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LARGE-TIME BLOWUP FOR A PERTURBATION OF THE CUBIC SZEGŐ EQUATION

HAIYAN XU

We consider the following Hamiltonian equation on a special manifold of rational functions:

$$i\partial_t u = \Pi(|u|^2 u) + \alpha(u|1), \quad \alpha \in \mathbb{R},$$

where Π denotes the Szegő projector on the Hardy space of the circle $\mathbb{S}^1$. The equation with $\alpha = 0$ was first introduced by Gérard and Grellier as a toy model for totally nondispersive evolution equations. We establish the following properties for this equation. For $\alpha < 0$, any compact subset of initial data leads to a relatively compact subset of trajectories. For $\alpha > 0$, there exist trajectories on which high Sobolev norms exponentially grow in time.

1. Introduction

The study on the long time behavior of solutions of Schrödinger type Hamiltonian equations is a central issue in the theory of dispersive nonlinear partial differential equations. For instance, Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2010] studied the cubic defocusing nonlinear Schrödinger equation,

$$i\partial_t u + \Delta u = \pm|u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^2. \tag{1-1}$$

In that paper, they constructed solutions with small H^s norm at the initial moment, which present a large Sobolev H^s norm at a sufficiently long time T. Guardia and Kaloshin [2012] improved this result by refining the estimates on the time T. Zaher Hani [2014] studied a version of the nonlinear Schrödinger equation obtained by canceling the least resonant part, and showed the existence of unbounded trajectories in high Sobolev norms. Hani, Pausader, Tzvetkov, and Visciglia [Hani et al. 2013] studied the nonlinear Schrödinger equation (1-1) on the spatial domain $\mathbb{R} \times \mathbb{T}^d$, and obtained global solutions to the defocusing and focusing problems (for any $d \geq 2$) with infinitely growing high Sobolev norms H^s.

Gérard and Grellier [2012a] achieved a related result by considering the following degenerate half wave equation on the one-dimensional torus:

$$i\partial_t u - |D|u = |u|^2 u. \tag{1-2}$$

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They found solutions with small Sobolev norms at initial time which become much larger as time grows. More precisely, there exist sequences of solutions u^n and t^n such that $\|u_0^n\|_{H^r} \to 0$ for any r, but

$$\|u^n(t^n)\|_{H^s} \sim \|u_0^n\|_{H^s}\left(\log \frac{1}{\|u_0^n\|_{H^s}}\right)^{2s-1}, \quad s > 1.$$

This result is a consequence of studies on the so-called *cubic Szegő equation*, introduced by Gérard and Grellier [2010; 2012b] as a model of nondispersive dynamics:

$$i\partial_t u = \Pi(|u|^2 u). \tag{1-3}$$

The above equation turns out to be the resonant part of the half wave equation (1-2). The operator Π, called the Szegő operator, is defined as a projector onto the nonnegative frequencies. If $u \in \mathscr{D}'(\mathbb{S}^1)$ is a distribution on the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, then

$$\Pi(u) = \Pi\left(\sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ik\theta}\right) = \sum_{k\geq 0} \hat{u}(k)e^{ik\theta}. \tag{1-4}$$

Notice that, on the Hilbert space $L^2(\mathbb{S}^1)$ endowed with the inner product

$$(u\,|\,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{ix})\overline{v(e^{ix})}\,dx, \tag{1-5}$$

Π is the orthogonal projector on the subspace $L^2_+(\mathbb{S}^1)$ defined by the conditions

$$\hat{u}(k) = 0 \quad \text{for all } k < 0.$$

Gérard and Grellier [2010; 2012b] studied the Szegő equation on the space

$$H^{1/2}_+(\mathbb{S}^1) := H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$$

and displayed two Lax pair structures for this completely integrable system. Moreover, they established an explicit formula of every solution with rational initial data [Gérard and Grellier 2013] and illustrated the large-time behavior of Sobolev norms of the solutions; for instance:

Theorem 1.1 [Gérard and Grellier 2010]. *Every solution u of (1-3) on*

$$\widetilde{\mathcal{M}}(1) := \left\{u = \frac{a+bz}{1-pz} : 0 \neq a \in \mathbb{C},\ b \in \mathbb{C},\ p \in \mathbb{C},\ |p| < 1,\ a + bp \neq 0\right\}$$

satisfies

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < \infty \quad \textit{for all } s > \tfrac{1}{2}.$$

However, there exists a family of Cauchy data u_0^ε in $\widetilde{\mathcal{M}}(1)$ which converges in $\widetilde{\mathcal{M}}(1)$ for the $C^\infty(\mathbb{S}^1)$ topology as $\varepsilon \to 0$, and $K > 0$ such that the corresponding solutions of (1-3) u^ε satisfy the following condition, for all $\varepsilon > 0$:

$$\textit{for some } t^\varepsilon > 0,\ \|u^\varepsilon(t^\varepsilon)\|_{H^s} \geq K(t^\varepsilon)^{2s-1} \textit{ as } t^\varepsilon \to \infty \quad \textit{for all } s > \tfrac{1}{2}.$$

Another result on this Szegő equation was obtained by Pocovnicu [2011b; 2011a], who studied this equation by replacing the circle $\mathbb{S}^1$ with the real line and got a polynomial growth of high Sobolev norms [Pocovnicu 2011a, Corollary 4], which says that there exists a solution u of the Szegő equation and a constant $C > 0$ such that $\|u(t)\|_{H^s} \geq C|t|^{2s-1}$ for sufficiently large $|t|$.

The aim of this article is to study the properties of global solutions for the following Hamiltonian equation on $L^2_+(\mathbb{S}^1)$, which is the cubic Szegő equation with a linear perturbation:

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u) + \alpha(u\,|\,1), & \alpha \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{1-6}$$

In view of (1-5),

$$(u\,|\,1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{ix})\,dx$$

is the average of u on $\mathbb{S}^1$.

Equation (1-6), called the α-Szegő equation, inherits three formal conservation laws:

$$\begin{aligned} &\text{mass:} && Q(u) := \int_{\mathbb{S}^1} |u|^2 \frac{d\theta}{2\pi} = \|u\|^2_{L^2}, \\ &\text{momentum:} && M(u) := (Du\,|\,u), \quad D := -i\partial_\theta = z\partial_z, \\ &\text{energy:} && E_\alpha(u) := \frac{1}{4}\int_{\mathbb{S}^1} |u|^4 \frac{d\theta}{2\pi} + \frac{1}{2}\alpha|(u\,|\,1)|^2. \end{aligned}$$

Slight modifications of the proof of the well-posedness result in [Gérard and Grellier 2010] lead to the result that the α-Szegő equation is globally well posed in $H^s_+(\mathbb{S}^1) = H^s(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$ for $s \geq \frac{1}{2}$:

Theorem 1.2. *Given $u_0 \in H^{1/2}_+(\mathbb{S}^1)$, there exists a unique global solution $u \in C(\mathbb{R}; H^{1/2}_+)$ of (1-6) with u_0 as the initial condition. Moreover, if $u_0 \in H^s_+(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C^\infty(\mathbb{R}; H^s_+)$. Furthermore, if $u_0 \in H^s_+(\mathbb{S}^1)$ with $s > 1$, the Wiener norm of u is bounded uniformly in time:*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_W := \sup_{t\in\mathbb{R}} \sum_{k=0}^{\infty} |\widehat{u(t)}(k)| \leq C_s \|u_0\|_{H^s}. \tag{1-7}$$

Now we present our main results. In our case with a perturbation term, it turns out that if $\alpha < 0$, the Sobolev norm stays bounded uniformly in time, while if $\alpha > 0$, it may grow exponentially fast:

Theorem 1.3. *Let $u_0 = b_0 + c_0 z/(1 - p_0 z)$, $c_0 \neq 0$, $|p_0| < 1$.*

For $\alpha < 0$, the Sobolev norm of the solution will stay bounded:

$$\|u(t)\|_{H^s} \leq C, \qquad C \text{ does not depend on time } t,\ s \geq 0. \tag{1-8}$$

For $\alpha > 0$, the solution u of the α-Szegő equation (1-6) has a Sobolev norm growing exponentially in time:

$$\|u(t)\|_{H^s} \simeq e^{C_{\alpha,s}|t|}, \qquad s > \tfrac{1}{2},\ C_{\alpha,s} > 0,\ |t| \to \infty \tag{1-9}$$

if and only if

$$E_\alpha = \tfrac{1}{4}Q^2 + \tfrac{1}{2}\alpha Q. \tag{1-10}$$

Remark 1.4. (1) Together with the results in [Gérard and Grellier 2010; 2012b], we now have a complete picture for the high Sobolev norm of the solutions to the α-Szegő equation. For $\alpha < 0$, it stays bounded (uniformly on time). For $\alpha > 0$, it turns out to have an exponential growth for some initial data satisfying the condition in Theorem 1.3. Finally, for $\alpha = 0$, the trajectories of the Szegő equation with rational initial data are quasiperiodic with instability of the H^s norm as in Theorem 1.1.

(2) Our result is in strong contrast with Bourgain's [1996] and Staffilani's [1997] results for the dispersive equations, which say that the dispersive equations admit polynomial upper bounds on Sobolev norms. Here, we give an example of exponential growth of Sobolev norms for a nondispersive model.

(3) The solutions to the α-Szegő equation admit an exponential upper bound of the Sobolev norms. Assuming $s > 1$, it is easy to solve (1-6) locally in time. More precisely, one has to solve the integral equation

$$u(t) = u_0 - i\int_0^t \big(\Pi(|u|^2u) + \alpha(u|1)\big)\,dt'.$$

Thus

$$\|u(t)\|_{H^s} \le \|u_0\|_{H^s} + c\int_0^t (1 + \|u(t')\|_W^2)\,\|u(t')\|_{H^s}\,dt',$$

since, by Theorem 1.2, the Wiener norm is uniformly bounded. Then, by Gronwall's inequality, we have

$$\|u(t)\|_{H^s} \le \|u_0\|_{H^s} e^{ct}.$$

This shows that (1-9) is the worst that can happen.

This paper is organized as follows. In Section 2, we prove that there exists a Lax pair for the α-Szegő equation based on Hankel operators. Then we define the manifolds $\mathscr{L}(k) := \{u : \operatorname{rk} K_u = k,\ k \in \mathbb{Z}^+\}$ with the shifted Hankel operator K_u. These manifolds are proved to be invariant by the flow and can be represented as sets of rational functions. In this paper we will just consider the solutions $u \in \mathscr{L}(1)$. We plan to address the other cases in a forthcoming work. In Section 3, we prove the large-time blowup result and the boundedness of the Wiener norm to show that our result is optimal. Furthermore, we provide an example which describes the energy cascade. Finally, we present some perspectives in Section 4.

2. The Lax pair structure

For $u \in E \subset \mathscr{D}'(\mathbb{S}^1)$, we define E_+ by canceling the negative Fourier modes of u:

$$E_+ = \{u \in E : \text{for all } k < 0, \hat{u}(k) = 0\}.$$

In particular, L^2_+ is the Hardy space of L^2 functions which extend to the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ as holomorphic functions

$$u(z) = \sum_{k\ge 0} \hat{u}(k) z^k, \qquad \sum_{k\ge 0} |\hat{u}(k)|^2 < \infty.$$

An element of L^2_+ can therefore be seen either as a square integrable function $u = u(e^{i\theta})$ on the circle with only nonnegative Fourier modes, or a holomorphic function $u = u(z)$ on the unit disc with square summable Taylor coefficients.

Using the Szegő projector defined as (1-4), we first introduce two important classes of operators on $L^2_+(\mathbb{S}^1)$, namely, the Hankel and Toeplitz operators.

Given $u \in H^{1/2}_+(\mathbb{S}^1)$, a Hankel operator $H_u : L^2_+ \to L^2_+$ is defined by

$$H_u(h) = \Pi(u\bar{h}).$$

Notice that H_u is $\mathbb{C}$-antilinear and symmetric with respect to the real scalar product $\operatorname{Re}(u|v)$. In fact, it satisfies

$$(H_u(h_1)\,|\,h_2) = (H_u(h_2)\,|\,h_1).$$

Moreover, H_u is a Hilbert-Schmidt operator with

$$\operatorname{Tr}(H_u^2) = \sum_{n=0}^{\infty} (n+1)|\hat{u}(n)|^2.$$

Given $b \in L^\infty(\mathbb{S}^1)$, a Toeplitz operator $T_b : L^2_+ \to L^2_+$ is defined by

$$T_b(h) = \Pi(bh).$$

T_b is $\mathbb{C}$-linear, bounded, and self-adjoint if and only if b is real valued.

The cubic Szegő equation was proved to admit two Lax pairs as follows:

Theorem 2.1 [Gérard and Grellier 2010, Theorem 3.1]. *Let $u \in C(\mathbb{R}, H^s(\mathbb{S}^1))$ for some $s > \frac{1}{2}$. The cubic Szegő equation*

$$i\partial_t u = \Pi(|u|^2 u) \tag{2-1}$$

has two Lax pairs (H_u, B_u) and (K_u, C_u), namely, if u solves (2-1), then

$$\frac{dH_u}{dt} = [B_u, H_u], \quad \frac{dK_u}{dt} = [C_u, K_u], \tag{2-2}$$

where

$$B_u = \frac{i}{2} H_u^2 - iT_{|u|^2}, \quad K_u := T_z^* H_u, \quad C_u = \frac{i}{2} K_u^2 - iT_{|u|^2}.$$

Corollary 2.2. *The perturbed Szegő equation (1-6) with $\alpha \neq 0$ still has one Lax pair (K_u, C_u).*

Proof of Corollary 2.2. We need an identity from [Gérard and Grellier 2013, Lemma 1]:

$$H_{\Pi(|u|^2 u)} = T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3. \tag{2-3}$$

Using (1-6) and (2-3),

$$\frac{dH_u}{dt} = H_{-i\Pi(|u|^2 u) - i\alpha(u|1)} = -i(T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3) - i\alpha(u\,|\,1)H_1.$$

Using the antilinearity of H_u, we deduce that

$$\frac{dH_u}{dt} = [B_u, H_u] - i\alpha(u\,|\,1)H_1, \tag{2-4}$$

which means that (H_u, B_u) is no longer a Lax pair. Fortunately, we have $T_z^* H_1 = 0$, which leads to the identity

$$\frac{dK_u}{dt} = [C_u, K_u]. \qquad \square$$

An important consequence of this Lax pair structure is the existence of finite dimensional submanifolds of $L^2_+(\mathbb{S}^1)$, which are invariant by the flow of (1-6). To describe these manifolds, Gérard and Grellier [2010, Appendix 4] proved a Kronecker-type theorem to the effect that the Hankel operator H_u is of finite rank k if and only if u is a rational function of the complex variable z with no poles in the unit disc and of the form $u(z) = A(z)/B(z)$ with $A \in \mathbb{C}_{k-1}[z]$, $B \in \mathbb{C}_k[z]$, $B(0) = 1$, $\deg A = k-1$ or $\deg B = k$, A and B having no common factors, and $B(z) \neq 0$ if $|z| \leq 1$. In fact, we can prove a similar theorem for our case.

Definition 2.3. Letting k be a positive integer, we define

$$\mathcal{L}(k) := \{u \in H_+^{1/2}(\mathbb{S}^1) : \operatorname{rk} K_u = k\}. \tag{2-5}$$

Due to the Lax pair structure, the manifolds $\mathcal{L}(k)$ are invariant by the flow.

Theorem 2.4. *The elements of $\mathcal{L}(k)$ are the rational functions $u = \dfrac{A(z)}{B(z)}$, where*

$$A, B \in \mathbb{C}_k[z], \quad A \wedge B = 1, \quad \deg A = k \text{ or } \deg B = k, \quad B^{-1}(\{0\}) \cap \overline{D} = \varnothing. \tag{2-6}$$

Here $A \wedge B = 1$ means A and B have no common factors.

Proof. Gérard and Grellier [2010, Appendix 4] proved that

$$\begin{aligned}\mathcal{M}(k+1) &= \{u : \operatorname{rk} H_u = k+1\} \\ &= \Big\{ u(z) = \frac{A(z)}{B(z)} : A \in \mathbb{C}_k[z],\, B \in \mathbb{C}_{k+1}[z],\, B(0) = 1, \\ &\qquad\qquad \deg A = k \text{ or } \deg B = k+1,\ A \wedge B = 1,\ B^{-1}(0) \cap \overline{D} = \varnothing \Big\}.\end{aligned}$$

For $u \in \mathcal{M}(k+1)$ we have $\dim \operatorname{Im} H_u = k+1$. Then $u, T_z^* u, \dots, (T_z^*)^{k+1} u$ are linearly dependent, that is, there exist C_ℓ, not all zero, such that $\sum_{\ell=0}^{k+1} C_\ell (T_z^*)^\ell u = 0$. We get

$$\sum_{\ell=0}^{k+1} C_\ell \hat{u}(\ell + n) = 0 \quad \text{for all } n \geq 0.$$

This is a recurrence equation for the sequence $\hat{u}$, and can be solved by using linear algebra. Define

$$P(X) = \sum_{\ell=0}^{k+1} C_\ell X^\ell = C \prod_{p \in \mathcal{P}} (X - p)^{m_p},$$

where $\mathcal{P} = \{p \in \mathbb{C} : P(p) = 0\}$ and m_p is the multiplicity of p. Then $(\hat{u}(n))_{n\geq 0}$ is a linear combination of the sequences

$$n^\ell p^{n-\ell},\ p \neq 0,\ 0 \leq \ell \leq m_p - 1 \quad \text{and} \quad \delta_{nm},\ p = 0,\ 0 \leq m \leq m_0 - 1.$$

Recall that

$$u(z) = \sum_{n\geq 0} \hat{u}(n) z^n \quad \text{for } |z| < 1.$$

Thus u is a linear combination of terms $\dfrac{1}{(1-pz)^{\ell+1}}$ with $0 < |p| < 1$ and $0 \leq \ell \leq m_p - 1$, and terms z^ℓ for $0 \leq \ell \leq m_0 - 1$.

Consequently, $u(z) = A(z)/B(z)$ with

$$\begin{aligned} &\deg A \leq k,\ \deg B = k+1 && \text{if } 0 \notin \mathcal{P}, \\ &\deg A = k,\ \deg B \leq k && \text{if } 0 \in \mathcal{P}. \end{aligned}$$

But $0 \in \mathcal{P}$ is equivalent to $1 \in \operatorname{Im} H_u$, or again to $\ker K_u \cap \operatorname{Im} H_u \neq \{0\}$, since $K_u = T_z^* H_u$ and $\operatorname{rk} H_u - 1 \leq \operatorname{rk} K_u \leq \operatorname{rk} H_u$. For $u \in \mathcal{L}(k)$ we have $\operatorname{rk} K_u = k$. Thus $u = A(z)/B(z)$ with

$$\begin{aligned} &\deg A \leq k-1,\ \deg B = k && \text{if } \operatorname{rk} H_u = \operatorname{rk} K_u = k, \\ &\deg A = k, \qquad \deg B \leq k && \text{if } \operatorname{rk} H_u = \operatorname{rk} K_u + 1 = k+1. \end{aligned}$$

The proof of the converse is similar. It follows that $\mathcal{L}(k) = \{u : \operatorname{rk} K_u = k+1\}$ contains precisely the quotients $u = A/B$, with A and B as in (2-6). □

3. Proof of the main theorem

We will now prove that the α-Szegő equation (1-6) has a large-time blowup as in Theorem 1.3. We also give an example to describe this phenomenon in terms of energy transfer to high frequencies. We start by proving the boundedness of the Wiener norm as in Theorem 1.2.

Proposition 3.1. *Assume $u_0 \in H^s_+(\mathbb{S}^1)$ with $s > 1$ and let u be the corresponding unique solution of (1-6). Then*

$$\|u(t)\|_W \leq C_s \|u_0\|_{H^s} \quad \textit{for all } t \in \mathbb{R}.$$

Proof. By Peller's theorem [2003], the regularity of u ensures that H_u is trace class and the trace norm of H_u is equivalent to the $B^1_{1,1}$ norm of u. Recall the definition of $B^s_{p,q}(\mathbb{S}^1)$.

Let $\chi \in C^\infty(\mathbb{R}^+)$ satisfy $\chi|_{t<1}(t) = 1$, $\chi|_{t>2}(t) = 0$, $0 \leq \chi \leq 1$. Set ψ as $\psi_0(t) = 1 - \chi(t)$, $\psi_j(t) = \chi(2^{-j+1}t) - \chi(2^{-j}t)$. Define the operator Δ_j for $f \in \mathcal{D}'(\mathbb{S}^1)$ as

$$\Delta_j f = \sum_{k\in\mathbb{Z}} \psi_j(k) \hat{f}(k) e^{ik\theta}.$$

Then the Besov space is defined as

$$B^s_{p,q}(\mathbb{S}^1) := \{u \in \mathcal{D}'(\mathbb{S}^1) : 2^{js} \|\Delta_j f\|_{L^p} \in l^q_j,\ 1 \leq p, q \leq +\infty,\ 0 \leq j \leq +\infty\},$$

with norm

$$\|u\|_{B^s_{p,q}(\mathbb{S}^1)} = \left(\sum_{j=0}^{+\infty}(2^{js}\|\Delta_j f\|_{L^p})^q\right)^{1/q}.$$

Observe that there exist C, $C_s > 0$ such that

$$\begin{aligned}\|u\|_{B^1_{1,1}} &= \sum_{j=0}^{+\infty} 2^j\|\Delta_j u\|_{L^1} \le C\sum_{j=0}^{+\infty} 2^j\|\Delta_j u\|_{L^2}\\ &\le C\left(\sum_{j=0}^{+\infty} 2^{2js}\|\Delta_j u\|^2_{L^2}\right)^{1/2}\left(\sum_{j=0}^{+\infty} 2^{2j(1-s)}\right)^{1/2} \le C_s\|u\|_{H^s} \quad \text{for all } s > 1. \end{aligned} \tag{3-1}$$

So, for $u \in H^s$ with $s > 1$, H_u is trace class, and

$$\operatorname{Tr}(|H_u|) \le C_s\|u\|_{H^s}.$$

Since $K_u = T_z^* H_u$, we have $K_u^2 = H_u^2 - (\cdot|u)u$, and so $\operatorname{Tr}(|K_u|) \le \operatorname{Tr}(|H_u|)$. Due to the Lax pair structure, we conclude that $K_{u(t)}$ is isospectral to K_{u_0}, an in particular $\operatorname{Tr}(|K_{u(t)}|) = \operatorname{Tr}(|K_{u_0}|)$. Therefore

$$\operatorname{Tr}(|K_{u(t)}|) \le C_s\|u_0\|_{H^s}.$$

Since $\|u\|_W = |\hat{u}(0)| + \sum_{n\ge1}|\hat{u}(n)|$ and $|\hat{u}(0)| \le \|u\|_{L^2}$, we just need to show that

$$\sum_{n\ge1}|\hat{u}(n)| \le C\operatorname{Tr}(|K_u|).$$

Let $\{e_n\}$ be an orthonormal basis of L^2_+. Then, for any bounded operator B,

$$\sum_n |(K_u e_n\,|\,Be_n)| \le \operatorname{Tr}(|K_u|)\|B\|.$$

Then we see that $\sum_{n\ge1}|\hat{u}(2n)| + \sum_{n\ge1}|\hat{u}(2n+1)| \le \operatorname{Tr}(|K_u|)$ by taking $B = T_z$ and $B = \mathrm{Id}$. This completes the proof. □

Remark 3.2. In fact, to prove the global well-posedness, it is natural to use the Brezis–Gallouët type estimate from [Gérard and Grellier 2010, Appendix 2]: for $s > \frac{1}{2}$,

$$\|u\|_W \le C_s\|u\|_{H^{1/2}}\left[\log\left(1 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}}\right)\right]^{\frac{1}{2}}.$$

This leads to a growth doubly exponential on time for the Sobolev norm of u. Fortunately, by the estimate in Proposition 3.1, we know the H^s norm of the solutions will admit an exponential on time upper bound for $s > 1$ (see Remark 1.4).

Now, let us start the large-time blowup theorem.

Theorem 3.3. *For $\alpha > 0$, we consider the solution of the Szegő equation (1-6) with initial data $u_0 \in \mathcal{L}(1)$.*

(1) *If the trajectory issued from u_0 is not relatively compact in $\mathcal{L}(1)$, then*

$$\left|b+\frac{\bar{p}c}{1-|p|^2}\right|=\sqrt{\alpha}, \tag{3-2}$$

or, equivalently,

$$E_\alpha=\tfrac{1}{4}Q^2+\tfrac{1}{2}\alpha Q. \tag{3-3}$$

(2) *If* (3-2) *holds, then*

$$\|u(t)\|_{H^s}\simeq e^{C_{\alpha,s}|t|},\quad s>\tfrac{1}{2},\ C_{\alpha,s}>0,\ |t|\to\infty. \tag{3-4}$$

Thus the equality (3-3), which is invariant by the flow, is a necessary and sufficient condition to cause large-time blowup.

Proof. First, since the trajectory of the solution is not relatively compact in $\mathcal{L}(1)$, the level set $L(u_0):=\{u\in\mathcal{L}(1):Q(u)=Q(u_0),\ M(u)=M(u_0),\ E_\alpha(u)=E_\alpha(u_0)\}$ is not compact in $\mathcal{L}(1)$.

We rewrite $u\in\mathcal{L}(1)$ as

$$u=b+\frac{cz}{1-pz}.$$

Then the conservation laws under the coordinates b,p,c are given as

$$Q=\|u\|_{L^2}^2=\frac{|c|^2}{1-|p|^2}+|b|^2,$$
$$M=(Du\,|\,u)=\frac{|c|^2}{(1-|p|^2)^2},$$
$$E_\alpha=\tfrac{1}{4}\|u\|_{L^4}^4+\tfrac{1}{2}\alpha|(u|1)|^2=\frac{1}{4}\left[|b|^4+\frac{4|b|^2|c|^2}{1-|p|^2}+\frac{|c|^4(1+|p|^2)}{(1-|p|^2)^3}+\frac{4|c|^2\operatorname{Re}(bp\bar{c})}{(1-|p|^2)^2}\right]+\tfrac{1}{2}\alpha|b|^2.$$

Now, $u\in\mathcal{L}(1)$ stays in a compact of $\mathcal{L}(1)$ if and only if $|b|\le C$, $1/C\le|c|\le C$, and $|p|\le k<1$ with some constant C and k. Otherwise, due to the formulas of mass Q and momentum M, there exist $t_n\to\infty$ such that $|c(t_n)|$ and $1-|p(t_n)|^2$ tend to 0 at the same order. Using the formula of Q and E_α, we have

$$|b(t_n)|^2\to Q,\quad \tfrac{1}{4}|b(t_n)|^4+\tfrac{1}{2}\alpha|b(t_n)|^2\to E_\alpha.$$

Since the limit should be unique,

$$E_\alpha=\tfrac{1}{4}Q^2+\tfrac{1}{2}\alpha Q.$$

Using the formula of mass and energy, (3-3) can be rewritten under coordinates of b,p,c as

$$|b|^2+\frac{|c|^2|p|^2}{(1-|p|^2)^2}+2\operatorname{Re}\frac{bp\bar{c}}{1-|p|^2}=\alpha.$$

Simplifying the left hand side, we get

$$\left|b+\frac{\bar{p}c}{1-|p|^2}\right|=\sqrt{\alpha}.$$

Now we turn to proving that (3-2) is sufficient to cause the exponential growth of Sobolev norms. Writing, as before,

$$u(t) = b(t) + \frac{c(t)z}{1-p(t)z},$$

the terms $\partial_t u$, $\Pi(|u|^2u)$, $(u|1)$ can be represented as linear combinations of 1, $\frac{z}{1-pz}$, $\frac{z^2}{(1-pz)^2}$:

$$\begin{cases} \partial_t u = \partial_t b + \partial_t c \dfrac{z}{1-pz} + \partial_t p \dfrac{z^2}{(1-pz)^2}, \\ \Pi(|u|^2u) = |b|^2 b + \dfrac{2b|c|^2}{1-|p|^2} + \dfrac{|c|^2 c\bar{p}}{1-|p|^2} \\ \qquad\qquad + \left[2|b|^2 c + \dfrac{2b|c|^2 p}{1-|p|^2} + \dfrac{1+|p|^2}{1-|p|^2}|c|^2 c\right]\dfrac{z}{1-pz} + \left[c^2\bar{b} + \dfrac{|c|^2 cp}{1-|p|^2}\right]\dfrac{z^2}{(1-pz)^2}, \\ (u|1) = b. \end{cases}$$

Then (1-6) reads

$$\begin{cases} i\partial_t b = |b|^2 b + \dfrac{2b|c|^2}{1-|p|^2} + \dfrac{|c|^2 c\bar{p}}{(1-|p|^2)^2} + \alpha b, \\ i\partial_t c = 2|b|^2 c + \dfrac{2b|c|^2 p}{1-|p|^2} + \dfrac{|c|^2 c}{(1-|p|^2)^2}, \\ i\partial_t p = c\bar{b} + \dfrac{|c|^2 p}{1-|p|^2}. \end{cases} \tag{3-5}$$

Using the second equation of (3-5), we obtain

$$\frac{d|c|}{dt} = \frac{2|c|}{1-|p|^2}\,\mathrm{Im}(bp\bar{c}). \tag{3-6}$$

This equality together with (3-2) gives us

$$\begin{aligned}
\left(\frac{d|c|}{|c|dt}\right)^2 &= \frac{4(\mathrm{Im}(bp\bar{c}))^2}{(1-|p|^2)^2} = \frac{4|bp\bar{c}|^2}{(1-|p|^2)^2} - \frac{4(\mathrm{Re}(bp\bar{c}))^2}{(1-|p|^2)^2} \\
&= \frac{4|bp\bar{c}|^2}{(1-|p|^2)^2} - \left[\alpha - |b|^2 - \frac{|c|^2|p|^2}{(1-|p|^2)^2}\right]^2 = \frac{4|bp\bar{c}|^2}{(1-|p|^2)^2} - \left[\alpha - |b|^2 - \frac{|c|^2}{(1-|p|^2)^2} + \frac{|c|^2}{1-|p|^2}\right]^2 \\
&= \frac{4|bp\bar{c}|^2}{(1-|p|^2)^2} - \left[\alpha - Q - M + 2\frac{|c|^2}{1-|p|^2}\right]^2 \\
&= \frac{4|bp\bar{c}|^2}{(1-|p|^2)^2} - \frac{4|c|^4}{(1-|p|^2)^2} - \frac{4|c|^2}{1-|p|^2}\left[\alpha - |b|^2 - \frac{|c|^2}{(1-|p|^2)^2} - \frac{|c|^2}{1-|p|^2}\right] - (\alpha - Q - M)^2 \\
&= \frac{4|b|^2|c|^2}{(1-|p|^2)^2} + \frac{4|c|^4}{(1-|p|^2)^3} - \alpha\frac{4|c|^2}{1-|p|^2} - (\alpha - Q - M)^2 \\
&= 4\left(|b|^2 + \frac{|c|^2}{1-|p|^2}\right)\frac{|c|^2}{(1-|p|^2)^2} - \alpha\frac{4|c|^2}{1-|p|^2} - (\alpha - Q - M)^2 \\
&= 4QM - 4\alpha\sqrt{M}|c| - (\alpha - Q - M)^2.
\end{aligned}$$

Thus

$$\left(\frac{d\log|c|}{dt}\right)^2 = -4\alpha\sqrt{M}|c| + 4QM - (\alpha - M - Q)^2.$$

Since $0 \le |c| \le 1$, it follows that $c_{\alpha,M,Q} \le \left(\frac{d\log|c|}{dt}\right)^2 \le C_{\alpha,M,Q}$, which leads to exponential decay in time for $|c|$:

$$|c|(t) \simeq |c(0)|e^{-C|t|}$$

with the positive constant C depending on α and M, Q.

Notice that $\hat{u}(k,t) = cp^{k-1}$ for $k \ge 1$. Using Fourier expansion, we obtain, as $|p|$ approaches 1,

$$\|u\|_{H^s}^2 \simeq \frac{|c|^2}{(1-|p|^2)^{2s+1}}.$$

Since $M(u) = |c|^2/(1-|p|^2)^2 =$ constant, we get $\|u\|_{H^s}^2 \simeq |c|^{-(2s-1)} \simeq e^{C(2s-1)|t|}$, which has an exponential growth as $s > \frac{1}{2}$. This completes the proof. □

Corollary 3.4. *We do not have the growth of H^s norms for small data in $\mathscr{L}(1)$. In other words, if $\|u(0)\|_{H_+^{1/2}} \ll \sqrt{\alpha}$, the higher Sobolev norm will never grow to infinity.*

Proof. $\|u(0)\|_{H_+^{1/2}} \ll \sqrt{\alpha}$. Then

$$\left|b + \frac{c\bar{p}}{1-|p|^2}\right| \le \sqrt{Q} + \sqrt{M} \lesssim \|u(0)\|_{H_+^{1/2}} \ll \sqrt{\alpha}.$$

According to the necessary and sufficient condition (3-2), there is no norm explosion. □

Remark 3.5. Consider a family of Cauchy data given by

$$u_0^\varepsilon = z + \varepsilon, \quad \varepsilon \in \mathbb{C} \text{ and } \varepsilon \ne \sqrt{\alpha}.$$

For the case $\alpha = 0$, Gérard and Grellier got the following instability of H^s norms:

$$\|u^\varepsilon(t^\varepsilon)\|_{H^s} \simeq \varepsilon^{-(2s-1)}, \quad s > \tfrac{1}{2}.$$

However, we do not have such an instability result for $\alpha > 0$. In fact, using Theorem 3.3, we know there exists a constant $C = C(\alpha)$ such that

$$\sup_{\varepsilon \ne \sqrt{\alpha}} \sup_{t\in\mathbb{R}} \|u^\varepsilon(t)\|_{H^s} < C.$$

Now we give an example to display the energy cascade in Theorem 3.3.

Theorem 3.6. *Given $\alpha > 0$,*

$$\begin{cases} i\partial_t u = \Pi(|u|^2u) + \alpha(u|1), & \\ u|_{t=0} = z + \sqrt{\alpha}, & z \in \mathbb{S}^1. \end{cases} \tag{3-7}$$

For all $s > \frac{1}{2}$, the above equation is globally well posed in H^s and the solution satisfies

$$\|u(t)\|_{H^s} \simeq e^{(2s-1)\sqrt{\alpha}t}, \quad t \to \infty.$$

Proof. Since $u_0 = z + \sqrt{\alpha}$, the conserved quantities are $Q = 1+\alpha$, $M = 1$, $E_\alpha = \frac{1}{4}(1+\alpha)(1+3\alpha)$. Thus $u_0 \in \mathcal{L}(1)$. So, by the proof of Theorem 3.3,

$$\left(\frac{d}{dt}|c|\right)^2 = 4\alpha|c|^2(1-|c|).$$

Together with the initial condition $|c|(0) = 1$, we get, for $t > 0$ (same strategy for $t < 0$),

$$\frac{d}{dt}|c| = -2\sqrt{\alpha}|c|\sqrt{1-|c|}, \tag{3-8}$$

and then

$$|c|(t) = \frac{4e^{2\sqrt{\alpha}t}}{(1+e^{2\sqrt{\alpha}t})^2}.$$

By (3-2), we get $\operatorname{Re}(bp\bar{c}) = |c|^2 - |c|$, and, by (3-6) and (3-8), we have $\operatorname{Im}(bp\bar{c}) = -\sqrt{\alpha}|c|\sqrt{1-|c|}$, so

$$bp\bar{c} = \operatorname{Re}(bp\bar{c}) + i\operatorname{Im}(bp\bar{c}) = |c|^2 - |c| - i\sqrt{\alpha}|c|\sqrt{1-|c|}.$$

The second equation of (3-5) can be simplified as follows:

$$\begin{cases} i\partial_t c = (1+2\alpha - 2i\sqrt{\alpha}\sqrt{1-|c|})c, \\ c(0) = 1. \end{cases}$$

Thus

$$c(t) = \frac{4e^{2\sqrt{\alpha}t}}{(1+e^{2\sqrt{\alpha}t})^2} e^{-i(1+2\alpha)t}. \tag{3-9}$$

Now we turn to calculating b and p. In fact, we only need to calculate their angles. Let us denote

$$b = |b|e^{i\theta(t)} = \sqrt{1+\alpha-|c|}e^{i\theta(t)}, \quad p = |p|e^{i\sigma(t)} = \sqrt{1-|c|}e^{i\sigma(t)}.$$

Then, using the differential equation on p, we get

$$\partial_t\sigma|p| = |c||p| + \operatorname{Re}(c\bar{b}e^{-i\sigma}) = |c||p| + \operatorname{Re}\left(\frac{c\bar{b}\bar{p}}{|p|}\right) = |c||p| + \frac{1}{|p|}(|c|^2 - |c|) = 0,$$

which means

$$\sigma(t) = \sigma(0).$$

Since

$$\begin{aligned} bp = \frac{c(bp\bar{c})}{|c|^2} &= (|c| - 1 - i\sqrt{\alpha}\sqrt{1-|c|})e^{-i(1+2\alpha)t} \\ &= \sqrt{(1+\alpha-|c|)(1-|c|)}\left(-\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} - i\frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}}\right)e^{-i(1+2\alpha)t}, \end{aligned}$$

$$e^{i(\theta+\sigma)} = \left(-\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} - i\frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}}\right)e^{-i(1+2\alpha)t},$$

and $e^{i\theta(0)} = 1$, we get

$$e^{i\sigma(t)} = e^{i\sigma(0)} = e^{i(\sigma(0)+\theta(0))} = -i.$$

Then

$$e^{i\theta(t)} = \left(-i\,\frac{\sqrt{1-|c|}}{\sqrt{1+\alpha-|c|}} + \frac{\sqrt{\alpha}}{\sqrt{1+\alpha-|c|}}\right)e^{-i(1+2\alpha)t}.$$

Finally, we have

$$\begin{aligned} p(t) &= -i\sqrt{1-|c|} = -i\,\frac{e^{2\sqrt{\alpha}t}-1}{e^{2\sqrt{\alpha}t}+1},\\ b(t) &= \left(\sqrt{\alpha} - i\,\frac{e^{2\sqrt{\alpha}t}-1}{e^{2\sqrt{\alpha}t}+1}\right)e^{-i(1+2\alpha)t}. \end{aligned} \tag{3-10}$$

Now we get the explicit formula for the solution $u(t) = b(t) + c(t)z/(1-p(t)z)$:

$$\begin{cases} b(t) = \left(\sqrt{\alpha} - i\,\dfrac{e^{2\sqrt{\alpha}t}-1}{e^{2\sqrt{\alpha}t}+1}\right)e^{-i(1+2\alpha)t},\\ c(t) = \dfrac{4e^{2\sqrt{\alpha}t}}{(1+e^{2\sqrt{\alpha}t})^2}e^{-i(1+2\alpha)t},\\ p(t) = -i\,\dfrac{e^{2\sqrt{\alpha}t}-1}{e^{2\sqrt{\alpha}t}+1}. \end{cases} \tag{3-11}$$

In this case, $M(u) = |c|^2/(1-|p|^2)^2 = 1$ and we get, for $t \to +\infty$,

$$\|u(t)\|_{H^s}^2 \simeq |c|^{-(2s-1)} \simeq Ce^{2(2s-1)\sqrt{\alpha}t}. \qquad \square$$

Remark 3.7. One can illustrate this instability of Sobolev norms from the viewpoint of transfer of energy to high frequencies. The Fourier coefficients for $u = b + cz/(1-pz)$ are

$$\hat{u}(k) = c(t)p(t)^{k-1} \quad \text{for all } k \geq 1.$$

Then

$$M(u) = 1 = \sum_{k\geq 1} |k|\,|\hat{u}(k)|^2 = \sum_{k\geq 1} |k|\,|c(t)|^2|p(t)|^{2(k-1)}.$$

With (3-11), we have

$$\sum_{k\geq 1}\left|\frac{1-e^{-2\sqrt{\alpha}t}}{1+e^{-2\sqrt{\alpha}t}}\right|^{2k} \frac{16|k|}{|(1+e^{-2\sqrt{\alpha}t})(1-e^{-2\sqrt{\alpha}t})|^2} = 1.$$

As $t \to \infty$, we get

$$\sum_{k\geq 1} 4|k|e^{-2\sqrt{\alpha}t}\exp\left(-4|k|e^{-2\sqrt{\alpha}t}\right) \sim \tfrac{1}{4},$$

so the main part of the summation is on the ks satisfying

$$|k| \sim e^{2\sqrt{\alpha}t}.$$

So as time increases, the main part of the energy concentrates on the Fourier modes as large as $e^{2\sqrt{\alpha}t}$.

On the other hand, from the viewpoint of the space variable, we find that as time grows to infinity, the energy will concentrate on one point. In fact, rewriting $z = e^{ix}$, we get

$$\begin{aligned}\left| u(t,x) - \sqrt{\alpha} - i\,\frac{1-e^{-2\sqrt{\alpha}t}}{1+e^{-2\sqrt{\alpha}t}} \right| &= \frac{|c(t)|}{|1-p(t)z|} = \frac{1-|p(t)|^2}{|1-p(t)z|} \sim \frac{1-|p(t)|}{|1-p(t)z|} \\ &\sim \frac{1}{\sqrt{2(e^{4\sqrt{\alpha}t}-1)(1-\sin x)+4}},\end{aligned}$$

which tends to 0 as $t \to \infty$ if and only if $x \neq \pi/2$. Therefore, as time tends to infinity, the value of $|u|$ will concentrate on the point $i \in \mathbb{S}^1$.

This example shows that the radius of analyticity of the solution of (1-6) may decay exponentially. This shows the optimality of the result in [Gérard et al. 2013].

Now, let us turn to the case $\alpha < 0$.

Theorem 3.8. *In the case $\alpha < 0$, for any given initial data $u_0 \in \mathcal{L}(1)$, let $u = (az+b)/(1-pz)$ be the corresponding solution of (1-6). Then there exists a constant $C = C(\alpha)$ such that, for all t,*

$$\|u(t)\|_{H^s} < C, \quad s \geq \tfrac{1}{2},$$

where the constant $C > 0$ is uniform for u_0 in a compact subset of $\mathcal{L}(1)$.

Proof. Assume for a contradiction that $u(t_n)$ leaves any compact subset of $\mathcal{L}(1)$. Then Theorem 3.3 leads to (3-3), or equivalently to the equality

$$\|u_0\|_{L^2}^4 - \|u_0\|_{L^4}^4 = 2\alpha(|(u_0\,|\,1)|^2 - \|u_0\|_{L^2}^2).$$

Via the Cauchy–Schwarz inequality and $\alpha < 0$, we get

$$\|u_0\|_{L^2} = \|u\|_{L^4} \quad \text{and} \quad |(u_0\,|\,1)| = \|u_0\|_{L^2}.$$

Then u_0 is a constant, which contradicts the fact that $u_0 \in \mathcal{L}(1)$. □

4. Further studies and open problems

In this paper, we just considered the data on the (complex) three-dimensional manifold

$$\mathcal{L}(1) := \{u : \operatorname{rk} K_u = 1\}.$$

It is of course natural to consider the higher-dimensional case, which will probably be much more complicated. Since we also have enough conservation laws for the case $\operatorname{rk} K_u = 2$, we have a conjecture that the system stays completely integrable for $\operatorname{rk} K_u \geq 2$. It would be interesting to know how the results of this paper extend to this bigger phase space. In particular, do small data generate large-time blowup of high Sobolev norms?

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HAIYAN XU: haiyan.xu@math.u-psud.fr
Laboratoire de Mathématique d'Orsay, Université Paris-Sud (XI), 91405 Paris Orsay, France

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