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TWO-PHASE PROBLEMS WITH DISTRIBUTED SOURCES: REGULARITY OF THE FREE BOUNDARY

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We investigate the regularity of the free boundary for a general class of two-phase free boundary problems with nonzero right-hand side. We prove that Lipschitz or flat free boundaries are $C^{1,\gamma}$. In particular, viscosity solutions are indeed classical.

1. Introduction and main results

In this paper we consider two phase free boundary problems governed by uniformly elliptic equations with distributed sources. Our purpose is to investigate the regularity of the free boundary under additional hypotheses such as flatness or Lipschitz continuity. A model problem we have in mind is:

$$\begin{cases} \Delta u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{1-1}$$

Here, as usual for any bounded domain $\Omega \subset \mathbb{R}^n$,

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) := \{x \in \Omega : u(x) \le 0\}^\circ,$$

and u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$.

Typical examples are the Prandtl–Batchelor model in fluid dynamics (see, e.g., [Batchelor 1956; Elcrat and Miller 1995]), where $f = \mathbf{1}_{\Omega^-(u)}$, the characteristic function of the negative phase, or the eigenvalue problem in magnetohydrodynamics (1,1) considered in [Friedman and Liu 1995], where $f = -\lambda u \mathbf{1}_{\Omega^-(u)}$. Other examples come from limits of singular perturbation problems with forcing term as in [Lederman and Wolanski 2006], where the authors analyze solutions to (1-1), arising in the study of flame propagation with nonlocal effects.

The homogeneous case $f \equiv 0$ was settled in the classical works of Caffarelli [1987; 1989]. A key step in these papers is the construction of a family of continuous sup-convolution deformations that act as comparison subsolutions.

The results in [Caffarelli 1987; 1989] have been widely generalized to different classes of homogeneous elliptic problems. See, for example, [Cerutti et al. 2004; Ferrari and Salsa 2007a; 2007b] for linear

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operators; [Argiolas and Ferrari 2009; Feldman 2001; 1997; Ferrari 2006; Wang 2000; 2002] for fully nonlinear operators; and [Lewis and Nyström 2010] for the p-Laplacian. All these papers follow the guidelines of [Caffarelli 1987; 1989].

De Silva [2011] introduced a new strategy to investigate inhomogeneous free boundary problems, motivated by a classical one phase problem in hydrodynamic. This method has been successfully applied in [De Silva and Roquejoffre 2012] to nonlocal one phase Bernoulli type problems, governed by the fractional Laplacian. For another application of the techniques in [De Silva 2011] see also [Leitão and Teixeira 2011].

Here we extend the method in [De Silva 2011] to two phase problems to prove that flat (see below) or Lipschitz free boundaries of (1-1) are $C^{1,\gamma}$.

In order to better emphasize the ideas involved, we first develop the regularity theory for free boundaries of viscosity solutions to problem (1-1) (see Section 2 for the relevant definitions), and then we extend our results to a more general class of free boundary problems. For simplicity, in order to avoid the machinery of L^p-viscosity solution, we assume that f is bounded in Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$. Our results may be extended to the case when f is merely bounded measurable.

We remark that in view of Theorem 4.5 in [Caffarelli et al. 2002], a viscosity solution to (1-1) is locally Lipschitz. In fact, as it can be easily checked, our viscosity solutions are also weak solutions in the sense of Definition 4.4 in that paper and both $\Delta u^{\pm} - f$ are nonnegative Radon measures.

We now state our first main results. Here constants depending only on n, $\|f\|_\infty$, and Lip(u) will be called universal.

Theorem 1.1 (flatness implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to (1-1) in B_1. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\bar{\delta} > 0$ such that, if*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\}, \tag{1-2}$$

with $0 \le \delta \le \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.1 still holds when (1-2) is replaced by other common flatness conditions (see page 296).

Theorem 1.2 (Lipschitz implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to (1-1) in B_1, with $0 \in F(u)$. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

The proof of Theorem 1.1 is based on an improvement of flatness, obtained via a compactness argument which linearizes the problem into a limiting one. The key tool is a geometric Harnack inequality that localizes the free boundary well, and allows the rigorous passage to the limit.

The main difficulty in the analysis comes from the case when u^- is degenerate, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of u to an "optimal" (two-plane) configuration. Thus one needs to work only with the positive phase u^+ to balance the situation in which u^+ highly predominates over u^- and the case in which u^- is not too small with respect to u^+.

Theorem 1.2 follows from Theorem 1.1 and the main result in [Caffarelli 1987], via a blow-up argument.

Sections 2–6 are devoted to the proof of the theorems above. In particular, in Section 2 we introduce the relevant definitions and some preliminary lemmas. In Section 3 we describe the linearized problem associated to (1-1). Section 4 is devoted to the proof of the Harnack inequality both in the nondegenerate and in the degenerate setting. In Section 5, we present the proof of the improvement of flatness lemmas. Section 6 contains the proof of the Theorem 1.1 and Theorem 1.2.

From Section 7 to Section 10 we deal with more general problems of the form

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x) & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{1-3}$$

with f bounded on Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$, and u Lipschitz continuous with $\operatorname{Lip}(u) \le L$. Here

$$\mathscr{L} = \sum_{i,j=1}^n a_{ij}(x) D_{ij} + \boldsymbol{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar\gamma}(\Omega),\ \boldsymbol{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic; that is, there exist $0 < \lambda \le \Lambda$ such that, for every $\xi \in \mathbb{R}^n$ and every $x \in \Omega$,

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2,$$

and

$$G(\eta, x) : [0, \infty) \times \Omega \to (0, \infty)$$

satisfies the following assumptions:

(H1) $G(\eta, \cdot) \in C^{0,\bar\gamma}(\Omega)$ uniformly in η; $G(\cdot, x) \in C^{1,\bar\gamma}([0, L])$ for every $x \in \Omega$.

(H2) $G'(\cdot, x) > 0$ with $G(0, x) \ge \gamma_0 > 0$ uniformly in x.

(H3) There exists $N > 0$ such that $\eta^{-N} G(\eta, x)$ is strictly decreasing in η, uniformly in x.

In this framework we prove the following main results. Here, a constant depending (possibly) on n, $\operatorname{Lip}(u)$, λ, Λ, $[a_{ij}]_{C^{0,\bar\gamma}}$, $\|\boldsymbol{b}\|_{L^\infty}$, $\|f\|_{L^\infty}$, $[G(\eta, \cdot)]_{C^{0,\bar\gamma}}$, γ_0 and N is called universal. The $C^{1,\bar\gamma}$ norm of $G(\cdot, x)$ may depend on x, and enters our proofs in a qualitative way only.

Theorem 1.3 (flatness implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to (1-3) in B_1, with $\operatorname{Lip}(u) \le L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \le L$ and G satisfies assumptions* (H1)–(H3). *There exists a universal constant $\bar\delta > 0$ such that, if*

$$\{x_n \le -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \delta\},$$

with $0 \le \delta \le \bar\delta$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.4 (Lipschitz implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to (1-3) in B_1, with $0 \in F(u)$ and $\operatorname{Lip}(u) \le L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \le L$ and G satisfies assumptions* (H1)–(H3). *If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

Further extensions can be achieved with small extra effort: there is no problem in extending our results to the case when $\boldsymbol{b}$ and f are merely bounded measurable. However, as already said of the prototype problem, we wish to avoid too many technicalities.

In Theorems 1.3 and 1.4 we need to assume the Lipschitz continuity of our solution unless the operator can be put into divergence form. Indeed, in this case an almost monotonicity formula is available (see [Matevosyan and Petrosyan 2011]) and under the assumption $G(\eta, x) \to \infty$, as $\eta \to \infty$ one can reproduce the proof of Theorem 4.5 in [Caffarelli et al. 2002], to recover the Lipschitz continuity of a viscosity solution. Observe that then $f = f(x, u, \nabla u)$ is allowed, with $f(x, \cdot, \cdot)$ locally bounded.

2. Compactness and localization lemmas

In this section, we state basic definitions and we prove some elementary lemmas. First we need the following standard notion.

Definition 2.1. Given $u, \varphi \in C(\Omega)$, we say that φ touches u from below at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$ and

$$u(x) \geq \varphi(x) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly from below. Touching (strictly) from above is defined similarly, replacing $\leq$ by $\geq$.

We retain the usual definition of C-viscosity sub/supersolutions and solutions of an elliptic PDE; see [Caffarelli and Cabré 1995], for example. Here is the definition of a viscosity solution to the problem (1-1):

Definition 2.2. Let u be a continuous function in Ω. We say that u is a *viscosity solution* to (1-1) in Ω if the following conditions are satisfied:

(i) $\Delta u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense.

(ii) Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ $(B = B_\delta(x_0))$ with $F(v) \in C^2$. If v touches u from below (resp. above) at $x_0 \in F(v)$, then

$$(v_\nu^+(x_0))^2 - (v_\nu^-(x_0))^2 \leq 1 \quad (\text{resp. } \geq 1).$$

For our arguments, it is convenient to introduce also the notion of comparison sub/supersolutions.

Definition 2.3. We say that $v \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1-1) in Ω if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied.

(i) $\Delta v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;

(ii) If $x_0 \in F(v)$, then

$$(v_\nu^+)^2 - (v_\nu^-)^2 > 1 \quad \big(\text{resp. } (v_\nu^+)^2 - (v_\nu^-)^2 < 1,\ v_\nu^+(x_0) \neq 0\big).$$

Notice that by the implicit function theorem, according to our definition the free boundary of a comparison sub/supersolution is C^2.

Remark 2.4. A strict comparison subsolution v cannot touch a viscosity solution u from below at any point in $F(u)\cap F(v)$. A strict comparison supersolution v cannot touch u from above at any point in $F(u)\cap F(v)$.

The next lemma shows that "δ-flat" viscosity solutions (in the sense of Theorem 1.1) enjoy non-degeneracy of the positive part δ-away from the free boundary:

Lemma 2.5. *Let u be a solution to (1-1) in B_2 with $\mathrm{Lip}(u)\le L$ and $\|f\|_{L^\infty}\le L$. If*

$$\{x_n\le g(x')-\delta\}\subset\{u^+=0\}\subset\{x_n\le g(x')+\delta\},$$

with g a Lipschitz function, $\mathrm{Lip}(g)\le L$, $g(0)=0$, *then*

$$u(x)\ge c_0(x_n-g(x')),\quad x\in\{x_n\ge g(x')+2\delta\}\cap B_{\rho_0},$$

for some $c_0,\rho_0>0$ depending on n, L as long as $\delta\le c_0$.

Proof. All constants in this proof will depend on n, L.

It suffices to show that our statement holds for $\{x_n\ge g(x')+C\delta\}$ for a possibly large constant C. Then one can apply the Harnack inequality to obtain the full statement.

We prove the statement above at $x=de_n$ (recall that $g(0)=0$). Precisely, we want to show that

$$u(de_n)\ge c_0d,\quad d\ge C\delta.$$

After rescaling, we reduce to proving that

$$u(e_n)\ge c_0$$

as long as $\delta\le 1/C$, and $\|f\|_\infty$ is sufficiently small. Let $\gamma>0$ and

$$w(x)=\frac{1}{2\gamma}(1-|x|^{-\gamma})$$

be defined on the closure of the annulus $B_2\setminus\bar{B}_1$ with $\|f\|_\infty$ small enough that

$$\Delta w<-\|f\|\quad\text{on } B_2\setminus\bar{B}_1.$$

Extend $w=0$ in B_1. Let

$$w_t(x)=w(x+te_n).$$

Notice that

$$((w_t)_\nu^+)^2-((w_t)_\nu^-)^2<1\quad\text{on } F(w_t)=\partial B_1(-te_n).\tag{2-1}$$

From our flatness assumption for $t=C(L)$ sufficiently large (depending on the Lipschitz constant of g), w_t is above u. We decrease t continuously and let $\bar{t}$ be the smallest t such that w_t is above u. Notice that $\bar{t}>0$.

Then, there is a touching point $z\in(\bar{B}_2\setminus B_1)-\bar{t}e_n$. Since $w_{\bar{t}}$ is a strict supersolution to $\Delta u=f$ in $(B_2\setminus\bar{B}_1)-\bar{t}e_n$ and (2-1) is satisfied, the touching point z can occur only on the $\eta:=\frac{1}{2\gamma}(1-2^{-\gamma})$ level set in the positive phase of u. From the bounds on $\bar{t}$ it follows $|z|\le C$ (C depending on L.)

Since u is Lipschitz continuous, we have $0 < u(z) = \eta \leq Ld(z, F(u))$; that is, a full ball around z of radius η/L is contained in the positive phase of u. Thus, for $\bar{\delta}$ small depending on η, L, we have $B_{\eta/2L}(z) \subset \{x_n \geq g(x') + 2\bar{\delta}\}$. Since $x_n = g(x') + 2\bar{\delta}$ is Lipschitz we can connect e_n and z with a chain of intersecting balls included in the positive side of u with radii comparable to $\eta/2L$. The number of balls depends on L. Then we can apply the Harnack inequality and obtain

$$u(e_n) \geq cu(z) = c_0,$$

as desired. □

Next, we state a compactness lemma. For its proof, we refer the reader to Section 7 where the analogue of this result for a more general class of operators and free boundary conditions is stated and proved (see Lemma 7.3).

Lemma 2.6. *Let u_k be a sequence of viscosity solutions to (1-1) with right-hand side f_k satisfying $\|f_k\|_{L^\infty} \leq L$. Assume $u_k \to u^*$ uniformly on compact sets, and $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance. Then*

$$-L \leq \Delta u^* \leq L \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

in the viscosity sense and u^* satisfies the free boundary condition

$$(u_\nu^{*+})^2 - (u_\nu^{*-})^2 = 1 \quad \text{on } F(u^*)$$

in the viscosity sense of Definition 2.2.

We are now ready to reformulate our main Theorem 1.1 using the two lemmas above. First, we denote by U_β the following one-dimensional function,

$$U_\beta(t) = \alpha t^+ - \beta t^-, \quad \beta \geq 0, \quad \alpha = \sqrt{1+\beta^2},$$

where

$$t^+ = \max\{t, 0\}, \quad t^- = -\min\{t, 0\}.$$

Then $U_\beta(x) = U_\beta(x_n)$ is the so-called two-plane solution to (1-1) when $f \equiv 0$.

Lemma 2.7. *Let u be a solution to (1-1) in B_1 with $\mathrm{Lip}(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. For any $\varepsilon > 0$ there exist $\bar{\delta}, \bar{r} > 0$ depending on ε, n, and L such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then

$$\|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r} \tag{2-2}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and $\bar{r}$ depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem (1-1) with right-hand side f_k such that $\mathrm{Lip}(u_k)$, $\|f_k\| \leq L$ and

$$\{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\}, \tag{2-3}$$

but the u_k do not satisfy the conclusion (2-2).

Then, up to a subsequence, the u_k converge uniformly on compacts to a function u^*. In view of (2-3) and the nondegeneracy of u_k^+ $2\delta_k$-away from the free boundary (Lemma 2.5), we can apply our compactness lemma and conclude that

$$-L \leq \Delta u^* \leq L \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

in the viscosity sense and also

$$(u_n^{*+})^2 - (u_n^{*-})^2 = 1 \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\}, \tag{2-4}$$

with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus,

$$u^* \in C^{1,\gamma}\big(B_{1/2} \cap \{x_n \geq 0\}\big) \cap C^{1,\gamma}\big(B_{1/2} \cap \{x_n \leq 0\}\big)$$

for all γ and in view of (2-4) we have that (for any $\bar{r}$ small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq C(n, L)\bar{r}^{1+\gamma},$$

with $\alpha^2 = 1 + \beta^2$. If $\bar{r}$ is chosen depending on ε so that

$$C(n, L)\bar{r}^{1+\gamma} \leq \frac{\varepsilon}{2}\bar{r},$$

since the u_k converge uniformly to u^* on $B_{1/2}$ we obtain that for all k large

$$\|u_k - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon\bar{r},$$

a contradiction. □

In view of Lemma 2.7, and after rescaling, our first main theorem (Theorem 1.1) follows from our second, which we now state:

Theorem 2.8. *Let u be a solution to (1-1) in B_1 with $\mathrm{Lip}(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \leq \bar{\varepsilon} \quad \textit{for some } 0 \leq \beta \leq L, \tag{2-5}$$

and

$$\{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\} \quad \textit{and} \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

The next lemma is elementary.

Lemma 2.9. *Let u be a continuous function. If, for $\eta > 0$ small, we have*

$$\|u - U_\beta\|_{L^\infty(B_2)} \leq \eta \quad \textit{for } 0 \leq \beta \leq L,$$

and

$$\{x_n \le -\eta\} \subset B_2 \cap \{u^+(x) = 0\} \subset \{x_n \le \eta\},$$

then

- *if* $\beta \ge \eta^{1/3}$, *then* $U_\beta(x_n - \eta^{1/3}) \le u(x) \le U_\beta(x_n + \eta^{1/3})$ *in* B_1;
- *if* $\beta < \eta^{1/3}$, *then* $U_0(x_n - \eta^{1/3}) \le u^+(x) \le U_0(x_n + \eta^{1/3})$ *in* B_1.

3. The linearized problem

This section is devoted to the study of the linearized problem associated with our free boundary problem (1-1), that is, the following boundary value problem ($\tilde{\alpha} \neq 0$):

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}. \end{cases} \tag{3-1}$$

Here $(\tilde{u}_n)^+$ (resp. $(\tilde{u}_n)^-$) denotes the derivative in the e_n direction of $\tilde{u}$ restricted to $\{x_n > 0\}$ (resp. $\{x_n < 0\}$).

We remark that Theorem 2.8 will follow, see Section 6, via a compactness argument from the regularity properties of viscosity solutions to (3-1).

Definition 3.1. A continuous function u is a viscosity solution to (3-1) if the following conditions are satisfied:

(i) $\Delta \tilde{u} = 0$ in $B_\rho \cap \{x_n \neq 0\}$, in the viscosity sense.

(ii) Let ϕ be a function of the form

$$\phi(x) = A + p x_n^+ - q x_n^- + B\, Q(x - y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then ϕ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_\rho$. Analogously, if

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q < 0,$$

then ϕ cannot touch u strictly from above at x_0.

We wish to prove the following regularity result for viscosity solutions to the linearized problem.

Theorem 3.2. *Let $\tilde{u}$ be a viscosity solution to (3-1) in $B_{1/2}$ such that $\|\tilde{u}\|_\infty \le 1$. There exists a universal constant $\bar{C}$ such that*

$$\left|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'} \tilde{u}(0) \cdot x' + \tilde{p} x_n^+ - \tilde{q} x_n^-)\right| \le \bar{C} r^2 \quad \text{in } B_r, \tag{3-2}$$

for all $r \le \frac{1}{4}$ and with $\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0$.

Before proving this, we first show that the problem (3-1) admits a classical solution:

Theorem 3.3. *Let h be a continuous function on ∂B_1. There exists a (unique) classical solution $\tilde v$ to (3-1) with $\tilde v = h$ on ∂B_1, that is, $\tilde v \in C^\infty(B_1 \cap \{x_n \geq 0\}) \cap C^\infty(B_1 \cap \{x_n \leq 0\})$. In particular, there exists a universal constant $\widetilde C$ such that*

$$\left|\tilde v(x) - \tilde v(\bar x) - \left(\nabla_{x'}\tilde v(\bar x)\cdot(x'-\bar x') + \tilde p(\bar x)x_n^+ - \tilde q(\bar x)x_n^-\right)\right| \leq \widetilde C \|\tilde v\|_{L^\infty} r^2 \quad \text{in } B_r(\bar x), \tag{3-3}$$

for all $r \leq \frac14$, $\bar x = (\bar x', 0) \in B_{1/2}$ and with $\tilde\alpha^2 \tilde p(\bar x) - \tilde\beta^2 \tilde q(\bar x) = 0$.

Proof. Let w be the harmonic function in $B_1 \cap \{x_n > 0\}$ such that

$$\begin{aligned} w &= 0 && \text{on } B_1 \cap \{x_n = 0\}, \\ w(x) &= h(x', x_n) - h(x', -x_n) && \text{on } \partial B_1 \cap \{x_n > 0\}. \end{aligned}$$

Then $w \in C^\infty(B_1 \cap \{x_n \geq 0\})$. Set

$$\phi(x') = w_n(x', 0), \qquad (x', 0) \in B_1.$$

Let

$$\tilde v_1(x) = w(x) + \tilde v_2(x', -x_n) \quad \text{in } \bar B_1 \cap \{x_n \geq 0\},$$

where $\tilde v_2$ is the solution to the problem

$$\begin{cases} \Delta \tilde v_2 = 0 & \text{in } B_1 \cap \{x_n < 0\}, \\ \tilde v_2 = h & \text{on } \partial B_1 \cap \{x_n < 0\}, \\ (\tilde v_2)_n = \tilde q \phi & \text{on } B_1 \cap \{x_n = 0\}, \end{cases}$$

with $\tilde q = \dfrac{\tilde\alpha^2}{\tilde\beta^2 + \tilde\alpha^2}$. Then it is easily verified that the function

$$\tilde v = \begin{cases} \tilde v_1 & \text{in } \bar B_1 \cap \{x_n \geq 0\}, \\ \tilde v_2 & \text{in } \bar B_1 \cap \{x_n \leq 0\} \end{cases}$$

is the unique classical solution to our problem and hence it satisfies the estimate (3-3) with

$$\tilde q(\bar x) = \tilde q \phi(\bar x), \qquad \tilde p(\bar x) = \tilde p \phi(\bar x), \qquad \tilde p = \frac{\tilde\beta^2}{\tilde\beta^2 + \tilde\alpha^2}. \qquad \square$$

Finally, to obtain our regularity result we only need to show the following fact.

Theorem 3.4. *Let $\tilde u$ be a viscosity solution to (3-1) in B_1 such that $\|\tilde u\|_\infty \leq 1$ and let $\tilde v$ be the classical solution to (3-1) in $B_{1/2}$ with boundary data $\tilde u$. Then $\tilde u = \tilde v$.*

Proof. We prove that $\tilde v \leq \tilde u$ in $B_{1/2}$. The opposite inequality is obtained in a similar way.

Let $\varepsilon > 0$, $t \in \mathbb{R}$ and set

$$\tilde v_{t,\varepsilon}(x) = \tilde v + \varepsilon |x_n| + \varepsilon x_n^2 - \varepsilon - t, \qquad x \in \bar B_{1/2}.$$

Since $\tilde u$ is bounded, for $t > 0$ large enough,

$$\tilde v_{t,\varepsilon} \leq \tilde u. \tag{3-4}$$

Let $\bar{t}$ be the smallest t such that (3-4) holds and let $\bar{x}$ be the first touching point. We want to show that $\bar{t} < 0$. Assume $\bar{t} \geq 0$. Since

$$\tilde{v}_{\bar{t},\varepsilon} < \tilde{u} \quad \text{on } \partial B_{1/2},$$

such touching point must belong to $B_{1/2}$. However,

$$\Delta \tilde{v}_{\bar{t},\varepsilon}(x) > 0 \quad \text{in } B_{1/2} \cap \{x_n \neq 0\},$$
$$\Delta \tilde{u} = 0 \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}.$$

Thus $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$. We claim that there exists a function ϕ of the form

$$\phi(x) = A + p x_n^+ - q x_n^- + B Q(x - y)$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \qquad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

such that ϕ touches $\tilde{v}_{\bar{t},\varepsilon}(x)$ strictly from below at $\bar{x}$. This would contradict the definition of viscosity solutions, hence $\bar{t} < 0$. In particular,

$$\tilde{v} + \varepsilon|x_n| + \varepsilon x_n^2 - \varepsilon < \tilde{u} \quad \text{on } B_{1/2},$$

and for ε going to 0 we obtain as desired

$$\tilde{v} \leq \tilde{u} \quad \text{on } B_{1/2}.$$

We are left with the proof of the claim. Define

$$\nu' = \nabla_{x'} \tilde{v}(\bar{x}),$$

and set

$$y' = \bar{x}' + \frac{\nu'}{B}, \qquad A = \tilde{v}(\bar{x}) - \varepsilon - \bar{t} - B Q(\bar{x} - y),$$

with $B > 0$ to be chosen later. In view of the estimate (3-3), to verify that in a small neighborhood of $\bar{x}$

$$\phi(x) < \tilde{v}_{\bar{t},\varepsilon}(x), \qquad x \neq \bar{x},$$

we need to show that we can find $B > 0$, p, q such that for $|x - \bar{x}| \neq 0$ small enough ($\widetilde{C}$ universal),

$$\frac{B}{2}(n-1)x_n^2 - \frac{B}{2}|x' - \bar{x}'|^2 + p x_n^+ - q x_n^- < (\tilde{p} + \varepsilon) x_n^+ - (\tilde{q} - \varepsilon) x_n^- - \widetilde{C}|x - \bar{x}|^2$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

(for simplicity we dropped the dependence of $\tilde{p}, \tilde{q}$ on $\bar{x}$).

It is then enough to choose

$$B = 4\widetilde{C}, \qquad p = \tilde{p} + \frac{\varepsilon}{2}, \qquad q = \tilde{q} - \frac{\varepsilon}{2}. \qquad \square$$

4. The Harnack inequality

In this section we prove our main tool, a Harnack-type inequality for solutions to our free boundary problem. The results contained here will allow us to pass to the limit in the compactness argument for our improvement of flatness lemmas in Section 5.

Throughout this section we consider a Lipschitz solution u to (1-1) with $\mathrm{Lip}(u) \leq L$.

We need to distinguish two cases, which we call the nondegenerate and the degenerate case.

Nondegenerate case. In this case our solution u is trapped between two translations of a "true" two-plane solution U_β that is with $\beta \neq 0$.

Theorem 4.1 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$ such that, if u satisfies at some point $x_0 \in B_2$*

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_2, \tag{4-1}$$

with

$$\|f\|_{L^\infty} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_\beta(x_n + a_1) \leq u(x) \leq U_\beta(x_n + b_1) \quad \text{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

Before giving the proof we deduce an important consequence.

If u satisfies (4-1) with, say $r = 1$, then we can apply the Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \leq u(x) \leq U_\beta(x_n + b_m) \quad \text{in } B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon,$$

for all m such that

$$(1-c)^m 20^m \varepsilon \leq \bar{\varepsilon}.$$

This implies that for all such m, the oscillation of the function

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon} & \text{in } B_2^+(u) \cup F(u), \\ \dfrac{u(x) - \beta x_n}{\beta \varepsilon} & \text{in } B_2^-(u), \end{cases}$$

in $B_r(x_0)$, $r = 20^{-m}$ is less than $(1-c)^m = 20^{-\gamma m} = r^\gamma$. Thus, the following corollary holds.

Corollary 4.2. *Let u be as in Theorem 4.1 satisfying (4-1) for $r = 1$. Then in $B_1(x_0)$ $\tilde{u}_\varepsilon$ has a Hölder modulus of continuity at x_0 outside the ball of radius $\varepsilon/\bar{\varepsilon}$; that is, for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,*

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality relies on the following lemma.

Lemma 4.3. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u(x) \geq U_\beta(x) \quad \text{in } B_1,$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2\beta, \quad 0 < \beta \leq L, \tag{4-2}$$

then if at $\bar{x} = \frac{1}{5}e_n$,

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon), \tag{4-3}$$

then

$$u(x) \geq U_\beta(x_n + c\varepsilon) \quad \text{in } \overline{B}_{1/2}, \tag{4-4}$$

for some universal c with $0 < c < 1$. Analogously, if $u(x) \leq U_\beta(x)$ in B_1 and $u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$, then

$$u(x) \leq U_\beta(x_n - c\varepsilon) \quad \text{in } \overline{B}_{1/2}.$$

Proof. We prove the first statement. For notational simplicity we drop the subindex β from U_β.

Let

$$w = c(|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}) \tag{4-5}$$

be defined in the closure of the annulus

$$A := B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x}).$$

The constant c is such that w satisfies the boundary conditions

$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$

Then, for a fixed $\gamma > n - 2$,

$$\Delta w \geq k(\gamma, n) = k(n) > 0, \quad 0 \leq w \leq 1 \text{ on } A.$$

Extend w to be equal to 1 on $B_{1/20}(\bar{x})$.

Notice that since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u \geq U$ in B_1, we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - U \geq 0$ and solves $\Delta(u - U) = f$ in $B_{1/10}(\bar{x})$ and we can apply the Harnack inequality to obtain

$$u(x) - U(x) \geq c(u(\bar{x}) - U(\bar{x})) - C\|f\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{4-6}$$

From the assumptions (4-2) and (4-3) we conclude that (for ε small enough)

$$u - U \geq \alpha c\varepsilon - C\alpha\varepsilon^2 \geq \alpha c_0\varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{4-7}$$

Now set $\psi = 1 - w$ and

$$v(x) = U(x_n - \varepsilon c_0\psi(x)), \quad x \in \bar{B}_{3/4}(\bar{x}),$$

and for $t \geq 0$,

$$v_t(x) = U(x_n - \varepsilon c_0\psi(x) + t\varepsilon), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Then,

$$v_0(x) = U(x_n - \varepsilon c_0\psi(x)) \leq U(x) \leq u(x), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Let $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed,

$$u(x) \geq v_{\bar{t}}(x) = U(x_n - \varepsilon c_0\psi + \bar{t}\varepsilon) \geq U(x_n + c\varepsilon) \quad \text{in } B_{1/2} \Subset B_{3/4}(\bar{x}),$$

with c universal. In the last inequality we used that $\|\psi\|_{L^\infty(B_{1/2})} < 1$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \bar{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\bar{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$,

$$v_{\bar{t}}(x) = U(x_n - \varepsilon c_0\psi(x) + \bar{t}\varepsilon) < U(x) \leq u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

We now show that $\tilde{x}$ cannot belong to the annulus A. Indeed,

$$\Delta v_{\bar{t}} \geq \beta\varepsilon c_0 k(n) > \varepsilon^2\beta \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough. Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = 1 + \varepsilon^2 c_0^2|\nabla\psi|^2 - 2\varepsilon c_0\psi_n \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A,$$

as long as

$$\psi_n < 0 \quad \text{on } F(v_{\bar{t}}) \cap A.$$

This can be easily verified from the formula for ψ (for ε small enough).

Thus, $v_{\bar{t}}$ is a strict subsolution to (1-1) in A which lies below u, hence by the definition of viscosity solution, $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \bar{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = U(\tilde{x}_n + \bar{t}\varepsilon) \le U(\tilde{x}) + \alpha\bar{t}\varepsilon < U(\tilde{x}) + \alpha c_0\varepsilon,$$

contradicting (4-7).

The proof of the second statement follows from a similar argument. □

Proof of Theorem 4.1. Assume without loss of generality that $x_0 = 0$, $r = 1$. We distinguish three cases.

Case 1: $a_0 < -\frac{1}{5}$. In this case it follows from (4-1) that $B_{1/10} \subset \{u < 0\}$ and

$$0 \le v(x) := \frac{u(x) - \beta(x_n + a_0)}{\beta\varepsilon} \le 1,$$

with

$$|\Delta v| \le \varepsilon \quad \text{in } B_{1/10}.$$

The desired claim follows from the standard Harnack inequality applied to the function v.

Case 2: $a_0 > \frac{1}{5}$. In this case it follows from (4-1) that $B_{1/5} \subset \{u > 0\}$ and

$$0 \le v(x) := \frac{u(x) - \alpha(x_n + a_0)}{\alpha\varepsilon} \le 1,$$

with

$$|\Delta v| \le \varepsilon \quad \text{in } B_{1/5}.$$

Again, the desired claim follows from the standard Harnack inequality for v.

Case 3: $|a_0| \le 1/5$. Assumption (4-1) gives that

$$U_\beta(x_n + a_0) \le u(x) \le U_\beta(x_n + a_0 + \varepsilon) \quad \text{in } B_1.$$

Assume that (the other case is treated similarly)

$$u(\bar{x}) \ge U_\beta\big(\bar{x}_n + a_0 + \tfrac{1}{2}\varepsilon\big), \quad \bar{x} = \tfrac{1}{5}e_n. \tag{4-8}$$

Set

$$v(x) := u(x - a_0 e_n), \quad x \in B_{4/5}.$$

Then the inequality above reads

$$U_\beta(x_n) \le v(x) \le U_\beta(x_n + \varepsilon) \quad \text{in } B_{4/5}.$$

From (4-8), we have

$$v(\bar{x}) \ge U_\beta\big(\bar{x}_n + \tfrac{1}{2}\varepsilon\big).$$

Then, by Lemma 4.3,

$$v(x) \ge U_\beta(x_n + c\varepsilon) \quad \text{in } B_{2/5},$$

which gives the desired improvement

$$u(x) \ge U_\beta(x + a_0 + c\varepsilon) \quad \text{in } B_{3/5}.$$ □

Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e., $\beta = 0$).

Theorem 4.4 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$, such that if u satisfies at some point $x_0 \in B_2$*

$$U_0(x_n + a_0) \le u^+(x) \le U_0(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_2, \tag{4-9}$$

with

$$\|u^-\|_{L^\infty} \le \varepsilon^2, \quad \|f\|_{L^\infty} \le \varepsilon^4,$$

and

$$b_0 - a_0 \le \varepsilon r,$$

for some $\varepsilon \le \bar{\varepsilon}$, then

$$U_0(x_n + a_1) \le u^+(x) \le U_0(x_n + b_1) \quad \text{in } B_{r/20}(x_0),$$

with

$$a_0 \le a_1 \le b_1 \le b_0, \quad b_1 - a_1 \le (1 - c)\varepsilon r,$$

and $0 < c < 1$ universal.

We can argue as in the nondegenerate case and get the following result.

Corollary 4.5. *Let u be as in Theorem 4.1 satisfying (4-9) for $r = 1$. Then in $B_1(x_0)$*

$$\tilde{u}_\varepsilon := \frac{u^+(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0 outside the ball of radius $\varepsilon/\bar{\varepsilon}$; that is, for all $x \in B_1(x_0)$ with $|x - x_0| \ge \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \le C|x - x_0|^\gamma.$$

The proof of the Harnack inequality can be deduced from the following lemma, as in the one-phase case [De Silva 2011].

Lemma 4.6. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \ge U_0(x) \quad \text{in } B_1,$$

with

$$\|u^-\|_{L^\infty} \le \varepsilon^2, \quad \|f\|_{L^\infty} \le \varepsilon^4, \tag{4-10}$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$u^+(\bar{x}) \ge U_0(\bar{x}_n + \varepsilon), \tag{4-11}$$

then

$$u^+(x) \ge U_0(x_n + c\varepsilon) \quad \text{in } \bar{B}_{1/2}, \tag{4-12}$$

for some universal c with $0 < c < 1$. Analogously, if $u^+(x) \le U_0(x)$ in B_1 and $u^+(\bar{x}) \le U_0(\bar{x}_n - \varepsilon)$, then

$$u^+(x) \le U_0(x_n - c\varepsilon) \quad \text{in } \bar{B}_{1/2}.$$

Proof. We prove the first statement. The proof follows the same line as in the nondegenerate case.

Since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u^+ \geq U_0$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - x_n \geq 0$ and solves $\Delta(u - x_n) = f$ in $B_{1/10}(\bar{x})$ and we can apply the Harnack inequality and the assumptions (4-10) and (4-11) to obtain that (for ε small enough)

$$u - x_n \geq c_0 \varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{4-13}$$

Let w be as in the proof of Lemma 4.3 and $\psi = 1 - w$. Set

$$v(x) = (x_n - \varepsilon c_0 \psi(x))^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x))^-, \quad x \in \bar{B}_{3/4}(\bar{x}),$$

and, for $t \geq 0$,

$$v_t(x) = (x_n - \varepsilon c_0 \psi + t\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x) + t\varepsilon)^-, \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Here C_1 is a universal constant to be made precise later. We claim that

$$v_0(x) = v(x) \leq u(x), \quad x \in \bar{B}_{3/4}(\bar{x}).$$

This is readily verified in the set where u is nonnegative using that $u \geq x_n^+$. To prove our claim in the set where u is negative we wish to use the following fact:

$$u^- \leq C x_n^- \varepsilon^2 \quad \text{in } B_{19/20},\ C \text{ universal.} \tag{4-14}$$

This estimate is easily obtained using that $\{u < 0\} \subset \{x_n < 0\}$, $\|u^-\|_\infty < \varepsilon^2$ and the comparison principle with the function w satisfying

$$\Delta w = -\varepsilon^4 \quad \text{in } B_1 \cap \{x_n < 0\}, \qquad w = u^- \quad \text{on } \partial(B_1 \cap \{x_n < 0\}).$$

Thus our claim immediately follows from the fact that for $x_n < 0$ and $C_1 \geq C$,

$$\varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x)) \leq C x_n \varepsilon^2.$$

Let $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed, it is easy to check that if

$$u(x) \geq v_{\bar{t}}(x) = (x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon)^- \quad \text{in } B_{3/4}(\bar{x}),$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } B_{1/2} \Subset B_{3/4}(\bar{x}),$$

with c universal, $c < c_0 \inf_{B_{1/2}} w$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \bar{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such a touching point can only occur on $\bar{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$

$$v_{\bar{t}}(x) = (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^- < u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

In the set where $u \geq 0$, this can be seen using that $u \geq x_n^+$, while in the set where $u < 0$ again we can use the estimate (4-14).

We now show that $\tilde{x}$ cannot belong to the annulus A. Indeed,

$$\Delta v_{\bar{t}} \geq \varepsilon^3 c_0 k(n) > \varepsilon^4 \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}}),$$

for ε small enough.

Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = (1 - \varepsilon^4 C_1^2)\bigl(1 + \varepsilon^2 c_0^2 |\nabla \psi|^2 - 2\varepsilon c_0 \psi_n\bigr) \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A,$$

as long as ε is small enough (as in the nondegenerate case one can check that $\inf_{F(v_{\bar{t}}) \cap A}(-\psi_n) > c > 0$, with c universal.) Thus, $v_{\bar{t}}$ is a strict subsolution to (1-1) in A which lies below u, hence by definition $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \bar{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = (\tilde{x}_n + \bar{t}\varepsilon) < \tilde{x}_n + c_0\varepsilon,$$

contradicting (4-13). □

5. Improvement of flatness

In this section we prove our key lemmas improving flatness. As in Section 4, we distinguish two cases.

Nondegenerate case. In this case our solution u is trapped between two translations of a two-plane solution U_β with $\beta \neq 0$. We plan to show that when we restrict to smaller balls, u is trapped between closer translations of another two-plane solution (in a different system of coordinates).

Lemma 5.1 (improvement of flatness). *Let u satisfy*

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1, 0 \in F(u), \tag{5-1}$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta.$$

If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then

$$U_{\beta'}\Bigl(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Bigr) \leq u(x) \leq U_{\beta'}\Bigl(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Bigr) \quad \text{in } B_r, \tag{5-2}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \widetilde{C}\varepsilon$, and $|\beta - \beta'| \leq \widetilde{C}\beta\varepsilon$ for a universal constant $\widetilde{C}$.

Proof. We divide the proof of this lemma into three steps.

Step 1: compactness. Fix $r \leq r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1-1) in B_1 with right-hand side f_k with L^∞ norm bounded by $\varepsilon_k^2 \beta_k$, such that

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{for } x \in B_1, 0 \in F(u_k), \tag{5-3}$$

with $L \geq \beta_k > 0$, but u_k does not satisfy the conclusion of the lemma, (5-2).

With $\alpha_k^2 = 1 + \beta_k^2$, set

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k), \\ \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Then (5-3) gives

$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1. \tag{5-4}$$

From Corollary 4.2, it follows that the function $\tilde{u}_k$ satisfies

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C|x - y|^\gamma, \tag{5-5}$$

for C universal, and

$$|x - y| \geq \varepsilon_k/\bar{\varepsilon}, \quad x, y \in B_{1/2}.$$

From (5-3) it clearly follows that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. This fact and (5-5) together with Ascoli–Arzelà give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$. Also, up to a subsequence we have

$$\beta_k \to \tilde{\beta} \geq 0,$$

and hence

$$\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}.$$

Step 2: limiting solution. We now show that $\tilde{u}$ solves the following linearized problem (transmission problem):

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{5-6}$$

Since

$$|\Delta u_k| \leq \varepsilon_k^2 \beta_k \quad \text{in } B_1^+(u_k) \cup B_1^-(u_k),$$

one easily deduces that $\tilde{u}$ is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that $\tilde{u}$ satisfies the boundary condition in (5-6) in the viscosity sense.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x-y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then we must show that $\tilde{\phi}$ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$ (the analogous statement from above follows with a similar argument).

Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let

$$\Gamma(x) = \frac{1}{n-2}\big[(|x'|^2 + |x_n - 1|^2)^{\frac{2-n}{2}} - 1\big]$$

and

$$\Gamma_k(x) = \frac{1}{B\varepsilon_k}\Gamma(B\varepsilon_k(x-y) + AB\varepsilon_k^2 e_n). \tag{5-7}$$

Now, set

$$\phi_k(x) = a_k\Gamma_k^+(x) - b_k\Gamma_k^-(x) + \alpha_k(d_k^+(x))^2\varepsilon_k^{3/2} + \beta_k(d_k^-(x))^2\varepsilon_k^{3/2},$$

where

$$a_k = \alpha_k(1+\varepsilon_k p), \quad b_k = \beta_k(1+\varepsilon_k q),$$

and $d_k(x)$ is the signed distance from x to $\partial B_{1/(B\varepsilon_k)}\Big(y + e_n\Big(\dfrac{1}{B\varepsilon_k} - A\varepsilon_k\Big)\Big)$.

Finally, let

$$\tilde{\phi}_k(x) = \begin{cases} \dfrac{\phi_k(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(\phi_k) \cup F(\phi_k), \\[2mm] \dfrac{\phi_k(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(\phi_k). \end{cases}$$

By Taylor's theorem,

$$\Gamma(x) = x_n + Q(x) + O(|x|^3), \quad x \in B_1;$$

thus it is easy to verify that

$$\Gamma_k(x) = A\varepsilon_k + x_n + B\varepsilon_k Q(x-y) + O(\varepsilon_k^2), \quad x \in B_1,$$

with the constant in $O(\varepsilon_k^2)$ depending on A, B, and $|y|$ (later this constant will depend also on p, q).

It follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ ($Q^y(x) = Q(x-y)$),

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k),$$

and analogously in $B_1^-(\phi_k)$,

$$\tilde{\phi}_k(x) = A + BQ^y + qx_n + A\varepsilon_k p + Bq\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly from below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at x_k. We thus get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is,

$$\begin{cases} \Delta\psi_k > \varepsilon_k^2\beta_k \geq \|f_k\|_\infty & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu^2 - (\psi_k^-)_\nu^2 > 1, & \text{on } F(\psi_k). \end{cases} \tag{5-8}$$

It is easily checked that, away from the free boundary,

$$\Delta\psi_k \geq \beta_k \varepsilon_k^{3/2} \Delta d_k^2(x + \varepsilon_k c_k e_n),$$

and the first condition in (5-8) is satisfied for k large enough.

Finally, since on the zero level set $|\nabla\Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to showing that

$$a_k^2 - b_k^2 > 1.$$

Using the definition of a_k, b_k we need to check that

$$(\alpha_k^2 p^2 - \beta_k^2 q^2)\varepsilon_k + 2(\alpha_k^2 p - \beta_k^2 q) > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Thus $\tilde{u}$ is a solution to the linearized problem.

Step 3: Contradiction. According to estimate (3-2), since $\tilde{u}(0) = 0$ we obtain that

$$|\tilde{u} - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq Cr^2, \quad x \in B_r,$$

with

$$\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0, \quad |\nu'| = |\nabla_{x'}\tilde{u}(0)| \leq C.$$

Thus, since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ (by slightly enlarging C) we get that

$$|\tilde{u}_k - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq Cr^2, \quad x \in B_r. \tag{5-9}$$

Now set

$$\beta_k' = \beta_k(1 + \varepsilon_k\tilde{q}), \quad \nu_k = \frac{1}{\sqrt{1 + \varepsilon_k^2|\nu'|^2}}(e_n + \varepsilon_k(\nu', 0)).$$

Then,

$$\alpha_k' = \sqrt{1 + \beta_k'^2} = \alpha_k(1 + \varepsilon_k\tilde{p}) + O(\varepsilon_k^2), \quad \nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2\tau, \quad |\tau| \leq C,$$

where to obtain the first equality we used that $\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0$ and hence

$$\frac{\beta_k^2}{\alpha_k^2}\tilde{q} = \tilde{p} + O(\varepsilon_k).$$

With these choices we can now show that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \quad \text{in } B_r,$$

where again we are using the notation

$$\widetilde{U}_{\beta_k'}(x) = \begin{cases} \dfrac{\widetilde{U}_{\beta_k'}(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(\widetilde{U}_{\beta_k'}) \cup F(\widetilde{U}_{\beta_k'}), \\ \dfrac{\widetilde{U}_{\beta_k'}(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(\widetilde{U}_{\beta_k'}). \end{cases}$$

This will clearly imply that

$$U_{\beta_k'}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) \le u_k(x) \le U_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \quad \text{in } B_r$$

and hence will lead to a contradiction.

In view of (5-9), we need to show that in B_r,

$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k - \varepsilon_k\frac{r}{2}\Big) \le (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2,$$
$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \ge (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

We show the second inequality. In the set where

$$x\cdot\nu_k + \varepsilon_k\frac{r}{2} < 0 \tag{5-10}$$

we have, by definition,

$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) = \frac{1}{\beta_k\varepsilon_k}\Big(\beta_k'\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) - \beta_k x_n\Big),$$

which from the formula for β_k', ν_k gives

$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \ge x'\cdot\nu' + \tilde{q}x_n + \frac{r}{2} - C_0\varepsilon_k.$$

Using (5-10) we then obtain

$$\widetilde{U}_{\beta_k'}\Big(x\cdot\nu_k + \varepsilon_k\frac{r}{2}\Big) \ge x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^- + \frac{r}{2} - C_1\varepsilon_k.$$

Thus to obtain the desired bound it suffices to fix $r_0 \le 1/(4C)$ and take k large enough.

The other case can be argued similarly. $\square$

Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution ($\beta = 0$). We prove below that in this setting only u^+ enjoys an improvement of flatness.

Lemma 5.2 (improvement of flatness). *Let u satisfy*

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \textit{in } B_1, 0 \in F(u), \tag{5-11}$$

with

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4 \quad \textit{and} \quad \|u^-\|_{L^\infty(B_1)} \le \varepsilon^2.$$

If $0 < r \le r_1$ for r_1 universal, and $0 < \varepsilon \le \varepsilon_1$ for some ε_1 depending on r, then

$$U_0\Big(x \cdot \nu_1 - r\frac{\varepsilon}{2}\Big) \le u^+(x) \le U_0\Big(x \cdot \nu_1 + r\frac{\varepsilon}{2}\Big) \quad \textit{in } B_r, \tag{5-12}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le C\varepsilon$ for a universal constant C.

Proof. We argue similarly as in the nondegenerate case.

Step 1: compactness. Fix $r \le r_1$ with r_1 universal (made precise in Step 3). Assume for a contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1-1) in B_1 with right-hand side f_k with L^∞ norm bounded by ε_k^4, such that

$$U_0(x_n - \varepsilon_k) \le u_k^+(x) \le U_0(x_n + \varepsilon_k) \quad \text{for } x \in B_1, 0 \in F(u_k), \tag{5-13}$$

with

$$\|u_k^-\|_\infty \le \varepsilon_k^2,$$

but u_k does not satisfy the conclusion (5-12) of the lemma. Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k).$$

Then (5-13) gives

$$-1 \le \tilde{u}_k(x) \le 1 \quad \text{for } x \in B_1^+(u_k) \cup F(u_k). \tag{5-14}$$

As in the nondegenerate case, it follows from Corollary 4.5 that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \ge 0\}$.

Step 2: limiting solution. We now show that $\tilde{u}$ solves the following Neumann problem:

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{5-15}$$

As before, the interior condition follows easily thus we focus on the boundary condition.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + p x_n + B Q(x - y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0),\ A \in \mathbb{R},\ B > 0$$

and $p > 0$. We must show that $\tilde{\phi}$ cannot touch u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$. Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let Γ_k and d_k be as in the proof of the nondegenerate case (see (5-7) and subsequent lines). Set

$$\phi_k(x) = a_k \Gamma_k^+(x) + (d_k^+(x))^2 \varepsilon_k^2, \quad a_k = (1 + \varepsilon_k p).$$

Let

$$\tilde{\phi}_k(x) = \frac{\phi_k(x) - x_n}{\varepsilon_k}.$$

As in the previous case, it follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ $(Q^y(x) = Q(x - y))$,

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2} \cap \{x_n \geq 0\}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly from below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at $x_k \in B_1^+(u_k) \cup F(u_k)$. We claim that x_k cannot belong to $B_1^+(u_k)$. Otherwise, in a small neighborhood N of x_k we would have

$$\Delta \psi_k > \varepsilon_k^4 \geq \|f_k\|_\infty = \Delta u_k, \quad \psi_k < u_k \text{ in } N \setminus \{x_k\}, \ \psi_k(x_k) = u_k(x_k),$$

a contradiction.

Thus $x_k \in F(u_k) \cap \partial B_{1/(B\varepsilon_k)}\big(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k)\big)$. For simplicity we set

$$\mathcal{B} := B_{1/(B\varepsilon_k)}\left(y + e_n\left(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k\right)\right).$$

Let N_ρ be a small neighborhood of x_k of size ρ. Since

$$\|u_k^-\|_\infty \leq \varepsilon_k^2, \quad u_k^+ \geq (x_n - \varepsilon_k)^+,$$

as in the proof of the Harnack inequality and using the fact that $x_k \in F(u_k) \cap \partial\mathcal{B}$, we can conclude by the comparison principle that

$$u_k^- \leq c\varepsilon_k^2 (d(x, \partial\mathcal{B}))^- \quad \text{in } N_{\frac{3}{4}\rho},$$

where d denotes again the signed distance from x to $\partial\mathcal{B}$.

Let

$$\Psi_k(x) = \begin{cases} \psi_k & \text{in } \mathcal{B}, \\ c\varepsilon_k^2(3d(x, \partial\mathcal{B}) + d^2(x, \partial\mathcal{B})) & \text{outside of } \mathcal{B}. \end{cases} \tag{5-16}$$

Then Ψ_k touches u_k strictly from below at $x_k \in F(u_k) \cap F(\Psi_k)$.

We will reach a contradiction if we show that

$$(\Psi_k^+)_\nu^2 - (\Psi_k^-)_\nu^2 > 1 \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that

$$a_k^2 - c\varepsilon_k^4 > 1, \quad \text{or again} \quad (1+\varepsilon_k p)^2 - c\varepsilon_k^4 > 1.$$

This holds for k large enough, since $p > 0$. We finally reached a contradiction.

Step 3: contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [De Silva 2011]. □

6. Proof of the main theorems

In this section we exhibit the proofs of our main results, Theorems 1.1 and 1.2. As already pointed out, Theorem 1.2 will follow via a blow-up analysis from the flatness result. Thus, first we present the proof of Theorem 1.1 based on the improvement of flatness lemmas of the previous section.

Proof of Theorem 1.1. To complete the analysis of the degenerate case, we need to deal with the situation when u is close to a one-plane solution and yet the size of u^- is not negligible. More precisely:

Lemma 6.1. *Let u solve (1-1) in B_2 with*

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4,$$

and let it satisfy

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \tag{6-1}$$

and

$$\|u^-\|_{L^\infty(B_2)} \le \bar{C}\varepsilon^2, \quad \|u^-\|_{L^\infty(B_1)} > \varepsilon^2,$$

for a universal constant $\bar{C}$. There is a universal $\varepsilon_2 > 0$ such that, if $\varepsilon \le \varepsilon_2$, the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2} u(\varepsilon^{1/2} x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \le u_\varepsilon(x) \le U_{\beta'}(x_n + C'\varepsilon^{1/2}),$$

with $\beta' \sim \varepsilon^2$ and $C' > 0$ depending on $\bar{C}$.

Proof. For notational simplicity we set

$$v = \frac{u^-}{\varepsilon^2}.$$

From our assumptions we can deduce that

$$F(v) \subset \{-\varepsilon \le x_n \le \varepsilon\},$$
$$v \ge 0 \quad \text{in } B_2 \cap \{x_n \le -\varepsilon\}, \quad v \equiv 0 \quad \text{in } B_2 \cap \{x_n > \varepsilon\}. \tag{6-2}$$

Also,

$$|\Delta v| \le \varepsilon^2 \quad \text{in } B_2 \cap \{x_n < -\varepsilon\},$$

and

$$\begin{aligned} 0 \le v \le \bar{C} \quad & \text{on } \partial B_2, && (6\text{-}3)\\ v(\bar{x}) > 1 \quad & \text{at some point } \bar{x} \text{ in } B_1. && (6\text{-}4) \end{aligned}$$

Thus, using comparison with the function w such that

$$\begin{aligned} \Delta w = -\varepsilon^2 \quad & \text{in } D := B_2 \cap \{x_n < \varepsilon\},\\ w = v \quad & \text{on } \partial D, \end{aligned}$$

we obtain that for some $k > 0$ universal

$$v \le k|x_n - \varepsilon| \quad \text{in } B_1. \tag{6-5}$$

This fact forces the point $\bar{x}$ in (6-4) to belong to $B_1 \cap \{x_n < -\varepsilon\}$ at a fixed distance δ from $x_n = -\varepsilon$.

Now, let w be the harmonic function in $B_1 \cap \{x_n < -\varepsilon\}$ such that

$$\begin{aligned} w = 0 \quad & \text{on } B_1 \cap \{x_n = -\varepsilon\},\\ w = v \quad & \text{on } \partial B_1 \cap \{x_n \le -\varepsilon\}. \end{aligned}$$

By the maximum principle we conclude that

$$w + \varepsilon^2(|x|^2 - 3) \le v \quad \text{on } B_1 \cap \{x_n < -\varepsilon\}.$$

Also, for ε small, in view of (6-5) we obtain that

$$w - k\varepsilon(|x|^2 - 3) \ge v \quad \text{on } \partial(B_1 \cap \{x_n < -\varepsilon\}),$$

and hence also in the interior. Thus we conclude that

$$|w - v| \le c\varepsilon \quad \text{in } B_1 \cap \{x_n < -\varepsilon\}. \tag{6-6}$$

In particular this is true at $\bar{x}$, which forces

$$w(\bar{x}) \ge 1/2. \tag{6-7}$$

By expanding w around $(0, -\varepsilon)$ we then obtain, say, in $B_{1/2} \cap \{x_n \le -\varepsilon\}$,

$$\big|w - a|x_n + \varepsilon|\big| \le C|x|^2 + C\varepsilon.$$

This combined with (6-6) gives that

$$\big|v - a|x_n + \varepsilon|\big| \le C\varepsilon \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \le -\varepsilon\}.$$

Moreover, in view of (6-7) and the fact that $\bar{x}$ occurs at a fixed distance from $\{x_n = -\varepsilon\}$ we deduce from the Hopf lemma that

$$a \ge c > 0,$$

with c universal. In conclusion (see (6-5)),

$$\begin{aligned} \big|u^- - b\varepsilon^2|x_n+\varepsilon|\big| &\le C\varepsilon^3 \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \le -\varepsilon\}, \\ u^- &\le b\varepsilon^2|x_n-\varepsilon| \quad \text{in } B_1, \end{aligned}$$

with b comparable to a universal constant.

Combining the two inequalities above and the assumption (6-1) we conclude that in $B_{\varepsilon^{1/2}}$

$$(x_n-\varepsilon)^+ - b\varepsilon^2(x_n-C\varepsilon)^- \le u(x) \le (x_n+\varepsilon)^+ - b\varepsilon^2(x_n+C\varepsilon)^-,$$

with $C>0$ universal and b larger than a universal constant. Rescaling, we obtain that in B_1

$$(x_n-\varepsilon^{1/2})^+ - \beta'(x_n-C\varepsilon^{1/2})^- \le u_\varepsilon(x) \le (x_n+\varepsilon^{1/2})^+ - \beta'(x_n+C\varepsilon^{1/2})^-,$$

with $\beta' \sim \varepsilon^2$. We finally need to check that this implies the desired conclusion in B_1

$$\alpha'(x_n-C\varepsilon^{1/2})^+ - \beta'(x_n-C\varepsilon^{1/2})^- \le u_\varepsilon(x) \le \alpha'(x_n+C\varepsilon^{1/2})^+ - \beta'(x_n+C\varepsilon^{1/2})^-,$$

with $\alpha'^2 = 1+\beta'^2 \sim 1+\varepsilon^4$. This clearly holds in B_1 for ε small, say by possibly enlarging C so that $C \ge 2$. □

We are finally ready to exhibit the proof of Theorem 2.8, which as already observed immediately gives the result of Theorem 1.1.

Proof of Theorem 2.8. Let us fix a universal constant $\bar r > 0$ such that

$$\bar r \le r_0, r_1, \tfrac{1}{16},$$

where r_0, r_1 are the universal constants in the improvement of flatness Lemmas 5.1 and 5.2. Also, let us fix a universal constant $\tilde\varepsilon > 0$ such that

$$2\tilde\varepsilon \le 2\varepsilon_0(\bar r), \varepsilon_1(\bar r), \widetilde{C}^{-1}, \varepsilon_2,$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \widetilde{C}$ are the constants in Lemmas 5.1, 5.2 and 6.1. Now, let

$$\bar\varepsilon = \tilde\varepsilon^3.$$

We distinguish two cases. For notational simplicity we assume that u satisfies our assumptions in the ball B_2 and $0 \in F(u)$.

Case 1: $\beta \ge \tilde\varepsilon$. In this case, in view of Lemma 2.9 and our choice of $\tilde\varepsilon$, we obtain that u satisfies the assumptions of Lemma 5.1, namely

$$U_\beta(x_n-\tilde\varepsilon) \le u(x) \le U_\beta(x_n+\tilde\varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \tag{6-8}$$

with $0 < \beta \le L$ and

$$\|f\|_{L^\infty(B_1)} \le \tilde\varepsilon^3 \le \tilde\varepsilon^2\beta.$$

Thus we can conclude that (with $\beta_1 = \beta'$)

$$U_{\beta_1}\Big(x\cdot\nu_1-\bar r\frac{\tilde\varepsilon}{2}\Big)\le u(x)\le U_{\beta_1}\Big(x\cdot\nu_1+\bar r\frac{\tilde\varepsilon}{2}\Big)\quad\text{in } B_{\bar r},$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le\widetilde C\tilde\varepsilon$, and $|\beta-\beta_1|\le\widetilde C\beta\tilde\varepsilon$. In particular, by our choice of $\tilde\varepsilon$ we have

$$\beta_1\ge\tilde\varepsilon/2.$$

We can therefore rescale and iterate the argument above. Precisely, for $k=0,1,2,\dots$, set

$$\rho_k=\bar r^k,\quad \varepsilon_k=2^{-k}\tilde\varepsilon,\quad u_k(x)=\frac{1}{\rho_k}u(\rho_k x),\quad f_k(x)=\rho_k f(\rho_k x).$$

Also, let β_k be the constants generated at each k-iteration, hence satisfying (with $\beta_0=\beta$)

$$|\beta_k-\beta_{k+1}|\le\widetilde C\beta_k\varepsilon_k.$$

Then we obtain by induction that each u_k satisfies

$$U_{\beta_k}(x\cdot\nu_k-\varepsilon_k)\le u_k(x)\le U_{\beta_k}(x\cdot\nu_k+\varepsilon_k)\quad\text{in } B_1, \tag{6-9}$$

with $|\nu_k|=1$, $|\nu_k-\nu_{k+1}|\le\widetilde C\tilde\varepsilon_k$ ($\nu_0=e_n$).

Case 2: $\beta<\tilde\varepsilon$. In view of Lemma 2.9 we conclude that

$$U_0(x_n-\tilde\varepsilon)\le u^+(x)\le U_0(x_n+\tilde\varepsilon)\quad\text{in } B_1.$$

Moreover, from the assumption (2-5) and the fact that $\beta<\tilde\varepsilon$ we also obtain that

$$\|u^-\|_{L^\infty(B_1)}<2\tilde\varepsilon.$$

Let ε' be given by $\varepsilon'^2=2\tilde\varepsilon$. Then u satisfies the assumptions of Lemma 5.2 on improvement of flatness in the degenerate case:

$$U_0(x_n-\varepsilon')\le u^+(x)\le U_0(x_n+\varepsilon')\quad\text{in } B_1,$$

with

$$\|f\|_{L^\infty(B_1)}\le(\varepsilon')^4,\quad \|u^-\|_{L^\infty(B_1)}<\varepsilon'^2.$$

We conclude that

$$U_0\Big(x\cdot\nu_1-\bar r\frac{\varepsilon'}{2}\Big)\le u^+(x)\le U_0\Big(x\cdot\nu_1+\bar r\frac{\varepsilon'}{2}\Big)\quad\text{in } B_{\bar r},$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le C\varepsilon'$ for a universal constant C. We now rescale as in the previous case and set, for $k=0,1,2,\dots$,

$$\rho_k=\bar r^k,\quad \varepsilon_k=2^{-k}\varepsilon',\quad u_k(x)=\frac{1}{\rho_k}u(\rho_k x),\quad f_k(x)=\rho_k f(\rho_k x).$$

We can iterate our argument and obtain that (with $|\nu_k|=1$, $|\nu_k-\nu_{k+1}|\le C\varepsilon_k$)

$$U_0(x\cdot\nu_k-\varepsilon_k)\le u_k^+(x)\le U_0(x\cdot\nu_k+\varepsilon_k)\quad\text{in } B_1, \tag{6-10}$$

as long as we can verify that

$$\|u_k^-\|_{L^\infty(B_1)}<\varepsilon_k^2.$$

Let $\bar{k}$ be the first integer $\bar{k} > 1$ for which this fails, that is,

$$\|u^-_{\bar{k}}\|_{L^\infty(B_1)} \geq \varepsilon^2_{\bar{k}} \quad \text{and} \quad \|u^-_{\bar{k}-1}\|_{L^\infty(B_1)} < \varepsilon^2_{\bar{k}-1}.$$

Also,

$$U_0(x\cdot \nu_{\bar{k}-1} - \varepsilon_{\bar{k}-1}) \leq u^+_{\bar{k}-1}(x) \leq U_0(x\cdot \nu_{\bar{k}-1} + \varepsilon_{\bar{k}_-1}) \quad \text{in } B_1.$$

As argued several times (see for example (4-14)), we can then conclude from the comparison principle that

$$u^-_{\bar{k}-1} \leq M|x_n - \varepsilon_{\bar{k}-1}|\varepsilon^2_{\bar{k}-1} \quad \text{in } B_{19/20},$$

for a universal constant $M > 0$. Thus, by rescaling we get that

$$\|u^-_{\bar{k}}\|_{L^\infty(B_2)} < \bar{C}\varepsilon^2_{\bar{k}},$$

with $\bar{C}$ universal (depending on the fixed $\bar{r}$). We obtain that $u_{\bar{k}}$ satisfies all the assumptions of Lemma 6.1 and hence the rescaling

$$v(x) = \varepsilon^{-1/2}_{\bar{k}} u_{\bar{k}}(\varepsilon^{1/2}_{\bar{k}} x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}_{\bar{k}}) \leq v(x) \leq U_{\beta'}(x_n + C'\varepsilon^{1/2}_{\bar{k}}),$$

with $\beta' \sim \varepsilon^2_{\bar{k}}$. Set $\eta = \bar{C}\varepsilon^{1/2}_{\bar{k}}$. Then v satisfies our free boundary problem in B_1 with right-hand side

$$g(x) = \varepsilon^{1/2}_{\bar{k}} f_{\bar{k}}(\varepsilon^{1/2}_{\bar{k}} x)$$

and the flatness assumption

$$U_{\beta'}(x_n - \eta) \leq v(x) \leq U_{\beta'}(x_n + \eta).$$

Since $\beta' \sim \varepsilon^2_{\bar{k}}$ with a universal constant,

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon^{1/2}_{\bar{k}}\varepsilon^4_{\bar{k}} \leq \eta^2\beta',$$

as long as $\tilde{\varepsilon} \leq C''$ depending on $\bar{C}$. In conclusion, choosing $\tilde{\varepsilon} \leq \varepsilon_0(\bar{r})^4/(2\bar{C}^4)$, v falls under the assumptions of Lemma 5.1 on improvement of flatness (nondegenerate) and we can use an iteration argument as in Case 1. □

Proof of Theorem 1.2. Although not strictly necessary, we use the following Liouville-type result for global viscosity solutions to a two-phase homogeneous free boundary problem, which could be of independent interest.

Lemma 6.2. *Let U be a global viscosity solution to*

$$\begin{cases} \Delta U = 0 & \text{in } \{U > 0\} \cup \{U \leq 0\}^0, \\ (U^+_\nu)^2 - (U^-_\nu)^2 = 1 & \text{on } F(U) := \partial\{U > 0\}. \end{cases} \tag{6-11}$$

Assume that $F(U) = \{x_n = g(x'), x' \in \mathbb{R}^{n-1}\}$ with $\operatorname{Lip}(g) \leq M$. Then g is linear and $U(x) = U_\beta(x)$ for some $\beta \geq 0$.

Proof. Assume for simplicity that $0 \in F(U)$. Also, balls (of radius ρ and centered at 0) in $\mathbb{R}^{n-1}$ are denoted by $\mathcal{B}_\rho$.

By the regularity theory in [Caffarelli 1987], since U is a solution in B_2, the free boundary $F(U)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n and on M. Thus,

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \le C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1,$$

with C depending only on n, M. Moreover, since U is a global solution, the rescaling

$$g_R(x') = \frac{1}{R} g(Rx'), \quad x' \in \mathcal{B}_2,$$

which preserves the same Lipschitz constant as g, satisfies the same inequality as above, that is,

$$|g_R(x') - g_R(0) - \nabla g_R(0) \cdot x'| \le C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

This reads,

$$|g(Rx') - g(0) - \nabla g(0) \cdot Rx'| \le CR|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

Thus,

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \le C\frac{1}{R^\alpha}|y'|^{1+\alpha}, \quad y' \in \mathcal{B}_R.$$

Passing to the limit as $R \to \infty$ we obtain the claim. □

Proof of Theorem 1.2. Let $\bar{\varepsilon}$ be the universal constant in Theorem 2.8. Consider the blow-up sequence

$$u_k(x) = \frac{u(\delta_k)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$. Each u_k solves (1-1) with right-hand side

$$f_k(x) = \delta_k f(\delta_k x)$$

and we have

$$\|f_k(x)\| \le \delta_k \|f\|_{L^\infty} \le \bar{\varepsilon} \quad \text{for } k \text{ large enough.}$$

Standard arguments (see for example [Alt et al. 1984]) using the uniform Lipschitz continuity of the u_k and the nondegeneracy of their positive part u_k^+ (see Lemma 2.5) imply that (up to a subsequence)

$$u_k \to \tilde{u} \quad \text{uniformly on compacts}$$

and

$$\{u_k^+ = 0\} \to \{\tilde{u} = 0\} \quad \text{in the Hausdorff distance.}$$

The blow-up limit $\tilde{u}$ solves the global homogeneous two-phase free boundary problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \le 0\}^0, \\ (\tilde{u}_\nu^+)^2 - (\tilde{u}_\nu^-)^2 = 1 & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases} \tag{6-12}$$

Since $F(u)$ is a Lipschitz graph in a neighborhood of 0, it follows from Lemma 6.2 that $\tilde{u}$ is a two-plane solution, $\tilde{u} = U_\beta$ for some $\beta \geq 0$. Thus, for k large enough,

$$\|u_k - U_\beta\|_{L^\infty} \leq \bar{\varepsilon} \quad \text{and} \quad \{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\}.$$

Therefore, we can apply our flatness theorem (Theorem 2.8) and conclude that $F(u_k)$, and hence $F(u)$, are smooth. □

Flatness and ε-monotonicity. The flatness results present in the literature (see [Caffarelli 1989], for instance), are often stated in terms of "ε-monotonicity" along a large cone of directions $\Gamma(\theta_0, e)$ of axis e and opening θ_0. Precisely, a function u is said to be ε-monotone ($\varepsilon > 0$ small) along the direction τ in the cone $\Gamma(\theta_0, e)$ if for every $\varepsilon' \geq \varepsilon$,

$$u(x + \varepsilon'\tau) \leq u(x).$$

A variant of Theorem 1.1 states the following.

Theorem 6.3. *Let u be a solution to (1-1) in B_1, $0 \in F(u)$. Suppose that u^+ is nondegenerate. Then there exist $\theta_0 < \pi/2$ and $\varepsilon_0 > 0$ such that if u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then u^+ is fully monotone in $B_{1/2}$ along any direction in $\Gamma(\theta_1, e_n)$ for some θ_1 depending on θ_0, ε_0. In particular $F(u)$ is the graph of a Lipschitz function.*

Geometrically, the ε-monotonicity of u^+ can be interpreted as ε-closeness of $F(u)$ to the graph of a Lipschitz function. Our flatness assumption requires ε-closeness of $F(u)$ to a hyperplane. While this looks like a somewhat stronger assumption, it is indeed a natural one since it is satisfied for example by rescaling of solutions around a "regular" point of the free boundary. Moreover, if $\|f\|_\infty$ is small enough, depending on ε, it is not hard to check that ε-flatness of $F(u)$ implies $c\varepsilon$-monotonicity of u^+ along the directions of a flat cone, for a c depending on its opening.

The proof of Theorem 6.3 follows immediately from the following elementary lemma:

Lemma 6.4. *Let u be a solution to (1-1) in B_1, with $0 \in F(u)$. Suppose that u^+ is Lipschitz and nondegenerate. Assume that u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$. Then there exist a radius $r_0 > 0$ and $\delta_0 > 0$ depending on ε_0, θ_0 such that u^+ is δ_0-flat in B_{r_0}, that is,*

$$\{x_n \leq -\delta_0\} \subset B_{r_0} \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta_0\}.$$

7. More general operators and free boundary conditions

The setup. In this section we analyze the free boundary problem (1-3), that is,

$$\begin{cases} \mathscr{L}u = f & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x) & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases} \tag{7-1}$$

where f is continuous in $\Omega^+(u) \cup \Omega^-(u)$ with $\|f\|_{L^\infty(\Omega)} \leq L$, and

$$\mathscr{L} = \sum_{i,j=1}^n a_{ij}(x) D_{ij} + \boldsymbol{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar{\gamma}}(\Omega), \boldsymbol{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic with constants $0 < \lambda \leq \Lambda$.

We recall that our assumptions on G are:

(H1) $G(\eta,\cdot) \in C^{0,\bar\gamma}(\Omega)$ uniformly in η; $G(\cdot,x) \in C^{1,\bar\gamma}([0,L])$ for every $x \in \Omega$.

(H2) $G'(\cdot,x) > 0$ with $G(0,x) \geq \gamma_0 > 0$ uniformly in x.

(H3) There exists $N > 0$ such that $\eta^{-N} G(\eta,x)$ is strictly decreasing in η, uniformly in x.

We assume that $0 \in F(u)$ and that $a_{ij}(0) = \delta_{ij}$. Also, for notational convenience we set

$$G_0(\beta) = G(\beta, 0).$$

Let U_β be the two-plane solution to (7-1) when $\mathscr{L} = \Delta$, $f \equiv 0$ and $G = G_0$, that is,

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-, \quad \beta \geq 0, \quad \alpha = G_0(\beta).$$

The following definitions parallel those in Section 2.

Definition 7.1. Let u be a continuous function in Ω. We say that u is a viscosity solution to (1-3) in Ω, if the following conditions are satisfied:

(i) $\mathscr{L}u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense.

(ii) Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ $(B = B_\delta(x_0))$ with $F(v) \in C^2$. If v touches u from below (resp. above) at $x_0 \in F(v)$, then

$$v_\nu^+(x_0) \leq G(v_\nu^-(x_0), x_0) \quad (\text{resp. } \geq).$$

Definition 7.2. We say that $v \in C(\Omega)$ is a C^2 strict (comparison) subsolution (resp. supersolution) to (7-1) in Ω, if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

(i) $\mathscr{L}v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$.

(ii) If $x_0 \in F(v)$, then

$$v_\nu^+(x_0) > G(v_\nu^-(x_0), x_0) \quad \big(\text{resp. } v_\nu^+(x_0) < G(v_\nu^-(x_0), x_0),\ \ v_\nu^+(x_0) \neq 0\big).$$

Observe that the free boundary of a strict comparison sub/supersolution is C^2.

From here after, most of the statements and proofs parallel those in Sections 2–6. Thus, we only point out the main differences as much as possible.

Compactness and localization. As for the problem (1-1), we prove some basic lemmas to reduce the statement of the flatness theorem to a proper normalized situation. We start with the compactness Lemma 2.6 which generalizes to operators of the form

$$\mathscr{L}_*^k = \sum a_{ij}^k D_{ij},$$

with $a_{ij}^k \in C^{0,\bar\gamma}$ uniformly elliptic with constants λ, Λ and free boundary conditions given by a G_k satisfying hypotheses (H1)–(H3).

Lemma 7.3. *Let u_k be a sequence of (Lipschitz) viscosity solutions to*

$$\begin{cases} |\mathcal{L}_*^k u_k| \le M & \text{in } \Omega^+(u_k) \cup \Omega^-(u_k), \\ (u_k^+)_\nu = G_k((u_k^-)_\nu, x) & \text{on } F(u_k). \end{cases} \tag{7-2}$$

Assume that

(i) $a_{ij}^k \to a_{ij}$, $u_k \to u^*$ *uniformly on compact sets*,

(ii) $G_k(\eta, \cdot) \to G(\eta, \cdot)$ *on compact sets, uniformly on* $0 \le \eta \le L = \mathrm{Lip}(u_k)$, *and*

(iii) $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ *in the Hausdorff distance.*

Then

$$\left|\sum a_{ij} D_{ij} u^*\right| \le M \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*),$$

and u^* satisfies the free boundary condition

$$(u^*)_\nu^+ = G((u^*)_\nu^-, x) \quad \text{on } F(u^*),$$

both in the viscosity sense.

Proof. Set

$$\mathcal{L}_* := \sum a_{ij} D_{ij}.$$

The proof that

$$|\mathcal{L}_* u^*| \le M \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

is standard. We show for example that

$$\mathcal{L}_* u^* + M \ge 0 \quad \text{in } \Omega^+(u^*).$$

Let $v \in C^2(\Omega^+(u^*))$ touch u^* from above at $\bar{x} \in \Omega^+(u^*)$ and assume by contradiction that

$$\mathcal{L}_* v(\bar{x}) + M < 0.$$

Without loss of generality we can assume that v touches u^* strictly from above; otherwise we replace v by

$$v + \frac{\eta}{2n\Lambda}|x - \bar{x}|^2,$$

with η small. Then, since $u_k \to u^*$ uniformly in compact sets and $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance, there exists $x_k \to \bar{x}$ and constants $c_k \to 0$ such that $v + c_k$ touches from above u_k at $x_k \in \Omega^+(u_k)$, for k large. Then, since $|\mathcal{L}_*^k u_k(x_k)| \le M$ we must have

$$\mathcal{L}_*^k v(x_k) + M \ge 0.$$

This implies, for $k \to \infty$,

$$\mathcal{L}_* v(\bar{x}) + M \ge 0,$$

which is a contradiction.

We now prove that the free boundary condition holds. Let $\bar{x} \in F(u^*)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ with $F(v) \in C^2$ touch u^* from above at $\bar{x} \in F(v)$.

Assume

$$v_\nu^+(\bar{x}) < G(v_\nu^-(\bar{x}), \bar{x}), \quad v_\nu^+(\bar{x}) \neq 0.$$

We distinguish two cases. For notational simplicity let $\nu(\bar{x}) = e_n$. If $v_n^-(\bar{x}) \neq 0$, we can assume that the free boundaries $F(v)$ and $F(u^*)$ touch strictly and that

$$\mathscr{L}_* v + M < 0 \quad \text{in } \Omega^+(v) \cup \Omega^-(v) \tag{7-3}$$

holds up to $F(v)$. Otherwise, in a small neighborhood of $\bar{x}$ we replace v with

$$\bar{v}(x) = v(x + \eta |x' - \bar{x}'|^2 e_n) + \eta\, |\mathrm{dist}(x, F(v))| - C\, \mathrm{dist}(x, F(v))^2 \quad (\eta \text{ small}, C \text{ large}).$$

Then, for a suitable $c_k \to 0$, $v(x + c_k e_n)$ touches from above u_k at x_k with $x_k \to \bar{x}$. Then, either for every (large) k we have $x_k \in \Omega^+(u_k) \cup \Omega^-(u_k)$ or there exists a subsequence, which we still call $\{x_k\}$, such that $x_k \in F(u_k)$ for every large k. Thus, either

$$\sum a_{ij}^k(x_k) D_{ij} v(x_k + c_k e_n) + M \geq 0$$

or

$$\bar{v}_{\nu_k}^+(x_k + c_k e_n) \geq G_k(v_{\nu_k}^-(x_k + c_k e_n), x_k),$$

and we easily reach a contradiction for k large.

If $v_n^-(\bar{x}) = 0$, we replace v^- with zero and argue as above for v^+. □

Lemma 2.5 on the nondegeneracy of the positive part δ-away from the free boundary continues to hold unaltered; only choose

$$w(x) = \frac{G_0(0)}{2\gamma}(1 - |x|^{-\gamma}).$$

The analogue of Lemma 2.7 is the following:

Lemma 7.4. *Let u be a Lipschitz solution to (1-3) in B_1, with $\mathrm{Lip}(u) \leq L$, $\|b\|_\infty$, $\|f\|_\infty \leq L$. For any $\varepsilon > 0$ there exist $\bar{\delta}, \bar{r} > 0$ such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then

$$\|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r}, \tag{7-4}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and $\bar{r}$ depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem (7-2) with $M = L + L^2$, such that $\mathrm{Lip}(u_k) \leq L$ and

$$\{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\}, \tag{7-5}$$

but the u_k do not satisfy the conclusion (7-4).

Then, up to a subsequence, the u_k converge uniformly on compact set to a function u^*. In view of (7-5) and the nondegeneracy of u_k^+, δ_k-away from the free boundary (see remark above), we can apply our compactness Lemma 7.3 and conclude that, for some $\tilde{\mathscr{L}} := \sum \tilde{a}_{ij} D_{ij}$ and $\widetilde{G}$ in our class,

$$|\tilde{\mathscr{L}}u^*| \le M \quad \text{in } B_{1/2} \cap \{x_n \ne 0\}$$

and

$$(u^*)_n^+ = \widetilde{G}((u^*)_n^-, x) \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\}, \tag{7-6}$$

in the viscosity sense, with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus, by L^p Schauder estimates, we have

$$u^* \in C^{1,\tilde{\gamma}}\big(B_{1/2} \cap \{x_n \ge 0\}\big) \cap C^{1,\tilde{\gamma}}\big(B_{1/2} \cap \{x_n \le 0\}\big)$$

for all $\tilde{\gamma} < 1$ and (for any $\bar{r}$ small)

$$\big\|u^* - (\alpha x_n^+ - \beta x_n^-)\big\|_{L^\infty(B_{\bar{r}})} \le C(n, L)\bar{r}^{1+\tilde{\gamma}},$$

with $\beta = (u^*)_n^-(0)$ and $\alpha = (u^*)_n^+(0) > 0$. Thus, from (7-6), we have $\alpha = \widetilde{G}_0(\beta)$.

Then we reach a contradiction as in Lemma 2.7. □

In view of the lemma above, after proper rescaling, Theorem 1.3 follows from the following result.

Theorem 7.5. *Let u be a Lipschitz solution to (1-3) in B_1, with $\mathrm{Lip}(u) \le L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \bar{\varepsilon}, \quad \textit{for some } 0 \le \beta \le L, \tag{7-7}$$
$$\{x_n \le -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \bar{\varepsilon}\},$$

and

$$[a_{ij}]_{C^{0,\bar{\gamma}}(B_1)} \le \bar{\varepsilon}, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \le \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \le \bar{\varepsilon},$$
$$[G(\eta, \cdot)]_{C^{0,\bar{\gamma}}(B_1)} \le \bar{\varepsilon} \quad \textit{for all } 0 \le \eta \le L,$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Linearized problem. The linearized problem becomes ($\tilde{\alpha} > 0$)

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \ne 0\}, \\ \tilde{\alpha}(\tilde{u})_n^+ - \tilde{\beta} G_0'(\tilde{\beta})(\tilde{u})_n^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}, \end{cases} \tag{7-8}$$

with $\tilde{\alpha} = G_0(\tilde{\beta})$.

Setting $\zeta^2 = \tilde{\alpha}$ and $\xi^2 = \tilde{\beta} G_0'(\tilde{\beta})$ we can write the free boundary condition as

$$\zeta^2 \tilde{u}_n^+ - \xi^2 \tilde{u}_n^- = 0.$$

Consequently, all the definitions and conclusions in Section 3 hold, in particular Theorems 3.2–3.4.

8. The nondegenerate case for general free boundary problems

In this section, we recover lemma on improvement of flatness in the nondegenerate case, that is, when the solution is trapped between parallel two-plane solutions U_β at ε distance, with $\beta > 0$. First we need the Harnack inequality.

The Harnack inequality. As in Section 4, the Harnack inequality follows from the following basic lemma.

Lemma 8.1. *Let u be a viscosity solution to (7-1). There exists a universal constant $\bar{\varepsilon} > 0$ such that, if u satisfies*

$$u(x) \geq U_\beta(x) \quad \textit{in } B_1,$$

with $0 < \beta \leq L$, and if furthermore we have

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \min\{G_0(\beta), \beta\}, \qquad \|\boldsymbol{b}\|_{L^\infty(B_1)} \leq \varepsilon^2, \tag{8-1}$$

$$\|G(\eta, x) - G_0(\eta)\|_{L^\infty(B_1)} \leq \varepsilon^2 \quad \textit{for all } 0 \leq \eta \leq L, \tag{8-2}$$

with $0 \leq \varepsilon \leq \bar{\varepsilon}$, then, if at $\bar{x} = \frac{1}{5} e_n$

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon), \tag{8-3}$$

then

$$u(x) \geq U_\beta(x_n + c\varepsilon) \quad \textit{in } \bar{B}_{1/2}, \tag{8-4}$$

for some universal $0 < c < 1$. Analogously, if $u(x) \leq U_\beta(x)$ in B_1 and $u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$, then

$$u(x) \leq U_\beta(x_n - c\varepsilon) \quad \textit{in } \bar{B}_{1/2}.$$

Proof. We argue as in the proof of Lemma 4.3 and we only point out the main differences.

By our assumptions, in $B_{1/10}(\bar{x}) \subset B_1^+(u)$, $u - U_\beta \geq 0$ solves

$$\mathcal{L}(u - U_\beta) = f - \alpha b_n.$$

Recall that $\alpha = G_0(\beta)$. By the Harnack inequality, we obtain in $\bar{B}_{1/20}(\bar{x})$

$$\begin{aligned} u(x) - U_\beta(x) &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C\|f - \alpha b_n\|_{L^\infty} \\ &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C(\|f\|_{L^\infty} + \alpha\|b\|_{L^\infty}). \end{aligned}$$

From (8-1), (8-3) and the inequality above we conclude that for ε small enough,

$$u - U_\beta \geq \alpha c\varepsilon - \alpha C\varepsilon^2 \geq c_0 \alpha \varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{8-5}$$

From (8-5) and the comparison principle it follows that for c_1 small universal

$$u - \alpha x_n \geq \alpha c_1 \varepsilon x_n, \qquad x \in \{x_n > 0\} \cap \bar{B}_{19/20}. \tag{8-6}$$

To prove this claim, let ϕ solve

$$\mathcal{L}\phi = 0 \quad \text{in } R := (B_1 \cap \{x_n > 0\}) \setminus \bar{B}_{1/20}(\bar{x}),$$

with boundary data

$$\phi = 0 \quad \text{on } \partial(B_1 \cap \{x_n > 0\}),$$
$$\phi = 1 \quad \text{on } \partial B_{1/20}(\bar{x}).$$

Then, by boundary Harnack,

$$\phi \geq c x_n \quad \text{in } \bar{R} \cap B_{19/20}.$$

We now compare $u - \alpha x_n$ with $\frac{1}{2}\alpha c_0 \phi \varepsilon - 8\alpha\varepsilon^2 x_n + 4\alpha\varepsilon^2 x_n^2$ in the domain R to obtain the desired conclusion.

We now proceed similarly as in Lemma 4.3, with w the function defined in (4-5). We compute

$$\begin{aligned}\sum a_{ij} D_{ij} w(x) &= \gamma(\gamma+2)|x-\bar{x}|^{-\gamma-4}\operatorname{Tr}(A(x-\bar{x})\otimes(x-\bar{x})) - \gamma|x-\bar{x}|^{-\gamma-2}\operatorname{Tr}(A)\\ &\geq \gamma(\gamma+2)|x-\bar{x}|^{-\gamma-2} n\lambda - \gamma|x-\bar{x}|^{-\gamma-2} n\Lambda\\ &= \gamma|x-\bar{x}|^{-\gamma-2} n((\gamma+2)\lambda - \Lambda).\end{aligned}$$

Then

$$\begin{aligned}\mathscr{L}w &\geq \gamma|x-\bar{x}|^{-\gamma-2} n((\gamma+2)\lambda-\Lambda) - \gamma\|\boldsymbol{b}\|_{L^\infty}|x-\bar{x}|^{-\gamma-1}\\ &= \gamma|x-\bar{x}|^{-\gamma-2}\big(n((\gamma+2)\lambda-\Lambda) - \|\boldsymbol{b}\|_{L^\infty}|x-\bar{x}|\big)\\ &\geq \gamma|x-\bar{x}|^{-\gamma-2}\big(n((\gamma+2)\lambda-\Lambda) - \|\boldsymbol{b}\|_{L^\infty}\big) \equiv k_0(\gamma, c_0, n, \lambda, \Lambda) > 0,\end{aligned}$$

as long as γ satisfies

$$n((\gamma+2)\lambda - \Lambda) - \|\boldsymbol{b}\|_{L^\infty} > 0.$$

Now set $\psi = 1 - w$ and for $x \in \bar{B}_{3/4}(\bar{x})$ define

$$v_t(x) = \alpha(1+c_1\varepsilon)(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^+ - \beta(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^-,$$

with $\delta > 0$ small to be made precise later, and c_1 the constant in (8-6).

Then, for $t = -c_1$ one can easily verify that

$$v_{-c_1} \leq U_\beta \leq u, \quad x \in \bar{B}_{3/4}(\bar{x}).$$

Let $\bar{t}$ be the largest $t \geq -c_1$ such that

$$v_t(x) \leq u(x) \quad \text{in } \bar{B}_{3/4}(\bar{x}),$$

and let $\tilde{x}$ be the first touching point. To guarantee that $\tilde{x}$ cannot belong to $\partial B_{3/4}$ when $\bar{t} < c_0\delta$ we use (8-6). Indeed if $x \in \partial B_{3/4}$ and $v_{\bar{t}}(x) \geq 0$ then $x_n > 0$ and in view of (8-6)

$$v_{\bar{t}}(x) = \alpha(1+c_1\varepsilon)(x_n - \varepsilon c_0\delta + \bar{t}\varepsilon) < \alpha(1+c_1\varepsilon)x_n \leq u(x).$$

If $v_{\bar{t}}(x) < 0$ we use that $u \geq U_\beta$ to reach again the conclusion that $v_{\bar{t}}(x) < u(x)$. To proceed as in Lemma 4.3 we now need to show that for $\bar{t} < c_0\delta$, $v_{\bar{t}}$ is a strict subsolution in the annulus A.

Indeed, in $A^+(v_{\bar{t}})$ in view of the assumption (8-1) and the computation above for $\mathscr{L}w$, we have

$$\mathscr{L}v_{\bar{t}} \geq \alpha(\varepsilon c_0\delta k_0 + b_n) \geq \varepsilon^2\min\{\alpha, \beta\} \geq \|f\|_\infty.$$

A similar estimate holds in $A^-(v_{\bar{t}})$. Thus

$$\mathcal{L} v_{\bar{t}} \geq f \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}}),$$

for ε small enough.

Also, since $\psi_n < -c$ on $F(v_{\bar{t}}) \cap A$, for ε small, we have

$$\kappa \equiv |e_n - \varepsilon c_0 \nabla \psi| = \left(1 - 2\varepsilon c_0 \delta \psi_n + \varepsilon^2 c_0^2 \delta^2 |\nabla \psi|^2\right)^{1/2} = 1 + \tilde{k} \delta \varepsilon,$$

with $\tilde{k}$ between two universal constants.

Then, on $F(v_{\bar{t}}) \cap A$, using (8-2), we can write, as long as ε is sufficiently small,

$$\begin{aligned}(v_{\bar{t}}^+)_\nu - G((v_{\bar{t}}^-)_\nu, x) &= \alpha(1 + c_1 \varepsilon)\kappa - G(\beta\kappa, x) \geq \alpha(1 + c_1 \varepsilon)\kappa - G_0(\beta\kappa) - \epsilon^2 \\ &> (1 + c_1 \varepsilon) G_0(\beta) - G_0(\beta)\kappa^N - \epsilon^2 \\ &\geq \varepsilon G_0(\beta)\left(\frac{c_1}{2} - N\tilde{k}\delta\right) > 0\end{aligned}$$

if $\delta < c_1/(2N\tilde{\kappa})$. We used that $G_0(\beta) \geq G_0(0) > 0$ and that $G_0(\beta\kappa) < G_0(\beta)\kappa^N$, since $\eta^{-N} G_0(\eta)$ is strictly decreasing.

Thus, $v_{\bar{t}}$ is a strict subsolution to (1-1) in A as desired. Hence $\bar{t} \geq c_0 \delta$ and we conclude as in the Laplacian case. □

With Lemma 8.1 at hand, the Harnack inequality and its corollary follow as in Section 4. We only state the corollary, since it is indeed the tool used in the proof of the improvement of flatness lemma in the next subsection.

Corollary 8.2. *Let u satisfy at some point $x_0 \in B_2$*

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \textit{in } B_1(x_0) \subset B_2, \tag{8-7}$$

for some $0 < \beta \leq L$, with

$$b_0 - a_0 \leq \varepsilon,$$

and let (8-1)–(8-2) hold, for $\varepsilon \leq \bar{\varepsilon}$, $\bar{\varepsilon}$ universal. Then in $B_1(x_0)$ (with $\alpha = G_0(\beta)$) we have

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon} & \textit{in } B_2^+(u) \cup F(u), \\[2ex] \dfrac{u(x) - \beta x_n}{\beta \varepsilon} & \textit{in } B_2^-(u), \end{cases}$$

has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$, that is, for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$,

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

Improvement of flatness. We now extend the basic induction step towards $C^{1,\gamma}$ regularity at 0. We argue as in the proof of Lemma 5.1.

Lemma 8.3. *Let u be solution of (1-3) and suppose that*

$$U_\beta(x_n-\varepsilon)\le u(x)\le U_\beta(x_n+\varepsilon)\quad \text{in } B_1, \tag{8-8}$$

with $0<\beta\le L$,

$$\|a_{ij}-\delta_{ij}\|_{L^\infty(B_1)}\le\varepsilon,\qquad \|f\|_{L^\infty(B_1)}\le\varepsilon^2\min\{G_0(\beta),\beta)\},\qquad \|\boldsymbol{b}\|_{L^\infty(B_1)}\le\varepsilon^2,$$

and

$$\|G(\eta,\cdot)-G_0(\eta)\|_{L^\infty(B_1)}\le\varepsilon^2\quad \text{for all}\quad 0\le\eta\le L.$$

If $0<r\le r_0$ for r_0 universal, and $0<\varepsilon\le\varepsilon_0$ for some ε_0 depending on r, then

$$U_{\beta'}\Big(x\cdot\nu_1-r\frac{\varepsilon}{2}\Big)\le u(x)\le U_{\beta'}\Big(x\cdot\nu_1+r\frac{\varepsilon}{2}\Big)\quad \text{in } B_r, \tag{8-9}$$

with $|\nu_1|=1$, $|\nu_1-e_n|\le\widetilde{C}\varepsilon$, and $|\beta-\beta'|\le\widetilde{C}\beta\varepsilon$ for a universal constant $\widetilde{C}$.

Proof. We divide the proof into three steps.

Step 1: compactness. We keep the same notation of Lemma 5.1. In this case, the sequence u_k is a solution of problem (1-3) for operators

$$\mathcal{L}^k=\sum_{ij}{a_{ij}}^kD_{ij}+\boldsymbol{b}^k\cdot\nabla,$$

where (with $\alpha_k=G_k(\beta_k,0)$)

$$\|a_{ij}^k-\delta_{ij}\|_{L^\infty}\le\varepsilon_k,\qquad \|f_k\|_{L^\infty}\le\varepsilon_k^2\min\{\alpha_k,\beta_k\},\qquad \|\boldsymbol{b}^k\|_{L^\infty}\le\varepsilon_k^2,$$

and

$$\|G_k(\eta,\cdot)-G_k(\eta,0)\|_\infty\le\varepsilon_k^2\quad \text{for all}\quad 0\le\eta\le L. \tag{8-10}$$

The normalized functions $\tilde u_k$ are defined by the same formula. Up to a subsequence, $G_k(\cdot,0)$ converges, locally uniformly, to some C^1-function $\widetilde{G}_0$, while $\beta_k\to\tilde\beta$ so that $\alpha_k\to\tilde\alpha=\widetilde{G}_0(\tilde\beta)$. Moreover, by Corollary 8.2 the graphs of $\tilde u_k$ converge in the Hausdorff distance to a Hölder continuous $\tilde u$.

Step 2: limiting solution. We show that $\tilde u$ solves

$$\begin{cases}\Delta\tilde u=0 & \text{in } B_{1/2}\cap\{x_n\ne0\},\\ \tilde\alpha\tilde u_n^+-\tilde\beta\widetilde{G}_0'(\tilde\beta)\tilde u_n^-=0 & \text{on } B_{1/2}\cap\{x_n=0\}.\end{cases} \tag{8-11}$$

We can write in $\Omega^+(u^k)$ (in $\Omega^-(u^k)$ replace α_k with β_k)

$$\sum a_{ij}^kD_{ij}\tilde u_k=\frac{1}{\alpha_k\varepsilon_k}\sum a_{ij}^kD_{ij}u_k=\frac{1}{\alpha_k\varepsilon_k}(-\alpha_k\boldsymbol{b}^k\cdot\nabla u_k+f^k)\equiv F^k,$$

where $|F^k|\le C\varepsilon_k$.

Thus

$$\Delta \tilde{u}_k = \sum_{i,j=1}^{n} (\delta_{ij} - a_{ij}^k) D_{ij} \tilde{u}_k + F^k.$$

Hence recalling that $\|a_{ij}^k - \delta_{ij}\|_\infty \le \varepsilon_k$, and from interior L^p Schauder estimates for second derivatives, we conclude that, for instance, $\Delta \tilde{u}_k \to 0$ in L^p on every compact set contained in $\Omega^+(\tilde{u}^k)$ or in $\Omega^-(\tilde{u}^k)$. This shows that $\tilde{u}$ is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that $\tilde{u}$ satisfies the transmission condition in (8-11) in the viscosity sense.

Again we argue by contradiction. Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + p x_n^+ - q x_n^- + B Q(x - y),$$

with

$$Q(x) = \tfrac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A, B > 0, \quad \tilde{\alpha} p - \tilde{\beta} \widetilde{G}_0'(\tilde{\beta}) q > 0,$$

and assume that $\tilde{\phi}$ touches u strictly from below at a point $x_0 = (x_0', 0) \in B_{1/2}$. As in Lemma 5.1, let

$$\phi_k = a_k \Gamma_k^+(x) - b_k \Gamma_k^-(x) + \alpha_k (d_k^+(x))^2 \varepsilon_k^{3/2} + \beta_k (d_k^-(x))^2 \varepsilon_k^{3/2},$$

where, we recall,

$$a_k = \alpha_k(1 + \varepsilon_k p), \quad b_k = \beta_k(1 + \varepsilon_k q),$$

and $d_k(x)$ is the signed distance from x to $\partial B_{1/(B\varepsilon_k)}\Big(y + e_n\Big(\dfrac{1}{B\varepsilon_k} - A\varepsilon_k\Big)\Big)$. Moreover,

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at x_k, with $c_k \to 0$, $x_k \to x_0$.

We get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is,

$$\begin{cases} \mathscr{L}^k \psi_k > f_k & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu - G_k((\psi_k^-)_\nu, x) > 0 & \text{on } F(\psi_k). \end{cases}$$

We have

$$|\nabla \Gamma_k| \le C, \quad |D_{ij} \Gamma_k| \le C\varepsilon_k, \quad |a_{ij} - \delta_{ij}| \le \varepsilon_k.$$

For k large enough, we can write, say in the positive phase of ψ_k,

$$\begin{aligned} \mathscr{L}_k \psi_k = (\mathscr{L}^k - \Delta)\psi_k + \Delta \psi_k &\ge -C\alpha_k \varepsilon_k^2 + \alpha_k \varepsilon_k^{3/2} \mathscr{L}^k d_k^2(x + \varepsilon c_k e_n) \\ &\ge c \min\{\alpha_k, \beta_k\} \varepsilon_k^{3/2} \ge \|f_k\|_{L^\infty}, \end{aligned}$$

and the first condition is satisfied. An analogous estimate holds in the negative phase.

Finally, since on the zero level set $|\nabla \Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to showing that

$$a_k - G_k(b_k, x) > 0.$$

Using the definition of a_k, b_k we need to check that

$$\alpha_k(1+\varepsilon_k p) - G_k(\beta_k(1+\varepsilon_k q), x) > 0.$$

From (8-10), it suffices to check that

$$\alpha_k(1+\varepsilon_k p) - G_k(\beta_k(1+\varepsilon_k q), 0) - \varepsilon_k^2 > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha} p - \tilde{\beta} \widetilde{G}_0'(\tilde{\beta}) q > 0.$$

Thus $\tilde{u}$ is a viscosity solution to the linearized problem.

Step 3: contradiction. According to estimate (3-2), since $\tilde{u}(0) = 0$ we obtain

$$|\tilde{u} - (x' \cdot \nu' + p x_n^+ - q x_n^-)| \le C r^2, \quad x \in B_r,$$

with

$$\tilde{\alpha} p - \tilde{\beta} \widetilde{G}_0'(\tilde{\beta}) q = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \le C.$$

Thus, since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ (by slightly enlarging C) we get

$$|\tilde{u}_k - (x' \cdot \nu' + p x_n^+ - q x_n^-)| \le C r^2, \quad x \in B_r.$$

Now set

$$\beta_k' = \beta_k(1+\varepsilon_k q), \quad \nu_k = \frac{1}{\sqrt{1+\varepsilon_k^2 |\nu'|^2}}(e_n + \varepsilon_k(\nu', 0)).$$

Then

$$\begin{aligned} \alpha_k' &= G_k(\beta_k(1+\varepsilon_k q), 0) = G_k(\beta_k, 0) + \beta_k G_k'(\beta_k, 0)\varepsilon_k q + O(\varepsilon_k^2) \\ &= \alpha_k\left(1 + \beta_k \frac{G_k'(\beta_k, 0)}{\alpha_k} q \varepsilon_k\right) + O(\varepsilon_k^2) = \alpha_k(1+\varepsilon_k p) + O(\varepsilon_k^2), \end{aligned}$$

since from the identity $\tilde{\alpha} p - \tilde{\beta} \tilde{G}_0'(\tilde{\beta}) q = 0$ we derive that

$$\beta_k \frac{G_k'(\beta_k, 0)}{\alpha_k} q = p + O(\varepsilon_k).$$

Moreover

$$\nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2 \tau, \quad |\tau| \le C.$$

With these choices, it follows as in Lemma 5.1 that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta_k'}\left(x \cdot \nu_k - \varepsilon_k \frac{r}{2}\right) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta_k'}\left(x \cdot \nu_k + \varepsilon_k \frac{r}{2}\right) \quad \text{in } B_r,$$

which leads to a contradiction. $\square$

9. The degenerate case for general free boundary problems

In this section, we recover the improvement of flatness lemma in the degenerate case, that is, when the negative part of u is negligible and the positive part is close to a one-plane solution ($\beta = 0$, $\alpha = G_0(0)$). First we need the Harnack inequality.

The Harnack inequality. As in Section 4, the Harnack inequality in the degenerate case is a consequence of the following basic lemma.

Lemma 9.1. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \geq U_0(x) \quad \text{in } B_1,$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|\boldsymbol{b}\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4, \tag{9-1}$$

$$\|G(\eta, \cdot) - G_0(\eta)\| \leq \varepsilon^2, \quad 0 \leq \eta \leq C\varepsilon^2, \tag{9-2}$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon), \tag{9-3}$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } \overline{B}_{1/2}, \tag{9-4}$$

for some universal c, with $0 < c < 1$. Analogously, if $u^+(x) \leq U_0(x)$ in B_1 and $u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon)$, then

$$u^+(x) \leq U_0(x_n - c\varepsilon) \quad \text{in } \overline{B}_{1/2}.$$

Proof. The proof is the same as for the model case in Lemma 4.6. To prove that

$$v_{\bar{t}}(x) = G_0(0)(x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon)^-, \quad x \in \overline{B}_{3/4}(\bar{x})$$

is a subsolution in the annulus A, we use the following computation:

$$\mathcal{L} v_{\bar{t}} \geq c_0 C_1 \varepsilon^3 \mathcal{L} w - C_1 \varepsilon^2 |b_n| \geq \varepsilon^3 K(n, \lambda, \Lambda) > \varepsilon^4 \geq \|f\|_\infty \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}}),$$

for ε small enough. Here we have used as in Lemma 8.1 that $\mathcal{L} w \geq k_0 > 0$.

Moreover, on $F(v_{\bar{t}}) \cap A$ we have

$$(v_{\bar{t}}^+)_\nu - G((v_{\bar{t}}^-)_\nu) = G_0(0)|e_n - \varepsilon c_0 \nabla\psi| - G(\varepsilon^2 C_1 |e_n - \varepsilon c_0 \nabla\psi|, x) \geq C\varepsilon|\psi_n| + O(\varepsilon^2) > 0,$$

as long as ε is small enough. ☐

We state here the corollary that can be deduced by the degenerate Harnack inequality.

Corollary 9.2. *Let u satisfy at some point $x_0 \in B_2$*

$$U_0(x_n + a_0) \leq u(x) \leq U_0(x_n + b_0) \quad \text{in } B_1(x_0) \subset B_2, \tag{9-5}$$

with

$$b_0 - a_0 \le \varepsilon,$$

and let (9-1)–(9-2) hold with $\varepsilon \le \bar\varepsilon$, *where* $\bar\varepsilon$ *is universal. Then in* $B_1(x_0)$

$$\tilde u_\varepsilon := \frac{u^+(x) - G_0(0)x_n}{\varepsilon G_0(0)}$$

has a Hölder modulus of continuity at x_0, *outside the ball of radius* $\varepsilon/\bar\varepsilon$, *that is, for all* $x \in B_1(x_0)$ *with* $|x - x_0| \ge \varepsilon/\bar\varepsilon$,

$$|\tilde u_\varepsilon(x) - \tilde u_\varepsilon(x_0)| \le C|x - x_0|^\gamma.$$

Improvement of flatness. We prove here the improvement of flatness in the degenerate setting. Recall that in this case one improves the flatness of u^+ only.

Lemma 9.3. *Let u satisfy*

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1, 0 \in F(u), \tag{9-6}$$

with

$$\|a_{ij} - \delta_{ij}\| \le \varepsilon, \quad \|f\|_{L^\infty(B_1)} \le \varepsilon^4, \quad \|\boldsymbol{b}\|_{L^\infty(B_1)} \le \varepsilon^2,$$
$$\|G(\eta,\cdot) - G_0(\eta)\|_{L^\infty} \le \varepsilon^2, \quad 0 \le \eta \le C\varepsilon^2,$$

and

$$\|u^-\|_{L^\infty(B_1)} \le \varepsilon^2.$$

If $0 < r \le r_1$ *for* r_1 *universal, and* $0 < \varepsilon \le \varepsilon_1$ *for some* ε_1 *depending on* r, *then*

$$U_0\Big(x\cdot\nu_1 - r\frac{\varepsilon}{2}\Big) \le u^+(x) \le U_0\Big(x\cdot\nu_1 + r\frac{\varepsilon}{2}\Big) \quad \text{in } B_r, \tag{9-7}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le C\varepsilon$ *for a universal constant* C.

Proof. Step 1: Compactness. As in Lemma 5.2, it follows from Corollary 9.2 that as $\varepsilon_k \to 0$ the graphs of the

$$\tilde u_k(x) = \frac{u_k(x) - G_k(0,0)x_n}{G_k(0,0)\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$

converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde u$ over $B_{1/2} \cap \{x_n \ge 0\}$. Here the u_k solve our free boundary problem (1-3) with coefficients a_{ij}^k, $\boldsymbol{b}^k$, right-hand side f_k and free boundary condition G_k satisfying the assumptions of the lemma for a subsequence of ε_k going to 0.

Step 2: limiting solution. One shows that $\tilde u$ solves the following Neumann problem

$$\begin{cases} \Delta \tilde u = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde u_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{9-8}$$

We can easily adapt the proof of Lemma 5.2, choosing

$$\phi_k(x) = a_k\Gamma_k^+(x) + (d_k^+(x))^2\varepsilon_k^{3/2}, \quad a_k = G_k(0,0)(1 + \varepsilon_k p).$$

and

$$\Psi_k(x) = \begin{cases} \phi_k(x + c_k \varepsilon_k e_n) & \text{in } \mathcal{B}, \\ c\varepsilon_k^2(3d(x, \partial\mathcal{B}) + d^2(x, \partial\mathcal{B})) & \text{outside of } \mathcal{B}, \end{cases} \tag{9-9}$$

with

$$\mathcal{B} := B_{1/(B\varepsilon_k)}\left(y + e_n\left(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k\right)\right).$$

To check the subsolution condition at the free boundary for the function $\Psi_k(x)$, we need that

$$(\Psi_k^+)_\nu > G_k((\Psi_k^-)_\nu, x) \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that $G_k(0, 0)(1 + \varepsilon_k p) - G_k(c\varepsilon_k^2, x) > 0$ for k large. Since $p > 0$, this follows immediately from the assumptions on G_k.

Step 3: contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [De Silva 2011]. □

10. Proofs of the main theorems for general free boundary problems

The proof of Theorem 1.3 and Theorem 1.4 follow the same scheme of the model case. In particular, for Theorem 1.3, we take care of choosing $\bar{r}^{\bar{\gamma}} < \frac{1}{16}$, say, while the other assumptions on $\bar{r}$ remain the same. Also, $\tilde{\varepsilon}$ may have to be smaller, depending on γ_0. The dichotomy degenerate/nondegenerate is handled through Lemma 6.1 which extends to the variable coefficients case, with minor changes in the proof.

In the proof of Theorem 1.4, the blow-up limit $\tilde{u}$ solves the following global homogeneous two-phase free boundary problem

$$\begin{cases} \Delta\tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \le 0\}^0, \\ \tilde{u}_\nu^+ = G_0(\tilde{u}_\nu^-) & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases} \tag{10-1}$$

Now, Lemma 6.2 holds with identical proof for the free boundary condition $U_\nu^+ = G_0(U_\nu^-)$, so that the proof of Theorem 1.4 does not present any further difficulty.

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