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ON THE ENERGY SUBCRITICAL, NONLINEAR WAVE EQUATION IN $\mathbb{R}^3$ WITH RADIAL DATA

RUIPENG SHEN

In this paper, we consider the wave equation in 3-dimensional space with an energy-subcritical nonlinearity, either in the focusing or defocusing case. We show that any radial solution of the equation which is bounded in the critical Sobolev space is globally defined in time and scatters. The proof depends on the compactness/rigidity argument, decay estimates for radial, "compact" solutions, gain of regularity arguments and the "channel of energy" method.

1. Introduction

In this paper we will consider the energy subcritical, nonlinear wave equation in $\mathbb{R}^3$ with radial initial data

$$\begin{cases} \partial_t^2 u - \Delta u = \pm |u|^{p-1} u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0 \in \dot{H}^{s_p}(\mathbb{R}^3), \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s_p - 1}(\mathbb{R}^3). \end{cases} \tag{1}$$

Here $3 < p < 5$ and

$$s_p = \frac{3}{2} - \frac{2}{p-1}.$$

The positive sign in the nonlinear term gives us the focusing case, while the negative sign indicates the defocusing case. The quantity

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2)\, dx \mp \frac{1}{p+1} \int_{\mathbb{R}^3} |u(x,t)|^{p+1}\, dx \tag{2}$$

is called the energy of the solution. The energy is a constant in the whole lifespan of the solution, as long as it is well-defined. Note that the energy can be a negative number in the focusing case.

Previous results in the energy-critical case. In the energy-critical case, namely $p = 5$, the initial data is in the energy space $\dot{H}^1 \times L^2$. This automatically guarantees the existence of the energy by the Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$. This kind of wave equations has been extensively studied. In the defocusing case, M. Grillakis [1990; 1992] proved the global existence and scattering of the solution with any $\dot{H}^1 \times L^2$ initial data. In the focusing case, however, the behavior of solutions is much more complicated. The solutions may scatter, blow up in finite time or even be independent of time. (See [Duyckaerts et al. 2013; Kenig and Merle 2008] for more details.) In particular, a solution independent of time is usually

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called a ground state or a soliton. This kind of solutions is actually the solutions of the elliptic equation $-\Delta W(x) = |W(x)|^{p-1}W(x)$. We can write down all the nontrivial radial solitons explicitly as

$$W(x) = \pm\frac{1}{\lambda^{1/2}}\left(1+\frac{|x|^2}{3\lambda^2}\right)^{-\frac{1}{2}}. \tag{3}$$

Here λ is an arbitrary positive parameter.

Energy subcritical case. We will consider the case $3 < p < 5$ in this paper; thus $1/2 < s_p < 1$. In this case the problem is critical in the space $\dot{H}^{s_p}(\mathbb{R}^3) \times \dot{H}^{s_p-1}(\mathbb{R}^3)$, because if $u(x,t)$ is a solution of (1) with initial data (u_0, u_1), then for any $\lambda > 0$, the function

$$\frac{1}{\lambda^{3/2-s_p}}u\left(\frac{x}{\lambda},\frac{t}{\lambda}\right)$$

is another solution of the (1) with the initial data

$$\left(\frac{1}{\lambda^{3/2-s_p}}u_0\left(\frac{x}{\lambda}\right), \frac{1}{\lambda^{5/2-s_p}}u_1\left(\frac{x}{\lambda}\right)\right),$$

which shares the same $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm as the original initial data (u_0, u_1). These scalings play an important role in our discussion of this problem.

Theorem 1.1 (main theorem). *Let u be a solution of the nonlinear wave equation (1) with radial initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ and a maximal lifespan I so that*

$$\sup_{t\in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} < \infty. \tag{4}$$

Then u is global in time ($I = \mathbb{R}$) and scatters; that is,

$$\|u(x,t)\|_{S(\mathbb{R})} < \infty, \quad \textit{or equivalently} \quad \|u(x,t)\|_{Y_{s_p}(\mathbb{R})} < \infty.$$

This is actually equivalent to saying that there exist two pairs (u_0^+, u_1^+) and (u_0^-, u_1^-) in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ such that

$$\lim_{t\to\pm\infty} \left\|\left(u(t) - S(t)(u_0^\pm, u_1^\pm), \partial_t u(t) - \partial_t S(t)(u_0^\pm, u_1^\pm)\right)\right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} = 0.$$

Here $S(t)(u_0^\pm, u_1^\pm)$ is the solution of the linear wave equation with the initial data $(u_0^\pm, u_1^\pm)$.

Please refer to Definition 2.4 for the S and Y_s norms. There are a couple of remarks on the main theorem.

- *The defocusing case.* As in the energy-critical case, we expect that the solutions always scatter as long as the initial data are in the critical Sobolev space. Besides the radial condition, the main theorem depends on the assumption (4), which is expected to be true for all solutions. Unfortunately, as far as the author knows, no one actually knows how to prove it without additional assumptions.

- *The focusing case.* In the focusing case, the solutions may blow up in finite time. (See Theorem 6.3, for instance.) Thus the assumption (4) is a meaningful and essential condition rather than a technical one. The main theorem gives us the following rough classification of radial solutions.

Proposition 1.2. *Let $u(t)$ be a solution of (1) in the focusing case with a maximal lifespan I and radial initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. Then one of the following holds for $u(x,t)$.*

(I) (*blow-up*) *The $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm of $(u(t), \partial_t u(t))$ blows up, namely*

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} = +\infty.$$

(II) (*scattering*) *If the upper bound of the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm above is finite instead, namely, the assumption (4) holds, then $u(t)$ is a global solution (i.e, $I = \mathbb{R}$) and scatters.*

Main idea in this paper. The main idea to establish Theorem 1.1 is to use the compactness/rigidity argument, namely to show:

(I) If the main theorem failed, it would break down at a minimal blow-up solution, which is almost periodic modulo scalings.

(II) The minimal blow-up solution is in the energy space.

(III) The minimal blow-up solution described above does not exist.

Step (I). The method of profile decomposition used here has been a standard way to deal with both the wave equation and the Schrödinger equation. Thus we will only give important statements instead of showing all the details. The other steps, however, depend on the specific problems. One could refer to [Bahouri and Gérard 1999] in order to understand what the profile decomposition is, and to [Kenig and Merle 2008; 2010; Killip and Visan 2010] in order to see why the profile decomposition leads to the existence of a minimal blow-up solution.

Step (II). We will combine the method used in my old paper [Shen 2011] and a method used in [Kenig and Merle 2011] on the supercritical case of the nonlinear wave equation in $\mathbb{R}^3$. The idea is to use the following fact. Given a radial solution $u(x,t)$ of the equation

$$\partial_t^2 u(x,t) - \Delta u(x,t) = F(x,t)$$

defined in the time interval I, if we define two functions $w, h : \mathbb{R}^+ \times I \to \mathbb{R}$, such that $w(|x|,t) = |x| u(x,t)$ and $h(|x|,t) = |x| F(x,t)$, then $w(r,t)$ is a solution of the one-dimensional wave equation $\partial_t^2 w(r,t) - \partial_r^2 w(r,t) = h(r,t)$. This makes it convenient to consider the integral

$$\int_{r_0 \pm t}^{4r_0 \pm t} |\partial_t w(r, t_0 + t) \mp \partial_r w(r, t_0 + t)|^2 \, dr.$$

as the parameter t changes.

Step (III). Given an energy estimate, all minimal blow-up solutions are not difficult to kill except for the soliton-like solutions in the focusing case. As I mentioned earlier, this kind of solutions actually exists in the energy-critical case. The ground states given in (3) are perfect examples. In the energy-subcritical case, however, the soliton does not exist at all. More precisely, none of the solutions of the corresponding elliptic equation is in the right space $\dot{H}^{s_p}$. This fact enables us to gain a contradiction by showing a soliton-like minimal blow-up solution must be a real soliton, which does not exist, using a new method introduced by T. Duyckaerts, C. E. Kenig and F. Merle. They classified all radial solutions of the energy-critical, focusing wave equation in their recent paper [Duyckaerts et al. 2013] using this "channel of energy" method.

Remark on the supercritical case. Simultaneously to this work, T. Duyckaerts et al. [2012] proved that results similar to ours also hold in the supercritical case $p > 5$ of the focusing wave equation, using the compactness/rigidity argument, a point-wise estimate on "compact" solutions obtained in [Kenig and Merle 2011] and the channel of energy method mentioned above.

2. Preliminary results

Notation. The following notation will be used throughout this paper.

- ($\lesssim$) The inequality $A \lesssim B$ means that there exists a constant c such that $A \leq cB$. A subscript on $\lesssim$ implies that the constant c depends on the parameter(s) indicated but nothing else.
- (the smooth frequency cutoff) We use $P_{<A}$ and $P_{>A}$ for the standard smooth frequency cutoff operators. In particular, we use the following notation on u for convenience:

$$u_{<A} \doteq P_{<A}u, \quad u_{>A} \doteq P_{>A}u.$$

- (notation for radial functions) If $u(x,t)$ is radial in the space, then $u(r,t)$ represents the value $u(x,t)$ when $|x| = r$.
- (linear wave evolution) Let $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ be a pair of initial data. Suppose $u(x,t)$ is the solution of the linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = u_1. \end{cases}$$

  We will use the following notation to represent this solution u:

$$S(t_0)(u_0, u_1) = u(t_0), \quad \mathbf{S}(t_0)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t_0) \\ \partial_t u(t_0) \end{pmatrix}.$$

- (method of center cutoff) Let $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0,r))$ be a pair of radial functions. We define (with $R > r$)

$$(\Psi_R v_0)(x) = \begin{cases} v_0(x) & \text{if } |x| > R, \\ v_0(R) & \text{if } |x| \leq R, \end{cases}$$

$$(\Psi_R v_1)(x) = \begin{cases} v_1(x) & \text{if } |x| > R, \\ 0 & \text{if } |x| \leq R. \end{cases}$$

Local theory with $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ initial data. In this section, we will review the theory for the Cauchy problem of the nonlinear wave equation (1) with initial data in the critical Sobolev space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. The same local theory works in both the focusing and defocusing cases. It can be also applied to the nonradial case.

Definition 2.1 (space-time norm). Let I be an interval of time. If $1 \le q, r < \infty$, the space-time norm is defined by

$$\|v(x,t)\|_{L^q L^r(I\times\mathbb{R}^3)} = \left(\int_I \left(\int_{\mathbb{R}^3} |v(x,t)|^r \, dx\right)^{q/r} dt\right)^{1/q},$$

$$\|v(x,t)\|_{L^\infty L^r(I\times\mathbb{R}^3)} = \inf\left\{ M > 0 : \left(\int_{\mathbb{R}^3} |v(x,t)|^r \, dx\right)^{1/r} < M, \text{ a.e. } t \in I \right\}.$$

This is used in the following Strichartz estimates.

Proposition 2.2 (generalized Strichartz inequalities; see Proposition 3.1 of [Ginibre and Velo 1995] — here we use the Sobolev version in $\mathbb{R}^3$). *Let $2 \le q_1, q_2 \le \infty$, $2 \le r_1, r_2 < \infty$ and $\rho_1, \rho_2, s \in \mathbb{R}$ with*

$$\begin{aligned} 1/q_i + 1/r_i &\le 1/2 \quad \textit{for } i = 1,2, \\ 1/q_1 + 3/r_1 &= 3/2 - s + \rho_1, \\ 1/q_2 + 3/r_2 &= 1/2 + s + \rho_2. \end{aligned}$$

Let u be the solution of the linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(x,t), & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3), \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3). \end{cases} \tag{5}$$

Then we have

$$\|(u(T), \partial_t u(T))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{\rho_1} u\|_{L^{q_1} L^{r_1}([0,T]\times\mathbb{R}^3)} \le C\left(\|(u_0,u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} + \|D_x^{-\rho_2} F(x,t)\|_{L^{\bar{q}_2} L^{\bar{r}_2}([0,T]\times\mathbb{R}^3)}\right).$$

The constant C does not depend on T.

Definition 2.3 (admissible pair). If $(q_1, r_1, s, \rho_1) = (q, r, m, 0)$ satisfies the conditions in Proposition 2.2, we say (q, r) is an m-admissible pair.

Definition 2.4. Fix $3 < p < 5$. We define the following norms with $s_p \le s \le 1$:

$$\begin{aligned} \|v(x,t)\|_{S(I)} &= \|v(x,t)\|_{L^{2(p-1)} L^{2(p-1)}(I\times\mathbb{R}^3)}, \\ \|v(x,t)\|_{W(I)} &= \|v(x,t)\|_{L^4 L^4(I\times\mathbb{R}^3)}, \\ \|v(x,t)\|_{Z_s(I)} &= \|v(x,t)\|_{L^{\frac{2}{s+1}} L^{\frac{2}{2-s}}(I\times\mathbb{R}^3)}, \\ \|v(x,t)\|_{Y_s(I)} &= \|v(x,t)\|_{L^{\frac{2p}{s+1-(2p-2)(s-s_p)}} L^{\frac{2p}{2-s}}(I\times\mathbb{R}^3)}. \end{aligned}$$

Remark 2.5. By the Strichartz estimates, we have if $u(x,t)$ is the solution of

$$\begin{cases} \partial_t^2 u - \Delta u = F(x,t), & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0 \in \dot{H}^s(\mathbb{R}^3), \\ \partial_t u|_{t=0} = u_1 \in \dot{H}^{s-1}(\mathbb{R}^3). \end{cases}$$

then

$$\|(u(T), \partial_t u(T))\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|u\|_{Y_s([0,T])} \le C(\|(u_0,u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F(x,t)\|_{Z_s([0,T])}).$$

Definition 2.6 (solutions). We say $u(t)(t \in I)$ is a solution of (1), if $(u(t), \partial_t u(t)) \in C(I; \dot{H}^{s_p} \times \dot{H}^{s_p-1})$, with finite norms $\|u\|_{S(J)}$ and $\|D_x^{s_p-1/2} u\|_{W(J)}$ for any bounded closed interval $J \subseteq I$ so that the integral equation

$$u(t) = S(t)(u_0,u_1) + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau$$

holds for all time $t \in I$. Here $S(t)(u_0,u_1)$ is the solution of the linear wave equation with initial data (u_0,u_1) and

$$F(u) = \pm |u|^{p-1} u.$$

Remark 2.7. We can take another way to define the solutions by substituting $S(I)$ and $W(I)$ norms by a single $Y_{s_p}(I)$ norm. Using the Strichartz estimates, these two definitions are equivalent to each other.

By the Strichartz estimate and a fixed-point argument, we have the following theorems. (Our argument is similar to those in a lot of earlier papers. See, for instance, [Lindblad and Sogge 1995; Kenig and Merle 2008] for more details.)

Theorem 2.8 (local solution). *For any initial data $(u_0,u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$, there is a maximal interval $(-T_-(u_0,u_1), T_+(u_0,u_1))$ in which the equation has a solution.*

Theorem 2.9 (scattering with small data). *There exists $\delta = \delta(p) > 0$ such that if the norm of the initial data $\|(u_0,u_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \delta$, then the Cauchy problem (1) has a global-in-time solution u with $\|u\|_{S(-\infty,+\infty)} < \infty$.*

Lemma 2.10 (standard finite blow-up criterion). *If $T_+ < \infty$, then $\|u\|_{S([0,T_+))} = \infty$.*

Theorem 2.11 (long-time perturbation theory; see [Colliander et al. 2008; Kenig and Merle 2008; 2006; 2011]). *Fix $3 < p < 5$. Let M, A, A' be positive constants. There exists $\varepsilon_0 = \varepsilon_0(M,A,A') > 0$ and $\beta > 0$ such that if $\varepsilon < \varepsilon_0$, then for any approximation solution $\tilde{u}$ defined on $\mathbb{R}^3 \times I$ $(0 \in I)$ and any initial data $(u_0,u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ satisfying*

$$\begin{gathered} (\partial_t^2 - \Delta)(\tilde{u}) - F(\tilde{u}) = e(x,t), \quad (x,t) \in \mathbb{R}^3 \times I, \\ \begin{cases} \sup_{t\in I} \|(\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \le A, \\ \|\tilde{u}\|_{S(I)} \le M, \\ \|D_x^{s_p-1/2} \tilde{u}\|_{W(J)} < \infty \quad \textit{for each } J \Subset I, \end{cases} \\ \|(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \le A', \\ \|D_x^{s_p-\frac{1}{2}} e\|_{L_I^{4/3} L_x^{4/3}} + \|S(t)(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{S(I)} \le \varepsilon, \end{gathered} \tag{6}$$

there exists a solution of (1) *defined in the interval I with the initial data (u_0, u_1) and satisfying*

$$\|u\|_{S(I)} \le C(M, A, A'),$$
$$\sup_{t\in I} \|(u(t), \partial_t u(t)) - (\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \le C(M, A, A')(A' + \varepsilon + \varepsilon^{\beta}).$$

Theorem 2.12 (perturbation theory with Y_{s_p} norm). *Fix $3 < p < 5$. Let M be a positive constant. There exists a constant $\varepsilon_0 = \varepsilon_0(M) > 0$ such that if $\varepsilon < \varepsilon_0$, then for any approximation solution $\tilde{u}$ defined on $\mathbb{R}^3 \times I$ $(0 \in I)$ and any initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ satisfying*

$$(\partial_t^2 - \Delta)(\tilde{u}) - F(\tilde{u}) = e(x,t), \quad (x,t) \in \mathbb{R}^3 \times I,$$
$$\|\tilde{u}\|_{Y_{s_p}(I)} < M, \quad \|(\tilde{u}(0), \partial_t \tilde{u}(0))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} < \infty,$$
$$\|e(x,t)\|_{Z_{s_p}(I)} + \|S(t)(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{Y_{s_p}(I)} \le \varepsilon,$$

there exists a solution $u(x,t)$ of (1) *defined in the interval I with the initial data (u_0, u_1) and satisfying*

$$\|u(x,t) - \tilde{u}(x,t)\|_{Y_{s_p}(I)} < C(M)\varepsilon.$$
$$\sup_{t\in I} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} - S(t) \begin{pmatrix} u_0 - \tilde{u}(0) \\ u_1 - \partial_t \tilde{u}(0) \end{pmatrix} \right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} < C(M)\varepsilon.$$

Remark 2.13. If K is a compact subset of the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, then there exists $T = T(K) > 0$ such that $T_+(u_0, u_1) > T(K)$ for any $(u_0, u_1) \in K$. This is a direct result from perturbation theory.

Local theory with more regular initial data. Let $s \in (s_p, 1]$. By a similar fixed-point argument we can obtain the following results.

Theorem 2.14 (local solution with $\dot{H}^s \times \dot{H}^{s-1}$ initial data). *If $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, then there is a maximal interval $(-T_-(u_0, u_1), T_+(u_0, u_1))$ in which the equation has a solution $u(x,t)$. In addition, we have*

$$T_-(u_0, u_1), T_+(u_0, u_1) > T_1 \doteq C_{s,p}(\|(u_0, u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}})^{-1/(s-s_p)},$$
$$\|u(x,t)\|_{Y_s([-T_1,T_1])} \le C_{s,p}\|(u_0, u_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}.$$

Theorem 2.15 (weak long-time perturbation theory). *Let $\tilde{u}$ be a solution of the equation* (1) *in the time interval $[0, T]$ with initial data $(\tilde{u}_0, \tilde{u}_1)$, so that*

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} < \infty, \quad \|\tilde{u}\|_{Y_s([0,T])} < M.$$

There exist two constants $\varepsilon_0(T, M), C(T, M) > 0$ such that if (u_0, u_1) is another pair of initial data with

$$\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s\times\dot{H}^{s-1}} < \varepsilon_0(T, M),$$

then there exists a solution u of the equation (1) *in the time interval $[0, T]$ with initial data (u_0, u_1) so that*

$$\|u - \tilde{u}\|_{Y_s([0,T])} \le C(T, M)\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s\times\dot{H}^{s-1}},$$
$$\sup_{t\in[0,T]} \|(u(t) - \tilde{u}(t), \partial_t u(t) - \partial_t \tilde{u}(t))\|_{\dot{H}^s\times\dot{H}^{s-1}} \le C(T, M)\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s\times\dot{H}^{s-1}}.$$

Technical results.

Lemma 2.16 (gluing of $\dot{H}^s$ functions). *Let $-1 \le s \le 1$. Suppose $f(x)$ is a tempered distribution defined on $\mathbb{R}^3$ such that* $(R > 0)$

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in B(0,2R), \\ f_2(x) & \text{for } x \in \mathbb{R}^3 \setminus B(0,R), \end{cases}$$

with $f_1, f_2 \in \dot{H}^s(\mathbb{R}^3)$. Then f is in the space $\dot{H}^s(\mathbb{R}^3)$ with

$$\|f\|_{\dot{H}^s(\mathbb{R}^3)} \le C(s)\big(\|f_1\|_{\dot{H}^s(\mathbb{R}^3)} + \|f_2\|_{\dot{H}^s(\mathbb{R}^3)}\big).$$

Proof. By a dilation we can always assume $R = 1$. Let $\phi(x)$ be a smooth, radial, nonnegative function such that

$$\phi(x) = \begin{cases} 1 & \text{for } x \in B(0,1), \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus B(0,2). \end{cases}$$

Let us define a linear operator: $P(f) = \phi(x) f$. We know this operator is bounded from $\dot{H}^1(\mathbb{R}^3)$ to $\dot{H}^1(\mathbb{R}^3)$, and from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Thus by an interpolation, this is a bounded operator from $\dot{H}^s$ to itself if $0 < s < 1$. By duality P is also bounded from $\dot{H}^s$ to itself if $-1 \le s \le 0$. In summary, P is a bounded operator from $\dot{H}^s$ to itself for each $-1 \le s \le 1$. Now we have

$$f = Pf_1 + f_2 - Pf_2$$

as a tempered distribution. Thus

$$\|f\|_{\dot{H}^s} \le \|Pf_1\|_{\dot{H}^s} + \|f_2\|_{\dot{H}^s} + \|Pf_2\|_{\dot{H}^s} \le (\|P\|_s + 1)(\|f_1\|_{\dot{H}^s} + \|f_2\|_{\dot{H}^s}). \qquad \square$$

Lemma 2.17. *Let $u(x,t)$ be a solution of the nonlinear wave equation (1) with the condition (4). Then for any $t_1, t_2 \in I$ and $t \in \mathbb{R}$, we have*

$$\left\| \begin{pmatrix} \displaystyle\int_{t_1}^{t_2} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau \\ \displaystyle -\int_{t_1}^{t_2} \cos((\tau - t)\sqrt{-\Delta}) F(u(\tau))\, d\tau \end{pmatrix} \right\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} \lesssim 1. \tag{7}$$

Proof. It follows directly from the identity

$$\begin{pmatrix} \displaystyle\int_{t_1}^{t_2} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau \\ \displaystyle -\int_{t_1}^{t_2} \cos((\tau - t)\sqrt{-\Delta}) F(u(\tau))\, d\tau \end{pmatrix} = S(t - t_1) \begin{pmatrix} u(t_1) \\ \partial_t u(t_1) \end{pmatrix} - S(t - t_2) \begin{pmatrix} u(t_2) \\ \partial_t u(t_2) \end{pmatrix}. \qquad \square$$

Lemma 2.18 (see Lemma 3.2 of [Kenig and Merle 2011]). *Let $1/2 < s < 3/2$. If $u(y)$ is a radial $\dot{H}^s(\mathbb{R}^3)$ function, then*

$$|u(y)| \lesssim_s \frac{1}{|y|^{\frac{3}{2} - s}} \|u\|_{\dot{H}^s}. \tag{8}$$

Remark 2.19. This actually means that a radial $\dot{H}^s$ function is uniformly continuous in $\mathbb{R}^3 \setminus B(0, R)$ if $R > 0$.

Lemma 2.20. *Let $r_1, r_2 > 0$ and $t_0, t_1 \in \mathbb{R}$ so that $r_1 + r_2 \le t_1 - t_0$. Suppose (u_0, u_1) is a weak limit in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$:*

$$\begin{aligned} u_0 &= \lim_{T\to+\infty} \int_{t_1}^{T} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t)\, dt, \\ u_1 &= -\lim_{T\to+\infty} \int_{t_1}^{T} \cos((t-t_0)\sqrt{-\Delta}) F(t)\, dt. \end{aligned} \tag{9}$$

Here $F(x,t)$ is a function defined in $[t_1, \infty) \times \mathbb{R}^3$ with a finite $Z_{s_p}([t_1, T])$ norm for each $T > t_1$. In addition, we have ($1/2 < s_1 \le 1$, $\chi(x,t)$ is the characteristic function of the region indicated)

$$S = \|\chi_{|x|>r_2+|t-t_1|}(x,t) F(x,t)\|_{L^1 L^{\frac{6}{5-2s_1}}([t_1,\infty)\times\mathbb{R}^3)} < +\infty. \tag{10}$$

Then there exists a pair $(\tilde{u}_0, \tilde{u}_1)$ with $\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1}\times\dot{H}^{s_1-1}(\mathbb{R}^3)} \le C_{s_1} S$ and

$$(u_0, u_1) = (\tilde{u}_0, \tilde{u}_1) \text{ in the ball } B(0, r_1).$$

Proof. Let us define

$$\begin{aligned} u_{0,T} &= \int_{t_1}^{T} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t)\, dt, & u_{1,T} &= -\int_{t_1}^{T} \cos((t-t_0)\sqrt{-\Delta}) F(t)\, dt, \\ \tilde{u}_{0,T} &= \int_{t_1}^{T} \frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} (\chi F(t))\, dt, & \tilde{u}_{1,T} &= -\int_{t_1}^{T} \cos((t-t_0)\sqrt{-\Delta}) (\chi F(t))\, dt. \end{aligned}$$

By the Strichartz estimates and the assumption (10), we know the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ converges strongly in $\dot{H}^{s_1} \times \dot{H}^{s_1-1}$ to a pair $(\tilde{u}_0, \tilde{u}_1)$ as $T \to +\infty$ so that

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1}\times\dot{H}^{s_1-1}(\mathbb{R}^3)} \le C_{s_1} S.$$

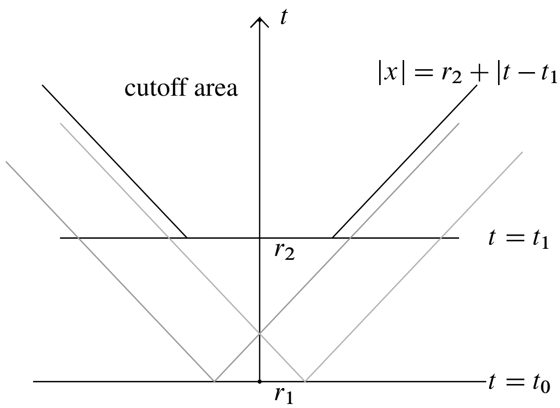

Figure 1. Illustration of proof.

In addition, we know the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ is the same as $(u_{0,T}, u_{1,T})$ in the ball $B(0, r_1)$ by the strong Huygens principle. Figure 1 shows the region where the value of $F(x,t)$ may affect the value of the integrals in the ball $B(0, r_1)$. This region is disjoint with the cutoff area if $r_1 + r_2 \le t_1 - t_0$. As a result, the pair $(\tilde{u}_{0,T}, \tilde{u}_{1,T})$ converges to (u_0, u_1) weakly in the ball $B(0, r_1)$ as the pair $(u_{0,T}, u_{1,T})$ does. Considering both strong and weak convergence, we conclude that

$$(u_0, u_1) = (\tilde{u}_0, \tilde{u}_1) \quad \text{in the ball } B(0, r_1). \qquad \square$$

3. Compactness process

As we stated in the first section, the standard technique here is to show that if the main theorem failed, there would be a special minimal blow-up solution. In addition, this solution is almost periodic modulo symmetries.

Definition 3.1. A solution $u(x,t)$ of (1) is almost periodic modulo symmetries if there exists a positive function $\lambda(t)$ defined on its maximal lifespan I such that the set

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. The function $\lambda(t)$ is called the frequency scale function, because the solution $u(t)$ at time t concentrates around the frequency $\lambda(t)$ by the compactness.

Remark 3.2. Here we use the radial condition, thus the only available symmetries are scalings. If we did not assume the radial condition, similar results would still hold but the symmetries would include translations besides scalings.

Existence of minimal blow-up solution.

Theorem 3.3 (minimal blow-up solution). *Assume that the main theorem failed. Then there would exist a solution $u(x,t)$ with a maximal lifespan I such that*

$$\sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \infty;$$

u blows up in the positive direction at time $T_+ \le +\infty$ with

$$\|u\|_{S([0,T_+))} = \infty.$$

In addition, u is almost periodic modulo scalings with a frequency scale function $\lambda(t)$. It is minimal in the following sense: if v is another solution with a maximal lifespan J and

$$\sup_{t \in J} \|(v(t), \partial_t v(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} < \sup_{t \in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}},$$

then v is a global solution in time and scatters.

The main tool to obtain this result is the profile decomposition. One can follow the general argument in [Kenig and Merle 2010], which deals with the cubic defocusing NLS under similar assumptions.

Three enemies. Since the frequency scale function $\lambda(t)$ plays an important role in the further discussion, it is helpful if we could make additional assumptions on this function. It turns out that we can reduce the whole problem into the following three special cases. This method of three enemies was introduced in R. Killip, T. Tao and M. Visan's paper [Killip et al. 2009].

Theorem 3.4 (three enemies). *Suppose our main theorem failed. Then there would exist a minimal blow-up solution u satisfying all the conditions we mentioned in the previous theorem, so that one of the following three assumptions on its lifespan I and frequency scale function $\lambda(t)$ holds*:

(I) (soliton-like case) $I=\mathbb{R}$ *and* $\lambda(t)\equiv 1$.

(II) (high-to-low frequency cascade) $I=\mathbb{R}$, $\lambda(t)\le 1$ *and*

$$\liminf_{t\to\pm\infty}\lambda(t)=0.$$

(III) (self-similar case) $I=\mathbb{R}^+$ *and* $\lambda(t)=1/t$.

The minimal blow-up solution u here could be different from the one we found in the previous theorem. But we can always manufacture a minimal blow-up solution in one of these three cases from the original one. One can follow the method used in [Killip et al. 2009] to verify this theorem.

Further compactness results. Fix a radial cutoff function $\varphi(x)\in C^\infty(\mathbb{R}^3)$ with the properties

$$\varphi(x)\begin{cases}=0 & \text{for } |x|\le 1/2,\\ \in[0,1] & \text{for } 1/2\le|x|\le 1,\\ =1 & \text{for } |x|\ge 1.\end{cases}$$

Given a minimal blow-up solution u mentioned above and its frequency scale function $\lambda(t)$, we have the following propositions by a compactness argument.

Proposition 3.5. *Let u be a minimal blow-up solution with a maximal lifespan I as above. There exist constants $d, C'>0$ and $C_1>1$ independent of t such that*:

(I) *The interval $[t-d\lambda^{-1}(t),t+d\lambda^{-1}(t)]\subseteq I$ for all $t\in I$. In addition, we have*

$$\frac{1}{C_1}\lambda(t)\le\lambda(t')\le C_1\lambda(t)\tag{11}$$

for each $t'\in[t-d\lambda^{-1}(t),t+d\lambda^{-1}(t)]$.

(II) *The following estimate holds for each s_p-admissible pair (q,r) and each $t\in I$*:

$$\|u\|_{L^qL^r([t-d\lambda^{-1}(t),t+d\lambda^{-1}(t)]\times\mathbb{R}^3)}\le C'.$$

Proposition 3.6. *Given $\varepsilon>0$, there exists $R_1=R_1(\varepsilon)>0$ such that the inequality*

$$\left\|\left(\varphi\left(\frac{x}{R\lambda^{-1}(t)}\right)u(t),\varphi\left(\frac{x}{R\lambda^{-1}(t)}\right)\partial_t u(t)\right)\right\|_{\dot H^{s_p}\times\dot H^{s_p-1}(\mathbb{R}^3)}\le\varepsilon$$

holds for each $t\in I$ and $R>R_1(\varepsilon)$.

Proposition 3.7. *There exists two constants $R_0, \eta_0 > 0$, such that the inequality*

$$\int_t^{t+d\lambda^{-1}(t)} \int_{|x|<R_0\lambda^{-1}(t)} \frac{|u(x,\tau)|^{p+1}}{|x|} \, dx\, d\tau \geq \lambda(t)^{2-2s_p}\eta_0$$

holds for each $t \in I$. (The constant d is the same constant we used in Proposition 3.5.)

Proof. By a compactness argument we obtain that there exist $R_0, \eta_0 > 0$, so that for all $t \in I$,

$$\int_0^d \int_{|x|<R_0} \frac{\left(\frac{1}{\lambda(t)^{2/(p-1)}} |u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|\right)^{p+1}}{|x|} \, dx\, d\tau \geq \eta_0.$$

This implies

$$\int_0^d \int_{|x|<R_0} \frac{|u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|^{p+1}}{\lambda^{-1}(t)|x|} \frac{dx\, d\tau}{\lambda(t)^{\frac{2(p+1)}{p-1}+1}} \geq \eta_0.$$

$$\frac{1}{\lambda(t)^{4/(p-1)-1}} \int_0^d \int_{|x|<R_0} \frac{|u(\lambda^{-1}(t)x, \lambda^{-1}(t)\tau + t)|^{p+1}}{\lambda^{-1}(t)|x|} \frac{dx\, d\tau}{\lambda(t)^4} \geq \eta_0.$$

$$\begin{aligned} \int_t^{t+d\lambda^{-1}(t)} \int_{|x|<R_0\lambda^{-1}(t)} \frac{|u(x,\tau)|^{p+1}}{|x|} \, dx\, d\tau &\geq \lambda(t)^{4/(p-1)-1}\eta_0 \\ &= \lambda(t)^{2-2s_p}\eta_0. \end{aligned} \tag{12}$$

This completes the proof. □

The Duhamel formula. The following formula will be frequently used in later sections.

Proposition 3.8 (Duhamel formula). *Let u be a minimal blow-up solution described above with a maximal lifespan $I = (T_-, \infty)$. Then we have*

$$\begin{aligned} u(t) &= \lim_{T\to+\infty} \int_t^T \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau, \\ \partial_t u(t) &= -\lim_{T\to+\infty} \int_t^T \cos((\tau-t)\sqrt{-\Delta}) F(u(\tau))\, d\tau; \\ u(t) &= \lim_{T\to T_-} \int_T^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau, \\ \partial_t u(t) &= \lim_{T\to T_-} \int_T^t \cos((t-\tau)\sqrt{-\Delta}) F(u(\tau))\, d\tau. \end{aligned}$$

Given a time $t \in I$, these limits are weak limits in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. If J is a closed interval compactly supported in I, then one can also understand the formula for $u(t)$ as a strong limit in the space $L^q L^r(J \times \mathbb{R}^3)$, as long as (q, r) is an s_p-admissible pair with $q \neq \infty$.

Remark 3.9. Actually we have

$$\begin{pmatrix} \int_t^T \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\,d\tau \\ -\int_t^T \cos((\tau-t)\sqrt{-\Delta}) F(u(\tau))\,d\tau \end{pmatrix} = \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - S(t-T)\begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix}. \tag{13}$$

Thus we only need to show the corresponding limit of the last term is zero in order to verify this formula. See Lemma A.2 in the appendix for details.

4. Energy estimate near infinity

In this section, we will prove the following theorem for a minimal blow-up solution $u(x,t)$. The method was previously used in the supercritical case of the equation. (See [Kenig and Merle 2011] for more details.) In the supercritical case, by the Sobolev embedding, the energy automatically exists at least locally in the space, for any given time $t \in I$. In the subcritical case, however, we need to use the approximation techniques.

Theorem 4.1 (energy estimate near infinity). *Let $u(x,t)$ be a minimal blow-up solution as we found in the previous section. Then $(u(x,t), \partial_t u(x,t)) \in \dot{H}^1 \times L^2(\mathbb{R}^3 \setminus B(0,r))$ for each $r > 0$, $t \in I$. Actually we have*

$$\int_{r<|x|<4r} (|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2)\,dx \le C r^{-2(1-s_p)}. \tag{14}$$

The constant C depends on p and $\sup_{t\in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}$ but nothing else.

Preliminary results.

Introduction to $w(r,t)$. Let $u(x,t)$ be a radial solution of the wave equation

$$\partial_t^2 u - \Delta u = F(x,t).$$

If we define $w(r,t), h(r,t) : \mathbb{R}^+ \times I \to \mathbb{R}$ so that

$$w(r,t) = ru(x,t), \quad h(r,t) = rF(x,t),$$

then we have $w(r,t)$ is the solution of the one-dimensional wave equation

$$\partial_t^2 w - \partial_r^2 w = h(r,t).$$

Lemma 4.2. *Let $(u(x,t_0), \partial_t u(x,t_0))$ be radial and in the energy space $\dot{H}^1 \times L^2$ locally. Then for any $0 < a < b < \infty$, we have that the identity*

$$\frac{1}{4\pi}\int_{a<|x|<b} (|\nabla u|^2 + |\partial_t u|^2)\,dx = \left(\int_a^b [(\partial_r w)^2 + (\partial_t w)^2]\,dr\right) + (au^2(a) - bu^2(b))$$

holds (if we take the value of the functions at time t_0).

Proof. By direct computation

$$\begin{aligned}
\int_a^b [(\partial_r w)^2 + (\partial_t w)^2]\,dr &= \int_a^b [(r\partial_r u + u)^2 + (r\partial_t u)^2]\,dr\\
&= \int_a^b [r^2(\partial_r u)^2 + u^2 + r^2(\partial_t u)^2]\,dr + \int_a^b 2ru\,\partial_r u\,dr\\
&= \int_a^b [r^2(\partial_r u)^2 + r^2(\partial_t u)^2 + u^2]\,dr + \int_a^b r\,d(u^2)\\
&= \int_a^b r^2[(\partial_r u)^2 + (\partial_t u)^2]\,dr + [ru^2]_a^b\\
&= \frac{1}{4\pi}\int_{a<|x|<b} (|\nabla u|^2 + |\partial_t u|^2)\,dx + bu^2(b) - au^2(a). \qquad \square
\end{aligned}$$

Lemma 4.3. *Let $w(r,t)$ be a solution to the equation*

$$\partial_t^2 w - \partial_r^2 w = h(r,t)$$

for $(r,t) \in \mathbb{R}^+ \times I$, so that $(w, \partial_t w) \in C(I; \dot{H}^1 \times L^2(R_1 < r < R_2))$ for any $0 < R_1 < R_2 < \infty$. Let us define

$$\begin{aligned}
z_1(r,t) &= \partial_t w(r,t) - \partial_r w(r,t),\\
z_2(r,t) &= \partial_t w(r,t) + \partial_r w(r,t).
\end{aligned}$$

Then we have (with $M > 0$)

$$\begin{aligned}
&\left|\left(\int_{r_0}^{4r_0} |z_1(r,t_0)|^2\,dr\right)^{\frac12} - \left(\int_{r_0+M}^{4r_0+M} |z_1(r,t_0+M)|^2\,dr\right)^{\frac12}\right|\\
&\qquad\qquad \le \left(\int_{r_0}^{4r_0}\left(\int_0^M h(r+t,t_0+t)\,dt\right)^2 dr\right)^{\frac12} \qquad (15)
\end{aligned}$$

if $t_0, t_0 + M \in I$, and

$$\begin{aligned}
&\left|\left(\int_{r_0}^{4r_0} |z_2(r,t_0)|^2\,dr\right)^{\frac12} - \left(\int_{r_0+M}^{4r_0+M} |z_2(r,t_0-M)|^2\,dr\right)^{\frac12}\right|\\
&\qquad\qquad \le \left(\int_{r_0}^{4r_0}\left(\int_0^M h(r+t,t_0-t)\,dt\right)^2 dr\right)^{\frac12} \qquad (16)
\end{aligned}$$

if $t_0, t_0 - M \in I$.

Proof. We will assume w has sufficient regularity, otherwise we only need to use the standard techniques of smooth approximation. Let us define

$$z(r,s) = (\partial_t - \partial_r)w(r+s, t_0+s).$$

We have

$$\partial_s z(r,s) = (\partial_t + \partial_r)(\partial_t - \partial_r) w(r+s, t_0+s) = h(r+s, t_0+s).$$

Thus

$$z(r,M) = z(r,0) + \int_0^M h(r+t, t_0+t)\, dt.$$

Applying the triangle inequality, we obtain the first inequality. The second inequality can be proved in a similar way. □

Smooth approximation.

Introduction. Let $u(x,t)$ be a minimal blow-up solution. Choose a smooth, nonnegative, radial function $\varphi(x,t)$ supported in the four-dimensional ball $B(0,1) \subset \mathbb{R}^4$ such that

$$\int_{\mathbb{R}^3\times\mathbb{R}} \varphi(x,t)\, dx\, dt = 1.$$

Let d be the number given in Proposition 3.5. If $\varepsilon < d$, we define (both the functions u and $F(u)$ are locally integrable)

$$\varphi_\varepsilon(x,t) = \frac{1}{\varepsilon^4}\varphi(x/\varepsilon, t/\varepsilon), \quad u_\varepsilon = u * \varphi_\varepsilon, \quad F_\varepsilon = F(u) * \varphi_\varepsilon.$$

This makes $u_\varepsilon(x,t)$ be a smooth solution of the linear wave equation

$$\partial_t^2 u_\varepsilon(x,t) - \Delta u_\varepsilon(x,t) = F_\varepsilon(x,t),$$

with the convergence (using the continuity of $(u(t), \partial_t u(t))$ in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$)

$$(u_\varepsilon(t_0), \partial_t u_\varepsilon(t_0)) \to (u(t_0), \partial_t u(t_0)) \quad \text{in the space } \dot{H}^{s_p} \times \dot{H}^{s_p-1} \text{ for each } t_0 \in I$$

and the estimate

$$\|(u_\varepsilon(t_0), \partial_t u_\varepsilon(t_0))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \le \sup_{t\in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \lesssim 1.$$

In addition, if $a - \varepsilon \in I$, we have

$$\|F_\varepsilon(x,t)\|_{Z_{s_p}([a,b])} < \infty.$$

Remark 4.4. We have to apply the smooth kernel on the whole nonlinear term, because if we just made the initial data smooth, we would not resume the compactness conditions of the minimal blow-up solution.

The Duhamel formula.

Lemma 4.5 (almost periodic property). *The set*

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u_\varepsilon\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}}\, \partial_t u_\varepsilon\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in [d+1, \infty) \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ for each fixed $\varepsilon < d$. The number d here is the constant we obtained in Proposition 3.5.

Proof. Given a sequence $\{t_n\}$ we could assume without loss of generality that

$$\lambda(t_n) \to \lambda_0 \in [0, 1],$$
$$\left(\frac{1}{\lambda(t_n)^{3/2-s_p}} u\left(\frac{x}{\lambda(t_n)}, t_n\right), \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t_n)}, t_n\right)\right) \to (u_0, u_1),$$

by extracting a subsequence if necessary. Let $\tilde{u}(x,t)$ be the solution of the equation (1) with initial data (u_0, u_1). By the long-time perturbation theory we know

$$\sup_{t\in[-d,d]} \left\| \begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u\left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)}\right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t_n)}, t_n + \frac{t}{\lambda(t_n)}\right) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} \right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}} \to 0.$$

This implies

$$\begin{aligned}
\begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u_\varepsilon\left(\frac{x}{\lambda(t_n)}, t_n\right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u_\varepsilon\left(\frac{x}{\lambda(t_n)}, t_n\right) \end{pmatrix} &= \left[\varphi_{\varepsilon\lambda(t_n)} * \begin{pmatrix} \frac{1}{\lambda(t_n)^{3/2-s_p}} u\left(\frac{\cdot}{\lambda(t_n)}, t_n + \frac{\cdot}{\lambda(t_n)}\right) \\ \frac{1}{\lambda(t_n)^{5/2-s_p}} \partial_t u\left(\frac{\cdot}{\lambda(t_n)}, t_n + \frac{\cdot}{\lambda(t_n)}\right) \end{pmatrix}\right]_{t=0} \\
&= \left[\varphi_{\varepsilon\lambda(t_n)} * \begin{pmatrix} \tilde{u} \\ \partial_t \tilde{u} \end{pmatrix}\right]_{t=0} + o(1) \\
&= \begin{cases} \left[\varphi_{\varepsilon\lambda_0} * \begin{pmatrix} \tilde{u} \\ \partial_t \tilde{u} \end{pmatrix}\right]_{t=0} + o(1) & \text{if } \lambda_0 > 0; \\ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + o(1) & \text{if } \lambda_0 = 0; \end{cases}
\end{aligned}$$

The error $o(1)$ tends to zero as $n \to \infty$ in the sense of the $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ norm. ☐

Lemma 4.6. *The Duhamel formula*

$$u_\varepsilon(t_0) = \int_{t_0}^{+\infty} \frac{\sin((\tau - t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(x,\tau)\, d\tau,$$
$$\partial_t u_\varepsilon(t_0) = -\int_{t_0}^{+\infty} \cos((\tau - t_0)\sqrt{-\Delta}) F_\varepsilon(x,\tau)\, d\tau.$$

still holds for u_ε in the sense of weak limit if $\varepsilon < d$ and $t_0 - \varepsilon \in I$. In the soliton-like or high-to-low frequency cascade case, we can also establish the Duhamel formula in the negative time direction.

Proof. This lemma can be proved in exactly the same way as the original Duhamel formula (see Lemma A.2). The key ingredient is the almost periodic property we have just obtained above. ☐

Decay of u_ε and F_ε at infinity.

Lemma 4.7. *If $|x| > 10\varepsilon$, we have*

$$|u_\varepsilon(x,t)| \leq \frac{C}{|x|^{2/(p-1)}}, \quad |F_\varepsilon(x,t)| \leq \frac{C}{|x|^{2p/(p-1)}}.$$

The constant C depends only on p and the upper bound $\sup_{t\in I} \|(u(t), \partial_t u(t))\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}$.

Proof. This comes from the estimate (8) and an easy computation. ☐

Uniform estimate on u_ε. In this subsection, we will prove the following lemma. It implies Theorem 4.1 immediately by a limit process. The functions $w_\varepsilon(r,t)$ and $z_{i,\varepsilon}(r,t)$ below are defined as described earlier using $u_\varepsilon(x,t)$.

Lemma 4.8. *Let $t_0 \in I$ and $r_0 > 0$. Then for sufficiently small ε, we have*

$$\int_{r_0<|x|<4r_0} (|\nabla u_\varepsilon(x,t_0)|^2 + |\partial_t u_\varepsilon(x,t_0)|^2)\, dx \le C r_0^{-2(1-s_p)}. \tag{17}$$

The constant C can be chosen in a way that it depends only on p and the upper bound

$$\sup_{t\in I} \|(u(t),\partial_t u(t))\|_{\dot H^{s_p}\times \dot H^{s_p-1}}.$$

Conversion to $w_\varepsilon(r,t)$. First choose $\varepsilon < \min\{r_0/10, d\}$. If the minimal blow-up solution is a self-similar one, we also require $\varepsilon < t_0/2$. Let us apply Lemmas 4.2 and 4.7. It is sufficient to show

$$\int_{r_0}^{4r_0} (|\partial_r w_\varepsilon(r,t_0)|^2 + |\partial_t w_\varepsilon(r,t_0)|^2)\, dr \le C r_0^{-2(1-s_p)}.$$

In other words,

$$\int_{r_0}^{4r_0} (|z_{1,\varepsilon}(r,t_0)|^2 + |z_{2,\varepsilon}(r,t_0)|^2)\, dr \le C r_0^{-2(1-s_p)}. \tag{18}$$

Expansion of $z_{1,\varepsilon}$. Let us break $(u_\varepsilon(t), \partial_t u_\varepsilon(t))$ into two pieces:

$$u_\varepsilon^{(1)}(t) = \int_t^{t_0+100r_0} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(\tau)\, d\tau,$$
$$\partial_t u_\varepsilon^{(1)}(t) = -\int_t^{t_0+100r_0} \cos((\tau-t)\sqrt{-\Delta}) F_\varepsilon(\tau)\, d\tau,$$

and

$$u_\varepsilon^{(2)}(t) = \int_{t_0+100r_0}^{\infty} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F_\varepsilon(\tau)\, d\tau,$$
$$\partial_t u_\varepsilon^{(2)}(t) = -\int_{t_0+100r_0}^{\infty} \cos((\tau-t)\sqrt{-\Delta}) F_\varepsilon(\tau)\, d\tau.$$

These are smooth functions, and we have

$$(u_\varepsilon(x,t_0), \partial_t u_\varepsilon(x,t_0)) = (u_\varepsilon^{(1)}(x,t_0), \partial_t u_\varepsilon^{(1)}(x,t_0)) + (u_\varepsilon^{(2)}(x,t_0), \partial_t u_\varepsilon^{(2)}(x,t_0)).$$

Defining $w_\varepsilon^{(j)}, z_{1,\varepsilon}^{(j)}$ accordingly for $j = 1, 2$, we have

$$z_{1,\varepsilon}(x,t_0) = z_{1,\varepsilon}^{(1)}(x,t_0) + z_{1,\varepsilon}^{(2)}(x,t_0).$$

Short-time contribution. We have $u_\varepsilon^{(1)}$ satisfies the wave equation

$$\begin{cases}\partial_t^2 u_\varepsilon^{(1)} - \Delta u_\varepsilon^{(1)} = F_\varepsilon(x,t), & (x,t)\in \mathbb{R}^3\times(t_0^-,+\infty),\\ u_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0 \in \dot{H}^{s_p}(\mathbb{R}^3),\\ \partial_t u_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0 \in \dot{H}^{s_p-1}(\mathbb{R}^3).\end{cases}$$

Thus $w_\varepsilon^{(1)}$ is a smooth solution of

$$\begin{cases}\partial_t^2 w_\varepsilon^{(1)} - \partial_r^2 w_\varepsilon^{(1)} = rF_\varepsilon(r,t), & (r,t)\in \mathbb{R}^+\times(t_0^-,+\infty),\\ w_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0,\\ \partial_t w_\varepsilon^{(1)}|_{t=t_0+100r_0} = 0.\end{cases}$$

Applying Lemmas 4.3 and 4.7, we obtain

$$\begin{aligned}\left(\int_{r_0}^{4r_0} |z_{1,\varepsilon}^{(1)}(r,t_0)|^2\,dr\right)^{\frac12} &\le \left(\int_{r_0}^{4r_0}\left(\int_0^{100r_0}(t+r)F_\varepsilon(t+r,t+t_0)\,dt\right)^2 dr\right)^{\frac12}\\ &\lesssim \left(\int_{r_0}^{4r_0}\left(\int_0^{100r_0}(t+r)\frac{1}{(t+r)^{2p/(p-1)}}\,dt\right)^2 dr\right)^{\frac12}\\ &\lesssim \left(\int_{r_0}^{4r_0}\left(\int_0^{100r_0}\frac{1}{(t+r)^{1+2/(p-1)}}\,dt\right)^2 dr\right)^{\frac12}\\ &\lesssim \left(\int_{r_0}^{4r_0}\frac{1}{r^{4/(p-1)}}\,dr\right)^{\frac12}\\ &\lesssim \frac{1}{r_0^{1-s_p}}.\end{aligned}$$

Long-time contribution. Let us define a cutoff function $\chi(x,t)$ to be the characteristic function of the region $\{(x,t) : |x| > t - t_0 - 50r_0\}$. By Lemma 4.7, we know

$$\begin{aligned}\|\chi F_\varepsilon\|_{L^1L^2([t_0+100r_0,\infty)\times\mathbb{R}^3)} &= \int_{t_0+100r_0}^{\infty}\left(\int_{|x|>t-t_0-50r_0}|F_\varepsilon|^2\,dx\right)^{\frac12}dt\\ &\lesssim \int_{t_0+100r_0}^{\infty}\left(\int_{|x|>t-t_0-50r_0}\frac{1}{|x|^{4p/(p-1)}}\,dx\right)^{\frac12}dt\\ &\lesssim \int_{t_0+100r_0}^{\infty}\left(\frac{1}{|t-t_0-50r_0|^{1+4/(p-1)}}\right)^{\frac12}dt\\ &\lesssim \int_{t_0+100r_0}^{\infty}\frac{1}{|t-t_0-50r_0|^{\frac12+2/(p-1)}}\,dt\\ &\lesssim \frac{1}{r_0^{1-s_p}}.\end{aligned}$$

Applying Lemma 2.20, we obtain

$$\int_{r_0<|x|<4r_0} (|\nabla u_\varepsilon^{(2)}(x,t_0)|^2 + |\partial_t u_\varepsilon^{(2)}(x,t_0)|^2)\,dx \lesssim r_0^{2(s_p-1)}.$$

Applying Lemma 4.2 and using the fact (plus (8))

$$\begin{aligned}\|(u_\varepsilon^{(2)}(t_0), \partial_t u_\varepsilon^{(2)}(t_0))\|_{\dot H^{s_p}\times \dot H^{s_p-1}} &= \left\| S(-100r_0)\begin{pmatrix} u_\varepsilon(t_0+100r_0) \\ \partial_t u_\varepsilon(t_0+100r_0)\end{pmatrix}\right\|_{\dot H^{s_p}\times \dot H^{s_p-1}} \\ &= \|(u_\varepsilon(t_0+100r_0), \partial_t u_\varepsilon(t_0+100r_0))\|_{\dot H^{s_p}\times \dot H^{s_p-1}} \\ &\le \sup_I \|(u,\partial_t u)\|_{\dot H^{s_p}\times \dot H^{s_p-1}} \lesssim 1,\end{aligned}$$

we obtain

$$\int_{r_0}^{4r_0} (|\partial_r w_\varepsilon^{(2)}(r,t_0)|^2 + |\partial_t w_\varepsilon^{(2)}(r,t_0)|^2)\,dr \lesssim r_0^{2(s_p-1)},$$

$$\int_{r_0}^{4r_0} |z_{1,\varepsilon}^{(2)}(r,t_0)|^2\,dr \lesssim r_0^{2(s_p-1)}.$$

Combining with the estimate for $z_{1,\varepsilon}^{(1)}$, we have

$$\int_{r_0}^{4r_0} |z_{1,\varepsilon}(r,t_0)|^2\,dr \lesssim r_0^{2(s_p-1)}.$$

Estimate of $z_{2,\varepsilon}$. We also need to consider $z_{2,\varepsilon}$. In the soliton-like case or the high-to-low frequency cascade case, this can be done in exactly the same way as $z_{1,\varepsilon}$. Now let us consider the self-similar case.

Lemma 4.9. *Let u be a self-similar minimal blow-up solution. If $t_0 \le 0.3r_0$, then $(u(t_0), \partial_t u(t_0))$ is in $\dot H^1 \times L^2(|x| > 0.9r_0)$ with*

$$\int_{|x|>0.9r_0} (|\nabla u(x,t_0)|^2 + |\partial_t u(x,t_0)|^2)\,dx \lesssim r_0^{2(s_p-1)}.$$

Proof. We have (the Duhamel formula)

$$\begin{aligned} u(t_0) &= \int_{0+}^{t_0} \frac{\sin((t_0-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(t)\,dt, \\ \partial_t u(t_0) &= \int_{0+}^{t_0} \cos((t_0-t)\sqrt{-\Delta}) F(t)\,dt,\end{aligned}$$

and

$$\begin{aligned} \tilde u_0 &= \int_{0+}^{t_0} \frac{\sin((t_0-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} \chi(|x|>0.5r_0) F(t)\,dt, \\ \tilde u_1 &= \int_{0+}^{t_0} \cos((t_0-t)\sqrt{-\Delta}) \chi(|x|>0.5r_0) F(t)\,dt.\end{aligned}$$

A straightforward computation shows $\|\chi F\|_{L^1 L^2((0^+,t_0)\times\mathbb{R}^3)} \lesssim r_0^{s_p-1}$. This means $(\tilde{u}_0, \tilde{u}_1)$ is in the space $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with a norm $\lesssim r_0^{s_p-1}$. By the strong Huygens principle we can repeat the argument we used in Lemma 2.20 and obtain

$$(u(t_0), \partial_t u(t_0)) = (\tilde{u}_0, \tilde{u}_1) \text{ in the region } \mathbb{R}^3 \setminus B(0, 0.9r_0). \qquad \square$$

Lemma 4.10. *Let u be a self-similar solution. If $t_0 \le 0.2r_0$ and $\varepsilon < t_0/2$, then we have*

$$\int_{r_0<|x|<4r_0} (|\nabla u_\varepsilon(x,t_0)|^2 + |\partial_t u_\varepsilon(x,t_0)|^2)\, dx \lesssim r_0^{2(s_p-1)}.$$

Proof. We have $\nabla u_\varepsilon = \varphi_\varepsilon * \nabla u$, thus $|\nabla u_\varepsilon| \le \varphi_\varepsilon * |\nabla u|$. Thus (we have $\varepsilon < 0.1r_0$)

$$\int_{r_0<|x|<4r_0} |\nabla u_\varepsilon(x,t_0)|^2\, dx \le \sup_{t\in[t_0-\varepsilon, t_0+\varepsilon]} \int_{0.9r_0<|x|<4.1r_0} |\nabla u(x,t)|^2\, dx \lesssim r_0^{2(s_p-1)}$$

by our previous lemma. The other term can be estimated using the same method. $\square$

Remark 4.11. By Lemmas 4.2 and 4.7, this lemma implies (if $t_0 \le 0.2r_0$ and $\varepsilon < t_0/2$)

$$\int_{r_0}^{4r_0} (|\partial_r w_\varepsilon(r,t_0)|^2 + |\partial_t w_\varepsilon(r,t_0)|^2)\, dr \lesssim r_0^{2(s_p-1)}. \tag{19}$$

In the self-similar case, let us recall that we always choose $\varepsilon < \min\{r_0/10, t_0/2, d\}$. By Lemma 4.10 and Remark 4.11, we only need to consider the case $t_0 > 0.2r_0$ in order to estimate $z_{2,\varepsilon}$. Applying Lemma 4.3, we have

$$\begin{aligned}\left(\int_{r_0}^{4r_0} |z_{2,\varepsilon}(r,t_0)|^2\, dr\right)^{\frac12} \\ \le \left(\int_{t_0+0.8r_0}^{t_0+3.8r_0} |z_{2,\varepsilon}(r, 0.2r_0)|^2\, dr\right)^{\frac12} + \left(\int_{r_0}^{4r_0}\left(\int_0^{t_0-0.2r_0} (t+r)F_\varepsilon(t+r, t_0-t)\, dt\right)^2 dr\right)^{\frac12}.\end{aligned}$$

The first term is dominated by $r_0^{s_p-1}$ because of (19). We can gain the same upper bound for the second term by a basic computation similar to the one we used for $z_{1,\varepsilon}$.

Conclusion. Now we combine the estimates for $z_{1,\varepsilon}$ and $z_{2,\varepsilon}$, thus concluding our Lemma 4.8.

Local energy estimate and its corollary. As mentioned earlier, we are able to establish Theorem 4.1 immediately by letting ε converge to zero. (See Lemma A.6 for details of this argument.) Furthermore, we can obtain the following proposition by applying Lemma 4.2 on u.

Proposition 4.12. *Let $u(x,t)$ be a minimal blow-up solution as above*; *we have*

$$\int_{r_0}^{4r_0} (|\partial_r w(r,t_0)|^2 + |\partial_t w(r,t_0)|^2)\, dr \lesssim r_0^{2(s_p-1)},$$

$$\int_{r_0}^{4r_0} (|z_1(r,t_0)|^2 + |z_2(r,t_0)|^2)\, dr \lesssim r_0^{2(s_p-1)}.$$

5. Recurrence process

In the previous section we found that the minimal blow-up solution is locally in the energy space. However, our goal is to gain a global energy estimate. This section features a recurrence process which helps us march toward higher regularity. We will prove the following lemma. Throughout the whole section we assume u satisfies all the conditions mentioned in the lemma.

Lemma 5.1. *Let $u(x,t)$ be a minimal blow-up solution of (1) as obtained in Section 3 (compactness process) with a frequency scale function $\lambda(t)$. In addition, the set K is precompact in the space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ for some number $s \in [s_p, 1)$:*

$$K = \left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I \right\}.$$

Then at least one of the following holds.

- *The solution u satisfies the energy estimate*
$$\|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} \lesssim (\lambda(t))^{1-s_p}.$$
- *The set K is also precompact in the space $\dot{H}^{s+0.98\sigma_2(p)} \times \dot{H}^{s-1+0.98\sigma_2(p)}$. Here the number $\sigma_2(p) > 0$ depends on nothing but p.*

Remark 5.2. The compactness of K immediately gives the estimate

$$\|u(t), \partial_t u(t)\|_{\dot{H}^s \times \dot{H}^{s-1}} \lesssim (\lambda(t))^{s-s_p}, \quad t \in I.$$

Setup and technical lemmas.

Definition 5.3. Let us define

$$S(A) = \sup_{t \in I} (\lambda(t))^{s_p - s} \|u_{>\lambda(t)A}\|_{Y_s([t, t+d\lambda^{-1}(t)])},$$
$$N(A) = \sup_{t \in I} (\lambda(t))^{s_p - s} \|P_{>\lambda(t)A} F(u)\|_{Z_s([t, t+d\lambda^{-1}(t)])}.$$

Proposition 5.4. *The functions $S(A)$ and $N(A)$ are universally bounded for all $A > 0$ with the limit*

$$\lim_{A \to +\infty} S(A) = 0.$$

Proof. By our assumptions on compactness and Proposition 3.5 part (I), we obtain that the set

$$\left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)}\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)}\right) \right) : \tau \in [0, d], t \in I \right\}$$

is precompact in the space $\dot{H}^s \times \dot{H}^{s-1}$. Applying either Proposition 3.5 part (II) (if $s = s_p$) or Theorem 2.14 (if $s > s_p$), we also have a bound independent of t:

$$\left\| \frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)}\right) \right\|_{Y_s([0,d])} \lesssim 1. \tag{20}$$

Combining these facts with perturbation theory (Theorem 2.12 if $s = s_p$, or Theorem 2.15 if $s > s_p$), we have

$$\left\{ \frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)}\right), \tau \in [0,d] : t \in I \right\}$$

is precompact in the space $Y_s([0,d])$. This immediately gives the uniform convergence for $t \in I$,

$$\left\| P_{>A} \frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t + \frac{\tau}{\lambda(t)}\right) \right\|_{Y_s([0,d])} \rightrightarrows 0, \quad \text{as } A \to \infty. \tag{21}$$

If we rescale the inequality (20) back, we obtain

$$(\lambda(t))^{s_p - s} \|u\|_{Y_s([t, t+d\lambda^{-1}(t)])} \lesssim 1 \Rightarrow (\lambda(t))^{s_p - s} \|F(u)\|_{Z_s([t,t+d\lambda^{-1}(t)])} \lesssim 1,$$

which implies that $S(A)$ and $N(A)$ are uniformly bounded. In a similar way we can show $S(A)$ converges to zero as $A \to \infty$, using the uniform convergence (21) above. □

Definition 5.5. Let us set

$$\Sigma(s,p) = s + 1 - (2p-2)(s - s_p)$$

for convenience. Thus the $Y_s(I)$ norm can also be written as $L^{2p/\Sigma(s,p)} L^{2p/(1-s)}(I \times \mathbb{R}^3)$ norm.

Lemma 5.6 (bilinear estimate). *Suppose u_i satisfies the linear wave equation on the time interval $I = [0,T]$, $i = 1,2$,*

$$\partial_t^2 u_i - \Delta u_i = F_i(x,t),$$

with the initial data $(u_i|_{t=0}, \partial_t u_i|_{t=0}) = (u_{0,i}, u_{1,i})$. Then

$$\begin{aligned} S &= \|(P_{>R} u_1)(P_{<r} u_2)\|_{L^{\frac{p}{\Sigma(s,p)}} L^{\frac{p}{2-s}}(I \times \mathbb{R}^3)} \\ &\lesssim \left(\frac{r}{R}\right)^{\sigma} \left(\|(u_{0,1}, u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F_1\|_{Z_s(I)}\right) \times \left(\|(u_{0,2}, u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F_2\|_{Z_s(I)}\right). \end{aligned}$$

Here the number σ is an arbitrary positive constant satisfying

$$\sigma \leq 3\left(\frac{1}{2} - \frac{\Sigma(s,p)}{2p} - \frac{2-s}{2p}\right), \quad \sigma < 3 \times \frac{2-s}{2p}. \tag{22}$$

Remark 5.7. We can actually choose

$$\sigma = \sigma(p) = \frac{3\min\{p-3, 1\}}{2p} > 0.$$

This constant $\sigma(p)$ depends on nothing but p. This fact plays an important role in our discussion.

Proof. By the Strichartz estimate

$$\begin{aligned} \|(P_{>R})u_1\|_{L^{\frac{2p}{\Sigma(s,p)}} L^{1/(\frac{2-s}{2p}+\frac{\sigma}{3})}} &\lesssim \|(D_x^{-\sigma} P_{>R} u_{0,1}, D_x^{-\sigma} P_{>R} u_{1,1})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{-\sigma} P_{>R} F_1\|_{Z_s(I)}, \\ \|(P_{<r})u_2\|_{L^{\frac{2p}{\Sigma(s,p)}} L^{1/(\frac{2-s}{2p}-\frac{\sigma}{3})}} &\lesssim \|(D_x^{\sigma} P_{<r} u_{0,2}, D_x^{\sigma} P_{<r} u_{1,2})\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|D_x^{\sigma} P_{<r} F_2\|_{Z_s(I)}. \end{aligned}$$

Our choice of σ makes sure that the pairs above are admissible. Thus we have

$$\begin{aligned}
&\|(P_{>R}u_1)(P_{<r}u_2)\|_{L^{\frac{p}{\Sigma(s,p)}}L^{\frac{p}{2-s}}}\\
&\lesssim \|(P_{>R})u_1\|_{L^{\frac{2p}{\Sigma(s,p)}}L^{1/(\frac{2-s}{2p}+\frac{\sigma}{3})}}\|(P_{<r})u_2\|_{L^{\frac{2p}{\Sigma(s,p)}}L^{1/(\frac{2-s}{2p}-\frac{\sigma}{3})}}\\
&\lesssim \left(\|(D_x^{-\sigma}P_{>R}u_{0,1},D_x^{-\sigma}P_{>R}u_{1,1})\|_{\dot H^s\times\dot H^{s-1}}+\|D_x^{-\sigma}P_{>R}F_1\|_{Z_s(I)}\right)\\
&\qquad\qquad\times\left(\|(D_x^{\sigma}P_{<r}u_{0,2},D_x^{\sigma}P_{<r}u_{1,2})\|_{\dot H^s\times\dot H^{s-1}}+\|D_x^{\sigma}P_{<r}F_2\|_{Z_s(I)}\right)\\
&\lesssim \left(\frac{1}{R}\right)^{\sigma}\left(\|(P_{>R}u_{0,1},P_{>R}u_{1,1})\|_{\dot H^s\times\dot H^{s-1}}+\|P_{>R}F_1\|_{Z_s(I)}\right)\\
&\qquad\qquad\times r^{\sigma}\left(\|(P_{<r}u_{0,2},P_{<r}u_{1,2})\|_{\dot H^s\times\dot H^{s-1}}+\|P_{<r}F_2\|_{Z_s(I)}\right)\\
&\lesssim \text{the right-hand side.}
\end{aligned}$$

□

Lemma 5.8. *Let $u(x,t)$ be a function defined on $I\times\mathbb{R}^3$, such that $\hat u$ is supported in the ball $B(0,r)$ for each $t\in I$. Then*

$$\|P_{>R}F(u(x,t))\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}(I\times\mathbb{R}^3)}\lesssim\left(\frac{r}{R}\right)^2\|u\|^p_{Y_s(I)}.$$

Proof. We have

$$\begin{aligned}
&\|P_{>R}F(u(x,t))\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}(I\times\mathbb{R}^3)}\\
&\qquad\lesssim\frac{1}{R^2}\|P_{>R}\Delta_xF(u(x,t))\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}(I\times\mathbb{R}^3)}\\
&\qquad\lesssim\frac{1}{R^2}\|\Delta_xF(u(x,t))\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}(I\times\mathbb{R}^3)}\\
&\qquad\lesssim\frac{1}{R^2}\|p(\Delta_xu)|u|^{p-1}+p(p-1)|\nabla_xu|^2|u|^{p-3}u\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}}\\
&\qquad\lesssim\frac{1}{R^2}\left(\|\Delta_xu\|_{Y_s(I)}\|u\|^{p-1}_{Y_s(I)}+\|\nabla_xu\|^2_{Y_s(I)}\|u\|^{p-2}_{Y_s(I)}\right)\\
&\qquad\lesssim\frac{r^2}{R^2}\|u\|^p_{Y_s(I)}.
\end{aligned}$$

□

Lemma 5.9. *Let $v(t)$ be a long-time contribution in the Duhamel formula*

$$v(t_0)=\int_{T_1}^{T_2}\frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(t))\,dt.$$

Then for any $t_0<T_1<T_2$, we have

$$\|v(t_0)\|_{L^\infty(\mathbb{R}^3)}\lesssim(T_1-t_0)^{-2/(p-1)}.$$

Proof. Using the explicit expression of the wave kernel in dimension 3, we obtain

$$\begin{aligned}\left|\left(\int_{T_1}^{T_2}\frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(t))\,dt\right)(x)\right| &= \left|\int_{T_1}^{T_2}\int_{|y-x|=t-t_0}\frac{1}{4\pi(t-t_0)}F(u(y,t))\,dS(y)\,dt\right|\\ &\lesssim \int_{T_1}^{T_2}\int_{|y-x|=t-t_0}\frac{1}{4\pi(t-t_0)}|u(y,t)|^p\,dS(y)\,dt\\ &\lesssim \int_{T_1}^{T_2}\int_{|y-x|=t-t_0}\frac{1}{(t-t_0)}\frac{1}{|y|^{\frac{2p}{p-1}}}\,dS(y)\,dt.\end{aligned}$$

In the last step, we use the estimate (8) for radial $\dot H^{s_p}$ functions. If $|x|\le\frac{1}{2}(T_1-t_0)$, then on the sphere for the integral we have

$$|y|\ge|t-t_0|-|x|\ge\frac{1}{2}(t-t_0).$$

Thus for these small x, we obtain

$$\begin{aligned}\left|\left(\int_{T_1}^{T_2}\frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(t))\,dt\right)(x)\right| &\lesssim \int_{T_1}^{T_2}\int_{|y-x|=t-t_0}\frac{1}{(t-t_0)}\frac{1}{(t-t_0)^{2p/(p-1)}}\,dS(y)\,dt\\ &\lesssim \int_{T_1}^{T_2}\int_{|y-x|=t-t_0}\frac{1}{(t-t_0)^{3+2/(p-1)}}\,dS(y)\,dt\\ &\lesssim \int_{T_1}^{T_2}\frac{(t-t_0)^2}{(t-t_0)^{3+2/(p-1)}}\,dt\\ &\lesssim \int_{T_1}^{T_2}\frac{1}{(t-t_0)^{1+2/(p-1)}}\,dt\\ &\lesssim (T_1-t_0)^{-2/(p-1)}.\end{aligned}$$

On the other hand, if $x\ge\frac{1}{2}(T_1-t_0)$, by the estimate on radial $\dot H^{s_p}$ functions (8) and Lemma 2.17, we have

$$\begin{aligned}\left|\left(\int_{T_1}^{T_2}\frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(t))\,dt\right)(x)\right| &\lesssim \frac{1}{|x|^{2/(p-1)}}\left\|\int_{T_1}^{T_2}\frac{\sin((t-t_0)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(t))\,dt\right\|_{\dot H^{s_p}}\\ &\lesssim \frac{1}{(T_1-t_0)^{2/(p-1)}}.\end{aligned}$$

Combining these two cases, we finish our proof. □

Lemma 5.10. *There exists a constant $\kappa=\kappa(p)\in(0,1)$ that depends only on p, so that for each $s\in[s_p,1)$, there exists an s-admissible pair (q,r), with $q\ne\infty$ and*

$$\frac{\Sigma(s,p)}{2p}=\kappa\cdot 0+(1-\kappa)\frac{1}{q},\qquad \frac{2-s}{2p}=\kappa\frac{3-2s}{6}+(1-\kappa)\frac{1}{r}.$$

Proof. We will choose $\kappa = 1-3/p \in (0, 0.4)$. Basic computation shows

$$\begin{aligned}
\frac{1}{q} &= \frac{\Sigma(s,p)}{2p(1-\kappa)} = \frac{s+1-(2p-2)(s-s_p)}{6} \in (0, 1/3);\\
\frac{1}{r} &= \frac{2-s}{2p(1-\kappa)} - \frac{\kappa}{1-\kappa} \times \frac{3-2s}{6} = \frac{2-s}{6} - \frac{\kappa}{1-\kappa} \times \frac{3-2s}{6}\\
&\in \left(\frac{2-s}{6} - \frac{2}{3} \times \frac{3-2s}{6}, \ \frac{2-s}{6}\right)\\
&\subseteq \left(\frac{s}{18}, \frac{2-s}{6}\right)\\
&\subseteq (1/36, 1/4).
\end{aligned}$$

Thus we can solve two positive real numbers q, r so that the two identities hold. In addition, we have $q \in (3, \infty)$ and $r \in (4, 36)$. Furthermore, by adding the identities together, we obtain

$$\frac{3-(2p-2)(s-s_p)}{2p} = \kappa \frac{3-2s}{6} + (1-\kappa)\left(\frac{1}{q} + \frac{1}{r}\right).$$

This implies

$$\frac{1}{q} + \frac{1}{r} < \frac{3-(2p-2)(s-s_p)}{2p(1-\kappa)} = \frac{3-(2p-2)(s-s_p)}{6} \le 1/2.$$

Using the same method, one can show $1/q + 3/r = 3/2 - s$. In summary, (q, r) is an s-admissible pair. □

Lemma 5.11. *Given any s-admissible pair (q, r) with $q < \infty$ and three times $t_0 < t_1 < t_2$ in the maximal lifespan I of u, we have*

$$\lim_{T\to\infty} \left\| \int_{t_2}^{T} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau \right\|_{L^q L^r([t_0,t_1]\times\mathbb{R}^3)} \le C(\lambda(t_2))^{s-s_p}.$$

The constant C does not depend on t_0, t_1 or t_2.

Proof. By Lemma A.5 and the identity

$$\int_{t_2}^{T} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau = S(t-t_2)(u(t_2), \partial_t u(t_2)) - S(t-T)(u(T), \partial_t u(T)),$$

we have

$$\lim_{T\to\infty} \int_{t_2}^{T} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau = S(t-t_2)(u(t_2), \partial_t u(t_2))$$

in the space $L^q L^r([t_0, t_1] \times \mathbb{R}^3)$. Thus

$$\begin{aligned}
\lim_{T\to\infty} \left\| \int_{t_2}^{T} \frac{\sin((\tau - t)\sqrt{-\Delta})}{\sqrt{-\Delta}} (u(\tau))\, d\tau \right\|_{L^q L^r([t_0,t_1])} &= \|S(t-t_2)(u(t_2), \partial_t u(t_2))\|_{L^q L^r([t_0,t_1]\times\mathbb{R}^3)}\\
&\lesssim \|(u(t_2), \partial_t u(t_2))\|_{\dot{H}^s \times \dot{H}^{s-1}}\\
&\lesssim (\lambda(t_2))^{s-s_p}.
\end{aligned}$$

□

Lemma 5.12. *Suppose $S(A)$ is a nonnegative function defined in $\mathbb{R}^+$ satisfying $S(A) \to 0$ as $A \to \infty$. In addition, there exist $0 < \alpha < \beta < 1$ and $l, \omega > 0$ with*

$$l\alpha + \beta > 1,$$

such that the inequality

$$S(A) \lesssim S(A^\beta) S^l(A^\alpha) + A^{-\omega} \tag{23}$$

is true for each sufficiently large A. Then

$$S(A) \lesssim A^{-\omega}$$

for each sufficiently large A.

Proof. Let us first choose two constants l^- and ω^-, which are slightly smaller than l and ω respectively, such that the inequality $l^-\alpha + \beta > 1$ still holds. By the conditions given, we can find a constant $A_0 \gg 1$, such that the following inequalities hold:

$$\begin{aligned} &S(A) \le \tfrac{1}{2} S(A^\beta) S^{l^-}(A^\alpha) + \tfrac{1}{2} A^{-\omega^-} && \text{if } A \ge A_0, \\ &S(A) < \tfrac{1}{2} && \text{if } A \ge A_0^\alpha. \end{aligned} \tag{24}$$

Using the second inequality above, we know the inequality

$$S(A) \le A^{-\omega_1} \tag{25}$$

holds for all $A \in [A_0^\alpha, A_0]$ if ω_1 is sufficiently small. Fix such a small constant $\omega_1 \le \omega^-$. We will show that the inequality (25) above holds for each $A \ge A_0^\alpha$ by an induction. We already know this is true for $A \in [A_0^\alpha, A_0]$. If $A \in [A_0, A_0^{1/\beta}]$, the inequality (24) implies

$$\begin{aligned} S(A) &\le \tfrac{1}{2} S(A^\beta) S^{l^-}(A^\alpha) + \tfrac{1}{2} A^{-\omega^-} \\ &\le \tfrac{1}{2} (A^\beta)^{-\omega_1} ((A^\alpha)^{-\omega_1})^{l^-} + \tfrac{1}{2} A^{-\omega^-} \\ &\le \tfrac{1}{2} (A^{-\omega_1})^{\beta + l^-\alpha} + \tfrac{1}{2} A^{-\omega_1} \\ &\le A^{-\omega_1}. \end{aligned}$$

Here we use the fact that $A^\alpha, A^\beta \in [A_0^\alpha, A_0]$ if A satisfies our assumption. Conducting an induction, we can show the inequality holds for each $A \in [A_0^{(1/\beta)^n}, A_0^{(1/\beta)^{n+1}}]$ if n is a nonnegative integer. In summary, the inequality (25) is true for each $A \ge A_0^\alpha$. Plugging this back in the original recurrence formula (23), we obtain for sufficiently large A,

$$S(A) \lesssim A^{-\omega_1(\beta + l\alpha)} + A^{-\omega} \lesssim A^{-\min\{\omega_1(\beta + l\alpha), \omega\}},$$

which indicates faster decay than $A^{-\omega_1}$. Iterating the argument if necessary, we gain the decay $S(A) \lesssim A^{-\omega}$ and finish the proof. $\square$

Recurrence formula. Under our setting in this section, given $0<\alpha<\beta<1$ and a small positive constant ε_1, we have the recurrence formula

$$N(A) \lesssim S(A^{\beta})S^{p-1}(A^{\alpha}) + A^{-(\beta-\alpha)\sigma(p)} + A^{-2(1-\beta)}, \tag{26}$$
$$S(A) \lesssim N(A^{1-\varepsilon_1}) + A^{-\sigma_1(p)} \tag{27}$$

for sufficiently large A. The constants $\sigma(p), \sigma_1(p)$ depend on p but nothing else.

Proof of (26). In the following argument, all the space-time norms are taken in $[t, t+d\lambda^{-1}(t)]\times\mathbb{R}^3$:

$$\begin{aligned}
\|P_{>\lambda(t)A}F(u)\|_{Z_s} &\lesssim \lambda(t)^{-(p-1)(s-s_p)}\|P_{>\lambda(t)A}F(u)\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}}\\
&\le \lambda(t)^{-(p-1)(s-s_p)}\|P_{>\lambda(t)A}F(u_{\le A^{\beta}\lambda(t)})\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}}\\
&\qquad + \lambda(t)^{-(p-1)(s-s_p)}\|P_{>\lambda(t)A}(F(u)-F(u_{\le A^{\beta}\lambda(t)}))\|_{L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}}\\
&= \lambda(t)^{-(p-1)(s-s_p)}(I_1+I_2).
\end{aligned}$$

By Lemma 5.8, we have

$$I_1 \lesssim \left(\frac{A^{\beta}}{A}\right)^2 \|u\|_{Y_s}^p \lesssim (\lambda(t))^{p(s-s_p)}A^{-2(1-\beta)}.$$

In order to estimate I_2, we have (all unmarked norms are $L^{\frac{2}{\Sigma(s,p)}}L^{\frac{2}{2-s}}([t, t+d\lambda^{-1}(t)]\times\mathbb{R}^3)$ norms)

$$\begin{aligned}
I_2 &\le \left\| P_{>\lambda(t)A}\left[u_{>A^{\beta}\lambda(t)}\int_0^1 F'(u_{\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau\right]\right\|\\
&\lesssim \left\| u_{>A^{\beta}\lambda(t)}\int_0^1 F'(u_{\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau\right\|\\
&\lesssim \|I_{2,1}\| + \|I_{2,2}\|.
\end{aligned}$$

Here

$$\begin{aligned}
I_{2,1} &= u_{>A^{\beta}\lambda(t)}\int_0^1 F'(u_{\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau\\
&\qquad\qquad - u_{>A^{\beta}\lambda(t)}\int_0^1 F'(u_{A^{\alpha}\lambda(t)<\cdot\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau,\\
I_{2,2} &= u_{>A^{\beta}\lambda(t)}\int_0^1 F'(u_{A^{\alpha}\lambda(t)<\cdot\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau.
\end{aligned}$$

We have

$$I_{2,1} = u_{>A^{\beta}\lambda(t)}u_{\le A^{\alpha}\lambda(t)} \times \int_0^1\int_0^1 F''(\tilde{\tau}u_{\le A^{\alpha}\lambda(t)} + u_{A^{\alpha}\lambda(t)<\cdot\le A^{\beta}\lambda(t)} + \tau u_{>A^{\beta}\lambda(t)})\,d\tau\,d\tilde{\tau}.$$

Applying the bilinear estimate (Lemma 5.6) on the term $u_{>A^\beta\lambda(t)}\, u_{\le A^\alpha\lambda(t)}$, we obtain

$$\begin{aligned}
\|I_{2,1}\| &\lesssim \|u_{>A^\beta\lambda(t)}u_{\le A^\alpha\lambda(t)}\|_{L^{\frac{p}{\Sigma(s,p)}}L^{\frac{p}{2-s}}}\\
&\quad\times\left\|\int_0^1\int_0^1 F''(\tilde\tau u_{\le A^\alpha\lambda(t)}+u_{A^\alpha\lambda(t)<\cdot\le A^\beta\lambda(t)}+\tau u_{>A^\beta\lambda(t)})\,d\tau\,d\tilde\tau\right\|_{L^{\frac{2p}{(p-2)\Sigma(s,p)}}L^{\frac{2p}{(p-2)(2-s)}}}\\
&\lesssim\left[\left(\frac{A^\alpha\lambda(t)}{A^\beta\lambda(t)}\right)^{\sigma(p)}(\lambda(t))^{2(s-s_p)}\right](\lambda(t))^{(p-2)(s-s_p)}\\
&\lesssim(\lambda(t))^{p(s-s_p)}A^{-(\beta-\alpha)\sigma(p)}.
\end{aligned}$$

On the other hand, we know that, for sufficiently large A,

$$\begin{aligned}
\|I_{2,2}\| &\lesssim \|u_{>A^\beta\lambda(t)}\|_{L^{\frac{2p}{\Sigma(s,p)}}L^{\frac{2p}{2-s}}}\left\|\int_0^1 F'(u_{A^\alpha\lambda(t)<\cdot\le A^\beta\lambda(t)}+\tau u_{>A^\beta\lambda(t)})\,d\tau\right\|_{L^{\frac{2p}{(p-1)\Sigma(s,p)}}L^{\frac{2p}{(p-1)(2-s)}}}\\
&\lesssim(\lambda(t))^{s-s_p}S(A^\beta)[(\lambda(t))^{(p-1)(s-s_p)}S^{p-1}(A^\alpha)]\\
&\lesssim(\lambda(t))^{p(s-s_p)}S(A^\beta)S^{p-1}(A^\alpha).
\end{aligned}$$

Collecting all terms above, we have

$$\|P_{>\lambda(t)A}(F(u))\|_{Z_s}\lesssim(\lambda(t))^{s-s_p}[S(A^\beta)S^{p-1}(A^\alpha)+A^{-(\beta-\alpha)\sigma(p)}+A^{-2(1-\beta)}].$$

Multiplying both sides by $(\lambda(t))^{s_p-s}$ and taking sup for all $t\in I$, we obtain the first inequality.

Definition 5.13. Given $t_0\in I$, define t_i recursively for $i\ge 1$ by

$$t_i=t_{i-1}+d\lambda^{-1}(t_{i-1}). \tag{28}$$

By the choice of d, all the t_i are in the maximal lifespan I. (See Proposition 3.5.)

Proof of (27). By the Strichartz estimate and the Duhamel formula (see Lemma A.5), we have

$$\begin{aligned}
\|u_{>\lambda(t_0)A}\|_{Y_s([t_0,t_1])} &= \left\|\int_t^\infty\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{Y_s([t_0,t_1])}\\
&\le\left\|\int_t^{t_2}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{Y_s([t_0,t_1])}\\
&\qquad+\liminf_{T\to\infty}\left\|\int_{t_2}^T\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{Y_s([t_0,t_1])}\\
&\lesssim\|P_{>\lambda(t_0)A}F(u)\|_{Z_s([t_0,t_2]\times\mathbb{R}^3)}\\
&\qquad+\liminf_{T\to\infty}\left\|\int_{t_2}^T\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{Y_s([t_0,t_1])}\\
&=I_1+I_2.
\end{aligned}$$

The first term can be dominated by

$$\begin{aligned} I_1 &\lesssim \|P_{>\lambda(t_0)A}F(u)\|_{Z_s([t_0,t_1]\times\mathbb{R}^3)} + \|P_{>\lambda(t_0)A}F(u)\|_{Z_s([t_1,t_2]\times\mathbb{R}^3)} \\ &\lesssim (\lambda(t_0))^{s-s_p}N(A) + (\lambda(t_1))^{s-s_p}N\left(\frac{\lambda(t_0)}{\lambda(t_1)}A\right) \\ &\lesssim (\lambda(t_0))^{s-s_p}N(A^{1-\varepsilon_1}) \end{aligned}$$

for any small positive number ε_1 and sufficiently large $A > A_0(u,\varepsilon_1)$, because $\lambda(t_0)$ and $\lambda(t_1)$ are comparable to each other by the local compactness result (11). Now let us consider the term I_2. First of all, by Lemma 2.17, we have

$$\begin{aligned} &\left\|\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{L^\infty L^2([t_0,t_1]\times\mathbb{R}^3)} \\ &\qquad\lesssim \frac{1}{(\lambda(t_0)A)^{s_p}}\left\|\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(\tau))\,d\tau\right\|_{L^\infty_{[t_0,t_1]}\dot{H}^{s_p}(\mathbb{R}^3)} \\ &\qquad\lesssim \frac{1}{(\lambda(t_0)A)^{s_p}}. \end{aligned}$$

Using Lemma 5.9, we are also able to obtain

$$\begin{aligned} &\left\|\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}P_{>\lambda(t_0)A}F(u(\tau))\,d\tau\right\|_{L^\infty L^\infty([t_0,t_1]\times\mathbb{R}^3)} \\ &\qquad\lesssim \left\|\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(\tau))\,d\tau\right\|_{L^\infty L^\infty([t_0,t_1]\times\mathbb{R}^3)} \\ &\qquad\lesssim (t_2-t_1)^{-2/(p-1)} \\ &\qquad\lesssim (\lambda(t_0))^{2/(p-1)}. \end{aligned}$$

By an interpolation between L^2 and L^∞, we have

$$\begin{aligned} &\left\|P_{>\lambda(t_0)A}\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(\tau))\,d\tau\right\|_{L^\infty L^{\frac{6}{3-2s}}([t_0,t_1]\times\mathbb{R}^3)} \\ &\qquad\le \|\cdot\|^{2s/3}_{L^\infty L^\infty([t_0,t_1]\times\mathbb{R}^3)}\|\cdot\|^{(3-2s)/3}_{L^\infty L^2([t_0,t_1]\times\mathbb{R}^3)} \\ &\qquad\lesssim [\lambda(t_0)^{2/(p-1)}]^{2s/3}[(\lambda(t_0)A)^{-s_p}]^{(3-2s)/3} \\ &\qquad= (\lambda(t_0))^{s-s_p}A^{\frac{-s_p(3-2s)}{3}}. \end{aligned}$$

Next, we will use the interpolation again to gain an estimate of the Y_s norm. Let (q,r) be the admissible pair given by Lemma 5.10. Applying Lemma 5.11, we have

$$\lim_{T\to\infty}\left\|\int_{t_2}^{T}\frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(\tau))\,d\tau\right\|_{L^qL^r([t_0,t_1]\times\mathbb{R}^3)} \lesssim (\lambda(t_2))^{s-s_p} \lesssim (\lambda(t_0))^{s-s_p}.$$

Using this fact and the construction of (q,r), we obtain

$$\begin{aligned}
I_2 &= \liminf_{T\to\infty} \left\| \int_{t_2}^{T} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau))\, d\tau \right\|_{Y_s([t_0,t_1])} \\
&\leq \liminf_{T\to\infty} \left(\left\| \int_{t_2}^{T} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau))\, d\tau \right\|_{L^\infty L^{\frac{6}{3-2s}}([t_0,t_1]\times\mathbb{R}^3)}^{\kappa(p)} \right. \\
&\qquad\qquad \left. \times \left\| \int_{t_2}^{T} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} P_{>\lambda(t_0)A} F(u(\tau))\, d\tau \right\|_{L^q L^r([t_0,t_1]\times\mathbb{R}^3)}^{1-\kappa(p)} \right) \\
&\lesssim \left[(\lambda(t_0))^{s-s_p} A^{\frac{-s_p(3-2s)}{3}} \right]^{\kappa(p)} \times \lim_{T\to\infty} \left\| \int_{t_2}^{T} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\, d\tau \right\|_{L^q L^r}^{1-\kappa(p)} \\
&\lesssim \left[(\lambda(t_0))^{s-s_p} A^{\frac{-s_p(3-2s)}{3}} \right]^{\kappa(p)} (\lambda(t_0))^{(s-s_p)(1-\kappa(p))} \\
&\lesssim (\lambda(t_0))^{s-s_p} A^{\frac{-s_p\kappa(p)(3-2s)}{3}} \\
&\lesssim (\lambda(t_0))^{s-s_p} A^{-\sigma_1(p)}.
\end{aligned}$$

Here $\sigma_1(p) = \kappa(p)/6$. It depends only on p. Combining our estimates on I_1 and I_2, we finish the proof of the second inequality.

Decay of $S(A)$ and $N(A)$ with applications.

Decay of $S(A)$ and $N(A)$. Plugging the first recurrence formula into the second one, we obtain

$$S(A) \lesssim S(A^{(1-\varepsilon_1)\beta}) S^{p-1}(A^{(1-\varepsilon_1)\alpha}) + A^{-\sigma(p)(1-\varepsilon_1)(\beta-\alpha)} + A^{-2(1-\varepsilon_1)(1-\beta)} + A^{-\sigma_1(p)}.$$

Choose α, β and ε_1 so that

$$(1-\varepsilon_1)\beta = 2/3, \quad (1-\varepsilon_1)\alpha = 1/3, \quad \varepsilon_1 = 1/10000. \tag{29}$$

Then we have

$$S(A) \lesssim S(A^{2/3}) S^{p-1}(A^{1/3}) + A^{-\sigma_2(p)}$$

for sufficiently large A. Here the positive number $\sigma_2(p)$, defined as

$$\sigma_2 = \min\{\sigma(p)/3, \sigma_1(p), 0.6\},$$

depends on p only. Applying Lemma 5.12, we have $S(A) \lesssim A^{-\sigma_2(p)}$ for sufficiently large A. Plugging this in the first recurrence formula, we have $N(A) \lesssim A^{-\sigma_2(p)}$ for large A. Observing that both $S(A)$ and $N(A)$ is uniformly bounded, we know these two decay estimates are actually valid for each $A > 0$. Now let us choose

$$s_1 = \min\left\{1, s + \frac{99}{100}\sigma_2(p)\right\},$$

and make the following definition.

Definition 5.14 (local contribution of the Duhamel formula). Assume $t' \in I$. Let us introduce the notation

$$v_{t'}(t) = \int_{t'}^{t'+d\lambda(t')^{-1}} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\,d\tau;$$
$$\partial_t v_{t'}(t) = -\int_{t'}^{t'+d\lambda(t')^{-1}} \cos((\tau-t)\sqrt{-\Delta}) F(u(\tau))\,d\tau.$$

Estimate on local contribution. Given any $t \le t'$ and integer $k \ge 0$, we know

$$\begin{aligned}
&\|P_{\lambda(t')2^k<\cdot<\lambda(t')2^{k+1}}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s_1}\times \dot H^{s_1-1}} \\
&\qquad\lesssim (\lambda(t')2^k)^{s_1-s} \|P_{\lambda(t')2^k<\cdot<\lambda(t')2^{k+1}}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s}\times \dot H^{s-1}} \\
&\qquad\lesssim (\lambda(t')2^k)^{s_1-s} \|P_{>\lambda(t')2^k}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s}\times \dot H^{s-1}} \\
&\qquad\lesssim (\lambda(t')2^k)^{s_1-s} \|P_{>\lambda(t')2^k} F(u)\|_{Z_s([t',t'+d\lambda(t')^{-1}])} \\
&\qquad\lesssim (\lambda(t')2^k)^{s_1-s} (\lambda(t'))^{s-s_p} N(2^k) \\
&\qquad\lesssim (\lambda(t'))^{s_1-s_p} (2^k)^{s_1-s-\sigma_2(p)}.
\end{aligned}$$

Summing for all $k \ge 0$, we have

$$\|P_{>\lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s_1}\times \dot H^{s_1-1}} \lesssim (\lambda(t'))^{s_1-s_p}.$$

Combining this with the estimate

$$\begin{aligned}
\|P_{\le\lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s_1}\times \dot H^{s_1-1}} &\lesssim (\lambda(t'))^{s_1-s_p} \|P_{\le\lambda(t')}(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s_p}\times \dot H^{s_p-1}} \\
&\lesssim (\lambda(t'))^{s_1-s_p},
\end{aligned}$$

we obtain

$$\|(v_{t'}(t), \partial_t v_{t'}(t))\|_{\dot H^{s_1}\times \dot H^{s_1-1}} \lesssim (\lambda(t'))^{s_1-s_p}. \tag{30}$$

Higher regularity. In this subsection we will show that $(u(x,t), \partial_t u(x,t)) \in \dot H^{s_1} \times \dot H^{s_1-1}(\mathbb{R}^3)$ for each $t \in I$. The idea is to deal with the "center" part and the "tail" part individually and then glue them together using Lemma 2.16.

Center estimate. Let us break the Duhamel formula into two pieces:

$$u^{(1)}(t) = \int_t^{t_1} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\,d\tau,$$
$$u^{(2)}(t) = \int_{t_1}^{\infty} \frac{\sin((\tau-t)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u(\tau))\,d\tau.$$

Let χ be the characteristic function of the region $\{(x,t) : |x| > d\lambda^{-1}(t_0)/2 + |t-t_1|\}$. We have

$$\begin{aligned}
\|\chi F(u(t))\|_{L^1 L^{\frac{6}{5-2s_1}}([t_1,\infty)\times\mathbb{R}^3)} &= \int_{t_1}^{\infty} \left(\int_{|x|>\frac{d\lambda^{-1}(t_0)}{2}+|t-t_1|} (F(u))^{\frac{6}{5-2s_1}}\, dx \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^{\infty} \left(\int_{|x|>\frac{d\lambda^{-1}(t_0)}{2}+|t-t_1|} \left(\frac{1}{|x|^{\frac{2p}{p-1}}} \right)^{\frac{6}{5-2s_1}} dx \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^{\infty} \left(\left| \frac{d\lambda^{-1}(t_0)}{2} + t - t_1 \right|^{-\frac{2p}{p-1}\frac{6}{5-2s_1}+3} \right)^{\frac{5-2s_1}{6}} dt \\
&\lesssim \int_{t_1}^{\infty} \left(\frac{d\lambda^{-1}(t_0)}{2} + t - t_1 \right)^{s_p - s_1 - 1} dt \\
&\lesssim \lambda(t_0)^{s_1 - s_p}.
\end{aligned}$$

By Lemma 2.20, there exists a pair $(\tilde{u}_0, \tilde{u}_1)$ such that

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^{s_1}\times\dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim \lambda(t_0)^{s_1-s_p},$$

$$(u^{(2)}(t_0), \partial_t u^{(2)}(t_0)) = (\tilde{u}_0, \tilde{u}_1) \quad \text{in } B\left(0, \frac{d\lambda^{-1}(t_0)}{2}\right).$$

This implies

$$(u(t_0), \partial_t u(t_0)) = (\tilde{u}_0 + u^{(1)}(t_0), \tilde{u}_1 + \partial_t u^{(1)}(t_0)) \quad \text{in } B\left(0, \frac{d\lambda^{-1}(t_0)}{2}\right). \tag{31}$$

By (30), we have

$$\|(u^{(1)}(t_0), \partial_t u^{(1)}(t_0))\|_{\dot{H}^{s_1}\times\dot{H}^{s_1-1}} \lesssim \lambda(t_0)^{s_1-s_p}.$$

Combining this with the $\dot{H}^{s_1} \times \dot{H}^{s_1-1}$ bound of $(\tilde{u}_0, \tilde{u}_1)$, we have

$$\|(\tilde{u}_0 + u^{(1)}(t_0), \tilde{u}_1 + \partial_t u^{(1)}(t_0))\|_{\dot{H}^{s_1}\times\dot{H}^{s_1-1}} \lesssim \lambda(t_0)^{s_1-s_p}. \tag{32}$$

Tail estimate. Let $(u_0', u_1') = \Psi_{d\lambda^{-1}(t_0)/4}(u(t_0), \partial_t u(t_0))$, and

$$\frac{1}{q} = \frac{1}{2} + \frac{1-s_1}{3}.$$

By Theorem 4.1, if $r \geq d\lambda^{-1}(t_0)/4$, we have

$$\begin{aligned}
\left(\int_{r<|x|<4r} (|\nabla u_0'|^q + |u_1'|^q)\, dx \right)^{1/q} &\lesssim \left(\int_{r<|x|<4r} (|\nabla u_0'|^2 + |u_1'|^2)\, dx \right)^{\frac{1}{2}} (r^3)^{\frac{1}{q}-\frac{1}{2}} \\
&\lesssim r^{-(1-s_p)} (r^3)^{(1-s_1)/3} \\
&\lesssim r^{-(s_1-s_p)}.
\end{aligned}$$

Letting $r = 4^k d\lambda^{-1}(t_0)/4$ and summing for all $k \geq 0$, we obtain that the pair (u_0', u_1') is in the space $\dot{W}^{1,q} \times L^q(\mathbb{R}^3)$ with

$$\|(u_0', u_1')\|_{\dot{W}^{1,q}\times L^q(\mathbb{R}^3)} \lesssim (d\lambda(t_0)^{-1}/4)^{-(s_1-s_p)} \lesssim (\lambda(t_0))^{s_1-s_p}.$$

By the Sobolev embedding, we have

$$\|(u_0', u_1')\|_{\dot{H}^{s_1}\times \dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim (\lambda(t_0))^{s_1-s_p}. \tag{33}$$

Combining the center estimate (32) and tail estimate (33) by Lemma 2.16, we have

$$\|(u(t_0), \partial_t u(t_0))\|_{\dot{H}^{s_1}\times \dot{H}^{s_1-1}(\mathbb{R}^3)} \lesssim (\lambda(t_0))^{s_1-s_p}. \tag{34}$$

Conclusion. Now we can finish our proof of Lemma 5.1.

- Case 1 ($s_1 = 1$) The inequality (34) is exactly the energy estimate we are looking for.
- Case 2 ($s_1 < 1$) This means $s_1 = s + 0.99\sigma_2(p)$. As a result, the set

$$K = \left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I \right\}$$

  is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$, and bounded in the space $\dot{H}^{s+0.99\sigma_2(p)} \times \dot{H}^{s-1+0.99\sigma_2(p)}$, thus it is also precompact in the space $\dot{H}^{s+0.98\sigma_2(p)} \times \dot{H}^{s-1+0.98\sigma_2(p)}$ by an interpolation.

6. Global energy estimate and its corollary

Repeat the recurrence process we described in the previous section starting from the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$. Each time we either obtain the global energy estimate below or gain additional regularity by $0.98\sigma_2(p)$. However, this number depends on p only. As a result, the process has to stop at $\dot{H}^1 \times L^2$ after finite steps.

Proposition 6.1 (global energy estimate). *Let $u(x,t)$ be a minimal blow-up solution. Then $(u(t), \partial_t u(t))$ is in the energy space for each $t \in I$ with*

$$\|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} \lesssim \lambda(t)^{1-s_p}. \tag{35}$$

By the local theory, we actually obtain

$$(u(t), \partial_t u(t)) \in C(I; \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)).$$

Remark 6.2. By Lemma 4.2, we have, for any $0 < a < b < \infty$,

$$(\partial_r w(t), \partial_t w(t)) \in C(I; L^2 \times L^2([a,b])).$$

Self-similar and high-to-low frequency cascade cases. In both two cases, we can choose $t_i \to \infty$ such that $\lambda(t_i) \to 0$. This implies

$$\int_{\mathbb{R}^3} (|\nabla u(x,t_i)|^2 + |\partial_t u(x,t_i)|^2)\, dx \to 0.$$

By the Sobolev embedding, we have

$$\|u\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \le \|u\|_{L^{\frac{3}{2}(p-1)}(\mathbb{R}^3)}^{p-1} \|u\|_{L^6(\mathbb{R}^3)}^2 \lesssim \|u\|_{\dot{H}^{s_p}(\mathbb{R}^3)}^{p-1} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^2. \tag{36}$$

This implies $\|u(t_i)\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \to 0$. Using the definition of energy we have $E(t_i) \to 0$. On the other hand, we know the energy is a constant. Therefore the energy must be zero.

- *Defocusing case.* It is nothing to say, because in this case an energy zero means that the solution is identically zero.
- *Focusing case.* We can still solve the problem using the following theorem. By the fact that the energy is zero, the theorem claims that u blows up in finite time in both time directions. But this is a contradiction with our assumption $T_+ = \infty$.

Theorem 6.3 (see Theorem 3.1 in [Killip et al. 2014], nonpositive energy implies blowup). *Let $(u_0, u_1) \in (\dot{H}^1 \times L^2) \cap (\dot{H}^{s_p} \times \dot{H}^{s_p-1})$ be initial data. Assume that (u_0, u_1) is not identically zero and satisfies $E(u_0, u_1) \le 0$. Then the maximal life-span solution to the nonlinear wave equation blows up both forward and backward in finite time.*

Soliton-like solutions in the defocusing case. Now let us consider the soliton-like solutions in the defocusing case. First we have a useful global integral estimate in the defocusing case.

Lemma 6.4 (see [Perthame and Vega 1999]; we use the 3-dimensional case). *Let u be a solution of (1) defined in a time interval $[0,T]$ with a finite energy*

$$E = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla_x u|^2 + \frac{1}{2}|\partial_t u|^2 + \frac{1}{p+1}|u(x)|^{p+1} \right) dx.$$

For any $R > 0$, we have

$$\begin{aligned}
&\frac{1}{2R}\int_0^T \int_{|x|<R} (|\nabla u|^2 + |\partial_t u|^2)\, dx\, dt + \frac{1}{2R^2}\int_0^T \int_{|x|=R} |u|^2\, d\sigma_R\, dt \\
&\quad + \frac{1}{2R}\frac{2p-4}{p+1}\int_0^T \int_{|x|<R} |u|^{p+1}\, dx\, dt + \frac{p-1}{p+1}\int_0^T \int_{|x|>R} \frac{|u|^{p+1}}{|x|}\, dx\, dt + \frac{2}{R^2}\int_{|x|<R} |u(T)|^2\, dx \\
&\qquad \le 2E.
\end{aligned}$$

Observing that each term on the left-hand side is nonnegative, we can obtain a uniform upper bound for the middle term in the second line above:

$$\int_0^T \int_{|x|>R} \frac{|u|^{p+1}}{|x|}\, dx\, dt \le \frac{2(p+1)}{p-1} E.$$

Letting R approach zero and T approach T_+, we have

$$\int_0^{T_+}\int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|}\,dx\,dt \le \frac{2(p+1)}{p-1}E. \tag{37}$$

The energy E here is finite by our estimate (36). On the other hand, recalling our local compactness result Proposition 3.7, we obtain ($T_+ = \infty$)

$$\int_0^{\infty}\int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|}\,dx\,dt = \infty.$$

This finishes our discussion in this case.

7. Further estimates in the soliton-like case

Let u be a soliton-like minimal blow-up solution. We will find additional decay of $u(x,t)$ as x tends to infinity. The method used here is similar to the one used in [Kenig and Merle 2011] for the supercritical case. Throughout this section $w(r,t)$, $h(r,t)$, $z_1(r,t)$ and $z_2(r,t)$ are defined as usual using $u(x,t)$. The argument in this section works in both the defocusing and focusing cases. But we are particularly interested in the focusing case, because the soliton-like solutions in the focusing case are the only solutions that still survive at this time.

Setup. Let $\varphi(x)$ be a smooth cutoff function in $\mathbb{R}^3$:

$$\varphi(x)\begin{cases} = 0 & \text{if } |x| \le \frac{1}{2}, \\ \in [0,1] & \text{if } \frac{1}{2} \le |x| \le 1, \\ = 1 & \text{if } |x| \ge 1. \end{cases}$$

Then by Proposition 3.6 (compactness of u), $\|\varphi(x/R)u(x,t)\|_{\dot H^{s_p}}$ converges to zero uniformly in t as $R \to \infty$. Thus we have a positive function $g(r)$ so that $g(r)$ decreases to zero as r increases to infinity with

$$\|\varphi(x/R)u(x,t)\|_{\dot H^{s_p}} \le g(R).$$

This means for each $|x| \ge R$, we have

$$|u(x,t)| = |\varphi(x/R)u(x,t)| \le C\frac{\|\varphi(\cdot/R)u(\cdot,t)\|_{\dot H^{s_p}}}{|x|^{2/(p-1)}} \le \frac{Cg(R)}{|x|^{2/(p-1)}}.$$

Definition 7.1.
$$f_\beta(r) = \sup_{t\in\mathbb{R},|x|\ge r} |x|^\beta |u(x,t)|$$

for $\beta \in [2/(p-1), 1)$ and $r > 0$.

This is a nonincreasing function of r defined from $\mathbb{R}^+$ to $[0,\infty)\cup\{\infty\}$. Consider the set

$$U = \{\beta \in [2/(p-1), 1) : f_\beta(r) \to 0 \text{ as } r \to \infty\}.$$

This is not empty, since $2/(p-1)$ is in U. Due to the estimate

$$|x|^\beta |u(x,t)| \le C_p |x|^{\beta - \frac{2}{p-1}} \|u(\cdot,t)\|_{\dot H^{s_p}},$$

we know if $\beta \in U$, then $f_\beta(r)$ is a bounded function. By the definition of f_β, we have

$$|u(x,t)| \leq \frac{f_\beta(r)}{|x|^\beta} \tag{38}$$

for any time $t \in \mathbb{R}$ and $|x| \geq r$. This is a meaningful inequality as long as $\beta \in U$.

Lemma 7.2. *Suppose u is a soliton-like minimal blow-up solution and $\beta \in U$. Then we have the local energy estimate on $w = ru$*

$$\left(\int_{r_0}^{4r_0} |\partial_t w(r,t_0)|^2 + |\partial_r w(r,t_0)|^2\, dr\right)^{\frac{1}{2}} \leq C_p \frac{f_\beta^p(r_0)}{r_0^{p\beta-5/2}} \tag{39}$$

for any $r_0 > 0$ and $t_0 \in \mathbb{R}$. The constant C_p depends on p only.

Proof. Applying Lemma 4.3 to w, we have

$$\left(\int_{r_0}^{4r_0} |z_1(r,t_0)|^2\, dr\right)^{\frac{1}{2}} \leq \left(\int_{r_0+M}^{4r_0+M} |z_1(r,t_0+M)|^2\, dr\right)^{\frac{1}{2}} + \left(\int_{r_0}^{4r_0}\left(\int_0^M h(r+t,t_0+t)\, dt\right)^2 dr\right)^{\frac{1}{2}}.$$

Let $M \to \infty$. Using Proposition 4.12 we have

$$\begin{aligned}
\left(\int_{r_0}^{4r_0} |z_1(r,t_0)|^2\, dr\right)^{\frac{1}{2}} &\leq \limsup_{M\to\infty}\left(\int_{r_0}^{4r_0}\left(\int_0^M (r+t)F(u(r+t,t_0+t))\, dt\right)^2 dr\right)^{\frac{1}{2}} \\
&\leq \limsup_{M\to\infty}\left(\int_{r_0}^{4r_0}\left(\int_0^M (r+t)\left(\frac{f_\beta(r_0)}{(r+t)^\beta}\right)^p dt\right)^2 dr\right)^{\frac{1}{2}} \\
&\lesssim_p \limsup_{M\to\infty}\left(\int_{r_0}^{4r_0}\left(\frac{f_\beta^p(r_0)}{r^{p\beta-2}}\right)^2 dr\right)^{\frac{1}{2}} \\
&\leq f_\beta^p(r_0)\left(\int_{r_0}^{4r_0}\frac{1}{r^{2p\beta-4}}\, dr\right)^{\frac{1}{2}} \\
&\lesssim_p f_\beta^p(r_0)\left(\frac{1}{r_0^{2p\beta-5}}\right)^{\frac{1}{2}} \\
&\leq f_\beta^p(r_0)\frac{1}{r_0^{p\beta-5/2}}.
\end{aligned}$$

Similarly we have

$$\left(\int_{r_0}^{4r_0} |z_2(r,t_0)|^2\, dr\right)^{\frac{1}{2}} \lesssim \frac{f_\beta^p(r_0)}{r_0^{p\beta-5/2}}.$$

Combining these two estimates we obtain the inequality (39). □

Recurrence formula.

Lemma 7.3. *The function f_β defined above satisfies the recurrence formula*

$$f_\beta(r_0) \le \frac{1}{2}\Big[\Big(\frac{3}{2}\Big)^{1-\beta} + \Big(\frac{1}{2}\Big)^{1-\beta}\Big] f_\beta\Big(\frac{r_0}{2}\Big) + C_p f_\beta^p\Big(\frac{r_0}{2}\Big) r_0^{2-(p-1)\beta}. \tag{40}$$

Proof. We know $w = ru$ is a solution to the one-dimensional wave equation

$$\partial_t^2 w - \partial_r^2 w = r|u|^{p-1}u.$$

Using the explicit formula to solve this equation, we obtain

$$\begin{aligned} r_0 u(r_0, t_0) &= \frac{1}{2}\Big[\Big(r_0 + \frac{r_0}{2}\Big)u\Big(r_0 + \frac{r_0}{2}, t_0 - \frac{r_0}{2}\Big) + \Big(r_0 - \frac{r_0}{2}\Big)u\Big(r_0 - \frac{r_0}{2}, t_0 - \frac{r_0}{2}\Big)\Big] \\ &\quad + \frac{1}{2}\int_{r_0 - \frac{r_0}{2}}^{r_0 + \frac{r_0}{2}} \partial_t w\Big(r, t_0 - \frac{r_0}{2}\Big)\, dr + \frac{1}{2}\int_0^{\frac{r_0}{2}} \int_{\frac{r_0}{2}+t}^{\frac{3r_0}{2}-t} r|u|^{p-1}u\Big(r, t_0 - \frac{r_0}{2} + t\Big)\, dr\, dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By Cauchy–Schwartz and Lemma 7.2, we have

$$\begin{aligned} |I_2| &\le \frac{1}{2}\Big(\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} \Big|\partial_t w\Big(r, t_0 - \frac{r_0}{2}\Big)\Big|^2 dr\Big)^{\frac{1}{2}} \Big(\int_{\frac{r_0}{2}}^{\frac{3r_0}{2}} 1\, dr\Big)^{\frac{1}{2}} \\ &\le C_p \frac{f_\beta^p(r_0/2)}{r_0^{p\beta - 5/2}} r_0^{1/2} \\ &= C_p f_\beta^p\Big(\frac{r_0}{2}\Big) r_0^{3-p\beta}. \end{aligned}$$

Next we estimate I_3 using the estimate (38)

$$|I_3| \le \frac{1}{2}\int_0^{\frac{r_0}{2}} \int_{\frac{r_0}{2}+t}^{\frac{3r_0}{2}-t} r\left(\frac{f_\beta(r_0/2)}{r^\beta}\right)^p dr\, dt \le C_p \int_0^{\frac{r_0}{2}} r_0^2 \frac{f_\beta^p(r_0/2)}{r_0^{p\beta}}\, dt \le C_p f_\beta^p\Big(\frac{r_0}{2}\Big) r_0^{3-p\beta}.$$

At the same time, we know

$$|I_1| \le \frac{1}{2}\Big[\frac{3r_0}{2}\frac{f_\beta(3r_0/2)}{(3r_0/2)^\beta} + \frac{r_0}{2}\frac{f_\beta(r_0/2)}{(r_0/2)^\beta}\Big] = \frac{1}{2}\Big[\Big(\frac{3}{2}\Big)^{1-\beta} f_\beta\Big(\frac{3r_0}{2}\Big) + \Big(\frac{1}{2}\Big)^{1-\beta} f_\beta\Big(\frac{r_0}{2}\Big)\Big] r_0^{1-\beta}.$$

Combining these three terms and dividing both sides of the inequality by $r_0^{1-\beta}$, we obtain (replace r_0 by r)

$$r^\beta |u(r, t_0)| \le \frac{1}{2}\Big[\Big(\frac{3}{2}\Big)^{1-\beta} f_\beta\Big(\frac{3r}{2}\Big) + \Big(\frac{1}{2}\Big)^{1-\beta} f_\beta\Big(\frac{r}{2}\Big)\Big] + C_p f_\beta^p\Big(\frac{r}{2}\Big) r^{2-(p-1)\beta}.$$

Observing that the right-hand side is a nonincreasing function of r, we apply $\sup_{r \ge r_0}$ on both sides and obtain

$$f_\beta(r_0) \le \frac{1}{2}\Big[\Big(\frac{3}{2}\Big)^{1-\beta} f_\beta\Big(\frac{3r_0}{2}\Big) + \Big(\frac{1}{2}\Big)^{1-\beta} f_\beta\Big(\frac{r_0}{2}\Big)\Big] + C_p f_\beta^p\Big(\frac{r_0}{2}\Big) r_0^{2-(p-1)\beta}. \tag{41}$$

This completes the proof because we know $f_\beta(3r_0/2) \le f_\beta(r_0/2)$. $\square$

Decay of $u(x,t)$.

Definition 7.4. Let us define $(2/(p-1) \le \beta < 1)$

$$g(\beta) = \tfrac{1}{2}\big[(\tfrac{3}{2})^{1-\beta} + (\tfrac{1}{2})^{1-\beta}\big] < 1.$$

Lemma 7.5. *If $\beta \in U$, then we have*

$$\left[\beta, \beta + \log_2 \frac{2}{1+g(\beta)}\right) \subseteq U.$$

Proof. Because $f_\beta(r) \to 0$ and $2-(p-1)\beta \le 0$, we know that there exists a large constant R, such that if $r_0 > R$, we have

$$C_p f_\beta^p\left(\frac{r_0}{2}\right) r_0^{2-(p-1)\beta} \le \frac{1-g(\beta)}{2} f_\beta\left(\frac{r_0}{2}\right).$$

Thus the inequality (40) gives, for $r_0 > R$,

$$f_\beta(r_0) \le \frac{g(\beta)+1}{2} f_\beta\left(\frac{r_0}{2}\right).$$

This implies

$$f_\beta(r) \le C r^{\log_2(\frac{g(\beta)+1}{2})}$$

for sufficiently large $r > R'$. As a result, for each $\beta_1 < \beta - \log_2(\frac{g(\beta)+1}{2}) \in (\beta, 1)$, we have

$$|x|^{\beta_1}|u(x,t)| \le f_\beta(|x|)|x|^{\beta_1-\beta} \le C|x|^{\beta_1-\beta+\log_2(\frac{g(\beta)+1}{2})} \to 0$$

as $|x| \to \infty$. This proves the lemma by our definition of f_{β_1} and U. □

Lemma 7.6. *Let U be defined as above. Then we have* $\sup U = 1$.

Proof. If this were false, we could assume $\sup U = \beta_0 < 1$. Then we have for each $\beta \in U$,

$$g(\beta) \le G_0 \doteq \max\left\{g(\beta_0), g\left(\frac{2}{p-1}\right)\right\} < 1$$

using the convexity of the function g. Thus $\log_2 \frac{2}{1+g(\beta)} \ge \log_2 \frac{2}{1+G_0} > 0$. By Lemma 7.5, we know

$$\left[\beta, \beta + \log_2 \frac{2}{1+G_0}\right) \subseteq U.$$

This gives us a contradiction as $\beta \to \sup U$. □

The following proposition is the main result of this section.

Proposition 7.7 (decay of u). *Let u be a soliton-like minimal blow-up solution. Then*

$$|u(x,t)| \le \frac{C_1}{|x|} \tag{42}$$

and

$$\int_{r<|x|<4r} (|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2)\, dx \le C_2 r^{-1}. \tag{43}$$

The constants C_1 and C_2 are independent of t, x or r.

Proof. Let β be a number slightly smaller than 1. Lemma 7.6 guarantees $\beta \in U$. As a result, we can obtain the following estimate using the conclusion of Lemma 7.2:

$$\int_{r_0}^{4r_0} |\partial_r w(r,t_0)|\,dr \le \left(\int_{r_0}^{4r_0} |\partial_r w(r,t_0)|^2\,dr\right)^{\frac{1}{2}} \left(\int_{r_0}^{4r_0} 1\,dr\right)^{\frac{1}{2}} \le \frac{C_p f_\beta^p(r_0)}{r_0^{p\beta-5/2}} r_0^{1/2} \le \frac{C_{p,\beta}}{r_0^{p\beta-3}}.$$

We can choose $\beta \in U$ so that $p\beta - 3 > 0$ by the fact $p > 3$. Thus we have

$$\int_1^\infty |\partial_r w(r,t_0)|\,dr \le C_{p,\beta}. \tag{44}$$

In addition, for $r \le 1$,

$$|w(r,t_0)| = r|u(r,t_0)| \le C_p \|u(t_0)\|_{\dot H^{s_p}} r^{1-\frac{2}{p-1}} \le C_p \|u(t_0)\|_{\dot H^{s_p}}.$$

Combining these two estimates above, we know that $|w(r,t)|$ is bounded by a universal constant C_1 for each pair (r,t). This gives us the first inequality in the conclusion by the definition $w = ru$. Plugging this in the definition of $f_\beta(r)$, we have

$$f_\beta(r_0) = \sup_{t\in\mathbb{R},|x|\ge r_0} |x|^\beta |u(x,t)| \le \sup_{t\in\mathbb{R},|x|\ge r_0} C_1 |x|^{\beta-1} = C_1 r_0^{\beta-1}.$$

Plugging this in (39), we obtain

$$\left(\int_{r_0}^{4r_0} |\partial_t w(r,t_0)|^2 + |\partial_r w(r,t_0)|^2\,dr\right)^{\frac{1}{2}} \lesssim \frac{1}{r_0^{p-5/2}}. \tag{45}$$

By Lemma 4.2, the combination of this estimate, Proposition 4.12 and the universal decay of u (42) indicates that the second inequality in the lemma is also true. □

8. Death of soliton-like solution

Solitons in the focusing case. In order to kill the soliton-like minimal blow-up solutions, we need to consider the solitons of the wave equation. It turns out that there does not exist any soliton for our equation. The elliptic equation

$$-\Delta W(x) = |W(x)|^{p-1} W(x) \tag{46}$$

does admit a lot of nontrivial radial solutions. However, none of these solutions is in the space $\dot H^{s_p}$. Among these solutions we are particularly interested in the following solution W_0 which satisfies the condition $W_0(x) \sim 1/|x|$.

Proposition 8.1. *The elliptic equation (46) has a solution $W_0(x)$ such that*:

- *$W_0(x)$ is a radial and smooth solution in $\mathbb{R}^3 \setminus \{0\}$.*
- *The point 0 is a singularity of $W_0(x)$.*
- *The solution $W_0(x)$ is **not** in the space $\dot H^{s_p}(\mathbb{R}^3)$.*

- *Its behavior near infinity is given by* $(|x| > R_0)$

$$\left|W_0(x) - \frac{1}{|x|}\right| \le \frac{C}{|x|^{p-2}}, \quad |\nabla W_0(x)| \le \frac{C}{|x|^2}. \tag{47}$$

The next section has a complete discussion of this solution.

Idea to deal with the soliton-like solutions. We will show there does not exist a soliton-like minimal blow-up solution in the focusing case. This conclusion is natural because there is actually no soliton. However, to prove this result is not an easy task. We will use a method developed by T. Duyckaerts et al. as I mentioned at the beginning of this paper. In [Duyckaerts et al. 2013] they use this method to prove the soliton resolution conjecture for radial solutions of the focusing, energy-critical wave equation. The idea is to show that our soliton-like solution has to be so close to the solitons $\pm W_0(x)$ or their rescaled versions that they must be exactly the same. But the soliton we mentioned above is not in the right Sobolev space. This is a contradiction. In order to achieve this goal, we have to be able to understand the behavior of a minimal blow-up solution if it is close to our soliton $W_0(x)$.

Preliminary results. We first recall a lemma proved in [Duyckaerts et al. 2011].

Lemma 8.2 (energy channel). *Let $(v_0, v_1) \in \dot{H}^1 \times L^2$ be a pair of radial initial data. Suppose $v(x,t)$ is the solution of the linear wave equation with the given initial data (v_0, v_1). Let $w(r,t) = rv(r,t)$ as usual. Then for any $R > 0$, either the inequality*

$$\int_{|x|>R+t} (|\nabla v(x,t)|^2 + |\partial_t v(x,t)|^2)\, dx \ge 2\pi \int_R^\infty (|\partial_r w(r,0)|^2 + |\partial_t w(r,0)|^2)\, dr$$

holds for all $t > 0$, or the inequality

$$\int_{|x|>R-t} (|\nabla v(x,t)|^2 + |\partial_t v(x,t)|^2)\, dx \ge 2\pi \int_R^\infty (|\partial_r w(r,0)|^2 + |\partial_t w(r,0)|^2)\, dr$$

holds for all $t < 0$.

Definition 8.3. Let us define $(R > 0)$

$$V_R(x,t) = \begin{cases} W_0(R+|t|) & \text{if } |x| \le R+|t|, \\ W_0(|x|) & \text{if } |x| > R+|t|. \end{cases} \tag{48}$$

Lemma 8.4. *The following space-time norms of $V_R(x,t)$ are both finite for $R > 0$:*

$$\|V_R\|_{Y_{s_p}(\mathbb{R})} < \infty; \quad \|V_R\|_{L^{2p/(p-3)}L^{2p}(\mathbb{R}\times\mathbb{R}^3)} < \infty.$$

Furthermore, if R is sufficiently large, we have the estimate

$$\|V_R\|_{Y_{s_p}(\mathbb{R})} \lesssim R^{\frac{1}{2}-s_p}; \quad \|V_R\|_{L^{2p/(p-3)}L^{2p}(\mathbb{R}\times\mathbb{R}^3)} \lesssim R^{-\frac{1}{2}}. \tag{49}$$

Proof. By the estimate (47) in Proposition 8.1, we have

$$|W_0(x)| \le \frac{C_R}{|x|} \quad \text{if } |x| \ge R.$$

Thus, if $3/r + 1/q < 1$,

$$\begin{aligned}
\|V_R\|_{L^qL^r(\mathbb{R}\times\mathbb{R}^3)} &= \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^3}|V_R(x,t)|^r\,dx\right)^{q/r}dt\right)^{1/q}\\
&\lesssim \left(\int_{\mathbb{R}}\left((R+|t|)^3|W_0(R+|t|)|^r + \int_{|x|>R+|t|}|W_0(x)|^r\,dx\right)^{q/r}dt\right)^{1/q}\\
&\lesssim C_R\left(\int_{\mathbb{R}}\left((R+|t|)^{3-r} + \int_{|x|>R+|t|}|x|^{-r}\,dx\right)^{q/r}dt\right)^{1/q}\\
&\lesssim_r C_R\left(\int_{\mathbb{R}}\left((R+|t|)^{3-r}\right)^{q/r}dt\right)^{1/q}\\
&\lesssim_{r,q} C_R(R^{(3-r)q/r+1})^{1/q}\\
&\lesssim_{r,q} C_R R^{\frac{3}{r}+\frac{1}{q}-1}.
\end{aligned}$$

This shows the norms in question are always finite. Furthermore, if R is sufficiently large, we can always choose $C_R = 2$. This finishes our proof by the computation above. □

Approximation theory.

Theorem 8.5. *Fix $3 < p < 5$. There exists a constant $\delta_0 > 0$, such that if $\delta < \delta_0$ and we have*

(i) *a function $V(x,t)$ with $\|V(x,t)\|_{Y_{s_p}(I)} < \delta$ (here I is a time interval containing 0), and*

(ii) *a pair of initial data (h_0, h_1) with*

$$\|(h_0,h_1)\|_{\dot H^1\times L^2(\mathbb{R}^3)} < \delta, \quad \|(h_0,h_1)\|_{\dot H^{s_p}\times \dot H^{s_p-1}(\mathbb{R}^3)} < \delta,$$

then the equation

$$\begin{cases} \partial_t^2 h - \Delta h = F(V+h) - F(V), & (x,t)\in\mathbb{R}^3\times I,\\ h|_{t=0} = h_0,\\ \partial_t h|_{t=0} = h_1\end{cases}$$

has a unique solution $h(x,t)$ on $I\times\mathbb{R}^3$ so that

$$\|h\|_{Y_{s_p}(I)} \le C_p\delta,$$
$$\sup_{t\in I}\|(h,\partial_t h) - (h_L,\partial_t h_L)\|_{\dot H^1\times L^2} \le C_p\delta^{p-1}\|(h_0,h_1)\|_{\dot H^1\times L^2}.$$

Here $(h_L, \partial_t h_L)$ is the solution of the linear wave equation with initial data (h_0, h_1).

Proof. In this proof, C_p represents a constant that depends on p only. In different places C_p may represent different constants. We will also write Y instead of $Y_{s_p}(I)$ for convenience. By the Strichartz estimates, we have

$$\|F(V+h) - F(V)\|_{Z_{s_p}} \le C_p\|h\|_Y(\|h\|_Y^{p-1} + \|V\|_Y^{p-1}),$$
$$\|F(V+h^{(1)}) - F(V+h^{(2)})\|_{Z_{s_p}} \le C_p\|h^{(1)} - h^{(2)}\|_Y(\|h^{(1)}\|_Y^{p-1} + \|h^{(2)}\|_Y^{p-1} + \|V\|_Y^{p-1}).$$

In addition, if we choose a 1-admissible pair $\left(\frac{4p}{9-p}, \frac{4p}{p-3}\right)$, we also have

$$\|F(V+h)-F(V)\|_{L^1L^2} \le C_p \|h\|_{L^{\frac{4p}{9-p}}L^{\frac{4p}{p-3}}} \left(\|h\|_Y^{p-1} + \|V\|_Y^{p-1}\right),$$

$$\|F(V+h^{(1)})-F(V+h^{(2)})\|_{L^1L^2} \le C_p \|h^{(1)}-h^{(2)}\|_{L^{\frac{4p}{9-p}}L^{\frac{4p}{p-3}}} \left(\|h^{(1)}\|_Y^{p-1} + \|h^{(2)}\|_Y^{p-1} + \|V\|_Y^{p-1}\right).$$

By a fixed point argument, if δ is sufficiently small, we have a unique solution $h(x,t)$ defined on $I \times \mathbb{R}^3$, so that

$$\|h\|_Y \le C_p \delta, \quad \|h\|_{L^{\frac{4p}{9-p}}L^{\frac{4p}{p-3}}} \le C_p \|(h_0,h_1)\|_{\dot{H}^1\times L^2}.$$

This immediately gives

$$\begin{aligned}\sup_{t\in I} \|(h,\partial_t h)-(h_L,\partial_t h_L)\|_{\dot{H}^1\times L^2} &\le C_p \|F(V+h)-F(V)\|_{L^1L^2}\\ &\le C_p \|h\|_{L^{\frac{4p}{9-p}}L^{\frac{4p}{p-3}}} \left(\|h\|_Y^{p-1} + \|V\|_Y^{p-1}\right)\\ &\le C_p \delta^{p-1} \|(h_0,h_1)\|_{\dot{H}^1\times L^2}.\end{aligned}$$

□

Match with $W_0(x)$. Using the estimate (45), we have

$$\int_{r_0}^{4r_0} |\partial_r w(r,t)|\, dr \lesssim \left(\int_{r_0}^{4r_0} |\partial_r w(r,t)|^2\, dr\right)^{\frac{1}{2}} r_0^{1/2} \lesssim \frac{1}{r_0^{p-3}}.$$

This means

$$\int_{r_0}^{\infty} |\partial_r w(r,t)|\, dr \lesssim \frac{1}{r_0^{p-3}}. \tag{50}$$

Thus we know the limit $\lim_{r\to\infty} w(r,t)$ exists for each t. In particular, the limit exists at $t=0$. There are two cases.

(I) If $\lim_{r\to\infty} w(r,0)=0$. Then in the rest of this section, set $W(x)=0$. By (50) we have

$$|w(r,0)| \lesssim \frac{1}{r^{p-3}}.$$

Thus

$$|u_0(x)-W(x)| = \frac{1}{|x|}\,|w(|x|,0)| \lesssim \frac{1}{|x|^{p-2}}.$$

(II) If $\lim_{r\to\infty} w(r,0) \neq 0$. Without loss, let us assume the limit is equal to 1. Otherwise we only need to apply some space-time dilation and/or multiplication by -1 on u. In the rest of this section, set $W(x)=W_0(x)$. By (50), we have

$$|w(r_0,0)-1| \le \int_{r_0}^{\infty} |\partial_r w(r,0)|\, dr \lesssim \frac{1}{r_0^{p-3}}.$$

Dividing this inequality by r_0, we have

$$\left|u_0(x) - \frac{1}{|x|}\right| \lesssim \frac{1}{|x|^{p-2}}.$$

Combining this with our estimate for $W_0(x)$, we have for large x

$$|u_0(x) - W(x)| \lesssim \frac{1}{|x|^{p-2}}.$$

Identity near infinity.

Theorem 8.6. *Let $W(x) = W_0(x)$ or $W(x) = 0$. Suppose $u(x,t)$ is a global radial solution of the equation (1) with initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ satisfying the following conditions.*

(I) *The following inequality holds for each $t \in \mathbb{R}$ and $r > 0$:*

$$\int_{r<|x|<4r} (|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2)\, dx \leq C_1 r^{-1}. \tag{51}$$

(II) *We have $u_0(x)$ and $W(x)$ are very close to each other as $|x|$ is large:*

$$|u_0(x) - W(x)| \lesssim \frac{1}{|x|^{p-2}}. \tag{52}$$

Then there exists $R_0 = R_0(C_1, p) \in (0, +\infty)$ such that the pair $(u_0(x) - W(x), u_1(x))$ is essentially supported in the ball $\bar{B}(0, R_0)$.

Remark 8.7. There are actually two separate theorems, and both can be proved in the same way. If $W(x) = W_0(x)$ (the primary case), then define V_{R_0} as usual in the proof below. Otherwise, if $W(x) = 0$, just make $V_{R_0} = 0$.

Proof. Consider the functions

$$g_0 = \Psi_R(u_0 - W), \quad g_1 = \Psi_R u_1, \quad G(r) = u_0(r) - W(r),$$

for $R \geq R_0$, where the constant R_0 is to be determined later. Choose a small constant $\delta = \delta(p)$, so that it is smaller than the constant δ_0 in Theorem 8.5 and guarantees the number $C_p \delta^{p-1}$ in the conclusion of that theorem is smaller than $\varepsilon(p)$, which is a small number determined later in the argument below. By the condition (51) and the properties of $W(x)$, we know $(R > 1)$

$$\int_{\mathbb{R}^3} (|\nabla g_0|^2 + g_1^2)\, dx \lesssim_{C_1,p} R^{-1};$$

$$\int_{\mathbb{R}^3} \left(|\nabla g_0|^{3(p-1)/(p+1)} + g_1^{3(p-1)/(p+1)}\right) dx \lesssim_{C_1,p} R^{-3(p-3)/(p+1)}.$$

As a result, if $R_0 = R_0(C_1, p)$ is sufficiently large, the following inequalities hold as long as $R \geq R_0$ (we use the Sobolev embedding in order to obtain the second inequality):

$$\|(g_0, g_1)\|_{\dot{H}^1 \times L^2} \leq \delta, \quad \|(g_0, g_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}} \leq \delta, \quad \|V_{R_0}\|_{Y_{s_p}(\mathbb{R})} \leq \delta.$$

Let g be the solution of

$$\partial_t^2 g - \Delta g = F(V_{R_0} + g) - F(V_{R_0})$$

with the initial data (g_0, g_1) and $\tilde{g}$ be the solution of the linear wave equation with the same initial data. On the other hand, we know $u(x,t) - W(x)$ is the solution of the equation

$$\partial_t^2 \tilde{u} - \Delta \tilde{u} = F(W + \tilde{u}) - F(W) \tag{53}$$

in the domain $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ with the initial data $(u_0 - W, u_1)$. Let K be the domain

$$K = \{(x,t) : |x| > |t| + R\}.$$

Considering the fact $W(x) = V_{R_0}(x,t)$ in the region K and the construction of (g_0, g_1), we have

$$u(x,t) - W(x) = g(x,t), \quad \partial_t u(x,t) = \partial_t g(x,t)$$

in the domain K by the finite speed of propagation. Using our assumption (51) and the decay of $W(x)$ at infinity and considering the identity above, we have

$$\lim_{t \to \pm\infty} \int_{|x|>|t|+R} (|\nabla g(x,t)|^2 + |\partial_t g(x,t)|^2)\, dx \to 0. \tag{54}$$

Using Lemma 8.2, without loss of generality, let us assume for all $t > 0$

$$\int_{|x|>R+t} (|\nabla \tilde{g}(x,t)|^2 + |\partial_t \tilde{g}(x,t)|^2)\, dx \geq 2\pi \int_R^\infty (|\partial_r (r g_0(r,0))|^2 + r^2 |g_1(r,0)|^2)\, dr.$$

That is

$$\int_{|x|>R+t} (|\nabla \tilde{g}(x,t)|^2 + |\partial_t \tilde{g}(x,t)|^2)\, dx \geq \frac{1}{2}\left(\int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx\right) - 2\pi R g_0^2(R).$$

Combining this with (54), we have

$$\liminf_{t\to\infty} \|(g(x,t), \partial_t g(x,t)) - (\tilde{g}, \partial_t \tilde{g})\|_{\dot{H}^1 \times L^2(|x|>R+t)} \geq \left(\frac{1}{2}\int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx - 2\pi R g_0^2(R)\right)^{\frac{1}{2}}.$$

On the other hand, we know that the inequality

$$\|(g(x,t), \partial_t g(x,t)) - (\tilde{g}, \partial_t \tilde{g})\|_{\dot{H}^1 \times L^2} \leq C_p \delta^{p-1} \|(g_0, g_1)\|_{\dot{H}^1 \times L^2} \leq \varepsilon(p) \|(g_0, g_1)\|_{\dot{H}^1 \times L^2}$$

holds for each $t \in \mathbb{R}$, by Theorem 8.5. Considering both inequalities above, we have

$$\frac{1}{2}\int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx - 2\pi R g_0^2(R) \leq \varepsilon^2(p) \int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx.$$

Thus

$$\int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx \leq \frac{4\pi}{1 - 2\varepsilon^2(p)} R g_0^2(R). \tag{55}$$

We have

$$\begin{aligned}|g_0(mR)-g_0(R)| &\le \int_R^{mR} |\partial_r g_0|\, dr \\ &\le \left(\int_R^{mR} |r\partial_r g_0|^2\, dr\right)^{\frac12}\left(\int_R^{mR} \frac{1}{r^2}\, dr\right)^{\frac12} \\ &\le \left(\frac{1}{4\pi}\int_{|x|>R} (|\nabla g_0|^2 + g_1^2)\, dx\right)^{\frac12}\left(\frac{1}{R}-\frac{1}{mR}\right)^{\frac12} \\ &\le \left(\frac{Rg_0^2(R)}{1-2\varepsilon^2(p)}\right)^{\frac12}\left(1-\frac{1}{m}\right)^{\frac12} R^{-\frac12} \\ &\le \left(\frac{1-1/m}{1-2\varepsilon^2(p)}\right)^{\frac12} |g_0(R)|.\end{aligned}$$

Since $p-2>1$, we can choose $k=k(p)\in\mathbb{Z}^+$ such that $(k+1)/k<p-2$. Let $m=2^k$. Since

$$(1-1/m)^{\frac12} < 1-\frac{1}{2m},$$

we can choose $\varepsilon(p)>0$ so small that

$$\left(\frac{1-1/m}{1-2\varepsilon^2(p)}\right)^{\frac12} \le 1-\frac{1}{2m} = 1-\frac{1}{2^{k+1}}.$$

Plugging this into our estimate above, we obtain

$$|g_0(2^k R)-g_0(R)| \le \left(1-\frac{1}{2^{k+1}}\right)|g_0(R)|.$$

Thus

$$|g_0(2^k R)| \ge \frac{1}{2^{k+1}}|g_0(R)|.$$

By the definition of g_0, this is the same as

$$|G(2^k R)| \ge \frac{1}{2^{k+1}}|G(R)|.$$

This inequality holds for all $R\ge R_0$. Now let us consider the value of $G(R_0)$. If $G(R_0)=0$, let us choose $R=R_0$. Plugging $g_0(R)$ back in (55), we have $(g_0,g_1)=(0,0)$. This means that (u_0-W,u_1) is supported in $\bar{B}(0,R_0)$ and finishes the proof. If $|G(R_0)|>0$, then we have

$$|G(2^{kn}R_0)| \ge \frac{1}{(2^{kn})^{(k+1)/k}}|G(R_0)| > 0$$

for each positive integer n. This contradicts the condition (52) because $(k+1)/k<p-2$ by our choice of k. □

Remark 8.8. If one feels uncomfortable about the singularity at zero in the equation (53), we could use the following center-cutoff version instead. Let φ be a smooth, radial, nonnegative function satisfying

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \geq 1, \\ \in [0,1] & \text{if } |x| \in (1/2, 1), \\ 0 & \text{if } |x| \leq 1/2. \end{cases}$$

Then $u(x,t) - \varphi(|x|/R_0)W_0(x)$ is a solution to the equation

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = F(\varphi(|x|/R_0)W_0 + \tilde{u}) + \Delta(\varphi(|x|/R_0)W_0(x)), (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ \tilde{u}|_{t=0} = u_0 - \varphi(|x|/R_0)W_0 \in \dot{H}^{s_p}(\mathbb{R}^3), \\ \partial_t \tilde{u}|_{t=0} = u_1 \in \dot{H}^{s_p - 1}(\mathbb{R}^3). \end{cases}$$

For any $T > 0$, we know

$$\|\varphi(|x|/R_0)W_0(x)\|_{Y_{s_p}([-T,T])} < \infty, \quad \|\Delta(\varphi(|x|/R_0)W_0(x))\|_{Z_{s_p}([-T,T])} < \infty.$$

In addition, the function $\Delta(\varphi(|x|/R_0)W_0(x)) = -F(W_0(x))$ in the region K. We can do the argument as usual in the proof above but avoid the singularity at zero with this new cutoff version of the equation (53). This method also works in the proof of Theorem 8.9, which will be introduced in the next subsection.

Application of the theorem. Now apply Theorem 8.6 to our soliton-like minimal blow-up solution. All the conditions are satisfied by our earlier argument. Thus $(u_0(x) - W(x), u_1(x))$ is supported in the closed ball of radius R_0 centered at the origin. In particular, because R_0 depends only on the constant C_1 and p, the same R_0 also works for other time t as long as the condition (52) is true at that time. But by the finite speed of propagation, we know $(u(x,t) - W(x), \partial_t u(x,t))$ is actually compactly supported in $\bar{B}(0, R_0 + |t|)$ at each time t. This means the condition (52) is always true at any given time. Thus the pair $(u(x,t) - W(x), \partial_t u(x,t))$ is essentially supported in the cylinder $\bar{B}(0, R_0) \times \mathbb{R}$.

Local radius analysis. Let us define the essential radius of the support of $(u(x,t) - W(x), \partial_t u(x,t))$ at time t as

$$R(t) = \min\{R \geq 0 : (u(x,t) - W(x), \partial_t u(x,t)) = (0,0) \text{ holds for } |x| > R\}.$$

This is well-defined for our minimal blow-up solution. Actually $R(t) \leq R_0$ holds for any $t \in \mathbb{R}$.

Theorem 8.9 (behavior of "compactly supported" solutions). *Let $W(x) = W_0(x)$ or $W(x) = 0$. Let $u(x,t)$ be a radial solution of the equation (1) in a time interval I containing 0, so that*

(I) $(u(x,t), \partial_t u(x,t)) \in C(I; \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$.

(II) *The pair $(u(x,0) - W(x), \partial_t u(x,0))$ is compactly supported with an essential radius of support $R(0) > R_1 > 0$.*

Then there exists a constant $\tau = \tau(R_1, p)$, such that

$$R(t) = R(0) + |t|$$

holds either for each $t \in [0, \tau] \cap I$ or for each $t \in [-\tau, 0] \cap I$.

Remark 8.10. If $W(x) = W_0(x)$ (the primary case), then define V_{R_1} as usual in the proof. Otherwise if $W(x) = 0$, just make $V_{R_1} = 0$. In this case we can choose $\tau = \infty$. In the proof we use the notation (u_0, u_1) for the initial data $(u(x,0), \partial_t u(x,0))$.

Proof. By Lemma 8.4, we have $\|V_{R_1}\|_{Y_{s_p}(\mathbb{R})} < \infty$. Thus we can choose $\tau = \tau(R_1, p) > 0$ such that $\|V_{R_1}\|_{Y_{s_p}([-\tau,\tau])} < \delta$. Here δ is a small constant so that we can apply Theorem 8.5 and make the number $C_p \delta^{p-1}$ less than $1/100$ in that theorem. If $\varepsilon < R(0) - R_1$, let us consider a pair of initial data (g_0, g_1) for each $R \in (R(0) - \epsilon, R(0))$,

$$g_0 = \Psi_R(u_0 - W), \quad g_1 = \Psi_R u_1.$$

This pair $(g_0(x), g_1(x))$ is nonzero by the definition of $R(0)$. By our assumptions on (u_0, u_1), we know the inequalities

$$\|(g_0, g_1)\|_{\dot{H}^1 \times L^2} < \delta, \quad \|(g_0, g_1)\|_{\dot{H}^{s_p} \times \dot{H}^{s_p - 1}} < \delta$$

hold for each $R \in (R(0) - \epsilon, R(0))$ as long as ε is sufficiently small. (In order to obtain the second inequality we use the Sobolev embedding.) Furthermore, we have

$$\begin{aligned}
|g_0(R)| &= \left| g_0(R(0)) - \int_R^{R(0)} \partial_r g_0(r)\, dr \right| \leq \int_R^{R(0)} |\partial_r g_0(r)|\, dr \\
&\leq \left(\int_R^{R(0)} r^2 |\partial_r g_0(r)|^2\, dr \right)^{\frac{1}{2}} \left(\int_R^{R(0)} \frac{1}{r^2}\, dr \right)^{\frac{1}{2}} \\
&\leq \left(\int_R^{R(0)} r^2 |\partial_r g_0(r)|^2\, dr \right)^{\frac{1}{2}} \left(\frac{R(0) - R}{R(0) R} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{\varepsilon}{R(0) R} \int_R^{R(0)} r^2 |\partial_r g_0(r)|^2\, dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$R g_0^2(R) \leq \frac{\varepsilon}{R(0)} \int_R^{R(0)} r^2 |\partial_r g_0(r)|^2\, dr \leq \frac{\varepsilon}{4\pi R(0)} \int_{R<|x|<R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2)\, dx.$$

If ε is sufficiently small, we can apply Lemma 4.2 to obtain

$$\int_R^{R(0)} \left[|\partial_r (r g_0(r))|^2 + r^2 g_1(r)^2 \right] dr \geq \frac{0.99}{4\pi} \int_{R<|x|<R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2)\, dx.$$

Let $\tilde{g}(x,t)$ be the solution to the linear wave equation with the initial data (g_0, g_1). By Lemma 8.2,

$$\begin{aligned}
\int_{|x|>R+|t|} \left(|\nabla \tilde{g}(x,t)|^2 + |\partial_t \tilde{g}(x,t)|^2 \right) dx &\geq 2\pi \int_R^{\infty} \left[|\partial_r (r g_0(r))|^2 + r^2 |g_1(r)|^2 \right] dr \\
&= 2\pi \int_R^{R(0)} \left[|\partial_r (r g_0(r))|^2 + r^2 |g_1(r)|^2 \right] dr \\
&\geq 0.49 \int_{R<|x|<R(0)} (|\nabla g_0(x)|^2 + |g_1(x)|^2)\, dx
\end{aligned}$$

holds either for each $t \geq 0$ or for each $t \leq 0$. Without loss of generality, let us choose $t \geq 0$; then we have

$$\|(\tilde{g}(x,t), \partial_t \tilde{g}(x,t))\|_{\dot{H}^1 \times L^2(|x|>R+t)} \geq 0.7 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}. \tag{56}$$

Let g be the solution of the equation

$$\begin{cases} \partial_t^2 g - \Delta g = F(V_{R_1} + g) - F(V_{R_1}), & (x,t) \in \mathbb{R}^3 \times [-\tau, \tau], \\ g|_{t=0} = g_0, \\ \partial_t g|_{t=0} = g_1. \end{cases}$$

By Theorem 8.5, we have

$$\|(g(x,t), \partial_t g(x,t)) - (\tilde{g}(x,t), \partial_t \tilde{g}(x,t))\|_{\dot{H}^1 \times L^2} \leq 0.01 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}$$

for each $t \in [-\tau, \tau]$. Combining this with (56), for $t \in [0, \tau]$ we obtain

$$\|(g(x,t), \partial_t g(x,t))\|_{\dot{H}^1 \times L^2(|x|>R+t)} \geq 0.69 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}. \tag{57}$$

In addition, we know $u(x,t) - W(x)$ is the solution of equation

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = F(W(x) + \tilde{u}) - F(W(x)), \\ \tilde{u}|_{t=0} = u_0 - W, \\ \partial_t \tilde{u}|_{t=0} = u_1 \end{cases}$$

in $(\mathbb{R}^3 \setminus \{0\}) \times I$. The initial data of these two equations mentioned above is the same in the region $\{x : |x| \geq R\}$ and the nonlinear part is the same function in the region

$$K = \{(x,t) : |x| > R + t,\ t \in [0, \tau] \cap I\}.$$

Thus by the finite speed of propagation, we have $g(x,t) = u(x,t) - W(x)$ and $\partial_t g(x,t) = \partial_t u(x,t)$ in K. Plugging this in (57), we obtain

$$\|(u(x,t) - W(x), \partial_t u(x,t))\|_{\dot{H}^1 \times L^2(|x|>R+t)} \geq 0.69 \|(g_0, g_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)}$$

for each $t \in I \cap [0, \tau]$. Since $R < R(0)$, we know the right-hand side of the inequality above is positive by the definition of essential radius of support. Thus we have

$$R(t) \geq R + |t| \tag{58}$$

for all $t \in [0, \tau] \cap I$. Letting $R \to R(0)^-$, we obtain $R(t) \geq R(0) + |t|$. By the finite speed of propagation, we have $R(t) = R(0) + |t|$. □

Remark 8.11. For each $R \in (R(0) - \varepsilon, R(0))$, we know that the inequality (58) above holds either in the positive or negative time direction. It may work in different directions as we choose different values of R. However, we can always choose a sequence $R_i \to R(0)^-$ such that the inequality works in the same time direction for all the R_i. This is sufficient for us to conclude the theorem.

End of soliton-like solution. Now let us show $R(0) = 0$. If it were not zero, let $R_1 = R(0)/2$, and then apply Theorem 8.9. We have (without loss of generality) $R(t) = R(0) + t$ for each $t \in [0, \tau]$. Applying Theorem 8.9 again at $t = \tau$, we obtain

$$R(t) = R(0) + \tau + (t - \tau) = R(0) + t$$

for $t \in [\tau, 2\tau]$, because

(i) The same constant τ works by the inequality $R(\tau) > R(0) > R_1$.

(ii) The theorem may only work in the positive time direction, since we know the radius of support $R(t)$ decreases in the other direction.

Repeating this process, we have for each $t > 0$,

$$R(t) = R(0) + t.$$

But it is impossible since $R(t)$ is uniformly bounded by R_0. Therefore we must have $R(0) = 0$. But this means either $u_0 = W_0(x) \notin \dot{H}^{s_p}(\mathbb{R}^3)$ or $(u_0, u_1) = (0, 0)$. This is a contradiction.

9. The solution of the elliptic equation

In this section we will consider the elliptic equation

$$-\Delta W(x) = |W(x)|^{p-1} W(x), \tag{59}$$

and prove Proposition 8.1. It has infinitely many solutions. For example,

$$W_1(x) = C|x|^{-2/(p-1)}$$

is a solution if we choose an appropriate constant C. Since we are interested in radial solutions of this elliptic equation, we can assume $W(x) = y(|x|)$. Here the function $y(r)$ satisfies the following equation in $(0, \infty)$:

$$y''(r) + \frac{2}{r} y'(r) + |y|^{p-1} y(r) = 0. \tag{60}$$

Let us first show that the solution $W_0(x)$ we mentioned earlier in this paper exists.

Existence of $W_0(x)$.

The idea. We are seeking a solution with the property $W_0(x) \simeq 1/|x|$ as x is large. That is equivalent to $y(r) \simeq 1/r$. Let us define $\rho(r) = r y(r)$; then $\rho(r)$ satisfies

$$\rho''(r) = -\frac{F(\rho)}{r^{p-1}}, \quad F(\rho) = |\rho|^{p-1}\rho.$$

We expect $\rho(r) \simeq 1$ for large r, thus let us assume $\rho(r) = \phi(r) + 1$. The corresponding equation for $\phi(r)$ is given as

$$\phi''(r) = -\frac{F(\phi + 1)}{r^{p-1}}.$$

We will show the following facts:

(I) This equation has a solution in the interval $[R,\infty)$ with boundary conditions at infinity $\phi(+\infty)=\phi'(+\infty)=0$, by a fixed-point argument.

(II) We can expand the domain of this solution to $\mathbb{R}^+$.

The fixed-point argument. Let us consider the metric space

$$K=\{\phi:\phi\in C([R,\infty);[-1,1]),\ \lim_{r\to+\infty}\phi(r)=0\}$$

with the distance $d(\phi_1,\phi_2)=\sup_r|\phi_1(r)-\phi_2(r)|$. One can check K is complete. Let us define a map $L:K\to K$ by

$$L(\phi)(r)=\int_r^\infty\left(\int_s^\infty\left(-\frac{F(\phi(t)+1)}{t^{p-1}}\right)dt\right)ds.$$

We have

$$|L(\phi)(r)|\le \int_r^\infty\left(\int_s^\infty\frac{2^p}{t^{p-1}}\,dt\right)ds \le\frac{C_p}{r^{p-3}},$$

$$|L(\phi_1)(r)-L(\phi_2)(r)|\le C_p\int_r^\infty\left(\int_s^\infty\frac{d(\phi_1,\phi_2)}{t^{p-1}}\,dt\right)ds\le C_p\frac{d(\phi_1,\phi_2)}{r^{p-3}}.$$

Thus if $R>R(p)$ is a sufficiently large number, then L is a contraction map from K to itself. As a result, there exists a unique fixed point $\phi_0(r)$. This gives us a classic smooth solution of the ODE in $[R,\infty)$. We have $\phi_0(r)\lesssim r^{3-p}$ and its derivative $\phi_0'(r)$ satisfies

$$|\phi_0'(r)|=\left|\int_r^\infty\frac{F(\phi_0(t)+1)}{t^{p-1}}\,dt\right|\le\frac{C_p}{r^{p-2}}.$$

Expansion of the solution. Now let us solve the ODE backward from $r=R$. We need to show it will never break down before we approach $r=0$. Actually we have

$$\frac{d}{dr}\left(\frac{|\phi_0+1|^{p+1}}{p+1}+\frac{r^{p-1}|\phi_0'|^2}{2}\right)=\frac{p-1}{2}r^{p-2}|\phi_0'|^2\ge 0.$$

Thus we have that the inequality

$$\frac{|\phi_0(r)+1|^{p+1}}{p+1}+\frac{r^{p-1}|\phi_0'(r)|^2}{2}\le\frac{|\phi_0(R)+1|^{p+1}}{p+1}+\frac{R^{p-1}|\phi_0'(R)|^2}{2}$$

holds for all $0<r\le R$ as long as the solution still exists at r. But this implies the solution will never break down at a positive r.

Properties of the solution. Now we can define

$$W_0(x)=\frac{\phi_0(|x|)+1}{|x|}.$$

This is a C^2, radial solution of our elliptic equation (59) for $|x| > 0$. Furthermore, we have for large x

$$\left|W_0(x) - \frac{1}{|x|}\right| = \frac{|\phi_0(|x|)|}{|x|} \leq \frac{C_p}{|x|^{p-2}}, \quad |\nabla W_0(x)| = \left|\frac{r\phi_0'(r) - \phi_0(r) - 1}{r^2}\right|_{r=|x|} \leq \frac{C_p}{|x|^2}.$$

Now the remaining task is to show $W_0(x)$ is not in the space $\dot{H}^{s_p}$. This implies $W_0(x)$ must have a singularity at 0. It turns out that it is not trivial. For instance, if we repeat the argument as above in the case $p = 5$, then the solution we obtain will be a smooth function in the whole space, as

$$W(x) = \frac{\sqrt{3}}{(1 + 3|x|^2)^{1/2}}.$$

Radial $\dot{H}^{s_p}$ solution does not exist. The following theorem shows that any nontrivial radial solution of our elliptic equation is not in the space $\dot{H}^{s_p}(\mathbb{R}^3)$. In particular, $W_0(x)$ is not in the space $\dot{H}^{s_p}(\mathbb{R}^3)$. Actually we have $\limsup_{x\to 0^+} |x|^\theta |W_0(x)| > 0$ by the argument below. This gives us a singularity at zero.

Theorem 9.1. *If $3 < p < 5$, then a radial $\dot{H}^{s_p}(\mathbb{R}^3)$ solution to the elliptic equation*

$$-\Delta W(x) = |W(x)|^{p-1} W(x)$$

must be the zero solution.

Remark 9.2. We always assume the function $y(r)$ has two continuous derivatives at any $r > 0$ in the proof below. Actually we can show any radial $\dot{H}^{s_p}$ solution of the elliptic equation must be in the space $C^2(\mathbb{R}^3 \setminus \{0\})$. First of all, a radial $\dot{H}^{s_p}$ function must be continuous except for $x = 0$. Using this fact and the regularity theory on the elliptic equation, we have the solution is C^2 except for $x = 0$.

Proof. The proof consists of three steps.

(I) (introduction to $r^\theta y(r)$) We assume $W(x) = y(|x|)$. The function $y(r)$ defined in $\mathbb{R}^+$ is a C^2 solution of

$$y''(r) + \frac{2}{r} y'(r) + |y|^{p-1} y(r) = 0.$$

Let us define another $C^2(\mathbb{R}^+)$ function

$$v(r) = r^\theta y(r), \quad \theta = \frac{2}{p-1}.$$

If $W(x) = y(|x|)$ is in the space $\dot{H}^{s_p}$, we then have $\lim_{r\to 0^+} v(r) = \lim_{r\to+\infty} v(r) = 0$ by Lemma A.7. Plugging $y(r) = r^{-\theta} v(r)$ in the equation for $y(r)$, we obtain an equation for $v(r)$,

$$r^2 v''(r) + \frac{2(p-3)}{p-1} r v'(r) - \frac{2(p-3)}{(p-1)^2} v(r) + |v|^{p-1} v(r) = 0.$$

Multiplying both sides by $v'(r)$, we obtain

$$\frac{d}{dr}\left(r^2 \frac{|v'(r)|^2}{2} - \frac{p-3}{(p-1)^2} v^2(r) + \frac{|v(r)|^{p+1}}{p+1}\right) = \frac{5-p}{p-1} r |v'(r)|^2 \geq 0. \tag{61}$$

(II) (the lower limit) If $v(r)$ is not the zero function, then the inequality

$$\liminf_{r\to+\infty} r^2|v'(r)|^2 > 0 \tag{62}$$

holds. If it failed, by considering the integral of (61) in the interval (ε, M) and letting $\varepsilon \to 0^+$ and $M \to +\infty$, we would have

$$\frac{5-p}{p-1}\int_0^\infty r|v'(r)|^2\,dr \le 0.$$

This means $v'(r) = 0$ everywhere, so $v(r) = 0$. But we assume it is not the zero function.

(III) (conclusion) If $W(x)$ were not identically zero, then $v(r)$ would be a nonzero function. By the limit (62), there exist $C > 0$ and $r_1 > 0$, such that if $r \in (r_1, \infty)$, the inequality $r^2|v'(r)|^2 > C$ holds. In other words, we have $|v'(r)| > \sqrt{C} r^{-1}$. This means $v'(r)$ does not change its sign in the interval (r_1, ∞) since it is a continuous function. Combining this fact with the lower bound of $|v'(r)|$, we know the limit of $v(r)$ does not exist at ∞. This gives us a contradiction. □

Further properties of the function $W_0(x)$. In this subsection, we will discover some additional properties of the soliton $W_0(x)$. Assume that $y(r)$ and $v(r)$ are defined in the same manner as the previous subsection.

- *$W_0(x)$ is a positive solution.* If this were not true, we could assume that $v(r_0) = 0$ for some $r_0 > 0$, because we know $v(r) > 0$ for sufficiently large r. Then by (61), we obtain

$$r^2\frac{|v'(r)|^2}{2} - \frac{p-3}{(p-1)^2}v^2(r) + \frac{|v(r)|^{p+1}}{p+1} \ge r_0^2\frac{|v'(r_0)|^2}{2} > 0 \tag{63}$$

for each $r > r_0$. However, the decay of $W_0(x)$ implies (if r is large) that

$$|v(r)| \lesssim r^{\theta-1}, \quad |v'(r)| = |\theta r^{\theta-1}y(r) + r^\theta y'(r)| \lesssim r^{\theta-2}.$$

This gives us a contradiction if we consider the limit of the left hand in the inequality (63) using these estimates.

- *$W_0(x)$ is smooth in $\mathbb{R}^3 \setminus \{0\}$.* Due to the fact that the function F is smooth in $\mathbb{R}^+$, a direct corollary follows that the function $W_0(x)$ is smooth everywhere except for $x = 0$.

Appendix

The Duhamel formula.

Lemma A.1. *Let $\frac{1}{2} < s \le 1$. If K is a compact subset of $\dot{H}^s \times \dot{H}^{s-1}$ with an s-admissible pair (q, r) so that $q \ne \infty$, then for each $\varepsilon > 0$, there exist two constants $M, \delta > 0$ such that*

$$\|S(t)(u_0,u_1)\|_{L^qL^r(J\times\mathbb{R}^3)} + \|S(t)(u_0,u_1)\|_{L^qL^r([M,\infty)\times\mathbb{R}^3)} + \|S(t)(u_0,u_1)\|_{L^qL^r((-\infty,M]\times\mathbb{R}^3)} < \varepsilon$$

holds for any $(u_0, u_1) \in K$ and any time interval J with a length $|J| \le \delta$.

Proof. Given $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, it is clear that we are able to find $M, \delta > 0$ so that the inequality holds for this particular pair of initial data and any interval J with a length $|J| \leq \delta$ by the fact $q < \infty$ and the Strichartz estimate

$$\|S(t)(u_0, u_1)\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^3)} < \infty.$$

If K is a finite set, then we can find M and δ so that they work for each pair in K by taking a maximum over all M and a minimum over all δ. In the general case, we can just choose a finite subset $\{(u_{0,i}, u_{1,i})\}_{i=1,2,\ldots,n}$ of K such that for each $(u_0, u_1) \in K$, there exists a positive integer i with $1 \leq i \leq n$ and $\|S(t)(u_0 - u_{0,i}, u_1 - u_{1,i})\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^3)} \leq C\|(u_0 - u_{0,i}, u_1 - u_{1,i})\|_{\dot{H}^s\times\dot{H}^{s-1}} < 0.01\varepsilon$; and then use our result for a finite subset. $\square$

Lemma A.2 (the Duhamel formula). *Let $u(x,t)$ be almost periodic modulo scaling in the interval $I = (T_-, \infty)$, namely the set*

$$K = \left\{ \left(\frac{1}{\lambda(t)^{3/2-s_p}} u\left(\frac{x}{\lambda(t)}, t\right), \frac{1}{\lambda(t)^{5/2-s_p}} \partial_t u\left(\frac{x}{\lambda(t)}, t\right) \right) : t \in I \right\}$$

is precompact in the space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. Then for any time $t_0 \in \mathbb{R}$, any bounded closed interval $[a,b]$ and any s_p-admissible pair (q,r) with $q < \infty$, we have

$$\lim_{T\to+\infty} \|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b]\times\mathbb{R}^3)} = 0,$$

$$\textit{weak} \lim_{T\to+\infty} S(t_0 - T)\begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} = 0.$$

Proof. We have

$$\begin{aligned}\|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b]\times\mathbb{R}^3)} &= \|S(t)(u(T), \partial_t u(T))\|_{L^q L^r([a-T,b-T]\times\mathbb{R}^3)} \\ &= \|S(t)(u_0^{(T)}, u_1^{(T)})\|_{L^q L^r([\lambda(T)(a-T),\lambda(T)(b-T)]\times\mathbb{R}^3)};\end{aligned}$$

here

$$(u_0^{(T)}, u_1^{(T)}) = \left(\frac{1}{\lambda(T)^{3/2-s_p}} u\left(\frac{\cdot}{\lambda(T)}, T\right), \frac{1}{\lambda(T)^{5/2-s_p}} \partial_t u\left(\frac{\cdot}{\lambda(T)}, T\right) \right).$$

Given $\varepsilon > 0$, let M, δ be the constants as in Lemma A.1. It is clear that if T is sufficiently large, we have either ($\lambda(T)$ is small)

$$\lambda(T)(b-T) - \lambda(T)(a-T) = (b-a)\lambda(T) < \delta,$$

or ($\lambda(T)$ is large)

$$\lambda(T)(b-T) < -M.$$

In either case, by Lemma A.1 we have $\|S(t-T)(u(T), \partial_t u(T))\|_{L^q L^r([a,b]\times\mathbb{R}^3)} < \varepsilon$. This completes the proof of the first limit. In order to obtain the second limit, we only need to choose $t_1 \in (t_0, +\infty)$, set $[a,b] = [t_0, t_1]$ and apply Lemma A.4 below using the first limit and the identity

$$S(t-t_0)\left[S(t_0 - T)\begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix} \right] = S(t-T)\begin{pmatrix} u(T) \\ \partial_t u(T) \end{pmatrix}.$$

$\square$

Remark A.3. We can obtain the similar result in the negative time direction using exactly the same argument. This implies the corresponding Duhamel formula in the negative time direction.

- Soliton-like case or high-to-low frequency cascade case

$$\lim_{T\to-\infty}\|S(t-T)(u(T),\partial_t u(T))\|_{L^qL^r([a,b]\times\mathbb{R}^3)}=0,$$

$$\text{weak}\lim_{T\to-\infty}S(t_0-T)\begin{pmatrix}u(T)\\ \partial_t u(T)\end{pmatrix}=0.$$

- Self-similar case (let $a, t_0 > 0$)

$$\lim_{T\to0^+}\|S(t-T)(u(T),\partial_t u(T))\|_{L^qL^r([a,b]\times\mathbb{R}^3)}=0,$$

$$\text{weak}\lim_{T\to0^+}S(t_0-T)\begin{pmatrix}u(T)\\ \partial_t u(T)\end{pmatrix}=0.$$

Lemma A.4. *Suppose that $\{(u_{0,n},u_{1,n})\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\dot H^s\times\dot H^{s-1}(\mathbb{R}^3)$ so that*

$$\lim_{n\to\infty}\|S(t)(u_{0,n},u_{1,n})\|_{L^qL^r([0,\mu]\times\mathbb{R}^3)}=0.$$

Here (q,r) is an s-admissible pair and μ is a positive constant. Then we have the weak limit in $\dot H^s\times\dot H^{s-1}(\mathbb{R}^3)$

$$(u_{0,n},u_{1,n})\rightharpoonup 0.$$

Proof. Let us suppose the conclusion were false. This means that there exists a subsequence (for which we use the same notation as the original sequence) that converges weakly to a nonzero limit $(\tilde u_0,\tilde u_1)$. We know the operator $P:\dot H^s\times\dot H^{s-1}\to L^qL^r([0,\mu]\times\mathbb{R}^3)$ defined by

$$P(u_0,u_1)=S(t)(u_0,u_1)$$

is bounded by the Strichartz estimate. This implies that we have the weak limit in $L^qL^r([0,\mu]\times\mathbb{R}^3)$

$$P(u_{0,n},u_{1,n})\rightharpoonup P(\tilde u_0,\tilde u_1).$$

On the other hand, we know $P(u_{0,n},u_{1,n})$ converges to zero strongly. Thus $P(\tilde u_0,\tilde u_1)=0$. This means $(\tilde u_0,\tilde u_1)=0$, which is a contradiction. □

Lemma A.5. *Assume $s\in[s_p,1]$. Let $u(x,t)$ be defined on $I=(T_-,\infty)$ and almost periodic modulo scalings in $\dot H^s\times\dot H^{s-1}(\mathbb{R}^3)$, namely the set*

$$K=\left\{\left(\frac{1}{\lambda(t)^{3/2-s_p}}u\left(\frac{x}{\lambda(t)},t\right),\frac{1}{\lambda(t)^{5/2-s_p}}\partial_t u\left(\frac{x}{\lambda(t)},t\right)\right):t\in I\right\}$$

is precompact in the space $\dot H^s\times\dot H^{s-1}(\mathbb{R}^3)$. In addition, $\lambda(t)\le1$ when t is large. Then, for any closed interval $[a,b]$ and any s-admissible pair (q,r) with $q<\infty$, we have

$$\lim_{T\to+\infty}\|S(t-T)(u(T),\partial_t u(T))\|_{L^qL^r([a,b]\times\mathbb{R}^3)}=0.$$

Proof. One could use the similar method as used in Lemma A.2 by observing

$$
\begin{aligned}
&\|S(t-T)(u(T),\partial_t u(T))\|_{L^q L^r([a,b]\times\mathbb{R}^3)}\\
&\qquad = \|S(t)(u(T),\partial_t u(T))\|_{L^q L^r([a-T,b-T]\times\mathbb{R}^3)}\\
&\qquad = (\lambda(T))^{s-s_p}\|S(t)(u_0^{(T)},u_1^{(T)})\|_{L^q L^r([\lambda(T)(a-T),\lambda(T)(b-T)]\times\mathbb{R}^3)}.
\end{aligned}
$$

Here

$$
(u_0^{(T)},u_1^{(T)}) = \left(\frac{1}{\lambda(T)^{3/2-s_p}}u\left(\frac{\cdot}{\lambda(T)},T\right),\frac{1}{\lambda(T)^{5/2-s_p}}\partial_t u\left(\frac{\cdot}{\lambda(T)},T\right)\right). \qquad \square
$$

Perturbation theory. In this subsection we will finish the proof of Theorem 2.12 and Theorem 2.15.

Proof of Theorem 2.12. Let us first prove the perturbation theory when M is sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x,t)$ to the equation (1) with the given initial data (u_0,u_1) and assume $[0,T]\subseteq I\cap I_1$. By the Strichartz estimate, we have

$$
\begin{aligned}
\|\tilde{u}-u\|_{Y_{s_p}([0,T])} &\le \|S(t)(u_0-\tilde{u}(0),u_1-\tilde{u}(0))\|_{Y_{s_p}([0,T])} + C_p\|e+F(\tilde{u})-F(u)\|_{Z_{s_p}([0,T])}\\
&\le \varepsilon + C_p\|e\|_{Z_{s_p}([0,T])} + C_p\|F(\tilde{u})-F(u)\|_{Z_{s_p}([0,T])}\\
&\le \varepsilon + C_p\varepsilon + C_p\|\tilde{u}-u\|_{Y_{s_p}([0,T])}\big(\|\tilde{u}\|^{p-1}_{Y_{s_p}([0,T])} + \|\tilde{u}-u\|^{p-1}_{Y_{s_p}([0,T])}\big)\\
&\le C_p\varepsilon + C_p\|\tilde{u}-u\|_{Y_{s_p}([0,T])}\big(M^{p-1} + \|\tilde{u}-u\|^{p-1}_{Y_{s_p}([0,T])}\big).
\end{aligned}
$$

By a continuity argument in T, there exist $M_0 = M_0(p)$ and $\varepsilon_0=\varepsilon_0(p)>0$ such that if $M\le M_0$ and $\varepsilon<\varepsilon_0$, we have

$$
\|\tilde{u}-u\|_{Y_{s_p}([0,T])}\le C_p\varepsilon.
$$

Observing that this estimate does not depend on the time T, we are actually able to conclude $I\subseteq I_1$ by the standard blow-up criterion and obtain

$$
\|\tilde{u}-u\|_{Y_{s_p}(I)}\le C_p\varepsilon.
$$

In addition, by the Strichartz estimate

$$
\begin{aligned}
&\sup_{t\in I}\left\|\begin{pmatrix}u(t)\\ \partial_t u(t)\end{pmatrix}-\begin{pmatrix}\tilde{u}(t)\\ \partial_t\tilde{u}(t)\end{pmatrix}-S(t)\begin{pmatrix}u_0-\tilde{u}(0)\\ u_1-\partial_t\tilde{u}(0)\end{pmatrix}\right\|_{\dot{H}^{s_p}\times\dot{H}^{s_p-1}}\\
&\qquad\le C_p\|F(u)-F(\tilde{u})-e\|_{Z_{s_p}(I)}\\
&\qquad\le C_p\big(\|e\|_{Z_{s_p}(I)}+\|F(u)-F(\tilde{u})\|_{Z_{s_p}(I)}\big)\\
&\qquad\le C_p\big[\varepsilon+\|u-\tilde{u}\|_{Y_{s_p}(I)}\big(\|\tilde{u}\|^{p-1}_{Y_{s_p}(I)}+\|u-\tilde{u}\|^{p-1}_{Y_{s_p}}\big)\big]\\
&\qquad\le C_p\varepsilon.
\end{aligned}
$$

This finishes the proof as M is sufficiently small. To deal with the general case, we can separate the time interval I into a finite number of subintervals $\{I_j\}$, so that $\|\tilde{u}\|_{Y_{s_p}(I_j)}<M_0$, and then iterate our argument above. $\square$

Proof of Theorem 2.15. Let us first prove the perturbation theory when M and T are sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x,t)$ to the equation (1) with the given initial data (u_0, u_1) and assume $[0, T_1] \subseteq [0, T] \cap I_1$. By the Strichartz estimate, we have

$$\begin{aligned}
\|\tilde{u} - u\|_{Y_s([0,T_1])} &\leq \|S(t)(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{Y_s([0,T_1])} + C_{s,p}\|F(\tilde{u}) - F(u)\|_{Z_s([0,T_1])} \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + C_{s,p}\|F(\tilde{u}) - F(u)\|_{Z_s([0,T_1])} \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
&\qquad + C_{s,p} T_1^{(p-1)(s-s_p)} \|F(\tilde{u}) - F(u)\|_{L^{\frac{2}{s+1-(2p-2)(s-s_p)}} L^{\frac{2}{2-s}}([0,T_1]\times\mathbb{R}^3)} \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
&\qquad + C_{s,p} T_1^{(p-1)(s-s_p)} \|\tilde{u} - u\|_{Y_s([0,T_1])} \big(\|\tilde{u} - u\|_{Y_s([0,T_1])}^{p-1} + \|\tilde{u}\|_{Y_s([0,T_1])}^{p-1}\big) \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
&\qquad + C_{s,p} T_1^{(p-1)(s-s_p)} \|\tilde{u} - u\|_{Y_s([0,T_1])} \big(\|\tilde{u} - u\|_{Y_s([0,T_1])}^{p-1} + M^{p-1}\big).
\end{aligned}$$

By a continuity argument in T_1, there exist $M_0 = M_0(s, p)$ and $\varepsilon_0 = \varepsilon_0(s, p) > 0$ such that if $M \leq M_0$, $T \leq 1$ and

$$\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq \varepsilon_0,$$

we have

$$\|\tilde{u} - u\|_{Y_s([0,T_1])} \leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

Observing that this estimate does not depend on the time T_1 as long as $T_1 \leq T \leq 1$, we are actually able to conclude $[0, T] \subseteq I_1$ by Theorem 2.14 and obtain

$$\|\tilde{u} - u\|_{Y_s([0,T])} \leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.$$

In addition, by the Strichartz estimate

$$\begin{aligned}
\sup_{t\in[0,T]} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1}} \hspace{-6em}& \\
&\leq \left\| S(t) \begin{pmatrix} u_0 - \tilde{u}_0 \\ u_1 - \tilde{u}_1 \end{pmatrix} \right\|_{\dot{H}^s \times \dot{H}^{s-1}} + C_{s,p}\|F(u) - F(\tilde{u})\|_{Z_s([0,T])} \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} \\
&\qquad + C_{s,p} T^{(p-1)(s-s_p)} \|\tilde{u} - u\|_{Y_s([0,T])} \big(\|\tilde{u} - u\|_{Y_s([0,T])}^{p-1} + \|\tilde{u}\|_{Y_s([0,T])}^{p-1}\big) \\
&\leq C_{s,p}\|(u_0 - \tilde{u}_0, u_1 - \tilde{u}_1)\|_{\dot{H}^s \times \dot{H}^{s-1}}.
\end{aligned}$$

This finishes the proof as M and T are sufficiently small. To deal with the general case, we can separate the time interval $[0, T]$ into a finite number of subintervals $\{I_j\}$, so that $\|\tilde{u}\|_{Y_s(I_j)} \leq M_0$ and $|I_j| \leq 1$, then iterate our argument above. □

Technical lemmas.

Lemma A.6. *Suppose that $(u_{0,\varepsilon}(x), u_{1,\varepsilon}(x))$ are radial, smooth pairs defined in $\mathbb{R}^3$ and converge to $(u_0(x), u_1(x))$ strongly in $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. In addition, we have*

$$\int_{r_0<|x|<4r_0} (|\nabla u_{0,\varepsilon}(x,t_0)|^2 + |u_{1,\varepsilon}(x,t_0)|^2)\,dx \le C$$

for each $\varepsilon < \varepsilon_0$. Then $(u_0(x), u_1(x))$ is in the space $\dot{H}^1 \times L^2(r < |x| < 4r)$ and satisfies

$$\int_{r_0<|x|<4r_0} (|\nabla u_0(x)|^2 + |u_1(x)|^2)\,dx \le C.$$

Proof. By the uniform bound of the integral, we can extract a sequence $\varepsilon_i \to 0$ so that $\partial_r u_{0,\varepsilon_i}(r)$ converges to $\tilde{u}_0'(r)$ weakly in $L^2(r_0, 4r_0)$, and u_{1,ε_i} converges to $\tilde{u}_1$ weakly in $L^2(r_0 < |x| < 4r_0)$. Define

$$\tilde{u}_0(r) = u_0(r_0) + \int_{r_0}^{r} \tilde{u}_0'(\tau)\,d\tau.$$

We have

$$\int_{r_0<|x|<4r_0} (|\nabla \tilde{u}_0(x)|^2 + |\tilde{u}_1(x)|^2)\,dx \le C.$$

By the strong and weak convergence, we have immediately $u_1 = \tilde{u}_1$ in the region $r_0 < |x| < 4r_0$. In order to conclude, we only need to show $u_0(r) = \tilde{u}_0(r)$. Observing $\int_{r_0}^{r_1} f(\tau)\,d\tau$ is a bounded linear functional in $L^2(r_0, 4r_0)$ for each $r_1 \in (r_0, 4r_0)$, we have

$$\begin{aligned}
\tilde{u}_0(r_1) &= u_0(r_0) + \int_{r_0}^{r_1} \tilde{u}_0'(\tau)\,d\tau \\
&= \lim_{i\to\infty} u_{0,\varepsilon_i}(r_0) + \lim_{i\to\infty} \int_{r_0}^{r_1} \partial_r u_{0,\varepsilon_i}(\tau)\,d\tau \\
&= \lim_{i\to\infty} \left(u_{0,\varepsilon_i}(r_0) + \int_{r_0}^{r_1} \partial_r u_{0,\varepsilon_i}(\tau)\,d\tau \right) \\
&= \lim_{i\to\infty} u_{0,\varepsilon_i}(r_1) \\
&= u_0(r_1).
\end{aligned}$$

This completes the proof. □

Lemma A.7. *Assume $\frac{1}{2} < s < \frac{3}{2}$. Given any radial $\dot{H}^s(\mathbb{R}^3)$ function f, we have*

$$\lim_{|x|\to 0^+} |x|^{\frac{3}{2}-s} f(x) = \lim_{|x|\to\infty} |x|^{\frac{3}{2}-s} f(x) = 0.$$

Proof. Let $s_1 \in (s, \frac{3}{2})$. Applying frequency cutoff techniques and using (8), we have

$$\begin{aligned}
|x|^{\frac{3}{2}-s} |(P_{>M} f)(x)| &\le C_s \|P_{>M} f\|_{\dot{H}^s}, \\
|x|^{\frac{3}{2}-s} |(P_{\le M} f)(x)| &\le C_{s_1} |x|^{s_1-s} \|P_{\le M} f\|_{\dot{H}^{s_1}},
\end{aligned}$$

for any fixed $M > 0$. Combining the higher and lower frequency parts, we obtain

$$\limsup_{|x|\to 0^+} |x|^{\frac{3}{2}-s}|f(x)| \le C_s \|P_{>M} f\|_{\dot{H}^s}.$$

This proves the first limit if we let $M \to +\infty$. We can prove the second limit in a similar way. □

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