|}{4\lambda_1^+ \|\partial_{x'}\kappa\|_{L^\infty}}$$

and

$$f_\pm \ge C\lambda \quad \text{if } \tau \ge |\xi'| \frac{\lambda_0^+}{4\lambda_1^+ \|\partial_{x'}\kappa\|_{L^\infty}} \text{ for } |x_n| \text{ sufficiently small.} \tag{4-19}$$

We may then consider the following cases.

(1) When $\tau\alpha_+ \le \sigma m^+(x',\xi')_{|x_n=0^+}$, arguing as in (4-5)–(4-6), we find that

$$\tau(\alpha_- + \beta x_n) - m_-(x',\xi')_{|x_n=0^-} \le -C\lambda,$$

if $|x_n|$ is sufficiently small. It follows that $\mathfrak{F}_-$ is elliptic negative if $\alpha_+/\|\kappa'\|_{L^\infty}$ is sufficiently large. In this region we may thus argue as we did in Section 4C.

(2) When

$$\frac{\lambda_0^+|\xi'|}{2\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}} \ge \tau \ge \frac{\sigma_0 m_+(x',\xi')}{\alpha_+},$$

the factorization is valid. Arguing as in (4-3)–(4-4), we find that

$$\tau(\alpha_+ + \beta x_n) - m_+(x',\xi') \ge C\lambda,$$

if $|x_n|$ is sufficiently small. It follows that $\mathfrak{F}_+$ is elliptic positive if $\alpha_+/\|\kappa'\|_{L^\infty}$ is sufficiently large. In this region we may thus argue as we did in Section 4B.

It is important to note that for β large and $\|\kappa'\|_{L^\infty}$ and $\|\kappa''\|_{L^\infty}$ sufficiently small, the weight functions $\Phi_\pm$ satisfy the (necessary and sufficient) subellipticity condition (2-26) with a loss of a half-derivative. Then the counterpart of Lemma 2.8 becomes, for $\|\kappa'\|_{L^\infty}$ sufficiently small,

$$|\mathfrak{f}_\pm| \le \delta\lambda \implies C^{-1}\tau \le |\xi'| \le C\tau \text{ and } \{\xi_n + s_\pm + \operatorname{Im}(\mathfrak{m}_\pm), \mathfrak{f}_\pm\} \ge C'\lambda,$$

for some $\delta > 0$ chosen sufficiently small. This allows us to then obtain the same results as those of Lemma 3.7 for the first-order factors $\mathcal{P}_{\mathfrak{F}\pm}$.

(3) Finally we consider the region

$$\tau \ge |\xi'| \frac{\lambda_0^+}{4\lambda_1^+\|\partial_{x'}\kappa\|_{L^\infty}}.$$

There the roots are no longer smooth, but we are well inside an elliptic region; with a perturbation argument, we may in fact disregard the contribution of κ.

By (4-18), we may write

$$\mathcal{P}_\pm \equiv \underbrace{a_{nn}^\pm\Big(\big[D_n + S_\pm(x,D') + i\tau\partial_n\varphi_\pm\big]^2 + \frac{b_{jk}^\pm}{a_{nn}^\pm} D_j D_k\Big)}_{P_\pm^0} + R_\pm, \tag{4-20}$$

with $R_\pm = R_{1,\pm}(x,D',\tau)D_n + R_{2,\pm}(x,D',\tau)$, where $R_{j,\pm} \in \mathrm{op}^w(\mathcal{S}_\tau^j)$, with $j = 1,2$, satisfy

$$\big\| R_{j,\pm}(x,D',\tau)u\big\| \le C\|\kappa'\|_{L^\infty}\|u\|_{L^2(\mathbb{R};\mathcal{H}^j)}. \tag{4-21}$$

The first term $P_{\pm}^{0}$ in (4-20) corresponds to the conjugated operator in the sections above, where the weight function only depends on the x_n variable. This term can be factored into two pseudodifferential first-order terms,

$$\mathcal{P}_{+}^{0} \equiv \mathcal{P}_{E+} a_{nn}^{+} \mathcal{P}_{F+}, \quad \mathcal{P}_{-}^{0} \equiv \mathcal{P}_{F-} a_{nn}^{-} \mathcal{P}_{E-}, \tag{4-22}$$

with the notation we introduced in Section 2C. In this third region we have $f_{\pm} \geq C\lambda$, by (4-19). Let $\chi_2 \in \mathcal{S}_{\tau}^{0}$ be a symbol that localizes in this region and set $\Xi_2 = \mathrm{op}^{w}(\chi_2)$.

For $\|\kappa'\|_{L^\infty}$ bounded with (4-23), we have

$$\big\| H_{\pm} R_{1,\pm} D_n \Xi_2 v_{\pm} \big\| \lesssim \tau^{k} \|\kappa'\|_{L^\infty} \big\| H_{\pm} D_n \Xi_2 v_{\pm} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k}} + C(\kappa) \| H_{\pm} D_n v_{\pm} \|, \tag{4-23}$$

$$\big\| H_{\pm} R_{2,\pm} D_n \Xi_2 v_{\pm} \big\| \lesssim \tau^{k} \|\kappa'\|_{L^\infty} \| H_{\pm} \Xi_2 v_{\pm} \|_{L^2(\mathbb{R};\mathcal{H}^{2-k}} + C(\kappa) \| H_{\pm} v_{\pm} \|, \tag{4-24}$$

for $k = 0$ or $\frac{1}{2}$.

On the one hand, arguing as in Section 4B, we have (see (4-14))

$$\begin{aligned} &\big\| H_{+} \mathcal{P}_{+}^{0} \Xi_2 v_{+} \big\| + \| H_{+} v_{+} \| \\ &\quad \gtrsim |\mathcal{V}_{2,+}|_{\mathcal{H}^{1/2}} + \big| \Xi_2 v_{+|x_n=0^+} \big|_{\mathcal{H}^{3/2}} + \| H_{+} \Xi_2 v_{+} \|_{L^2(\mathbb{R};\mathcal{H}^2)} + \big\| H_{+} \Xi_2 D_n v_{+} \big\|_{L^2(\mathbb{R};\mathcal{H}^1)}, \end{aligned} \tag{4-25}$$

where $\mathcal{V}_{2,\pm}$ is given as in (4-10).

On the other hand, with Lemma 3.4, we have

$$\begin{aligned} \big\| H_{-} \mathcal{P}_{-}^{0} \Xi_2 v_{-} \big\|_{L^2(\mathbb{R};\mathcal{H}^{-k})} + \| H_{-} v_{-} \| + \| H_{-} D_n v_{-} \| + \big| \mathcal{V}_{2,-} + i a_{nn}^{-} M_{-} \Xi_2 v_{-|x_n=0^-} \big|_{\mathcal{H}^{1/2-k}} \\ \gtrsim \big\| H_{-} \mathcal{P}_{E-} \Xi_2 v_{-} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}, \end{aligned}$$

for $k = 0$ or $\frac{1}{2}$, which gives

$$\begin{aligned} \big\| H_{-} \mathcal{P}_{-}^{0} \Xi_2 v_{-} \big\| + \tau^{k} \| H_{-} v_{-} \| + \tau^{k} \| H_{-} D_n v_{-} \| + \tau^{k} \big| \mathcal{V}_{2,-} + i a_{nn}^{-} M_{-} \Xi_2 v_{-|x_n=0^-} \big|_{\mathcal{H}^{1/2-k}} \\ \gtrsim \tau^{k} \big\| H_{-} \mathcal{P}_{E-} \Xi_2 v_{-} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}. \end{aligned}$$

Combining this with Lemma 3.2, we obtain

$$\begin{aligned} \big\| H_{-} \mathcal{P}_{-}^{0} \Xi_2 v_{-} \big\| + \tau^{k} \Big(\| H_{-} v_{-} \| + \| H_{-} D_n v_{-} \| + |\mathcal{V}_{2,-}|_{\mathcal{H}^{1/2-k}} + \big| \Xi_2 v_{-|x_n=0^-} \big|_{\mathcal{H}^{3/2-k}} \Big) \\ \gtrsim \tau^{k} \| H_{-} \Xi_2 v_{-} \|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} + \tau^{k} \big\| H_{+} \Xi_2 D_n v_{-} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})}. \end{aligned} \tag{4-26}$$

Now, from the transmission conditions (4-9)–(4-11), by adding ε(4-26) + (4-25) we obtain, for ε small,

$$\begin{aligned} &\big\| H_{+} \mathcal{P}_{+}^{0} \Xi_2 v_{+} \big\| + \big\| H_{-} \mathcal{P}_{-}^{0} \Xi_2 v_{-} \big\| + \tau^{k} \big(|\theta_{\varphi}|_{\mathcal{H}^{3/2-k}} + |\Theta_{\varphi}|_{\mathcal{H}^{1/2-k}} + |v_{|x_n=0^+}|_{\mathcal{H}^{1/2-k}} \big) \\ &\quad + \tau^{k} \big(\| H_{-} v_{-} \| + \| H_{-} D_n v_{-} \| \big) + \| H_{+} v_{+} \| + \| H_{+} D_n v_{+} \| \\ &\qquad \gtrsim \tau^{k} \Big(\big| \Xi_2 D_n v_{-|x_n=0^-} \big|_{\mathcal{H}^{1/2-k}} + \big| \Xi_2 D_n v_{+|x_n=0^+} \big|_{\mathcal{H}^{1/2-k}} \\ &\qquad\qquad + \big| \Xi_2 v_{-|x_n=0^-} \big|_{\mathcal{H}^{3/2-k}} + \big| \Xi_2 v_{+|x_n=0^+} \big|_{\mathcal{H}^{3/2-k}} + \| \Xi_2 v \|_{L^2(\mathbb{R};\mathcal{H}^{2-k})} \\ &\qquad\qquad + \big\| H_{-} \Xi_2 D_n v_{-} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} + \big\| H_{+} \Xi_2 D_n v_{+} \big\|_{L^2(\mathbb{R};\mathcal{H}^{1-k})} \Big). \end{aligned} \tag{4-27}$$

With (4-23)–(4-24), we see that the same estimate holds for $\mathcal{P}_\pm$ in place of $\mathcal{P}^0_\pm$ for $\|\kappa'\|_{L^\infty}$ chosen sufficiently small. This estimate is of the same quality as those obtained in the two other regions.

Summing up, we have obtained three microlocal overlapping regions and estimates in each of them. The three regions are illustrated in Figure 5. As we did above, we make sure that the preliminary cut-off region of Section 3A does not interact with the overlapping zones by choosing ϵ sufficiently small (see (3-5) and Lemma 3.1).

The overlap of the regions allows us to use a partition of unity argument, and we can conclude as in Section 4D.

5. Necessity of the geometric assumption on the weight function

Considering the operator $\mathcal{L}_\tau$ given by (1-23), we may wonder about the relevance of conditions (1-28) to derive a Carleman estimate. In the simple model and weight used here, it turns out that we can show that condition (1-28) is necessary for an estimate to hold. For simplicity, we consider a *piecewise constant* case $c = H_+ c_+ + H_- c_-$ as in Section 1E.

Theorem 5.1. *Let us assume that* (1-29) *is violated*, *that is*,

$$\frac{\alpha_+}{\alpha_-} < \frac{m_+(\xi'_0)}{m_-(\xi'_0)} \text{ for some } \xi'_0 \in \mathbb{R}^{n-1} \setminus 0. \tag{5-1}$$

Then, *for any neighborhood V of the origin*, $C > 0$, *and* $\tau_0 > 0$, *there exist*

$$v = H_+ v_+ + H_- v_-, \quad v_\pm \in \mathscr{C}_c^\infty(\mathbb{R}^n),$$

satisfying the transmission conditions (1-21)–(1-22) *at* $x_n = 0$, *and* $\tau \geq \tau_0$ *such that*

$$\operatorname{supp}(v) \subset V \quad \textit{and} \quad C\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \leq \|v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})}.$$

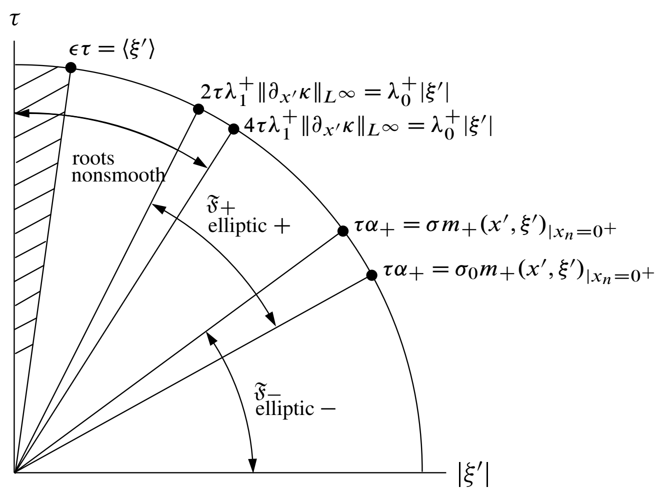

Figure 5. The overlapping microlocal regions in the case of a convex weight function.

To prove Theorem 5.1, we wish to construct a function v, depending on the parameter τ, such that $\|\mathcal{L}_\tau v\|_{L^2} \ll \|v\|_{L^2}$ as τ becomes large. The existence of such a quasimode v obviously ruins any hope of obtaining a Carleman estimate for the operator $\mathcal{L}$ with a weight function satisfying (5-1). The remainder of this section is devoted to this construction.

We set

$$(\mathcal{M}_\tau u)(\xi', x_n) = H_+(x_n)c_n^+(D_n + ie_+)(D_n + if_+)u_+ + H_-(x_n)c_n^-(D_n + ie_-)(D_n + if_-)u_-, \quad (5\text{-}2)$$

that is, the action of the operator $\mathcal{L}_\tau$ given in (1-23) in the Fourier domain with respect to x'. Observe that the terms in each product commute here. We start by constructing a quasimode for $\mathcal{M}_\tau$, that is, functions $u_\pm(\xi', x_n)$ compactly supported in the x_n variable and in a conic neighborhood of ξ'_0 in the variable ξ' with $\|\mathcal{M}_\tau u\|_{L^2} \ll \|u\|_{L^2}$, so that u is nearly an eigenvector of $\mathcal{M}_\tau$ for the eigenvalue 0.

Condition (5-1) implies that there exists $\tau_0 > 0$ such that

$$\frac{m_-(\xi'_0)}{\alpha_-} < \tau_0 < \frac{m_+(\xi'_0)}{\alpha_+} \implies \tau_0\alpha_+ - m_+(\xi'_0) < 0 < \tau_0\alpha_- - m_-(\xi'_0).$$

By homogeneity, we may in fact choose (τ_0, ξ'_0) such that $\tau_0^2 + |\xi'_0|^2 = 1$. We thus have, using the notation in (1-23),

$$f_+(x_n = 0) = \tau\alpha_+ - m_+(\xi') < 0 < f_-(x_n = 0) = \tau\alpha_- - m_-(\xi'),$$

for (τ, ξ') in a conic neighborhood Γ of (τ_0, ξ'_0) in $\mathbb{R} \times \mathbb{R}^{n-1}$. Let $\chi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$, $0 \le \chi_1 \le 1$, with $\chi_1 \equiv 1$ in a neighborhood of 0, such that $\operatorname{supp}(\psi) \subset \Gamma$ with

$$\psi(\tau, \xi') = \chi_1\left(\frac{\tau}{(\tau^2 + |\xi'|^2)^{1/2}} - \tau_0\right)\chi_1\left(\left|\frac{\xi'}{(\tau^2 + |\xi'|^2)^{1/2}} - \xi'_0\right|\right).$$

We thus have

$$f_+(x_n = 0) \le -C\tau, \quad C'\tau \le f_-(x_n = 0) \quad \text{in } \operatorname{supp}(\psi).$$

Let $(\tau, \xi') \in \operatorname{supp}(\psi)$. We can solve the equations

$$\begin{aligned}
\big(D_n + if_+(x_n, \xi'))\big)q_+ &= 0 \quad \text{on } \mathbb{R}_+, \quad f_+(x_n, \xi') = \tau\varphi'(x_n) - m_+(\xi') = f_+(0) + \tau\beta x_n,\\
\big(D_n + if_-(x_n, \xi'))\big)q_- &= 0 \quad \text{on } \mathbb{R}_-, \quad f_-(x_n, \xi') = \tau\varphi'(x_n) - m_-(\xi') = f_-(0) + \tau\beta x_n,\\
\big(D_n + ie_-(x_n, \xi'))\big)\tilde{q}_- &= 0 \quad \text{on } \mathbb{R}_-, \quad e_-(x_n, \xi') = \tau\varphi'(x_n) + m_-(\xi') = e_-(0) + \tau\beta x_n,
\end{aligned}$$

that is,

$$\begin{aligned}
q_+(\xi', x_n) &= Q_+(\xi', x_n)q_+(\xi', 0), \quad Q_+(\xi', x_n) = e^{x_n(f_+(0) + \tau\beta x_n/2)},\\
q_-(\xi', x_n) &= Q_-(\xi', x_n)q_-(\xi', 0), \quad Q_-(\xi', x_n) = e^{x_n(f_-(0) + \tau\beta x_n/2)},\\
\tilde{q}_-(\xi', x_n) &= \tilde{Q}_-(\xi', x_n)\tilde{q}_-(\xi', 0), \quad \tilde{Q}_-(\xi', x_n) = e^{x_n(e_-(0) + \tau\beta x_n/2)}.
\end{aligned}$$

Since $f_+(0) < 0$, a solution of the form of q_+ is a good idea on $x_n \ge 0$ as long as $\tau\beta x_n + 2f_+(0) \le 0$, that is, $x_n \le 2|f_+(0)|/\tau\beta$. Similarly, as $f_-(0) > 0$ (resp. $e_-(0) > 0$), a solution of the form of q_- (resp.

$\tilde{q}_-$) is a good idea on $x_n \le 0$ as long as $\tau\beta x_n + 2f_-(0) \ge 0$ (resp. $\tau\beta x_n + 2e_-(0) \ge 0$). To secure this, we introduce a cut-off function $\chi_0 \in \mathscr{C}_c^\infty((-1,1);[0,1])$, equal to 1 on $\left[-\frac{1}{2},\frac{1}{2}\right]$, and for $\gamma \ge 1$ we define

$$u_+(\xi',x_n) = Q_+(\xi',x_n)\psi(\tau,\xi')\chi_0\left(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\right) \tag{5-3}$$

and

$$u_-(\xi',x_n) = aQ_-(\xi',x_n)\psi(\tau,\xi')\chi_0\left(\frac{\tau\beta\gamma x_n}{f_-(0)}\right) + b\tilde{Q}_-(\xi',x_n)\psi(\tau,\xi')\chi_0\left(\frac{\tau\beta\gamma x_n}{e_-(0)}\right), \tag{5-4}$$

with $a,b \in \mathbb{R}$ and

$$u(\xi',x_n) = H_+(x_n)u_+(\xi',x_n) + H_-(x_n)u_-(\xi',x_n).$$

The factor γ is introduced to control the size of the support in the x_n direction. Observe that we can satisfy the transmission condition (1-21)–(1-22) by choosing the coefficients a and b. Transmission condition (1-21) implies

$$a + b = 1. \tag{5-5}$$

Transmission condition (1-22) and the equations satisfied by Q_+, Q_- and $\tilde{Q}_-$ imply

$$c_+m_+ = c_-(a-b)m_-. \tag{5-6}$$

In particular, note that $a - b \ge 0$, which gives $a \ge \frac{1}{2}$.

Lemma 5.2. *For τ sufficiently large, we have*

$$\|\mathcal{M}_\tau u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \le C(\gamma^2+\tau^2)\gamma\tau^{n-1}e^{-C'\tau/\gamma}$$

and

$$\|u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \ge C\tau^{n-2}\big(1 - e^{-C'\tau/\gamma}\big).$$

See Section AB.3 for a proof.

We now introduce

$$v_\pm(x',x_n) = (2\pi)^{-(n-1)}\chi_0\big(\big|\tau^{1/2}x'\big|\big)\check{\hat{u}}_\pm(x',x_n) = (2\pi)^{-(n-1)}\chi_0\big(\big|\tau^{1/2}x'\big|\big)\hat{u}_\pm(-x',x_n),$$

that is, a localized version of the inverse Fourier transform (in x') of $u_\pm$. The functions $v_\pm$ are smooth and compactly supported in $\mathbb{R}^{n-1}_\pm \times \mathbb{R}$ and they satisfy transmission conditions (1-21)–(1-22). We set $v(x',x_n) = H_+(x_n)v_+(x',x_n) + H_-(x_n)v_-(x',x_n)$. In fact, we have the following estimates.

Lemma 5.3. *Let $N \in \mathbb{N}$. For τ sufficiently large, we have*

$$\|\mathcal{L}_\tau v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \le C(\gamma^2+\tau^2)\gamma\tau^{n-1}e^{-C'\tau/\gamma} + C_{\gamma,N}\tau^{-N}$$

and

$$\|v\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \ge C\tau^{n-2}\big(1 - e^{-C'\tau/\gamma}\big) - C_{\gamma,N}\tau^{-N}.$$

See Section AB.4 for a proof.

We may now conclude the proof of Theorem 5.1. In fact, if V is an arbitrary neighborhood of the origin, we choose τ and γ sufficiently large that $\operatorname{supp}(v) \subset V$. We then keep γ fixed. The estimates of Lemma 5.3 show that

$$\|\mathcal{L}_\tau v\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \|v\|^{-1}_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \underset{\tau\to\infty}{\longrightarrow} 0.$$

Remark 5.4. As opposed to the analogy we give at the beginning of Section 1F, the construction of this quasimode does not simply rely on one of the first-order factors. The transmission conditions are responsible for this fact. The construction relies on the factor $D_n + if_+$ in $x_n \geq 0$, that is, a one-dimensional space of solutions (see (5-3)), and on both factors $D_n + if_-$ and $D_n + ie_-$ in $x_n \geq 0$, that is, a two-dimensional space of solutions (see (5-4)). See also (5-5) and (5-6).

Appendix

AA. *A few facts on pseudodifferential operators.*

AA.1. *Standard classes and Weyl quantization.* We define for $m \in \mathbb{R}$ the class of tangential symbols $\mathcal{S}^m$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ such that for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1}} \langle\xi'\rangle^{-m+|\beta|} \left|(\partial_x^\alpha \partial_{\xi'}^\beta a)(x,\xi')\right| < \infty, \tag{A-1}$$

with $\langle\xi'\rangle^2 = 1 + |\xi'|^2$. The quantities on the left-hand side are called the seminorms of the symbol a. For $a \in \mathcal{S}^m$, let $\operatorname{op}(a)$ be the operator defined on $\mathscr{S}(\mathbb{R}^n)$ by

$$(\operatorname{op}(a)u)(x', x_n) = a(x, D')u(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} a(x', x_n, \xi')\hat{u}(\xi', x_n)\, d\xi' (2\pi)^{1-n}, \tag{A-2}$$

with $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where $\hat{u}$ is the partial Fourier transform of u with respect to the variable x'. For all $(k, s) \in \mathbb{Z} \times \mathbb{R}$, we have

$$\operatorname{op}(a) : H^k\big(\mathbb{R}_{x_n}; H^{s+m}(\mathbb{R}^{n-1}_{x'})\big) \to H^k\big(\mathbb{R}_{x_n}; H^s(\mathbb{R}^{n-1}_{x'})\big) \quad \text{continuously}, \tag{A-3}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta|\leq\mu(k,s,m,n)}$, where $\mu : \mathbb{Z}\times\mathbb{R}\times\mathbb{R}\times\mathbb{N} \to \mathbb{N}$.

We shall also use the *Weyl quantization* of a, denoted by $\operatorname{op}^w(a)$ and given by the formula

$$\begin{aligned}(\operatorname{op}^w(a)u)(x', x_n) &= a^w(x, D')u(x', x_n)\\ &= \iint_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} a\Big(\frac{x'+y'}{2}, x_n, \xi'\Big) u(y', x_n)\, dy'\, d\xi' (2\pi)^{1-n}.\end{aligned} \tag{A-4}$$

Property (A-3) holds as well for $\operatorname{op}^w(a)$. A nice feature of the Weyl quantization that we use in this article is the simple relationship with adjoint operators with the formula

$$(\operatorname{op}^w(a))^* = \operatorname{op}^w(\bar{a}), \tag{A-5}$$

so that for a real-valued symbol $a \in \mathcal{S}^m$, we have $(\mathrm{op}^w(a))^* = \mathrm{op}^w(a)$. We have also, for $a_j \in \mathcal{S}^{m_j}$, $j = 1, 2$,

$$\mathrm{op}^w(a_1)\mathrm{op}^w(a_2) = \mathrm{op}^w(a_1 \sharp a_2), \quad a_1 \sharp a_2 \in \mathcal{S}^{m_1+m_2}, \tag{A-6}$$

with, for any $N \in \mathbb{N}$,

$$(a_1 \sharp a_2)(x, \xi) - \sum_{j<N} \Big(\frac{i\sigma(D_{x'}, D_{\xi'}; D_{y'}, D_{\eta'})}{2} \Big)^j \frac{a_1(x, \xi) a_2(y, \eta)}{j!} \Big|_{(y,\eta)=(x,\xi)} \in \mathcal{S}^{m-N}, \tag{A-7}$$

where σ is the symplectic two-form, that is, $\sigma(x, \xi; y, \eta) = y \cdot \xi - x \cdot \eta$. In particular,

$$\mathrm{op}^w(a_1)\mathrm{op}^w(a_2) = \mathrm{op}^w(a_1 a_2) + \mathrm{op}^w(r_1), \qquad r_1 \in \mathcal{S}^{m_1+m_2-1}, \tag{A-8}$$

$$\text{with } r_1 = \frac{1}{2i}\{a_1, a_2\} + r_2, \qquad r_2 \in \mathcal{S}^{m_1+m_2-2}, \tag{A-9}$$

$$\big[\mathrm{op}^w(a_1), \mathrm{op}^w(a_2)\big] = \mathrm{op}^w\Big(\frac{1}{i}\{a_1, a_2\}\Big) + \mathrm{op}^w(r_3), \qquad r_3 \in \mathcal{S}^{m_1+m_2-3}, \tag{A-10}$$

where $\{a_1, a_2\}$ is the Poisson bracket. Also, for $b_j \in \mathcal{S}^{m_j}$, $j = 1, 2$, both real-valued, we have

$$\big[\mathrm{op}^w(b_1), i\,\mathrm{op}^w(b_2)\big] = \mathrm{op}^w(\{b_1, b_2\}) + \mathrm{op}^w(s_3), \quad s_3 \text{ real-valued} \in \mathcal{S}^{m_1+m_2-3}. \tag{A-11}$$

Lemma A.1. *Let $a \in \mathcal{S}^1$ be such that $a(x, \xi') \geq \mu\langle\xi'\rangle$, with $\mu \geq 0$. Then there exists $C > 0$ such that*

$$\mathrm{op}^w(a) + C \geq \mu\langle D'\rangle, \quad (\mathrm{op}^w(a))^2 + C \geq \mu^2\langle D'\rangle^2.$$

Proof. The first statement follows from the sharp Gårding inequality [Hörmander 1985a, Chapters 18.1,18.5] applied to the nonnegative first-order symbol $a(x, \xi') - \mu\langle\xi'\rangle$; also, $(\mathrm{op}^w(a))^2 = \mathrm{op}^w(a^2) + \mathrm{op}^w(r)$ with $r \in \mathcal{S}^0$, so that the Fefferman–Phong inequality [Hörmander 1985a, Chapter 18.5] applied to the second-order $a^2 - \mu^2\langle\xi'\rangle^2$ implies the result. ☐

AA.2. *Semiclassical pseudodifferential calculus with a large parameter.* We let $\tau \in \mathbb{R}$ be such that $\tau \geq \tau_0 \geq 1$. We set $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. We define, for $m \in \mathbb{R}$, the class of symbols $\mathcal{S}^m_\tau$ as the smooth functions on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ depending on the parameter τ such that for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}$,

$$N_{\alpha\beta}(a) = \sup_{\substack{(x,\xi')\in\mathbb{R}^n\times\mathbb{R}^{n-1} \\ \tau\geq\tau_0}} \lambda^{-m+|\beta|} \big|(\partial_x^\alpha \partial_{\xi'}^\beta a)(x, \xi', \tau)\big| < \infty. \tag{A-12}$$

Note that $\mathcal{S}^0_\tau \subset \mathcal{S}^0$. The associated operators are defined by (A-2). We can introduce Sobolev spaces and Sobolev norms which are adapted to the scaling large parameter τ. Let $s \in \mathbb{R}$; we set

$$\|u\|_{\mathcal{H}^s} := \|\Lambda^s u\|_{L^2(\mathbb{R}^{n-1})}, \quad \text{with } \Lambda^s := \mathrm{op}(\lambda^s),$$

and

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^{n-1}) := \big\{u \in \mathscr{S}'(\mathbb{R}^{n-1}) : \|u\|_{\mathcal{H}^s} < \infty\big\}.$$

The space $\mathcal{H}^s$ is algebraically equal to the classical Sobolev space $H^s(\mathbb{R}^{n-1})$, whose norm is denoted by $\| \,.\, \|_{H^s}$. For $s \geq 0$, we have

$$\|u\|_{\mathcal{H}^s} \sim \tau^s \|u\|_{L^2(\mathbb{R}^{n-1})} + \|\langle D'\rangle^s u\|_{L^2(\mathbb{R}^{n-1})}.$$

If $a \in \mathcal{S}^m_\tau$ then, for all $(k,s) \in \mathbb{Z} \times \mathbb{R}$, we have

$$\operatorname{op}(a) : H^k\big(\mathbb{R}_{x_n}; \mathcal{H}^{s+m}\big) \to H^k\big(\mathbb{R}_{x_n}; \mathcal{H}^s\big(\mathbb{R}^{n-1}_{x'}\big)\big) \quad \text{continuously,} \tag{A-13}$$

and the norm of this mapping depends only on $\{N_{\alpha\beta}(a)\}_{|\alpha|+|\beta|\leq \mu(k,s,m,n)}$, where $\mu : \mathbb{Z}\times\mathbb{R}\times\mathbb{R}\times\mathbb{N} \to \mathbb{N}$.

For the calculus with a large parameter, we shall also use the Weyl quantization of (A-4). The formulae (A-5)–(A-11) hold as well, with $\mathcal{S}^m$ everywhere replaced by $\mathcal{S}^m_\tau$. We shall often use the Gårding inequality as stated in the following lemma.

Lemma A.2. *Let $a \in \mathcal{S}^m_\tau$ such that* $\operatorname{Re} a \geq C\lambda^m$. *Then*

$$\operatorname{Re}\big(\operatorname{op}^w(a)u, u\big) \gtrsim \|u\|^2_{L^2(\mathbb{R};\mathcal{H}^{m/2})},$$

for τ sufficiently large.

Proof. The proof follows from the sharp Gårding inequality [Hörmander 1985a, Chapters 18.1 and 18.5] applied to the nonnegative symbol $a - C\lambda^m$. ☐

Definition A.3. The essential support of a symbol $a \in \mathcal{S}^m_\tau$, denoted by $\operatorname{esssupp}(a)$, is the complement of the largest open set of $\mathbb{R}\times\mathbb{R}^{n-1}\times\{\tau \geq 1\}$ where the estimates for $\mathcal{S}^{-\infty}_\tau = \cap_{m\in\mathbb{R}}\mathcal{S}^m_\tau$ hold.

For technical reasons we shall often need the following result.

Lemma A.4. *Let $m, m' \in \mathbb{R}$ and $a_1(x,\xi') \in \mathcal{S}^m$ and $a_2(x,\xi',\tau) \in \mathcal{S}^{m'}_\tau$ such that the essential support of a_2 is contained in a region where $\langle\xi'\rangle \gtrsim \tau$. Then*

$$\operatorname{op}^w(a_1)\operatorname{op}^w(a_2) = \operatorname{op}^w(b_1), \quad \operatorname{op}^w(a_2)\operatorname{op}^w(a_1) = \operatorname{op}^w(b_2),$$

with $b_1, b_2 \in \mathcal{S}^{m+m'}_\tau$. Moreover, the asympotic series of (A-7) *is also valid for these cases* (*with $\mathcal{S}^m$ replaced by $\mathcal{S}^m_\tau$*).

Proof. As the essential support is invariant when we change quantization, we may simply use the standard quantization in the proof. With a_1 and a_2 satisfying the assumption listed above, we thus consider $\operatorname{op}(a_1)\operatorname{op}(a_2)$. For fixed τ, the standard composition formula applies, and we have (see [Hörmander 1985a, Section 18.1] or [Alinhac and Gérard 2007])

$$(a_1 \circ a_2)(x,\xi',\tau) = (2\pi)^{1-n} \iint e^{-iy'\cdot\eta'} a_1(x,\xi'-\eta') a_2\big(x'-y', x_n, \xi', \tau\big)\, dy'\, d\eta'.$$

Properties of oscillatory integrals (see, for example, [Alinhac and Gérard 2007, Appendices I.8.1 and I.8.2]) give, for some $k \in \mathbb{N}$,

$$\big|(a_1 \circ a_2)(x,\xi',\tau)\big| \leq C \sup_{\substack{|\alpha|+|\beta|\leq k \\ (y',\eta')\in R^{2n-2}}} \langle (y',\eta')\rangle^{-|m|} \big|\partial^\alpha_{y'}\partial^\beta_{\eta'} a_1(x,\xi'-\eta') a_2\big(x'-y', x_n, \xi', \tau\big)\big|.$$

In a region $\langle\xi'\rangle \gtrsim \tau$ that contains the essential support of a_2, we have $\langle\xi'\rangle \sim \lambda$. With the Peetre inequality, we thus obtain

$$\left|(a_1 \circ a_2)(x,\xi',\tau)\right| \lesssim \langle\eta'\rangle^{-|m|}\langle\xi'-\eta'\rangle^{m}\lambda^{m'} \lesssim \langle\xi'\rangle^{m}\lambda^{m'} \lesssim \lambda^{m+m'}.$$

In a region $\langle\xi'\rangle \lesssim \tau$ outside of the essential support of a_2, we find, for any $\ell \in \mathbb{N}$,

$$\left|(a_1 \circ a_2)(x,\xi',\tau)\right| \lesssim \langle\eta'\rangle^{-|m|}\langle\xi'-\eta'\rangle^{m}\lambda^{-\ell} \lesssim \langle\xi'\rangle^{m}\lambda^{-\ell} \lesssim \lambda^{m-\ell}.$$

In the whole phase space we thus obtain $\left|(a_1 \circ a_2)(x,\xi')\right| \lesssim \lambda^{m+m'}$. The estimation of

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta (a_1 \circ a_2)(x,\xi',\tau)\right|$$

can be done similarly to give

$$\left|\partial_x^\alpha \partial_{\xi}^\beta (a_1 \circ a_2)(x,\xi',\tau)\right| \lesssim \lambda^{m+m'-|\beta|}.$$

Hence $a_1 \circ a_2 \in \mathcal{S}_\tau^{m+m'}$. We also obtain the asymptotic series (following the references cited above)

$$(a_1 \circ a_2)(x,\xi',\tau) - \sum_{j<N} \frac{(iD_\xi \cdot D_y)^j a_1(x,\xi)a_2(y,\eta,\tau)}{j!}\Big|_{(y,\eta)=(x,\xi)} \in \mathcal{S}_\tau^{m+m'-N},$$

where each term is respectively in $\mathcal{S}_\tau^{m+m'-j}$ by the arguments given above. From this series, the corresponding Weyl quantization series follows.

For the second result, considering the adjoint operator $\left(\mathrm{op}(a_2)\mathrm{op}(a_1)\right)^*$ yields a composition of operators as in the first case. The second result thus follows from the first one. □

Remark A.5. The symbol class and calculus we have introduced in this section can be written as $\mathcal{S}_\tau^m = \mathcal{S}(\lambda^m, g)$ in the sense of the Weyl–Hörmander calculus [Hörmander 1985a, Sections 18.4–18.6] with the phase-space metric $g = |dx|^2 + |d\xi|^2/\lambda^2$.

AB. *Proofs of some intermediate results.*

AB.1. *Proof of Lemma 2.8.* For simplicity we remove the $\pm$ notation here. We first prove that there exist $C > 0$ and $\eta > 0$ such that

$$|q_2| \le \eta\tau^2 \text{ and } |q_1| \le \eta\tau^2 \implies \{q_2, q_1\} \ge C\tau^3. \tag{A-14}$$

We set

$$\tilde{q}_2 = (\xi_n + s)^2 + \frac{b_{jk}}{a_{nn}}\xi_j\xi_k - (\varphi')^2, \quad \tilde{q}_1 = \varphi'(\xi_n + s).$$

We have $q_j(x,\xi) = \tau^2\tilde{q}_j(x,\xi/\tau)$. Observe next that we have $\{q_2,q_1\}(x,\xi) = \tau^3\{\tilde{q}_2,\tilde{q}_1\}(x,\xi/\tau)$. We thus have $\tilde{q}_2 = 0$ and $\tilde{q}_1 = 0 \implies \{\tilde{q}_2,\tilde{q}_1\} > 0$. As $\tilde{q}_2(x,\xi) = 0$ and $\tilde{q}_1(x,\xi) = 0$ yield a compact set for (x,ξ) (recall that x lies in a compact set K here), for some $C > 0$, we have

$$\tilde{q}_2 = 0 \text{ and } \tilde{q}_1 = 0 \implies \{\tilde{q}_2,\tilde{q}_1\} > C.$$

This remains true locally, that is, for some $C' > 0$ and $\eta > 0$,

$$|\tilde{q}_2| \le \eta \text{ and } |\tilde{q}_1| \le \eta \implies \{\tilde{q}_2, \tilde{q}_1\} > C'.$$

Then (A-14) follows.

We note that $q_2^{\pm} = 0$ and $q_1^{\pm} = 0$ imply $\tau \sim |\xi'|$. Hence, for τ sufficiently large, we have (2-25). We thus obtain

$$q_2^{\pm} = 0 \text{ and } q_1^{\pm} = 0 \iff \xi_n + s_{\pm} = 0 \text{ and } \tau\varphi'_{\pm} = m_{\pm}.$$

Let us assume that $|f| \le \delta\lambda$ with δ small and $\lambda^2 = 1 + \tau^2 + |\xi'|^2$. Then

$$\tau \lesssim |\xi'| \lesssim \tau. \tag{A-15}$$

We set $\xi_n = -s$, that is, we choose $q_1 = 0$. A direct computation yields

$$\{q_2, q_1\} = \tau e \varphi' \{\xi_n + s, f\} + \tau f \varphi' \{\xi_n + s, e\} \quad \text{if } \xi_n + s = 0.$$

With (2-25), we have $|q_2| \le C\delta\tau^2$. For δ small, by (A-14) we have $\{q_2, q_1\} \ge C\tau^3$. Since $f\tau\varphi'\{\xi_n + s, e\} \le C\delta\tau^3$, we obtain $e\tau\varphi'\{\xi_n + s, f\} \ge C\tau^3$, with $C > 0$, for δ sufficiently small. With (A-15), we have $\tau \lesssim e \lesssim \tau$ and the result follows. ☐

AB.2. *Proof of Lemma 3.1.* We set $s = 2\ell + 1$ and $\omega_1 = \mathrm{op}(\psi_\epsilon)\omega$. We write

$$\begin{aligned}
2\,\mathrm{Re}\big(\mathcal{P}_{F+}\omega_1, iH_+\tau^s\omega_1\big) &= \big(i[D_n, H_+]\omega_1, \tau^s\omega_1\big) + 2\big(F_+\omega_1, H_+\tau^s\omega_1\big) \\
&= \tau^s \big|\omega_{1\,|x_n=0^+}\big|^2_{L^2(\mathbb{R}^{n-1})} + 2\big(\tau^{s+1}\varphi'\omega_1, H_+\omega_1\big) - 2\big(\tau^s M_+\omega_1, H_+\omega_1\big) \\
&\ge \tau^s \big|\omega_{1\,|x_n=0^+}\big|^2_{L^2(\mathbb{R}^{n-1})} + 2\big(\tau^{s+1}C_0\omega_1, H_+\omega_1\big) - 2C_1\tau^s \|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{1/2}(\mathbb{R}^{n-1}))},
\end{aligned}$$

by (3-4). We have

$$\begin{aligned}
&2\big(\tau^{s+1}C_0\omega_1, H_+\omega_1\big) - 2C_1\tau^s \|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{1/2}(\mathbb{R}^{n-1}))} \\
&\qquad = 2\tau^s(2\pi)^{1-n}\int_0^\infty\int_{\mathbb{R}^{n-1}} \big(C_0\tau - C_1\langle\xi'\rangle\big)\big|\psi_\epsilon(\tau,\xi')\hat{\omega}(\xi', x_n)\big|^2\, d\xi'\, dx_n.
\end{aligned}$$

As $\tau \ge C\langle\xi\rangle/\epsilon$ in $\mathrm{supp}(\psi_\epsilon)$, for ϵ sufficiently small we have

$$\begin{aligned}
&2\big(\tau^{s+1}C_0\omega_1, H_+\omega_1\big) - 2C_1\tau^s \|H_+\omega_1\|^2_{L^2(\mathbb{R};H^{1/2}(\mathbb{R}^{n-1}))} \\
&\qquad \gtrsim \int_0^\infty\int_{\mathbb{R}^{n-1}} \lambda^{s+1}\big|\psi_\epsilon(\tau,\xi')\hat{\omega}(\xi', x_n)\big|^2\, d\xi'\, dx_n \gtrsim \|H_+\omega_1\|^2_{L^2(\mathbb{R};\mathcal{H}^{\ell+1})}.
\end{aligned}$$

Similarly, we find $\tau^s\big|\omega_{1\,|x_n=0^+}\big|^2_{L^2(\mathbb{R}^{n-1})} \gtrsim \big|\omega_{1\,|x_n=0^+}\big|^2_{\mathcal{H}^{\ell+1/2}}$. The result for $\mathcal{P}_{E+}$ follows from the Young inequality. The proof is identical for $\mathcal{P}_{F+}$.

On the other side of the interface we write

$$\begin{aligned}
2\,\mathrm{Re}\big(H_-\mathcal{P}_{F-}\omega_1, iH_-\tau^s\omega_1\big) &= \big(i[D_n, H_-]\omega_1, \tau^s\omega_1\big) + 2\big(F_-\omega_1, H_-\tau^s\omega_1\big) \\
&= -\tau^s \big|\omega_{1\,|x_n=0^-}\big|^2_{L^2(\mathbb{R}^{n-1})} + 2\big(\tau^{s+1}\varphi'\omega_1, H_-\omega_1\big) - 2\big(\tau^s M_-\omega_1, H_-\omega_1\big),
\end{aligned}$$

which yields a boundary contribution with the opposite sign. ☐

AB.3. *Proof of Lemma 5.2.* Let $(\tau,\xi')\in\operatorname{supp}(\psi)$. We choose τ sufficiently large that, through $\operatorname{supp}(\psi)$, $|\xi'|$ is itself sufficiently large that the symbol $m_\pm$ is homogeneous — see (2-15).

We set

$$y_+(\xi',x_n)=Q_+(\xi',x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big),$$
$$y_-(\xi',x_n)=aQ_-(\xi',x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big)+b\tilde{Q}_-(\xi',x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big).$$

On the one hand, we have $i(D_n+if_+)y_+=\dfrac{\tau\beta\gamma}{|f_+(0)|}Q_+(\xi',x_n)\chi_0'\Big(\dfrac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)$ and

$$(\mathcal{M}_\tau y_+)(\xi',x_n)=2\tau\beta\gamma c_+m_+\frac{Q_+(\xi',x_n)}{|f_+(0)|}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)-(\tau\beta\gamma)^2c_+\frac{Q_+(\xi',x_n)}{|f_+(0)|^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big),$$

as $D_n+ie_+=D_n+i(f_++2m_+)$, so that

$$\int_0^{+\infty}\big|(\mathcal{M}_\tau y_+)(\xi',x_n)\big|^2\,dx_n\le 8c_+^2m_+^2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\int_0^{+\infty}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2e^{x_n(2f_+(0)+\tau\beta x_n)}\,dx_n$$
$$+2c_+^2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^4\int_0^{+\infty}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2e^{x_n(2f_+(0)+\tau\beta x_n)}\,dx_n.$$

On the support of $\chi_0^{(j)}(\tau\beta\gamma x_n/|f_+(0)|)$, $j=1,2$, we have $|f_+(0)|/(2\tau\beta\gamma)\le x_n\le|f_+(0)|/(\tau\beta\gamma)$, and in particular $2f_+(0)+\tau\beta\gamma x_n\le-|f_+(0)|$, which gives

$$\int_0^{+\infty}\big|(\mathcal{M}_\tau y_+)(\xi',x_n)\big|^2\,dx_n$$
$$\le c_+^2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\Big(8m_+^2\|\chi_0'\|_{L^\infty}^2+2\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\|\chi_0''\|_{L^\infty}^2\Big)\int_{\frac{|f_+(0)|}{2\tau\beta\gamma}\le x_n\le\frac{|f_+(0)|}{\tau\beta\gamma}}e^{-|f_+(0)|x_n}\,dx_n$$
$$\le c_+^2\frac{\tau\beta\gamma}{|f_+(0)|}\Big(4m_+^2\|\chi_0'\|_{L^\infty}^2+\Big(\frac{\tau\beta\gamma}{f_+(0)}\Big)^2\|\chi_0''\|_{L^\infty}^2\Big)e^{-\frac{f_+(0)^2}{2\tau\beta\gamma}}.$$

Similarly, we have

$$(\mathcal{M}_\tau y_-)(\xi',x_n)=2\tau\beta\gamma c_-m_-\Big(\frac{aQ_-(\xi',x_n)}{f_-(0)}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big)-b\frac{\tilde{Q}_-(\xi',x_n)}{e_-(0)}\chi_0'\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\Big)$$
$$-c_-(\tau\beta\gamma)^2\Big(a\frac{Q_-(\xi',x_n)}{f_-(0)^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big)+b\frac{\tilde{Q}_-(\xi',x_n)}{e_-(0)^2}\chi_0''\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\Big),$$

and because of the support of $\chi_0^{(j)}\Big(\dfrac{\tau\beta\gamma x_n}{f_-(0)}\Big)$, resp. $\chi_0^{(j)}\Big(\dfrac{\tau\beta\gamma x_n}{e_-(0)}\Big)$, $j=1,2$, for $x_n\le0$, we obtain

$$\int_{-\infty}^0\big|(\mathcal{M}_\tau y_-)(\xi',x_n)\big|^2\,dx_n\le 2c_-^2\frac{\tau\beta\gamma a^2}{f_-(0)}\Big(4m_-^2\|\chi_0'\|_{L^\infty}^2+\|\chi_0''\|_{L^\infty}^2\Big(\frac{\tau\beta\gamma}{f_-(0)}\Big)^2\Big)e^{-\frac{f_-(0)^2}{2\tau\beta\gamma}}$$
$$+2c_-^2\frac{\tau\beta\gamma b^2}{e_-(0)}\Big(4m_-^2\|\chi_0'\|_{L^\infty}^2+\|\chi_0''\|_{L^\infty}^2\Big(\frac{\tau\beta\gamma}{e_-(0)}\Big)^2\Big)e^{-\frac{e_-(0)^2}{2\tau\beta\gamma}}.$$

Now we have $(\mathcal{M}_\tau u)(\xi', x_n) = \psi(\tau, \xi')(\mathcal{M}_\tau y)(\xi', x_n)$. As $|\xi'| \sim \tau$ in $\operatorname{supp}(\psi)$, we obtain

$$\|\mathcal{M}_\tau u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \le C(\gamma^2+\tau^2)\gamma e^{-C'\tau/\gamma} \int_{\mathbb{R}^{n-1}} \psi(\tau,\xi')^2 \, d\xi'.$$

With the change of variable $\xi' = \tau\eta$, we find

$$\int_{\mathbb{R}^{n-1}} \psi(\tau,\xi')^2 \, d\xi' = C\tau^{n-1}, \tag{A-16}$$

which gives the first result.

On the other hand, observe now that

$$\begin{aligned}\|y_+\|^2_{L^2(\mathbb{R}_+)} &= \int_0^{+\infty} Q_+(\xi', x_n)^2 \chi_0\Big(\frac{\tau\beta\gamma x_n}{|f_+(0)|}\Big)^2 dx_n \\ &\ge \int_{0\le \frac{\tau\beta\gamma x_n}{|f_+(0)|}\le 1/2} e^{x_n(2f_+(0)+\tau\beta x_n)} \, dx_n = \frac{|f_+(0)|}{\tau\beta\gamma} \int_0^{1/2} e^{2t\frac{|f_+(0)|}{\tau\beta\gamma}\left(f_+(0)+t\frac{|f_+(0)|}{2\gamma}\right)} dt \\ &\ge \frac{|f_+(0)|}{\tau\beta\gamma} \int_0^{1/2} e^{-2t\frac{|f_+(0)|^2}{\tau\beta\gamma}} dt = \frac{1}{2|f_+(0)|}\Big(1-e^{-\frac{|f_+(0)|^2}{\tau\beta\gamma}}\Big).\end{aligned}$$

We also have

$$\begin{aligned}\|y_-\|^2_{L^2(\mathbb{R}_-)} &= \int_{-\infty}^0 \Big(aQ_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{f_-(0)}\Big) + b\tilde{Q}_-(\xi', x_n)\chi_0\Big(\frac{\tau\beta\gamma x_n}{e_-(0)}\Big)\Big)^2 dx_n \\ &\ge \int_{-1/2\le \frac{\tau\beta\gamma x_n}{f_-(0)}\le 0} e^{x_n(2f_-(0)+\tau\beta x_n)} \big(a+be^{x_n(e_-(0)-f_-(0))}\big)^2 \, dx_n,\end{aligned}$$

and as $e_-(0) - f_-(0) = 2m_- \ge 0$ and $a+b=1$ and $a \ge \frac{1}{2}$, we have $a + be^{x_n(e_-(0)-f_-(0))} \ge \frac{1}{2}$, and thus obtain

$$\|y_-\|^2_{L^2(\mathbb{R}_-)} \ge \frac{1}{4} \int_{-1/2\le \frac{\tau\beta\gamma x_n}{f_-(0)}\le 0} e^{x_n(2f_-(0)+\tau\beta x_n)} \, dx_n \ge \frac{1}{8f_-(0)}\Big(1-e^{-\frac{|f_-(0)|^2}{\tau\beta\gamma}}\Big),$$

arguing as above. As a result, using (A-16), we have

$$\|u\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R})} \ge C\tau^{n-2}\big(1-e^{-C'\tau/\gamma}\big). \qquad \square$$

AB.4. *Proof of Lemma 5.3.* We start with the second result. We set

$$z_+ = \big(1-\chi_0\big(|\tau^{1/2}x'|\big)\big)\check{\hat{u}}_+(x', x_n), \quad \text{for } x_n \ge 0.$$

We shall prove that for all $N \in \mathbb{N}$, we have $\|z_+\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_+)} \le C_{\gamma,N}\tau^{-N}$.

From the definition of χ_0, we find

$$\|z_+\|^2_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_+)} \le \int_{|\tau^{1/2}x'|\ge 1/2} \int_{\mathbb{R}_+} |\hat{u}_+(x', x_n)|^2 \, dx' \, dx_n.$$

Recalling the definition of u_+ and performing the change of variable $\xi' = \tau\eta$, we obtain

$$\hat{u}_+(x', x_n) = \tau^{n-1} \int_{\mathbb{R}^{n-1}} e^{i\tau\phi} \tilde{\psi}(\eta) \chi_0\Big(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\Big) d\eta,$$

where the complex phase function is given by

$$\phi = -x' \cdot \eta - i x_n \Big(\tilde{f}_+(\eta) + \frac{\beta x_n}{2}\Big), \quad \text{with } \tilde{f}_+(\eta) = \alpha_+ - m_+(\eta)$$

and

$$\tilde{\psi}(\eta) = \chi_1\Big(\frac{1}{(1+|\eta|^2)^{1/2}} - \tau_0\Big) \chi_1\Big(\Big|\frac{\eta}{(1+|\eta|^2)^{1/2}} - \xi_0'\Big|\Big).$$

Here τ is chosen sufficiently large that m_+ is homogeneous. Observe that $\tilde{\psi}$ has a compact support independent of τ and that $\tilde{f}_+(\eta) + \beta x_n/2 \leq -C < 0$ in the support of the integrand.

We place ourselves in the neighborhood of a point x' such that $|\tau^{1/2} x'| \geq \frac{1}{2}$. Up to a permutation of the variables, we may assume that $|\tau^{1/2} x_1| \geq C$. We then introduce the differential operator

$$L = \tau^{-1} \frac{\partial_{\eta_1}}{-i x_1 - x_n \partial_{\eta_1} m_+(\eta)},$$

which satisfies $L e^{i\tau\phi} = e^{i\tau\phi}$. We thus have

$$\hat{u}_+(x', x_n) = \tau^{n-1} \int_{\mathbb{R}^{n-1}} e^{i\tau\phi} (L^t)^N \Big(\tilde{\psi}(\eta) \chi_0\Big(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\Big)\Big) d\eta,$$

and we find

$$\big|\hat{u}_+(x', x_n)\big| \leq C_N \frac{\tau^{n-1}\gamma^N}{|\tau x_1|^N} e^{-C\tau x_n}.$$

More generally, for $|\tau^{1/2} x'| \geq \frac{1}{2}$ we have

$$\big|\hat{u}_+(x', x_n)\big| \leq C_N \frac{\tau^{n-1}\gamma^N}{|\tau x'|^N} e^{-C\tau x_n}.$$

Then we obtain

$$\begin{aligned}
&\int_{|\tau^{1/2}x'| \geq 1/2} \int_{\mathbb{R}_+} \big|\hat{u}_+(x', x_n)\big|^2 \, dx' \, dx_n \\
&\qquad \leq C_N^2 \gamma^{2N} \tau^{2n-2} \Big(\int_{|\tau^{1/2}x'| \geq 1/2} \frac{1}{|\tau x'|^{2N}} \, dx'\Big) \Big(\int_{\mathbb{R}_+} e^{-2C\tau x_n} \, dx_n\Big) \\
&\qquad \leq C_N' \gamma^{2N} \tau^{(3/2)n - N - 5/2} \int_{|x'| \geq 1/2} \frac{1}{|x'|^{2N}} \, dx'.
\end{aligned}$$

Similarly, setting $z_- = \big(1 - \chi_0\big(|\tau^{1/2} x'|\big)\big)\check{\hat{u}}_-(x', x_n)$ for $x_n \leq 0$, we get $\|z_-\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_-)} \leq C_{\gamma,N} \tau^{-N}$. The second result thus follows from Lemma 5.2.

For the first result we write

$$\mathcal{L}_\tau v_\pm = (2\pi)^{-(n-1)} \chi_0\big(|\tau^{1/2} x'|\big) \mathcal{L}_\tau \check{\hat{u}}_\pm + (2\pi)^{-(n-1)} \big[\mathcal{L}_\tau, \chi_0\big(|\tau^{1/2} x'|\big)\big] \check{\hat{u}}_\pm.$$

The first term is estimated, using Lemma 5.2, as

$$(2\pi)^{-(n-1)/2}\|\mathcal{L}_\tau \check{\hat{u}}_\pm\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_\pm)} = \|\mathcal{M}_\tau u_\pm\|_{L^2(\mathbb{R}^{n-1}\times\mathbb{R}_\pm)}.$$

Observe that $\mathcal{L}_\tau$ is a *differential* operator; the commutator is thus a first-order differential operator in x' with support in a region $|\tau^{1/2}x'| \geq C$, because of the behavior of χ_1 near 0. The coefficients of this operator depend on τ polynomially. The zero-order terms can be estimated as we did for z_+ above with an additional $\tau^{3/2}$ factor.

For the first-order term, observe that we have

$$\partial_{x'_j}\check{\hat{u}}_+(x',\tau) = \tau^n \int_{\mathbb{R}^{n-1}} \eta_j e^{i\tau\left(x'\cdot\eta - i x_n\left(\tilde{f}_+(\eta)+\frac{\beta x_n}{2}\right)\right)}\tilde{\psi}(\eta)\chi_0\left(\frac{\beta\gamma x_n}{|\tilde{f}_+(\eta)|}\right)d\eta.$$

We thus obtain similar estimates as above with an additional $\tau^{3/2}$ factor. This concludes the proof. □

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