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THE GLOBAL STABILITY OF THE MINKOWSKI SPACETIME SOLUTION TO THE EINSTEIN-NONLINEAR SYSTEM IN WAVE COORDINATES

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We study the coupling of the Einstein field equations of general relativity to a family of nonlinear electromagnetic field equations. The family comprises all covariant electromagnetic models that satisfy the following criteria: (i) they are derivable from a sufficiently regular Lagrangian; (ii) they reduce to the standard Maxwell model in the weak-field limit; (iii) their corresponding energy-momentum tensors satisfy the dominant energy condition. Our main result is a proof of the global nonlinear stability of the $(1+3)$-dimensional Minkowski spacetime solution to the coupled system for any member of the family, which includes the standard Maxwell model. This stability result is a consequence of a small-data global existence result for a reduced system of equations that is equivalent to the original system in our wave-coordinate gauge. Our analysis of the spacetime metric components is based on a framework recently developed by Lindblad and Rodnianski, which allows us to derive suitable estimates for tensorial systems of quasilinear wave equations with nonlinearities that satisfy the weak null condition. Our analysis of the electromagnetic fields, which satisfy quasilinear first-order equations that have a special null structure, is based on an extension of a geometric energy-method framework developed by Christodoulou together with a collection of pointwise decay estimates for the Faraday tensor developed in the article. We work directly with the electromagnetic fields and thus avoid the use of electromagnetic potentials.

1. Introduction

The Einstein field equations are the fundamental equations of general relativity. They connect the *Einstein tensor* $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, which contains information about the curvature of spacetime[1] $(\mathfrak{M}, g_{\mu\nu})$, to the energy-momentum-stress-density tensor (energy-momentum tensor for short) $T_{\mu\nu}$, which contains information about the matter present in $\mathfrak{M}$. Here, $g_{\mu\nu}$ is the *spacetime metric*, $R_{\mu\nu}$ is the *Ricci curvature tensor* of $g_{\mu\nu}$, and $R = (g^{-1})^{\kappa\lambda} R_{\kappa\lambda}$ is the *scalar curvature* of $g_{\mu\nu}$. In this article, we show the stability of

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[1] By spacetime, we mean a four-dimensional time-orientable Lorentzian manifold $\mathfrak{M}$ together with a Lorentzian metric $g_{\mu\nu}$ of signature $(-,+,+,+)$.

the $(1+3)$-dimensional Minkowski spacetime solution of the Einstein-nonlinear electromagnetic system, which takes the following form relative to an arbitrary coordinate system:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.0.1a}$$

$$(d\mathscr{F})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{1.0.1b}$$

$$(d\mathscr{M})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{1.0.1c}$$

Above, $T_{\mu\nu}$ (see (3.5.4a)) is one of the energy-momentum tensors corresponding to a family of nonlinear models of electromagnetism, d denotes the exterior derivative operator, the two-form $\mathscr{F}_{\mu\nu}$ denotes the *Faraday tensor*, the two-form $\mathscr{M}_{\mu\nu}$ denotes the *Maxwell tensor*, and $\mathscr{M}_{\mu\nu}$ is connected to $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ through a constitutive relation (see (3.2.4)). We make the following three assumptions concerning the electromagnetic matter model:

(1) Its Lagrangian ${}^{\star}\mathscr{L}$ is a scalar-valued function of the two electromagnetic invariants[2]

$$\digamma_{(1)} \overset{\text{def}}{=} \tfrac{1}{2} (g^{-1})^{\kappa\mu} (g^{-1})^{\lambda\nu} \mathscr{F}_{\kappa\lambda} \mathscr{F}_{\mu\nu} \quad \text{and} \quad \digamma_{(2)} \overset{\text{def}}{=} \tfrac{1}{4} (g^{-1})^{\kappa\mu} (g^{-1})^{\lambda\nu} \mathscr{F}_{\kappa\lambda} {}^{\star}\mathscr{F}_{\mu\nu},$$

where $\star$ denotes the Hodge duality operator corresponding to $g_{\mu\nu}$.

(2) The energy-momentum tensor $T_{\mu\nu}$ corresponding to ${}^{\star}\mathscr{L}$ satisfies the *dominant energy condition* (sufficient conditions on ${}^{\star}\mathscr{L}$ are given in (3.3.4a)–(3.3.4b) below).

(3) ${}^{\star}\mathscr{L}$ is a sufficiently differentiable function of $(\digamma_{(1)}, \digamma_{(2)})$, and its Taylor expansion around (0, 0) agrees with that of the linear[3] Maxwell–Maxwell[4] equations to first order; i.e., ${}^{\star}\mathscr{L}(\digamma_{(1)}, \digamma_{(2)}) = -\tfrac{1}{2}\digamma_{(1)} + O^{\ell+2}(|(\digamma_{(1)}, \digamma_{(2)})|^2)$, where $\ell \geq 10$ is an integer; see Section 2.13 regarding the notation $O^{\ell+2}(\,\cdot\,)$.

The fundamental results in [Fourès-Bruhat 1952; Choquet-Bruhat and Geroch 1969] together imply that the system (1.0.1a)–(1.0.1c) has an initial-value problem formulation in which suitably regular initial data launch a unique maximal globally hyperbolic development. Roughly speaking, the maximal globally hyperbolic development, which is uniquely determined up to isomorphism, is the largest possible solution to the equations that is uniquely determined by the data. However, the results cited are abstract in the sense that they do not provide any detailed quantitative information about the global structure of the maximal globally hyperbolic development. In particular, the results do not address the question of whether the resulting spacetime $(\mathfrak{M}, g_{\mu\nu})$ is geodesically complete. The main goal of this article is to provide a detailed qualitative and quantitative description of the global structure of maximal globally hyperbolic developments launched by data near that of the most fundamental solution to (1.0.1a)–(1.0.1c): the vacuum Minkowski spacetime. We briefly summarize our main results here. They are rigorously stated and proved in Section 16.

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[2]Throughout the article, we use Einstein's summation convention in that repeated indices are summed over.

[3]By "linear", we mean that the familiar electromagnetic equations of Maxwell are linear on any *fixed* spacetime background $(\mathscr{M}, g_{\mu\nu})$; the coupled Einstein–Maxwell system is highly nonlinear.

[4]Throughout the article, we use the terminology "Maxwell–Maxwell" equations in place of the more common terminology "Maxwell" equations. The justification is that Maxwell's theory is based on the electromagnetic equations (1.0.1b)–(1.0.1c) *and* the constitutive relation $\mathscr{M} = {}^{\star}\mathscr{F}$; in a general covariant nonlinear electromagnetic theory, such as the ones considered in this article, the equations (1.0.1b)–(1.0.1c) survive while the constitutive relation differs from that of Maxwell.

Main results. The vacuum Minkowski spacetime solution $\widetilde{g}_{\mu\nu} \stackrel{\text{def}}{=} \mathrm{diag}(-1, 1, 1, 1)$ and $\widetilde{\mathscr{F}}_{\mu\nu} \stackrel{\text{def}}{=} 0$ $(\mu, \nu = 0, 1, 2, 3)$ to the system (1.0.1a)–(1.0.1c) is globally stable. In particular, small perturbations of the trivial initial data corresponding to $(\widetilde{g}_{\mu\nu}, \widetilde{\mathscr{F}}_{\mu\nu})$ have maximal globally hyperbolic developments that are geodesically complete. Furthermore, the perturbed solution converges to the Minkowski spacetime solution as the evolution progresses. These conclusions are consequences of a small-data global existence result plus decay estimates for solutions to the *reduced* system (3.7.1a)–(3.7.1c) under the wave-coordinate gauge condition $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu} = 0$ $(\mu = 0, 1, 2, 3)$, where $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu}$ is a contracted Christoffel symbol of $g_{\mu\nu}$. Furthermore, relative to the wave-coordinate system that we construct (i.e., a coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$ such that $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu} = 0$ $(\mu = 0, 1, 2, 3)$), the system (1.0.1a)–(1.0.1c) is equivalent to the reduced system.

We recall the following standard facts (see, e.g., [Christodoulou 2008; Wald 1984]) concerning the initial data for the system (1.0.1a)–(1.0.1c), which we refer to as "abstract" initial data. The abstract initial data consist of a three-dimensional manifold Σ_0 together with the following fields on Σ_0: a Riemannian metric $\mathring{\underline{g}}_{jk}$, a symmetric type-$\binom{0}{2}$ tensor field $\mathring{K}_{jk}$, and a pair of electromagnetic one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ $(j, k = 1, 2, 3)$. Furthermore, viable data must satisfy the *Gauss*, *Codazzi*, and *electromagnetic constraint* equations, which are respectively given by

$$\mathring{\underline{R}} - \mathring{K}_{ab}\mathring{K}^{ab} + [(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}]^2 = 2T(\widehat{N}, \widehat{N})|_{\Sigma_0}, \tag{1.0.2a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{K}_{bj} - (\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_j\mathring{K}_{ab} = T\Big(\widehat{N}, \frac{\partial}{\partial x^j}\Big)\Big|_{\Sigma_0} \quad (j = 1, 2, 3), \tag{1.0.2b}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{D}}_b = 0, \tag{1.0.3a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{B}}_b = 0. \tag{1.0.3b}$$

In the above expressions, the indices are lowered and raised with $\mathring{\underline{g}}_{jk}$ and $(\mathring{\underline{g}}^{-1})^{jk}$, $\mathring{\underline{R}}$ denotes the scalar curvature of $\mathring{\underline{g}}_{jk}$, $\mathring{\underline{\mathscr{D}}}$ denotes the Levi-Civita connection corresponding to $\mathring{\underline{g}}_{jk}$, and $\widehat{N}^\mu$ is the future-directed unit g-normal to Σ_0 (viewed as an embedded Riemannian submanifold of $(\mathfrak{M}, g_{\mu\nu})$). The one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ together form a geometric decomposition of $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$, and the right-hand sides of (1.0.2a)–(1.0.2b) can be computed (in principle) in terms of $\mathring{\underline{g}}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_j$ alone; see Section 9.2 for more details concerning the relationship of $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ to $\mathscr{F}_{\mu\nu}|_{\Sigma_0}$. The dominant energy condition manifests itself along Σ_0 as the inequalities $T(\widehat{N}, \widehat{N}) \geq 0$ and $T(\widehat{N}, \widehat{N})^2 - (\mathring{\underline{g}}^{-1})^{ab}T(\widehat{N}, \partial/\partial x^a)T(\widehat{N}, \partial/\partial x^b) \geq 0$.

In this article, we consider the case $\Sigma_0 = \mathbb{R}^3$. We will construct spacetimes of the form $\mathfrak{M} = I \times \mathbb{R}^3$, where I is a time interval and Σ_0 is a spacelike Cauchy hypersurface in $(\mathfrak{M}, g_{\mu\nu})$. The constraints (1.0.2a)–(1.0.2b) are necessary to ensure that (1.0.1a) can be satisfied along Σ_0 while the constraints (1.0.3a)–(1.0.3b) are necessary to ensure that the electromagnetic equations (1.0.1b)–(1.0.1c) can be satisfied along Σ_0. Our stability criteria for the abstract initial data include both decay assumptions at spatial infinity and smallness assumptions. We provide here a description of our decay assumptions at spatial infinity, which are based on the assumptions of [Lindblad and Rodnianski 2010]. Our smallness assumptions will be discussed in detail in Section 10.

Assumptions on the abstract initial data. We assume that there exists a global coordinate chart $x = (x^1, x^2, x^3)$ on $\Sigma_0 = \mathbb{R}^3$, a real number $\kappa > 0$, and an integer $\ell \geq 10$ such that (with $r \stackrel{\text{def}}{=} |x| \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and $j, k = 1, 2, 3$)

$$\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}, \tag{1.0.4a}$$

$$\mathring{\underline{h}}^{(0)}_{jk} = \chi(r)\frac{2M}{r}\delta_{jk}, \tag{1.0.4b}$$

$$\mathring{\underline{h}}^{(1)}_{jk} = o^{\ell+1}(r^{-1-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4c}$$

$$\mathring{K}_{jk} = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4d}$$

$$\mathring{\mathfrak{D}}_j = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4e}$$

$$\mathring{\mathfrak{B}}_j = o^{\ell}(r^{-2-\kappa}) \qquad \text{as } r \to \infty, \tag{1.0.4f}$$

where the meaning of $o^{\ell}(\,\cdot\,)$ is described in Section 2.13. The cut-off function $\chi(\,\cdot\,)$ in (1.0.4b) is defined in (4.2.1).

The parameter M in (1.0.4a), which is known as the *ADM mass*, is constrained by the following requirements: according to the *positive mass theorem* of Schoen and Yau [1979; 1981] and Witten [1981], under the assumption that $T_{\mu\nu}$ satisfies the dominant energy condition, the only solutions $\mathring{\underline{g}}_{jk}$ and $\mathring{K}_{jk}$ to the constraint equations (1.0.2a)–(1.0.2b) that have an expansion of the form (1.0.4a) with the asymptotic behavior (1.0.4b)–(1.0.4d) either have (i) $M > 0$ or (ii) $M = 0$, in which case the Riemannian manifold $(\Sigma_0, \mathring{\underline{g}}_{jk})$ embeds isometrically into Minkowski spacetime with second fundamental form $\mathring{K}_{jk}$. The groundbreaking work of Christodoulou and Klainerman [1993] (which is discussed further in Section 1.1.1) demonstrated the stability of the Minkowski spacetime solution to the Einstein-vacuum equations in the case that the initial data are *strongly asymptotically flat*, which corresponds to the parameter range $\kappa \geq \frac{1}{2}$ in the above expansions. Our work here, which relies on the alternate framework developed by Lindblad and Rodnianski [2010] (see Section 1.1.1), allows for the parameter range $\kappa > 0$.

In this article, we do not consider the issue of solving the constraint equations. The standard method for solving the constraint equations is called the conformal method. For a detailed discussion of this method, see, e.g., [Choquet-Bruhat and York 1980]. Roughly speaking, in this approach, part of the data can be specified freely, and the constraint equations imply nonlinear elliptic PDEs for the remaining part. To the best of our knowledge, under the restrictions on ${}^{\star}\mathscr{L}$ described at the beginning of Section 1, there are presently no rigorous results concerning the construction of initial data on the manifold $\mathbb{R}^3$ that satisfy the constraints. However, we remark that, for the Einstein-vacuum equations $T_{\mu\nu} \equiv 0$, initial data that satisfy the constraints and that coincide with the standard Schwarzschild data (written here relative to isotropic coordinates)

$$\mathring{\underline{g}}_{jk} = \left(1 + \frac{M}{2r}\right)^4 \delta_{jk} \quad (j, k = 1, 2, 3), \tag{1.0.5a}$$

$$\mathring{K}_{jk} = 0 \qquad (j, k = 1, 2, 3) \tag{1.0.5b}$$

outside of the unit ball centered at the origin were shown to exist in [Chruściel and Delay 2002a; 2002b; Corvino 2000]. We remark that the stability of the Minkowski spacetime solution to the Einstein-vacuum equations for such data follows from the methods of the aforementioned works [Christodoulou and Klainerman 1993], [Lindblad and Rodnianski 2010] (and its precursor [2005]), and also from the *conformal method* approach of Friedrich [1986] (this is not the same conformal method that was mentioned above in connection with the constraint equations).

Remark 1.1. The only role of the dominant energy condition in this article is to ensure the physical condition $M \geq 0$; we assume this physical condition throughout the article. However, only the smallness of $|M|$ is needed to prove our global stability result; the sign of M does not enter into our stability analysis for solutions to the evolution equations. In particular, if there existed small initial data with small negative ADM mass M, we would still be able to prove that the corresponding solution to the evolution equations exists globally. Similarly, if we made the replacement $T_{\mu\nu} \to -T_{\mu\nu}$ in the reduced equations (3.7.1a)–(3.7.1c), we could still prove a small-data global existence result.

1.1. *Comparison with previous work.*

1.1.1. *Mathematical comparisons.* Our result is an extension of a large and growing hierarchy of global stability results for the $(1+3)$-dimensional Minkowski spacetime solution to the Einstein equations. The hierarchy began with the celebrated work of Christodoulou and Klainerman [1993], who proved stability in the case of the Einstein-vacuum equations (i.e., $T_{\mu\nu} \equiv 0$). Klainerman and Nicolò [2003] gave a second proof of this result using alternate (but related) techniques. Both of these proofs used a manifestly covariant framework for the formulation of the equations and the derivation of estimates. However, mathematically speaking, the closest relatives to the present article are the seminal works [2005; 2010], in which Lindblad and Rodnianski developed a technically simpler framework for showing the stability of the Minkowski spacetime solution of the Einstein-scalar field system using a *wave-coordinate* gauge. As we previously mentioned, a wave-coordinate gauge is a coordinate system in which the contracted Christoffel symbols $(g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu}$ completely vanish. Relative to such a coordinate system, the Einstein-vacuum equations are equivalent to a *reduced* system comprising quasilinear wave equations for the components $g_{\mu\nu}$; in the present article, the analogous equation is (3.7.1a). In her celebrated result [1952], Choquet-Bruhat used wave coordinates to prove local well-posedness for the Einstein equations. However, because of the logarithmic divergences discussed below in Section 1.2.4 and because of the delicate nonlinearities in the Einstein equations, it was unexpected (see, e.g., [Choquet-Bruhat 1973]) that the wave-coordinate approach of [Lindblad and Rodnianski 2005; 2010] for proving the global stability of Minkowski spacetime is in fact viable. We remark that although the decay estimates of [Lindblad and Rodnianski 2005; 2010] are not as precise as those of [Christodoulou and Klainerman 1993; Klainerman and Nicolò 2003], these works are *much* shorter than their predecessors yet are robust enough to allow for modifications, including the presence of the nonlinear electromagnetic fields examined in this article. We also remark that many of the technical results we need are contained in [Lindblad and Rodnianski 2005; 2010], and we will often direct the reader to these works for their proofs.

Other stability results in this vein include [2000], in which Zipser extended the framework of [Christodoulou and Klainerman 1993] to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system, and [2007], in which Bieri weakened the assumptions of [Christodoulou and Klainerman 1993] on the decay of the initial data at spatial infinity. We also mention the work [2008] (see also [2006; 2009]), in which Loizelet used the framework of [Lindblad and Rodnianski 2005; 2010] to demonstrate the stability of the Minkowski spacetime solution of the Einstein-scalar field-Maxwell system in $1+n$ ($n \geq 3$) dimensions. Moreover, in spacetimes of dimension $1+n$, with $n \geq 5$ odd, it has been shown [Choquet-Bruhat et al. 2006] that a conformal method (distinct from the one used by Friedrich) can be used to show the stability of the Minkowski spacetime solution to the Einstein–Maxwell system for initial data that coincide with the standard Schwarzschild data outside of a compact set. Roughly speaking, a conformal method is a way of mapping a global existence problem into a local existence problem by working with rescaled solution variables. When a conformal method is viable, it tends to give very precise information concerning the asymptotics of the global solutions. In particular, the results of [Choquet-Bruhat et al. 2006] provide a more detailed description of the asymptotics than the results of [Loizelet 2008].

We now compare the amount of regularity and decay that we require on the data to the amount required in the alternate frameworks. The Christodoulou and Klainerman [1993], Zipser [2000], and Klainerman and Nicolò [2003] proofs required two derivatives on the curvature (i.e., four derivatives on the metric). Furthermore, the initial metric was required to be strongly asymptotically flat in the sense described above. Zipser's proof required (in addition) three derivatives on the Faraday tensor. Bieri's [2007] proof required only one derivative on the curvature (i.e., three derivatives on the metric), and it allowed for very slow decay of the data at spatial infinity: $\mathring{\underline{g}}_{jk} = \delta_{jk} + o^3(r^{-1/2})$ and $\mathring{K}_{jk} = o^2(r^{-3/2})$. The present article is less efficient: we require 11 derivatives on the metric and 10 derivatives on the Faraday tensor. We also require asymptotic flatness in the sense of (1.0.4a)–(1.0.4f), which is in between the decay required by Christodoulou and Klainerman and Bieri. Our assumptions are similar to the ones made by Lindblad and Rodnianski [2010] and Loizelet [2008]. For example, in $n \geq 3$ spatial dimensions, Loizelet's proof required $7 + 2\lfloor (n+2)/2 \rfloor$ derivatives of the metric. The main focus of the Lindblad–Rodnianski wave-coordinate approach is on providing a technically simpler approach to the proof of stability as opposed to a proof that closes at a low regularity level. There are at least two ways in which the wave-coordinate approach is suboptimal from the point of view of the number of derivatives. The first is that all product nonlinearities are estimated in L^2 on constant-time hypersurfaces from only $L^2 - L^\infty$ estimates with no use of intermediate L^p norms, norms on other hypersurfaces,[5] or Calderón–Zygmund theory. That is, all nonlinear products are estimated in spatial L^2 by bounding the factor with the most derivatives on it in L^2 and all other factors in L^∞. For quadratic terms, this means that we must be able to bound approximately half of the total number of derivatives in L^∞. This approach stands in contrast to the approaches of [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007], where, e.g., intermediate L^p norms and other hypersurface integrals played an important role in the analysis. The second source of suboptimality comes from the version of the weighted Klainerman–Sobolev inequality that we use (see Section 1.2.7 and (1.2.10)). This

[5] As is explained in Section 1.2.6, our proof of global stability also makes use of the positivity of certain time integrals of the L^2 integrals (i.e., positive spacetime integrals) that arise in our energy identities.

inequality allows one to estimate a weighted L^∞ norm of a function by weighted L^2 norms of *up-to-order-three* weighted derivatives. The reason that three derivatives are used (instead of the familiar two derivatives of standard Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$) is that this allows one to avoid putting more than one derivative on the weight function (the at-most-two other derivatives are rotations, which pass through the weight function); see the proof given in [Lindblad and Rodnianski 2010, Proposition 14.1].

We also emphasize the following point: the techniques used in this article to analyze the electromagnetic fields differ in a fundamental way from those used by Loizelet [2008]. Our methods are closer in spirit to (though distinct from) the methods used by Zipser [2000]. More specifically, Loizelet [2008] analyzed the standard Maxwell–Maxwell equations through the use of a four-potential[6] A_μ satisfying the *Lorenz gauge* condition $(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa A_\lambda = 0$, where $\mathscr{D}$ is the Levi-Civita connection corresponding to $g_{\mu\nu}$. In Loizelet's analysis of the Maxwell–Maxwell equations, the Lorenz gauge leads to a diagonal system of semilinear-in-A_μ wave equations for the components A_μ. Furthermore, these equations can be analyzed by using the same techniques that are used in the study of the components of the metric (see (3.7.1a)) and the scalar field. In particular, in Loizelet's analysis, Lemma 12.2 can be used to deduce suitable weighted energy estimates for the components $\nabla_\mu A_\nu$. In contrast, as discussed in [Speck 2012], it is not clear that the Lorenz gauge can be used to analyze the kinds of quasilinear-in-$\mathscr{F}$ electromagnetic field equations (1.0.1c) studied in this article. More specifically, it is not clear that the Lorenz gauge in general leads to a hyperbolic formulation of the electromagnetic equations that is suitable for deriving the kinds of L^2 energy estimates needed for our analysis. For this reason, throughout this article, we work directly with the Faraday tensor. In particular, as described in detail in Section 8, we use Christodoulou's [2000] geometric framework to construct *energy currents* that can be used to derive the kinds of L^2 estimates needed in our analysis. Using these methods, we prove Lemma 12.1, which compensates for the fact that Lemma 12.2 is not generally available for controlling the electromagnetic quantities. We remark that there is another advantage to working directly with the Faraday tensor: *our smallness condition for stability depends only on the physical field variables and not on auxiliary mathematical quantities such as the components* $\nabla_\mu A_\nu$.

Now roughly speaking, the reason that we are able to prove our main stability result is because, in our wave-coordinate gauge, the nonlinear terms have a special algebraic structure, which Lindblad and Rodnianski [2003] have labeled *the weak null condition*; see Section 1.2.5 for additional details. We remark that, in order for small-data global existence to hold, it is essential that the quadratic nonlinearities have special structure: John's [1981] blow-up result shows that quadratic perturbations[7] of the homogeneous linear wave equation in $(1+3)$-dimensional Minkowski spacetime (of which our equations (1.2.4a) below are an example) *can sometimes lead to finite-time blow-up even for arbitrarily small data*. Now by definition, a system of PDEs satisfies the weak null condition if the corresponding *asymptotic system* has small-data global solutions. Roughly, the asymptotic system is obtained by keeping only the quadratic terms with both factors involving derivatives that are transversal to the outgoing Minkowskian null cones and the related linear term that drives their evolution along those cones (see the discussion in Section 1.2.4); the discarded terms are expected to decay faster than the remaining terms. The general philosophy is that,

[6]Recall that a four-potential is a one-form A_μ such that $\mathscr{F}_{\mu\nu} = (dA)_{\mu\nu}$.

[7]In [John 1981], it was shown that both semilinear and quasilinear quadratic perturbations can lead to small-data blow-up.

if the asymptotic system has small-data global existence, then one should be hopeful that the original system does too. Lindblad and Rodnianski [2010] showed that the asymptotic system corresponding to the Einstein-scalar field system in wave coordinates has global solutions for small (i.e., near-Minkowskian) data. Although we do not carry out such an analysis in this article, we remark that it can be checked that the asymptotic system[8] corresponding to the Einstein-nonlinear electromagnetic system in wave coordinates also has global solutions for small data. This was our original motivation for pursuing the present work.

The aforementioned weak null condition is a generalization of the classic *null condition* of Klainerman [1986] (see also [Christodoulou 1986]), in which the quadratic nonlinearities are *standard null forms* (which are defined below in the statement of Lemma 3.8). We remark that standard null forms have a very favorable structure and are completely discarded when one forms the asymptotic system. By now, there is a very large body of global existence and almost-global existence results that are based on the analysis of nonlinearities that satisfy generalizations of Klainerman's null condition. This includes the global stability results for the Einstein equations mentioned above but also many other results; there are far too many to list exhaustively, but we mention the following as examples: [Katayama 2005; Klainerman and Sideris 1996; Lindblad 2004; 2008; Metcalfe and Sogge 2007; Metcalfe et al. 2005; Sideris 1996; Speck 2012].

1.1.2. *Connections to the "divergence" problem.* One of the most important unresolved issues in physics is that of the so-called "divergence problem". In the setting of classical electrodynamics on the Minkowski spacetime background, this problem manifests itself as the unhappy fact that the standard Maxwell–Maxwell equations with *point-charge* sources (i.e., delta-function source terms modeling the point charges) together with the *Lorentz force law*[9] (which is supposed to drive the motion of the point charges) do not form a well-defined system of equations. This is because the theory dictates that the Lorentz force at the location of a point charge is "infinite in all directions" so that the charge's motion is ill-defined. A further symptom of the divergence problem in this theory is that the energy of a static point charge is infinite. Moreover, our present-day flagship model of quantum electrodynamics (QED), which is based on a quantization of the classical Maxwell–Dirac field equations, has not yet fixed the crux of the problem; similar manifestations of the divergence problem arise in QED; see [Kiessling 2004a; 2004b] for a detailed discussion of these issues.

Now in [2004a; 2004b], Kiessling has taken a preliminary step in the direction of resolving the divergence problem by reconsidering classical electrodynamics in Minkowski spacetime. One of Kiessling's primary strategies is to follow the lead of Max Born [1933] by replacing the standard Maxwell–Maxwell equations with a suitable nonlinear system, the hope being that it will be possible to make rigorous mathematical sense of the motion of point charges in the nonlinear theory. Kiessling's leading candidate is the Maxwell–Born–Infeld (MBI) model of classical electromagnetism, which was proposed by Born and Infeld [1934] based on Born's [1933] earlier ideas. The electromagnetic Lagrangian for this model is

$$ {}^{\star}\mathscr{L}_{(\mathrm{MBI})} \stackrel{\mathrm{def}}{=} \frac{1}{\beta^4} - \frac{1}{\beta^4}\big(1+\beta^4 \digamma_{(1)} - \beta^8 \digamma_{(2)}^2\big)^{1/2} = \frac{1}{\beta^4} - \frac{1}{\beta^4}(\det{}_g(g+\beta^2\mathscr{F}))^{1/2}, \tag{1.1.1} $$

[8] To obtain this asymptotic system, one also discards the quadratic terms containing the fast-decaying null components $\alpha[\mathscr{F}]$, $\rho[\mathscr{F}]$, and $\sigma[\mathscr{F}]$ of the Faraday tensor; see Section 1.2.4.

[9] Recall that the Lorentz force is $F_{\mathrm{Lorentz}} = q[E + v \times B]$, where q is the charge associated to the point charge, E is the electric field, v is the instantaneous point charge velocity, and B is the magnetic induction.

where $\beta > 0$ denotes *Born's "aether" constant*. We point out that, as verified in, e.g., [Speck 2012], this Lagrangian satisfies the assumptions (3.3.3a) and (3.3.4a)–(3.3.4b) so that the main results of this article apply to the MBI model. Now it turns out that it was not enough for Kiessling to simply replace the standard Maxwell–Maxwell equations with the Maxwell–Born–Infeld equations, for such a modification fails to fix the problem of the Lorentz force being ill-defined at the location of the point charge. On the other hand, in MBI theory on the Minkowski spacetime background, there exist *Lipschitz-continuous* electromagnetic potentials corresponding to solutions to the field equations with a single static point-charge source. Kiessling observed that this level of regularity is (just barely) sufficient for a relativistic version of Hamilton–Jacobi theory to be well-defined. He thus proposed a new relativistic Hamilton–Jacobi "guiding law" of motion for the point charges (see [Kiessling 2004a] for the details).

Kiessling's interest in the Maxwell–Born–Infeld system was further motivated by results contained in [Boillat 1970; Plebański 1970], which show that it is the unique[10] theory of classical electromagnetism that is derivable from a Lagrangian and that satisfies the following five postulates (see also the discussions in [Białynicki-Birula 1983; Kiessling 2004a]):

(i) The field equations transform covariantly under the Poincaré group.

(ii) The field equations are covariant under a Weyl (gauge) group.

(iii) The electromagnetic energy surrounding a stationary point charge is finite.

(iv) The field equations reduce to the standard Maxwell–Maxwell equations in the weak field limit.

(v) The solutions to the field equations are not birefringent.

We remark that the standard Maxwell–Maxwell system satisfies all of the above postulates except for (iii) and that the MBI system was shown to satisfy (iii) by Born [1933]. Physically, postulate (v) is equivalent to the statement that the "speed of light propagation" is independent of the polarization of the wave fields. Mathematically, this is the postulate that there is only a single *null cone*[11] associated to the electromagnetic equations; in a typical theory of classical electromagnetism, the causal structure of the electromagnetic equations is more complicated than the structure corresponding to a single null cone (see [Speck 2012] for a detailed discussion of this issue in the context of the Maxwell–Born–Infeld equations on the Minkowski spacetime background).

We can now clarify the connection of the present article to Kiessling's work. First, as noted in [2004a], Kiessling expects that his theory can be generalized to the case of a curved spacetime through a coupling to the Einstein equations. Next, we mention that although the Maxwell–Born–Infeld system is Kiessling's leading candidate for an electromagnetic model, he is also considering other models. In particular, by relaxing postulate (v) above, a relaxation that in principle could be supported by experimental evidence, one is led to consider a larger family of electromagnetic models. Now one basic criterion for any viable electromagnetic model is that small, nearly linear-Maxwellian electromagnetic fields in near-Minkowski spacetimes should not lead to a severe breakdown in the structure of spacetime or other degenerate

[10] More precisely, there is a one-parameter family of such theories indexed by $\beta > 0$.

[11] In general, this "light cone" does not have to coincide with the gravitational null cone although it *does* in the case of the standard Maxwell–Maxwell equations.

behavior. *The present work confirms this criterion* for a large family of electromagnetic models coupled to the Einstein equations, including the Maxwell–Born–Infeld system and many other models that fall under the scope of Kiessling's program.

1.2. *Discussion of the analysis.*

1.2.1. *The splitting of the spacetime metric and setting up the equations.* As in [Lindblad and Rodnianski 2005; 2010], in order to analyze the spacetime metric, we split it into the following three pieces (where we view $h_{\mu\nu}^{(1)}$ as the "new unknown metric variable"):

$$g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.2.1a}$$

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.2.1b}$$

$$h_{\mu\nu}^{(0)} \overset{\text{def}}{=} \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.2.1c}$$

$$\left(h_{\mu\nu}^{(0)}\big|_{t=0} = \chi(r)\frac{2M}{r}\delta_{\mu\nu},\ \partial_t h_{\mu\nu}^{(0)}\big|_{t=0} = 0\right),$$

where $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric and the function χ plays several roles that will be discussed in Section 1.2.9. Above and throughout, $\chi(z)$ is a fixed cut-off function that satisfies

$$\chi \in C^\infty, \qquad \chi \equiv 1 \text{ for } z \geq \tfrac{3}{4}, \quad \text{and} \quad \chi \equiv 0 \text{ for } z \leq \tfrac{1}{2}. \tag{1.2.2}$$

We remark that, here and throughout the rest of the article, unless we explicitly indicate otherwise, all indices on all tensors are lowered and raised with the Minkowski metric $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ *and its inverse* $(m^{-1})^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ (*as is explained in Section 2.2, we use the symbol # whenever we raise indices with* g^{-1}). Furthermore, as in [Lindblad and Rodnianski 2005; 2010], we work in a wave-coordinate system, which is a coordinate system in which the contracted Christoffel symbols $\Gamma^{\mu} \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda}\Gamma_{\kappa\ \lambda}^{\ \mu}$ (see (3.0.2d)) of $g_{\mu\nu}$ satisfy

$$\Gamma^{\mu} = 0 \quad (\mu = 0, 1, 2, 3). \tag{1.2.3}$$

We remark that several equivalent definitions of the wave-coordinate gauge (1.2.3) are discussed in Section 3.1 and that the viability of the wave-coordinate gauge for proving local well-posedness for the system (1.0.1a)–(1.0.1c) (which is a rather standard result based on the fundamental ideas of [Fourès-Bruhat 1952]) is discussed in Section 4.3.

As is discussed in detail in Section 3.7, in a wave-coordinate system (t, x), the equations (1.0.1a)–(1.0.1c) are equivalent to the *reduced equations*

$$\widetilde{\Box}_g h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu} - \widetilde{\Box}_g h_{\mu\nu}^{(0)} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.2.4a}$$

$$\nabla_\lambda \mathcal{F}_{\mu\nu} + \nabla_\mu \mathcal{F}_{\nu\lambda} + \nabla_\nu \mathcal{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{1.2.4b}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathcal{F}_{\kappa\lambda} = \mathfrak{F}^{\nu} \qquad (\nu = 0, 1, 2, 3), \tag{1.2.4c}$$

where $\widetilde{\Box}_g = (g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ is the reduced wave operator corresponding to $g_{\mu\nu}$, ∇ is the Levi-Civita connection corresponding to the *Minkowski metric* $m_{\mu\nu}$,

$$N^{\#\mu\nu\kappa\lambda} \stackrel{\text{def}}{=} \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa} - h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa} - (m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big) + N_{\triangle}^{\#\mu\nu\kappa\lambda},$$

$N_{\triangle}^{\#\mu\nu\kappa\lambda} = O^{\ell}(|(h, \mathscr{F})|^2)$ is a quadratic error term that depends on the chosen model of nonlinear electromagnetism, and $\mathfrak{H}_{\mu\nu}$ and $\mathfrak{F}^{\nu}$ are inhomogeneous terms that depend in part on the chosen model of nonlinear electromagnetism.

The question of the stability of the Minkowski spacetime solution to (1.0.1a)–(1.0.1c) has thus been reduced to two subquestions: (i) show that the reduced system (1.2.4a)–(1.2.4c), where the unknowns are viewed to be $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$, has small-data global existence (if the ADM mass M is sufficiently small) and (ii) show that the resulting spacetime $(\mathbb{R}^{1+3},\ g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)})$ is geodesically complete. The second question is very much related to the first, for as in [Lindblad and Rodnianski 2005, Section 16] and [Loizelet 2008, Section 9], the question of geodesic completeness can be answered if one has sufficiently detailed information about the asymptotic behavior of $h_{\mu\nu}^{(1)}$; our stability theorem (see Section 16) provides sufficient information. Therefore, the main focus of this article is (i).

1.2.2. *The smallness condition.* Our smallness condition on the abstract initial data is stated in terms of the ADM mass M and a weighted Sobolev norm of the field data $\underline{\nabla}_i \mathring{\underline{h}}_{jk}^{(1)}$, $\mathring{K}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_k$. More specifically, in order to deduce global existence, we will require that

$$E_{\ell;\gamma}(0) + M < \varepsilon_{\ell}, \tag{1.2.5}$$

where $\varepsilon_{\ell} > 0$ is a sufficiently small positive number, $E_{\ell;\gamma}(0) \geq 0$ is defined by

$$E_{\ell;\gamma}^2(0) \stackrel{\text{def}}{=} \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^{\ell}}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^{\ell}}^2, \tag{1.2.6}$$

the weighted Sobolev norm $\|\cdot\|_{H_{1/2+\gamma}^{\ell}}$ is defined in Definition 10.1 below, $0 < \gamma < \frac{1}{2}$ is a constant, and $\ell \geq 10$ is an integer. The condition $\ell \geq 10$ is needed for various weighted Sobolev embedding results, including the weighted Klainerman–Sobolev inequality (1.2.10), and the results stated in Appendix A. In the above expressions, $\underline{\nabla}$ is the Levi-Civita connection corresponding to the Euclidean metric[12] $\underline{m}_{jk} \stackrel{\text{def}}{=} \text{diag}(1, 1, 1)$. Note that the assumed fall-off conditions (1.0.4c)–(1.0.4f) guarantee the existence of a constant $0 < \gamma < \frac{1}{2}$ such that $E_{\ell;\gamma}(0) < \infty$.

Although the norm (1.2.6) is useful for expressing the small-data global existence condition in terms of quantities inherent to the data, from the perspective of analysis, a more useful quantity is the energy $\mathscr{E}_{\ell;\gamma;\mu}(t) \geq 0$, which is defined by

$$\mathscr{E}_{\ell;\gamma;\mu}^2(t) \stackrel{\text{def}}{=} \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} \big\{|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|^2\big\} w(q)\, d^3x, \tag{1.2.7}$$

[12]Throughout the article, we use the symbol $\underline{m}$ to denote both the Euclidean metric $\underline{m}_{jk} \stackrel{\text{def}}{=} \text{diag}(1, 1, 1)$ on $\mathbb{R}^3$ and the first fundamental form $\underline{m}_{\mu\nu} \stackrel{\text{def}}{=} \text{diag}(0, 1, 1, 1)$ of the constant time hypersurfaces Σ_t viewed as embedded hypersurfaces of Minkowski spacetime; this double-use of notation should not cause any confusion.

where ∇ denotes the Levi-Civita connection corresponding to the *full Minkowski spacetime metric*, $q \stackrel{\text{def}}{=} |x| - t$ is a null coordinate, the weight function $w(q)$ is defined by

$$w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{1.2.8}$$

γ is from (1.2.6), and $0 < \mu < \frac{1}{2}$ is a fixed constant. In the above expression, $\mathscr{Z} \stackrel{\text{def}}{=} \{\partial_\mu, x_\mu \partial_\nu - x_\nu \partial_\mu, x^\kappa \partial_\kappa\}_{0 \le \mu \le \nu \le 3}$ is a subset of the conformal Killing fields of Minkowski spacetime, I is a vector field multi-index, $\nabla_{\mathscr{Z}}^I$ represents iterated Minkowski covariant differentiation with respect to vector fields in $\mathscr{Z}$, and $\mathscr{L}_{\mathscr{Z}}^I$ represents iterated Lie differentiation with respect to vector fields in $\mathscr{Z}$. The significance of the set $\mathscr{Z}$ is that it is needed for the weighted Klainerman–Sobolev inequality (1.2.10), which is discussed below.

Remark 1.2. The presence of the parameter $\mu > 0$ in (1.2.8) might seem unnecessary as $1 + (1+|q|)^{-2\mu} \approx 1$. However, as is explained in Section 1.2.6, the presence of $\mu > 0$ ensures that $w'(q) > 0$, an inequality that plays a key role in our energy estimates.

1.2.3. *Overall strategy of the proof.* The overall strategy is to deduce a hierarchy of Gronwall-amenable inequalities for the energies $\mathscr{E}_{k;\gamma;\mu}(t)$ $(0 \le k \le \ell)$; this is accomplished in (16.2.5) below. The net effect is that, under the assumptions that $E_{\ell;\gamma}(0) + M \le \varepsilon$ and ε is sufficiently small, we are able to deduce the following a priori estimate for the solution, which is valid during its classical lifetime:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \le c_\ell \varepsilon (1 + t)^{\tilde{c}_\ell \varepsilon}. \tag{1.2.9}$$

In the above inequality, c_ℓ and $\tilde{c}_\ell$ are positive constants. Now it is a standard result in the theory of hyperbolic PDEs that, if ε is sufficiently small, then an a priori estimate of the form (1.2.9) implies that the solution exists for $(t, x) \in (-\infty, \infty) \times \mathbb{R}^3$; see Proposition 14.1 for more details. Furthermore, as shown in [Lindblad and Rodnianski 2005; Loizelet 2008], if ε is sufficiently small, then it also follows that the spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)})$ is geodesically complete. *The main goal of this article is therefore to derive* (1.2.9).

1.2.4. *Geometry and null decompositions.* Let us now describe the tools used to derive (1.2.9). First and foremost, as mentioned above in Section 1.1.1, the reason we are able to prove our stability result is that the reduced equations (1.2.4a)–(1.2.4c) have a special algebraic structure and satisfy (in the language of Lindblad and Rodnianski) the *weak null condition*. Now in order to see the special structure of the terms in the reduced equations, we use the strategy of Lindblad and Rodnianski and decompose them into their *Minkowskian null components*; we refer to this as a *Minkowskian null decomposition*. We emphasize the following point: *the Minkowskian geometry is not the "correct" geometry to use for analyzing the equations, for the actual characteristics of the system correspond to the null cones of the spacetime metric $g_{\mu\nu}$ and the characteristics of the nonlinear electromagnetic equations* (*which in general do not have to coincide with the gravitational null cones*). *However, the errors that we make in using the Minkowskian geometry* (*which has the advantage of being simple*) *are controllable.*

We stress that the strategy of using the Minkowskian geometry to prove a global stability result for the Minkowski spacetime solution (in wave coordinates) was a novel (and unexpectedly viable) feature of [Lindblad and Rodnianski 2005; 2010]. The previous works [Christodoulou and Klainerman 1993; Zipser

2000; Klainerman and Nicolò 2003; Bieri 2007] used foliations of spacetime built out of outgoing null cones of the actual spacetime metric $g_{\mu\nu}$, which logarithmically diverge from the corresponding outgoing cones of Minkowski spacetime as $t \to \infty$. The use of the actual geometry allowed these authors to derive sharp estimates for the asymptotic behavior of the perturbed solution. However, this approach required an enormous effort. In addition to (i) constructing geometric foliations, the authors also had to (ii) carefully decompose every term relative to a g-null frame, (iii) construct vector fields (with controllable deformation tensors) for commuting the equations, and (iv) use an elaborate collection of elliptic estimates to control the foliations. At the expense of reduced precision, the Lindblad–Rodnianski approach eliminates many of these difficulties: the Minkowskian geometry is very easy to "construct", one only has to carefully decompose "the important terms" relative to the Minkowskian null frame, the vector field differential operators are prespecified, and no elliptic estimates are needed since the foliations are prespecified.

Let us briefly recall the meaning of a Minkowskian null decomposition; a more detailed description is offered in Section 5. The notion of a Minkowskian null decomposition is intimately connected to the following spacetime subsets: the *outgoing Minkowskian null cones* $C_q^+ \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\}$, the *ingoing Minkowskian null cones* $C_s^- \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}$, the *constant time slices* $\Sigma_t \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t\}$, and the *Euclidean spheres* $S_{r,t} \stackrel{\text{def}}{=} \{(\tau, y) \mid t = \tau,\ |y| = r\}$. Observe that the *null coordinate* $q \stackrel{\text{def}}{=} |x| - t$ associated to the spacetime point with coordinates (t, x) is constant on the outgoing cones and the null coordinate $s \stackrel{\text{def}}{=} |x| + t$ is constant on the ingoing cones. These coordinates will be used throughout the article to describe the rates of decay of various quantities. With $\omega^j \stackrel{\text{def}}{=} x^j/r$ $(j = 1, 2, 3)$, we also define the *ingoing Minkowskian null geodesic vector field* $\underline{L}^\mu \stackrel{\text{def}}{=} (1, -\omega^1, -\omega^2, -\omega^3)$, which satisfies $m_{\kappa\lambda}\underline{L}^\kappa\underline{L}^\lambda = 0$ and is tangent to the C_s^-, and the *outgoing Minkowskian null geodesic vector field* $L^\mu \stackrel{\text{def}}{=} (1, \omega^1, \omega^2, \omega^3)$, which satisfies $m_{\kappa\lambda}L^\kappa L^\lambda = 0$ and $m_{\kappa\lambda}\underline{L}^\kappa L^\lambda = -2$ and is tangent to the C_q^+. Furthermore, in a neighborhood of each nonzero spacetime point p, there exists a locally defined pair of m-orthonormal vector fields e_1 and e_2 that are tangent to the family of Euclidean spheres and m-orthogonal to $\underline{L}$ and L. The set $\mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$, which spans the tangent space at each point, is known as a *Minkowskian null frame*. In the discussion that follows, we will also make use of the set $\mathcal{T} \stackrel{\text{def}}{=} \{L, e_1, e_2\}$, which is the subset consisting of only those frame vectors tangent to the C_q^+, and the set $\mathcal{L} \stackrel{\text{def}}{=} \{L\}$.

Given any two-form $\mathcal{F}$, we can decompose it into its Minkowskian null components $\underline{\alpha}[\mathcal{F}]$, $\alpha[\mathcal{F}]$, $\rho[\mathcal{F}]$, and $\sigma[\mathcal{F}]$, where $\underline{\alpha}$ and α are two-forms m-tangent[13] to the spheres $S_{r,t}$ and ρ and σ are scalars. More specifically, we define $\underline{\alpha}_A = \mathcal{F}_{A\underline{L}}$, $\alpha_A = \mathcal{F}_{AL}$, $\rho = \frac{1}{2}\mathcal{F}_{\underline{L}L}$, and $\sigma = \mathcal{F}_{12}$, where $A \in \{1, 2\}$ and we have abbreviated $\mathcal{F}_{A\underline{L}} \stackrel{\text{def}}{=} e_A^\kappa \underline{L}^\lambda \mathcal{F}_{\kappa\lambda}$, etc. Similarly, we can decompose the tensor $h_{\mu\nu}$ into its null components h_{LL}, $h_{\underline{L}L}$, h_{LT}, etc., where T stands for any of the vectors in $\mathcal{T}$. We are now ready to discuss one of the major themes running throughout this article: the rates of decay of the various null components of $\mathcal{F}$ and h are distinguished by the kinds of contractions taken against the null frame vectors. In particular, contractions against L, e_1, and e_2 are associated with favorable decay, with L being the most favorable, while contractions against $\underline{L}$ are associated with unfavorable decay. Similarly, differentiation in the directions L, e_1, and e_2 are associated with creating *additional favorable decay* in the null coordinate s while differentiation in the direction $\underline{L}$ is associated with creating less favorable additional decay in q

[13]By m-tangent, we mean that their vector duals relative to the Minkowski metric are tangent to the $S_{r,t}$.

(see Lemma 6.16 for a precise version of this claim). Equivalently, the operator $\overline{\nabla}$ creates favorable decay in s while ∇ only creates decay in q. Here and throughout, $\overline{\nabla}$ is the null frame projection (of the derivative component only when $\overline{\nabla}$ is applied to a tensor field) of the Minkowski connection ∇ onto the outgoing Minkowski null cones (i.e., $\overline{\nabla}$ projects away the $\underline{L}$ component of ∇). From this point of view, the most dangerous terms in the equations are $\underline{\alpha}$ and $h_{\underline{L}\underline{L}}$ and the $\partial_q \sim \nabla_{\underline{L}}$ derivatives (see Section 2.7) of these quantities. We recommend that at this point the reader should examine the conclusions of Propositions 15.6 and 15.7 to get a feel for the kind of decay properties possessed by the various null components.

The main idea behind the Minkowskian null decomposition is that it can be used to show the following fact: *the worst possible combinations of terms, from the point of view of decay rates, are not present in the reduced equations* (1.2.4a)–(1.2.4c). This special algebraic structure, which is of central importance in our small-data global existence proof, is examined in detail in Propositions 11.1–11.4. As revealed in [Lindblad and Rodnianski 2003; 2005; 2010], this special algebraic structure is highly tensorial in nature. A related fact is that various null components of the lower-order derivatives of the solution exhibit a partially decoupled behavior. Moreover, this partial decoupling allows us to derive a hierarchy of "upgraded pointwise decay" estimates for the lower-order derivatives. These estimates, which play an essential role in the proof of our main theorem, provide bounds that *are stronger than the bounds implied by the size of* $\mathcal{E}_{\ell;\gamma;\mu}(t)$. This critical issue is discussed in more detail in Section 1.2.11.

1.2.5. *The special structure of the nonlinearities involving the Faraday tensor.* We now briefly summarize the special structures that allow us to extend the results of [Lindblad and Rodnianski 2010] to include small electromagnetic fields. We emphasize the following point: because of our assumptions on the electromagnetic Lagrangian, all of the important nonlinearities (from the point of view of small-data global existence) are the quadratic ones that are present in the case of the standard Maxwell–Maxwell Lagrangian ${}^{\star}\mathscr{L}_{(\mathrm{Maxwell})} = -\frac{1}{2}\mathfrak{z}_{(1)}$; all of the other electromagnetic theories that are covered by our main theorem introduce cubic and higher-order nonlinearities into the PDEs that are relatively easy to control. We first discuss how the electromagnetic fields couple into the equations for the components of the metric term $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$. The presence of the electromagnetic fields introduces only one important nonlinear term into these equations: the main $\mathscr{F}$-containing quadratic term $\mathfrak{Q}_{\mu\nu}^{(2;h)}(\mathscr{F}, \mathscr{F})$ on the right-hand side of (3.7.2a). A null decomposition reveals that this term has only one dangerous component involving the product $|\underline{\alpha}|^2$: $\mathfrak{Q}_{\underline{L}\underline{L}}^{(2;h)}$, which will be shown to decay like $|\mathfrak{Q}^{(2;h)}|_{\underline{L}\underline{L}} \lesssim \varepsilon^2(1+t)^{-2}$ (see inequalities (11.2.7e) and (15.3.3)). All other null components of $\mathfrak{Q}^{(2;h)}$ have a negligible effect on the dynamics because at least one of their factors is a "good" null component of $\mathscr{F}$ (see inequalities (11.2.7d) and (15.3.4c)); these quadratic terms therefore decay rapidly. Furthermore, a null decomposition of the wave equations (3.7.1a) reveals that the dangerous component only directly influences the behavior of the metric perturbation component $|\nabla h|_{\underline{L}\underline{L}}$. The main point is that Lindblad and Rodnianski [2010] were able to close their proof even though they allowed $|\nabla h|_{\underline{L}\underline{L}}$ to decay at a slower rate than the other null components of ∇h. The decay rate $|\mathfrak{Q}^{(2;h)}|_{\underline{L}\underline{L}} \lesssim \varepsilon^2(1+t)^{-2}$, though relatively slow, still allows us to prove the same estimates for $|\nabla h|_{\underline{L}\underline{L}}$ and $|h|_{\underline{L}\underline{L}}$ as in [ibid.] (see Proposition 15.6 and note the presence of the growing $\ln(1+t)$ factor in (15.3.2b) compared to the other estimates).

We now discuss the nonlinearities present in the electromagnetic field equations for the components of $\mathscr{F}$. There are three important nonlinear terms: the main quadratic terms $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ and $\mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F})$ from (3.7.3a) and the main quadratic term $\mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ from (3.7.2b). The terms $\mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F})$ and $\mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ have a very favorable null structure (all quadratic factors involve either a good tangential derivative $\overline{\nabla}$ or a good component of $\mathscr{F}$) and therefore have a negligible effect on the dynamics (see inequalities (11.2.7h)–(11.2.7i)). Furthermore, this special structure survives upon commuting the equations with $\mathscr{L}^{I}_{\mathscr{Z}}$. In contrast, the term $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ has a less favorable null structure and must be handled with care. For example, if X is any one-form, then in order to bound $|X_{\nu}\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})|$, one must in particular bound $|X||h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ (see inequality (11.2.7f)). The product $|h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ is only expected to decay like $\varepsilon^{2}(1+t)^{-2}$ thanks to the presence of the worst null components of $\nabla\mathscr{F}$ (the worst null component combination in the product $|h|_{\mathscr{L}\mathscr{L}}|\nabla\mathscr{F}|$ is the magnitude of the product $\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_{\nu}$, which is discussed below in the third paragraph of Section 1.2.11). The main reason that we are able to handle the difficult term $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ is that the wave-coordinate condition allows one to derive independent estimates for $|h|_{\mathscr{L}\mathscr{L}}$ that are just good enough to close the proof of stability; this is discussed in more detail in Section 1.2.10. Another difficulty is that some of this structure is destroyed after one commutes $\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F})$ with $\mathscr{L}^{I}_{\mathscr{Z}}$. In particular, the commuted term $|\mathscr{L}^{I}_{\mathscr{Z}}h|_{\mathscr{L}\mathscr{L}}$ must be carefully analyzed, for Lie differentiation results in the presence of some potentially dangerous lower-order terms. These terms are discussed in more detail at the end of Section 1.2.12.

1.2.6. *Energy inequalities and the canonical stress.* The first major analytical step in deriving the all-important Gronwall-amenable estimate (16.2.5) (which is the main ingredient in the derivation of the a priori estimate (1.2.9)) is to deduce the energy inequalities of Lemmas 12.1 and 12.2, which respectively provide L^2 estimates for solutions to the electromagnetic *equations of variation* and L^2 estimates for solutions to quasilinear wave equations whose principal operator agrees with that of (1.2.4a) (i.e., whose principal operator is $\widetilde{\Box}_{g}$). The equations of variation are linear (in the principal term) PDEs that are satisfied by the derivatives of solutions $\mathscr{F}$ to (1.2.4b)–(1.2.4c). Specifically, the equations of variation are the PDEs (8.1.1a)–(8.1.1b). As is explained below, these equations come into play because we require L^2 estimates for higher-order derivatives of $h^{(1)}$ and $\mathscr{F}$ in order to close our global existence argument. We will comment mainly on the estimates for the electromagnetic equations of variation since the estimates of Lemma 12.2 are perhaps more familiar to the reader and in any case are explained in detail in [Lindblad and Rodnianski 2010, Lemma 6.1 and Proposition 6.2]. Our proof of Lemma 12.1 is based on the construction of a suitable *energy current* $\dot{J}^{\mu} \overset{\text{def}}{=} -\dot{Q}^{\mu}{}_{\nu}X^{\nu}$, where $\dot{Q}^{\mu}{}_{\nu}$ is the *canonical stress*. $\dot{Q}^{\mu}{}_{\nu} = \dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}}, \dot{\mathscr{F}}]$ is a tensor field that depends quadratically on the *variations* $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}^{I}_{\mathscr{Z}}\mathscr{F}$, $X^{\nu} \overset{\text{def}}{=} w(q)\delta^{\nu}_{0}$ ($\nu = 0, 1, 2, 3$) is a "multiplier vector field", and $w(q)$ is the weight function defined in (1.2.8). The end result is provided by inequality (12.2.1) below. Although at first glance inequality (12.2.1) may appear to be a standard energy inequality, one of the most important features of this particular energy current is that it provides the *additional positive* spacetime integral $\int_{t_1}^{t_2}\int_{\Sigma_{\tau}}(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)w'(q)\,d^3x\,d\tau$ on the left-hand side of (12.2.1); here, $\dot{\alpha}$, $\dot{\rho}$, and $\dot{\sigma}$ are the "favorable" null components of the two-form $\dot{\mathscr{F}}$. This additional positive quantity, which is analogous to the quantity $\int_{t_1}^{t_2}\int_{\Sigma_{\tau}}|\overline{\nabla}\phi|^2 w'(q)\,d^3x\,d\tau$ on the left-hand side of (12.2.4) that was exploited by Lindblad and Rodnianski, is one of the key advantages afforded by our use of a weight function of the form (1.2.8). Its availability is directly related to the fact that we have better integrated control over the quadratic terms $|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2$ than we do over the term $|\dot{\underline{\alpha}}|^2$. The spacetime integral plays a key role in our derivation of the energy inequality (16.2.5).

Let us now make a few comments concerning the canonical stress and the construction of the energy current $\dot{J}^\mu$ introduced above. A very detailed description is located in [Christodoulou 2000; Speck 2012], so we confine ourselves here to its two most salient features. The canonical stress (see (8.2.2)) plays the role of an energy-momentum-type tensor for the electromagnetic equations of variation. Because these (linear-in-$\dot{\mathscr{F}}$) equations depend on the "background" $\mathscr{F}_{\mu\nu}$ in addition to the linearized variables $\dot{\mathscr{F}}_{\mu\nu}$, it is *not* the case that $\mathscr{D}_\mu(\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]) = 0$; this is in contrast to the property $(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa T_{\lambda\nu} = 0$ (see (3.5.3)) enjoyed by the energy-momentum tensor. However, we now point out the first key property of the canonical stress: $\nabla_\mu(\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}])$ is lower-order in the sense that it does not depend on $\nabla_\lambda\dot{\mathscr{F}}_{\mu\nu}$; by using the equations of variation for substitution, the $\nabla_\lambda\dot{\mathscr{F}}_{\mu\nu}$ terms can be replaced with inhomogeneous terms (see (8.2.4)). It is already important to appreciate the availability of this nontrivial quadratic-in-$\dot{\mathscr{F}}$ quantity whose divergence can be expressed in terms of only $\mathscr{F}$, $\nabla\mathscr{F}$, $\dot{\mathscr{F}}$, and inhomogeneous terms. The availability of such a quantity is not a feature inherent to all systems of equations,[14] but is instead related to the symmetry properties of the indices of the principal terms (i.e., the terms on the left-hand side) in equations (8.1.1a)–(8.1.1b), which themselves are related to the fact that the original nonlinear electromagnetic equations are derivable from a Lagrangian.

The second key property enjoyed by the canonical stress is that of integrated positivity upon contraction against certain pairs (ξ, X) consisting of a one-form ξ and a vector field X. More precisely, for certain hypersurfaces Σ, there exist choices of (ξ, X) such that ξ is normal to Σ (in the sense of covector-vector annihilation) and such that the quantity $\int_\Sigma \dot{Q}^\mu{}_\nu \xi_\mu X^\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]\, d\Sigma$ is bounded from below by the square of an L^2-type norm of $\dot{\mathscr{F}}$ along Σ. This is a general fact that holds for all electromagnetic equations of variation that are *regularly hyperbolic* in the sense of [Christodoulou 2000]. However, in the present article, a stronger condition than integrated positivity holds: for certain pairs (ξ, X), $\dot{Q}^\mu{}_\nu\xi_\mu X^\nu[\,\cdot\,,\cdot\,]$ is in fact a *positive-definite* quadratic form in $\dot{\mathscr{F}}$. We remark that this stronger property concerns the structure of the quadratic form $\dot{Q}^\mu{}_\nu\xi_\mu X^\nu[\,\cdot\,,\cdot\,]$ and therefore has nothing to do with whether $\dot{\mathscr{F}}$ satisfies the equations of variation.

The two key properties are analogous to (but distinct from) the positivity properties of an energy-momentum tensor satisfying the dominant energy condition and the positivity properties of the Bel–Robinson tensor (which played a central role in [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007]). As is explained in [Christodoulou 2000; Speck 2012], the set of pairs (ξ, X) leading to integrated positivity is intimately connected to the *hyperbolicity of and the geometry of the electromagnetic equations* and to the speeds and directions of propagation in the system. In this article, the only hypersurfaces that we integrate over are the constant-time hypersurfaces Σ_t and the only pair (ξ, X) that we use is $\xi_\mu = -\delta_\mu^0$, and $X^\nu = w(q)\delta_0^\nu$. The special positivity properties stemming from this choice of (ξ, X), and in particular the availability of the additional positive spacetime integral $\int_{t_1}^{t_2}\int_{\Sigma_\tau}(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)w'(q)\, d^3x\, d\tau$ mentioned above, are derived in Lemma 12.1. We emphasize that

[14] However, such quantities do in fact exist for all scalar wave equations.

our derivation of this additional spacetime integral is not just a consequence of the second key property; rather, our derivation requires $\dot{\mathscr{F}}$ to be a solution to the equations of variation.

1.2.7. *Weighted Klainerman–Sobolev inequalities.* Based on the energy inequalities of Proposition 12.3, which are relatively straightforward consequences of Lemmas 12.1 and 12.2, it is clear that most of the hard work in deriving the estimate (16.2.5) goes into estimating the integrals involving the inhomogeneous terms $\mathfrak{I}$ and $\dot{\mathfrak{F}}$ on the right-hand sides of (12.2.6) and (12.2.8). In particular, we attempt to summarize here the origin of the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ that appear in (16.2.5) and that are of central importance in our derivation of the fundamental a priori energy estimate (1.2.9). Roughly speaking, these factors arise from a collection of pointwise decay estimates that we will soon explain. The first tools of interest to us along these lines are the weighted Klainerman–Sobolev inequalities, which allow us to deduce pointwise decay estimates for functions $\phi \in C_0^\infty(\mathbb{R}^3)$ in terms of weighted L^2 estimates for ϕ and its Minkowskian covariant derivatives with respect to vector fields $Z \in \mathscr{Z}$. More specifically (see also Appendix B), the weighted Klainerman–Sobolev inequalities state that (with $q \overset{\text{def}}{=} |x| - t$)

$$(1+t+|x|)[(1+|q|)w(q)]^{1/2}|\phi(t,x)| \le C \sum_{|I|\le 3} \|w^{1/2}(q)\nabla_{\mathscr{Z}}^I \phi(t,\cdot)\|_{L^2}. \tag{1.2.10}$$

We refer to these estimates as "*weak pointwise decay estimates*" since they have nothing to do with the special structure of the Einstein-nonlinear electromagnetic equations; a major theme permeating this article is that, in order to close our global existence bootstrap argument, the estimate (1.2.10) needs to be upgraded using the special structure of the equations. Inequality (1.2.10) can therefore be viewed as a preliminary estimate that will play a role in the proof of the upgraded estimates.

The form of the inequalities (1.2.10) raises several important issues. First, in order to apply the weighted Klainerman–Sobolev inequalities to $h^{(1)}$, we have to achieve L^2 control over the quantities $w^{1/2}(q)\nabla_{\mathscr{Z}}^I h^{(1)}$. To this end, we have to study the equations satisfied by the quantities $\nabla_{\mathscr{Z}}^I h^{(1)}$. In order to derive these equations, we have to commute the operator $\nabla_{\mathscr{Z}}^I$ through the reduced wave operator term $\widetilde{\Box}_g h^{(1)}$. Lindblad and Rodnianski accomplished this commutation through the use of *modified covariant derivatives* $\widehat{\nabla}_Z$, which are equal to ordinary covariant derivatives plus a scalar multiple (depending on $Z \in \mathscr{Z}$) of the identity; see Definition 6.5. The main advantage of these operators is that $\widehat{\nabla}_Z \Box_m - \Box_m \nabla_Z = 0$, where $\Box_m \overset{\text{def}}{=} (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ denotes the wave operator of the Minkowski metric; see Lemma 6.13. Therefore, $\nabla_{\mathscr{Z}}^I h^{(1)}$ is a solution to the equation $\widetilde{\Box}_g \nabla_{\mathscr{Z}}^I h^{(1)} = \widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\Box}_g h^{(1)} + H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\nabla_{\mathscr{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{Z}}^I(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(1)})$, where $\widetilde{\Box}_g h^{(1)}$ is equal to the inhomogeneous term on the right-hand side of (1.2.4a) above and $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu} = -h^{\mu\nu} + O(|h|^2)$. We remark that the analysis of the commutator term $H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\nabla_{\mathscr{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathscr{Z}}^I(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(1)})$, which was performed in [Lindblad and Rodnianski 2010] (see also Proposition 7.1 and Lemma 16.11), is among the most challenging work encountered. Rather than repeat this analysis and the discussion behind it, which is thoroughly explained and carried out in [Lindblad and Rodnianski 2010], we will instead focus on the analogous difficulties that arise in our analysis of $\mathscr{F}$. We do, however, point out the role that the Hardy inequalities of Proposition C.1 play in the analysis of $h^{(1)}$: they are used to estimate a weighted L^2 norm of $\nabla_{\mathscr{Z}}^I h^{(1)}$ by a weighted L^2 norm of $\nabla\nabla_{\mathscr{Z}}^I h^{(1)}$. The main point is that $\nabla_{\mathscr{Z}}^I h^{(1)}$

is *not* directly controlled in L^2 by the energy while $\nabla\nabla_{\mathscr{Z}}^I h^{(1)}$ is. The cost of applying the Hardy inequalities is powers of $1+|q|$, which are always sufficiently available thanks to our use of the weight $w(q)$.

1.2.8. *The role of Lie derivatives.* The next important issue concerning the weighted Klainerman–Sobolev inequality (1.2.10) is that it is more convenient to work with Lie derivatives of $\mathscr{F}$ rather than covariant derivatives of $\mathscr{F}$; note that our definition (1.2.7) of our energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ involves Lie derivatives of $\mathscr{F}$. According to inequality (6.5.22) below, inequality (1.2.10) remains valid if we replace the operators $\nabla_{\mathscr{Z}}^I$ with $\mathscr{L}_{\mathscr{Z}}^I$. However, as in the case of the $\nabla_{\mathscr{Z}}^I h^{(1)}$, we have to study the equations satisfied by the $\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}$. Now on the one hand, Lemma 6.8 shows that the operator $\mathscr{L}_Z$ can be commuted through the Minkowski connection ∇ in (1.2.4b). On the other hand, to commute Lie derivatives through (1.2.4c), it is convenient to work with *modified Lie derivatives* $\widehat{\mathscr{L}}_Z$, which are equal to ordinary Lie derivatives plus a scalar multiple[15] (depending on $Z\in\mathscr{Z}$) of the identity; see Definition 6.5. Unlike covariant derivatives, these operators have favorable commutation properties with the linear Maxwell–Maxwell term $\nabla_\mu\mathscr{F}^{\mu\nu}$, which is the leading term in (1.2.4c). More specifically, $\widehat{\mathscr{L}}_Z[((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa})\nabla_\mu\mathscr{F}_{\kappa\lambda}] = [(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}]\nabla_\mu\mathscr{L}_Z\mathscr{F}_{\kappa\lambda}$; see Lemma 6.14. As is captured by Proposition 8.1, these operators are also useful for differentiating the nonlinear equation (1.2.4c); the error terms generated have a favorable null structure that is captured in Proposition 11.4.

1.2.9. *The tensor field $h_{\mu\nu}^{(0)}$.* Let us now discuss the ideas behind the Lindblad–Rodnianski splitting of the metric defined in (1.2.1a)–(1.2.1c). We first note that because of the $2M/r$ ADM mass term present in $h_{\mu\nu}^{(0)}$, substituting the tensor field $h_{\mu\nu}\overset{\text{def}}{=}h_{\mu\nu}^{(0)}+h_{\mu\nu}^{(1)}$ in place of $h_{\mu\nu}^{(1)}$ in the definition of the energy would lead to $\mathscr{E}_{\ell;\gamma;\mu}(0)=\infty$. Thus, as a practical matter, the introduction of $h_{\mu\nu}^{(1)}$ allows us to work with a quantity of finite energy. Now according to the discussion in [Lindblad and Rodnianski 2010], the precise form $h_{\mu\nu}^{(0)}=\chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu}$ was determined by making an "educated" guess concerning the contribution of the ADM mass term $\chi(r)(2M/r)\delta_{jk}$, which is present in the data, to the solution. The term $h_{\mu\nu}^{(0)}$ manifests itself in the reduced equations as the $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$ inhomogeneous term on the right-hand side of the reduced equation (1.2.4a). Because of the identity $\Box_m(1/r)=0$ for $r>0$, where $\Box_m=(m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ is the Minkowski wave operator, it follows that the main contribution of the term $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$ comes from the "interior" region $\{(t,x)\mid \frac{1}{2}<r/t<\frac{3}{4}\}$; this is because the derivatives of $\chi(z)$ are supported in the interval $[\frac{1}{2},\frac{3}{4}]$. Now in the interior region, the quantities $1+|q|$ and $1+s$ are uniformly comparable. Thus, the weighted Klainerman–Sobolev inequality (1.2.10) predicts strong decay for the solution in this region, and consequently, one can derive suitable weighted Sobolev bounds for the inhomogeneity $\widetilde{\Box}_g h_{\mu\nu}^{(0)}$; see Lemma 16.10 for a precise statement of this estimate.

1.2.10. *The wave-coordinate condition.* Before expanding our discussion of the pointwise decay estimates, we will discuss the analytic role of the *wave-coordinate* condition $\nabla_\nu[\sqrt{|\det g|}(g^{-1})^{\mu\nu}]=0$ $(\mu=0,1,2,3)$, which plays *multiple* roles in this article. First, it hyperbolizes the Einstein equations. Second, it allows us to replace certain unfavorable nonlinear terms from the equations (1.0.1a)–(1.0.1c) with more favorable ones; the culmination of this procedure is exactly the reduced system (1.2.4a)–(1.2.4c). Finally, the wave-coordinate condition also allows us to deduce several *independent and improved estimates*, both

[15] The multiple is $2c_Z$, where c_Z is the multiple corresponding to the modified covariant derivative $\widehat{\nabla}_Z$.

at both the pointwise level and the L^2 level, for the components h_{LL} and h_{LT}. As we will see, these improved estimates are central to the structure of the proof of Theorem 16.1, and our stability argument would not close without them. More specifically, as shown in [Lindblad and Rodnianski 2010], a null decomposition of the wave-coordinate condition leads to the algebraic inequalities

$$|\nabla h|_{\mathcal{L}\mathcal{T}} \lesssim |\overline{\nabla} h| + |h||\nabla h|, \tag{1.2.11a}$$

$$|\nabla \nabla_Z h|_{\mathcal{L}\mathcal{L}} \lesssim |\overline{\nabla} \nabla_Z h| + \sum_{|I_1|+|I_2|\leq 1} |\nabla_{\mathcal{Z}}^{I_1} h||\nabla \nabla_{\mathcal{Z}}^{I_2} h| \quad (Z \in \mathcal{Z}), \tag{1.2.11b}$$

where $\overline{\nabla}$ is the null frame projection of ∇ (the derivative component only) onto the outgoing Minkowski cones. Note that the right-hand side of (1.2.11a) involves only favorable derivatives of h and quadratic error terms while the left-hand side involves *all* derivatives of h, including the dangerous $\nabla_{\underline{L}}$ derivative. Generalizations of (1.2.11a) for $\nabla_{\mathcal{Z}}^I h$ are stated in Proposition 11.1. We remark that it is important to note in these generalizations that the estimates for $|\nabla \nabla_Z h|_{\mathcal{L}\mathcal{L}}$ are stronger than what can be proved for $|\nabla \nabla_Z h|_{\mathcal{L}\mathcal{T}}$.

1.2.11. *Upgraded pointwise decay estimates.* We now discuss the full collection of *upgraded pointwise decay estimates* (see Propositions 15.5–15.7 below), which are of central importance in closing the global existence bootstrap argument. For as mentioned above, the weighted Klainerman–Sobolev estimates (1.2.10) are not sufficient to close the argument. We remark that the reasons that we truly need the upgraded pointwise decay estimates are discussed in more detail at beginning of Section 15. Aside from the components h_{LL} and h_{LT}, which are controlled by the wave-coordinate condition, there is a relatively strong coupling between the evolution of the remaining components of h and the evolution of the dangerous $\underline{\alpha}[\mathcal{F}]$ component of the Faraday tensor. Therefore, our proofs of the upgraded estimates (and Proposition 15.7 in particular) have a hierarchical structure; i.e., the order in which they are proved is very important. Although we don't provide a complete description of all of the subtleties of this hierarchy in this introduction, we do provide a preliminary description of some of its salient features. We first emphasize the following important feature: most null components of h, the $\underline{\alpha}$ null component of $\mathcal{F}$, and the components $\nabla_Z h_{LL}$ (for $Z \in \mathcal{Z}$) have better t-decay properties than their higher-order-derivative counterparts; this is the content of Proposition 15.6. Roughly speaking, the reason for this discrepancy is that the nondifferentiated reduced equations have a more favorable algebraic structure than the differentiated reduced equations. This feature will be particularly important during our global existence argument, for the principal terms (from the point of view of differentiability) in the Leibniz expansion of the operator $\nabla_{\mathcal{Z}}^I$ acting on a quadratic term are of the form $u\nabla_{\mathcal{Z}}^I v$ and similarly for the operator $\mathcal{L}_{\mathcal{Z}}^I$. Consequently, the strong pointwise decay property of the nondifferentiated quantity, which is represented by u, is a crucially important ingredient of the derivation of the $C\varepsilon \int_0^t (1+\tau)^{-1} \mathcal{E}_{k;\gamma;\mu}^2(\tau)\, d\tau$ term on the right-hand side of (16.2.5). We emphasize that our stability proof would not go through if this term were replaced with $C\varepsilon \int_0^t (1+\tau)^{-1+C\varepsilon} \mathcal{E}_{k;\gamma;\mu}^2(\tau)\, d\tau$.

The derivation of the upgraded pointwise decay estimates for the Faraday tensor begins with Proposition 9.3, which provides a null decomposition of the electromagnetic equations of variation, and Proposition 11.4, which provides a null decomposition of the inhomogeneous terms that result after differentiating the reduced electromagnetic equations with modified Lie derivatives. The net effect is that the null components of the *lower-order* Lie derivatives of $\mathscr{F}$ satisfy ordinary differential inequalities[16] (which we loosely refer to as ODEs) along ingoing and outgoing cones (see Proposition 11.5), and furthermore, the inhomogeneous terms appearing on the right-hand side of the ODEs can be inductively controlled (see the proofs of Propositions 15.5–15.7). We remark that this analysis of the lower-order derivatives of $\mathscr{F}$ involves a loss of several derivatives because the right-hand sides of the ODEs depend on the higher-order derivatives of $\mathscr{F}$, which are pointwise bounded via the weighted Klainerman–Sobolev estimates (1.2.10). We stress that this loss of differentiability is not a concern because we only need to analyze the lower-order derivatives of $\mathscr{F}$ in this fashion. Similar remarks apply for our analysis of the upgraded pointwise decay estimates for h, which are briefly described below. It is important to distinguish between two classes of ODEs that play a role in this analysis. The first class consists of ODEs for rescaled versions of the null components $(\dot{\alpha}, \dot{\rho}, \dot{\sigma}) \stackrel{\text{def}}{=} (\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}], \rho[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}], \sigma[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}])$ and involves differentiation in the direction of the null generators of the *ingoing Minkowskian cones*; i.e., the principal part of the ODEs is $\nabla_{\underline{L}}$. We remark that this point of view represents a rather crude treatment of (9.1.8b)–(9.1.8d), but because of the favorable decay properties of the inhomogeneities, this approach is sufficient to conclude the desired estimates: by integrating back towards the Cauchy hypersurface Σ_0 in the direction $-\underline{L}$, we are able to deduce t-decay for $\alpha[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]$, $\rho[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]$, and $\sigma[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]$ from t-decay of the inhomogeneous terms at the expense of a loss of decay in q. We remark that the proof of the upgraded estimates for these components happens in two stages. We refer to the first-stage estimates, which are proved in Proposition 15.5, as the "initial upgraded" pointwise decay estimates. These first-stage estimates follow from using the weighted Klainerman–Sobolev estimates to bound the inhomogeneous terms in the ODEs. The second-stage upgraded estimates, which we refer to as "fully upgraded" pointwise decay estimates, are proved at the end of Proposition 15.7 after all of the other upgraded pointwise decay estimates for the remaining components of the lower-order derivatives of h and $\mathscr{F}$ have been proved. For at this point in the upgraded hierarchy, we will have better pointwise control over the inhomogeneous terms in the ODEs than that afforded by the weighted Klainerman–Sobolev estimates.

The next class consists of ODEs for rescaled versions of the null component $\dot{\underline{\alpha}} \stackrel{\text{def}}{=} \underline{\alpha}[\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}]$. Notice that (see (9.1.8a)), unlike the other null components, $\dot{\underline{\alpha}}$ does *not* satisfy an ODE that to 0-th order involves differentiation in the direction of $\underline{L}$. Instead, at first sight, it might appear that one should reason in analogy with the first class and view (9.1.8a) as an ODE in the direction of L with inhomogeneous terms. However, the desired decay estimates do *not* close at this level. Instead, one must also consider the effect of the quadratic term $-\not{m}_{\nu}{}^{\lambda} h^{\mu\kappa} \nabla_{\mu} \dot{\mathscr{F}}_{\kappa\lambda}$. A null decomposition of this term reveals that it contains the dangerous term $\frac{1}{4} h_{LL} \nabla_{\underline{L}} \dot{\underline{\alpha}}_{\nu}$, which decays too slowly to be treated as an inhomogeneous term in the ODE satisfied by $\dot{\underline{\alpha}}$. To remedy this difficulty, we introduce the vector field $\Lambda = L + \frac{1}{4} h_{LL} \underline{L}$, which can be viewed as a first-order correction to the Minkowski outgoing null direction arising from the presence of a nonzero tensor field h in the expansion $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$. Note that, for these upgraded pointwise decay estimates for the lower-order Lie derivatives, we do not bother to correct for the fact that the electromagnetic model is not necessarily the Maxwell–Maxwell model; the deviation from the Maxwell–Maxwell model comprises cubic terms, which we can treat as small inhomogeneities. We may thus view (9.1.8a) as an ODE in the direction of Λ with inhomogeneous terms; this is exactly the point of

[16]More precisely, the null components satisfy transport equations with small sources.

view emphasized in Proposition 11.5. Because we have a sufficiently strong independent decay estimates for h_{LL} and for the inhomogeneities, this approach is sufficient to achieve the desired estimates.

Our analysis of the upgraded pointwise decay estimates for the metric-related quantities h and $h^{(1)}$ closely mirrors the analysis in [Lindblad and Rodnianski 2010]. Hence, we will not discuss them in full detail here but instead refer the reader to the discussion in [ibid.]. The estimates can be divided into three classes, the first one being the estimates (15.3.1a) and (15.3.1b) for $|\nabla h|_{\mathscr{L}\mathscr{T}}$, $|\nabla\nabla_Z h|_{\mathscr{L}\mathscr{L}}$, $|h|_{\mathscr{L}\mathscr{T}}$, and $|\nabla_Z h|_{\mathscr{L}\mathscr{L}}$. As was suggested above, the first-class estimates are consequences of the additional special algebraic structure that follows from the wave-coordinate condition together with the weighted Klainerman–Sobolev inequality. The second class consists of the estimates (15.3.2a) and (15.3.2b) for $|\nabla h|_{\mathscr{T}\mathscr{N}}$ and $|\nabla h|$. These estimates heavily rely on the decay estimates of Lemma 13.2 and Corollary 13.3 below, which were proved in [Lindblad and Rodnianski 2010] and which are of independent interest. The lemma and its corollary can be viewed as a second-order counterpart to the ODE estimates for the Faraday tensor discussed in the previous paragraphs. It is important to note that the hypotheses of the lemma and its corollary are satisfied *as a consequence* of the independent upgraded pointwise decay estimates provided by the wave-coordinate condition. The third class consists of the estimates (15.3.4a), (15.3.4b), and (15.3.4c) for $|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|$, $|\nabla_{\mathscr{Z}}^I h^{(1)}|$, and $|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|$ (related estimates for the tensor field h also hold). Their derivation is similar in spirit to the derivation of the second-class estimates, but the inductive proof we give is highly coupled to the simultaneous derivation of analogous upgraded pointwise decay estimates for $|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|$, which were discussed two paragraphs ago.

1.2.12. *Lie differentiation, Minkowski-covariant differentiation, and null structure.* We make some final comments concerning the relationship between Lie derivatives and Minkowski-covariant derivatives. On the one hand, because we commute the equations satisfied by $h^{(1)}$ with the operators $\nabla_{\mathscr{Z}}^I$, our analysis of $h^{(1)}$ naturally requires us to estimate the quantities $|\nabla_{\mathscr{Z}}^I h|$, $|\nabla_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{L}}$, $|\nabla_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{T}}$, etc. Furthermore, as discussed above, the quantities $|\nabla_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{L}}$ and $|\nabla_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{T}}$ have a distinguished role in view of their connection to the wave-coordinate condition. On the other hand, because we commute the electromagnetic equations with Lie derivatives, we will have to confront the terms $|\mathscr{L}_{\mathscr{Z}}^I h|$, $|\mathscr{L}_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{L}}$, $|\mathscr{L}_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{T}}$, etc. In order to bridge the gap between Lie derivative estimates and covariant derivative estimates, we provide Proposition 6.19, the proof of which relies on the special algebraic-geometric structure of the vector fields in $\mathscr{Z}$. Proposition 6.19 is an especially important ingredient in the null decomposition estimate (11.1.11b). As an example of the role played by this proposition, we cite the estimate (6.5.23c), which reads

$$|\mathscr{L}_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{L}} \lesssim |\nabla_{\mathscr{Z}}^I h|_{\mathscr{L}\mathscr{L}} + \underbrace{\sum_{|J|\le|I|-1} |\nabla_{\mathscr{Z}}^J h|_{\mathscr{L}\mathscr{T}}}_{\text{absent if } |I|=0} + \underbrace{\sum_{|J'|\le|I|-2} |\nabla_{\mathscr{Z}}^{J'} h|}_{\text{absent if } |I|\le 1}.$$

This shows that, in the translation from Lie derivatives to covariant derivatives, the error terms that arise in the analysis of the $|\cdot|_{\mathscr{L}\mathscr{L}}$ seminorm are either 1 degree lower in order *and* controllable by the wave-coordinate condition (i.e., the terms with $|J|\le|I|-1$) or are 2 degrees lower in order (i.e., the terms with $|J'|\le|I|-2$). This fact, and others similar to it, play an essential role in allowing our hierarchy of estimates to unfold in a viable order.

1.3. ***Outline of the article.*** The remainder of the article is organized as follows.

- In Section 2, we provide for convenience a summary of the notation that is used throughout the article.
- In Section 3, we discuss the Einstein-nonlinear electromagnetic equations in detail. We also introduce our wave-coordinate condition and our assumptions on the electromagnetic Lagrangian. Next, we derive a reduced system of equations, which is equivalent to the system of interest in our wave-coordinate gauge. In Section 3.7, we summarize the version of the reduced equations that we work with for most of the article.
- In Section 4, we construct initial data for the reduced system from the abstract initial data in a manner compatible with the wave-coordinate condition. We also sketch a proof of the fact that the wave-coordinate condition is preserved by the flow of the reduced equations.
- In Section 5, we introduce the notion of a Minkowskian null frame and discuss the corresponding null decomposition of various tensor fields.
- In Section 6, we introduce the differential operators that will be used throughout the remainder of the article, including modified Lie derivatives and modified covariant derivatives with respect to a special subset $\mathscr{Z}$ of Minkowskian conformal Killing fields. We also provide a collection of lemmas that relate the various operators.
- In Section 7, we provide a preliminary algebraic expression for the equations satisfied by $\nabla_{\mathscr{Z}}^{I} h^{(1)}$, where $h^{(1)}$ is a solution to the reduced equations.
- In Section 8, we introduce the electromagnetic equations of variation, which are a linearized version of the electromagnetic equations. We also provide a preliminary algebraic expression for the inhomogeneous terms in the equations of variation satisfied by $\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}$, where $\mathscr{F}$ is a solution to the reduced equations. We then introduce the canonical stress tensor and use it to construct an energy current that will be used to control weighted Sobolev norms of $\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}$.
- In Section 9, we perform two decompositions of the electromagnetic equations, including a null decomposition of the electromagnetic equations of variation and a decomposition of the electromagnetic equations into constraint equations and evolution equations for the Minkowskian one-forms E, D, B, and H. In order to connect these one-forms to the abstract initial data, we also introduce the geometric electromagnetic one-forms $\mathfrak{E}$, $\mathfrak{D}$, $\mathfrak{B}$, and $\mathfrak{H}$.
- In Section 10, we introduce our smallness condition on the abstract initial data. We then prove that this smallness condition guarantees that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ of the corresponding solution to the reduced equations is small at $t = 0$; it is this smallness of $\mathscr{E}_{\ell;\gamma;\mu}(0)$ that will lead to a global solution of the reduced equations.
- In Section 11, we provide algebraic estimates for the inhomogeneities in the reduced equations under the assumption that the wave-coordinate condition holds. We also derive ordinary differential inequalities for the null components of $\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}$ and provide algebraic estimates for the corresponding inhomogeneities.
- In Section 12, we prove weighted energy estimates for solutions to the electromagnetic equations of variation. We also recall some results of [Lindblad and Rodnianski 2010] that provide analogous weighted

energy estimates for both scalar wave equations and tensorial systems of wave equations with principal part $(g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$.

- In Section 13, we recall some results of [Lindblad and Rodnianski 2010] that provide pointwise decay estimates for both scalar wave equations and tensorial systems of wave equations with principal part $(g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$.
- In Section 14, we state a basic local well-posedness result and continuation principle for the reduced equations. The continuation principle will be used in Section 16 in order to deduce small-data global existence for the reduced equations from a suitable bound on the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$.
- In Section 15, we introduce our bootstrap assumption on the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$. We then use this assumption to deduce a collection of pointwise decay estimates for solutions to the reduced equations under the assumption that the wave-coordinate condition holds.
- In Section 16, we prove our main results. The results are separated into two theorems. In Theorem 16.1, we use the decay estimates proved in Section 15 to derive a "strong" a priori estimate for the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$; *the proof of this theorem is the centerpiece of the article*. Theorem 16.3, which is our main theorem demonstrating the stability of Minkowski spacetime, is then an easy consequence of Theorem 16.1 and the continuation principle of Section 14. Both of these theorems rely upon the assumption that the wave-coordinate condition holds.

2. Notation

For convenience, in this section, we collect some of the important notation that is introduced throughout the article.

2.1. ***Constants.*** We use the symbols c, $\tilde{c}$, C, and $\widetilde{C}$ to denote generic *positive* constants that are free to vary from line to line. In general, they can depend on many quantities, but in the small-solution regime that we consider in this article, they can be chosen uniformly. Sometimes it is illuminating to explicitly indicate one of the quantities $\mathfrak{Q}$ that a constant depends on; we do by writing, e.g., $C_{\mathfrak{Q}}$. If A and B are two quantities, then we often write

$$A \lesssim B$$

to mean that "there exists a uniform constant $C > 0$ such that $A \leq CB$". Furthermore, if $A \lesssim B$ and $B \lesssim A$, then we often write

$$A \approx B.$$

2.2. ***Indices.***

- Lowercase Latin indices a, b, j, k, etc., take on the values 1, 2, or 3.
- Greek indices κ, λ, μ, ν, etc., take on the values 0, 1, 2, or 3.
- Primed indices κ', λ', etc., are used in the same way as unprimed indices.
- Uppercase Latin indices A, B, etc., take on the values 1 or 2 and are used to enumerate the two Minkowski-orthonormal null frame vectors tangent to the spheres $S_{r,t}$.

- As a convention, the tensor fields $\mathscr{F}_{\mu\nu}$, $\mathscr{M}_{\mu\nu}$, $R_{\mu\nu}$, $T_{\mu\nu}$, $\epsilon_{\mu\nu\kappa\lambda}$, and $N_{\mu\nu\kappa\lambda}$ are assumed to "naturally" have all of their indices downstairs, and unless indicated otherwise, all indices on all tensors are lowered and raised with the Minkowski metric $m_{\mu\nu}$ and its inverse $(m^{-1})^{\mu\nu}$; e.g., $T^{\mu\nu} = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}T_{\kappa\lambda}$.
- The symbol # is used to indicate that all indices of a given tensor field have been raised with g^{-1}; e.g., $T^{\#\mu\nu} = (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}T_{\kappa\lambda}$.
- Repeated indices are summed over.

2.3. ***Coordinates.***

- $\{x^\mu\}_{\mu=0,1,2,3}$ denotes the wave-coordinate system.
- $t = x^0$, $x = (x^1, x^2, x^3)$.
- $q = r - t$ and $s = r + t$ are the null coordinates of the spacetime point (t, x), where $r = |x|$.
- $q_- = 0$ if $q \geq 0$, and $q_- = |q|$ if $q < 0$.
- $\omega^j = x^j/r$ $(j = 1, 2, 3)$.

2.4. ***Surfaces.*** Relative to the wave-coordinate system:

- $C_s^- = \{(\tau, y) \mid |y| + \tau = s\}$ are the ingoing Minkowskian null cones.
- $C_q^+ = \{(\tau, y) \mid |y| - \tau = q\}$ are the outgoing Minkowskian null cones.
- $\Sigma_t = \{(\tau, y) \mid \tau = t\}$ are the constant Minkowskian time slices.
- $S_{r,t} = \{(\tau, y) \mid \tau = t,\ |y| = r\}$ are the Euclidean spheres.

2.5. ***Metrics and volume forms.***

- $m_{\mu\nu}$ denotes the standard Minkowski metric on $\mathbb{R}^{1+3}$; $m_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ in our wave-coordinate system.
- $\underline{m}$ denotes the Minkowskian first fundamental form of Σ_t; $\underline{m}_{\mu\nu} = \text{diag}(0, 1, 1, 1)$ in our wave-coordinate system.
- $\not{m}$ denotes the Minkowskian first fundamental form of $S_{r,t}$; relative to an arbitrary coordinate system, $\not{m}_{\mu\nu} = m_{\mu\nu} + \frac{1}{2}(L_\mu \underline{L}_\nu + \underline{L}_\mu L_\nu)$, where $\underline{L}$ and L are defined in Section 2.9.
- $g_{\mu\nu}$ denotes the spacetime metric.
- $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ is the splitting of the spacetime metric into the Minkowski metric $m_{\mu\nu}$, the Schwarzschild tail $h_{\mu\nu}^{(0)} = \chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu}$, and the remainder $h_{\mu\nu}^{(1)}$.
- $h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$.
- $(g^{-1})^{\mu\nu} = (m^{-1})^{\mu\nu} + H_{(0)}^{\mu\nu} + H_{(1)}^{\mu\nu}$ is the splitting of the inverse spacetime metric into the inverse Minkowski metric $(m^{-1})^{\mu\nu}$, the Schwarzschild tail $H_{(0)}^{\mu\nu} = -\chi(r/t)\chi(r)(2M/r)\delta^{\mu\nu}$, and the remainder $H_{(1)}^{\mu\nu}$.
- $H^{\mu\nu} = H_{(0)}^{\mu\nu} + H_{(1)}^{\mu\nu}$.
- $\underline{\mathring{g}}$ denotes the first fundamental form of the Cauchy hypersurface Σ_0 relative to the spacetime metric g.

- $\mathring{\underline{g}}_{jk} = \delta_{jk} + \chi(r)(2M/r)\delta_{jk} + \mathring{\underline{h}}_{jk}^{(1)}$ is the splitting of $\mathring{\underline{g}}_{jk}$ into the Euclidean metric δ_{jk}, the Schwarzschild tail $\chi(r)(2M/r)\delta_{jk}$, and the remainder $\mathring{\underline{h}}_{jk}^{(1)}$.
- $\upsilon_{\mu\nu\kappa\lambda} = |\det m|^{1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the Minkowski metric m; $[\mu\nu\kappa\lambda]$ is totally antisymmetric with normalization $[0123] = 1$; $|\det m|^{1/2} = 1$ in our wave-coordinate system.
- $\epsilon_{\mu\nu\kappa\lambda} = |\det g|^{1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the spacetime metric g.
- $\epsilon^{\#\mu\nu\kappa\lambda} = -|\det g|^{-1/2}[\mu\nu\kappa\lambda]$ denotes the volume form of the spacetime metric g with all of the indices raised with g^{-1}.
- $\underline{\upsilon}_{\nu\kappa\lambda} = [0\nu\kappa\lambda]$ denotes the Euclidean volume form of the surfaces Σ_t viewed as embedded Riemannian submanifolds of Minkowski spacetime equipped with the wave-coordinate system.
- $\underline{\upsilon}_{ijk} = [ijk]$ denotes the Euclidean volume form of the surfaces Σ_t viewed as a Riemannian 3-manifold equipped with the standard Euclidean coordinate system.
- $\not{\upsilon}_{\mu\nu} = \upsilon_{\mu\nu\kappa\lambda}\underline{L}^{\kappa} L^{\lambda}$ denotes the Euclidean volume form of the spheres $S_{r,t}$.

2.6. *Hodge duals.* For an arbitrary two-form $\mathscr{F}_{\mu\nu}$:

- ${}^{\star}\mathscr{F}_{\mu\nu} = \frac{1}{2} g_{\mu\mu'} g_{\nu\nu'} \epsilon^{\#\mu'\nu'\kappa\lambda} \mathscr{F}_{\kappa\lambda} = -\frac{1}{2}|\det g|^{-1/2} g_{\mu\mu'} g_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda}$ denotes the Hodge dual of $\mathscr{F}_{\mu\nu}$ with respect to the spacetime metric $g_{\mu\nu}$.
- ${}^{\circledast}\mathscr{F}_{\mu\nu} = \frac{1}{2}\upsilon_{\mu\nu}{}^{\kappa\lambda}\mathscr{F}_{\kappa\lambda} = -\frac{1}{2}|\det\, m|^{-1/2} m_{\mu\mu'} m_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda}$ denotes the Hodge dual of $\mathscr{F}_{\mu\nu}$ with respect to the Minkowski metric $m_{\mu\nu}$. In our wave-coordinate system, $|\det m|^{-1/2} = 1$.

2.7. *Derivatives.*

- ∇ denotes the Levi-Civita connection corresponding to m.
- $\mathscr{D}$ denotes the Levi-Civita connection corresponding to g.
- $\mathring{\underline{\mathscr{D}}}$ denotes the Levi-Civita connection corresponding to $\mathring{\underline{g}}$.
- $\underline{\nabla}$ denotes the Levi-Civita connection corresponding to $\underline{m}$.
- $\not{\nabla}$ denotes the Levi-Civita connection corresponding to $\not{m}$.
- $\overline{\nabla}$ denotes the null frame projection of ∇ onto the outgoing Minkowski null cones; i.e., $\overline{\nabla}_{\mu} = \overline{\pi}_{\mu}{}^{\kappa}\nabla_{\kappa}$, where $\overline{\pi}_{\mu}{}^{\nu} = \delta_{\mu}^{\nu} + \frac{1}{2}L_{\mu}\underline{L}^{\nu}$ projects vectors X^{μ} onto the outgoing Minkowski null cones.
- In our wave-coordinate system $\{x^{\mu}\}_{\mu=0,1,2,3}$, $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and $\nabla_{\mu} = \nabla_{(\partial/\partial x^{\mu})}$.
- In our wave-coordinate system, $\partial_r = \omega^a \partial_a$ denotes the radial derivative, where $\omega^j = x^j/r$.
- In our wave-coordinate system, $\partial_s = \frac{1}{2}(\partial_r + \partial_t)$ and $\partial_q = \frac{1}{2}(\partial_r - \partial_t)$ denote the null derivatives; ∂_q denotes partial differentiation at fixed s and fixed angle $x/|x|$ while ∂_s denotes partial differentiation at fixed q and fixed angle $x/|x|$.
- If X is a vector field and ϕ is a function, then $X\phi = X^{\kappa}\,\partial_{\kappa}\phi$.
- ∇_X denotes the differential operator $X^{\kappa}\nabla_{\kappa}$.
- $\underline{\nabla}_X$ denotes the differential operator $X^{\kappa}\underline{\nabla}_{\kappa}$.
- $\not{\nabla}_X$ denotes the differential operator $X^{\kappa}\,\not{\nabla}_{\kappa}$.

- $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X.
- $[X, Y]^\mu = (\mathcal{L}_X Y)^\mu = X^\kappa\,\partial_\kappa Y^\mu - Y^\kappa\,\partial_\kappa X^\mu$ denotes the Lie bracket of the vector fields X and Y.
- For $Z \in \mathcal{Z}$, $\widehat{\nabla}_Z = \nabla_Z + c_Z$ denotes the modified covariant derivative, where the constant c_Z is defined in Section 2.8.
- For $Z \in \mathcal{Z}$, $\widehat{\mathcal{L}}_Z = \mathcal{L}_Z + 2c_Z$ denotes the modified Lie derivative, where the constant c_Z is defined in Section 2.8.
- $\nabla^I U$, $\underline{\nabla}^I U$, $\nabla_{\mathcal{Z}}^I U$, $\widehat{\nabla}_{\mathcal{Z}}^I U$, $\mathcal{L}_{\mathcal{Z}}^I U$, and $\widehat{\mathcal{L}}_{\mathcal{Z}}^I U$ respectively denote an $|I|$-th order iterated Minkowski covariant derivative, iterated Euclidean (spatial) covariant derivative, iterated Minkowski $\mathcal{Z}$-covariant derivative, iterated modified Minkowski $\mathcal{Z}$-covariant derivative, iterated $\mathcal{Z}$-Lie derivative, and iterated modified $\mathcal{Z}$-Lie derivative of the tensor field U.
- $\Box_m = (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ denotes the standard Minkowski wave operator.
- $\widetilde{\Box}_g = (g^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ denotes the reduced wave operator corresponding to the spacetime metric g. Note that ∇ is the *Minkowskian* connection.

2.8. *Minkowskian conformal Killing fields.* Relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3} = (t, x)$:

- $\partial_\mu = \frac{\partial}{\partial x^\mu}$ $(\mu = 0, 1, 2, 3)$ denotes a translation vector field.
- $\Omega_{jk} = x_j \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^j}$ $(1 \le j < k \le 3)$ denotes a rotation vector field.
- $\Omega_{0j} = -t\frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial t}$ $(j = 1, 2, 3)$ denotes a Lorentz boost vector field.
- $S = x^\kappa \frac{\partial}{\partial x^\kappa}$ denotes the scaling vector field.
- $\mathcal{O} = \{\Omega_{jk}\}_{1\le j<k\le 3}$ are the rotational Minkowskian Killing fields.
- $\mathcal{Z} = \big\{\frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S\big\}_{0\le\mu\le\nu\le 3}$.
- For $Z \in \mathcal{Z}$, ${}^{(Z)}\pi_{\mu\nu} = \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = c_Z m_{\mu\nu}$ is the Minkowskian deformation tensor of Z, where c_Z is a constant.
- Commutation properties with the Maxwell–Maxwell term:

$$\widehat{\mathcal{L}}_{\mathcal{Z}}^I\big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{F}_{\kappa\lambda}\big) = \big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\kappa\lambda}.$$

- Commutation properties with the Minkowski wave operator $\Box_m = (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$:

$$[\Box_m, \partial_\mu] = [\Box_m, \Omega_{\mu\nu}] = 0, \qquad [\Box_m, S] = 2\Box_m, \qquad [\nabla_Z, \Box_m] = -c_Z\Box_m, \quad \text{and} \quad \Box_m\nabla_Z\phi = \widehat{\nabla}\Box_m\phi.$$

2.9. *Minkowskian null frames.*

- $\underline{L} = \partial_t - \partial_r$ denotes the Minkowskian null geodesic vector field transversal to the C_q^+; it generates the cones C_s^-.
- $L = \partial_t + \partial_r$ denotes the Minkowskian null geodesic vector field generating the cones C_q^+.
- e_A $(A = 1, 2)$ denotes Minkowski-orthonormal vector fields spanning the tangent space of the spheres $S_{r,t}$.
- The set $\mathcal{L} = \{L\}$ contains only L.

- The set $\mathcal{T} = \{L, e_1, e_2\}$ denotes the frame vector fields tangent to the C_q^+.
- The set $\mathcal{N} = \{\underline{L}, L, e_1, e_2\}$ denotes the entire Minkowski null frame.

2.10. ***Minkowskian null frame decomposition.***

- For an arbitrary vector field X and frame vector field $N \in \mathcal{N}$, we define $X_N = X_\kappa N^\kappa$, where $X_\mu = m_{\mu\kappa} X^\kappa$.
- For an arbitrary vector field X, $X = X^\kappa \partial_\kappa = X^L L + X^{\underline{L}} \underline{L} + X^A e_A$, where $X^L = -\frac{1}{2} X_{\underline{L}}$, $X^{\underline{L}} = -\frac{1}{2} X_L$, and $X^A = X_A$.
- For an arbitrary pair of vector fields X and Y,

$$m(X, Y) = m_{\kappa\lambda} X^\kappa X^\lambda = X^\kappa Y_\kappa = -\tfrac{1}{2} X_L Y_{\underline{L}} - \tfrac{1}{2} X_{\underline{L}} Y_L + X_A Y_A.$$

If $\mathcal{F}_{\mu\nu}$ is any two-form, its Minkowskian null components are:

- $\underline{\alpha}_\mu = \not{m}_\mu{}^\nu \mathcal{F}_{\nu\lambda} \underline{L}^\lambda$.
- $\alpha_\mu = \not{m}_\mu{}^\nu \mathcal{F}_{\nu\lambda} L^\lambda$.
- $\rho = \frac{1}{2} \mathcal{F}_{\kappa\lambda} \underline{L}^\kappa L^\lambda$.
- $\sigma = \frac{1}{2} \not{\upsilon}^{\kappa\lambda} \mathcal{F}_{\kappa\lambda}$.

2.11. ***Electromagnetic decompositions.*** If $\mathcal{F}_{\mu\nu}$ is any two-form, ${}^\star\mathcal{M}_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} \big(\frac{\partial {}^\star\mathcal{L}}{\partial \mathcal{F}_{\kappa\lambda}} - \frac{\partial {}^\star\mathcal{L}}{\partial \mathcal{F}_{\lambda\kappa}}\big)$ and $\widehat{N}^\mu$ is the future-directed unit g-normal to Σ_t, then its electromagnetic components are:

- $\mathfrak{E}_\mu = \mathcal{F}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{B}_\mu = -{}^\star\mathcal{F}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{D}_\mu = -{}^\star\mathcal{M}_{\mu\kappa} \widehat{N}^\kappa$.
- $\mathfrak{H}_\mu = -\mathcal{M}_{\mu\kappa} \widehat{N}^\kappa$.

If $\mathcal{F}_{\mu\nu}$ is any two-form, then relative to the wave-coordinate system, its Minkowskian electromagnetic components are:

- $E_\mu = \mathcal{F}_{\mu 0}$.
- $B_\mu = -{}^\circledast\mathcal{F}_{\mu 0}$.
- $D_\mu = -{}^\circledast\mathcal{M}_{\mu 0}$.
- $H_\mu = -\mathcal{M}_{\mu 0}$.

2.12. ***Seminorms and energies.*** For an arbitrary type-$\binom{0}{2}$ tensor field $P_{\mu\nu}$ and $\mathcal{V}, \mathcal{W} \in \{\mathcal{L}, \mathcal{T}, \mathcal{N}\}$:

- $|P|_{\mathcal{V}\mathcal{W}} = \sum_{V \in \mathcal{V},\, W \in \mathcal{W}} |V^\kappa W^\lambda P_{\kappa\lambda}|$.
- $|\nabla P|_{\mathcal{V}\mathcal{W}} = \sum_{N \in \mathcal{N},\, V \in \mathcal{V},\, W \in \mathcal{W}} |V^\kappa W^\lambda N^\gamma \nabla_\gamma P_{\kappa\lambda}|$.
- $|\overline{\nabla} P|_{\mathcal{V}\mathcal{W}} = \sum_{T \in \mathcal{T},\, V \in \mathcal{V},\, W \in \mathcal{W}} |V^\kappa W^\lambda T^\gamma \nabla_\gamma P_{\kappa\lambda}|$.
- $|P| = |P|_{\mathcal{N}\mathcal{N}}$.
- $|\nabla P| = |\nabla P|_{\mathcal{N}\mathcal{N}}$.
- $|\overline{\nabla} P| = |\overline{\nabla} P|_{\mathcal{N}\mathcal{N}}$.

- We use similar notation for an arbitrary tensor field U of type $\binom{n}{m}$.

For an arbitrary tensor field U defined on the Euclidean space Σ_0 with Euclidean coordinate system $x = (x^1, x^2, x^3)$:

- $\|U\|_{L^2}^2 = \int_{x \in \mathbb{R}^3} |U(x)|^2 \, d^3x$ is the square of the standard spatial L^2 norm of U.
- $\|U\|_{L^\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^3} |U(x)|$ is the standard spatial L^∞ norm of U.
- $\|U\|_{H_\eta^\ell}^2 = \sum_{|I| \leq \ell} \int_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2 \, d^3x$ is the square of a weighted Sobolev norm of U.
- $\|U\|_{C_\eta^\ell}^2 = \sum_{|I| \leq \ell} \operatorname{ess\,sup}_{x \in \mathbb{R}^3} (1 + |x|^2)^{(\eta + |I|)} |\underline{\nabla}^I U(x)|^2$ is the square of a weighted L^∞ norm of U.

For arbitrary abstract initial data $(\mathring{\underline{h}}_{jk}^{(1)}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ on the manifold $\mathbb{R}^3$:

- $E_{\ell;\gamma}^2(0) = \|\underline{\nabla} \mathring{\underline{h}}^{(1)}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{K}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{D}}\|_{H_{1/2+\gamma}^\ell}^2 + \|\mathring{\mathfrak{B}}\|_{H_{1/2+\gamma}^\ell}^2$ is the square of the norm of the abstract initial data.

For an arbitrary symmetric type-$\binom{0}{2}$ tensor field $h_{\mu\nu}^{(1)}$ and an arbitrary two-form $\mathscr{F}_{\mu\nu}$:

- $\mathscr{E}_{\ell;\gamma;\mu}^2(t) = \sup_{0 \leq \tau \leq t} \sum_{|I| \leq \ell} \int_{\Sigma_\tau} (|\nabla \nabla_{\mathscr{Z}}^I h^{(1)}|^2 + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|^2) w(q) \, d^3x$ is the square of the energy of the pair $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$.

2.13. $O^\ell(\cdot)$ and $o^\ell(\cdot)$.

- Given an ℓ-times continuously differentiable function $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m)$ depending on the tensorial quantities $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$, we write $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m) = O^\ell(|\mathfrak{Q}_1|^{p_1} \cdots |\mathfrak{Q}_k|^{p_k}; \mathfrak{Q}_{k+1}, \ldots, \mathfrak{Q}_m)$ if we can decompose $f(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m) = \sum_{i=1}^n P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k) \tilde{f}_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m)$, where n is a positive integer, each $P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k)$ is a polynomial in the components of $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k$ that satisfies $|P_i(\mathfrak{Q}_1, \ldots, \mathfrak{Q}_k)| \lesssim |\mathfrak{Q}_1|^{p_1} \cdots |\mathfrak{Q}_k|^{p_k}$ on a neighborhood of the origin, and $\tilde{f}_i(\cdot)$ is ℓ-times continuously differentiable on a neighborhood of the origin.
- Given an ℓ-times continuously differentiable function $f(x)$, if $\lim_{r \to \infty} |\underline{\nabla}^I f(x)| / r^{a + |I|} = 0$ for $|I| \leq \ell$, we write $f(x) = o^\ell(r^{-a})$.

2.14. *Fixed constants.* The fixed constants ℓ, δ, γ, μ, γ', and μ' are subject to the following constraints:

- To prove our global stability theorem, we assume that ℓ is an integer satisfying $\ell \geq 10$.
- $0 < \delta < \frac{1}{4}$.
- $0 < \delta < \gamma < \frac{1}{2}$.
- $0 < \gamma' < \gamma - \delta$.
- $0 < \delta < \mu' < \frac{1}{2}$.
- $0 < \mu < \frac{1}{2} - \mu'$.

2.15. *Weights.*

- $w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0 \end{cases}$ is the energy estimate weight function.
- $\varpi = \varpi(q) = \begin{cases} (1 + |q|)^{1+\gamma'} & \text{if } q > 0, \\ (1 + |q|)^{1/2-\mu'} & \text{if } q < 0 \end{cases}$ is the pointwise decay estimate weight function.

3. The Einstein-nonlinear electromagnetic system in wave coordinates

In this section, we discuss (1.0.1a)–(1.0.1c) in detail. We also discuss our assumptions on the electromagnetic Lagrangian and introduce our wave-coordinate gauge. We then derive a reduced system of equations that is equivalent to the system (1.0.1a)–(1.0.1c) in the wave-coordinate gauge. Finally, we summarize the results by providing the version (3.7.1a)–(3.7.1c) of the reduced equations, which will be used throughout the remainder of the article. In particular, in this version, we distinguish between principal terms, which require a careful treatment, and "error terms", which are, from the point of view of decay rates, relatively easy to estimate.

In this article, we consider the $(1+3)$-dimensional electrogravitational system (1.0.1a)–(1.0.1c), which we restate here for convenience:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{3.0.1a}$$

$$(d\mathcal{F})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.0.1b}$$

$$(d\mathcal{M})_{\lambda\mu\nu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{3.0.1c}$$

We remark that the spacetimes we consider will always have the manifold structure $I \times \mathbb{R}^3$ for some "time" interval I. The energy-momentum tensor $T_{\mu\nu}$ is given below in (3.5.4a), while $\mathcal{M}_{\mu\nu}$ is related to $(g_{\mu\nu}, \mathcal{F}_{\mu\nu})$ via the constitutive relation (3.2.4). The precise forms of $T_{\mu\nu}$ and $\mathcal{M}_{\mu\nu}$ depend on the chosen model of electromagnetism, which, as is discussed in detail in Section 3.2, we assume is a Lagrangian-derived model subject to the restrictions (3.3.3a) and (3.3.4a)–(3.3.4b) below. We recall (see, e.g., [Christodoulou 2008; Wald 1984]) the following relationships between the *spacetime metric* $g_{\mu\nu}$, the *Riemann curvature tensor*[17] $R_{\mu\kappa\nu}{}^{\lambda}$, the *Ricci tensor* $R_{\mu\nu}$, the *scalar curvature* R, and the *Christoffel symbols* $\Gamma_{\mu}{}^{\kappa}{}_{\nu}$, which are valid in an arbitrary coordinate system:

$$R_{\mu\kappa\nu}{}^{\lambda} \stackrel{\text{def}}{=} \partial_\kappa \Gamma_{\mu}{}^{\lambda}{}_{\nu} - \partial_\mu \Gamma_{\kappa}{}^{\lambda}{}_{\nu} + \Gamma_{\kappa}{}^{\lambda}{}_{\beta}\Gamma_{\mu}{}^{\beta}{}_{\nu} - \Gamma_{\mu}{}^{\lambda}{}_{\beta}\Gamma_{\kappa}{}^{\beta}{}_{\nu}, \tag{3.0.2a}$$

$$R_{\mu\nu} \stackrel{\text{def}}{=} R_{\mu\kappa\nu}{}^{\kappa} = \partial_\kappa \Gamma_{\mu}{}^{\kappa}{}_{\nu} - \partial_\mu \Gamma_{\kappa}{}^{\kappa}{}_{\nu} + \Gamma_{\kappa}{}^{\kappa}{}_{\lambda}\Gamma_{\mu}{}^{\lambda}{}_{\nu} - \Gamma_{\mu}{}^{\lambda}{}_{\kappa}\Gamma_{\lambda}{}^{\kappa}{}_{\nu}, \tag{3.0.2b}$$

$$R \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda} R_{\kappa\lambda}, \tag{3.0.2c}$$

$$\Gamma_{\mu}{}^{\kappa}{}_{\nu} \stackrel{\text{def}}{=} \tfrac{1}{2}(g^{-1})^{\kappa\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{3.0.2d}$$

We also recall the following symmetry properties:

$$R_{\mu\nu} = R_{\nu\mu}, \tag{3.0.3}$$

$$\Gamma_{\mu}{}^{\kappa}{}_{\nu} = \Gamma_{\nu}{}^{\kappa}{}_{\mu}. \tag{3.0.4}$$

We note for future use that taking the trace with respect to g of each side of (3.0.1a) implies that

$$R = -(g^{-1})^{\kappa\lambda} T_{\kappa\lambda}. \tag{3.0.5}$$

Hence, (3.0.1a) is equivalent to

[17] Under our sign convention, $\mathscr{D}_\mu \mathscr{D}_\nu X_\kappa - \mathscr{D}_\nu \mathscr{D}_\mu X_\kappa = R_{\mu\nu\kappa}{}^{\lambda} X_\lambda$.

$$R_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} (g^{-1})^{\kappa\lambda} T_{\kappa\lambda}. \tag{3.0.1a$'$}$$

Furthermore, we note that the twice-contracted Bianchi identities (see, e.g., [Wald 1984]) are the relation (see Section 2.2 concerning our use of the notation #)

$$\mathscr{D}_\mu (R^{\#\mu\nu} - \tfrac{1}{2} (g^{-1})^{\mu\nu} R) = 0 \quad (\nu = 0, 1, 2, 3) \tag{3.0.6}$$

so that by (3.0.1a) $T_{\mu\nu}$ necessarily satisfies the following divergence-free condition:

$$\mathscr{D}_\mu T^{\#\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3). \tag{3.0.7}$$

In the above expressions, $\mathscr{D}$ denotes the Levi-Civita connection corresponding to $g_{\mu\nu}$.

3.1. *Wave coordinates.* In this article, we use the framework developed in [Lindblad and Rodnianski 2005; 2010] and work in a *wave-coordinate* system, which is defined to be a coordinate system in which

$$\Gamma^\mu \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda} \Gamma_{\kappa\ \lambda}^{\ \mu} = 0 \quad (\mu = 0, 1, 2, 3). \tag{3.1.1a}$$

The condition (3.1.1a) is also known as *harmonic gauge* or *de Donder gauge*. It is easy to check that the condition (3.1.1a) is equivalent to the conditions

$$g_{\mu\nu} (g^{-1})^{\kappa\lambda} \Gamma_{\kappa\ \lambda}^{\ \nu} = 0 \quad (\mu = 0, 1, 2, 3), \tag{3.1.1b}$$

$$(g^{-1})^{\kappa\lambda}\, \partial_\kappa g_{\lambda\mu} - \tfrac{1}{2} (g^{-1})^{\kappa\lambda}\, \partial_\mu g_{\kappa\lambda} = 0 \quad (\mu = 0, 1, 2, 3), \tag{3.1.1c}$$

$$\partial_\nu \big[\sqrt{|\det g|} (g^{-1})^{\mu\nu} \big] = 0 \quad (\mu = 0, 1, 2, 3). \tag{3.1.1d}$$

We also note that condition (3.1.1d) follows from the identity

$$\Gamma^\mu \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda} \Gamma_{\kappa\ \lambda}^{\ \mu} = - \frac{1}{\sqrt{|\det g|}}\, \partial_\nu \big[\sqrt{|\det g|} (g^{-1})^{\mu\nu} \big] \quad (\mu = 0, 1, 2, 3), \tag{3.1.2}$$

which holds in any coordinate system. Furthermore, if the wave-coordinate system is also interpreted to be a coordinate system in which the Minkowski metric takes the form $m_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, then all coordinate derivatives ∂ can be interpreted as covariant derivatives ∇, where ∇ is the Levi-Civita connection corresponding to the Minkowski metric. *Throughout the article, we will often take this point of view because it allows for a covariant interpretation of all of our equations.*

We remark that the use of wave coordinates in the study of the Einstein equations goes back at least to the work of de Donder [1921]. However, it was not until Choquet-Bruhat's [1952] fundamental work that it became clear that the Einstein equations are fundamentally hyperbolic in nature and that wave coordinates can be used to prove local well-posedness. See Section 4.3 for further discussion on the viability of using wave coordinates to analyze the system (3.0.1a)–(3.0.1c).

3.2. *The Lagrangian formulation of nonlinear electromagnetism.* In this section, we recall some standard facts concerning a classical electromagnetic field theory in a Lorentzian spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu})$. Our goal is to explain the origin of (3.0.1b)–(3.0.1c). We remark that, for our purposes in this section, we may assume that the spacetime is known. The fundamental quantity in such a classical electromagnetic field theory is the *Faraday tensor* $\mathscr{F}_{\mu\nu}$, which is an antisymmetric type-$\binom{0}{2}$ tensor field (i.e., a two-form). We assume the *Faraday–Maxwell law*, which is the postulate that $\mathscr{F}_{\mu\nu}$ is closed:

$$(d\mathscr{F})_{\lambda\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.1}$$

where d denotes the exterior derivative operator.

We restrict our attention to covariant theories of nonlinear electromagnetism arising from a Lagrangian $\mathscr{L}$. In such a theory, the Hodge dual[18] ${}^{\star}\mathscr{L}$ of $\mathscr{L}$ is a scalar-valued function of the two invariants of the Faraday tensor, which we denote by $\text{ϟ}_{(1)}$ and $\text{ϟ}_{(2)}$:

$$
\begin{aligned}
{}^{\star}\mathscr{L} &= {}^{\star}\mathscr{L}(\text{ϟ}_{(1)}, \text{ϟ}_{(2)}), && (3.2.2a)\\
\text{ϟ}_{(1)} = \text{ϟ}_{(1)}[\mathscr{F}] &\stackrel{\text{def}}{=} \tfrac{1}{2}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu}, && (3.2.2b)\\
\text{ϟ}_{(2)} = \text{ϟ}_{(2)}[\mathscr{F}] &\stackrel{\text{def}}{=} \tfrac{1}{4}(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}{}^{\star}\mathscr{F}_{\mu\nu} = \tfrac{1}{8}\epsilon^{\#\kappa\lambda\mu\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu}. && (3.2.2c)
\end{aligned}
$$

Throughout the article, we use $\star$ to denote the Hodge duality operator corresponding to the spacetime metric $g_{\mu\nu}$:

$${}^{\star}\mathscr{F}^{\#\mu\nu} \stackrel{\text{def}}{=} \tfrac{1}{2}\epsilon^{\#\mu\nu\kappa\lambda}\mathscr{F}_{\kappa\lambda}. \tag{3.2.3}$$

Here, $\epsilon^{\#\mu\nu\kappa\lambda}$ is totally antisymmetric with normalization $\epsilon^{\#0123} = -|\det g|^{-1/2}$ while $\epsilon_{\mu\nu\kappa\lambda}$ is totally antisymmetric with normalization $\epsilon_{0123} = |\det g|^{1/2}$. See Section 2.2 concerning our use of the notation #. We remind the reader that our main results are derived for a class of Lagrangians that satisfy certain assumptions; these assumptions are listed in (3.3.3a) and (3.3.4a)–(3.3.4b) below.

We now introduce the *Maxwell tensor* $\mathscr{M}_{\mu\nu}$, a two-form whose Hodge dual ${}^{\star}\mathscr{M}_{\mu\nu}$ is defined by

$${}^{\star}\mathscr{M}^{\#\mu\nu} \stackrel{\text{def}}{=} \frac{\partial\,{}^{\star}\mathscr{L}}{\partial\mathscr{F}_{\mu\nu}} - \frac{\partial\,{}^{\star}\mathscr{L}}{\partial\mathscr{F}_{\nu\mu}}. \tag{3.2.4}$$

We also postulate that $\mathscr{M}_{\mu\nu}$ is closed:

$$(d\mathscr{M})_{\lambda\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{3.2.5}$$

Taken together, (3.2.1) and (3.2.5) are the electromagnetic equations for $\mathscr{F}_{\mu\nu}$ corresponding to ${}^{\star}\mathscr{L}$.

We remark for future use that it is straightforward to verify that (3.2.1) is equivalent to any of

$$
\begin{aligned}
\mathscr{D}_{\lambda}\mathscr{F}_{\mu\nu} + \mathscr{D}_{\nu}\mathscr{F}_{\lambda\mu} + \mathscr{D}_{\mu}\mathscr{F}_{\nu\lambda} &= 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), && (3.2.6a)\\
\nabla_{\lambda}\mathscr{F}_{\mu\nu} + \nabla_{\nu}\mathscr{F}_{\lambda\mu} + \nabla_{\mu}\mathscr{F}_{\nu\lambda} &= 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), && (3.2.6b)\\
\mathscr{D}_{\mu}{}^{\star}\mathscr{F}^{\#\mu\nu} &= 0 \qquad\quad (\nu = 0, 1, 2, 3), && (3.2.6c)\\
\nabla_{\mu}{}^{\circledast}\mathscr{F}^{\mu\nu} &= 0 \qquad\quad (\nu = 0, 1, 2, 3) && (3.2.6d)
\end{aligned}
$$

[18] For brevity, we often refer to ${}^{\star}\mathscr{L}$ as the Lagrangian.

and that (3.2.5) is equivalent to any of

$$\mathscr{D}_\lambda \mathscr{M}_{\mu\nu} + \mathscr{D}_\nu \mathscr{M}_{\lambda\mu} + \mathscr{D}_\mu \mathscr{M}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.7a}$$

$$\nabla_\lambda \mathscr{M}_{\mu\nu} + \nabla_\nu \mathscr{M}_{\lambda\mu} + \nabla_\mu \mathscr{M}_{\nu\lambda} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.2.7b}$$

$$\mathscr{D}_\mu {}^\star\mathscr{M}^{\#\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3), \tag{3.2.7c}$$

$$\nabla_\mu {}^\circledast\mathscr{M}^{\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3). \tag{3.2.7d}$$

In the above formulas, $\circledast$ denotes the Hodge duality operator corresponding to the Minkowski metric $m_{\mu\nu}$; this operator is defined in Section 2.6.

We state as a lemma the following identities, which will be used for various computations. We leave the proof as a simple exercise for the reader.

Lemma 3.1 (Basic identities). *The following identities hold*:

$$\frac{\partial|\det g|}{\partial g_{\mu\nu}} = |\det g|(g^{-1})^{\mu\nu}, \tag{3.2.8a}$$

$$\frac{\partial(g^{-1})^{\kappa\lambda}}{\partial g_{\mu\nu}} = -(g^{-1})^{\kappa\mu}(g^{-1})^{\lambda\nu}, \tag{3.2.8b}$$

$$\text{ϟ}_{(2)}^2 = |\det \mathscr{F}||\det g|^{-1}, \tag{3.2.8c}$$

$$(g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - (g^{-1})^{\kappa\lambda\star}\mathscr{F}_{\mu\kappa}{}^\star\mathscr{F}_{\nu\lambda} = \text{ϟ}_{(1)} g_{\mu\nu}, \tag{3.2.8d}$$

$$(g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}{}^\star\mathscr{F}_{\nu\lambda} = \text{ϟ}_{(2)} g_{\mu\nu}, \tag{3.2.8e}$$

$$\frac{\partial \text{ϟ}_{(1)}}{\partial g_{\mu\nu}} = -g_{\kappa\lambda}\mathscr{F}^{\#\mu\kappa}\mathscr{F}^{\#\nu\lambda}, \tag{3.2.8f}$$

$$\frac{\partial \text{ϟ}_{(2)}}{\partial g_{\mu\nu}} = -\tfrac{1}{2}\text{ϟ}_{(2)}(g^{-1})^{\mu\nu}, \tag{3.2.8g}$$

$$\frac{\partial \text{ϟ}_{(1)}}{\partial \mathscr{F}_{\mu\nu}} = \mathscr{F}^{\#\mu\nu}, \tag{3.2.8h}$$

$$\frac{\partial \text{ϟ}_{(2)}}{\partial \mathscr{F}_{\mu\nu}} = \tfrac{1}{2}{}^\star\mathscr{F}^{\#\mu\nu}, \tag{3.2.8i}$$

$$\frac{\partial \mathscr{F}^{\#\mu\nu}}{\partial \mathscr{F}_{\kappa\lambda}} = (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}, \tag{3.2.8j}$$

$$\frac{\partial {}^\star\mathscr{F}^{\#\mu\nu}}{\partial \mathscr{F}_{\kappa\lambda}} = \tfrac{1}{2}\epsilon^{\#\mu\nu\kappa\lambda}, \tag{3.2.8k}$$

$$\mathscr{D}_\mu \text{ϟ}_{(1)} = \mathscr{F}^{\#\kappa\lambda}\mathscr{D}_\mu\mathscr{F}_{\kappa\lambda} \quad (\mu = 0, 1, 2, 3), \tag{3.2.8l}$$

$$\mathscr{D}_\mu \text{ϟ}_{(2)} = \tfrac{1}{2}{}^\star\mathscr{F}^{\#\kappa\lambda}\mathscr{D}_\mu\mathscr{F}_{\kappa\lambda} \quad (\mu = 0, 1, 2, 3), \tag{3.2.8m}$$

$${}^\star\mathscr{M}^{\#\mu\nu} = 2\frac{\partial^\star\mathscr{L}}{\partial\text{ϟ}_{(1)}}\mathscr{F}^{\#\mu\nu} + \frac{\partial^\star\mathscr{L}}{\partial\text{ϟ}_{(2)}}{}^\star\mathscr{F}^{\#\mu\nu}. \tag{3.2.8n}$$

3.3. ***Assumptions on the electromagnetic Lagrangian.*** The standard Maxwell–Maxwell equations correspond to the Lagrangian

$$ {}^{\star}\mathscr{L}_{(\mathrm{Maxwell})} = -\tfrac{1}{2}ϟ_{(1)}, \tag{3.3.1} $$

which by (3.2.8n) leads to the relationship

$$ \mathscr{M}_{\mu\nu}^{(\mathrm{Maxwell})} = {}^{\star}\mathscr{F}_{\mu\nu}. \tag{3.3.2} $$

Roughly speaking, we will assume that our electromagnetic Lagrangian is a covariant perturbation of ${}^{\star}\mathscr{L}_{(\mathrm{Maxwell})}$. More precisely, we make the following assumptions concerning our Lagrangian ${}^{\star}\mathscr{L}$:

Assumptions. We assume that, in a neighborhood of $(0, 0)$, ${}^{\star}\mathscr{L}$ is an $(\ell + 2)$-times (where $\ell \geq 10$) continuously differentiable function of the invariants $(ϟ_{(1)}, ϟ_{(2)})$ that can be expanded as follows:

$$ {}^{\star}\mathscr{L} = {}^{\star}\mathscr{L}_{(\mathrm{Maxwell})} + O^{\ell+2}(|(ϟ_{(1)}, ϟ_{(2)})|^2). \tag{3.3.3a} $$

The notation $O^{\ell+2}(\,\cdot\,)$ is defined in Section 2.13.

We also assume that the corresponding energy-momentum tensor $T_{\mu\nu}$, which is defined below in (3.5.1), satisfies the *dominant energy condition*, which is the assumption that

$$ T_{\kappa\lambda}X^{\kappa}Y^{\lambda} \geq 0 \tag{3.3.3b} $$

whenever the following conditions are satisfied:

- X and Y are both timelike (i.e., $g_{\kappa\lambda}X^{\kappa}X^{\lambda} < 0$ and $g_{\kappa\lambda}Y^{\kappa}Y^{\lambda} < 0$).
- X and Y are g-future-directed.

As discussed in, e.g., [Gibbons and Herdeiro 2001], sufficient conditions for the dominant energy condition to hold are

$$ \frac{\partial {}^{\star}\mathscr{L}}{\partial ϟ_{(1)}} < 0, \tag{3.3.4a} $$

$$ {}^{\star}\mathscr{L} - ϟ_{(1)}\frac{\partial {}^{\star}\mathscr{L}}{\partial ϟ_{(1)}} - ϟ_{(2)}\frac{\partial {}^{\star}\mathscr{L}}{\partial ϟ_{(2)}} \leq 0. \tag{3.3.4b} $$

We remark that it is straightforward to verify the sufficiency of these conditions by using (3.5.4b) below and that condition (3.3.4b) is equivalent to the nonpositivity of the trace of the energy-momentum tensor corresponding to ${}^{\star}\mathscr{L}$. Furthermore, we recall that the trace vanishes in the case of the standard Maxwell–Maxwell model.

Remark 3.2. We make the $(\ell+2)$-times differentiability assumption because we will need to differentiate the equations (3.3.7) below ℓ times in order to prove our main stability theorem.

We will now derive an equivalent version of the electromagnetic equations that will be used throughout the remainder of the article. The final form, which is valid only in a wave-coordinate system, is given below in Lemma 3.4. To begin, we use (3.2.6c), (3.2.7c), and (3.2.8n) to compute that the following equation holds:

$$-2\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}}\mathscr{D}_{\mu}\mathscr{F}^{\#\mu\nu} - 2\mathscr{F}^{\#\mu\nu}\mathscr{D}_{\mu}\Big(\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}}\Big) - {}^{\star}\mathscr{F}^{\#\mu\nu}\mathscr{D}_{\mu}\Big(\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(2)}}\Big) = 0. \tag{3.3.5}$$

Furthermore, from the chain rule and the fact that $\mathscr{D}_{\mu}\phi = \nabla_{\mu}\phi$ for scalar-valued functions ϕ, it follows from (3.3.5) and (3.2.8l)–(3.2.8m) that

$$\begin{aligned} -2\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}}\mathscr{D}_{\mu}\mathscr{F}^{\#\mu\nu} - \Big(2\mathscr{F}^{\#\mu\nu}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}^2} + {}^{\star}\mathscr{F}^{\#\mu\nu}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}\,\partial \digamma_{(2)}}\Big)\nabla_{\mu}\digamma_{(1)} \\ - \Big(2\mathscr{F}^{\#\mu\nu}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}\,\partial \digamma_{(2)}} + {}^{\star}\mathscr{F}^{\#\mu\nu}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(2)}^2}\Big)\nabla_{\mu}\digamma_{(2)} = 0. \end{aligned} \tag{3.3.6}$$

We note for future use that (3.3.6) can be expressed as

$$N^{\#\mu\nu\kappa\lambda}\mathscr{D}_{\mu}\mathscr{F}_{\kappa\lambda} = 0 \quad (\nu = 0, 1, 2, 3), \tag{3.3.7}$$

where the tensor field $N^{\#\mu\nu\kappa\lambda}$ is defined by

$$\begin{aligned} N^{\#\mu\nu\kappa\lambda} \stackrel{\text{def}}{=} -\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}}\big((g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda} - (g^{-1})^{\mu\lambda}(g^{-1})^{\nu\kappa}\big) - 2\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}^2}\mathscr{F}^{\#\mu\nu}\mathscr{F}^{\#\kappa\lambda} \\ - \frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(1)}\partial \digamma_{(2)}}\big(\mathscr{F}^{\#\mu\nu}{}^{\star}\mathscr{F}^{\#\kappa\lambda} + {}^{\star}\mathscr{F}^{\#\mu\nu}{}^{\star}\mathscr{F}^{\#\kappa\lambda}\big) - \frac{1}{2}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \digamma_{(2)}^2}{}^{\star}\mathscr{F}^{\#\mu\nu}{}^{\star}\mathscr{F}^{\#\kappa\lambda}. \end{aligned} \tag{3.3.8}$$

We also note that $N^{\#\mu\nu\kappa\lambda}$ has the following symmetry properties, which will play an important role during our construction of suitable energies for $\mathscr{F}_{\mu\nu}$ (and in particular during our proof of Lemma 8.5):

$$N^{\#\nu\mu\kappa\lambda} = -N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.3.9a}$$

$$N^{\#\mu\nu\lambda\kappa} = -N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.3.9b}$$

$$N^{\#\kappa\lambda\mu\nu} = N^{\#\mu\nu\kappa\lambda} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3). \tag{3.3.9c}$$

The moral reason that the above properties are satisfied is that $N^{\#\mu\nu\kappa\lambda}$ is closely related to the Hessian of ${}^{\star}\mathscr{L}$ (with respect to $\mathscr{F}$):

$$N^{\#\mu\nu\kappa\lambda} = -\frac{1}{2}\frac{\partial^2 {}^{\star}\mathscr{L}}{\partial \mathscr{F}_{\mu\nu}\,\partial \mathscr{F}_{\kappa\lambda}} + \frac{1}{2}\frac{\partial {}^{\star}\mathscr{L}}{\partial \digamma_{(2)}}\epsilon^{\#\mu\nu\kappa\lambda}. \tag{3.3.10}$$

We have added the last term on the right-hand side of (3.3.10) in order to cancel a term appearing in the Hessian; this is permissible because (3.2.6a) implies that this term does not contribute to (3.3.7).

Our next goal is to formulate a "reduced" electromagnetic equation that is equivalent to (3.3.7) in a wave-coordinate system. We also decompose the reduced equation into the principal terms and error terms of an equation involving the Minkowski connection ∇. This is accomplished in Lemma 3.4 below. Before proving this lemma, we first provide the following preliminary lemma, whose simple proof is left to the reader:

Lemma 3.3 (Expansions). *Assume that the electromagnetic Lagrangian* $^{\star}\mathscr{L}$ *satisfies* (3.3.3a). *Then in terms of the expansion* $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ *from* (1.2.1a) *and with* $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, *we have*

$$\begin{aligned}
H^{\mu\nu} &= -h^{\mu\nu} + O^{\infty}(|h|^2) \\
&= -h^{\mu\nu} + O^{\infty}(|H|^2), && (3.3.11a) \\
\nabla_\lambda (g^{-1})^{\mu\nu} &= -(g^{-1})^{\mu\mu'}(g^{-1})^{\nu\nu'}\nabla_\lambda h_{\mu'\nu'} \\
&= -(m^{-1})^{\mu\mu'}(m^{-1})^{\nu\nu'}\nabla_\lambda h_{\mu'\nu'} + O^{\infty}(|h||\nabla h|), && (3.3.11b) \\
|\det g| &= 1 + (m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^{\infty}(|h|^2) \\
&= 1 - m_{\kappa\lambda}H^{\kappa\lambda} + O^{\infty}(|H|^2), && (3.3.11c) \\
|\det g|^{1/2} &= 1 + \tfrac{1}{2}(m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^{\infty}(|h|^2) \\
&= 1 - \tfrac{1}{2}m_{\kappa\lambda}H^{\kappa\lambda} + O^{\infty}(|H|^2), && (3.3.11d) \\
|\det g|^{-1/2} &= 1 - \tfrac{1}{2}(m^{-1})^{\kappa\lambda}h_{\kappa\lambda} + O^{\infty}(|h|^2) \\
&= 1 + \tfrac{1}{2}m_{\kappa\lambda}H^{\kappa\lambda} + O^{\infty}(|H|^2), && (3.3.11e) \\
\epsilon^{\#\mu\nu\kappa\lambda} &= -(1 + O^{\infty}(|h|))[\mu\nu\kappa\lambda], && (3.3.11f) \\
\epsilon_{\mu\nu\kappa\lambda} &= (1 + O^{\infty}(|h|))[\mu\nu\kappa\lambda], && (3.3.11g) \\
\mathscr{F}^{\#\mu\nu} &= \mathscr{F}^{\mu\nu} + O^{\infty}(|h||\mathscr{F}|) \stackrel{\text{def}}{=} (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda} + O^{\infty}(|h||\mathscr{F}|), && (3.3.11h) \\
{}^{\star}\mathscr{F}_{\mu\nu} &= {}^{\circledast}\mathscr{F}_{\mu\nu} + O^{\infty}(|h||\mathscr{F}|) \stackrel{\text{def}}{=} -\tfrac{1}{2}m_{\mu\mu'}m_{\nu\nu'}[\mu'\nu'\kappa\lambda]\mathscr{F}_{\kappa\lambda} + O^{\infty}(|h||\mathscr{F}|), && (3.3.11i) \\
\digamma_{(1)} &= \tfrac{1}{2}(m^{-1})^{\kappa\mu}(m^{-1})^{\lambda\nu}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\nu} + O^{\infty}(|h||\mathscr{F}|^2), && (3.3.11j) \\
\digamma_{(2)} &= -\tfrac{1}{8}[\mu\nu\kappa\lambda]\mathscr{F}_{\mu\nu}\mathscr{F}_{\kappa\lambda} + O^{\infty}(|h||\mathscr{F}|^2), && (3.3.11k) \\
{}^{\star}\mathscr{L} &= -\tfrac{1}{4}(m^{-1})^{\eta\kappa}(m^{-1})^{\zeta\lambda}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta} + O^{\ell+2}(|h||\mathscr{F}|^2) + O^{\ell+2}(|\mathscr{F}|^4; h), && (3.3.11l) \\
\nabla\digamma_{(i)} &= O^{\infty}(|\mathscr{F}||\nabla\mathscr{F}|) + O^{\infty}(|\nabla h||\mathscr{F}|^2; h) + O^{\infty}(|h||\mathscr{F}||\nabla\mathscr{F}|), && (3.3.11m) \\
\mathscr{M}_{\mu\nu} &= {}^{\circledast}\mathscr{F}_{\mu\nu} + O^{\ell+1}(|h||\mathscr{F}|) + O^{\ell+1}(|\mathscr{F}|^3; h). && (3.3.11n)
\end{aligned}$$

In (3.3.11f)–(3.3.11g), $[\mu\nu\kappa\lambda]$ *is totally antisymmetric with normalization* $[0123] = 1$, $\star$ *denotes the Hodge duality operator corresponding to the spacetime metric* $g_{\mu\nu}$, *and* $\circledast$ *denotes the Hodge duality operator corresponding to the Minkowski metric* $m_{\mu\nu}$. *Furthermore*, *the notation* $O(\cdot)$ *is defined in Section 2.13.*

3.4. ***The reduced electromagnetic equations.*** In this section, we provide the aforementioned decomposition of the reduced electromagnetic equations.

Lemma 3.4 (The reduced electromagnetic equations). *Assume that the wave-coordinate condition* (3.1.1a) *holds. Then in terms of the expansion* (1.2.1a), *the system of electromagnetic equations* (3.2.1) *and* (3.3.7) *is equivalent to the following reduced system of equations*:

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0, \tag{3.4.1a}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + O^{\ell}(|h||\nabla h||\mathscr{F}|) + O^{\ell}(|\nabla h||\mathscr{F}|^2; h), \tag{3.4.1b}$$

where

$$\begin{aligned} N^{\#\mu\nu\kappa\lambda} = \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big) \\ + \tfrac{1}{2}\big(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}\big) \\ + \tfrac{1}{2}\big(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big) \ + \ N^{\#\mu\nu\kappa\lambda}_{\triangle}, \end{aligned} \tag{3.4.2}$$

$$\mathfrak{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\nu'}(m^{-1})^{\lambda\lambda'}(\nabla_\mu h_{\nu'\lambda'})\mathscr{F}_{\kappa\lambda}. \tag{3.4.3}$$

Furthermore,

$$N^{\#\mu\nu\kappa\lambda}_{\triangle} = O^{\ell}\big(|(h, \mathscr{F})|^2\big), \tag{3.4.4}$$

and like $N^{\#\mu\nu\kappa\lambda}$, *the tensor field* $N^{\#\mu\nu\kappa\lambda}_{\triangle}$ *also possesses the symmetry properties* (3.3.9a)–(3.3.9c).

Remark 3.5. Equations (3.4.1a)–(3.4.3) are equivalent to (3.2.1) and (3.3.7) only in a wave-coordinate system. Hence, we refer to (3.4.1a)–(3.4.3) as the "reduced" electromagnetic equations.

Proof. We use the assumption (3.3.3a) and the Leibniz rule to expand (3.3.6) and apply the results of Lemma 3.3, arriving at the following expansion:

$$\mathscr{D}_\mu \mathscr{F}^{\#\mu\nu} + \widetilde{N}^{\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} = O^{\ell}(|h||\nabla h||\mathscr{F}|) + O^{\ell}(|\nabla h||\mathscr{F}|^2; h), \tag{3.4.5}$$

where $\widetilde{N}^{\mu\nu\kappa\lambda} = O^{\ell}(|(h, \mathscr{F})|^2)$. Let us now decompose the $\mathscr{D}_\mu \mathscr{F}^{\#\mu\nu}$ term. Using the antisymmetry of $\mathscr{F}^{\#\mu\nu}$, the symmetry of the Christoffel symbol $\Gamma_{\mu\ \lambda}^{\ \nu}$ under the exchanges $\mu \leftrightarrow \lambda$, the identity $\Gamma_{\kappa\ \mu}^{\ \kappa} = (1/\sqrt{|\det g|})\nabla_\mu(\sqrt{|\det g|})$, and the wave-coordinate condition $\nabla_\mu[\sqrt{|\det g|}(g^{-1})^{\mu\kappa}] = 0$ ($\kappa = 0, 1, 2, 3$), we have that

$$\begin{aligned} \mathscr{D}_\mu \mathscr{F}^{\#\mu\nu} &= \nabla_\mu \mathscr{F}^{\#\mu\nu} + \Gamma_{\kappa\ \mu}^{\ \kappa}\mathscr{F}^{\#\mu\nu} + \Gamma_{\mu\ \lambda}^{\ \nu}\mathscr{F}^{\#\mu\lambda} \\ &= \nabla_\mu\big[(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}\big] + \left[\frac{1}{\sqrt{|\det g|}}\nabla_\mu(\sqrt{|\det g|})\right](g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda} \\ &= \frac{1}{\sqrt{|\det g|}}\nabla_\mu\big[\sqrt{|\det g|}(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}\big] \\ &= (g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} + \big[(g^{-1})^{\mu\kappa}\nabla_\mu(g^{-1})^{\nu\lambda}\big]\mathscr{F}_{\kappa\lambda}. \end{aligned} \tag{3.4.6}$$

Using (3.3.11a), we conclude that the term $(g^{-1})^{\mu\kappa}(g^{-1})^{\nu\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda}$ on the right-hand side of (3.4.6) can be expressed as the terms in parentheses on the right-hand side of (3.4.2) plus $O^{\ell}(|h^2|)\nabla_\mu \mathscr{F}_{\kappa\lambda}$.

Similarly, using (3.3.11b), we conclude that the term $[(g^{-1})^{\mu\kappa}\nabla_\mu(g^{-1})^{\nu\lambda}]\mathscr{F}_{\kappa\lambda}$ on the right-hand side of (3.4.6) is equal to $-\mathfrak{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + O^{\ell}(|h||\nabla h||\mathscr{F}|)$, where $\mathfrak{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F})$ is defined in (3.4.3). Combining these expansions with (3.4.5), we arrive at (3.4.1b)–(3.4.4).

The fact that $N^{\#\mu\nu\kappa\lambda}_{\triangle}$ possesses the symmetry properties (3.3.9a)–(3.3.9c) follows trivially from the fact that both $N^{\#\mu\nu\kappa\lambda}$ and the term in parentheses on the right-hand side of (3.4.2) have these properties. □

Remark 3.6. With the help of the identity (3.1.2), the above proof shows that the reduced equation (3.4.1b) is obtained by adding the inhomogeneous term $-\Gamma^{\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}$ to the right-hand side of (3.3.7). That is, (3.4.1b) is equivalent to

$$N^{\#\mu\nu\kappa\lambda}\mathscr{D}_{\mu}\mathscr{F}_{\kappa\lambda} = -\Gamma^{\kappa}(g^{-1})^{\nu\lambda}\mathscr{F}_{\kappa\lambda}. \tag{3.4.7}$$

We will use this fact in our proof of Proposition 4.2.

3.5. ***The energy-momentum tensor.*** In this section, we discuss the energy-momentum tensor $T_{\mu\nu}$ appearing on the right-hand side of (3.0.1a). We recall that the energy-momentum tensor for an electromagnetic Lagrangian field theory is defined as follows:

$$T^{\#\mu\nu} \stackrel{\text{def}}{=} 2\frac{\partial^{\star}\mathscr{L}}{\partial g_{\mu\nu}} + (g^{-1})^{\mu\nu\star}\mathscr{L}. \tag{3.5.1}$$

It follows trivially from the definition (3.5.1) that $T_{\mu\nu}$ is symmetric:

$$T_{\mu\nu} = T_{\nu\mu} \quad (\mu, \nu = 0, 1, 2, 3). \tag{3.5.2}$$

Furthermore, we recall that, if $\mathscr{F}_{\mu\nu}$ is a solution to the (nonreduced) electromagnetic equations (3.0.1b)–(3.0.1c), then

$$\mathscr{D}_{\mu}T^{\#\mu\nu} = 0 \quad (\nu = 0, 1, 2, 3). \tag{3.5.3}$$

For the class of electromagnetic energy-momentum tensors considered in this article, we can use the chain rule and Lemma 3.1 to express $T_{\mu\nu}$ as follows:

$$T_{\mu\nu} = -2\frac{\partial^{\star}\mathscr{L}}{\partial ϟ_{(1)}}(g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - ϟ_{(2)}\frac{\partial^{\star}\mathscr{L}}{\partial ϟ_{(2)}}g_{\mu\nu} + g_{\mu\nu}{}^{\star}\mathscr{L} \tag{3.5.4a}$$

$$= -2\frac{\partial^{\star}\mathscr{L}}{\partial ϟ_{(1)}}T_{\mu\nu}^{(\text{Maxwell})} + \tfrac{1}{4}T g_{\mu\nu}, \tag{3.5.4b}$$

where

$$T_{\mu\nu}^{(\text{Maxwell})} \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \tfrac{1}{2}ϟ_{(1)}g_{\mu\nu} \tag{3.5.5}$$

is the energy-momentum tensor corresponding to the standard Maxwell–Maxwell equations and

$$T \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}T_{\kappa\lambda} = 4\left({}^{\star}\mathscr{L} - ϟ_{(1)}\frac{\partial^{\star}\mathscr{L}}{\partial ϟ_{(1)}} - ϟ_{(2)}\frac{\partial^{\star}\mathscr{L}}{\partial ϟ_{(2)}}\right) \tag{3.5.6}$$

is the trace of $T_{\mu\nu}$ with respect to $g_{\mu\nu}$. Furthermore, from (3.5.4a) and the expansions of Lemma 3.3, it follows that

$$T_{\mu\nu} = (m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} - \tfrac{1}{4}m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\zeta}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta} + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.5.7}$$

We now compute the right-hand side of (3.0.1a′). First, taking the trace of (3.5.7) with respect to g, we compute that

$$(g^{-1})^{\kappa\lambda}T_{\kappa\lambda} = O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{3.5.8}$$

Combining (3.5.7) and (3.5.8) and using the expansion (1.2.1a), we have that the right-hand side of (3.0.1a′) can be expressed as follows:

$$T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} (g^{-1})^{\kappa\lambda} T_{\kappa\lambda} = (m^{-1})^{\kappa\lambda} \mathscr{F}_{\mu\kappa} \mathscr{F}_{\nu\lambda} - \tfrac{1}{4} m_{\mu\nu} (m^{-1})^{\kappa\eta} (m^{-1})^{\lambda\zeta} \mathscr{F}_{\kappa\lambda} \mathscr{F}_{\eta\zeta} + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \quad (3.5.9)$$

To conclude this section, we note for future use that, if $\mathscr{F}_{\mu\nu}$ is a solution to the inhomogeneous system

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \quad (3.5.10a)$$

$$N^{\#\mu\nu\kappa\lambda} \mathscr{D}_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{J}^\nu \qquad (\nu = 0, 1, 2, 3), \quad (3.5.10b)$$

then with the help of Lemma 3.1, it can be shown that the following identity holds:

$$(g^{-1})^{\kappa\lambda} \mathscr{D}_\kappa T_{\lambda\nu} = \mathfrak{J}^\kappa \mathscr{F}_{\nu\kappa} \quad (\nu = 0, 1, 2, 3). \quad (3.5.11)$$

We will use this fact in our proof of Proposition 4.2 (which shows that the wave-coordinate gauge is preserved by the flow of the reduced equations), where $\mathfrak{J}^\nu$ will be equal to the right-hand side of (3.4.7). We also remark that (3.5.3) corresponds to the special case $\mathfrak{J}^\nu = 0$ $(\nu = 0, 1, 2, 3)$.

3.6. ***The modified Ricci tensor.*** Throughout the remainder of this article, we perform the standard wave-coordinate system procedure (see, e.g., [Wald 1984]) of replacing the Ricci tensor $R_{\mu\nu}$ in the Einstein field equation (3.0.1a) with a modified Ricci tensor $\widetilde{R}_{\mu\nu}$. As we will soon see, this replacement transforms equations (3.0.1a) into a system of quasilinear wave equations.

Definition 3.7. We define the *modified Ricci tensor* $\widetilde{R}_{\mu\nu}$ of the metric $g_{\mu\nu}$ as follows:

$$\widetilde{R}_{\mu\nu} \overset{\text{def}}{=} R_{\mu\nu} - \tfrac{1}{2}(g_{\kappa\nu} \mathscr{D}_\mu \Gamma^\kappa + g_{\kappa\mu} \mathscr{D}_\nu \Gamma^\kappa) + u_{\mu\nu\kappa}(g, g^{-1}, \partial g) \Gamma^\kappa, \quad (3.6.1)$$

where the Ricci tensor $R_{\mu\nu}$ is defined in (3.0.2b) and the "gauge term" $u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$ is a smooth function of g, g^{-1}, and ∂g that will be discussed in Lemma 3.8. We remark that, for purposes of covariant differentiation by $\mathscr{D}$ in (3.6.1), the Γ^μ are treated as the components of a vector field.

In the next lemma, we provide an algebraic decomposition of the modified Ricci tensor.

Lemma 3.8 (Decomposition of the modified Ricci tensor [Lindblad and Rodnianski 2005, Lemmas 3.1 and 3.2]). *For a suitable choice of the gauge term $u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$, the modified Ricci tensor $\widetilde{R}_{\mu\nu}$ of the metric $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}$ can be decomposed as follows*:

$$\widetilde{R}_{\mu\nu} = -\tfrac{1}{2}\big(\widetilde{\Box}_g g_{\mu\nu} - \mathscr{P}(\nabla_\mu h, \nabla_\nu h) - \mathfrak{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h)\big) + O^\infty(|h||\nabla h|^2), \quad (3.6.2)$$

where

$$\widetilde{\Box}_g \overset{\text{def}}{=} (g^{-1})^{\kappa\lambda} \nabla_\kappa \nabla_\lambda \quad (3.6.3)$$

is the reduced wave operator *corresponding to $g_{\mu\nu}$ and the quadratic terms $\mathscr{P}(\nabla_\mu\,\cdot\,, \nabla_\nu\,\cdot\,)$ and $\mathfrak{Q}^{(1;h)}_{\mu\nu}(\,\cdot\,,\,\cdot\,)$ are defined by their action on tensor fields $\Pi_{\mu\nu}$, $\Theta_{\mu\nu}$, and $h_{\mu\nu}$ as follows*:

$$\mathscr{P}(\nabla_\mu \Pi, \nabla_\nu \Theta) \stackrel{\text{def}}{=} \tfrac{1}{4}(\nabla_\mu \Pi_\kappa{}^\kappa)(\nabla_\nu \Theta_\lambda{}^\lambda) - \tfrac{1}{2}(\nabla_\mu \Pi^{\kappa\lambda})(\nabla_\nu \Theta_{\kappa\lambda}), \tag{3.6.4}$$

$$\begin{aligned}\mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) \stackrel{\text{def}}{=} (m^{-1})^{\lambda\lambda'} \mathfrak{Q}_0(\nabla h_{\lambda\mu}, \nabla h_{\lambda'\nu}) \\ - (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'} \mathfrak{Q}_{\kappa\lambda'}(\nabla h_{\lambda\mu}, \nabla h_{\kappa'\nu}) \\ + (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'} \mathfrak{Q}_{\mu\kappa}(\nabla h_{\kappa'\lambda'}, \nabla h_{\lambda\nu}) \\ + (m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'} \mathfrak{Q}_{\nu\kappa}(\nabla h_{\kappa'\lambda'}, \nabla h_{\lambda\mu}) \\ + \tfrac{1}{2}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'} \mathfrak{Q}_{\lambda'\mu}(\nabla h_{\kappa\kappa'}, \nabla h_{\lambda\nu}) \\ + \tfrac{1}{2}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\lambda\lambda'} \mathfrak{Q}_{\lambda'\nu}(\nabla h_{\kappa\kappa'}, \nabla h_{\lambda\mu}).\end{aligned} \tag{3.6.5}$$

The bilinear forms $\mathfrak{Q}_0(\cdot\,,\cdot)$ *and* $\mathfrak{Q}_{\mu\nu}(\cdot\,,\cdot)$, *which appear on the right-hand side of* (3.6.5), *are known as the* standard null forms. *They are defined through their action on the derivatives of scalar-valued functions* ψ *and* χ *by*

$$\mathfrak{Q}_0(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (m^{-1})^{\kappa\lambda}(\nabla_\kappa \psi)(\nabla_\lambda \chi), \tag{3.6.6a}$$

$$\mathfrak{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (\nabla_\mu \psi)(\nabla_\nu \chi) - (\nabla_\nu \psi)(\nabla_\mu \chi). \tag{3.6.6b}$$

Proof. This decomposition is carried out in Lemmas 3.1 and 3.2 of [Lindblad and Rodnianski 2005]. □

We conclude this section by observing that (3.0.1a′), (3.5.9), and (3.6.2) together imply that under the wave-coordinate condition (3.1.1a), and under the assumption (3.3.3a) on the Lagrangian, the Einstein field equation (3.0.1a) is equivalent to the following equation:

$$\begin{aligned}\widetilde{\Box}_g g_{\mu\nu} = \mathscr{P}(\nabla_\mu h, \nabla_\nu h) + \mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) - 2(m^{-1})^{\kappa\lambda} \mathscr{F}_{\mu\kappa}\mathscr{F}_{\nu\lambda} + \tfrac{1}{2} m_{\mu\nu}(m^{-1})^{\kappa\eta}(m^{-1})^{\lambda\zeta} \mathscr{F}_{\kappa\lambda}\mathscr{F}_{\eta\zeta} \\ + O^\infty(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h).\end{aligned} \tag{3.6.7}$$

3.7. ***Summary of the reduced system.*** In this section, we summarize the above results by stating the form of the reduced Einstein-nonlinear electromagnetic system that we work with for most of the remainder of the article, namely (3.7.1a)–(3.7.1c); the derivation of this version of the reduced equations follows easily from the previous results of Section 3. We remind the reader that the reduced equations are obtained by adding the inhomogeneous term $-\Gamma^\kappa (g^{-1})^{\nu\lambda} \mathscr{F}_{\kappa\lambda}$ to the right-hand side of (3.3.7) and by substituting the modified Ricci tensor in place of the Ricci tensor in (3.0.1a). Furthermore, in a wave-coordinate system, the reduced system is equivalent to the system (3.0.1a)–(3.0.1c) (see Proposition 4.2).

Reduced system. The reduced system (where $g_{\mu\nu} = m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$ and the unknowns are viewed to be $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$) can be expressed as

$$\widetilde{\Box}_g h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu} - \widetilde{\Box}_g h_{\mu\nu}^{(0)} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{3.7.1a}$$

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{3.7.1b}$$

$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda} = \mathfrak{F}^\nu \qquad (\nu = 0, 1, 2, 3), \tag{3.7.1c}$$

where $\widetilde{\Box}_g \stackrel{\text{def}}{=} (g^{-1})^{\kappa\lambda}\nabla_\kappa \nabla_\lambda$ is the *reduced wave operator* corresponding to $g_{\mu\nu}$.

The quantities $\mathfrak{H}_{\mu\nu}$, $N^{\#\mu\nu\kappa\lambda}$, and $\mathfrak{F}^{\nu}$ can be decomposed into principal terms and error terms (which are denoted with a "$\triangle$") as follows:

$$\mathfrak{H}_{\mu\nu} = \mathscr{P}(\nabla_\mu h, \nabla_\nu h) + \mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h) + \mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{F}) + \mathfrak{H}^{\triangle}_{\mu\nu}, \tag{3.7.2a}$$

$$\mathfrak{F}^{\nu} = \mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) + \mathfrak{F}^{\nu}_{\triangle}, \tag{3.7.2b}$$

$$\begin{aligned} N^{\#\mu\nu\kappa\lambda} &= \tfrac{1}{2}((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}) \\ &\quad + \tfrac{1}{2}(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}) \\ &\quad + \tfrac{1}{2}(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}) + N^{\#\mu\nu\kappa\lambda}_{\triangle}, \end{aligned} \tag{3.7.2c}$$

where $\mathscr{P}(\nabla_\mu h, \nabla_\nu h)$ is defined in (3.6.4), $\mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h)$ is defined in (3.6.5), and

$$\mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G}) = -2(m^{-1})^{\kappa\lambda}\mathscr{F}_{\mu\kappa}\mathscr{G}_{\nu\lambda} + \tfrac{1}{2}m_{\mu\nu}(m^{-1})^{\kappa\lambda}(m^{-1})^{\lambda\kappa}\mathscr{F}_{\kappa\lambda}\mathscr{G}_{\kappa\lambda}, \tag{3.7.2d}$$

$$\mathscr{Q}^{\nu}_{(2;\mathscr{F})}(\nabla h, \mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\lambda\lambda'}(m^{-1})^{\nu\nu'}(\nabla_\mu h_{\nu'\lambda'})\mathscr{F}_{\kappa\lambda}, \tag{3.7.2e}$$

$$\mathfrak{H}^{\triangle}_{\mu\nu} = O^{\infty}(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h), \tag{3.7.2f}$$

$$\mathfrak{F}^{\nu}_{\triangle} = O^{\ell}(|h||\nabla h||\mathscr{F}|) + O^{\ell}(|\nabla h||\mathscr{F}|^2; h), \tag{3.7.2g}$$

$$N^{\#\mu\nu\kappa\lambda}_{\triangle} = O^{\ell}(|(h, \mathscr{F})|^2). \tag{3.7.2h}$$

Furthermore, the left-hand side of (3.7.1c) can be expressed as

$$\begin{aligned} N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda} &= \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \mathscr{F}_{\kappa\lambda} \\ &\quad - \mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F}) - \mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F}) + N^{\#\mu\nu\kappa\lambda}_{\triangle}\nabla_\mu\mathscr{F}_{\kappa\lambda}, \end{aligned} \tag{3.7.3a}$$

where

$$\mathscr{P}^{\nu}_{(\mathscr{F})}(h, \nabla\mathscr{F}) = (m^{-1})^{\mu\mu'}(m^{-1})^{\kappa\kappa'}(m^{-1})^{\nu\lambda}h_{\mu'\kappa'}\nabla_\mu\mathscr{F}_{\kappa\lambda}, \tag{3.7.3b}$$

$$\mathscr{Q}^{\nu}_{(1;\mathscr{F})}(h, \nabla\mathscr{F}) = (m^{-1})^{\mu\kappa}(m^{-1})^{\nu\nu'}(m^{-1})^{\lambda\lambda'}h_{\nu'\lambda'}\nabla_\mu\mathscr{F}_{\kappa\lambda}. \tag{3.7.3c}$$

More precisely, (3.7.1a) follows from (3.6.7) and the expansions (1.2.1a)–(1.2.1b) while (3.7.1b)–(3.7.1c) were derived in Lemma 3.4.

4. The initial-value problem

In this section, we discuss the abstract initial data and the constraint equations for the Einstein-nonlinear electromagnetic system. We then use the abstract initial data to construct initial data for the reduced equations that satisfy the wave-coordinate condition at $t = 0$. Finally, we sketch a proof of the well-known fact that the wave-coordinate condition is satisfied by the solution to the reduced equations launched by this data; this result shows that the wave-coordinate gauge is a viable gauge for studying the Einstein-nonlinear electromagnetic system.

4.1. ***The abstract initial data.*** The initial-value problem formulation of the Einstein equations goes back to the seminal work by Fourès-Bruhat [1952]. In this article, initial data for the Einstein-nonlinear electromagnetic system consist of the 3-dimensional manifold $\Sigma_0 = \mathbb{R}^3$ together with the following fields on Σ_0: a Riemannian metric $\mathring{\underline{g}}_{jk}$, a symmetric two-tensor $\mathring{K}_{jk}$, and a pair of one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$. After we construct the ambient Lorentzian spacetime $(\mathfrak{M}, g_{\mu\nu})$, $\mathring{\underline{g}}_{jk}$ and $\mathring{K}_{jk}$ will respectively be the first and second fundamental forms of Σ_0 while $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$, which are defined below in Section 9.2, will be an electromagnetic decomposition of $\mathcal{F}_{\mu\nu}|_{\Sigma_0}$ into a pair of one-forms that are both m-tangent and g-tangent to Σ_0.

It is well-known that one cannot consider arbitrary data for the Einstein-nonlinear electromagnetic system. The data are subject to the following constraints:

$$\mathring{\underline{R}} - \mathring{K}_{ab}\mathring{K}^{ab} + \big[(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}\big]^2 = 2T(\widehat{N}, \widehat{N})|_{\Sigma_0}, \tag{4.1.1a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{K}_{bj} - (\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_j\mathring{K}_{ab} = T\Big(\widehat{N}, \frac{\partial}{\partial x^j}\Big)\Big|_{\Sigma_0} \quad (j = 1, 2, 3), \tag{4.1.1b}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{D}}_b = 0, \tag{4.1.2a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{B}}_b = 0, \tag{4.1.2b}$$

where $\mathring{\underline{\mathscr{D}}}$ is the Levi-Civita connection corresponding to $\mathring{\underline{g}}_{jk}$, $\mathring{\underline{R}}$ is the scalar curvature of $\mathring{\underline{g}}_{jk}$, $T_{\mu\nu}$ is defined in (3.5.4a), and $\widehat{N}^\mu$ is the future-directed unit g-normal to Σ_0. The right-hand sides of (4.1.1a)–(4.1.1b) can (in principle) be computed in terms of and $\mathring{\underline{g}}_{jk}$, $\mathring{\mathfrak{D}}_j$, and $\mathring{\mathfrak{B}}_j$ with the help of the relations (9.2.3), which connect these quantities to $\mathcal{F}_{\mu\nu}|_{\Sigma_0}$. In (4.1.1a)–(4.1.1b), indices are lowered and raised with the Riemannian metric $\mathring{\underline{g}}_{jk}$ and its inverse $(\mathring{\underline{g}}^{-1})^{jk}$. The constraints (4.1.1a)–(4.1.1b) are respectively known as the *Gauss* and *Codazzi* equations while (4.1.2a)–(4.1.2b) are known as the *electromagnetic constraints*. They relate the fields present in the ambient spacetime $(\mathfrak{M}, g_{\mu\nu}, \mathcal{F}_{\mu\nu})$ (which has to be constructed) to the fields induced on an embedded Riemannian hypersurface (which will be $(\Sigma_0, \mathring{\underline{g}}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ after construction). Without providing the rather standard details (see, e.g., [Christodoulou 2008]), we remark that they are consequences of the following assumptions:

- Σ_0 is a spacelike submanifold of the spacetime manifold $\mathfrak{M}$.
- $\mathring{\underline{g}}_{jk}$ is the first fundamental form of Σ_0, and $\mathring{K}_{jk}$ is the second fundamental form of Σ_0.
- The Einstein-nonlinear electromagnetic system is satisfied along Σ_0.
- Along Σ_0 (viewed as a subset of $\mathfrak{M}$), $\mathfrak{B}_\mu = -{}^\star\mathcal{F}_{\mu\kappa}\widehat{N}^\kappa$ and $\mathfrak{D}_\mu = -{}^\star\mathcal{M}_{\mu\kappa}\widehat{N}^\kappa$.

We recall that, under the above assumptions, $\mathring{\underline{g}}$ and $\mathring{K}$ are defined by

$$\mathring{\underline{g}}|_p(X, Y) = g|_p(X, Y) \qquad \forall X, Y \in T_p\Sigma_0, \tag{4.1.3}$$

$$\mathring{K}|_p(X, Y) = g|_p(\mathscr{D}_X\widehat{N}, Y) \quad \forall X, Y \in T_p\Sigma_0, \tag{4.1.4}$$

where $\widehat{N}$ is the future-directed unit g-normal[19] to Σ_0 at p and $\mathscr{D}$ is the Levi-Civita connection corresponding to g. Furthermore, if X and Y are vector fields tangent to Σ_0, then

$$\mathscr{D}_X Y = \mathring{\underline{\mathscr{D}}}_X Y + \mathring{K}(X, Y)\widehat{N}. \tag{4.1.5}$$

[19]Under the assumptions of Section 4.2, it follows that, at every point $p \in \Sigma_0$, $\widehat{N}^\mu = (A^{-1}, 0, 0, 0)$, where A is defined by (4.2.2).

We also remind the reader that our stability theorem requires the hypothesis that the abstract initial data decay at spatial infinity according to the rates (1.0.4a)–(1.0.4f).

4.2. *The initial data for the reduced equations.* We assume that we are given "abstract" initial data $(\underline{\mathring{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ on the manifold $\mathbb{R}^3$ for the Einstein equations as discussed in the previous section. In this section, we will use this data to construct data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ for the reduced equations (3.7.1a)–(3.7.1c) that satisfy the wave-coordinate condition $\Gamma^\mu|_{t=0} = 0$. We begin by recalling that $\chi(z)$ is a fixed cut-off function with the following properties:

$$\chi \in C^\infty, \qquad \chi \equiv 1 \text{ for } z \geq \tfrac{3}{4}, \quad \text{and} \quad \chi \equiv 0 \text{ for } z \leq \tfrac{1}{2}. \tag{4.2.1}$$

We then define the function $A(x^1, x^2, x^3) \geq 0$ by

$$A^2 \stackrel{\text{def}}{=} 1 - \frac{2M}{r}\chi(r) \quad \text{and} \quad r \stackrel{\text{def}}{=} |x|. \tag{4.2.2}$$

We define the data for the spacetime metric $g_{\mu\nu}$ by

$$g_{00}|_{t=0} = -A^2, \qquad g_{0j}|_{t=0} = 0, \qquad g_{jk}|_{t=0} = \underline{\mathring{g}}_{jk}, \tag{4.2.3a}$$

$$\begin{aligned} &\partial_t g_{00}|_{t=0} = 2A^3(\underline{\mathring{g}}^{-1})^{ab}\mathring{K}_{ab}, \\ &\partial_t g_{0j}|_{t=0} = A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_a\underline{\mathring{g}}_{bj} - \tfrac{1}{2}A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_j\underline{\mathring{g}}_{ab} - A\,\partial_j A, \\ &\partial_t g_{jk}|_{t=0} = 2A\mathring{K}_{jk} \end{aligned} \tag{4.2.3b}$$

and the data for the Faraday tensor $\mathscr{F}_{\mu\nu}$ by

$$\mathscr{F}_{j0}|_{t=0} = \mathring{E}_j \quad \text{and} \quad \mathscr{F}_{jk}|_{t=0} = [ijk]\mathring{B}_i. \tag{4.2.4}$$

The one-forms $\mathring{E}_j$ and $\mathring{B}_j$ can be expressed in terms of $\underline{\mathring{h}}_{jk}$ and the one-forms $\mathring{\mathfrak{D}}_j$ and $\mathring{\mathfrak{B}}_j$ appearing in the constraint equations (4.1.2a)–(4.1.2b) by using the relations (9.2.3) and (9.2.4) below. The precise form of these relations depends on the choice of Lagrangian ${}^\star\mathscr{L}$, but in the small-data regime, the estimates (9.2.7) (9.2.8a), and (9.2.8b) hold.

We now state the main result of this section.

Lemma 4.1 (Wave-coordinate condition holds at $t = 0$). *Suppose that the initial data* $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *for the reduced equations are constructed from abstract initial data* $(\underline{\mathring{g}}_{jk}, \mathring{K}_{jk})$ $(j, k = 1, 2, 3)$ *as described above. Then the wave-coordinate condition holds initially*:

$$\Gamma^\mu|_{t=0} \quad (\mu = 0, 1, 2, 3). \tag{4.2.5}$$

Proof. Lemma 4.1 follows from the expression (3.1.1c), the definitions (4.2.3a)–(4.2.3b), and straightforward calculations. □

Note that the above definitions induce the following data for the spacetime metric "remainder" piece $h^{(1)}_{\mu\nu}$, which is defined by (1.2.1a)–(1.2.1c):

$$h_{00}^{(1)}|_{t=0} = 0, \qquad h_{0j}^{(1)}|_{t=0} = 0, \qquad h_{jk}^{(1)}|_{t=0} = \mathring{\underline{h}}_{jk}^{(1)}, \tag{4.2.6a}$$

$$\begin{aligned} \partial_t h_{00}^{(1)}|_{t=0} &= 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}, \\ \partial_t h_{0j}^{(1)}|_{t=0} &= A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a \mathring{\underline{g}}_{bj} - \tfrac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j \mathring{\underline{g}}_{ab} - A\,\partial_j A, \\ \partial_t h_{jk}^{(1)}|_{t=0} &= 2A\mathring{K}_{jk}. \end{aligned} \tag{4.2.6b}$$

Similarly, the following data are induced in $h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$, which is defined in (1.2.1b):

$$h_{00}|_{t=0} = \chi(r)\frac{2M}{r}, \qquad h_{0j}|_{t=0} = 0, \qquad h_{jk}|_{t=0} = \chi(r)\frac{2M}{r}\delta_{jk} + \mathring{\underline{h}}_{jk}^{(1)}, \tag{4.2.7a}$$

$$\begin{aligned} \partial_t h_{00}|_{t=0} &= 2A^3(\mathring{\underline{g}}^{-1})^{ab}\mathring{K}_{ab}, \\ \partial_t h_{0j}|_{t=0} &= A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_a \mathring{\underline{g}}_{bj} - \tfrac{1}{2}A^2(\mathring{\underline{g}}^{-1})^{ab}\,\partial_j \mathring{\underline{g}}_{ab} - A\,\partial_j A, \\ \partial_t h_{jk}|_{t=0} &= 2A\mathring{K}_{jk}. \end{aligned} \tag{4.2.7b}$$

We will make use of these facts in our proof of Proposition 10.4 below.

4.3. ***Preservation of the wave-coordinate gauge.*** In this section, we sketch a proof of the fact that, if the reduced data are constructed from abstract data as described in Section 4.2, then the wave-coordinate condition $\Gamma^\mu = 0$ is preserved by the flow of the reduced equations. This result requires the assumption that the abstract data satisfy the constraints (4.1.1a)–(4.1.2b). To simplify the discussion, we assume in this section that the data are smooth. However, the result also holds in the regularity class we use during our global existence proof. We remark that this result is quite standard and that we have included it only for convenience.

Proposition 4.2 (Preservation of the wave-coordinate gauge). *Suppose that* $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *are smooth initial data for the reduced equations* (3.7.1a)–(3.7.1c) *that are constructed from abstract initial data satisfying the constraints* (4.1.1a)–(4.1.2b) *as described in Section 4.2. In particular, by Lemma 4.1, the wave-coordinate condition* $\Gamma^\mu|_{t=0}$ *holds. Assume further that the reduced data are small enough so that they lie within the regime of hyperbolicity*[20] *of the reduced equations. Let* $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ *be the corresponding smooth solution to the reduced equations that is launched by the data. Let* $T > 0$, *and assume that the reduced solution exists on the slab* $[0, T) \times \mathbb{R}^3$ *and lies within the regime of hyperbolicity of the reduced equations. Then* $\Gamma^\mu \equiv 0$ *for* $[t, x) \in [0, T) \times \mathbb{R}^3$.

Sketch of proof. Our goal is to show that under the assumptions of the proposition, whenever we have a smooth solution to the reduced equations (3.7.1a)–(3.7.1c) on $[0, T) \times \mathbb{R}^3$, the corresponding Γ^μ satisfy a *homogeneous-in-*Γ^μ system of wave equations with principal part equal to $(g^{-1})^{\kappa\lambda}\,\partial_\kappa\,\partial_\lambda$ and with trivial initial data $\Gamma^\mu|_{t=0} = \partial_t\Gamma^\mu|_{t=0} = 0$. The conclusion that $\Gamma^\mu \equiv 0$ for $(t, x) \in [0, T) \times \mathbb{R}^3$ then follows from a standard uniqueness theorem for such wave equations that is based on energy estimates (see, e.g., [Hörmander 1997; Sogge 2008] for ideas on how to prove such a theorem). To derive the equations satisfied by the Γ^μ, we will view Γ^μ as a vector field for purposes of covariant

[20] Since our electromagnetic equations are perturbations of the standard Maxwell–Maxwell equations, there will always be such a regime.

differentiation. We first recall (see Remark 3.6) that (3.6.2) is obtained by adding the gauge term $-\frac{1}{2}(g_{\kappa\nu}\mathscr{D}_\mu\Gamma^\kappa + g_{\kappa\mu}\mathscr{D}_\nu\Gamma^\kappa) + u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa$ to the expression (3.0.2b) for $R_{\mu\nu}$. Consequently, it follows that, for a solution to the reduced equations (3.7.1a)–(3.7.1c), we have that

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} - T_{\mu\nu} = \tfrac{1}{2}(g_{\kappa\nu}\mathscr{D}_\mu\Gamma^\kappa + g_{\kappa\mu}\mathscr{D}_\nu\Gamma^\kappa) - u_{\mu\nu\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa \\ - \tfrac{1}{2}g_{\mu\nu}\mathscr{D}_\kappa\Gamma^\kappa + \tfrac{1}{2}g_{\mu\nu}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta. \quad (4.3.1)$$

We note that the left-hand side of (4.3.1) is simply the difference of the left-hand and right-hand sides of the Einstein equation (1.0.1a).

We now apply $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda$ to each side of (4.3.1), use the Bianchi identity $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$, the fact that $(g^{-1})^{\nu\lambda}\mathscr{D}_\lambda T_{\mu\nu} = -\Gamma^\kappa(g^{-1})^{\beta\lambda}\mathscr{F}_{\kappa\lambda}\mathscr{F}_{\mu\beta}$ (see Remark 3.6 and (3.5.11)), and the curvature relation $\mathscr{D}_\mu\mathscr{D}_\kappa\Gamma^\kappa = \mathscr{D}_\kappa\mathscr{D}_\mu\Gamma^\kappa - R_{\mu\kappa}\Gamma^\kappa$, and expand the covariant derivatives in terms of coordinate derivatives and Christoffel symbols to deduce that the Γ^μ are solutions to the following *hyperbolic* system of wave equations that is *homogeneous in* Γ^μ:

$$(g^{-1})^{\kappa\lambda}\,\partial_\kappa\,\partial_\lambda\Gamma^\mu = A^{\mu\kappa}{}_\lambda(g, g^{-1}, \partial g, \partial\partial g)\,\partial_\kappa\Gamma^\lambda + B^\mu{}_\kappa(g, g^{-1}, \partial g, \partial\partial g, \mathscr{F})\Gamma^\kappa \quad (\mu = 0, 1, 2, 3), \quad (4.3.2)$$

where the $A^{\mu\kappa}{}_\lambda(g(t,x), g^{-1}(t,x), \partial g(t,x), \partial\partial g(t,x))$ and $B^\mu{}_\kappa(g(t,x), g^{-1}(t,x), \partial g(t,x), \partial\partial g(t,x), \mathscr{F}(t,x))$ are smooth functions of (t, x).

To complete our sketch of the proof, it remains to show that $\partial_t\Gamma^\mu|_{t=0} = 0$. Since the abstract initial data $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ are assumed to satisfy the constraint equations (4.1.1a)–(4.1.1b), it follows that the left-hand side of (4.3.1) is equal to 0 at $t = 0$ after contracting[21] against $\widehat{N}^\mu\widehat{N}^\nu$ or $\widehat{N}^\mu X^\nu$, where $\widehat{N}^\mu$ is the future-directed unit g-normal to Σ_0 and X^μ is any vector tangent to Σ_0.

Recalling that $\widehat{N}^\mu|_{t=0} = A^{-1}\delta_0^\mu$ and choosing $X^\nu = \delta_j^\nu$, it therefore follows that the right-hand side must also be equal to 0 at $t = 0$ upon contraction (where $j = 1, 2, 3$ in (4.3.3b)):

$$\big(g_{\kappa 0}\mathscr{D}_t\Gamma^\kappa - u_{00\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa - \tfrac{1}{2}g_{00}\mathscr{D}_\lambda\Gamma^\lambda \\ + \tfrac{1}{2}g_{00}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta\big)\big|_{t=0} = 0, \quad (4.3.3a)$$

$$\big(\tfrac{1}{2}(g_{\kappa j}\mathscr{D}_t\Gamma^\kappa + g_{\kappa 0}\mathscr{D}_j\Gamma^\kappa) - u_{0j\kappa}(g, g^{-1}, \partial g)\Gamma^\kappa - \tfrac{1}{2}g_{0j}\mathscr{D}_\lambda\Gamma^\lambda \\ + \tfrac{1}{2}g_{0j}(g^{-1})^{\kappa\lambda}u_{\kappa\lambda\delta}(g, g^{-1}, \partial g)\Gamma^\delta\big)\big|_{t=0} = 0. \quad (4.3.3b)$$

Expanding the covariant differentiation in (4.3.3a)–(4.3.3b) in terms of coordinate derivatives and Christoffel symbols and using (4.2.3a) plus the fact that the initial data were constructed so as to satisfy $\Gamma^\mu|_{t=0} = 0$, it is straightforward to verify that $\partial_t\Gamma^\mu$ must *also necessarily* be trivial at $t = 0$:

$$\partial_t\Gamma^\mu|_{t=0} = 0 \quad (\mu = 0, 1, 2, 3). \quad (4.3.4)$$

This completes our sketch of a proof of the proposition. □

[21] In fact, one derives the constraint equations by assuming that these contractions are 0 at $t = 0$.

5. Geometry and the Minkowskian null frame

In this section, we introduce the families of ingoing Minkowskian null cones C_s^-, outgoing Minkowskian light cones C_q^+, constant Minkowskian time slices Σ_t, and Euclidean spheres $S_{r,t}$. We then discuss the well-known notion of a Minkowskian null frame, which allows us to geometrically decompose the tangent space at p as a direct sum $T_p\mathbb{R}^{1+3} = \text{span}\{\underline{L}|_p\} \oplus \text{span}\{L|_p\} \oplus T_p S_{r,t}$. These decompositions allow us to geometrically decompose tensor fields. In Section 5.3, we provide a full description of the null decomposition of a two-form $\mathscr{F}$ into its *Minkowskian null components*. This decomposition will be essential to our subsequent analysis of the decay properties of the Faraday tensor. In Section 9.1, we will derive equations for these null components under the assumption that $\mathscr{F}$ is a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c). In Section 15, we will use the equations for the null components to deduce "upgraded" pointwise decay estimates for the lower-order Lie derivatives of $\mathscr{F}$; these estimates are essential for closing our global existence bootstrap argument in Section 16. We refer the reader to Section 1.2.4 for discussion on how our use of Minkowskian decompositions compares and contrasts against other decompositions that have been used by other authors in the context of the stability of Minkowski spacetime.

5.1. ***The Minkowskian null frame.*** Before proceeding, we introduce the subsets C_q^+, C_s^-, Σ_t, and $S_{r,t}$.

Definition 5.1. In our wave-coordinate system (t, x), we define the *outgoing Minkowski null cones* C_q^+, *ingoing Minkowski null cones* C_s^-, *constant Minkowskian time slices* Σ_t, and Euclidean spheres $S_{r,t}$ as

$$C_q^+ \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| - \tau = q\}, \tag{5.1.1a}$$

$$C_s^- \stackrel{\text{def}}{=} \{(\tau, y) \mid |y| + \tau = s\}, \tag{5.1.1b}$$

$$\Sigma_t \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t\}, \tag{5.1.1c}$$

$$S_{r,t} \stackrel{\text{def}}{=} \{(\tau, y) \mid \tau = t,\ |y| = r\}. \tag{5.1.1d}$$

In the above formulas, $y \stackrel{\text{def}}{=} (y^1, y^2, y^3)$ and $|y| \stackrel{\text{def}}{=} \sqrt{(y^1)^2 + (y^2)^2 + (y^2)^2}$.

We also introduce the following vector fields, which play a fundamental role throughout this article:

Definition 5.2. We define the *ingoing Minkowski-null geodesic vector field* $\underline{L}$ and the *outgoing Minkowski-null geodesic vector field* L by

$$\underline{L}^\mu = (1, -\omega^1, -\omega^2, -\omega^3), \tag{5.1.2a}$$

$$L^\mu = (1, \omega^1, \omega^2, \omega^3), \tag{5.1.2b}$$

where $\omega^j \stackrel{\text{def}}{=} x^j/r$. By "Minkowski-null", we mean that $m(\underline{L}, \underline{L}) = m(L, L) = 0$. Note that $\underline{L}$ is tangent to the ingoing cones C_s^-, that L is tangent to the outgoing cones C_q^+, and that $\underline{L}$ and L are both m-orthogonal to the $S_{r,t}$. By "Minkowski-geodesic", we mean that $\nabla_{\underline{L}}\underline{L} = \nabla_L L = 0$.

Note that

$$\underline{L} = \partial_t - \partial_r, \tag{5.1.3a}$$

$$L = \partial_t + \partial_r. \tag{5.1.3b}$$

We now recall the definitions of the Minkowskian first fundamental forms of the surfaces Σ_t and $S_{r,t}$.

Definition 5.3. The *Minkowskian first fundamental* forms of the surfaces Σ_t and $S_{r,t}$ are respectively defined to be the following intrinsic metrics:

$$\underline{m}_{\mu\nu} \stackrel{\text{def}}{=} \operatorname{diag}(0, 1, 1, 1), \tag{5.1.4a}$$

$$\not{m}_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + \tfrac{1}{2}(\underline{L}_\mu L_\nu + L_\mu \underline{L}_\nu). \tag{5.1.4b}$$

Recall that $\underline{m}|_p(X, Y) = m|_p(X, Y)$ for $X, Y \in T_p\Sigma_t$ and $\not{m}(X, Y) = m(X, Y)$ for $X, Y \in T_p S_{r,t}$. Note also that the tensor fields $\underline{m}_\mu{}^\nu$ and $\not{m}_\mu{}^\nu$ respectively m-orthogonally project onto the Σ_t and the $S_{r,t}$.

We now define a related tensor field corresponding to the outgoing Minkowski null cones C_q^+.

Definition 5.4. The tensor field $\overline{\pi}_\mu{}^\nu$, which projects vectors X^μ onto the outgoing cones C_q^+, is defined as

$$\overline{\pi}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu^\nu + \tfrac{1}{2} L_\mu \underline{L}^\nu. \tag{5.1.5}$$

Note in particular that $\overline{\pi}_\mu{}^\nu \underline{L}^\mu = 0$ while $\overline{\pi}_\mu{}^\nu X^\mu = X^\nu$ whenever X is tangent to C_q^+.

Furthermore, we recall the definitions of the Minkowskian volume forms of Minkowski spacetime and of the surfaces Σ_t and $S_{r,t}$.

Definition 5.5. The *Minkowskian volume forms* of Minkowski spacetime, the surfaces Σ_t, and the Euclidean spheres $S_{r,t}$ are respectively defined relative to our wave-coordinate system as follows:

$$\upsilon_{\mu\nu\kappa\lambda} \stackrel{\text{def}}{=} [\mu\nu\kappa\lambda], \tag{5.1.6a}$$

$$\underline{\upsilon}_{\nu\kappa\lambda} \stackrel{\text{def}}{=} \upsilon_{0\nu\kappa\lambda}, \tag{5.1.6b}$$

$$\not{\upsilon}_{\mu\nu} \stackrel{\text{def}}{=} \upsilon_{\mu\nu\kappa\lambda} \underline{L}^\kappa L^\lambda, \tag{5.1.6c}$$

where $[\mu\nu\kappa\lambda]$ is totally antisymmetric with normalization $[0123] = 1$.

We also recall what it means for a spacetime tensor field to be m-tangent to the surfaces Σ_t or $S_{r,t}$.

Definition 5.6. Let U be a type-$\binom{n}{m}$ spacetime tensor field. We say that U is m-tangent to the time slices Σ_t if

$$U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \underline{m}_{\mu_1}{}^{\mu_1'} \cdots \underline{m}_{\mu_m}{}^{\mu_m'} \underline{m}_{\nu_1'}{}^{\nu_1} \cdots \underline{m}_{\nu_n'}{}^{\nu_n} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{5.1.7}$$

Equivalently, U is m-tangent to the Σ_t if and only if every wave-coordinate component of U containing a 0 index vanishes.

Similarly, we say that U is m-tangent to the spheres $S_{r,t}$ if

$$U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \not{m}_{\mu_1}{}^{\mu_1'} \cdots \not{m}_{\mu_m}{}^{\mu_m'} \not{m}_{\nu_1'}{}^{\nu_1} \cdots \not{m}_{\nu_n'}{}^{\nu_n} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{5.1.8}$$

Equivalently, U is m-tangent to the spheres $S_{r,t}$ if and only if any contraction of any index of U with either $\underline{L}$ or L vanishes.

We are now ready to introduce the notion of a *Minkowskian null frame*. We complement the vector fields $\underline{L}$ and L with a locally defined pair of m-orthogonal vector fields e_1 and e_2 that are tangent to the spheres $S_{r,t}$ and therefore m-orthogonal to $\underline{L}$ and L. The resulting collection of vector fields $\mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$ is known as *Minkowskian null frame*. It spans the tangent space $T_p\mathbb{R}^{1+3}$ at each point p where it is defined.

We leave the proof of the following lemma, which summarizes some of the important properties of the geometric quantities introduced in this section, as an exercise for the reader:

Lemma 5.7 (Null frame field properties). *The following identities hold*:

$$\nabla_L L = \nabla_{\underline{L}} \underline{L} = 0, \tag{5.1.9a}$$

$$\nabla_L \underline{L} = \nabla_{\underline{L}} L = 0, \tag{5.1.9b}$$

$$L^\kappa \underline{L}_\kappa = -2, \tag{5.1.9c}$$

$$e_A^\kappa L_\kappa = e_A^\kappa \underline{L}_\kappa = 0 \qquad (A = 1, 2), \tag{5.1.9d}$$

$$m_{\kappa\lambda} e_A^\kappa e_B^\lambda = \delta_{AB} \qquad (A, B = 1, 2), \tag{5.1.9e}$$

$$\nabla_{\underline{L}} \not{m}_{\mu\nu} = \nabla_L \not{m}_{\mu\nu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3), \tag{5.1.10}$$

$$\nabla_{\underline{L}} \not{\upsilon}_{\mu\nu} = \nabla_L \not{\upsilon}_{\mu\nu} = 0 \qquad (\mu, \nu = 0, 1, 2, 3). \tag{5.1.11}$$

See Definition 6.4 concerning our use of notation in these formulas.

Later in the article, we will see that the decay rates of the null components (see Section 5.3) of h and $\mathcal{F}$ are distinguished according to the kinds of contractions of $\mathcal{F}$ taken against $\underline{L}$, L, e_1, and e_2. With these ideas in mind, we introduce the following sets of vector fields:

$$\mathcal{L} \stackrel{\text{def}}{=} \{L\}, \qquad \mathcal{T} \stackrel{\text{def}}{=} \{L, e_1, e_2\}, \quad \text{and} \quad \mathcal{N} \stackrel{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}. \tag{5.1.12}$$

In order to measure the size of the contractions of various tensors and their covariant derivatives against vectors belonging to the sets $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{N}$, we introduce the following definitions:

Definition 5.8. If $\mathcal{V}$ and $\mathcal{W}$ denote any two of the above sets and P is a type-$\binom{0}{2}$ tensor, then we define the following pointwise seminorms:

$$|P|_{\mathcal{V}\mathcal{W}} \stackrel{\text{def}}{=} \sum_{V\in\mathcal{V},\ W\in\mathcal{W}} |V^\kappa W^\lambda P_{\kappa\lambda}|, \tag{5.1.13a}$$

$$|\nabla P|_{\mathcal{V}\mathcal{W}} \stackrel{\text{def}}{=} \sum_{N\in\mathcal{N},\ V\in\mathcal{V},\ W\in\mathcal{W}} |V^\kappa W^\lambda N^\gamma \nabla_\gamma P_{\kappa\lambda}|, \tag{5.1.13b}$$

$$|\overline{\nabla} P|_{\mathcal{V}\mathcal{W}} \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T},\ V\in\mathcal{V},\ W\in\mathcal{W}} |V^\kappa W^\lambda T^\gamma \nabla_\gamma P_{\kappa\lambda}|. \tag{5.1.13c}$$

We often use the abbreviations $|P| \stackrel{\text{def}}{=} |P|_{\mathcal{N}\mathcal{N}}$, $|\nabla P| \stackrel{\text{def}}{=} |\nabla P|_{\mathcal{N}\mathcal{N}}$, and $|\overline{\nabla} P| \stackrel{\text{def}}{=} |\overline{\nabla} P|_{\mathcal{N}\mathcal{N}}$.

The above definition generalizes in an obvious way to arbitrary type-$\binom{n}{m}$ tensor fields $U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}$. Observe that, for any such tensor field, the following inequalities hold *in our wave-coordinate system*:

$$|U| \approx \sum_{\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n=0}^{3} |U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}|. \tag{5.1.14}$$

5.2. ***Minkowskian null frame decomposition of a tensor field.*** For an arbitrary vector field X and frame vector field $N \in \mathcal{N}$, we define

$$X_N \stackrel{\text{def}}{=} X_\kappa N^\kappa, \quad \text{where } X_\mu \stackrel{\text{def}}{=} m_{\mu\kappa} X^\kappa. \tag{5.2.1}$$

The components X_N are known as the *Minkowskian null components* of X. In the sequel, we often abbreviate

$$X_A \stackrel{\text{def}}{=} X_{e_A} \quad \text{and} \quad \nabla_A \stackrel{\text{def}}{=} \nabla_{e_A}, \quad \text{etc.} \tag{5.2.2}$$

It follows from (5.2.1) that

$$X = X^\kappa\, \partial_\kappa = X^L L + X^{\underline{L}} \underline{L} + X^A e_A, \tag{5.2.3}$$

$$X^L = -\tfrac{1}{2} X_{\underline{L}}, \qquad X^{\underline{L}} = -\tfrac{1}{2} X_L, \qquad X^A = X_A. \tag{5.2.4}$$

Furthermore, it is easy to check that

$$m(X, Y) \stackrel{\text{def}}{=} m_{\kappa\lambda} X^\kappa X^\lambda = X^\kappa Y_\kappa = -\tfrac{1}{2} X_L Y_{\underline{L}} - \tfrac{1}{2} X_{\underline{L}} Y_L + \delta^{AB} X_A Y_B. \tag{5.2.5}$$

The above null decomposition of a vector field generalizes in the obvious way to higher-order tensor fields. In the next section, we provide a detailed version of the null decomposition of two-forms $\mathcal{F}$ since this decomposition is needed for our derivation of decay estimates later in the article; see, e.g., Propositions 9.3 and 11.5.

5.3. ***The detailed Minkowskian null decomposition of a two-form.***

Definition 5.9. Given any two-form $\mathcal{F}$, we define its *Minkowskian null components* to be the following pair of one-forms $\underline{\alpha}_\mu$ and α_μ and the following pair of scalars ρ and σ:

$$\underline{\alpha}_\mu \stackrel{\text{def}}{=} \not{m}_\mu{}^\nu \mathcal{F}_{\nu\lambda} \underline{L}^\lambda \quad (\mu = 0, 1, 2, 3), \tag{5.3.1a}$$

$$\alpha_\mu \stackrel{\text{def}}{=} \not{m}_\mu{}^\nu \mathcal{F}_{\nu\lambda} L^\lambda \quad (\mu = 0, 1, 2, 3), \tag{5.3.1b}$$

$$\rho \stackrel{\text{def}}{=} \tfrac{1}{2} \mathcal{F}_{\kappa\lambda} \underline{L}^\kappa L^\lambda, \tag{5.3.1c}$$

$$\sigma \stackrel{\text{def}}{=} \tfrac{1}{2} \not{\upsilon}^{\kappa\lambda} \mathcal{F}_{\kappa\lambda}. \tag{5.3.1d}$$

It is a simple exercise to check that $\underline{\alpha}_\mu$ and α_μ are m-tangent to the spheres $S_{r,t}$:

$$\underline{\alpha}_\kappa \underline{L}^\kappa = 0, \qquad \underline{\alpha}_\kappa L^\kappa = 0, \tag{5.3.2a}$$

$$\alpha_\kappa \underline{L}^\kappa = 0, \qquad \alpha_\kappa L^\kappa = 0. \tag{5.3.2b}$$

Furthermore, relative to the null frame $\mathcal{N} \overset{\text{def}}{=} \{\underline{L}, L, e_1, e_2\}$, we have that

$$\underline{\alpha}_A = \mathscr{F}_{A\underline{L}} \qquad (A = 1, 2), \tag{5.3.3a}$$

$$\alpha_A = \mathscr{F}_{AL} \qquad (A = 1, 2), \tag{5.3.3b}$$

$$\rho = \tfrac{1}{2}\mathscr{F}_{\underline{L}L}, \tag{5.3.3c}$$

$$\sigma = \mathscr{F}_{12}. \tag{5.3.3d}$$

In terms of the seminorms introduced in Definition 5.8, it follows that

$$|\mathscr{F}| \approx |\mathscr{F}|_{\mathcal{N}\mathcal{N}} \approx |\underline{\alpha}| + |\alpha| + |\rho| + |\sigma|, \tag{5.3.4a}$$

$$|\mathscr{F}|_{\mathcal{L}\mathcal{N}} \approx |\alpha| + |\rho|, \tag{5.3.4b}$$

$$|\mathscr{F}|_{\mathcal{T}\mathcal{T}} \approx |\alpha| + |\sigma|. \tag{5.3.4c}$$

The null components of ${}^{\circledast}\mathscr{F}$ (the Minkowskian Hodge duality operator $\circledast$ is defined in Section 2.6) can be expressed in terms of the above null components of $\mathscr{F}$. Denoting the null components[22] of ${}^{\circledast}\mathscr{F}$ by ${}^{\odot}\underline{\alpha}$, ${}^{\odot}\alpha$, ${}^{\odot}\rho$, and ${}^{\odot}\sigma$, we leave it as a simple exercise for the reader to check that

$${}^{\odot}\underline{\alpha}_A = -\underline{\alpha}^B \not{\upsilon}_{BA} \quad (A = 1, 2), \tag{5.3.5a}$$

$${}^{\odot}\alpha_A = \alpha^B \not{\upsilon}_{BA} \qquad (A = 1, 2), \tag{5.3.5b}$$

$${}^{\odot}\rho = \sigma, \tag{5.3.5c}$$

$${}^{\odot}\sigma = -\rho. \tag{5.3.5d}$$

6. Differential operators

In this section, we introduce a collection of differential operators that will be used throughout the remainder of the article. In order to define these operators, we also introduce subsets $\mathbb{O}$ and $\mathfrak{Z}$ of Minkowskian conformal Killing fields. Finally, we prove a collection of lemmas that expose useful properties of these operators and that illustrate various relationships between them.

6.1. *Covariant derivatives.* As previously mentioned, throughout the article, ∇ denotes the Levi-Civita connection of the Minkowski metric m. Let $\underline{m}$ and $\not{m}$ be the first fundamental forms of the Σ_t and $S_{r,t}$ as defined in Definition 5.3, and let $\underline{\nabla}$ and $\not{\nabla}$ be their corresponding Levi-Civita connections. We state as a lemma the following well-known identities, which relate the connections $\underline{\nabla}$ and $\not{\nabla}$ to ∇:

Lemma 6.1 (Relationships between connections). *If U is any type-$\binom{n}{m}$ tensor field m-tangent to the Σ_t, then*

$$\underline{\nabla}_\lambda U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \underline{m}_\lambda{}^{\lambda'} \underline{m}_{\mu_1}{}^{\mu_1'} \cdots \underline{m}_{\mu_m}{}^{\mu_m'} \underline{m}_{\nu_1'}{}^{\nu_1} \cdots \underline{m}_{\nu_n'}{}^{\nu_n} \nabla_{\lambda'} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{6.1.1}$$

[22] We use the symbol $\odot$ in order to avoid confusion with the Minkowskian Hodge duality operator $\circledast$; i.e., it is not true that ${}^{\circledast}(\underline{\alpha}[\mathscr{F}]) = \underline{\alpha}[{}^{\circledast}\mathscr{F}]$.

Similarly, if U is any type-$\binom{n}{m}$ tensor field m-tangent to $S_{r,t}$, then

$$\nabla\!\!\!/_{\lambda} U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = {m\!\!\!/}_{\lambda}{}^{\lambda'} {m\!\!\!/}_{\mu_1}{}^{\mu_1'} \cdots {m\!\!\!/}_{\mu_m}{}^{\mu_m'} {m\!\!\!/}_{\nu_1'}{}^{\nu_1} \cdots {m\!\!\!/}_{\nu_n'}{}^{\nu_n} \nabla_{\lambda'} U_{\mu_1'\cdots\mu_m'}{}^{\nu_1'\cdots\nu_n'}. \tag{6.1.2}$$

We recall the following fundamental properties of the connections ∇, $\underline{\nabla}$, and $\nabla\!\!\!/$:

$$\nabla_\lambda m_{\mu\nu} = 0 = \nabla_\lambda (m^{-1})^{\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.1.3a}$$

$$\underline{\nabla}_\lambda \underline{m}_{\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.1.3b}$$

$$\nabla\!\!\!/_\lambda {m\!\!\!/}_{\mu\nu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{6.1.3c}$$

We will also make use of the projection of the operator ∇ onto the favorable directions, i.e., the directions tangent to the outgoing Minkowski cones C_q^+.

Definition 6.2. If U is any type-$\binom{n}{m}$ spacetime tensor field, then we define the projected Minkowskian covariant derivative $\overline{\nabla} U$ by

$$\overline{\nabla}_\lambda U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \overline{\pi}_\lambda{}^{\lambda'} \nabla_{\lambda'} U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}, \tag{6.1.4}$$

where the null frame projection $\overline{\pi}_\mu{}^\nu$ is defined in (5.1.5).

Remark 6.3. Note that only the λ component is projected onto the outgoing cones so that the tensor field $\overline{\nabla}_\lambda U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}$ need not be m-tangent to the outgoing Minkowski cones.

Definition 6.4. If X is any vector field, then we define the covariant derivative operators ∇_X and $\nabla\!\!\!/_X$ by

$$\nabla_X \stackrel{\text{def}}{=} X^\kappa \nabla_\kappa, \tag{6.1.5a}$$

$$\nabla\!\!\!/_X \stackrel{\text{def}}{=} X^\kappa \nabla\!\!\!/_\kappa. \tag{6.1.5b}$$

6.2. ***Minkowskian conformal Killing fields.*** In this section, we introduce the special set of vector fields $\mathcal{Z}$ that appears in the definition (1.2.7) of our energy $\mathcal{E}_{\ell;\gamma;\mu}(t)$ and in the weighted Klainerman–Sobolev inequality (1.2.10). We begin by recalling that a *Minkowskian conformal Killing field* is a vector field Z such that

$$\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = {}^{(Z)}\phi m_{\mu\nu} \tag{6.2.1}$$

for some function ${}^{(Z)}\phi(t, x)$. The tensor field

$${}^{(Z)}\pi_{\mu\nu} \stackrel{\text{def}}{=} \nabla_\mu Z_\nu + \nabla_\nu Z_\mu \tag{6.2.2}$$

is known as the *Minkowskian deformation tensor* of Z. If ${}^{(Z)}\pi_{\mu\nu} = 0$, then Z is known as a *Minkowskian Killing field*. We also recall that the conformal Killing fields of the Minkowski metric $m_{\mu\nu}$ form a Lie algebra under the Lie bracket $[\cdot, \cdot]$ (see (6.3.1)). The Lie algebra is generated by the following 15 vector fields (see, e.g., [Christodoulou 2008]):

(i) the four *translations* $\partial_\mu = \frac{\partial}{\partial x^\mu}$ $(\mu = 0, 1, 2, 3)$,

(ii) the three *rotations* $\Omega_{jk} \stackrel{\text{def}}{=} x_j \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^j}$ $(1 \le j < k \le 3)$,

(iii) the three *Lorentz boosts* $\Omega_{0j} \stackrel{\text{def}}{=} -t \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial t}$ $(j = 1, 2, 3)$,

(iv) the *scaling* vector field $S \stackrel{\text{def}}{=} x^\kappa \frac{\partial}{\partial x^\kappa}$, and

(v) the four *acceleration* vector fields $K_\mu \stackrel{\text{def}}{=} -2x_\mu S + g_{\kappa\lambda} x^\kappa x^\lambda \frac{\partial}{\partial x^\mu}$ $(\mu = 0, 1, 2, 3)$.

It can be checked that the translations, rotations, and Lorentz boosts are in fact Killing fields of $m_{\mu\nu}$.

Two subsets of the above conformal Killing fields will play a prominent role in the remainder of the article, namely the rotations $\mathscr{O}$ and a larger set $\mathscr{Z}$, which are defined by

$$\mathscr{O} \stackrel{\text{def}}{=} \{\Omega_{jk}\}_{1\le j<k\le 3}, \tag{6.2.3a}$$

$$\mathscr{Z} \stackrel{\text{def}}{=} \left\{ \frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S \right\}_{0\le\mu\le\nu\le 3}. \tag{6.2.3b}$$

The vector fields in $\mathscr{Z}$ satisfy a strong version of the relation (6.2.1). That is, if $Z \in \mathscr{Z}$, then

$$\nabla_\mu Z_\nu = {}^{(Z)}c_{\mu\nu}, \tag{6.2.4}$$

where the components ${}^{(Z)}c_{\mu\nu}$ are *constants* in our wave-coordinate system. In particular, we compute for future use that

$$\nabla_\mu S_\nu = m_{\mu\nu}, \tag{6.2.5a}$$

$$\nabla_\mu (\Omega_{\kappa\lambda})_\nu = m_{\mu\kappa} m_{\nu\lambda} - m_{\mu\lambda} m_{\nu\kappa}. \tag{6.2.5b}$$

We note in addition that if $Z \in \mathscr{Z}$ then there exists a *constant* c_Z such that

$$\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = c_Z m_{\mu\nu}. \tag{6.2.6}$$

Furthermore, by contracting each side of (6.2.6) against $(m^{-1})^{\mu\nu}$, we deduce that

$$c_Z = \tfrac{1}{4} {}^{(Z)}\pi_\kappa^{\ \kappa} = \tfrac{1}{2} {}^{(Z)}c_\kappa^{\ \kappa}. \tag{6.2.7}$$

6.3. *Lie derivatives.* As mentioned in Section 1.2.3, it is convenient to use Lie derivatives to differentiate the electromagnetic equations (3.7.1b)–(3.7.1c). In this section, we recall some basic facts concerning Lie derivatives.

We recall that, if X and Y are any pair of vector fields, then relative to an arbitrary coordinate system their *Lie bracket* $[X, Y]$ can be expressed as

$$[X, Y]^\mu = X^\kappa \partial_\kappa Y^\mu - Y^\kappa \partial_\kappa X^\mu. \tag{6.3.1}$$

Furthermore, we have that

$$\mathscr{L}_X Y = [X, Y], \tag{6.3.2}$$

where $\mathscr{L}$ denotes the *Lie derivative operator*. Given a tensor field U of type $\binom{0}{m}$ and vector fields $Y_{(1)}, \ldots, Y_{(m)}$, the Leibniz rule for $\mathscr{L}$ implies that (6.3.2) generalizes as follows:

$$(\mathscr{L}_X U)(Y_{(1)}, \ldots, Y_{(m)}) = X\{U(Y_{(1)}, \ldots, Y_{(m)})\} - \sum_{i=1}^{n} U(Y_{(1)}, \ldots, Y_{(i-1)}, [X, Y_{(i)}], Y_{(i+1)}, \ldots, Y_{(m)}). \tag{6.3.3}$$

Using Lemma 6.7 below, we see that the left-hand side of (6.2.6) is equal to the Lie derivative of the Minkowski metric. It therefore follows that if $Z \in \mathscr{Z}$ then

$$\mathscr{L}_Z m_{\mu\nu} = c_Z m_{\mu\nu}, \tag{6.3.4a}$$

$$(\mathscr{L}_Z m^{-1})^{\mu\nu} = -c_Z (m^{-1})^{\mu\nu}, \tag{6.3.4b}$$

where the constant c_Z is defined in (6.2.6).

6.4. ***Modified covariant and modified Lie derivatives.*** It will be convenient for us to work with *modified Minkowski covariant derivatives* $\widehat{\nabla}_Z$ and *modified Lie derivatives*[23] $\widehat{\mathscr{L}}_Z$.

Definition 6.5. For $Z \in \mathscr{Z}$, we define the modified Minkowski covariant derivative $\widehat{\nabla}_Z$ by

$$\widehat{\nabla}_Z \stackrel{\text{def}}{=} \nabla_Z + c_Z, \tag{6.4.1}$$

where c_Z denotes the constant from (6.2.6).

For each vector field $Z \in \mathscr{Z}$, we define the modified Lie derivative $\widehat{\mathscr{L}}_Z$ by

$$\widehat{\mathscr{L}}_Z \stackrel{\text{def}}{=} \mathscr{L}_Z + 2c_Z, \tag{6.4.2}$$

where c_Z denotes the constant from (6.2.6).

The crucial features of the above definitions are captured by Lemmas 6.13 and 6.14 below. The first shows that, for each $Z \in \mathscr{Z}$, $\widehat{\nabla}_Z \Box_m \phi = \Box_m \nabla_Z \phi$, where $\Box_m = (m^{-1})^{\kappa\lambda}\nabla_\kappa \nabla_\lambda$ is the *Minkowski* wave operator. The second shows that

$$\widehat{\mathscr{L}}_Z\big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \mathscr{F}_{\kappa\lambda}\big) = \big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu \mathscr{L}_Z \mathscr{F}_{\kappa\lambda}.$$

Furthermore, Lemma 6.8 shows that $\mathscr{L}_Z \nabla_{[\lambda}\mathscr{F}_{\mu\nu]} = \nabla_{[\lambda}\mathscr{L}_Z \mathscr{F}_{\mu\nu]}$, where $[\,\cdot\,]$ denotes antisymmetrization. These commutation identities suggest that the operators $\widehat{\nabla}_Z$ and $\widehat{\mathscr{L}}_Z$ are potentially useful operators for differentiating the nonlinear equations (3.7.1a) and (3.7.1b)–(3.7.1c), respectively. This suggestion is borne out in Propositions 11.4 and 11.6, which show that the inhomogeneous terms generated by differentiating the nonlinear equations have a special algebraic structure, a structure that will be exploited during our global existence bootstrap argument.

6.5. ***Vector-field algebra.*** We introduce here some notation that will allow us to compactly express iterated derivatives. If $\mathscr{A}$ is one of the sets from (6.2.3a)–(6.2.3b), then we label the vector fields in $\mathscr{A}$ as $Z^{\iota_1}, \dots, Z^{\iota_d}$, where d is the cardinality of $\mathscr{A}$. Then for any multi-index $I = (\iota_1, \dots, \iota_k)$ of length k, where each $\iota_i \in \{1, 2, \dots, d\}$, we make the following definition:

Definition 6.6. The iterated derivative operators are defined by

$$\nabla_{\mathscr{A}}^I \stackrel{\text{def}}{=} \nabla_{Z^{\iota_1}} \circ \cdots \circ \nabla_{Z^{\iota_k}}, \tag{6.5.1a}$$

$$\widehat{\nabla}_{\mathscr{A}}^I \stackrel{\text{def}}{=} \widehat{\nabla}_{Z^{\iota_1}} \circ \cdots \circ \widehat{\nabla}_{Z^{\iota_k}}, \tag{6.5.1b}$$

$$\mathscr{L}_{\mathscr{A}}^I \stackrel{\text{def}}{=} \mathscr{L}_{Z^{\iota_1}} \circ \cdots \circ \mathscr{L}_{Z^{\iota_k}}, \tag{6.5.1c}$$

$$\widehat{\mathscr{L}}_{\mathscr{A}}^I \stackrel{\text{def}}{=} \widehat{\mathscr{L}}_{Z^{\iota_1}} \circ \cdots \circ \widehat{\mathscr{L}}_{Z^{\iota_k}}, \quad \text{etc.} \tag{6.5.1d}$$

[23] Note that these are not the same modified Lie derivatives that appear in [Christodoulou and Klainerman 1993; Zipser 2000; Klainerman and Nicolò 2003; Bieri 2007].

Similarly, if $I = (\mu_1, \dots, \mu_k)$ is a coordinate multi-index of length k, where $\mu_1, \dots, \mu_k \in \{0, 1, 2, 3\}$ and U is a tensor field, then we use shorthand notation such as

$$\nabla^I U \overset{\text{def}}{=} \nabla_{\mu_1} \cdots \nabla_{\mu_k} U, \quad \text{etc.} \tag{6.5.2}$$

Under the above conventions, the Leibniz rule can be written as, e.g.,

$$\mathscr{L}_{\mathfrak{Z}}^I(UV) = \sum_{I_1+I_2=I} (\mathscr{L}_{\mathfrak{Z}}^{I_1} U)(\mathscr{L}_{\mathfrak{Z}}^{I_2} V), \quad \text{etc.,} \tag{6.5.3}$$

where by a sum over $I_1 + I_2 = I$ we mean a sum over all order-preserving partitions of the index I into two multi-indices. That is, if $I = (\iota_1, \dots, \iota_k)$, then $I_1 = (\iota_{i_1}, \dots, \iota_{i_a})$ and $I_2 = (\iota_{i_{a+1}}, \dots, \iota_{i_k})$, where $i_1, \dots, i_k$ is any reordering of the integers $1, \dots, k$ such that $i_1 < \cdots < i_a$ and $i_{a+1} < \cdots < i_k$.

The next standard lemma provides a useful expression relating Lie derivatives to covariant derivatives.

Lemma 6.7 (Lie derivatives in terms of covariant derivatives [Wald 1984, p. 441]). *Let X be a vector field, and let U be a tensor field of type $\binom{n}{m}$. Then $\mathscr{L}_X U$ can be expressed in terms of covariant derivatives of U and X as follows*:

$$\begin{aligned}\mathscr{L}_X U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} = \nabla_X U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n} + U_{\kappa\mu_2\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\nabla_{\mu_1}X^\kappa + \cdots + U_{\mu_1\cdots\mu_{m-1}\kappa}{}^{\nu_1\cdots\nu_n}\nabla_{\mu_m}X^\kappa \\ - U_{\mu_1\cdots\mu_m}{}^{\kappa\nu_2\cdots\nu_n}\nabla_\kappa X^{\nu_1} - \cdots - U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_{n-1}\kappa}\nabla_\kappa X^{\nu_n}.\end{aligned} \tag{6.5.4}$$

The next lemma shows that the operators $\mathscr{L}_Z$ and $\widehat{\mathscr{L}}_Z$ commute with ∇ if $Z \in \mathfrak{Z}$.

Lemma 6.8 ($\mathscr{L}_Z$ and ∇ commute). *Let ∇ denote the Levi-Civita connection corresponding to the Minkowski metric m, and let I be a $\mathfrak{Z}$-multi-index. Let $\widehat{\mathscr{L}}_{\mathfrak{Z}}^I$ be the iterated modified Lie derivative from Definitions 6.5 and 6.6. Then*

$$[\nabla, \mathscr{L}_{\mathfrak{Z}}^I] = 0 \quad \textit{and} \quad [\nabla, \widehat{\mathscr{L}}_{\mathfrak{Z}}^I] = 0. \tag{6.5.5}$$

In an arbitrary coordinate system, equations (6.5.5) *are equivalent to the following relations, which hold for all type-$\binom{n}{m}$ tensor fields U*:

$$\begin{aligned}\nabla_\mu\{\mathscr{L}_{\mathfrak{Z}}^I U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\} &= \mathscr{L}_{\mathfrak{Z}}^I\{\nabla_\mu U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\}, \\ \nabla_\mu\{\widehat{\mathscr{L}}_{\mathfrak{Z}}^I U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\} &= \widehat{\mathscr{L}}_{\mathfrak{Z}}^I\{\nabla_\mu U_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\}.\end{aligned} \tag{6.5.6}$$

Proof. The relation (6.5.5) can be shown via induction in $|I|$ by using (6.5.4) and the fact that $\nabla\nabla Z = 0$. $\square$

The next lemma captures the commutation properties of vector fields $Z \in \mathfrak{Z}$.

Lemma 6.9 (Lie bracket relations [Christodoulou and Klainerman 1990, p. 139]). *Relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, the vector fields belonging to the subset $\mathfrak{Z} \overset{\text{def}}{=} \{\frac{\partial}{\partial x^\mu}, \Omega_{\mu\nu}, S\}_{0\le\mu\le\nu\le 3}$ of the Minkowskian conformal Killing fields satisfy the following commutation relations, where ${}^{(Z)}c_\mu{}^\kappa$ is defined in* (6.2.4):

$$\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0 = {}^{(\partial/\partial x^\nu)}c_\mu^{\ \kappa}\frac{\partial}{\partial x^\kappa} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{6.5.7a}$$

$$\left[\frac{\partial}{\partial x^\lambda}, \Omega_{\mu\nu}\right] = m_{\lambda\mu}\frac{\partial}{\partial x^\nu} - m_{\lambda\nu}\frac{\partial}{\partial x^\mu} = {}^{(\Omega_{\mu\nu})}c_\lambda^{\ \kappa}\frac{\partial}{\partial x^\kappa} \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.5.7b}$$

$$\left[\frac{\partial}{\partial x^\mu}, S\right] = \frac{\partial}{\partial x^\mu} = {}^{(S)}c_\mu^{\ \kappa}\frac{\partial}{\partial x^\kappa} \qquad (\mu = 0, 1, 2, 3), \tag{6.5.7c}$$

$$[\Omega_{\kappa\lambda}, \Omega_{\mu\nu}] = m_{\kappa\mu}\Omega_{\nu\lambda} - m_{\kappa\nu}\Omega_{\mu\lambda} + m_{\lambda\mu}\Omega_{\kappa\nu} - m_{\lambda\nu}\Omega_{\kappa\mu} \quad (\kappa, \lambda, \mu, \nu = 0, 1, 2, 3), \tag{6.5.7d}$$

$$[\Omega_{\mu\nu}, S] = 0 \qquad (\mu, \nu = 0, 1, 2, 3). \tag{6.5.7e}$$

We now provide the following simple commutation lemma:

Lemma 6.10 (∇_Z and $\nabla_{\partial/\partial x^\mu}$ commutation relations). *Let $Z \in \mathfrak{Z}$. Then relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, the differential operators $\nabla_{\partial/\partial x^\mu}$ and ∇_Z satisfy the following commutation relations*:

$$[\nabla_{\partial/\partial x^\mu}, \nabla_Z] = {}^{(Z)}c_\mu^{\ \kappa}\frac{\partial}{\partial x^\kappa}, \tag{6.5.8}$$

where ${}^{(Z)}c_\mu^{\ \kappa}$ is defined in (6.2.4).

Proof. The relation (6.5.8) follows from Lemma 6.9 and the identity $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$, which holds for all pairs of vector fields X and Y; this identity holds because of the torsion-free property of the connection ∇ and because the Riemann curvature tensor of the Minkowski metric $m_{\mu\nu}$ completely vanishes. □

The next lemma shows that the operators ∇ and $\nabla_{\mathfrak{Z}}^I$ commute up to lower-order terms.

Lemma 6.11 (∇ and $\nabla_{\mathfrak{Z}}^I$ commutation inequalities). *Let U be a type-$\binom{n}{m}$ tensor field, and let I be a $\mathfrak{Z}$-multi-index. Then the following inequality holds*:

$$|\nabla_{\mathfrak{Z}}^I \nabla U| \lesssim |\nabla \nabla_{\mathfrak{Z}}^I U| + \sum_{|J| \le |I|-1} |\nabla \nabla_{\mathfrak{Z}}^J U|. \tag{6.5.9}$$

Proof. Using (5.1.14), we have that

$$|\nabla_{\mathfrak{Z}}^I \nabla U| \approx \sum_{\mu=0}^{3} |\nabla_{\mathfrak{Z}}^I \nabla_{\partial/\partial x^\mu} U|. \tag{6.5.10}$$

We therefore repeatedly apply Lemma 6.10 to deduce that there exist constants $C_{I;J}^\nu$ such that

$$\nabla_{\mathfrak{Z}}^I \nabla_{\partial/\partial x^\mu} U = \nabla_{\partial/\partial x^\mu} \nabla_{\mathfrak{Z}}^I U + \sum_{|J| \le |I|-1} \sum_{\nu=0}^{3} C_{I;J}^\nu \nabla_{\partial/\partial x^\nu} \nabla_{\mathfrak{Z}}^J U. \tag{6.5.11}$$

Inequality (6.5.9) now follows from applying (5.1.14) to each side of (6.5.11). □

The next lemma provides some important differential identities.

Lemma 6.12 (Geometric differential identities). *Let $\underline{L}$ and L be the Minkowski-null geodesic vector fields defined in* (5.1.2a)–(5.1.2b), *and let $O \in \mathbb{O}$. Then the vector fields $\underline{L}$, L, and O mutually commute*:

$$[\underline{L}, L] = 0, \qquad [\underline{L}, O] = 0, \quad \textit{and} \quad [L, O] = 0. \tag{6.5.12}$$

Furthermore, let $\upsilon_{\kappa\lambda\mu\nu}$, $\not{h}_{\mu\nu}$, *and* $\not{\psi}_{\mu\nu}$ *denote the tensor fields defined in* (5.1.4b), (5.1.6a), *and* (5.1.6c). *Then*

$$\mathcal{L}_O \upsilon_{\kappa\lambda\mu\nu} = 0, \tag{6.5.13a}$$

$$\mathcal{L}_O \not{h}_{\mu\nu} = 0, \tag{6.5.13b}$$

$$\mathcal{L}_O \not{\psi}_{\mu\nu} = 0. \tag{6.5.13c}$$

Proof. Equation (6.5.12) can be checked by performing straightforward calculations and using the definitions (5.1.2a)–(5.1.2b) of $\underline{L}$ and L, the definitions of the rotations $O \in \mathbb{O}$ given at the beginning of Section 6.2, and the Lie bracket formula (6.3.1). Equation (6.5.13a) follows from the well-known identity $\mathcal{L}_X \upsilon_{\kappa\lambda\mu\nu} = \frac{1}{2}{}^{(X)}\pi^{\beta}{}_{\beta}\upsilon_{\kappa\lambda\mu\nu}$, where ${}^{(X)}\pi_{\mu\nu}$ is defined in (6.2.2), together with the fact that $\mathcal{L}_O m_{\mu\nu} = {}^{(O)}\pi_{\mu\nu} = 0$ (i.e., that O is a Killing field of $m_{\mu\nu}$). Equations (6.5.13b) and (6.5.13c) then follow from definitions (5.1.4b) and (5.1.6c) and the identities (6.5.12)–(6.5.13a). □

The next lemma shows that the modified covariant derivatives $\widehat{\nabla}_{\mathcal{Z}}^I$ have favorable commutation properties with the Minkowski wave operator.

Lemma 6.13 ($\widehat{\nabla}_{\mathcal{Z}}^I$ and $\square_m$ commutation properties). *Let I be a $\mathcal{Z}$-multi-index, and let ϕ be any function. Let $\widehat{\nabla}_{\mathcal{Z}}^I$ be the iterated modified Minkowski covariant derivative operator from Definitions 6.5 and 6.6, and let $\square_m \overset{\text{def}}{=} (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ denote the Minkowski wave operator. Then*

$$\widehat{\nabla}_{\mathcal{Z}}^I \square_m \phi = \square_m \nabla_{\mathcal{Z}}^I \phi. \tag{6.5.14}$$

Proof. Using the symmetry of the tensor field $\nabla_\kappa\nabla_\lambda\phi$ together with (6.1.3a), (6.2.6), and definition (6.4.1), we compute that

$$\begin{aligned}
\square_m \nabla_Z \phi = (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda(Z^\zeta\nabla_\zeta\phi) &= \nabla_Z\square_m\phi + 2(\nabla^\kappa Z^\lambda)\nabla_\lambda\nabla_\kappa\phi \\
&= \nabla_Z\square_m\phi + (\nabla^\kappa Z^\lambda + \nabla^\lambda Z^\kappa)\nabla_\kappa\nabla_\lambda\phi \\
&= \nabla_Z\square_m\phi + c_Z\square_m\phi \\
&\overset{\text{def}}{=} \widehat{\nabla}_Z\square_m\phi.
\end{aligned} \tag{6.5.15}$$

This proves (6.5.14) in the case $|I| = 1$. The general case now follows inductively. □

The next lemma shows that the modified Lie derivative $\mathcal{L}_{\mathcal{Z}}^I$ operator has favorable commutation properties with the linear Maxwell–Maxwell term $\nabla_\mu\mathcal{F}^{\mu\nu} = \frac{1}{2}[(m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}]\nabla_\mu\mathcal{F}_{\kappa\lambda}$.

Lemma 6.14 (Commutation properties of $\widehat{\mathcal{L}}_{\mathcal{Z}}^I$ with a linear Maxwell–Maxwell term). *Let I be a $\mathcal{Z}$-multi-index, and let $\mathcal{F}$ be a two-form. Let $\widehat{\mathcal{L}}_{\mathcal{Z}}^I$ be the iterated modified Lie derivative from Definitions 6.5 and 6.6. Then*

$$\begin{aligned}
&\widehat{\mathcal{L}}_{\mathcal{Z}}^I\Big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{F}_{\kappa\lambda}\Big) \\
&\qquad = \big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\kappa\lambda}.
\end{aligned} \tag{6.5.16}$$

Proof. Let $Z \in \mathcal{Z}$. By the Leibniz rule, (6.3.4b), and Lemma 6.8, we have that

$$\begin{aligned}
&\mathcal{L}_Z\Big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{F}_{\kappa\lambda}\Big)\\
&\qquad=-2c_Z\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{F}_{\kappa\lambda}\\
&\qquad\qquad+\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{L}_Z\mathcal{F}_{\kappa\lambda}. \qquad (6.5.17)
\end{aligned}$$

It thus follows from Definition 6.5 that

$$\begin{aligned}
&\widehat{\mathcal{L}}_Z\Big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{F}_{\kappa\lambda}\Big)\\
&\qquad=\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda}-(m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_\mu\mathcal{L}_Z\mathcal{F}_{\kappa\lambda}. \qquad (6.5.18)
\end{aligned}$$

This implies (6.5.16) in the case $|I|=1$. The general case now follows inductively. □

The next lemma shows that some of the differential operators we have introduced commute with the null decomposition of a two-form.

Lemma 6.15 (Differential operators that commute with the null decomposition). *Let $\mathcal{F}$ be a two-form, and let $\underline{\alpha}$, α, ρ, and σ be its Minkowskian null components. Let $O\in\mathbb{O}$ be any of the rotational Minkowskian Killing fields Ω_{jk} $(1\le j<k\le 3)$. Then $\mathcal{L}_O\underline{\alpha}[\mathcal{F}]=\underline{\alpha}[\mathcal{L}_O\mathcal{F}]$, $\mathcal{L}_O\alpha[\mathcal{F}]=\alpha[\mathcal{L}_O\mathcal{F}]$, $\mathcal{L}_O\rho[\mathcal{F}]=\rho[\mathcal{L}_O\mathcal{F}]$, and $\mathcal{L}_O\sigma[\mathcal{F}]=\sigma[\mathcal{L}_O\mathcal{F}]$. An analogous result holds for the operators $\nabla_{\underline{L}}$ and ∇_L; i.e., $\mathcal{L}_O$, $\nabla_{\underline{L}}$, and ∇_L commute with the null decomposition of $\mathcal{F}$.*

Proof. Lemma 6.15 follows from Definition 5.9, Lemmas 5.7 and 6.12, and the fact that $\mathcal{L}_O m_{\mu\nu}=(\mathcal{L}_O m^{-1})^{\mu\nu}=0$. □

The next lemma shows that *weighted* covariant derivatives can be controlled by covariant derivatives with respect to vector fields $Z\in\mathcal{Z}$.

Lemma 6.16 (Weighted pointwise differential operator inequalities [Lindblad and Rodnianski 2010, Lemma 5.1]). *For any tensor field U and any two-tensor Π, we have the following pointwise estimates (where $|\overline{\nabla}^2U|\stackrel{\text{def}}{=}|\overline{\nabla}\,\overline{\nabla}U|$):*

$$(1+t+|q|)|\overline{\nabla}U|+(1+|q|)|\nabla U|\lesssim\sum_{|I|\le 1}|\nabla_{\mathcal{Z}}^I U|, \qquad (6.5.19a)$$

$$|\overline{\nabla}^2U|+r^{-1}|\overline{\nabla}U|\lesssim r^{-1}(1+t+|q|)^{-1}\sum_{|I|\le 2}|\nabla_{\mathcal{Z}}^I U|, \qquad (6.5.19b)$$

$$|\Pi^{\kappa\lambda}\nabla_\kappa\nabla_\lambda U|\lesssim\big((1+t+|q|)^{-1}|\Pi|+(1+|q|)^{-1}|\Pi|_{\mathcal{L}\mathcal{L}}\big)\sum_{|I|\le 1}|\nabla\nabla_{\mathcal{Z}}^I U|. \qquad (6.5.19c)$$

The next lemma shows that rotational Lie derivatives can be used to approximate weighted $S_{r,t}$-intrinsic covariant derivatives.

Lemma 6.17 (Weighted covariant derivatives approximated by rotational Lie derivatives [Speck 2012, Lemma 8.0.5]). *Let U be any tensor field m-tangent to the spheres $S_{r,t}$ and $k\ge 0$ be any integer. Then with $r\stackrel{\text{def}}{=}|x|$, we have that*

$$\sum_{|I|\le k}r^{|I|}|\nabla\!\!\!/^{I}U|\approx\sum_{|I|\le k}|\mathcal{L}_{\mathbb{O}}^I U|. \qquad (6.5.20)$$

Corollary 6.18. *Let $\mathscr{F}$ be a two-form, and let $\underline{\alpha}[\mathscr{F}]$, $\alpha[\mathscr{F}]$, $\rho[\mathscr{F}]$, and $\sigma[\mathscr{F}]$ denote its Minkowskian null components. Then with $r = |x|$, we have that*

$$r|\nabla\!\!\!/\,\underline{\alpha}[\mathscr{F}]| \lesssim \sum_{|I|\leq 1} |\underline{\alpha}[\mathcal{L}_{\mathscr{Z}}^I \mathscr{F}]|. \tag{6.5.21}$$

Furthermore, analogous inequalities hold for $\alpha[\mathscr{F}]$, $\rho[\mathscr{F}]$, and $\sigma[\mathscr{F}]$.

Proof. Inequality (6.5.21) follows from Lemmas 6.15 and 6.17. □

Finally, the following proposition provides pointwise inequalities relating various Lie and covariant derivative operators under various contraction seminorms:

Proposition 6.19 (Lie derivative and Minkowski covariant derivative comparison inequalities). *Let U be a tensor field. Then*

$$\sum_{|I|\leq k} |\mathcal{L}_{\mathscr{Z}}^I U| \approx \sum_{|I|\leq k} |\nabla_{\mathscr{Z}}^I U|. \tag{6.5.22}$$

Furthermore, let P be a symmetric or an antisymmetric type-$\binom{0}{2}$ tensor field. Then the following inequalities hold:

$$\sum_{|I|\leq k} |\nabla \mathcal{L}_{\mathscr{Z}}^I P| \lesssim \sum_{|I|\leq k} |\nabla \nabla_{\mathscr{Z}}^I P|, \tag{6.5.23a}$$

$$\sum_{|I|\leq k} |\overline{\nabla} \mathcal{L}_{\mathscr{Z}}^I P| \lesssim \sum_{|I|\leq k} |\overline{\nabla} \nabla_{\mathscr{Z}}^I P|, \tag{6.5.23b}$$

$$|\mathcal{L}_{\mathscr{Z}}^I P|_{\mathcal{L}\mathcal{L}} \lesssim |\nabla_{\mathscr{Z}}^I P|_{\mathcal{L}\mathcal{L}} + \underbrace{\sum_{|J|\leq |I|-1} |\nabla_{\mathscr{Z}}^J P|_{\mathcal{L}\mathcal{T}}}_{\textit{absent if } |I|=0} + \underbrace{\sum_{|J'|\leq |I|-2} |\nabla_{\mathscr{Z}}^{J'} P|}_{\textit{absent if } |I|\leq 1}, \tag{6.5.23c}$$

$$|\nabla \mathcal{L}_{\mathscr{Z}}^I P|_{\mathcal{L}\mathcal{L}} \lesssim |\nabla \nabla_{\mathscr{Z}}^I P|_{\mathcal{L}\mathcal{L}} + \underbrace{\sum_{|J|\leq |I|-1} |\nabla_{\mathscr{Z}}^J P|_{\mathcal{L}\mathcal{T}}}_{\textit{absent if } |I|=0} + \underbrace{\sum_{|J'|\leq |I|-2} |\nabla \nabla_{\mathscr{Z}}^{J'} P|}_{\textit{absent if } |I|\leq 1}, \tag{6.5.23d}$$

$$|\nabla P|_{\mathcal{L}\mathcal{N}} + |\nabla P|_{\mathcal{T}\mathcal{T}} \lesssim (1+|q|)^{-1} \sum_{|I|\leq 1} \big(|\mathcal{L}_{\mathscr{Z}}^I P|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathscr{Z}}^I P|_{\mathcal{T}\mathcal{T}}\big) + (1+t+|q|)^{-1} \sum_{|I|\leq 1} |\mathcal{L}_{\mathscr{Z}}^I P|. \tag{6.5.23e}$$

Proof. Inequality (6.5.22) follows inductively from (6.2.4) and (6.5.4).

To prove the remaining inequalities, for each $Z \in \mathscr{Z}$, we define the contraction operator $\mathscr{C}_Z$ by

$$(\mathscr{C}_Z P)_{\mu\nu} \overset{\text{def}}{=} P_{\kappa\nu} {}^{(Z)}c_\mu{}^\kappa + P_{\mu\kappa} {}^{(Z)}c_\nu{}^\kappa, \tag{6.5.24}$$

where the covariantly constant tensor field ${}^{(Z)}c_\mu{}^\kappa$ is defined in (6.2.4). It follows from definition (6.5.24) and Lemma 6.7 that

$$\mathcal{L}_{\mathscr{Z}} P = \nabla_Z P + \mathscr{C}_Z P. \tag{6.5.25}$$

Since each $Z \in \mathscr{Z}$ is a conformal Killing field and since $L^\mu L^\nu m_{\mu\nu} = 0$, it follows that $L^\mu L_\nu {}^{(Z)}c_\mu{}^\nu = 0$. Also using the fact that each ${}^{(Z)}c_\mu{}^\nu$ is a constant, we have that

$$|\mathscr{C}_Z P|_{\mathcal{L}\mathcal{L}} \lesssim |P|_{\mathcal{L}\mathcal{T}}, \tag{6.5.26}$$

$$|\mathscr{C}_Z P| \lesssim |P|. \tag{6.5.27}$$

If $I = (\iota_1, \ldots, \iota_k)$ is a $\mathcal{Z}$-multi-index with $1 \le |I| = k$, then using the fact that the components ${}^{(Z)}c_{\mu}{}^{\kappa}$ are constants, we have that

$$\begin{aligned}
\mathcal{L}_{\mathcal{Z}}^I P &\stackrel{\text{def}}{=} \mathcal{L}_{Z^{\iota_1}} \circ \cdots \circ \mathcal{L}_{Z^{\iota_k}} P \\
&= (\nabla_{Z^{\iota_1}} + \mathscr{C}_{Z^{\iota_1}}) \circ \cdots \circ (\nabla_{Z^{\iota_k}} + \mathscr{C}_{Z^{\iota_k}}) P \\
&= \nabla_{\mathcal{Z}}^I P + \sum_{i=1}^{k} \mathscr{C}_{Z^{\iota_i}} \circ \nabla_{Z^{\iota_1}} \circ \cdots \circ \nabla_{Z^{\iota_{i-1}}} \circ \nabla_{Z^{\iota_{i+1}}} \circ \cdots \circ \nabla_{Z^{\iota_k}} P + \overbrace{\sum_{\substack{I_1+I_2=I \\ |I_2|\le k-2}} \mathscr{C}_{\mathcal{Z}}^{I_1} \nabla_{\mathcal{Z}}^{I_2} P}^{\text{absent if } k=1}.
\end{aligned} \tag{6.5.28}$$

Inequality (6.5.23a) now follows from applying ∇ to each side of (6.5.28), from using the fact that the operator ∇ commutes through the operators $\mathscr{C}_Z$, and from (6.5.27). Inequality (6.5.23b) follows from similar reasoning. Inequalities (6.5.23c) and (6.5.23d) also follow from similar reasoning together with (6.5.26).

To prove (6.5.23e), we first observe that, by (6.5.19a) and (6.5.22), we have that

$$\begin{aligned}
|\nabla P|_{\mathcal{L}\mathcal{N}} + |\nabla P|_{\mathcal{T}\mathcal{T}} &\lesssim |\nabla_{\underline{L}} P|_{\mathcal{L}\mathcal{N}} + |\nabla_{\underline{L}} P|_{\mathcal{T}\mathcal{T}} + |\overline{\nabla} P| \\
&\lesssim |\nabla_{\underline{L}} P|_{\mathcal{L}\mathcal{N}} + |\nabla_{\underline{L}} P|_{\mathcal{T}\mathcal{T}} + (1+t+|q|)^{-1} \sum_{|I|\le 1} |\mathcal{L}_{\mathcal{Z}}^I P|.
\end{aligned} \tag{6.5.29}$$

Therefore, from (6.5.29), we see that to prove (6.5.23e) it suffices to prove that the following inequality holds for any symmetric or antisymmetric type-$\binom{0}{2}$ tensor field P:

$$|\nabla_{\underline{L}} P|_{\mathcal{L}\mathcal{N}} + |\nabla_{\underline{L}} P|_{\mathcal{T}\mathcal{T}} \lesssim (1+|q|)^{-1} \sum_{|I|\le 1} \big(|\mathcal{L}_{\mathcal{Z}}^I P|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I P|_{\mathcal{T}\mathcal{T}} \big). \tag{6.5.30}$$

To this end, we use the vector fields $S = x^{\kappa}\,\partial_{\kappa}$ and $\Omega_{0j} = -t\,\partial_j - x_j\,\partial_t$ to decompose

$$\underline{L} = -q^{-1}(S + \omega^a \Omega_{0a}) \quad \text{and} \quad \omega^a \stackrel{\text{def}}{=} x^a/r, \tag{6.5.31}$$

which implies that

$$-q \nabla_{\underline{L}} P_{\mu\nu} = \nabla_S P_{\mu\nu} + \omega^a \nabla_{\Omega_{0a}} P_{\mu\nu}. \tag{6.5.32}$$

Using (6.2.5a), (6.2.5b), and (6.5.4), we compute that

$$\nabla_S P_{\mu\nu} = \mathcal{L}_S P_{\mu\nu} - 2P_{\mu\nu}, \tag{6.5.33}$$

$$\omega^a \nabla_{\Omega_{0a}} P_{\mu\nu} = \omega^a \mathcal{L}_{\Omega_{0a}} P_{\mu\nu} - \tfrac{1}{2}\big(\underline{L}_{\mu} L^{\kappa} P_{\kappa\nu} - L_{\mu}\underline{L}^{\kappa} P_{\kappa\nu} + \underline{L}_{\nu} L^{\kappa} P_{\mu\kappa} - L_{\nu}\underline{L}^{\kappa} P_{\mu\kappa}\big). \tag{6.5.34}$$

Inserting these two identities into (6.5.32), we conclude that

$$\begin{aligned}
-q \nabla_{\underline{L}} P_{\mu\nu} = \mathcal{L}_S P_{\mu\nu} + \omega^a \mathcal{L}_{\Omega_{0a}} P_{\mu\nu} - 2P_{\mu\nu}& \\
- \tfrac{1}{2}\big(\underline{L}_{\mu} L^{\kappa} P_{\kappa\nu} - L_{\mu}\underline{L}^{\kappa} P_{\kappa\nu} + \underline{L}_{\nu} L^{\kappa} P_{\mu\kappa} - L_{\nu}\underline{L}^{\kappa} P_{\mu\kappa}\big)&.
\end{aligned} \tag{6.5.35}$$

Contracting (6.5.35) against the sets $\mathscr{L}\mathscr{N}$ and $\mathscr{T}\mathscr{T}$, we see that

$$|q||\nabla_{\underline{L}} P|_{\mathscr{L}\mathscr{N}} + |q||\nabla_{\underline{L}} P|_{\mathscr{T}\mathscr{T}} \lesssim \sum_{|I|\leq 1} \big(|\mathscr{L}_{\mathfrak{Z}}^I P|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathfrak{Z}}^I P|_{\mathscr{T}\mathscr{T}} \big). \tag{6.5.36}$$

Furthermore, by decomposing

$$\underline{L} = \partial_t - \partial_r = \partial_t - \omega^a \, \partial_a \tag{6.5.37}$$

and using the fact that ${}^{(\partial/\partial t)}c_{\mu}{}^{\nu} = {}^{(\partial/\partial x^j)}c_{\mu}{}^{\nu} = 0$ (where ${}^{(Z)}c_{\mu\nu}$ is defined in (6.2.4)), we deduce that

$$\nabla_{\underline{L}} P_{\mu\nu} = \mathscr{L}_{\partial/\partial t} P_{\mu\nu} - \omega^a \mathscr{L}_{\partial/\partial x^a} P_{\mu\nu}. \tag{6.5.38}$$

Contracting (6.5.38) against the sets $\mathscr{L}\mathscr{N}$ and $\mathscr{T}\mathscr{T}$, we have that

$$|\nabla_{\underline{L}} P|_{\mathscr{L}\mathscr{N}} + |\nabla_{\underline{L}} P|_{\mathscr{T}\mathscr{T}} \lesssim \sum_{|I|=1} \big(|\mathscr{L}_{\mathfrak{Z}}^I P|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathfrak{Z}}^I P|_{\mathscr{T}\mathscr{T}} \big). \tag{6.5.39}$$

Adding (6.5.36) and (6.5.39), we arrive at inequality (6.5.30). This completes our proof of (6.5.23e). □

7. The reduced equation satisfied by $\nabla_{\mathfrak{Z}}^I h^{(1)}$

In this short section, we assume that $h_{\mu\nu}^{(1)}$ is a solution to the reduced equation (3.7.1a). We provide a proposition that gives a preliminary description of the inhomogeneities in the equation satisfied by $\nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)}$.

Proposition 7.1 (Inhomogeneities for $\nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)}$). *Suppose that $h_{\mu\nu}^{(1)}$ is a solution to the reduced equation* (3.7.1a), *and let I be any $\mathfrak{Z}$-multi-index. Then $\nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)}$ is a solution to the inhomogeneous system*

$$\widetilde{\Box}_g \nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu}^{(1;I)}, \tag{7.0.1}$$

$$\begin{aligned} \mathfrak{H}_{\mu\nu}^{(1;I)} &= \widehat{\nabla}_{\mathfrak{Z}}^I \mathfrak{H}_{\mu\nu} - \widehat{\nabla}_{\mathfrak{Z}}^I \widetilde{\Box} h_{\mu\nu}^{(0)} - \big(\widehat{\nabla}_{\mathfrak{Z}}^I \widetilde{\Box}_g h_{\mu\nu}^{(1)} - \widetilde{\Box}_g \nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)} \big) \\ &= \widehat{\nabla}_{\mathfrak{Z}}^I \mathfrak{H}_{\mu\nu} - \widehat{\nabla}_{\mathfrak{Z}}^I \widetilde{\Box} h_{\mu\nu}^{(0)} - \big(\widehat{\nabla}_{\mathfrak{Z}}^I \big(H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda h_{\mu\nu}^{(1)} \big) - H^{\kappa\lambda} \nabla_\kappa \nabla_\lambda \nabla_{\mathfrak{Z}}^I h_{\mu\nu}^{(1)} \big). \end{aligned} \tag{7.0.2}$$

Proof. Proposition 7.1 follows from differentiating each side of (3.7.1a) with modified covariant derivatives $\widehat{\nabla}_{\mathfrak{Z}}^I$ and applying Lemma 6.13. □

8. The equations of variation, the canonical stress, and electromagnetic energy currents

In this section, we introduce the electromagnetic equations of variation, which are linearized versions of the reduced electromagnetic equations. The significance of the equations of variation is the following: if $\mathscr{F}$ is a solution to the reduced electromagnetic equations (3.7.1b)–(3.7.1c), then $\mathscr{L}_{\mathfrak{Z}}^I \mathscr{F}$ is a solution to the equations of variation. We then provide a preliminary description of the structure of the inhomogeneous terms in the equations of variation satisfied by $\mathscr{L}_{\mathfrak{Z}}^I \mathscr{F}$. Additionally, we introduce the canonical stress tensor field and use it to construct energy currents. The energy currents are vector fields that will be used in the divergence theorem to derive weighted energy estimates for solutions to the equations of variation; this analysis is carried out in Section 12.

8.1. ***Equations of variation.*** The equations of variation in the unknowns $\dot{\mathscr{F}}_{\mu\nu}$ are the linearization[24] of (3.7.1b)–(3.7.1c) around a background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$. More specifically, the equations of variation are the system

$$\nabla_\lambda \dot{\mathscr{F}}_{\mu\nu} + \nabla_\mu \dot{\mathscr{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathscr{F}}_{\lambda\mu} = \dot{\mathfrak{F}}_{\lambda\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{8.1.1a}$$

$$N^{\#\mu\nu\kappa\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu \quad (\nu = 0, 1, 2, 3), \tag{8.1.1b}$$

where $N^{\#\mu\nu\kappa\lambda}$ is the $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$-dependent tensor field defined in (3.7.2c) and $\dot{\mathfrak{F}}_{\lambda\mu\nu}$ and $\dot{\mathfrak{F}}^\nu$ are inhomogeneous terms that are specified in Proposition 8.1. In this article, the equations of variation will arise when we differentiate the reduced equations (3.7.1b)–(3.7.1c) with modified Lie derivatives. In particular, $\dot{\mathscr{F}}$ will be equal to $\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu}$. The next proposition, which is a companion of Proposition 7.1, provides a preliminary expression of the inhomogeneous terms that arise in the study of the equations of variation satisfied by $\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu}$. We remark that the proof of the proposition uses lemmas that are proved in Section 11.

Proposition 8.1 (Inhomogeneities for $\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu}$). *If $\mathscr{F}_{\mu\nu}$ is a solution to the reduced electromagnetic equations* (3.7.1b)–(3.7.1c) *and I is a $\mathscr{Z}$-multi-index, then $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu}$ is a solution to the equations of variation* (8.1.1a)–(8.1.1b) (*corresponding to the background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$*) *with inhomogeneous terms $\dot{\mathfrak{F}}_{\lambda\mu\nu} \overset{\text{def}}{=} \mathfrak{F}_{\lambda\mu\nu}^{(I)}$ and $\dot{\mathfrak{F}}^\nu \overset{\text{def}}{=} \mathfrak{F}_{(I)}^\nu$, where*

$$\mathfrak{F}_{\lambda\mu\nu}^{(I)} = 0, \tag{8.1.2a}$$

$$\mathfrak{F}_{(I)}^\nu = \widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu + \big(N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{Z}}^I \big(N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda}\big)\big). \tag{8.1.2b}$$

Furthermore, there exist constants $\widetilde{C}_{1;I_1,I_2}$, $\widetilde{C}_{2;I_1,I_2}$, $\widetilde{C}_{\mathscr{P};I_1,I_2}$, $\widetilde{C}_{\mathfrak{F}_\triangle;J}$, and $\widetilde{C}_{N_\triangle^\#;I_1,I_2}$ such that

$$\widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu = \sum_{|I_1|+|I_2| \leq |I|} \widetilde{C}_{2;I_1,I_2} \mathfrak{Q}_{(2;\mathscr{F})}^\nu (\nabla \mathscr{L}_{\mathscr{Z}}^{I_1} h, \mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}) + \sum_{|J| \leq |I|} \widetilde{C}_{\mathfrak{F}_\triangle;J} \mathscr{L}_{\mathscr{Z}}^J \mathfrak{F}_\triangle^\nu, \tag{8.1.3a}$$

$$\begin{aligned} N^{\#\mu\nu\kappa\lambda} &\nabla_\mu \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{Z}}^I (N^{\#\mu\nu\kappa\lambda} \nabla_\mu \mathscr{F}_{\kappa\lambda}) \\ &= \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{\mathscr{P};I_1,I_2} \mathscr{P}_{(\mathscr{F})}^\nu (\mathscr{L}_{\mathscr{Z}}^{I_1} h, \nabla \mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}) + \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{1;I_1,I_2} \mathfrak{Q}_{(1;\mathscr{F})}^\nu (\mathscr{L}_{\mathscr{Z}}^{I_1} h, \nabla \mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}) \\ &\quad + \sum_{\substack{|I_1|+|I_2| \leq |I| \\ |I_2| \leq |I|-1}} \widetilde{C}_{N_\triangle^\#;I_1,I_2} (\mathscr{L}_{\mathscr{Z}}^{I_1} N_\triangle^{\#\mu\nu\kappa\lambda}) \nabla_\mu \mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}_{\kappa\lambda}. \end{aligned} \tag{8.1.3b}$$

In the above formulas, $\mathfrak{F}_\triangle^\nu$ and $N_\triangle^{\#\mu\nu\kappa\lambda}$ are the error terms appearing in (3.7.2g) *and* (3.7.2h), *respectively, while $\mathscr{P}_{(\mathscr{F})}^\nu(\cdot,\cdot)$ and $\mathfrak{Q}_{(i;\mathscr{F})}^\nu(\cdot,\cdot)$ ($i = 1, 2$ and $\nu = 0, 1, 2, 3$) are the quadratic forms defined in* (3.7.3b), (3.7.3c), *and* (3.7.2e), *respectively.*

Proof. To prove (8.1.2a), we first recall (3.7.1b), which states that $\mathscr{F}_{\mu\nu}$ is a solution to $\nabla_{[\kappa} \mathscr{F}_{\mu\nu]} = 0$, where $[\cdot]$ denotes antisymmetrization. From (6.5.5), it therefore follows that

$$0 = \mathscr{L}_{\mathscr{Z}}^I \nabla_{[\lambda} \mathscr{F}_{\mu\nu]} = \nabla_{[\lambda} \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu]}, \tag{8.1.4}$$

which is the desired result.

[24] More precisely, the equations of variation are linear in $\dot{\mathscr{F}}$.

To derive (8.1.2b), we conclude that $\widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}(N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}) = \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}\mathfrak{F}^{\nu}$ by simply differentiating each side of (8.1.1b) with $\widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}$. Trivial algebraic manipulation then leads to the fact that $N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}_{\kappa\lambda} = \mathfrak{F}_{(I)}^{\nu}$, where $\mathfrak{F}_{(I)}^{\nu}$ is defined by (8.1.2b).

Equation (8.1.3a) follows from (3.7.2b), Definition 6.5 of $\widehat{\mathscr{L}}_{Z}$, and Lemma 11.8, which is proved in Section 11.2.

To prove (8.1.3b), we first recall the decomposition (3.7.3a):

$$N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda} = \tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathscr{F}_{\kappa\lambda} - \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathscr{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) + N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}. \quad (8.1.5)$$

The commutator term arising from the first term on the right-hand side of (8.1.5) vanishes. More specifically, we use (6.5.16) to conclude that

$$\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}\Big(\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\mathscr{F}_{\kappa\lambda}\Big) = 0. \quad (8.1.6)$$

Therefore, it follows from (8.1.5) and (8.1.6) that

$$N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}(N^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}) = \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}\mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathscr{P}_{(\mathscr{F})}^{\nu}(h, \nabla\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}) + \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}\mathscr{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{F}) - \mathscr{Q}_{(1;\mathscr{F})}^{\nu}(h, \nabla\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}) + N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{L}_{\mathfrak{Z}}^{I}\mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathfrak{Z}}^{I}(N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\mathscr{F}_{\kappa\lambda}). \quad (8.1.7)$$

The expression (8.1.3b) now follows from (8.1.7), the Leibniz rule, Definition 6.5 of $\widehat{\mathscr{L}}_{Z}$, Lemma 6.8, and Lemma 11.8 below. □

8.2. *The canonical stress.* The notion of the *canonical stress tensor field* $\dot{Q}^{\mu}{}_{\nu}$ in the context of PDE energy estimates was introduced by Christodoulou [2000]. As explained in Section 1.2.6, from the point of view of energy estimates, it plays the role of an energy-momentum-type tensor for the equations of variation. Its two key properties are (i) its divergence is lower-order (in the sense of the number of derivatives falling on the variations $\dot{\mathscr{F}}_{\mu\nu}$) and (ii) contraction against certain pairs (ξ, X) consisting of a one-form ξ_{μ} and a vector field X^{ν} leads to an energy density that can be used derive L^2 control of solutions $\dot{\mathscr{F}}_{\mu\nu}$ to the equations of variation. As we will see, property (i) is captured by Lemma 8.5 and (8.3.3) while property (ii) is captured by (8.3.2), (12.2.1), and (12.2.8). In order to explain the origin of the canonical stress, we first define the *linearized Lagrangian*; our definition is modeled after the definition given by Christodoulou [2000].

Definition 8.2. Given an electromagnetic Lagrangian $\mathscr{L}[\,\cdot\,]$ (as described in Section 3.2) and a "background" $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$, we define the linearized Lagrangian by

$$\dot{\mathscr{L}} \stackrel{\text{def}}{=} -\tfrac{1}{4}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}, \quad (8.2.1)$$

where $N^{\#\zeta\eta\kappa\lambda}$ is the $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$-dependent tensor field defined in (3.3.8).

Remark 8.3. $\dot{\mathscr{L}}$ is equal to $\frac{1}{2}(\partial^{2\star}\mathscr{L}[h,\mathscr{F}]/(\partial\mathscr{F}_{\zeta\eta}\,\partial\mathscr{F}_{\kappa\lambda}))\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}$ up to a correction term $\frac{1}{4}(\partial^{\star}\mathscr{L}/\partial\text{ⅎ}_{(2)})\cdot\epsilon^{\#\mu\nu\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}$ corresponding to the term $\frac{1}{2}(\partial^{\star}\mathscr{L}/\partial\text{ⅎ}_{(2)})\epsilon^{\#\mu\nu\kappa\lambda}$ from (3.3.10).

The merit of Definition 8.2 is the following: the principal part (from the point of view of number of derivatives) of the Euler–Lagrange equations (assuming that we view $(h,\mathscr{F})$ as a known background and $\dot{\mathscr{F}}$ to be the unknowns and that an appropriately defined action[25] is stationary with respect to closed variations of $\dot{\mathscr{F}}$) corresponding to $\dot{\mathscr{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$ is identical to the principal part of the electromagnetic equations of variation (8.1.1b); i.e., $\dot{\mathscr{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$ generates the principal part of the linearized equations.

Definition 8.4. Given a linearized Lagrangian $\dot{\mathscr{L}}[\dot{\mathscr{F}};h,\mathscr{F}]$, the canonical stress tensor field $\dot{Q}^{\mu}{}_{\nu}$ is defined as follows:

$$\dot{Q}^{\mu}{}_{\nu}=\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]\stackrel{\text{def}}{=}-2\frac{\partial\dot{\mathscr{L}}}{\partial\dot{\mathscr{F}}_{\mu\zeta}}\dot{\mathscr{F}}_{\nu\zeta}+\delta^{\mu}_{\nu}\dot{\mathscr{L}}=N^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\tfrac{1}{4}\delta^{\mu}_{\nu}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda},\tag{8.2.2}$$

where $N^{\#\mu\nu\kappa\lambda}$ is defined in (3.3.8).

Note that, in contrast to the energy-momentum tensor $T_{\mu\nu}$, $\dot{Q}_{\mu\nu}\stackrel{\text{def}}{=}m_{\mu\kappa}\dot{Q}^{\kappa}{}_{\nu}$ is in general not symmetric. We use the notation $\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]$ whenever we want to emphasize the quadratic dependence of $\dot{Q}^{\mu}{}_{\nu}$ on $\dot{\mathscr{F}}$.

Because of our assumption (3.3.3a) concerning the Lagrangian, $\dot{Q}^{\mu}{}_{\nu}$ is equal to the energy-momentum tensor (in $\dot{\mathscr{F}}$) for the standard Maxwell–Maxwell equations in Minkowski spacetime plus small corrections. More precisely, we insert the decomposition (3.7.2c) of $N^{\#\mu\zeta\kappa\lambda}$ into the right-hand side of (8.2.2) and perform simple computations, thereby arriving at the following decomposition of $\dot{Q}^{\mu}{}_{\nu}$:

$$\begin{aligned}\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]=&\overbrace{\dot{\mathscr{F}}^{\mu\zeta}\dot{\mathscr{F}}_{\nu\zeta}-\tfrac{1}{4}\delta^{\mu}_{\nu}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}^{\zeta\eta}}^{\substack{\text{terms from linear Maxwell–Maxwell}\\ \text{equations in Minkowski spacetime}}}\ \overbrace{-h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_{\nu}{}^{\zeta}-h^{\kappa\lambda}\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{\nu\lambda}+\tfrac{1}{2}\delta^{\mu}_{\nu}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}}^{\substack{\text{corrections to Minkowskian linear}\\ \text{Maxwell–Maxwell equations arising from } h}}\\&+\underbrace{N_{\triangle}^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\tfrac{1}{4}\delta^{\mu}_{\nu}N_{\triangle}^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}}_{\text{error terms}}.\end{aligned}\tag{8.2.3}$$

The next lemma captures the lower-order divergence property enjoyed by $\dot{Q}^{\mu}{}_{\nu}$.

Lemma 8.5 (Divergence of the canonical stress). *Let $\dot{\mathscr{F}}_{\mu\nu}$ be a solution to the equations of variation* (8.1.1a)–(8.1.1b) *corresponding to the background $(h_{\mu\nu},\mathscr{F}_{\mu\nu})$, and let $\dot{\mathfrak{F}}_{\lambda\mu\nu}$ and $\dot{\mathfrak{F}}^{\nu}$ be the inhomogeneous terms from the right-hand sides of* (8.1.1a)–(8.1.1b). *Let $\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]$ be the canonical stress tensor field defined in* (8.2.2). *Then*

$$\begin{aligned}\nabla_{\mu}\big(\dot{Q}^{\mu}{}_{\nu}[\dot{\mathscr{F}},\dot{\mathscr{F}}]\big)&=-\tfrac{1}{2}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathfrak{F}}_{\nu\kappa\lambda}+\dot{\mathscr{F}}_{\nu\eta}\dot{\mathfrak{F}}^{\eta}+(\nabla_{\mu}N^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\tfrac{1}{4}(\nabla_{\nu}N^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\\&=-\tfrac{1}{2}N^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathfrak{F}}_{\nu\kappa\lambda}+\dot{\mathscr{F}}_{\nu\eta}\dot{\mathfrak{F}}^{\eta}-(\nabla_{\mu}h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_{\nu}{}^{\zeta}-(\nabla_{\mu}h^{\kappa\lambda})\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{\nu\lambda}\\&\quad+\tfrac{1}{2}(\nabla_{\nu}h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}+(\nabla_{\mu}N_{\triangle}^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{\nu\zeta}-\tfrac{1}{4}(\nabla_{\nu}N_{\triangle}^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}.\end{aligned}\tag{8.2.4}$$

Proof. To obtain (8.2.4), we use (8.1.1a)–(8.1.1b), the expansion (3.7.2c), and the properties (3.3.9a)–(3.3.9c) (which are also satisfied by the tensor field $N_{\triangle}^{\#\mu\zeta\kappa\lambda}$). □

[25] A suitable action $\mathscr{A}_{\mathfrak{C}}[\dot{\mathscr{F}}]$ is, e.g., of the form $\mathscr{A}_{\mathfrak{C}}[\dot{\mathscr{F}}]\stackrel{\text{def}}{=}\int_{\mathfrak{C}\in\mathfrak{M}}\dot{\mathscr{L}}[\dot{\mathscr{F}};h,\mathscr{F}]\,d^{4}x$, where $\mathfrak{C}$ is a compact subset of spacetime.

8.3. ***Electromagnetic energy currents.*** In this section, we introduce the energy currents that will be used to derive the weighted energy estimate (12.2.1) for a solution $\dot{\mathscr{F}}$ to the equations of variation (8.1.1a)–(8.1.1b).

Definition 8.6. Let $h_{\mu\nu}$ be a symmetric type-$\binom{0}{2}$ tensor field, and let $\mathscr{F}_{\mu\nu}$ and $\dot{\mathscr{F}}_{\mu\nu}$ be a pair of two-forms. Let $w(q)$ be the weight defined in (12.1.1), and let $X^\nu \stackrel{\text{def}}{=} w(q)\delta_0^\nu$ be the "multiplier" vector field. We define the *energy current* $\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}]$ corresponding to the variation $\dot{\mathscr{F}}_{\mu\nu}$ and the background $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$ to be the vector field

$$\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}] \stackrel{\text{def}}{=} -\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]X^\nu = -w(q)\dot{Q}^\mu{}_0[\dot{\mathscr{F}}, \dot{\mathscr{F}}], \tag{8.3.1}$$

where $\dot{Q}^\mu{}_\nu[\dot{\mathscr{F}}, \dot{\mathscr{F}}]$ is the canonical stress tensor field from (8.2.2).

Lemma 8.7 (Positivity of $\dot{J}^0_{(h,\mathscr{F})}$). *Let $\dot{J}^\mu_{(h,\mathscr{F})}[\dot{\mathscr{F}}]$ be the energy current defined in* (8.3.1). *Then*

$$\dot{J}^0_{(h,\mathscr{F})} = \tfrac{1}{2}|\dot{\mathscr{F}}|^2 w(q) + \big(O^\infty(|h|; \mathscr{F}) + O^\ell(|(h, \mathscr{F})|^2)\big)|\dot{\mathscr{F}}|^2 w(q). \tag{8.3.2}$$

Furthermore, if $\dot{\mathscr{F}}_{\mu\nu}$ is a solution to the equations of variation (8.1.1a)–(8.1.1b) *with inhomogeneous terms $\dot{\mathfrak{F}}_{\lambda\mu\nu} \equiv 0$, then the Minkowskian divergence of $\dot{J}_{(h,\mathscr{F})}$ can be expressed as follows*:

$$\begin{aligned}
\nabla_\mu \dot{J}^\mu_{(h,\mathscr{F})} = -\tfrac{1}{2}w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2) - w(q)\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{F}}^\eta \\
- w(q)\big(-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big) \\
- w'(q)\big(-L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big) \\
- w(q)\big((\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big) \\
- w'(q)\big(L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big),
\end{aligned} \tag{8.3.3}$$

where $\dot{\alpha} \stackrel{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \stackrel{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, and $\dot{\sigma} \stackrel{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ are the "favorable" Minkowskian null components of $\dot{\mathscr{F}}$ defined in Section 5.3.

Remark 8.8. The term $\frac{1}{2}w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)$ appearing on the right-hand side of (8.3.3) is of central importance for closing the bootstrap argument during our global existence proof. It manifests itself as the additional positive spacetime integral $\int_0^t \int_{\Sigma_\tau} (|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}})w'(q)\,d^3x\,d\tau$ on the left-hand side of (12.2.1) below and provides a means for controlling some of the spacetime integrals that emerge in Section 16.4.

Proof. Equation (8.3.2) follows from (8.2.3), simple calculations, and (3.7.2h).

To prove (8.3.3), we first recall that since $q = r - t$ it follows that $\nabla_\mu q = L_\mu$, where L is defined in (5.1.2b). Hence, we have that $\nabla_\mu w(q) = w'(q)L_\mu$. Using this fact, (8.2.3), and (8.2.4), we calculate that

$$\begin{aligned}\nabla_\mu \dot{J}^\mu_{(h,\mathscr{F})} = -w(q)\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{J}}^\eta - w(q)\big(-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0^{\ \zeta} - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu_{\ \kappa}\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda^{\ \eta}\big)\\ - w(q)\big((\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big)\\ - w'(q)\underbrace{\big(L_\mu \dot{\mathscr{F}}^{\mu\zeta}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}^{\kappa\lambda}\big)}_{(|\dot{\alpha}|^2+\dot{\rho}^2+\dot{\sigma}^2)/2}\\ - w'(q)\big(-L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0^{\ \zeta} - L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu_{\ \kappa}\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda^{\ \eta}\big)\\ - w'(q)\big(L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big). \qquad (8.3.4)\end{aligned}$$

The expression (8.3.3) thus follows. □

9. Decompositions of the electromagnetic equations

In this section we perform two decompositions of the electromagnetic equations. The first is a null decomposition of the equations of variation, which will be used in Section 15 to derive pointwise decay estimates for the lower-order Lie derivatives of $\mathscr{F}_{\mu\nu}$. The second is a decomposition of the electromagnetic equations into constraint and evolution equations for the Minkowskian one-forms E_μ and B_μ, which are respectively known as the electric field and magnetic induction. This decomposition will be used in Section 10 to prove that our smallness condition on the abstract data necessarily implies a smallness condition on the initial energy $\mathscr{E}_{\ell;\gamma;\mu}(0)$ of the corresponding solution to the reduced equations. We remark that the Minkowskian one-forms D_μ and H_μ, which are respectively known as the electric displacement and the magnetic field, and also the geometric electromagnetic one-forms $\mathfrak{E}_\mu$, $\mathfrak{B}_\mu$, $\mathfrak{D}_\mu$, and $\mathfrak{H}_\mu$ will play a role in the discussion.

9.1. ***The Minkowskian null decomposition of the electromagnetic equations of variation.*** In this section, we decompose the equations of variation into equations for the null components of $\dot{\mathscr{F}}$. The main advantage of our decomposition, which is given in Proposition 9.3, is that the terms in each equation can be separated into two classes: (i) a derivative of a null component in a "nearly Minkowski-null" direction[26] and (ii) the error terms. Although from the point of view of differentiability some of the error terms are higher-order, it will turn out that all error terms are lower-order in terms of decay rates. In this way, the equations can be viewed as *ordinary differential inequalities* with inhomogeneous terms (which we loosely refer to as ODEs) for the null components of $\dot{\mathscr{F}}$. This point of view is realized in Proposition 11.5. The key point is that the ODEs we derive are amenable to Gronwall estimates. In Section 15, we will use this line of argument to derive pointwise decay estimates for the null components of the lower-order Lie derivatives of a solution $\mathscr{F}$ to the electromagnetic equations (3.7.1b)–(3.7.1c). These estimates will be an improvement over what can be deduced from the weighted Klainerman–Sobolev inequality (B.4) alone; see the beginning of Section 15 for additional details regarding this improvement.

We begin the analysis by using (3.7.2c) to write the equations of variation (8.1.1a)–(8.1.1b) in the following form:

[26] By "nearly Minkowski-null", we mean vectors that are nearly parallel to $\underline{L}$ or L with some corrections coming from the presence of a nonzero h in the case of the vector field L.

$$\nabla_{\lambda}\dot{\mathscr{F}}_{\mu\nu} + \nabla_{\mu}\dot{\mathscr{F}}_{\nu\lambda} + \nabla_{\nu}\dot{\mathscr{F}}_{\lambda\mu} = 0, \tag{9.1.1a}$$

$$\begin{aligned} &\tfrac{1}{2}\big((m^{-1})^{\mu\kappa}(m^{-1})^{\nu\lambda} - (m^{-1})^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\dot{\mathscr{F}}_{\kappa\lambda} \\ &\quad + \tfrac{1}{2}\big(-h^{\mu\kappa}(m^{-1})^{\nu\lambda} + h^{\mu\lambda}(m^{-1})^{\nu\kappa}\big)\nabla_{\mu}\dot{\mathscr{F}}_{\kappa\lambda} \\ &\quad + \tfrac{1}{2}\big(-(m^{-1})^{\mu\kappa}h^{\nu\lambda} + (m^{-1})^{\mu\lambda}h^{\nu\kappa}\big)\nabla_{\mu}\dot{\mathscr{F}}_{\kappa\lambda} + N_{\triangle}^{\#\mu\nu\kappa\lambda}\nabla_{\mu}\dot{\mathscr{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^{\nu}. \end{aligned} \tag{9.1.1b}$$

In our calculations below, we will make use of the identities

$$\nabla_A \underline{L} = -r^{-1}e_A \quad \text{and} \quad \nabla_A L = r^{-1}e_A, \tag{9.1.2}$$

which can be directly calculated in our wave-coordinate system by using (5.1.2a)–(5.1.2b). We will also make use of the identity

$$\begin{aligned} \nabla\!\!\!/_A e_B &= \nabla_A e_B + \tfrac{1}{2}m(\nabla_A e_B, \underline{L})L + \tfrac{1}{2}m(\nabla_A e_B, L)\underline{L} \\ &= \nabla_A e_B - \tfrac{1}{2}m(e_B, \nabla_A \underline{L})L - \tfrac{1}{2}m(e_B, \nabla_A L)\underline{L} \\ &= \nabla_A e_B + \tfrac{1}{2}r^{-1}\delta_{AB}(L - \underline{L}), \end{aligned} \tag{9.1.3}$$

which follows from (6.1.2) and (9.1.2).

Furthermore, if U is a type-$\binom{0}{m}$ tensor field and $X_{(i)}$ $(1 \le i \le m)$ and Y are vector fields, then by the Leibniz rule we have that

$$\begin{aligned} \nabla_Y\{U(X_{(1)}, \dots, X_{(m)})\} = (\nabla_Y U)(X_{(1)}, \dots, X_{(m)}) + U(\nabla_Y X_{(1)}, X_{(2)}, \dots, X_{(m)}) \\ + \dots + U(X_{(1)}, X_{(2)}, \dots, \nabla_Y X_{(m)}). \end{aligned} \tag{9.1.4}$$

Similarly, if U is m-tangent to the spheres $S_{r,t}$, then

$$\begin{aligned} \nabla\!\!\!/_{e_A}\{U(e_{B_{(1)}}, \dots, e_{B_{(m)}})\} = (\nabla\!\!\!/_A U)(e_{B_{(1)}}, \dots, e_{B_{(m)}}) + U(\nabla\!\!\!/_A e_{B_{(1)}}, e_{B_{(2)}}, \dots, e_{B_{(m)}}) \\ + \dots + U(e_{B_{(1)}}, e_{B_{(2)}}, \dots, \nabla\!\!\!/_A e_{B_{(m)}}). \end{aligned} \tag{9.1.5}$$

Applying (9.1.4) and (9.1.5) to $\mathscr{F}$ and using (9.1.2), (9.1.3), and (5.3.5a)–(5.3.5d), we compute (as in [Christodoulou and Klainerman 1990, p. 161]) the following identities, which we state as a lemma:

Lemma 9.1 (Contracted derivatives expressed in terms of the null components [Christodoulou and Klainerman 1990, p. 161]). *Let $\mathscr{F}$ be a two-form, and let $\underline{\alpha}$, α, ρ, and σ be its Minkowskian null components. Then the following identities hold*:

$$\nabla_A \mathscr{F}_{B\underline{L}} = \nabla\!\!\!/_A \underline{\alpha}_B - r^{-1}(\rho\delta_{AB} + \sigma \not\epsilon_{AB}), \tag{9.1.6a}$$

$$\nabla_A \mathscr{F}_{BL} = \nabla\!\!\!/_A \alpha_B - r^{-1}(\rho\delta_{AB} - \sigma \not\epsilon_{AB}), \tag{9.1.6b}$$

$$\nabla_A {}^{\circledast}\mathscr{F}_{B\underline{L}} = -\not\epsilon_{CB}\, \nabla\!\!\!/_A \underline{\alpha}_C - r^{-1}(\sigma\delta_{AB} - \rho \not\epsilon_{AB}), \tag{9.1.6c}$$

$$\nabla_A {}^{\circledast}\mathscr{F}_{BL} = \not\epsilon_{CB}\, \nabla\!\!\!/_A \alpha_C - r^{-1}(\sigma\delta_{AB} + \rho \not\epsilon_{AB}), \tag{9.1.6d}$$

$$\tfrac{1}{2}\nabla_A \mathscr{F}_{\underline{L}L} = \nabla\!\!\!/_A \rho + \tfrac{1}{2}r^{-1}(\underline{\alpha}_A + \alpha_A), \tag{9.1.6e}$$

$$\tfrac{1}{2}\nabla_A {}^{\circledast}\mathscr{F}_{\underline{L}L} = \nabla\!\!\!/_A \sigma + \tfrac{1}{2}r^{-1}(-\not\epsilon_{BA}\underline{\alpha}_B + \not\epsilon_{BA}\alpha_B), \tag{9.1.6f}$$

$$\nabla_A \mathscr{F}_{BC} = \not\epsilon_{BC}\big(\nabla\!\!\!/_A \sigma + \tfrac{1}{2}r^{-1}(-\not\epsilon_{DA}\underline{\alpha}_D + \not\epsilon_{DA}\alpha_D)\big). \tag{9.1.6g}$$

In all of our expressions, contractions are taken after differentiating; *e.g.*, $\nabla_A \mathscr{F}_{BL} \overset{\text{def}}{=} e_A^{\mu} e_B^{\kappa} \underline{L}^{\lambda} \nabla_{\mu} \mathscr{F}_{\kappa\lambda}$.

Remark 9.2. The identities in Lemma 9.1 can be reinterpreted as identities for spacetime tensors that are m-tangent to the spheres $S_{r,t}$. That is, they can be rephrased in terms of our wave-coordinate frame with the help of the projection $\not{m}_{\mu}{}^{\nu}$ and the spherical volume form $\not{\upsilon}_{\mu}{}^{\nu}$ defined in (5.1.4b) and (5.1.6c), respectively. For example, (9.1.6a) is equivalent to the following equation:

$$\not{m}_{\mu}{}^{\mu'}\not{m}_{\nu}{}^{\nu'}\underline{L}^{\kappa}\nabla_{\kappa}\mathscr{F}_{\nu'\kappa} = \not{m}_{\mu}{}^{\mu'}\not{m}_{\nu}{}^{\nu'}\nabla_{\mu'}\underline{\alpha}_{\nu'} - r^{-1}(\rho\not{m}_{\mu\nu} + \sigma\not{\upsilon}_{\mu\nu}). \tag{9.1.7}$$

We will use the spacetime-coordinate-frame version of the identities in our proof of Proposition 9.3.

We now derive equations for the null components of a solution $\dot{\mathscr{F}}$ to (9.1.1a)–(9.1.1b).

Proposition 9.3 (Minkowskian null decomposition of the equations of variation). *Let $\dot{\mathscr{F}}$ be a solution to the equations of variation* (9.1.1a)–(9.1.1b), *and let* $\dot{\underline{\alpha}} \stackrel{\text{def}}{=} \underline{\alpha}[\dot{\mathscr{F}}]$, $\dot{\alpha} \stackrel{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \stackrel{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, *and* $\dot{\sigma} \stackrel{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ *denote its Minkowskian null components. Assume that the source term $\dot{\mathfrak{F}}_{\lambda\mu\nu}$ on the right-hand side of* (9.1.1a) *vanishes.*[27] *Then the following equations are satisfied by the null components*:

$$\begin{aligned}
\nabla_L\dot{\underline{\alpha}}_\nu + r^{-1}\dot{\underline{\alpha}}_\nu + \not{m}_\nu{}^\kappa\nabla_\kappa\dot{\rho} - \not{\upsilon}_\nu{}^\kappa\nabla_\kappa\dot{\sigma} - \overbrace{\not{m}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\not{m}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\not{m}_{\nu\nu'}(m^{-1})^{\mu\kappa}h^{\nu'\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\not{m}_{\nu\lambda}\mathscr{Q}^\lambda_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}& \\
+\not{m}_{\nu\nu'}N_\triangle^{\#\mu\nu'\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \not{m}_{\nu\nu'}\dot{\mathfrak{F}}^{\nu'}&, \quad (9.1.8a)\\
\nabla_{\underline{L}}\dot{\alpha}_\nu - r^{-1}\dot{\alpha}_\nu - \not{m}_\nu{}^\kappa\nabla_\kappa\dot{\rho} - \not{\upsilon}_\nu{}^\kappa\nabla_\kappa\dot{\sigma} - \overbrace{\not{m}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\not{m}_{\nu\lambda}\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\not{m}_{\nu\nu'}(m^{-1})^{\mu\kappa}h^{\nu'\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\not{m}_{\nu\lambda}\mathscr{Q}^\lambda_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}& \\
+\not{m}_{\nu\nu'}N_\triangle^{\#\mu\nu'\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \not{m}_{\nu\nu'}\dot{\mathfrak{F}}^{\nu'}&, \quad (9.1.8b)\\
\nabla_{\underline{L}}\dot{\rho} - 2r^{-1}\dot{\rho} + \not{m}^{\mu\nu}\nabla_\mu\dot{\underline{\alpha}}_\nu - \overbrace{\underline{L}^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\underline{L}_\lambda\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} - \overbrace{\underline{L}_\nu(m^{-1})^{\mu\kappa}h^{\nu\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{\underline{L}_\nu\mathscr{Q}^\nu_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}& \\
+\underline{L}_\nu N_\triangle^{\#\mu\nu\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = \underline{L}_\lambda\dot{\mathfrak{F}}^\lambda&, \quad (9.1.8c)\\
\nabla_{\underline{L}}\dot{\sigma} - 2r^{-1}\dot{\sigma} + \not{\upsilon}^{\mu\nu}\nabla_\mu\dot{\underline{\alpha}}_\nu = 0&, \quad (9.1.8d)\\
\nabla_L\dot{\rho} + 2r^{-1}\dot{\rho} - \not{m}^{\mu\nu}\nabla_\mu\dot{\alpha}_\nu + \overbrace{L^\lambda h^{\mu\kappa}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{L_\lambda\mathscr{P}^\lambda_{(\mathscr{F})}(h,\nabla\dot{\mathscr{F}})} + \overbrace{L_\nu(m^{-1})^{\mu\kappa}h^{\nu\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda}}^{L_\nu\mathscr{Q}^\nu_{(1;\mathscr{F})}(h,\nabla\dot{\mathscr{F}})}& \\
-L_\nu N_\triangle^{\#\mu\nu\kappa\lambda}\nabla_\mu\dot{\mathscr{F}}_{\kappa\lambda} = -L_\lambda\dot{\mathfrak{F}}^\lambda&, \quad (9.1.8e)\\
\nabla_L\dot{\sigma} + 2r^{-1}\dot{\sigma} + \not{\upsilon}^{\mu\nu}\nabla_\mu\dot{\underline{\alpha}}_\nu = 0&. \quad (9.1.8f)
\end{aligned}$$

In the above expressions, the quadratic terms $\mathscr{P}^\lambda_{(\mathscr{F})}(h, \nabla\dot{\mathscr{F}})$ and $\mathscr{Q}^\lambda_{(1;\mathscr{F})}(h, \nabla\dot{\mathscr{F}})$ are as defined in Section 3.7.

Remark 9.4. Note that in the above equations, we have that, e.g., $\not{m}_\nu{}^\kappa\nabla_\kappa = \not{m}_\nu{}^\kappa\not{\nabla}_\kappa$ and $\not{\upsilon}_\nu{}^\kappa\nabla_\kappa = \not{\upsilon}_\nu{}^\kappa\not{\nabla}_\kappa$ so that these operators only involve favorable angular derivatives.

Proof. To obtain (9.1.8a) and (9.1.8b), we contract (9.1.1a) against $\underline{L}^\lambda L^\mu e_A^\nu$ and (9.1.1b) against $(e_A)_\nu$ and use Lemma 9.1 plus Remark 9.2 to deduce that

[27] By Proposition 8.1, this assumption holds for the variations $\dot{\mathscr{F}}$ of interest in this article.

$$\nabla_L \underline{\alpha}_\nu - \nabla_{\underline{L}} \alpha_\nu + 2 \not{h}_\nu^{\ \nu'} \nabla_{\nu'} \rho + r^{-1}(\underline{\alpha}_\nu + \alpha_\nu) = 0, \tag{9.1.9}$$

$$\begin{aligned} &\nabla_L \underline{\alpha}_\nu + \nabla_{\underline{L}} \alpha_\nu - 2 \not{\upsilon}_\nu^{\ \kappa} \nabla_\kappa \sigma + r^{-1}(\underline{\alpha}_\nu - \alpha_\nu) \\ &\quad - 2 \not{h}_\nu^{\ \lambda} h^{\mu\kappa} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} - 2 \not{h}_{\nu\nu'} (m^{-1})^{\mu\kappa} h^{\nu'\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} + \not{h}_{\nu\nu'} N_{\triangle}^{\#\mu\nu'\kappa\lambda} \nabla_\mu \dot{\mathscr{F}}_{\kappa\lambda} = 2 \not{h}_{\nu\nu'} \dot{\mathfrak{F}}^{\nu'}. \end{aligned} \tag{9.1.10}$$

Adding the two above equations gives (9.1.8a) while subtracting the first from the second gives (9.1.8b).

Similarly, to deduce (9.1.8d), we contract (9.1.1a) against $\underline{L}^\lambda e_A^{\ \mu} e_B^{\ \nu}$ and then contract against $\not{\upsilon}_{AB}$; to deduce (9.1.8f), we contract (9.1.1a) against $L^\lambda e_A^{\ \mu} e_B^{\ \nu}$ and then against $\not{\upsilon}_{AB}$; to deduce (9.1.8c), we contract (9.1.1b) against $\underline{L}_\nu$; and to deduce (9.1.8e), we contract (9.1.1b) against $-L_\nu$. □

9.2. *Electromagnetic one-forms.* In this section, we introduce the one-forms $\mathfrak{E}$, $\mathfrak{B}$, $\mathfrak{D}$, and $\mathfrak{H}$, which are derived from a geometric decomposition of $\mathscr{F}$ that depends on the spacetime metric $g_{\mu\nu}$. We also introduce the one-forms E, B, D, and H, which are derived from a Minkowskian decomposition of $\mathscr{F}$. We then derive an equivalent version of the electromagnetic equations, namely constraint and electromagnetic evolution equations for the Minkowskian one-forms. These quantities play a role only in Section 10, where they are used to connect the smallness of the abstract initial data to the smallness of the energy of the corresponding reduced solution at $t = 0$. Furthermore, we show that the abstract one-forms $\mathring{\mathfrak{D}}$ and $\mathring{\mathfrak{B}}$ satisfy the constraints (1.0.3a)–(1.0.3b) if and only if the corresponding Minkowskian one-forms $\mathring{D}$ and $\mathring{B}$ satisfy a Minkowskian version of the constraints.

We will perform our electromagnetic decompositions of the equations with the help of two versions of the (nonreduced) electromagnetic equations, namely (3.2.6a) and (3.2.7a) and (3.2.6b) and (3.2.7b). We restate them here for convenience:

$$\mathscr{D}_\lambda \mathscr{F}_{\mu\nu} + \mathscr{D}_\mu \mathscr{F}_{\nu\lambda} + \mathscr{D}_\nu \mathscr{F}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{9.2.1a}$$

$$\mathscr{D}_\lambda \mathscr{M}_{\mu\nu} + \mathscr{D}_\mu \mathscr{M}_{\nu\lambda} + \mathscr{D}_\nu \mathscr{M}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{9.2.1b}$$

$$\nabla_\lambda \mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{F}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{9.2.2a}$$

$$\nabla_\lambda \mathscr{M}_{\mu\nu} + \nabla_\mu \mathscr{M}_{\nu\lambda} + \nabla_\nu \mathscr{M}_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu = 0, 1, 2, 3). \tag{9.2.2b}$$

Before decomposing the equations, we first define the aforementioned geometric electromagnetic one-forms.

Definition 9.5. Let $\widehat{N}^\mu = \widehat{N}^\mu(t, x)$ denote the future-directed unit g-normal to the hypersurface Σ_t. Then in components relative to an arbitrary coordinate system, we define the following one-forms:

$$\mathfrak{E}_\mu = \mathscr{F}_{\mu\kappa} \widehat{N}^\kappa, \qquad \mathfrak{B}_\mu = -{}^\star\mathscr{F}_{\mu\kappa} \widehat{N}^\kappa, \qquad \mathfrak{D}_\mu = -{}^\star\mathscr{M}_{\mu\kappa} \widehat{N}^\kappa, \quad \text{and} \quad \mathfrak{H}_\mu = -\mathscr{M}_{\mu\kappa} \widehat{N}^\kappa. \tag{9.2.3}$$

Note that, in the above expressions, $\star$ denotes the Hodge duality operator corresponding to the spacetime metric g.

We now define the Minkowskian electromagnetic one-forms.

Definition 9.6. In components relative to the wave-coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, we define the *electric field E*, the *magnetic induction B*, the *electric displacement D*, and the *magnetic field H* by

$$E_\mu = \mathscr{F}_{\mu 0}, \qquad B_\mu = -{}^\circledast\mathscr{F}_{\mu 0}, \qquad D_\mu = -{}^\circledast\mathscr{M}_{\mu 0}, \quad \text{and} \quad H_\mu = -\mathscr{M}_{\mu 0}. \tag{9.2.4}$$

Note that in the above expressions, $\circledast$ denotes the Hodge duality operator corresponding to the Minkowski metric m.

Observe that (9.2.4) implies that

$$\mathscr{F}_{jk} = [ijk]B_i, \qquad B_j = \tfrac{1}{2}[jab]\mathscr{F}_{ab}, \quad \text{and} \quad D_j = \tfrac{1}{2}[jab]\mathscr{M}_{ab} \qquad (j, k = 1, 2, 3). \tag{9.2.5}$$

Remark 9.7. Our definition of B coincides with the one commonly found in the physics literature, but it has the opposite sign convention of the definition given in [Christodoulou and Klainerman 1990].

It follows from the antisymmetry of $\mathscr{F}_{\mu\nu}$ and $\mathscr{M}_{\mu\nu}$ that E_μ, B_μ, D_μ, and H_μ are m-tangent to the hyperplanes Σ_t; i.e., we have that $E_0 = B_0 = D_0 = H_0 = 0$. We may therefore view these four quantities as one-forms that are intrinsic to Σ_t. Similarly, we have that $\mathfrak{E}_\mu \widehat{N}^\mu = \mathfrak{B}_\mu \widehat{N}^\mu = \mathfrak{D}_\mu \widehat{N}^\mu = \mathfrak{H}_\mu \widehat{N}^\mu = 0$.

From the assumption (3.3.3a) on the electromagnetic Lagrangian, (3.3.11n), Definition 9.6, (9.2.5), and the implicit-function theorem, we deduce that, when all of the fields are sufficiently small, we have (see Section 2.13 for the definition of $O^{\ell+1}(\,\cdot\,)$):

$$D = E + O^{\ell+1}(|h||(E, B)|) + O^{\ell+1}(|(E, B)|^3; h), \tag{9.2.6a}$$
$$H = B + O^{\ell+1}(|h||(E, B)|) + O^{\ell+1}(|(E, B)|^3; h), \tag{9.2.6b}$$
$$E = D + O^{\ell+1}(|h||(D, B)|) + O^{\ell+1}(|(D, B)|^3; h), \tag{9.2.6c}$$
$$H = B + O^{\ell+1}(|h||(D, B)|) + O^{\ell+1}(|(D, B)|^3; h). \tag{9.2.6d}$$

We now assume that the reduced initial data $(g_{\mu\nu}|_{\Sigma_0}, \partial_t g_{\mu\nu}|_{\Sigma_0}, \mathscr{F}_{0j}|_{\Sigma_0} = \mathring{E}_j, \mathscr{F}_{jk}|_{\Sigma_0} = [ijk]\mathring{B}_i)$ have been constructed from the abstract initial data $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ in the manner described in Section 4.2. In particular, we recall that $\widehat{N}^\nu|_{\Sigma_0} = A^{-1}\delta_0^\nu$, where $A \overset{\text{def}}{=} \sqrt{1 - (2M/r)\chi(r)}$. Consequently, we can use (3.3.11i) and (4.2.7a) to deduce that

$$\mathring{E} = \mathring{D} + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big) + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|(\mathring{D}, \mathring{B})|^3; \chi(r)M/r; \mathring{h}^{(1)}\big). \tag{9.2.7}$$

Using also Definitions 9.5 and 9.6, we infer that the following relations hold:

$$\mathring{B} = \mathring{\mathfrak{B}} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \chi(r)M/r\big), \tag{9.2.8a}$$
$$\mathring{D} = \mathring{\mathfrak{D}} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{\mathfrak{D}}, \mathring{\mathfrak{B}})|; \chi(r)M/r\big), \tag{9.2.8b}$$
$$\mathring{\mathfrak{B}} = \mathring{B} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big), \tag{9.2.8c}$$
$$\mathring{\mathfrak{D}} = \mathring{D} + O^{\ell+1}\big(|\chi(r)M/r||(\mathring{D}, \mathring{B})|; \mathring{h}^{(1)}\big) + O^{\ell+1}\big(|\mathring{h}^{(1)}||(\mathring{D}, \mathring{B})|; \chi(r)M/r\big). \tag{9.2.8d}$$

Remark 9.8. Logically speaking, the ADM mass M (and hence also the components of the unit normal vector $\widehat{N}|_{\Sigma_0}$) is only well-defined *after* one has solved the abstract Einstein constraint equations (1.0.2a)–(1.0.3b).

The main goal of this section is to deduce the following proposition, which is a decomposition of the electromagnetic equations into *constraint* equations and *evolution* equations:

Proposition 9.9 (Electromagnetic constraint and evolution equations). *Under the assumption* (3.3.3a) *on* ${}^{\star}\mathscr{L}$, *the* (*nonreduced*) *electromagnetic equations* (9.2.2a)–(9.2.2b) *are equivalent to pairs of constraint equations and evolution equations that have the following structure* (*the precise details depend on the choice of electromagnetic Lagrangian* ${}^{\star}\mathscr{L}$):

Constraint equations

$$(\underline{m}^{-1})^{ab}\underline{\nabla}_a D_b = 0, \tag{9.2.9a}$$

$$(\underline{m}^{-1})^{ab}\underline{\nabla}_a B_b = 0, \tag{9.2.9b}$$

Evolution equations

$$\partial_t B_j = -[jab]\underline{\nabla}_a E_b, \tag{9.2.10a}$$

$$\begin{aligned}\partial_t E_j = [jab]\underline{\nabla}_a B_b &+ O^{\ell}\big(|h||\underline{\nabla}(E,B)|;(E,B)\big)\\ &+ O^{\ell}\big(|(E,B)|^2|\underline{\nabla}(E,B)|;h\big) + O^{\ell}\big(|\nabla h||(E,B)|;h\big).\end{aligned} \tag{9.2.10b}$$

Furthermore, assume that the reduced initial data $(g_{\mu\nu}|_{\Sigma_0}, \partial_t g_{\mu\nu}|_{\Sigma_0}, \mathscr{F}_{0j}|_{\Sigma_0} = \mathring{E}_j, \mathscr{F}_{jk}|_{\Sigma_0} = [ijk]\mathring{B}_i)$ *have been constructed from the abstract initial data* $(\mathring{\underline{g}}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ *in the manner described in Section 4.2. Then* (9.2.9a)–(9.2.9b) *hold for* $\mathring{D}$ *and* $\mathring{B}$ *along* Σ_0 *if and only if the following equations hold along* Σ_0:

Abstract constraint equations

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{D}}_b = 0, \tag{9.2.11a}$$

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a\mathring{\mathfrak{B}}_b = 0. \tag{9.2.11b}$$

In the above expressions, $\mathring{\underline{g}}_{jk}$ *is the first fundamental form of* Σ_0 *and* $\mathring{\underline{\mathscr{D}}}$ *is the Levi-Civita connection corresponding to* $\mathring{\underline{g}}_{jk}$.

Remark 9.10. In (9.2.9a)–(9.2.9b), $(\underline{m}^{-1})^{ab}\underline{\nabla}_a$ is the standard Euclidean divergence operator while in equations (9.2.10a)–(9.2.10b) $[jab]\underline{\nabla}_a$ is the standard Euclidean curl operator.

Remark 9.11. With the help of (9.2.16)–(9.2.17) below, it is straightforward to check that, if a classical solution to the evolution equations satisfies the constraints at $t = 0$, then it necessarily satisfies the constraints (9.2.9a)–(9.2.9b) at all later times (as long as it persists).

Proof. We first show that (9.2.9b) and (9.2.11b) follow from either (9.2.1a) or (9.2.2a) (which are equivalent) and that (9.2.9b) holds if and only if (9.2.11b) holds. To this end, we first note that, since $\widehat{N}^{\mu}$ is the future-directed unit g-normal to Σ_t and $g_{\mu\nu} = \mathring{\underline{g}}_{\mu\nu} - \widehat{N}_{\mu}\widehat{N}_{\nu}$ along Σ_0, the following identities hold for any one-form X_{μ} g-tangent to Σ_0 and any two-form $P_{\mu\nu}$:

$$(\mathring{\underline{g}}^{-1})^{ab}\mathring{\underline{\mathscr{D}}}_a X_b = (g^{-1})^{\kappa\lambda}\mathscr{D}_{\kappa}X_{\lambda} - X_{\lambda}\widehat{N}^{\kappa}\mathscr{D}_{\kappa}\widehat{N}^{\lambda}, \tag{9.2.12}$$

$$(g^{-1})^{\kappa\lambda}P_{\lambda\nu}\mathscr{D}_{\kappa}\widehat{N}^{\nu} = P_{\lambda\nu}\widehat{N}^{\nu}\widehat{N}^{\kappa}\mathscr{D}_{\kappa}\widehat{N}^{\lambda}. \tag{9.2.13}$$

Using (9.2.12) and (9.2.13) with $X_\mu \stackrel{\text{def}}{=} \mathfrak{B}_\mu$ and $P_{\mu\nu} \stackrel{\text{def}}{=} {}^\star\mathscr{F}_{\mu\nu}$, we compute that the following identities hold along Σ_0:

$$\begin{aligned}(\underline{\mathring{g}}^{-1})^{ab}\underline{\mathring{\mathscr{D}}}_a\mathfrak{B}_b &= (g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa\mathfrak{B}_\lambda - \mathfrak{B}_\lambda\widehat{N}^\kappa\mathscr{D}_\kappa\widehat{N}^\lambda \\ &= -(g^{-1})^{\kappa\lambda}\mathscr{D}_\kappa({}^\star\mathscr{F}_{\lambda\nu}\widehat{N}^\nu) + {}^\star\mathscr{F}_{\lambda\nu}\widehat{N}^\nu\widehat{N}^\kappa\mathscr{D}_\kappa\widehat{N}^\lambda \\ &= -\tfrac{1}{2}g_{\nu\nu'}\widehat{N}^{\nu'}\epsilon^{\#\mu\nu\kappa\lambda}\mathscr{D}_\mu\mathscr{F}_{\kappa\lambda}. \end{aligned} \tag{9.2.14}$$

Identities analogous to (9.2.14) hold if we make the replacements $(\underline{\mathring{g}}^{-1}, g, \underline{\mathring{\mathscr{D}}}, \mathscr{D}, \star, \widehat{N}^\mu, \epsilon^{\#\mu\nu\kappa\lambda}, \mathfrak{B}) \to (\underline{m}^{-1}, m, \underline{\nabla}, \nabla, \circledast, \hat{n}^\mu, \upsilon^{\mu\nu\kappa\lambda}, B)$, where $\hat{n}^\mu(t,x)$ is the future-directed Minkowskian unit normal to Σ_t. Now by (9.2.14) and the Minkowskian analogy of (9.2.14), (9.2.9b) and (9.2.11b) follow from either (9.2.1a) or (9.2.2a) since either (9.2.1a) or (9.2.2a) is sufficient to guarantee that the right-hand side of (9.2.14) is 0. Furthermore, since $g_{\nu\nu'}\widehat{N}^{\nu'}$ and $m_{\nu\nu'}\hat{n}^{\nu'}$ are proportional along Σ_0, since $\epsilon^{\#\mu\nu\kappa\lambda}$ and $\upsilon^{\mu\nu\kappa\lambda}$ are proportional, and since the Christoffel symbols of $\mathscr{D}$ and ∇ are symmetric in their two lower indices, it follows that

$$g_{\nu\nu'}\widehat{N}^{\nu'}\epsilon^{\#\mu\nu\kappa\lambda}\mathscr{D}_\mu\mathscr{F}_{\kappa\lambda}|_{\Sigma_0} = 0 \iff m_{\nu\nu'}\hat{n}^{\nu'}\upsilon^{\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda}|_{\Sigma_0} = 0. \tag{9.2.15}$$

Hence, (9.2.9b) holds along Σ_0 if and only if (9.2.11b) holds along Σ_0. The derivation of (9.2.9a) and (9.2.11a) along Σ_0 from (9.2.1b) or (9.2.2b) and the proof of the equivalence of (9.2.9a) and (9.2.11a) along Σ_0 are similar.

We now set $\lambda = 0$, $\mu = a$, and $\nu = b$ in (9.2.2a), contract against the Euclidean volume form $[jab]$, and use (9.2.4)–(9.2.5) to deduce that

$$\partial_t B_j = -[jab]\underline{\nabla}_a E_b. \tag{9.2.16}$$

Similarly, we set $\lambda = 0$, $\mu = a$, and $\nu = b$ in (9.2.2b), contract against $[jab]$, and use (9.2.4)–(9.2.5) to deduce that

$$\partial_t D_j = [jab]\underline{\nabla}_a H_b. \tag{9.2.17}$$

Finally, we use (9.2.16), (9.2.17), and (9.2.6a)–(9.2.6b) to deduce (9.2.10a)–(9.2.10b). □

10. The smallness condition on the abstract data

In this section, we assume that we are given abstract initial data $(\underline{\mathring{g}}_{jk} = \delta_{jk} + \underline{\mathring{h}}^{(0)}_{jk} + \underline{\mathring{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ on the manifold $\mathbb{R}^3$ satisfying the constraint equations (4.1.1a)–(4.1.2b). Our goal is to describe in detail the smallness condition on $(\underline{\mathring{h}}^{(0)}_{jk}, \underline{\mathring{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ that will lead to global existence for the reduced system (3.7.1a)–(3.7.1c) under the assumption that its initial data $(g_{\mu\nu}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ are constructed from the abstract initial data as described in Section 4.2. Recall that our global existence argument is heavily based on the analysis of $\mathscr{E}_{\ell;\gamma;\mu}(t)$, which is the energy defined in (1.2.7). In particular, $\mathscr{E}_{\ell;\gamma;\mu}(0)$ must be sufficiently small in order for us to close the argument. The energy depends on *both normal and tangential* Minkowskian covariant derivatives of the quantities $(\nabla_\lambda h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ at $t = 0$. On the other hand, our smallness condition will be expressed in terms of the ADM mass M and $E_{\ell;\gamma}(0)$, which is a weighted Sobolev norm of $(\underline{\nabla}_i\underline{\mathring{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ depending only on *tangential* derivatives of the abstract data. More specifically, our smallness condition is expressed in terms of the weighted Sobolev norms $\|\cdot\|_{H^{\ell}_{1/2+\gamma}}$ introduced in Definition 10.1. The main result of this section is contained in Proposition 10.4, which shows that, if $E_{\ell;\gamma}(0)+M$ is sufficiently small and $(h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ is the corresponding solution to the reduced equations, then $\mathcal{E}_{\ell;\gamma;\mu}(0) \lesssim E_{\ell;\gamma}(0)+M$. Thus, Proposition 10.4 allows us to deduce the smallness of $\mathcal{E}_{\ell;\gamma;\mu}(0)$ from the smallness of quantities that depend exclusively on the abstract initial data.

We begin by introducing the weighted Sobolev norm discussed in the above paragraph.

Definition 10.1. Let $U(x)$ be a tensor field defined along the Euclidean space $\mathbb{R}^3$. Then for any integer $\ell \geq 0$ and any real number η, we define the H^{ℓ}_{η} norm of U by

$$\|U\|^2_{H^{\ell}_{\eta}} \stackrel{\text{def}}{=} \sum_{|I|\leq\ell} \int_{x\in\mathbb{R}^3} (1+|x|^2)^{(\eta+|I|)} |\underline{\nabla}^I U(x)|^2 \, d^3x. \tag{10.0.1}$$

We also introduce the following norm, which can be controlled in terms of a suitable H^{ℓ}_{η} norm via a Sobolev embedding result; see Proposition A.1.

Definition 10.2. Let $U(x)$ be a tensor field defined along the Euclidean space $\mathbb{R}^3$. Then for any integer $\ell \geq 0$ and any real number η, we define the C^{ℓ}_{η} norm of U by

$$\|U\|^2_{C^{\ell}_{\eta}} \stackrel{\text{def}}{=} \sum_{|I|\leq\ell} \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} (1+|x|^2)^{(\eta+|I|)} |\underline{\nabla}^I U(x)|^2. \tag{10.0.2}$$

We are now ready to introduce our norm $E_{\ell;\gamma}(0) \geq 0$ on the abstract initial data. Recall that, as discussed in Section 4.1, the data are the following four fields on $\mathbb{R}^3$: $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \underbrace{\mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}}_{\mathring{\underline{h}}_{jk}}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$.

Definition 10.3. The norm $E_{\ell;\gamma}(0) \geq 0$ of the abstract initial data is defined by

$$E^2_{\ell;\gamma}(0) \stackrel{\text{def}}{=} \|\underline{\nabla}\mathring{\underline{h}}^{(1)}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{K}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{\mathfrak{D}}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{\mathfrak{B}}\|^2_{H^{\ell}_{1/2+\gamma}}. \tag{10.0.3}$$

The smallness condition. Our smallness condition for global existence is

$$E_{\ell;\gamma}(0) + M \leq \varepsilon_{\ell}, \tag{10.0.4}$$

where ε_{ℓ} is a sufficiently small positive number.

Recall that the energy $\mathcal{E}_{\ell;\gamma;\mu}(t) \geq 0$ is defined by

$$\mathcal{E}^2_{\ell;\gamma;\mu}(t) \stackrel{\text{def}}{=} \sup_{0\leq\tau\leq t} \sum_{|I|\leq\ell} \int_{\Sigma_\tau} \big(|\nabla\nabla^I_{\mathcal{Z}} h^{(1)}|^2 + |\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|^2\big) w(q) \, d^3x, \tag{10.0.5}$$

where ∇ denotes the *full Minkowski spacetime* covariant derivative operator and the weight $w(q)$ is defined in (12.1.1). The dependence on γ and μ in $\mathcal{E}_{\ell;\gamma;\mu}$ is through $w(q)$. The next proposition, which is the main result of this section, shows that the smallness of $\mathcal{E}_{\ell;\gamma;\mu}(0)$ follows from the smallness of $E_{\ell;\gamma}(0)+M$:

Proposition 10.4 (The smallness of the initial energy). *Let* $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}^{(0)}_{jk} + \mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ *be abstract initial data on the manifold* $\mathbb{R}^3$ *for the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c). *Assume that the abstract initial data satisfy the constraints* (1.0.2a)–(1.0.3b) *and* *that they are asymptotically flat in the sense that* (1.0.4a)–(1.0.4f) *hold. Let* $(g_{\mu\nu}|_{t=0} = m_{\mu\nu} + h^{(0)}_{\mu\nu}|_{t=0} + h^{(1)}_{\mu\nu}|_{t=0},\ \partial_t g_{\mu\nu}|_{t=0} = \partial_t h^{(0)}_{\mu\nu}|_{t=0} + \partial_t h^{(1)}_{\mu\nu}|_{t=0},\ \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *be the corresponding initial data for the reduced system* (3.7.1a)–(3.7.1c) *as defined in Section 4.2, and let* $(g_{\mu\nu} = m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}, \mathscr{F}_{\mu\nu})$ *be the solution to the reduced system launched by this data. Let* $\ell \geq 10$ *be an integer. In particular, by Proposition 4.2, the wave-coordinate condition* (3.1.1a) *is satisfied by the reduced solution. Then there exist a constant* $\varepsilon_0 > 0$ *and a constant* $C_\ell > 0$ *such that, if* $E_{\ell;\gamma}(0) + M \leq \varepsilon \leq \varepsilon_0$, *then*

$$\mathcal{E}_{\ell;\gamma;\mu}(0) \leq C_\ell \{E_{\ell;\gamma}(0) + M\} \leq C_\ell \varepsilon. \tag{10.0.6}$$

Remark 10.5. Note that $q \geq 0$ holds at $t = 0$. Therefore, $\mathcal{E}_{\ell;\gamma;\mu}(0)$ does not depend on the constant μ.

The proof of Proposition 10.4 starts on page 845. We first establish some technical lemmas.

Lemma 10.6 (Energy in terms of $h^{(1)}$, E, and B). *Let* $\mathscr{F}_{\mu\nu}$ *be a two-form, let the pair of one-forms* (E_μ, B_μ) *be its Minkowskian electromagnetic decomposition as defined in Section 9.2, and let* $h^{(1)}_{\mu\nu}$ *be an arbitrary type-*$\binom{0}{2}$ *tensor field. Let* $\mathcal{E}_{\ell;\gamma;\mu}(t)$ *be the energy defined in* (10.0.5). *Then*

$$\mathcal{E}^2_{\ell;\gamma;\mu}(t) \approx \sup_{0\leq\tau\leq t} \sum_{|I|\leq\ell} \int_{\Sigma_\tau} \big(|\nabla\nabla^I_{\mathcal{Z}} h^{(1)}|^2 + |\nabla^I_{\mathcal{Z}} E|^2 + |\nabla^I_{\mathcal{Z}} B|^2\big) w(q)\, d^3x. \tag{10.0.7}$$

Proof. Equation (10.0.7) easily follows from (6.5.22) and the identity $|\nabla^I_{\mathcal{Z}} \mathscr{F}|^2 = 2|\nabla^I_{\mathcal{Z}} E|^2 + 2|\nabla^I_{\mathcal{Z}} B|^2$, the verification of which we leave to the reader. □

Lemma 10.7. *The following estimates hold for any* ℓ*-times differentiable spacetime tensor field* $U(t, x)$ *defined in a neighborhood of* $\Sigma_0 \overset{\text{def}}{=} \{(t, x) \mid t = 0\}$, *where* $w(q)$ *is the weight defined in* (12.1.1):

$$\begin{aligned}\Big(\sum_{|I|\leq\ell} w^{1/2}(q)|\nabla^I_{\mathcal{Z}} U|\Big)\Big|_{\Sigma_0} &\approx \Big(\sum_{|I|\leq\ell} (1+r)^{1/2+\gamma+|I|} |\nabla^I U|\Big)\Big|_{\Sigma_0} \\ &\approx \Big(\sum_{|J|+k\leq\ell} (1+r)^{1/2+\gamma+|J|+k} |\partial_t^k \underline{\nabla}^J U|\Big)\Big|_{\Sigma_0}.\end{aligned} \tag{10.0.8}$$

The same estimates hold if $\nabla^I_{\mathcal{Z}}$ *is replaced with* $\mathcal{L}^I_{\mathcal{Z}}$. *The notation* $|_{\Sigma_0}$ *is meant to indicate that the estimates only hold along* Σ_0.

Proof. By iterating the identity $\frac{\partial}{\partial x^\mu} = (x^\kappa \Omega_{\kappa\mu} + x_\mu S)/qs$ and noting that $q = r = s$ along Σ_0, we deduce that

$$(1+r)^{|I|} |\nabla^I U| \lesssim \sum_{|J|\leq|I|} |\nabla^J_{\mathcal{Z}} U|. \tag{10.0.9}$$

It thus follows from the definition (12.1.1) of $w(q)$ that

$$\Big(\sum_{|I|\leq\ell} (1+r)^{1/2+\gamma+|I|} |\nabla^I U|\Big)\Big|_{\Sigma_0} \lesssim \Big(\sum_{|I|\leq\ell} w^{1/2}(q) |\nabla^I_{\mathcal{Z}} U|\Big)\Big|_{\Sigma_0}. \tag{10.0.10}$$

On the other hand, the opposite inequality follows easily from expanding the operator $\nabla^I_{\mathcal{Z}}$ and using the Leibniz rule plus (6.2.4). This proves the first $\approx$ in (10.0.8). The second $\approx$ is trivial. We have thus established (10.0.8). To establish the same estimates with the operator $\mathscr{L}_{\mathfrak{Z}}^I$ in place of $\nabla_{\mathfrak{Z}}^I$, we simply use (6.5.22). □

Corollary 10.8. *Under the assumptions of Lemma 10.6, we have that*

$$\mathcal{E}^2_{\ell;\gamma;\mu}(0) \approx \sum_{k+|I|\le\ell} \int_{\mathbb{R}^3} (1+|x|)^{1+2\gamma+2(k+|I|)} \big(|\partial_t^k \underline{\nabla}^I \, \partial_t h^{(1)}|^2(0,x) + |\underline{\nabla}^I \underline{\nabla} h^{(1)}|^2(0,x)\big)\, d^3x \\ + \int_{\mathbb{R}^3} (1+|x|)^{1+2\gamma+2(k+|I|)} \big(|\partial_t^k \underline{\nabla}^I E|^2(0,x) + |\partial_t^k \underline{\nabla}^I B|^2(0,x)\big)\, d^3x. \quad (10.0.11)$$

Proof. Corollary 10.8 follows easily from Lemmas 10.6 and 10.7. □

Lemma 10.9. *Assume the hypotheses of Proposition 10.4. Let $k \ge 1$ and $\ell \ge 10$ be integers, and let J be a $\underline{\nabla}$-multi-index. Assume that $|J| + k \le \ell$. Define the arrays V, $V^{(0)}$, $V^{(1)}$, W, $W^{(0)}$, and $W^{(1)}$ by*

$$V \overset{\text{def}}{=} (h, \underline{\nabla} h, \partial_t h, E, B) = V^{(0)} + V^{(1)}, \quad (10.0.12a)$$

$$V^{(0)} \overset{\text{def}}{=} (h^{(0)}, \underline{\nabla} h^{(0)}, \partial_t h^{(0)}, 0, 0), \quad (10.0.12b)$$

$$V^{(1)} \overset{\text{def}}{=} (h^{(1)}, \underline{\nabla} h^{(1)}, \partial_t h^{(1)}, E, B), \quad (10.0.12c)$$

$$W \overset{\text{def}}{=} (0, \underline{\nabla} h, \partial_t h, E, B) = W^{(0)} + W^{(1)}, \quad (10.0.12d)$$

$$W^{(0)} \overset{\text{def}}{=} (0, \underline{\nabla} h^{(0)}, \partial_t h^{(0)}, 0, 0), \quad (10.0.12e)$$

$$W^{(1)} \overset{\text{def}}{=} (0, \underline{\nabla} h^{(1)}, \partial_t h^{(1)}, E, B). \quad (10.0.12f)$$

In the above expressions, the tensor fields $h^{(0)}_{\mu\nu}$ and $h^{(1)}_{\mu\nu}$ are defined by (1.2.1a)–(1.2.1c) *while the electromagnetic one-forms E_μ and B_μ are defined in* (9.2.4). *Assume further that $|V^{(1)}| + M \le \varepsilon$. Then if ε is sufficiently small, $\partial_t^k \underline{\nabla}^J W^{(1)}$ can be written as the following finite linear combination*:

$$\underline{\nabla}^J \partial_t^k W^{(1)} = \sum \text{terms}, \quad (10.0.13)$$

where each term *can be written as*

$$\text{term} = \sum_{\substack{|I_1|+\cdots+|I_s|\le|J|+k \\ 0\le|I_1|,\dots,|I_s|}} F_{(I_1,\dots,I_s;J;k;s)}(t,x) \times \mathcal{M}_{(I_1,\dots,I_s;J;k;s)}(V)[\underline{\nabla}^{I_1} W^{(1)}, \dots, \underline{\nabla}^{I_s} W^{(1)}], \quad (10.0.14)$$

and:

(i) *The array-valued functions $\mathcal{M}_{(I_1,\dots,I_s;J;k;s)}(V)[\underline{\nabla}^{I_1} W^{(1)}, \dots, \underline{\nabla}^{I_s} W^{(1)}]$ are continuous in a neighborhood of $V = 0$ and are multilinear in the arguments $[\underline{\nabla}^{I_1} W^{(1)}, \dots, \underline{\nabla}^{I_s} W^{(1)}]$.*

(ii) *If $s = 0$ (i.e., if there are no multilinear arguments $[\,\cdot\,]$), the array-valued functions $F_{(I_1,\dots,I_s;J;k;s)}(t,x)$ are smooth and satisfy $|F_{(I_1,\dots,I_s;J;k;s)}(t,x)| \lesssim M(1+t+|x|)^{-(3+|J|+k)}$, where M is the ADM mass.*

(iii) *When $s \ge 1$, $|F_{(I_1,\dots,I_s;J;k;s)}(t,x)| \lesssim (1+t+|x|)^{-d}$, where $d \ge |J| + k - (|I_1| + \cdots + |I_s|) - (s-1)$.*

Proof. We first claim that we can write the reduced system (3.7.1a)–(3.7.1c) as a finite linear combination

$$\partial_t W^{(1)} = \sum \text{terms}, \quad (10.0.15a)$$

where each term can be written in the form

$$\text{term} = \sum_{|I|=1} \mathcal{M}_{(I;0;1;1)}(V)[\underline{\nabla}^I W^{(1)}] + \mathcal{M}_{(0;0;1;2)}(V)[W^{(1)}, W^{(1)}] \\ + f_{(0;0;1;1)}(t,x)\,\mathcal{M}_{(0;0;1;1)}(V)[W^{(1)}] + f_{(0;0;1;0)}(t,x)\,\mathcal{M}_{(0;0;1;0)}(V). \quad (10.0.15b)$$

Above, the functions $\mathcal{M}_{(\cdot)}(V)[\cdot]$, which depend on the $(\ell+2)$-times continuously differentiable Lagrangian ${}^{\star}\mathscr{L}$ for the electromagnetic equations, have the properties stated in the conclusions of the theorem. In addition, $f_{(0;0;1;1)}(t,x)$ and $f_{(0;0;1;0)}(t,x)$ are smooth functions satisfying $|\nabla^I f_{(0;0;1;1)}(t,x)| \lesssim (1+t+|x|)^{-(2+|I|)}$ and $|\nabla^I f_{(0;0;1;0)}(t,x)| \lesssim M(1+t+|x|)^{-(3+|I|)}$ for any ∇-multi-index I. Let us accept the claim (10.0.15b) for now; we will briefly discuss the derivation of (10.0.15b) at the end of the proof. We also note that

$$\partial_t V = \partial_t W^{(1)} + \Pi_1 W^{(1)} + \partial_t V^{(0)}, \quad (10.0.16)$$

$$\underline{\nabla} V = \underline{\nabla} W^{(1)} + \Pi_2 W^{(1)} + \underline{\nabla} V^{(0)}, \quad (10.0.17)$$

where $V^{(0)}(t,x)$ satisfies $|\nabla^I \partial_t V^{(0)}(t,x)| + |\nabla^I \underline{\nabla} V^{(0)}(t,x)| \lesssim (1+t+|x|)^{-(2+|I|)}$ for any ∇-multi-index I (see Lemma 15.1), $\Pi_1 W^{(1)} \stackrel{\text{def}}{=} (\partial_t h^{(1)}, 0, 0, 0, 0)$, and $\Pi_2 W^{(1)} \stackrel{\text{def}}{=} (\underline{\nabla} h^{(1)}, 0, 0, 0, 0)$. Now with the help of (10.0.16)–(10.0.17), the chain rule, and the Leibniz rule, we repeatedly partially differentiate (10.0.15b) with respect to time and spatial derivatives, using the resulting equations to replace time derivatives with spatial derivatives, thereby inductively arriving at an expression of the form (10.0.14) verifying the properties (i)–(iii). The properties (ii)–(iii) capture the fact that each additional differentiation of $\partial_t W^{(1)}$ either (a) creates an additional decay factor of $(1+t+|x|)^{-1}$ (when the derivative falls on one of the $f_{\dots}(t,x)$), (b) increases one of the powers $|I_j|$ (when the derivative is spatial and falls on one of the multilinear factors $[\dots, \underline{\nabla}^{I_j} W^{(1)}, \dots]$), or (c) increases s by one (when the derivative falls on $\mathcal{M}(V)$, thereby creating an additional multilinear factor of $\nabla W^{(1)}$ via the chain rule).

We now return to the issue of expressing $\partial_t W^{(1)}$ in the form (10.0.15a)–(10.0.15b). We will make repeated use of the splitting $h = h^{(0)} + h^{(1)}$, where $h^{(0)}$ is the smooth function of (t,x) with the decay properties (15.1.1a), which are proved in Section 15.1. We first note that $\partial_t E$ and $\partial_t B$ can be expressed in the desired form by using (9.2.10a)–(9.2.10b) together with the splitting of h and the properties (15.1.1a). We remark that, although (9.2.10a)–(9.2.10b) are *nonreduced* electromagnetic equations, they are nonetheless satisfied by virtue of the fact that the wave-coordinate condition holds and the fact that the reduced and nonreduced equations are equivalent under that condition. Next, we note that the quantities $\partial_t \underline{\nabla} h_{\mu\nu}^{(1)}$ can be expressed in the desired form through the trivial identity $\partial_t \underline{\nabla} h_{\mu\nu}^{(1)} = \underline{\nabla} \partial_t h_{\mu\nu}^{(1)}$. The quantities $\partial_t^2 h_{\mu\nu}^{(1)}$ can be expressed in the desired form by using (3.7.1a) to isolate them. We remark that the $\mathcal{M}_{I;0;1;1}(V)[\underline{\nabla}^I W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the spatial derivatives and mixed spacetime derivatives of $h^{(1)}$ contained in the term $\widetilde{\square}_g h_{\mu\nu}^{(1)}$ on the left-hand side of (3.7.1a). Furthermore, the $\mathcal{M}_{0;0;1;2}(V)[W^{(1)}, W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the quadratic and higher-order-in-$W^{(1)}$ terms on the right-hand sides of (3.7.1a) and (9.2.10b) while the $f_{0;0;1;1}(t,x)\mathcal{M}_{0;0;1;1}(V)[W^{(1)}]$ term on the right-hand side of (10.0.15b) arises from the $h^{(0)}$- and $\nabla h^{(0)}$-containing factors that arise from the terms on the right-hand sides of (3.7.1a) and (9.2.10b) that contain a linear factor of h or ∇h. Finally, the $f_{0;0;1;0}(t,x)\mathcal{M}_{0;0;1;0}(V)$ term on the right-hand side of (10.0.15b) arises from the $\widetilde{\square}_g h^{(0)}_{\mu\nu}$ term on the right-hand side of (3.7.1a) and from the $O(|\nabla h^{(0)}|^2)$ terms arising from splitting the $O(|\nabla h|^2)$ terms on the right-hand side of (3.7.1a). □

Corollary 10.10. *Assume the hypotheses of Proposition 10.4, which include the smallness condition* $E_{\ell;\gamma}(0) + M \le \varepsilon$. *Let* $k \ge 0$ *be an integer, let* J *be a* $\underline{\nabla}$ *multi-index, and assume that* $|J| + k \le \ell$. *Let* $V(t,x), \dots, W^{(1)}(t,x)$ *be the array-valued functions defined in* (10.0.12a)–(10.0.12f), *let* $\mathring{V}(x) = V(0,x), \dots, \mathring{W}^{(1)}(x) = W^{(1)}(0,x)$, *and assume that* $\|\mathring{V}^{(1)}\|_{L^\infty} + \|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}} \le \varepsilon$. *Then if* ε *is sufficiently small, the following inequality holds*:

$$\big\|(1+|x|)^{1/2+\gamma+|J|+k}\underline{\nabla}^J\,\partial_t^k W^{(1)}(0,x)\big\|_{L^2} \lesssim \|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}} + M. \tag{10.0.18}$$

Proof. We first consider the case $s=0$ in (10.0.14). Then using that $|F_{(0;J;k;0)}(t,x)| \lesssim M(1+|x|)^{-(3+|J|+k)}$ (i.e., property (ii) from the conclusions of Lemma 10.9) and recalling that $0<\gamma<\frac{1}{2}$, we deduce that

$$\begin{aligned}
&\big\|(1+|x|)^{1/2+\gamma+|J|+k}F_{(0;J;k;0)}(0,x)\mathcal{M}_{(0;J;k;0)}(\mathring{V}(x))\big\|^2_{L^2}\\
&\qquad= \int_{x\in\mathbb{R}^3}(1+|x|)^{1+2\gamma+2|J|+2k}|F_{(0;J;k;0)}(0,x)\mathcal{M}_{(0;J;k;0)}(\mathring{V}(x))|^2\,d^3x\\
&\qquad\lesssim M^2\int_{x\in\mathbb{R}^3}(1+|x|)^{2\gamma-5}\,d^3x \lesssim M^2.
\end{aligned}\tag{10.0.19}$$

For the case $s\ge 1$, we first use Proposition A.1 to deduce that, for all $\underline{\nabla}$-indices K with $|K|\le \ell-2$, we have

$$|\underline{\nabla}^K\mathring{W}^{(1)}(x)| \lesssim (1+|x|)^{-(|K|+1)}\|\mathring{W}^{(1)}\|_{H^{|K|+2}_{1/2+\gamma}}. \tag{10.0.20}$$

Then (without loss of generality assuming $|I_1|\le|I_2|\le\cdots\le|I_s|$) we use $|F_{(I_1,\dots,I_s;J;k;s)}(t,x)| \lesssim (1+t+|x|)^{-(|J|+k-(|I_1|+\cdots+|I_s|)-(s-1))}$ (i.e., property (iii)), together with (10.0.20), to deduce

$$\begin{aligned}
&\big\|(1+|x|)^{1/2+\gamma+|J|+k}F_{(I_1,\dots,I_s;J;k;s)}(0,x)\times\mathcal{M}_{(I_1,\dots,I_s;J;k;s)}(\mathring{V}(x))[\underline{\nabla}^{I_1}\mathring{W}^{(1)}(x),\dots,\underline{\nabla}^{I_s}\mathring{W}^{(1)}(x)]\big\|_{L^2}\\
&\quad\lesssim \bigg\|(1+|x|)^{|I_1|+\cdots+|I_{s-1}|+(s-1)}\prod_{i=1}^{s-1}\underline{\nabla}^{I_i}\mathring{W}^{(1)}(x)\bigg\|_{L^\infty}\times\big\|(1+|x|)^{1/2+\gamma+|I_s|}\underline{\nabla}^{I_s}\mathring{W}^{(1)}(x)\big\|_{L^2}\\
&\quad\lesssim \big\|(1+|x|)^{1/2+\gamma+|I_s|}\underline{\nabla}^{I_s}\mathring{W}^{(1)}(x)\big\|_{L^2}\lesssim\|\mathring{W}^{(1)}\|_{H^\ell_{1/2+\gamma}}.
\end{aligned}\tag{10.0.21}$$

Combining (10.0.19) and (10.0.21), we arrive at (10.0.18). □

We are now ready for the proof of the proposition.

Proof of Proposition 10.4. We first stress that *the estimates derived in this proof are valid under the assumption that* ε *is sufficiently small*. Recall that $g_{\mu\nu}(t,x) = m_{\mu\nu} + \chi(r/t)\chi(r)(2M/r)\delta_{\mu\nu} + h^{(1)}_{\mu\nu}(t,x)$. Also recall that, according to the assumptions of the proposition, we have (see (4.2.6a)–(4.2.6b)) the following relations (where we slightly abuse matrix notation):

$$h^{(1)}(0,x)=\begin{pmatrix}0 & 0\\ 0 & \underline{\mathring{h}}^{(1)}_{jk}\end{pmatrix}, \tag{10.0.22a}$$

$$\partial_t h^{(1)}(0,x)=\begin{pmatrix}2A^3(\underline{\mathring{g}}^{-1})^{ab}\mathring{K}_{ab} & A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_a\underline{\mathring{g}}_{bj}-\frac{1}{2}A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_j\underline{\mathring{g}}_{ab}-A\,\partial_j A\\ A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_a\underline{\mathring{g}}_{bj}-\frac{1}{2}A^2(\underline{\mathring{g}}^{-1})^{ab}\,\partial_j\underline{\mathring{g}}_{ab}-A\partial_j A & 2A\mathring{K}_{jk}\end{pmatrix}, \tag{10.0.22b}$$

where $A(x)=\sqrt{1-(2M/r)\chi(r)}$ and $\underline{\mathring{g}}_{jk}(x)=\delta_{jk}+(2M/r)\chi(r)\delta_{jk}+\underline{\mathring{h}}^{(1)}_{jk}(x)$. Note that $(\underline{\mathring{g}}^{-1})^{jk}=\delta^{jk}+O^{\infty}(|(M/r)\chi(r)|;\underline{\mathring{h}}^{(1)})+O^{\infty}(|\underline{\mathring{h}}^{(1)}|;(M/r)\chi(r))$. Our immediate objectives are to relate $\|\mathring{E}\|_{H^{\ell}_{1/2+\gamma}}$ and $\|\partial_t h^{(1)}(0,\cdot)\|_{H^{\ell}_{1/2+\gamma}}$ to the inherent quantities $\|\underline{\nabla}\underline{\mathring{h}}\|_{H^{\ell}_{1/2+\gamma}}$, $\|\mathring{K}\|_{H^{\ell}_{1/2+\gamma}}$, $\|\mathring{\mathfrak{D}}\|_{H^{\ell}_{1/2+\gamma}}$, $\|\mathring{\mathfrak{B}}\|_{H^{\ell}_{1/2+\gamma}}$, and M. To this end, we first observe that the following estimates hold for sufficiently small M:

$$\Big|\underline{\nabla}^I\Big(\frac{M}{r}\chi(r)\Big)\Big|\lesssim M(1+r)^{-(1+|I|)}, \tag{10.0.23}$$

$$|A(x)|\lesssim 1, \tag{10.0.24}$$

$$|\underline{\nabla}^I A(x)|\lesssim M(1+r)^{-(1+|I|)}\quad(|I|\geq 1). \tag{10.0.25}$$

With the help of (10.0.22a)–(10.0.22b), the decay estimates (10.0.23)–(10.0.25), the Leibniz rule, Corollary A.4, the definition of $\|\cdot\|_{H^{\ell}_{1/2+\gamma}}$, and the assumption $0<\gamma<\frac{1}{2}$, it is straightforward to check that

$$\|\partial_t h^{(1)}(0,\cdot)\|_{H^{\ell}_{1/2+\gamma}}\lesssim\|\underline{\nabla}\underline{\mathring{h}}^{(1)}\|_{H^{\ell}_{1/2+\gamma}}+\|\mathring{K}\|_{H^{\ell}_{1/2+\gamma}}+M. \tag{10.0.26}$$

Furthermore, from (9.2.8a)–(9.2.8d) and Corollary A.4, it follows that

$$\|\mathring{D}\|_{H^{\ell}_{1/2+\gamma}}+\|\mathring{B}\|_{H^{\ell}_{1/2+\gamma}}\approx\|\mathring{\mathfrak{D}}\|_{H^{\ell}_{1/2+\gamma}}+\|\mathring{\mathfrak{B}}\|_{H^{\ell}_{1/2+\gamma}}. \tag{10.0.27}$$

Similarly, from (9.2.6a) and (9.2.6c), we have that

$$\|\mathring{E}\|_{H^{\ell}_{1/2+\gamma}}+\|\mathring{B}\|_{H^{\ell}_{1/2+\gamma}}\approx\|\mathring{D}\|_{H^{\ell}_{1/2+\gamma}}+\|\mathring{B}\|_{H^{\ell}_{1/2+\gamma}}. \tag{10.0.28}$$

By (10.0.26), (10.0.27), (10.0.28), and Proposition A.1, it follows that, if $E_{\ell;\gamma}(0)+M$ is sufficiently small, then the smallness conditions[28] for $\|\mathring{V}^{(1)}\|_{L^{\infty}}$ and $\|\mathring{W}^{(1)}\|_{H^{\ell}_{1/2+\gamma}}$ in the hypotheses of Lemma 10.9 and Corollary 10.10 hold. Therefore, combining Corollaries 10.8 and 10.10, (10.0.26), (10.0.27), and (10.0.28), we deduce that, if ε is sufficiently small, then

$$\begin{aligned}\mathscr{E}^2_{\ell;\gamma;\mu}(0)&\lesssim\|\underline{\nabla}\underline{\mathring{h}}^{(1)}\|^2_{H^{\ell}_{1/2+\gamma}}+\|\partial_t h^{(1)}(0,\cdot)\|^2_{H^{\ell}_{1/2+\gamma}}+\|\mathring{E}\|^2_{H^{\ell}_{1/2+\gamma}}+\|\mathring{B}\|^2_{H^{\ell}_{1/2+\gamma}}+M^2\\&\lesssim\|\underline{\nabla}\underline{\mathring{h}}^{(1)}\|^2_{H^{\ell}_{1/2+\gamma}}+\|\mathring{K}\|^2_{H^{\ell}_{1/2+\gamma}}+\|\mathring{\mathfrak{D}}\|^2_{H^{\ell}_{1/2+\gamma}}+\|\mathring{\mathfrak{B}}\|^2_{H^{\ell}_{1/2+\gamma}}+M^2\\&\overset{\text{def}}{=}E^2_{\ell;\gamma}(0)+M^2.\end{aligned} \tag{10.0.29}$$

This concludes our proof of Proposition 10.4. □

[28] As in the Lindblad–Rodnianski proof of Corollary 15.3 below, the *smallness* condition $|h^{(1)}(0,x)|\lesssim\varepsilon(1+r)^{-1-\gamma}$ follows from integrating the smallness condition $|\partial_r h^{(1)}(0,x)|\lesssim\varepsilon(1+r)^{-2-\gamma}$, which is a consequence of Proposition A.1, from spatial infinity, and from using the decay assumption (1.0.4c) for $|\mathring{h}^{(1)}(x)|$ at spatial infinity.

11. Algebraic estimates of the nonlinearities

In this section, we provide algebraic estimates for the inhomogeneous terms that arise from commuting the reduced equations (3.7.1a)–(3.7.1c) with various differential operators. We also use the equations of Proposition 9.3 to derive ordinary differential inequalities for the null components of $\dot{\mathcal{F}} = \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}$. Furthermore, we provide algebraic estimates for the inhomogeneous terms appearing on the right-hand sides of these inequalities. Many of the estimates derived in this section rely on the wave coordinate condition.

11.1. ***Statement and proofs of the propositions.*** The proofs of the propositions given in this section use the results of a collection of technical null-structure lemmas, which we relegate to the end of the section. We begin by quoting the following proposition, which is central to many of the estimates. The basic idea is the following: many of our estimates would break down if we could not achieve good control of the components h_{LL} and h_{LT}. Amazingly, as shown in [Lindblad and Rodnianski 2005; 2010], the wave-coordinate condition allows for *independent, improved* estimates of exactly these components.

Proposition 11.1 (Algebraic consequences of the wave coordinate condition [Lindblad and Rodnianski 2010, Proposition 8.2]). *Let g be a Lorentzian metric satisfying the wave-coordinate condition* (3.1.1a) *relative to the coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$. Let I be a $\mathcal{Z}$-multi-index, and assume that $|\nabla_{\mathcal{Z}}^J h| \leq \varepsilon$ holds for all $\mathcal{Z}$-multi-indices J satisfying $|J| \leq \lfloor |I|/2 \rfloor$, where $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. Then if ε is sufficiently small, the following pointwise estimates hold for the tensor $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$:*

$$|\nabla \nabla_{\mathcal{Z}}^I H|_{\mathcal{L}\mathcal{T}} \lesssim \sum_{|J| \leq |I|} |\overline{\nabla} \nabla_{\mathcal{Z}}^J H| + \underbrace{\sum_{|J| \leq |I|-1} |\nabla \nabla_{\mathcal{Z}}^J H|}_{\textit{absent if } |I|=0} + \sum_{|I_1|+|I_2| \leq |I|} |\nabla_{\mathcal{Z}}^{I_1} H| |\nabla \nabla_{\mathcal{Z}}^{I_2} H|, \tag{11.1.1a}$$

$$|\nabla \nabla_{\mathcal{Z}}^I H|_{\mathcal{L}\mathcal{L}} \lesssim \sum_{|J| \leq |I|} |\overline{\nabla} \nabla_{\mathcal{Z}}^J H| + \underbrace{\sum_{|J| \leq |I|-2} |\nabla \nabla_{\mathcal{Z}}^J H|}_{\textit{absent if } |I| \leq 1} + \sum_{|I_1|+|I_2| \leq |I|} |\nabla_{\mathcal{Z}}^{I_1} H| |\nabla \nabla_{\mathcal{Z}}^{I_2} H|. \tag{11.1.1b}$$

Furthermore, analogous estimates hold for the tensor $h_{\mu\nu}$.

The next lemma provides an analogous version of the proposition for the "remainder" pieces of $(g^{-1})^{\mu\nu}$ and $g_{\mu\nu}$.

Lemma 11.2 (Algebraic/analytic consequences of the wave-coordinate condition; slight extension of [Lindblad and Rodnianski 2010, Lemma 15.4]). *Let g be a Lorentzian metric satisfying the wave-coordinate condition* (3.1.1a) *relative to the coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$, and let $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Let $k \geq 0$ be an integer, and assume that there is a constant ε such that $|\nabla_{\mathcal{Z}}^J h| \leq \varepsilon$ holds for all $\mathcal{Z}$-multi-indices J satisfying $|J| \leq \lfloor k/2 \rfloor$, where $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. Let*

$$H_{(1)}^{\mu\nu} \stackrel{\text{def}}{=} H^{\mu\nu} - H_{(0)}^{\mu\nu} \quad \textit{and} \quad H_{(0)}^{\mu\nu} \stackrel{\text{def}}{=} -\chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta^{\mu\nu}, \tag{11.1.2}$$

where $H_{(1)}^{\mu\nu}$ is the tensor obtained by subtracting the Schwarzschild part $H_{(0)}^{\mu\nu}$ from $H^{\mu\nu}$, and $\chi_0\left(\frac{1}{2} < z < \frac{3}{4}\right)$ denotes the characteristic function of the interval $\left[\frac{1}{2}, \frac{3}{4}\right]$. Assume further that $M \leq \varepsilon$. Then if ε is sufficiently *small, the following pointwise estimates hold*:

$$\sum_{|I|\leq k}|\nabla\nabla_{\mathcal{Z}}^{I}H_{(1)}|_{\mathcal{L}\mathcal{L}}+\sum_{|J|\leq k-1}|\nabla\nabla_{\mathcal{Z}}^{J}H_{(1)}|_{\mathcal{L}\mathcal{T}}$$
$$\lesssim \sum_{|I|\leq k}|\overline{\nabla}\nabla_{\mathcal{Z}}^{I}H_{(1)}|+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-2}|\nabla_{\mathcal{Z}}^{I}H_{(1)}||\nabla\nabla_{\mathcal{Z}}^{I}H_{(1)}|$$
$$+\sum_{|I_1|+|I_2|\leq k}|\nabla_{\mathcal{Z}}^{I_1}H_{(1)}||\nabla\nabla_{\mathcal{Z}}^{I_2}H_{(1)}|+\underbrace{\sum_{|J'|\leq k-2}|\nabla\nabla_{\mathcal{Z}}^{J'}H_{(1)}|}_{\text{absent if } k\leq 1}$$
$$+\varepsilon(1+t+|q|)^{-2}\chi_0\Big(\frac{1}{2}<\frac{r}{t}<\frac{3}{4}\Big)+\varepsilon^2(1+t+|q|)^{-3}. \qquad (11.1.3)$$

Additionally, let

$$h_{\mu\nu}^{(1)}\stackrel{\text{def}}{=}h_{\mu\nu}-h_{\mu\nu}^{(0)} \quad \textit{and} \quad h_{\mu\nu}^{(0)}\stackrel{\text{def}}{=}\chi\Big(\frac{r}{t}\Big)\chi(r)\frac{2M}{r}\delta_{\mu\nu}, \qquad (11.1.4)$$

where $h_{\mu\nu}^{(1)}$ is the tensor field obtained by subtracting the Schwarzschild part $h_{\mu\nu}^{(0)}$ from $h_{\mu\nu}$. Then an estimate analogous to (11.1.2) *holds if we replace the tensor field $H_{(1)}$ with the tensor field $h^{(1)}$.*

Proof. The estimates for the tensor field $H_{(1)}^{\mu\nu}$ were proved as [Lindblad and Rodnianski 2010, Lemma 15.4]. The analogous estimates for the tensor field $h_{\mu\nu}^{(1)}$ follow from those for $H_{(1)}^{\mu\nu}$ together with the fact that $H_{(1);\mu\nu}=-h_{\mu\nu}^{(1)}+O^{\infty}(|h^{(0)}+h^{(1)}|^2)$ and the decay estimates for $h^{(0)}$ provided by Lemma 15.1 below. □

We now provide the following proposition, which captures the algebraic structure of the inhomogeneous term $\mathfrak{H}_{\mu\nu}$ appearing on the right-hand side of the reduced equation (3.7.1a).

Proposition 11.3 (Algebraic estimates of $\mathfrak{H}_{\mu\nu}$ and $\nabla_{\mathcal{Z}}^{I}\mathfrak{H}_{\mu\nu}$; extension of [Lindblad and Rodnianski 2010, Proposition 9.8]). *Let $\mathfrak{H}_{\mu\nu}$ be the inhomogeneous term on the right-hand side of the reduced equation* (3.7.1a), *and assume that the wave-coordinate condition* (3.1.1a) *holds. Then*

$$|\mathfrak{H}|_{\mathcal{T}\mathcal{N}}\lesssim|\overline{\nabla}h||\nabla h|+(|\mathcal{F}|_{\mathcal{L}\mathcal{N}}+|\mathcal{F}|_{\mathcal{T}\mathcal{T}})|\mathcal{F}|+O^{\infty}(|h||\nabla h|^2)+O^{\ell+1}(|h||\mathcal{F}|^2)+O^{\ell+1}(|\mathcal{F}|^3;h), \qquad (11.1.5a)$$
$$|\mathfrak{H}|\lesssim|\nabla h|^2_{\mathcal{T}\mathcal{N}}+|\overline{\nabla}h||\nabla h|+|\mathcal{F}|^2+O^{\infty}(|h||\nabla h|^2)+O^{\ell+1}(|h||\mathcal{F}|^2)+O^{\ell+1}(|\mathcal{F}|^3;h). \qquad (11.1.5b)$$

In addition, assume that there exists an $\varepsilon>0$ such that $|\nabla_{\mathcal{Z}}^{J}h|+|\mathcal{L}_{\mathcal{Z}}^{J}\mathcal{F}|\leq\varepsilon$ holds for all $\mathcal{Z}$-multi-indices $|J|\leq\lfloor|I|/2\rfloor$. Then if ε is sufficiently small, the following pointwise estimates hold:

$$|\nabla_{\mathcal{Z}}^{I}\mathfrak{H}|\lesssim\sum_{|I_1|+|I_2|\leq|I|}\big(|\nabla\nabla_{\mathcal{Z}}^{I_1}h|_{\mathcal{T}\mathcal{N}}|\nabla\nabla_{\mathcal{Z}}^{I_2}h|_{\mathcal{T}\mathcal{N}}+|\overline{\nabla}\nabla_{\mathcal{Z}}^{I_1}h||\nabla\nabla_{\mathcal{Z}}^{I_2}h|\big)+\sum_{|I_1|+|I_2|\leq|I|}|\mathcal{L}_{\mathcal{Z}}^{I_1}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|$$
$$+\underbrace{\sum_{|I_1|+|I_2|\leq|I|-2}|\nabla\nabla_{\mathcal{Z}}^{I_1}h||\nabla\nabla_{\mathcal{Z}}^{I_2}h|}_{\text{absent if } |I|\leq 1}+\sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\nabla_{\mathcal{Z}}^{I_1}h||\nabla\nabla_{\mathcal{Z}}^{I_2}h||\nabla\nabla_{\mathcal{Z}}^{I_3}h|$$
$$+\sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\nabla_{\mathcal{Z}}^{I_1}h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}|+\sum_{|I_1|+|I_2|+|I_3|\leq|I|}|\mathcal{L}_{\mathcal{Z}}^{I_1}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}|. \qquad (11.1.5c)$$

Proof. Using (3.7.2a), we can decompose $\mathfrak{H}_{\mu\nu}$ into

$$\mathfrak{H}_{\mu\nu} = (\mathrm{i})_{\mu\nu} + (\mathrm{ii})_{\mu\nu} + (\mathrm{iii})_{\mu\nu} + (\mathrm{iv})_{\mu\nu}, \tag{11.1.6}$$

where

$$(\mathrm{i})_{\mu\nu} \stackrel{\text{def}}{=} \mathscr{P}(\nabla_\mu h, \nabla_\nu h), \tag{11.1.7}$$

$$(\mathrm{ii})_{\mu\nu} \stackrel{\text{def}}{=} \mathfrak{Q}^{(1;h)}_{\mu\nu}(\nabla h, \nabla h), \tag{11.1.8}$$

$$(\mathrm{iii})_{\mu\nu} \stackrel{\text{def}}{=} \mathfrak{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{F}), \tag{11.1.9}$$

$$(\mathrm{iv})_{\mu\nu} \stackrel{\text{def}}{=} O^{\infty}(|h||\nabla h|^2) + O^{\ell+1}(|h||\mathscr{F}|^2) + O^{\ell+1}(|\mathscr{F}|^3; h). \tag{11.1.10}$$

We will analyze each of the four pieces separately.

The facts that $|(\mathrm{i})|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) and that $|(\mathrm{i})| \lesssim$ the right-hand side of (11.1.5b) follow from Proposition 11.1, (11.2.7a), and (11.2.7b). The fact that $|\nabla_{\mathscr{Z}}^I(\mathrm{i})| \lesssim$ the right-hand side of (11.1.5c) follows from Proposition 11.1, (11.2.2c), and (11.2.7a).

The facts that $|(\mathrm{ii})|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) and that $|(\mathrm{ii})| \lesssim$ the right-hand side of (11.1.5b) both follow from (11.2.7c). The fact that $|\nabla_{\mathscr{Z}}^I(\mathrm{ii})| \lesssim$ the right-hand side of (11.1.5c) follows from (11.2.2a) and (11.2.7c).

The fact that $|(\mathrm{iii})|_{\mathcal{T}\mathcal{N}} \lesssim$ the right-hand side of (11.1.5a) follows from (11.2.7d) while the fact that $|(\mathrm{iii})| \lesssim$ the right-hand side of (11.1.5b) follows from (11.2.7e). The fact that $|\nabla_{\mathscr{Z}}^I(\mathrm{iii})| \lesssim$ the right-hand side of (11.1.5c) follows from (6.5.22), (11.2.2b), and (11.2.7e).

The desired estimates for term (iv) follow easily with the help of the Leibniz rule and (6.5.22). □

The next proposition captures the special algebraic structure of the reduced inhomogeneous term $\mathfrak{F}^{\nu}_{(I)}$ defined in (8.1.2b).

Proposition 11.4 (Algebraic estimates of $\mathfrak{F}^{\nu}_{(I)}$). *Let $\mathfrak{F}^{\nu}$ be the inhomogeneous term* (3.7.2b) *in the reduced electromagnetic equations, let I be a $\mathscr{Z}$-multi-index with $|I| = k$, and let X_ν be any one-form. In addition, assume that there exists an $\varepsilon > 0$ such that $|\nabla_{\mathscr{Z}}^J h| + |\mathcal{L}_{\mathscr{Z}}^J \mathscr{F}| \le \varepsilon$ holds for all $\mathscr{Z}$-multi-indices $|J| \le \lfloor k/2 \rfloor$. Then if ε is sufficiently small, the following pointwise estimates hold*:

$$\begin{aligned}
&|X_\nu \widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu| \\
&\lesssim \sum_{|I_1|+|I_2|\le k} |X||\overline{\nabla}\nabla_{\mathscr{Z}}^{I_1} h||\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}| + \sum_{|I_1|+|I_2|\le k} |X||\nabla\nabla_{\mathscr{Z}}^{I_1} h|\big(|\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\
&\qquad + \sum_{|I_1|+|I_2|+|I_3|\le k} |X||\nabla_{\mathscr{Z}}^{I_1} h||\nabla\nabla_{\mathscr{Z}}^{I_2} h||\mathcal{L}_{\mathscr{Z}}^{I_3}\mathscr{F}| + \sum_{|I_1|+|I_2|+|I_3|\le k} |X||\nabla\nabla_{\mathscr{Z}}^{I_1} h||\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}||\mathcal{L}_{\mathscr{Z}}^{I_3}\mathscr{F}| \\
&\lesssim (1+t+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\le k+1 \\ |I_2|\le k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}| + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\le k+1 \\ |I_2|\le k}} |X||\nabla_{\mathscr{Z}}^{I_1} h|\big(|\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\le k+1 \\ |I_2|,|I_3|\le k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\nabla_{\mathscr{Z}}^{I_2} h||\mathcal{L}_{\mathscr{Z}}^{I_3}\mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\le k+1 \\ |I_2|,|I_3|\le k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\mathcal{L}_{\mathscr{Z}}^{I_2}\mathscr{F}||\mathcal{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|.
\end{aligned} \tag{11.1.11a}$$

In addition, the same estimates hold if we replace modified Lie derivatives $\widehat{\mathscr{L}}_{\mathscr{Z}}^I$ *with standard Lie derivatives* $\mathscr{L}_{\mathscr{Z}}^I$.

Furthermore, let $N^{\#\mu\nu\kappa\lambda}$ *be the tensor field from the reduced electromagnetic equation* (3.7.1c). *Then if* ε *is sufficiently small and* $k \geq 1$, *the following pointwise commutator estimate holds:*

$$\begin{aligned}
&\left|X_\nu\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{Z}}^I (N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda})\big)\right| \\
&\quad \lesssim (1+|q|)^{-1} \sum_{|I'|=k,\ |J|\leq 1} |X||\nabla_{\mathscr{Z}}^{I'} h|_{\mathcal{L}\mathcal{L}} |\mathscr{L}_{\mathscr{Z}}^J \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{|J|\leq 1,\ |I'|=k} |X||\nabla_{\mathscr{Z}}^{J} h|_{\mathcal{L}\mathcal{L}} |\mathscr{L}_{\mathscr{Z}}^{I'} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{|I'|=k} |X||h|_{\mathcal{L}\mathcal{T}} |\mathscr{L}_{\mathscr{Z}}^{I'} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X||\nabla_{\mathscr{Z}}^{I_1} h|\big(|\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\
&\qquad + (1+t+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|,|I_2|\leq k}} |X|_{\mathcal{L}}|\nabla_{\mathscr{Z}}^{I_1} h||\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|\leq k-1,\ |I_2|\leq k-1}} |X||\nabla_{\mathscr{Z}}^{I_1} h|_{\mathcal{L}\mathcal{L}}|\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|\leq k \\ |I_1|\leq k-1,\ |I_2|\leq k-1}} |X||\nabla_{\mathscr{Z}}^{I_1} h|_{\mathcal{L}\mathcal{T}}|\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \underbrace{\sum_{\substack{|I_1|+|I_2|\leq k-1 \\ |I_1|\leq k-2,\ |I_2|\leq k-1}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}|}_{\textit{absent if } k=1} \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\nabla_{\mathscr{Z}}^{I_2} h||\mathscr{L}_{\mathscr{Z}}^{I_3} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X||\nabla_{\mathscr{Z}}^{I_1} h||\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_3} \mathscr{F}| \\
&\qquad + (1+|q|)^{-1} \sum_{\substack{|I_1|+|I_2|+|I_3|\leq k+1 \\ |I_1|,|I_2|,|I_3|\leq k}} |X||\mathscr{L}_{\mathscr{Z}}^{I_1} \mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_2} \mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_3} \mathscr{F}|. \qquad (11.1.11b)
\end{aligned}$$

Proof. To derive (11.1.11a), we first appeal to the relation (8.1.3a), which shows that we have to estimate principal terms of the form $X_\nu \mathfrak{Q}^\nu_{(2;\mathcal{F})}(\nabla \mathcal{L}^{I_1}_{\mathcal{Z}} h, \mathcal{L}^{I_2}_{\mathcal{Z}} \mathcal{F})$ and error terms of the form $X_\nu \mathcal{L}^{J}_{\mathcal{Z}} \mathfrak{F}^\nu_\triangle$. The desired estimates for the principal terms follow from the null-structure estimate (11.2.7i) together with inequalities (6.5.22), (6.5.23a), and (6.5.23b), which allow us to estimate Lie derivatives of h in terms of covariant derivatives of h. The error terms can easily be bounded by the right-hand side of (11.1.11a), where we use Lemma 6.16 to derive the second inequality in (11.1.11a).

Inequality (11.1.11b) can be proved in a similar fashion with the help of the relation (8.1.3b). In this case, there are two kinds of principal terms that have to be estimated: $X_\nu \mathcal{P}^\nu_{(\mathcal{F})}(\mathcal{L}^{I_1}_{\mathcal{Z}} h, \nabla \mathcal{L}^{I_2}_{\mathcal{Z}} \mathcal{F})$ and $X_\nu \mathfrak{Q}^\nu_{(1;\mathcal{F})}(\mathcal{L}^{I_1}_{\mathcal{Z}} h, \nabla \mathcal{L}^{I_2}_{\mathcal{Z}} \mathcal{F})$ while the error terms are of the form $X_\nu(\mathcal{L}^{I_1}_{\mathcal{Z}} N^{\#\mu\nu\kappa\lambda}_\triangle)\nabla_\mu \mathcal{L}^{I_2}_{\mathcal{Z}} \mathcal{F}_{\kappa\lambda}$. The error terms can be estimated as in the previous paragraph. The principal terms can be bounded by using the null-structure estimates (11.2.7f) and (11.2.7h). As in the previous paragraph, we use (6.5.22) and (6.5.23c) to estimate Lie derivatives of h in terms of covariant derivatives of h. □

As discussed at the beginning of Section 9.1, the null components of the lower-order Lie derivatives of $\mathcal{F}$ satisfy ordinary differential inequalities with controllable inhomogeneous terms. The next proposition provides convenient algebraic expressions for the inhomogeneities. In Section 15, these algebraic expressions will be combined with preliminary pointwise decay estimates to deduce upgraded pointwise decay estimates for the null components of $\mathcal{F}$ and its lower-order Lie derivatives.

Proposition 11.5 (Ordinary differential inequalities for $\underline{\alpha}[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, $\alpha[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, $\rho[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, and $\sigma[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$). *Let $\mathcal{F}$ be a solution to the reduced electromagnetic equations* (3.7.1b)–(3.7.1c), *and let $\underline{\alpha}$, α, ρ, and σ denote its Minkowskian null components. Let $\Lambda \overset{\text{def}}{=} L + \frac{1}{4}h_{LL}\underline{L}$, and assume that $|h| + |\mathcal{F}| \le \varepsilon$ holds. Then if ε is sufficiently small, the following pointwise estimate holds*:

$$\begin{aligned}
r^{-1}|\nabla_\Lambda(r\underline{\alpha})| \lesssim r^{-1}|h|_{\mathcal{L}\mathcal{L}}|\underline{\alpha}| &+ \sum_{|I|\le 1} r^{-1}\big(|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|_{\mathcal{T}\mathcal{T}}\big) + \sum_{|I_1|+|I_2|\le 1} r^{-1}|\nabla^{I_1}_{\mathcal{Z}} h||\mathcal{L}^{I_2}_{\mathcal{Z}}\mathcal{F}| \\
&+ \sum_{|I|\le 1}(1+|q|)^{-1}|h|\big(|\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}|_{\mathcal{T}\mathcal{T}}\big) \\
&+ \sum_{|I_1|+|I_2|+|I_3|\le 1}(1+|q|)^{-1}|\nabla^{I_1}_{\mathcal{Z}} h||\nabla^{I_2}_{\mathcal{Z}} h||\mathcal{L}^{I_3}_{\mathcal{Z}}\mathcal{F}| \\
&+ \sum_{|I_1|+|I_2|+|I_3|\le 1}(1+|q|)^{-1}|\nabla^{I_1}_{\mathcal{Z}} h||\mathcal{L}^{I_2}_{\mathcal{Z}}\mathcal{F}||\mathcal{L}^{I_3}_{\mathcal{Z}}\mathcal{F}| \\
&+ \sum_{|I_1|+|I_2|+|I_3|\le 1}(1+|q|)^{-1}|\mathcal{L}^{I_1}_{\mathcal{Z}}\mathcal{F}||\mathcal{L}^{I_2}_{\mathcal{Z}}\mathcal{F}||\mathcal{L}^{I_3}_{\mathcal{Z}}\mathcal{F}|. \quad (11.1.12)
\end{aligned}$$

Similarly, for each $\mathcal{Z}$-multi-index I, let $\underline{\alpha}[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, $\alpha[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, $\rho[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$, and $\sigma[\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}]$ denote the Minkowskian null components of $\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}$. Furthermore, let $\varpi(q)$ be any differentiable function of q. Assume that $|\nabla^I_{\mathcal{Z}} h| + |\mathcal{L}^I_{\mathcal{Z}}\mathcal{F}| \le \varepsilon$ holds for $|I| \le \lfloor k/2\rfloor$. Then if ε is sufficiently small, the following pointwise estimates also hold:

$$
\begin{aligned}
\sum_{|I|\leq k} r^{-1}\big|\nabla_{\Lambda}\big(r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]\big)\big| \lesssim & \sum_{|I|\leq k} r^{-1}\varpi(q)|h|_{\mathcal{L}\mathcal{L}}\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]\big| + \sum_{|I|\leq k}\varpi'(q)|h|_{\mathcal{L}\mathcal{L}}\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]\big| \\
& + \underbrace{\sum_{|I|\leq k,\ |J|\leq 1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^I h|_{\mathcal{L}\mathcal{L}}\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^J\mathcal{F}]\big|}_{\text{absent if } k\leq 1} \\
& + \underbrace{\sum_{|J|\leq 1,\ |I|\leq k}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^J h|_{\mathcal{L}\mathcal{L}}\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]\big|}_{\text{absent if } k=0} \\
& + \underbrace{\sum_{|I|\leq k}\varpi(q)(1+|q|)^{-1}|h|_{\mathcal{L}\mathcal{T}}\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]\big|}_{\text{absent if } k=0} \\
& + \underbrace{\sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|\leq k-1,\ |I_2|\leq k-1}}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h|\big|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}]\big|}_{\text{absent if } k=0} \\
& + \sum_{|I|\leq |k|+1}\varpi(q)r^{-1}\big(|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|_{\mathcal{T}\mathcal{T}}\big) \\
& + \sum_{|I_1|+|I_2|\leq k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h|\big(|\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}|_{\mathcal{T}\mathcal{T}}\big) \\
& + \sum_{|I_1|+|I_2|\leq k+1}\varpi(q)(1+t+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\nabla_{\mathcal{Z}}^{I_2} h||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}\varpi(q)(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}\varpi(q)(1+|q|)^{-1}|\mathcal{L}_{\mathcal{Z}}^{I_1}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}|, \qquad (11.1.13a)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{|I|\leq k} r^{-1}\big|\nabla_{\underline{L}}(r\alpha[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}])\big| \lesssim & \sum_{|I|\leq k+1} r^{-1}|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}| + \sum_{\substack{|I_1|+|I_2|\leq k+1 \\ |I_1|\leq k}}(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\nabla_{\mathcal{Z}}^{I_2} h||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1} h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\
& + \sum_{|I_1|+|I_2|+|I_3|\leq k+1}(1+|q|)^{-1}|\mathcal{L}_{\mathcal{Z}}^{I_1}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}|, \qquad (11.1.13b)
\end{aligned}
$$

$$\begin{aligned}\sum_{|I|\leq k} r^{-2}\big|\nabla_{\underline{L}}(r^2\rho[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}])\big| \lesssim \sum_{|I|\leq k+1} r^{-1}|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}| &+ \sum_{\substack{|I_1|+|I_2|\leq k+1\\|I_1|\leq k}} (1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1}h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}| \\ &+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1}h||\nabla_{\mathcal{Z}}^{I_2}h||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\ &+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\nabla_{\mathcal{Z}}^{I_1}h||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}| \\ &+ \sum_{|I_1|+|I_2|+|I_3|\leq k+1} (1+|q|)^{-1}|\mathcal{L}_{\mathcal{Z}}^{I_1}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}||\mathcal{L}_{\mathcal{Z}}^{I_3}\mathcal{F}|, \quad (11.1.13c)\end{aligned}$$

$$\sum_{|I|\leq k} r^{-2}\big|\nabla_{\underline{L}}(r^2\sigma[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}])\big| \lesssim \sum_{|I|\leq k+1} r^{-1}|\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|. \quad (11.1.13d)$$

Proof. Our proof of (11.1.12) is based on decomposing the terms in (9.1.8a), where $\dot{\underline{\alpha}}_\nu \stackrel{\text{def}}{=} \underline{\alpha}_\nu[\mathcal{F}]$, $\dot{\mathfrak{F}}^{\nu'} = \mathfrak{F}^{\nu'}$, etc., in the equation. We remind the reader that this equation is a consequence of performing a Minkowskian null decomposition on the electromagnetic equations (3.7.1b)–(3.7.1c). Here, $\mathfrak{F}^{\nu'}$ is defined in (3.7.2b). We begin by noting that the first two terms in (9.1.8a) can be written as $r^{-1}\nabla_L(r\underline{\alpha})$. We then remove the dangerous $-\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu$ component from the quadratic term $\not{m}_{\nu\lambda}\mathcal{P}^\lambda_{(\mathcal{F})}(h, \nabla\mathcal{F}) \stackrel{\text{def}}{=} \not{m}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\mathcal{F}_{\kappa\lambda}$ on the left-hand side of (9.1.8a) and add it to the $r^{-1}\nabla_L(r\underline{\alpha}_\nu)$ term. From the fact that $\nabla_\Lambda r = 1 - \frac{1}{4}h_{LL}$, it follows that the resulting sum can be written as $r^{-1}\nabla_\Lambda(r\underline{\alpha}_\nu) + \frac{1}{4}r^{-1}h_{LL}\underline{\alpha}_\nu$. We then put the $\frac{1}{4}r^{-1}h_{LL}\underline{\alpha}_\nu$ term on the right-hand side of (11.1.12) as the first inhomogeneous term; all the remaining terms in (9.1.8a) will also be placed on the right-hand side of (11.1.12). The left-over terms in $\mathcal{P}^\nu_{(\mathcal{F})}(h, \nabla\mathcal{F})$ (after the dangerous component $\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}^\nu$ has been removed) are denoted by $\widetilde{\mathcal{P}}^\nu_{(\mathcal{F})}(h, \nabla\mathcal{F})$ in Lemma 11.10 below. Now by (11.2.7g), with $X_{\nu'} \stackrel{\text{def}}{=} \not{m}_{\nu\nu'}$ (so that $|X|_{\mathcal{L}} = 0$), it follows that the left-over terms $X_{\nu'}\widetilde{\mathcal{P}}^{\nu'}_{(\mathcal{F})}(h, \nabla\mathcal{F})$ are bounded by the right-hand side of (11.1.12). The terms $\not\nabla\rho$ and $\not\nabla\sigma$ appearing on the left-hand side of (9.1.8a) (see Remark 9.4) can be bounded by the second term on the right-hand side of (11.1.12) via Corollary 6.18. The remaining terms in (11.1.12) that need to be bounded can be expressed as $X_{\nu'}\mathfrak{Q}^{\nu'}_{(1;\mathcal{F})}(h, \nabla\mathcal{F})$, $X_{\nu'}N_\triangle^{\#\beta\nu'\kappa\lambda}\nabla_\beta\mathcal{F}_{\kappa\lambda}$, and $X_{\nu'}\mathfrak{F}^{\nu'}$. The first of these can be bounded by using (11.2.7h) and the third with (11.1.11a) (in the case $|I| = 0$) while the second (with the help of Lemma 6.16) contributes to the cubic terms on the right-hand side of (11.1.12).

Our proof of (11.1.13a) is similar but more elaborate. To begin, we differentiate the electromagnetic equations with the iterated modified Lie derivative $\widehat{\mathcal{L}}_{\mathcal{Z}}^I$ to obtain the equations of variation (8.1.1a)–(8.1.1b) for $\dot{\mathcal{F}}_{\mu\nu} \stackrel{\text{def}}{=} \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\mu\nu}$ with inhomogeneous terms $\dot{\mathfrak{F}}^\nu = \mathfrak{F}^\nu_{(I)}$, where $\mathfrak{F}^\nu_{(I)}$ is defined in (8.1.2b). We then perform a null decomposition of the equations of variation, obtaining (9.1.8a) with $\dot{\underline{\alpha}}_\nu \stackrel{\text{def}}{=} \underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]$, $\dot{\mathfrak{F}}^{\nu'} \stackrel{\text{def}}{=} \mathfrak{F}^{\nu'}_{(I)}$, etc. Next, we multiply (9.1.8a) by $\varpi(q)$, use the identities $\nabla_\Lambda r = 1 - \frac{1}{4}h_{LL}$ and $\nabla_\Lambda q = -\frac{1}{2}h_{LL}$, and argue as above, removing the dangerous $-\frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]$ component from the quadratic term $\not{m}_{\nu\lambda}\mathcal{P}^\lambda_{(\mathcal{F})}(h, \nabla\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}) \stackrel{\text{def}}{=} \not{m}_\nu{}^\lambda h^{\mu\kappa}\nabla_\mu\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\kappa\lambda}$ and denoting the remaining terms by $\not{m}_{\nu\lambda}\widetilde{\mathcal{P}}^\lambda_{(\mathcal{F})}(h, \nabla\mathcal{L}_{\mathcal{Z}}^I\mathcal{F})$, to deduce that $\varpi(q)(\nabla_L\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}] + \frac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}] + r^{-1}\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]) = r^{-1}\nabla_\Lambda(r\varpi(q)\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]) + \frac{1}{4}r^{-1}\varpi(q)h_{LL}\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}] + \frac{1}{2}\varpi'(q)h_{LL}\underline{\alpha}_\nu[\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}]$. The first of these three terms is the only term on the left-hand side of (11.1.13a) while the last two are brought over to the right-hand side of (11.1.13a). To bound $\not{m}_{\nu\nu'}\mathfrak{F}^{\nu'}_{(I)}$ by the right-hand side of (11.1.13a), we again set $X_{\nu'} \stackrel{\text{def}}{=} \not{m}_{\nu\nu'}$ (so that $|X|_{\mathcal{L}} = 0$); the desired bound then follows from (11.1.11a) and (11.1.11b) together with repeated use of the inequality $|\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}| \lesssim |\underline{\alpha}[\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}]| + |\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}|_{\mathscr{L}\mathscr{N}} + |\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}|_{\mathscr{T}\mathscr{T}}$. The terms $\varpi(q)\,\nabla\!\!\!/\,\rho[\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}]$ and $\varpi(q)\,\nabla\!\!\!/\,\sigma[\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}]$ appearing on the left-hand side of (9.1.8a) (see Remark 9.4) can be bounded by the seventh sum on the right-hand side of (11.1.13a) with the help of Corollary 6.18. The remaining three terms on the left-hand side of (9.1.8a) to be estimated are $X_{\nu'}\widetilde{\mathscr{P}}_{(\mathscr{F})}^{\nu'}(h, \nabla\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F})$, $X_{\nu'}\mathscr{Q}_{(1;\mathscr{F})}^{\nu'}(h, \nabla\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F})$, and $X_{\nu'}N_{\triangle}^{\#\beta\nu'\kappa\lambda}\nabla_{\beta}\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}_{\kappa\lambda}$. The first of these can be bounded by using (11.2.7g) and the second with (11.2.7h) while the third (with the help of Lemma 6.16) contributes to the cubic terms on the right-hand side of (11.1.12).

The proofs of (11.1.13b)–(11.1.13d), which are based on an analysis of (9.1.8b)–(9.1.8d), are similar but much simpler. We will provide a brief overview of how to derive (11.1.13b); we then leave the remaining details to the reader. To begin, as in the previous paragraph, we differentiate the electromagnetic equations with the iterated modified Lie derivative $\widehat{\mathscr{L}}_{\mathfrak{Z}}^I$ and null-decompose the equations of variation. We use the same notation as in the previous paragraph and also the notation $\dot{\alpha}_{\nu} \stackrel{\text{def}}{=} \alpha_{\nu}[\mathscr{L}_{\mathfrak{Z}}^I\mathscr{F}]$. To derive inequality (11.1.13b), we will manipulate the equation (9.1.8b) satisfied by $\dot{\alpha}_{\nu}$. First, we rewrite the first two terms on the left-hand side of (9.1.8b) as $r^{-1}\nabla_{\underline{L}}(r\dot{\alpha})$. This term is the only one that appears on the left-hand side of (11.1.13b); all other terms are placed on the right-hand side. The only thing that remains to be discussed is how to bound these other terms from (9.1.8b) by the right-hand side of (11.1.13b). These terms separate into two classes: the linear terms involving angular derivatives $\nabla\!\!\!/$ and the remaining nonlinear terms. As in the previous paragraph, the linear terms can be suitably bounded by the first sum on the right-hand side of (11.1.13b) thanks to Corollary 6.18. With the help of Lemma 6.16, the nonlinear terms can all bounded in the crudest possible fashion by estimates of, e.g., the form

$$\sum_{|I|\le k}|\nabla_{\mathfrak{Z}}^I(U\nabla V)| \lesssim (1+|q|)^{-1}\sum_{\substack{|I_1|+|I_2|\le k+1\\|I_1|\le k}}|\nabla_{\mathfrak{Z}}^{I_1}U||\nabla_{\mathfrak{Z}}^{I_2}V|. \qquad \square$$

The next proposition provides pointwise estimates for the challenging commutator term $\widetilde{\square}_g\nabla_{\mathfrak{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathfrak{Z}}^I\widetilde{\square}_g h^{(1)}$ from the right-hand side of (7.0.1).

Proposition 11.6 (Algebraic estimates of $[\widetilde{\square}_g, \nabla_{\mathfrak{Z}}^I]$ [Lindblad and Rodnianski 2010, Proposition 5.3]). *Let $g_{\mu\nu}$ be a Lorentzian metric, $h_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$, and $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - m^{\mu\nu}$. Let $\widetilde{\square}_g \stackrel{\text{def}}{=} \square_m + H^{\kappa\lambda}\nabla_{\kappa}\nabla_{\lambda}$, and let I be a $\mathfrak{Z}$-multi-index with $1 \le |I|$. Let $\widehat{\nabla}_{\mathfrak{Z}}^I$ denote the modified Minkowskian covariant derivative operator defined in* (6.4.1). *Assume that there is a constant ε such that $|\nabla_{\mathfrak{Z}}^J h| \le \varepsilon$ holds for all $\mathfrak{Z}$-multi-indices J satisfying $|J| \le \lfloor |I|/2 \rfloor$. Then if ε is sufficiently small, the following pointwise estimate holds:*

$$\begin{aligned}
|\widetilde{\square}_g\nabla_{\mathfrak{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathfrak{Z}}^I\widetilde{\square}_g h^{(1)}| \lesssim{}& (1+t+|q|)^{-1}\sum_{|K|\le|I|}\;\sum_{|J|+(|K|-1)_+\le|I|}|\nabla\nabla_{\mathfrak{Z}}^K h^{(1)}||\nabla_{\mathfrak{Z}}^J H| \\
&+ (1+|q|)^{-1}\sum_{|K|\le|I|}\;\sum_{|J|+(|K|-1)_+\le|I|}|\nabla\nabla_{\mathfrak{Z}}^K h^{(1)}||\nabla_{\mathfrak{Z}}^J H|_{\mathscr{L}\mathscr{L}} \\
&+ (1+|q|)^{-1}\sum_{|K|\le|I|}\;\sum_{|J'|+(|K|-1)_+\le|I|-1}|\nabla\nabla_{\mathfrak{Z}}^K h^{(1)}||\nabla_{\mathfrak{Z}}^{J'} H|_{\mathscr{L}\mathscr{T}} \\
&+ \underbrace{(1+|q|)^{-1}\sum_{|K|\le|I|}\;\sum_{|J''|+(|K|-1)_+\le|I|-2}|\nabla\nabla_{\mathfrak{Z}}^K h^{(1)}||\nabla_{\mathfrak{Z}}^{J''} H|}_{\textit{absent if } |I|\le 1 \textit{ or } |K|=|I|}, \qquad (11.1.15)
\end{aligned}$$

where $(|K|-1)_+ \overset{\text{def}}{=} 0$ *if* $|K|=0$ *and* $(|K|-1)_+ \overset{\text{def}}{=} |K|-1$ *if* $|K|\geq 1$.

Corollary 11.7 (Algebraic estimates of $|\widetilde{\Box}_g \nabla_{\mathcal{Z}}^I h^{(1)}|$). *Assume that* $h_{\mu\nu}^{(1)}$ $(\mu, \nu = 0, 1, 2, 3)$ *is a solution to the reduced equation* (3.7.1a). *Then under the assumptions of Proposition 11.6, we have that*

$$
\begin{aligned}
|\widetilde{\Box}_g \nabla_{\mathcal{Z}}^I h^{(1)}| \lesssim |\widehat{\nabla}_{\mathcal{Z}}^I \mathfrak{H}| + |\widehat{\nabla}_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}| &+ (1+t+|q|)^{-1} \sum_{|K|\leq|I|} \sum_{|J|+(|K|-1)_+\leq|I|} |\nabla\nabla_{\mathcal{Z}}^K h^{(1)}||\nabla_{\mathcal{Z}}^J H| \\
&+ (1+|q|)^{-1} \sum_{|K|\leq|I|} \sum_{|J|+(|K|-1)_+\leq|I|} |\nabla\nabla_{\mathcal{Z}}^K h^{(1)}||\nabla_{\mathcal{Z}}^J H|_{\mathcal{L}\mathcal{L}} \\
&+ (1+|q|)^{-1} \sum_{|K|\leq|I|} \sum_{|J'|+(|K|-1)_+\leq|I|-1} |\nabla\nabla_{\mathcal{Z}}^K h^{(1)}||\nabla_{\mathcal{Z}}^{J'} H|_{\mathcal{L}\mathcal{T}} \\
&+ \underbrace{(1+|q|)^{-1} \sum_{|K|\leq|I|} \sum_{|J''|+(|K|-1)_+\leq|I|-2} |\nabla\nabla_{\mathcal{Z}}^K h^{(1)}||\nabla_{\mathcal{Z}}^{J''} H|}_{\text{absent if } |I|\leq 1 \text{ or } |K|=|I|}. \qquad (11.1.16)
\end{aligned}
$$

Proof. Simply use Proposition 7.1 to decompose $\widetilde{\Box}_g \nabla_{\mathcal{Z}}^I h^{(1)} = \widehat{\nabla}_{\mathcal{Z}}^I \mathfrak{H} - \widehat{\nabla}_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)} + (\widetilde{\Box}_g \nabla_{\mathcal{Z}}^I h^{(1)} - \widehat{\nabla}_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(1)})$ and apply Proposition 11.6. □

11.2. ***Null-structure lemmas.*** In this section, we provide the lemmas that are used in the proofs of some of the previous propositions. We will make repeated use of the following decompositions of the Minkowski metric and its inverse:

$$m_{\mu\nu} = -\tfrac{1}{2} L_\mu \underline{L}_\nu - \tfrac{1}{2} \underline{L}_\mu L_\nu + \not{m}_{\mu\nu}, \qquad (11.2.1a)$$

$$(m^{-1})^{\mu\nu} = -\tfrac{1}{2} L^\mu \underline{L}^\nu - \tfrac{1}{2} \underline{L}^\mu L^\nu + \not{m}^{\mu\nu}, \qquad (11.2.1b)$$

where $\not{m}_{\mu\nu}$ is the Euclidean first fundamental form of the spheres $S_{r,t}$ defined in (5.1.4b).

We begin with a lemma that shows that the essential algebraic structure of the quadratic terms appearing on the right-hand sides of the reduced equations (3.7.1a)–(3.7.1c) is preserved under differentiation.

Lemma 11.8 (Leibniz rules for the quadratic terms). *Let* $\mathfrak{Q}_0(\nabla\psi, \nabla\chi)$ *and* $\mathfrak{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ *denote the standard null forms defined in* (3.6.6a)–(3.6.6b), *and let* $\mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$, $\mathfrak{Q}_{\mu\nu}^{(2;h)}(\mathcal{F}, \mathcal{F})$, $\mathcal{P}(\nabla_\mu h, \nabla_\nu h)$, $\mathcal{P}_{(\mathcal{F})}^{\nu}(\nabla h, \mathcal{F})$, $\mathfrak{Q}_{(1;\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$, *and* $\mathfrak{Q}_{(2;\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$ *denote the quadratic terms defined in* (3.6.5), (3.7.2d), (3.6.4), (3.7.3b), (3.7.3c), *and* (3.7.2e), *respectively. Let I be a $\mathcal{Z}$-multi-index. Then there exist constants* $C_{I_1,I_2;\mu\nu}^{\kappa\lambda\gamma\gamma'\delta\delta'}$, $C_{I_1,I_2;\mu\nu}^{0;\gamma\gamma'\delta\delta'}$, C_{I_1,I_2}, $C_{\mathcal{P};I_1,I_2;\mu\nu}^{\kappa\lambda}$, $C_{\mathcal{P};I_1,I_2}$, *and* $C_{i;I_1,I_2}$ *such that*

$$
\begin{aligned}
\nabla_{\mathcal{Z}}^I \mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h) = \sum_{|I_1|+|I_2|\leq|I|} & C_{I_1,I_2;\mu\nu}^{\kappa\lambda\gamma\gamma'\delta\delta'} \mathfrak{Q}_{\kappa\lambda}(\nabla\nabla_{\mathcal{Z}}^{I_1} h_{\gamma\gamma'}, \nabla\nabla_{\mathcal{Z}}^{I_2} h_{\delta\delta'}) \\
&+ \sum_{|I_1|+|I_2|<|I|} C_{I_1,I_2;\mu\nu}^{0;\gamma\gamma'\delta\delta'} \mathfrak{Q}_0(\nabla\nabla_{\mathcal{Z}}^{I_1} h_{\gamma\gamma'}, \nabla\nabla_{\mathcal{Z}}^{I_2} h_{\delta\delta'}), \qquad (11.2.2a)
\end{aligned}
$$

$$\nabla_{\mathcal{Z}}^I \mathfrak{Q}_{\mu\nu}^{(2;h)}(\mathcal{F}, \mathcal{F}) = \sum_{|I_1|+|I_2|\leq|I|} C_{I_1,I_2} \mathfrak{Q}_{\mu\nu}^{(2;h)}(\nabla_{\mathcal{Z}}^{I_1}\mathcal{F}, \nabla_{\mathcal{Z}}^{I_2}\mathcal{F}), \qquad (11.2.2b)$$

$$\nabla_{\mathcal{Z}}^I \mathcal{P}(\nabla_\mu h, \nabla_\nu h) = \sum_{|I_1|+|I_2|\leq|I|} C_{\mathcal{P};I_1,I_2;\mu\nu}^{\kappa\lambda} \mathcal{P}(\nabla_\kappa \nabla_{\mathcal{Z}}^{I_1} h, \nabla_\lambda \nabla_{\mathcal{Z}}^{I_2} h), \qquad (11.2.2c)$$

$$\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{P}_{(\mathcal{F})}^{\nu}(\nabla h, \mathcal{F}) = \sum_{|I_1|+|I_2|\leq|I|} C_{\mathcal{P};I_1,I_2}\mathcal{P}_{(\mathcal{F})}^{\nu}(\nabla\mathcal{L}_{\mathcal{Z}}^{I_1}h, \mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}), \tag{11.2.2d}$$

$$\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{Q}_{(i;\mathcal{F})}^{\nu}(h, \nabla\mathcal{F}) = \sum_{|I_1|+|I_2|\leq|I|} C_{i;I_1,I_2}\mathcal{Q}_{(i;\mathcal{F})}^{\nu}(\mathcal{L}_{\mathcal{Z}}^{I_1}h, \nabla\mathcal{L}_{\mathcal{Z}}^{I_2}\mathcal{F}) \quad (i = 1, 2). \tag{11.2.2e}$$

Proof. By pure calculation, if $Z \in \mathcal{Z}$, then the following identity holds for the standard null form $\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$:

$$\nabla_Z\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) = \mathcal{Q}_{\mu\nu}(\nabla\nabla_Z\psi, \nabla\chi) + \mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\nabla_Z\chi) - {}^{(Z)}c_{\mu}{}^{\kappa}\mathcal{Q}_{\kappa\nu}(\nabla\psi, \nabla\chi) - {}^{(Z)}c_{\nu}{}^{\kappa}\mathcal{Q}_{\mu\kappa}(\nabla\psi, \nabla\chi), \tag{11.2.3}$$

where ${}^{(Z)}c_{\mu\nu}$ is the covariantly constant tensor field defined in (6.2.4). A similar identity holds for the standard null form $\mathcal{Q}_0(\nabla\psi, \nabla\chi)$. Equation (11.2.2a) now follows inductively from these facts and the Leibniz rule since $\mathcal{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$ is a linear combination of standard null forms. Equation (11.2.2c) follows similarly. Equation (11.2.2b) follows trivially from definition (3.7.2d) and the Leibniz rule. Equations (11.2.2d) and (11.2.2e) follow from (6.3.4b), Lemma 6.8, and the Leibniz rule. □

The next lemma concerns the null structure of the standard null forms.

Lemma 11.9 (Null structure estimates for the standard null forms). *Let* $\mathcal{Q}_0(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (m^{-1})^{\kappa\lambda}(\nabla_\kappa\psi)\cdot(\nabla_\lambda\chi)$ *and* $\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi) \stackrel{\text{def}}{=} (\nabla_\mu\psi)(\nabla_\nu\chi) - (\nabla_\nu\psi)(\nabla_\mu\chi)$ *denote the standard null forms defined in* (3.6.6a)–(3.6.6b). *Then*

$$|\mathcal{Q}_0(\nabla\psi, \nabla\chi)| + |\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)| \lesssim |\overline{\nabla}\psi||\nabla\chi| + |\overline{\nabla}\chi||\nabla\psi|. \tag{11.2.4}$$

Proof. The estimate (11.2.4) for $\mathcal{Q}_0$ easily follows from using (11.2.1b) to decompose $(m^{-1})^{\kappa\lambda}$. To obtain the estimates for $\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$, we first consider the $\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ to be components of a 2-covariant tensor $\mathcal{Q}(\nabla\psi, \nabla\chi)$. Inequality (11.2.4) is equivalent to the following inequality:

$$|\mathcal{Q}(\nabla\psi, \nabla\chi)|_{\mathcal{N}\mathcal{N}} \lesssim |\overline{\nabla}\psi||\nabla\chi| + |\overline{\nabla}\chi||\nabla\psi|. \tag{11.2.5}$$

Contracting $\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$ against frame vectors $N^\mu, N^\nu \in \mathcal{N}$, we see that the only component on the left-hand side of (11.2.5) that could pose any difficulty is $\underline{L}^\mu\underline{L}^\nu\mathcal{Q}_{\mu\nu}(\nabla\psi, \nabla\chi)$. But the antisymmetry of the $\mathcal{Q}_{\mu\nu}(\cdot, \cdot)$ implies that this component is 0. □

The next lemma addresses the null structure of some of the terms appearing in the reduced equations (3.7.1a)–(3.7.1c).

Lemma 11.10 (Null structure estimates for the reduced equations). *Let* $\mathcal{P}(\nabla_\mu\Pi, \nabla_\nu\Theta)$, $\mathcal{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$, $\mathcal{Q}_{\mu\nu}^{(2;h)}(\mathcal{F}, \mathcal{G})$, $\mathcal{P}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$, $\mathcal{Q}_{(1;\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$, *and* $\mathcal{Q}_{(2;\mathcal{F})}^{\nu}(\nabla h, \mathcal{F})$ *be the quadratic forms defined in Section 3.7, and define the quadratic form* $\widetilde{\mathcal{P}}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$ *by removing the* $\nabla_{\underline{L}}\underline{\alpha}^\nu[\mathcal{F}]$*-containing component of* $\mathcal{P}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F})$:

$$\widetilde{\mathcal{P}}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F}) \stackrel{\text{def}}{=} \mathcal{P}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F}) - \tfrac{1}{4}h_{LL}\,\not{m}^{\nu\nu'}\nabla_{\underline{L}}\mathcal{F}_{\underline{L}\nu'} = \mathcal{P}_{(\mathcal{F})}^{\nu}(h, \nabla\mathcal{F}) + \tfrac{1}{4}h_{LL}\nabla_{\underline{L}}\underline{\alpha}^\nu[\mathcal{F}]. \tag{11.2.6}$$

Let X_ν be any one-form, let $\Pi_{\mu\nu}$ and $\Theta_{\mu\nu}$ be symmetric type-$\binom{0}{2}$ tensor fields, and let $\mathscr{F}_{\mu\nu}$ and $\mathscr{G}_{\mu\nu}$ be two-forms. Then the following pointwise inequalities hold:

$$|\mathscr{P}(\nabla_\mu \Pi, \nabla_\nu \Theta)| \lesssim |\nabla\Pi|_{\mathcal{T}\mathcal{N}}|\nabla\Theta|_{\mathcal{T}\mathcal{N}} + |\nabla\Pi|_{\mathcal{L}\mathcal{L}}|\nabla\Theta| + |\Pi||\nabla\Theta|_{\mathcal{L}\mathcal{L}} \quad (\mu, \nu = 0, 1, 2, 3), \tag{11.2.7a}$$

$$\sum_{T\in\mathcal{T},\ N\in\mathcal{N}} |T^\mu N^\nu \mathscr{P}(\nabla_\mu \Pi, \nabla_\nu \Theta)| \lesssim |\overline{\nabla}\Pi||\nabla\Theta|, \tag{11.2.7b}$$

$$|\mathscr{Q}^{(1;h)}_{\mu\nu}(\nabla\Pi, \nabla\Theta)| \lesssim |\overline{\nabla}\Pi||\nabla\Theta| + |\nabla\Pi||\overline{\nabla}\Theta| \quad (\mu, \nu = 0, 1, 2, 3), \tag{11.2.7c}$$

$$\sum_{T\in\mathcal{T},\ N\in\mathcal{N}} |T^\mu N^\nu \mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G})| \lesssim (|\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathscr{F}|_{\mathcal{T}\mathcal{T}})|\mathscr{G}| + |\mathscr{F}|(|\mathscr{G}|_{\mathcal{L}\mathcal{N}} + |\mathscr{G}|_{\mathcal{T}\mathcal{T}}), \tag{11.2.7d}$$

$$|\mathscr{Q}^{(2;h)}_{\mu\nu}(\mathscr{F}, \mathscr{G})| \lesssim |\mathscr{F}||\mathscr{G}| \quad (\mu, \nu = 0, 1, 2, 3), \tag{11.2.7e}$$

$$\begin{aligned} |X_\nu \mathscr{P}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F})| &\lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|\big(|\nabla\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\nabla\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) + |X||h|_{\mathcal{L}\mathcal{L}}|\nabla\mathscr{F}| + |X|_{\mathcal{L}}|h||\nabla\mathscr{F}| \\ &\lesssim (1+t+|q|)^{-1}\sum_{|I|\le 1}|X||h||\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}| \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X||h|\big(|\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X||h|_{\mathcal{L}\mathcal{L}}|\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}| \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X|_{\mathcal{L}}|h||\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|, \end{aligned} \tag{11.2.7f}$$

$$\begin{aligned} |X_\nu \widetilde{\mathscr{P}}^\nu_{(\mathscr{F})}(h, \nabla\mathscr{F})| &\lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|\big(|\nabla\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\nabla\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) + |X|_{\mathcal{L}}|h||\nabla\mathscr{F}| \\ &\lesssim (1+t+|q|)^{-1}\sum_{|I|\le 1}|X||h||\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}| \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X||h|\big(|\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X|_{\mathcal{L}}|h||\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|, \end{aligned} \tag{11.2.7g}$$

$$\begin{aligned} |X_\nu \mathscr{Q}^\nu_{(1;\mathscr{F})}(h, \nabla\mathscr{F})| &\lesssim |X||h||\overline{\nabla}\mathscr{F}| + |X||h|\big(|\nabla\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\nabla\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big) \\ &\lesssim (1+t+|q|)^{-1}\sum_{|I|\le 1}|X||h||\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}| \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X||h|\big(|\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}^I_{\mathcal{Z}}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big), \end{aligned} \tag{11.2.7h}$$

$$\begin{aligned} |X_\nu \mathscr{Q}^\nu_{(2;\mathscr{F})}(\nabla h, \mathscr{F})| &\lesssim |X||\overline{\nabla}h||\mathscr{F}| + |X||\nabla h||\mathscr{F}|_{\mathcal{L}\mathcal{N}} \\ &\lesssim (1+t+|q|)^{-1}\sum_{|I|\le 1}|X||\nabla^I_{\mathcal{Z}}h||\mathscr{F}| \\ &\quad + (1+|q|)^{-1}\sum_{|I|\le 1}|X||\nabla^I_{\mathcal{Z}}h|\big(|\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big). \end{aligned} \tag{11.2.7i}$$

Proof. Inequality (11.2.7c) follows directly from Lemma 11.9 since $\mathfrak{Q}_{\mu\nu}^{(1;h)}(\nabla h, \nabla h)$ is a linear combination of standard null forms. Inequality (11.2.7e) is trivial while (11.2.7a), (11.2.7b), and the first inequalities in (11.2.7d)–(11.2.7i) are straightforward to verify by using (11.2.1a)–(11.2.1b). The second inequalities in (11.2.7d)–(11.2.7i) then follow from the first ones, Lemma 6.16, and Proposition 6.19. □

The next lemma addresses the null structure of some of the cubic terms on the right-hand side of (12.2.4).

Lemma 11.11 (Null structure estimates for quasilinear wave equations [Lindblad and Rodnianski 2010, Lemma 4.2]). *Let Π be a type-$\binom{0}{2}$ tensor field, and let ϕ be a scalar function. Then the following inequalities hold*:

$$|\Pi^{\kappa\lambda}(\nabla_\kappa \phi)(\nabla_\lambda \phi)| \lesssim |\Pi|_{\mathcal{L}\mathcal{L}}|\nabla\phi|^2 + |\Pi||\overline{\nabla}\phi||\nabla\phi|, \tag{11.2.8a}$$

$$|L_\kappa \Pi^{\kappa\lambda}\nabla_\lambda\phi| \lesssim |\Pi|_{\mathcal{L}\mathcal{L}}|\nabla\phi| + |\Pi||\overline{\nabla}\phi|, \tag{11.2.8b}$$

$$|(\nabla_\kappa \Pi^{\kappa\lambda})\nabla_\lambda\phi| \lesssim |\nabla\Pi|_{\mathcal{L}\mathcal{L}}|\nabla\phi| + |\overline{\nabla}\Pi||\nabla\phi| + |\nabla\Pi||\overline{\nabla}\phi|, \tag{11.2.8c}$$

$$|\Pi^{\kappa\lambda}\nabla_\kappa\nabla_\lambda\phi| \lesssim |\Pi|_{\mathcal{L}\mathcal{L}}|\nabla\nabla\phi| + |\overline{\nabla}\nabla\phi|. \tag{11.2.8d}$$

The following lemma addresses the null structure of some of the cubic terms on the right-hand side of (12.2.8):

Lemma 11.12 (Null structure estimates for the terms appearing in the divergence of the electromagnetic energy currents). *Let $h_{\mu\nu}$ be a type-$\binom{0}{2}$ tensor field, and let $\mathcal{F}_{\mu\nu}$ be a two-form. Then the following inequalities hold*:

$$|(\nabla_\mu h^{\mu\kappa})\mathcal{F}_{\kappa\zeta}\mathcal{F}_0^{\ \zeta}| \lesssim |\nabla h|_{\mathcal{L}\mathcal{L}}|\mathcal{F}|^2 + |\overline{\nabla} h||\mathcal{F}|^2 + |\nabla h||\mathcal{F}|(|\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}), \tag{11.2.9a}$$

$$|(\nabla_\mu h^{\kappa\lambda})\mathcal{F}^\mu_{\ \kappa}\mathcal{F}_{0\lambda}| \lesssim |\overline{\nabla} h||\mathcal{F}|^2 + |\nabla h||\mathcal{F}|(|\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}), \tag{11.2.9b}$$

$$|(\nabla_t h^{\kappa\lambda})\mathcal{F}_{\kappa\eta}\mathcal{F}_\lambda^{\ \eta}| \lesssim |\nabla h|_{\mathcal{L}\mathcal{L}}|\mathcal{F}|^2 + |\nabla h||\mathcal{F}|(|\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}), \tag{11.2.9c}$$

$$|L_\mu h^{\mu\kappa}\mathcal{F}_{\kappa\zeta}\mathcal{F}_0^{\ \zeta}| \lesssim |h|_{\mathcal{L}\mathcal{L}}|\mathcal{F}|^2 + |h||\mathcal{F}|(|\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}), \tag{11.2.9d}$$

$$|L_\mu h^{\kappa\lambda}\mathcal{F}^\mu_{\ \kappa}\mathcal{F}_{0\lambda}| \lesssim |h||\mathcal{F}||\mathcal{F}|_{\mathcal{L}\mathcal{N}}, \tag{11.2.9e}$$

$$|h^{\kappa\lambda}\mathcal{F}_{\kappa\eta}\mathcal{F}_\lambda^{\ \eta}| \lesssim |h|_{\mathcal{L}\mathcal{L}}|\mathcal{F}|^2 + |h||\mathcal{F}|(|\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{F}|_{\mathcal{T}\mathcal{T}}). \tag{11.2.9f}$$

Proof. It is straightforward to derive inequalities (11.2.9a)–(11.2.9f) by using (11.2.1a). □

12. Weighted energy estimates for the electromagnetic equations of variation and for systems of nonlinear wave equations in a curved spacetime

In this section, we prove weighted energy estimates for the electromagnetic equations of variation

$$\nabla_\lambda \dot{\mathcal{F}}_{\mu\nu} + \nabla_\mu \dot{\mathcal{F}}_{\nu\lambda} + \nabla_\nu \dot{\mathcal{F}}_{\lambda\mu} = \dot{\mathfrak{F}}_{\lambda\mu\nu} \quad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{12.0.1a}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \dot{\mathcal{F}}_{\kappa\lambda} = \dot{\mathfrak{F}}^\nu \qquad (\nu = 0, 1, 2, 3). \tag{12.0.1b}$$

Our estimates complement the weighted energy estimates proved in [Lindblad and Rodnianski 2010] for the inhomogeneous wave equation

$$\widetilde{\square}_g \phi = \mathfrak{I} \tag{12.0.2}$$

and for tensorial systems of inhomogeneous wave equations with principal part $\widetilde{\square}_g$:

$$\widetilde{\square}_g \phi_{\mu\nu} = \mathfrak{I}_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3). \tag{12.0.3}$$

12.1. ***The energy estimate weight function $w(q)$.*** As in [Lindblad and Rodnianski 2010], our energy estimates will involve the weight function $w(q)$ defined by

$$w = w(q) = \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{12.1.1}$$

where the constants γ and μ are subject to the restrictions stated in Section 2.14.

Observe that the following inequalities follow from the definition (12.1.1):

$$w' \leq 4(1 + |q|)^{-1} w \leq 16\mu^{-1}(1 + q_-)^{2\mu} w', \tag{12.1.2}$$

where $q_- = 0$ if $q \geq 0$ and $q_- = |q|$ if $q < 0$.

12.2. ***Weighted energy estimates.*** We begin by deriving weighted energy estimates for the electromagnetic equations of variation.

Lemma 12.1 (Weighted energy estimates for $\dot{\mathscr{F}}$). *Assume that $\dot{\mathscr{F}}_{\mu\nu}$ is a solution to the equations of variation* (8.1.1a)–(8.1.1b) *corresponding to the background* $(h_{\mu\nu}, \mathscr{F}_{\mu\nu})$, *where* $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$. *Let* $\dot{\alpha} \overset{\text{def}}{=} \alpha[\dot{\mathscr{F}}]$, $\dot{\rho} \overset{\text{def}}{=} \rho[\dot{\mathscr{F}}]$, *and* $\dot{\sigma} \overset{\text{def}}{=} \sigma[\dot{\mathscr{F}}]$ *denote the "favorable" Minkowskian null components of $\dot{\mathscr{F}}$ as defined in Definition 5.9. Assume that* $|h| + |\mathscr{F}| \leq \varepsilon$. *Then if ε is sufficiently small and* $t_1 \leq t_2$, *the following integral inequality holds*:

$$\begin{aligned}
&\int_{\Sigma_{t_2}} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2\big) w'(q)\, d^3x\, d\tau \\
&\qquad \lesssim \int_{\Sigma_{t_1}} |\dot{\mathscr{F}}|^2 w(q)\, d^3x + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{F}}^{\eta}\big| w(q)\, d^3x\, d\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^{\zeta} - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}\big| w(q)\, d^3x\, d\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^{\zeta} + L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^{\mu}{}_{\kappa}\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_{\lambda}{}^{\eta}\big| w'(q)\, d^3x\, d\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|(\nabla_\mu N_{\triangle}^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_{\triangle}^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big| w(q)\, d^3x\, d\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|L_\mu N_{\triangle}^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_{\triangle}^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big| w'(q)\, d^3x\, d\tau.
\end{aligned} \tag{12.2.1}$$

Proof. It follows from (8.3.2) that, if ε is sufficiently small, we have that

$$\tfrac{1}{4}|\dot{\mathscr{F}}|^2 w(q) \leq \dot{J}^0_{(h,\mathscr{F})} \leq |\dot{\mathscr{F}}|^2 w(q). \tag{12.2.2}$$

From (8.3.3) and the divergence theorem, it follows that

$$
\begin{aligned}
\int_{\Sigma_{t_2}} \dot{J}^0_{(h,\mathscr{F})}\, d^3x &+ \frac{1}{2}\int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)(|\dot{\alpha}|^2 + \dot{\rho}^2 + \dot{\sigma}^2)\, d^3x\, d\tau \\
&= \int_{\Sigma_{t_1}} \dot{J}^0_{(h,\mathscr{F})}\, d^3x - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\dot{\mathscr{F}}_{0\eta}\dot{\mathfrak{F}}^\eta\, d^3x\, d\tau \\
&\quad - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\big\{-(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - (\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big\}\, d^3x\, d\tau \\
&\quad - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)\big\{-L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta - L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} - \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big\}\, d^3x\, d\tau \\
&\quad - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w(q)\big\{(\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big\}\, d^3x\, d\tau \\
&\quad - \int_{t_1}^{t_2}\int_{\Sigma_\tau} w'(q)\big\{L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big\}\, d^3x\, d\tau, \qquad (12.2.3)
\end{aligned}
$$

which, with the help of (12.2.2), implies (12.2.1). □

We now recall the analogous lemma proved in [Lindblad and Rodnianski 2010] for solutions to the inhomogeneous wave equation in curved spacetime.

Lemma 12.2 (Weighted energy estimates for a scalar wave equation [Lindblad and Rodnianski 2010, Lemma 6.1]). *Assume that the scalar-valued function ϕ is a solution to the equation $\widetilde{\Box}_g\phi = \mathfrak{I}$, and let $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Assume that the metric $g_{\mu\nu}$ is such that $|H| \le \frac{1}{2}$. Then*

$$
\begin{aligned}
\int_{\Sigma_{t_2}} |\nabla\phi|^2 w(q)\, d^3x &+ 2\int_{t_1}^{t_2}\int_{\Sigma_\tau} |\overline{\nabla}\phi|^2 w'(q)\, d^3x\, d\tau \\
&\le 4\int_{\Sigma_{t_1}} |\nabla\phi|^2 w(q)\, d^3x + 4\int_{t_1}^{t_2}\int_{\Sigma_\tau} |\mathfrak{I}_\kappa\nabla_t\phi^\kappa|\, w(q)\, d^3x\, d\tau \\
&\quad + 4\int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|(\nabla_\nu H^{\nu\lambda})(\nabla_\lambda\phi)(\nabla_t\phi) - \tfrac{1}{2}(\nabla_t H^{\lambda\kappa})(\nabla_\lambda\phi)(\nabla_\kappa\phi)\big|\, w(q)\, d^3x\, d\tau \\
&\quad + 4\int_{t_1}^{t_2}\int_{\Sigma_\tau} \big|\underbrace{(\omega_j H^{j\lambda} - H^{0\lambda})}_{L_\kappa H^{\kappa\lambda}}(\nabla_t\phi)(\nabla_\lambda\phi) + \tfrac{1}{2}H^{\lambda\kappa}(\nabla_\lambda\phi)(\nabla_\kappa\phi)\big|\, w'(q)\, d^3x\, d\tau. \qquad (12.2.4)
\end{aligned}
$$

We now extend the results of the previous lemmas by estimating (under assumptions that are compatible with our global stability theorem) some of the cubic terms on the right-hand sides of (12.2.1) and (12.2.4).

Proposition 12.3 (Weighted energy estimates for the reduced equations; extension of [Lindblad and Rodnianski 2010, Proposition 6.2]). *Let ϕ be a solution to $\widetilde{\Box}_g\phi = \mathfrak{I}$ for the metric $g_{\mu\nu}$, and let $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$. Let γ and μ be positive constants satisfying the restrictions described* *in Section 2.14. Assume that the following pointwise estimates hold for* $(t,x)\in[0,T)\times\mathbb{R}^3$:

$$(1+|q|)^{-1}|H|_{LL}+|\nabla H|_{LL}+|\overline{\nabla}H|\le C\varepsilon(1+t+|q|)^{-1}, \tag{12.2.5a}$$

$$(1+|q|)^{-1}|H|+|\nabla H|\le C\varepsilon(1+t+|q|)^{-1/2}(1+|q|)^{-1/2}(1+q_-)^{-\mu}, \tag{12.2.5b}$$

where $q_-=0$ *if* $q\ge 0$ *and* $q_-=|q|$ *if* $q<0$. *Then there exists a constant* $C_1>0$ *such that, if* $0<\varepsilon\le\mu/C_1$, *then the following integral inequality holds for* $t\in[0,T)$:

$$\int_{\Sigma_t}|\nabla\phi|^2 w(q)\,d^3x+\int_0^t\int_{\Sigma_\tau}|\overline{\nabla}\phi|^2 w'(q)\,d^3x\,d\tau \lesssim \int_{\Sigma_0}|\nabla\phi|^2 w(q)\,d^3x+\int_0^t\int_{\Sigma_\tau}\Big(\frac{C\varepsilon|\nabla\phi|^2}{1+\tau}+|\mathfrak{I}||\nabla\phi|\Big)w(q)\,d^3x\,d\tau. \tag{12.2.6}$$

Furthermore, let $\dot{\mathscr{F}}_{\mu\nu}$ *be a solution to the electromagnetic equations of variation* (8.1.1a)–(8.1.1b) *corresponding to the background* $(h_{\mu\nu},\mathscr{F}_{\mu\nu})$, *where* $h_{\mu\nu}\stackrel{\text{def}}{=}g_{\mu\nu}-m_{\mu\nu}$. *Assume that the following pointwise estimates hold for* $(t,x)\in[0,T)\times\mathbb{R}^3$:

$$(1+|q|)^{-1}|h|_{\mathcal{L}\mathcal{L}}+|\nabla h|_{\mathcal{L}\mathcal{L}}+|\overline{\nabla}h|+|\mathscr{F}|\le C\varepsilon(1+t+|q|)^{-1}, \tag{12.2.7a}$$

$$(1+|q|)^{-1}|h|+|\nabla h|+|\nabla\mathscr{F}|\le C\varepsilon(1+t+|q|)^{-1/2}(1+|q|)^{-1/2}(1+q_-)^{-\mu}, \tag{12.2.7b}$$

where $q_-=0$ *if* $q\ge 0$ *and* $q_-=|q|$ *if* $q<0$. *Then there exists a constant* $C_1>0$ *such that, if* $0<\varepsilon\le\mu/C_1$, *then the following integral inequality holds for* $t\in[0,T)$:

$$\int_{\Sigma_t}|\dot{\mathscr{F}}|^2 w(q)\,d^3x+\int_0^t\int_{\Sigma_\tau}\big(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}}+|\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau \lesssim \int_{\Sigma_0}|\dot{\mathscr{F}}|^2 w(q)\,d^3x+\varepsilon\int_0^t\int_{\Sigma_\tau}\frac{|\dot{\mathscr{F}}|^2}{1+\tau}w(q)\,d^3x\,d\tau+\int_0^t\int_{\Sigma_\tau}|\dot{\mathscr{F}}_{0\kappa}\dot{\mathfrak{F}}^\kappa|w(q)\,d^3x\,d\tau. \tag{12.2.8}$$

Remark 12.4. Proposition 12.3 will not be used until the proof of Theorem 16.1, where it plays a key role; see Section 16.2. We also remark that the hypotheses of the proposition are implied by the hypotheses of the theorem; see Section 2.14 and Remark 16.2.

Proof. Inequality (12.2.6) was proved as Proposition 6.2 of [Lindblad and Rodnianski 2010]. The proof was based on using Lemma 11.11 to estimate the inhomogeneous terms on the right-hand side of (12.2.4). Rather than reproving this inequality, we only give the proof of (12.2.8), which is based on (12.2.1) and uses related ideas.

We commence with the proof of (12.2.8), our goal being to deduce suitable pointwise bounds for some of the terms appearing on the right-hand side of (12.2.1). For the cubic terms, we use Lemma 11.12, the hypotheses of the proposition, and the inequality $|ab|\lesssim a^2+b^2$ to conclude that

$$\begin{aligned}\big|(\nabla_\mu h^{\mu\kappa})\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^{\zeta}-(\nabla_\mu h^{\kappa\lambda})\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda}+\tfrac12(\nabla_t h^{\kappa\lambda})\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big| &\lesssim (|\nabla h|_{\mathcal{L}\mathcal{L}}+|\overline{\nabla}h|)|\dot{\mathscr{F}}|^2+|\nabla h||\dot{\mathscr{F}}|(|\dot{\mathscr{F}}|_{\mathcal{L}\mathcal{N}}+|\dot{\mathscr{F}}|_{\mathcal{T}\mathcal{T}})\\ &\lesssim \varepsilon(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2+\varepsilon(1+|q|)^{-1}(1+q_-)^{-2\mu}\big(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}}+|\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big)\end{aligned} \tag{12.2.9}$$

and

$$\big|L_\mu h^{\mu\kappa}\dot{\mathscr{F}}_{\kappa\zeta}\dot{\mathscr{F}}_0{}^\zeta + L_\mu h^{\kappa\lambda}\dot{\mathscr{F}}^\mu{}_\kappa\dot{\mathscr{F}}_{0\lambda} + \tfrac{1}{2}h^{\kappa\lambda}\dot{\mathscr{F}}_{\kappa\eta}\dot{\mathscr{F}}_\lambda{}^\eta\big| \lesssim |h|_{\mathcal{L}\mathcal{L}}|\dot{\mathscr{F}}|^2 + |h||\dot{\mathscr{F}}|(|\dot{\mathscr{F}}|_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|_{\mathcal{T}\mathcal{T}}) \lesssim \varepsilon(1+|q|)(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2 + \varepsilon(1+q_-)^{-2\mu}\big(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big). \quad (12.2.10)$$

For the higher-order terms, we use (3.7.2h), the hypotheses of the proposition, and the inequality $|ab| \lesssim a^2 + b^2$ to deduce that

$$\big|(\nabla_\mu N_\triangle^{\#\mu\zeta\kappa\lambda})\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} - \tfrac{1}{4}(\nabla_t N_\triangle^{\#\zeta\eta\kappa\lambda})\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big| \lesssim \big(|(h,\mathscr{F})||(\nabla h, \nabla\mathscr{F})|\big)|\dot{\mathscr{F}}|^2 \lesssim \varepsilon(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2 \quad (12.2.11)$$

and

$$\big|L_\mu N_\triangle^{\#\mu\zeta\kappa\lambda}\dot{\mathscr{F}}_{\kappa\lambda}\dot{\mathscr{F}}_{0\zeta} + \tfrac{1}{4}N_\triangle^{\#\zeta\eta\kappa\lambda}\dot{\mathscr{F}}_{\zeta\eta}\dot{\mathscr{F}}_{\kappa\lambda}\big| \lesssim |(h,\mathscr{F})|^2|\dot{\mathscr{F}}|^2 \lesssim \varepsilon(1+|q|)(1+t+|q|)^{-1}|\dot{\mathscr{F}}|^2. \quad (12.2.12)$$

Inserting (12.2.9)–(12.2.12) into the right-hand side of (12.2.1) and using (12.1.2), we have that

$$\int_{\Sigma_t}|\dot{\mathscr{F}}|^2 w(q)\,d^3x + \int_0^t\int_{\Sigma_\tau}\big(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau \le C\int_{\Sigma_0}|\dot{\mathscr{F}}|^2 w(q)\,d^3x + C_1\varepsilon\int_0^t\int_{\Sigma_\tau}\Big(\frac{|\dot{\mathscr{F}}|^2}{1+\tau}w(q) + \big(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}}\big)\frac{w'(q)}{\mu}\Big)\,d^3x\,d\tau + C\int_0^t\int_{\Sigma_\tau}|\dot{\mathscr{F}}_{0\kappa}\dot{\mathfrak{F}}^\kappa|w(q)\,d^3x\,d\tau. \quad (12.2.13)$$

Now if $C_1\varepsilon/\mu$ is sufficiently small, we can absorb the $C_1\varepsilon\int_0^t\int_{\Sigma_\tau}[(|\dot{\mathscr{F}}|^2_{\mathcal{L}\mathcal{N}} + |\dot{\mathscr{F}}|^2_{\mathcal{T}\mathcal{T}})w'(q)/\mu]\,d^3x\,d\tau$ term on the right-hand side of (12.2.13) into the second term on the left-hand side at the expense of increasing the constants C. Inequality (12.2.8) thus follows. □

13. Pointwise decay estimates for wave equations in a curved spacetime

In this section, we state a lemma and a corollary proved in [Lindblad and Rodnianski 2010]. They allow one to deduce pointwise decay estimates for solutions to inhomogeneous wave equations (e.g., for the $h_{\mu\nu}$). The main advantage of these estimates is that, if one has good control over the inhomogeneous terms, then the pointwise decay estimates provided by the lemma and its corollary are *improvements over what can be deduced from the weighted Klainerman–Sobolev inequalities* of Proposition B.1. In particular, the lemma and its corollary play a fundamental role in the proofs of Propositions 15.6 and 15.7. See the beginning of Section 15 for additional details regarding this improvement.

Remark 13.1. The Faraday tensor analogs of Lemma 13.2 and Corollary 13.3 are contained in the estimates of Proposition 11.5. More specifically, the analogous inequalities would arise from integrating (in the direction of the first-order vector field differential operators on the left-hand sides of the inequalities) the inequalities in the proposition. We will carry out these integrations in Section 15, which will allow us to derive improved pointwise decay estimates for the lower-order Lie derivatives of the Faraday tensor (improved relative to what can be deduced from the weighted Klainerman–Sobolev inequalities of Proposition B.1).

13.1. ***The decay estimate weight function $\varpi(q)$.*** As in [Lindblad and Rodnianski 2010], our decay estimates will involve the following weight function $\varpi(q)$, which is chosen to complement the energy-estimate weight function $w(q)$ defined in (12.1.1):

$$\varpi = \varpi(q) = \begin{cases} (1+|q|)^{1+\gamma'} & \text{if } q > 0, \\ (1+|q|)^{1/2-\mu'} & \text{if } q < 0, \end{cases} \tag{13.1.1}$$

where $0 < \delta < \mu' < \frac{1}{2} - \mu$ and $0 < \gamma' < \gamma - \delta$ are fixed constants. Its complementary role will become apparent in Section 15.

13.2. ***Pointwise decay estimates.*** We now state the lemma concerning pointwise decay estimates for solutions to inhomogeneous quasilinear wave equations.

Lemma 13.2 (Pointwise decay estimates for solutions to a scalar wave equation [Lindblad and Rodnianski 2010, Lemma 7.1]). *Let ϕ be a solution of the scalar wave equation* (13.2.1)

$$\widetilde{\Box}_g \phi = \mathfrak{I} \tag{13.2.1}$$

on a curved background with metric $g_{\mu\nu}$. Assume that the tensor $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ obeys the following estimates:

$$|H| \le \varepsilon', \qquad \int_0^\infty (1+t)^{-1} \|H(t,\cdot)\|_{L^\infty(D_t)}\, dt \le \tfrac{1}{4}, \quad \textit{and} \quad |H|_{\mathcal{L}\mathcal{T}} \le \varepsilon'(1+t+|x|)^{-1}(1+|q|) \tag{13.2.2}$$

in the region

$$D_t \stackrel{\text{def}}{=} \{x \mid t/2 < |x| < 2t\} \tag{13.2.3}$$

for $t \in [0, T)$. Then with $\alpha \stackrel{\text{def}}{=} \max(1+\gamma', \frac{1}{2} - \mu')$, the following pointwise estimate holds for $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$\begin{aligned}(1+t+|q|)\varpi(q)|\nabla\phi| \lesssim \sup_{0\le\tau\le t} \sum_{|I|\le 1} \|\varpi(q)\nabla_{\mathcal{Z}}^I \phi(\tau,\cdot)\|_{L^\infty} &+ \int_{\tau=0}^t \varepsilon' \alpha \|\varpi(q)\nabla\phi(\tau,\cdot)\|_{L^\infty}\, d\tau \\ &+ \int_{\tau=0}^t (1+\tau)\|\varpi(q)\mathfrak{I}(\tau,\cdot)\|_{L^\infty(D_\tau)}\, d\tau \\ &+ \int_{\tau=0}^t \sum_{|I|\le 2} (1+\tau)^{-1} \|\varpi(q)\nabla_{\mathcal{Z}}^I \phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\, d\tau. \end{aligned} \tag{13.2.4}$$

We now state the following corollary, which provides similar decay estimates for the null components of tensorial systems of wave equations:

Corollary 13.3 (Pointwise decay estimates for solutions to a system of tensorial wave equations [Lindblad and Rodnianski 2010, Corollary 7.2]). *Let $\phi_{\mu\nu}$ be a solution of the system*

$$\widetilde{\Box}_g \phi_{\mu\nu} = \mathfrak{I}_{\mu\nu} \tag{13.2.5}$$

on a curved background with a metric $g_{\mu\nu}$. *Assume that the tensor* $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ *obeys the following estimates*:

$$|H| \leq \frac{\varepsilon'}{4}, \qquad \int_0^\infty (1+t)^{-1} \|H(t,\cdot)\|_{L_\infty(D_t)}\, dt \leq \varepsilon', \quad \textit{and} \quad |H|_{\mathcal{L}\mathcal{T}} \leq \frac{\varepsilon'}{4}(1+t+|q|)^{-1}(1+|q|) \quad (13.2.6)$$

in the region

$$D_t \stackrel{\text{def}}{=} \{x \mid t/2 < |x| < 2t\} \quad (13.2.7)$$

for $t \in [0, T)$. *Then for any* $\mathcal{U}, \mathcal{V} \in \{\mathcal{L}, \mathcal{T}, \mathcal{N}\}$ *and with* $\alpha \stackrel{\text{def}}{=} \max(1+\gamma', \frac{1}{2} - \mu')$, *the following pointwise estimate holds for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$\begin{aligned}(1+t+|q|)\varpi(q)|\nabla\phi|_{\mathcal{U}\mathcal{V}} \lesssim \sup_{0\leq\tau\leq t} \sum_{|I|\leq 1} \|\varpi(q)\nabla_{\mathcal{Z}}^I \phi(\tau,\cdot)\|_{L^\infty} + \int_{\tau=0}^t \varepsilon'\alpha \|\varpi(q)|\nabla\phi(\tau,\cdot)|_{\mathcal{U}\mathcal{V}}\|_{L^\infty}\, d\tau \\ + \int_{\tau=0}^t (1+\tau)\|\varpi(q)|\mathfrak{I}(\tau,\cdot)|_{\mathcal{U}\mathcal{V}}\|_{L^\infty(D_\tau)}\, d\tau \\ + \sum_{|I|\leq 2} \int_{\tau=0}^t (1+\tau)^{-1} \|\varpi(q)\nabla_{\mathcal{Z}}^I \phi(\tau,\cdot)\|_{L^\infty(D_\tau)}\, d\tau. \quad (13.2.8)\end{aligned}$$

14. Local well-posedness and the continuation principle for the reduced equations

In this short section, we state for convenience a standard proposition concerning local well-posedness and a continuation principle for the reduced equations (3.7.1a)–(3.7.1c). The continuation principle shows that a suitable a priori bound on the energy of the solution implies global existence. It therefore plays a fundamental role in our global stability argument of Section 16.

Proposition 14.1 (Local well-posedness and the continuation principle). *Let* $(h^{(1)}_{\mu\nu}|_{t=0}, \partial_t h^{(1)}_{\mu\nu}|_{t=0}, \mathcal{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *be initial data for the reduced equations* (3.7.1a)–(3.7.1c) *constructed from abstract initial data* $(\mathring{\underline{h}}^{(1)}_{jk}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ *on the manifold* $\mathbb{R}^3$ *satisfying the constraints* (4.1.1a)–(4.1.2b) *as described in Section 4.2. Assume that the data are asymptotically flat in the sense of* (1.0.4a)–(1.0.4f). *Let* $\ell \geq 4$ *be an integer, and let* $\gamma > 0$ *and* $\mu > 0$ *be constants satisfying the restrictions stated in Section 2.14. Assume that* $E_{\ell;\gamma}(0) < \varepsilon$, *where* $E_{\ell;\gamma}(0)$ *is the norm of the abstract data defined in* (10.0.3). *Then if* ε *is sufficiently small*,[29] *these data launch a unique classical solution to the reduced equations existing on a nontrivial maximal spacetime slab* $[0, T_{\max}) \times \mathbb{R}^3$. *The energy* $\mathcal{E}_{\ell;\gamma;\mu}(t)$ *of the solution, which is defined in* (1.2.7), *satisfies* $\mathcal{E}_{\ell;\gamma;\mu}(0) \lesssim \varepsilon$ *and is continuous on* $[0, T_{\max})$. *Furthermore, either* $T_{\max} = \infty$, *or one of the following two "breakdown" scenarios must occur*:

(i) $\lim_{t\uparrow T_{\max}} \mathcal{E}_{\ell;\gamma;\mu}(t) = \infty$.

(ii) *The solution escapes the regime of hyperbolicity of the reduced equations.*

Remark 14.2. The classification of the two breakdown scenarios is known as a *continuation principle*.

[29] This smallness assumption ensures that the reduced data lie within the regime of hyperbolicity of the reduced equations.

Remark 14.3. Note that, in order to deduce global existence, Proposition 14.1 shows that it suffices to derive an a priori bound on $\mathscr{E}_{4;\gamma;\mu}(t)$ together with a bound ensuring that the solution remains in the regime of hyperbolicity. However, our methods do not allow us to derive an a priori bound for $\mathscr{E}_{4;\gamma;\mu}(t)$ alone; our derivation of upgraded pointwise estimates (see Section 15), which are essential for our derivation of an a priori energy estimate, requires that we work with $\mathscr{E}_{\ell;\gamma;\mu}(t)$ for $\ell \geq 10$.

The main ingredients in the proof of Proposition 14.1 are Lemmas 12.1 and 12.2, which provide weighted energy estimates for linearized versions of the reduced equations. Based on the availability of these estimates, the proof is rather standard, and we omit the details. Readers may consult, e.g., [Hörmander 1997, Chapter VI; Majda 1984, Chapter 2; Shatah and Struwe 1998, Chapter 5; Sogge 2008, Chapter 1; Speck 2009b; Taylor 1996, Chapter 16] for details concerning local existence and, e.g., [Hörmander 1997, Chapter VI; Sogge 2008, Chapter 1; Speck 2009a] for the ideas behind the continuation principle.

15. The fundamental energy bootstrap assumption and pointwise decay estimates for the reduced equations

In this section, we introduce our fundamental bootstrap assumption (15.0.1) for the energy of a solution to the reduced equations. Under this assumption, we derive a collection of pointwise decay estimates that will play a crucial role in the proof of Theorem 16.1. In particular, these decay estimates are used to deduce the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ in (16.2.10), which are essential for deriving the a priori energy bound (16.1.8). The decay estimates can be roughly divided into two classes: the weak pointwise decay estimates and the upgraded pointwise decay estimates. The weak decay estimates are consequences of the weighted Klainerman–Sobolev inequality (1.2.10). These estimates inherit a loss of approximately $(1+t)^{\delta}$ relative to what is needed to prove our main result. We remark that δ is a fixed small constant that is independent of the data while ε is connected to the size of the data. The loss comes from the loss we allow in our energy bootstrap assumption. Roughly speaking, if one tried to prove global stability using only the weak estimates, then the factors $(1+\tau)^{-1}$ and $(1+\tau)^{-1+C\varepsilon}$ in (16.2.10) would have to be replaced with $(1+\tau)^{-1+\delta}$; this loss of approximately $(1+t)^{\delta}$ would completely destroy the viability of our approach. The purpose of the upgraded pointwise decay estimates is precisely to eliminate some of this loss for the lower-order derivatives of the solution. The upgraded estimates are derived using the weak estimates and the special structure of the equations in wave coordinates; that is, many of the estimates we derive in this section rely upon the wave-coordinate condition.

We recall that the spacetime metric $g_{\mu\nu}$ is split into the pieces $g_{\mu\nu} = m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}$ and that the energy $\mathscr{E}_{\ell;\gamma;\mu}(t)$ (see (1.2.7)) is a functional of $(h^{(1)}, \mathscr{F})$. Our main bootstrap assumption for the energy is

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^{\delta}, \tag{15.0.1}$$

where $\ell \geq 10$ is an integer, $0 < \gamma < \frac{1}{2}$ is a fixed constant, δ is a fixed constant satisfying both $0 < \delta < \frac{1}{4}$ and $0 < \delta < \gamma$, $0 < \mu < \frac{1}{2}$ is a fixed constant (all of which will be chosen during the proof of Theorem 16.3), and ε is a small positive number whose required smallness is adjusted (as many times as necessary) during the derivation of our inequalities. With the help of (6.5.22), inequality (15.0.1) implies the following more explicit consequence of the energy bootstrap assumption:

$$\sum_{|I|\le \ell}\big(\|w^{1/2}(q)\nabla\nabla_{\mathfrak{Z}}^{I}h^{(1)}\|_{L^2}+\|w^{1/2}(q)\mathcal{L}_{\mathfrak{Z}}^{I}\mathcal{F}\|_{L^2}\big)\le C\varepsilon(1+t)^{\delta}. \tag{15.0.2}$$

In the remaining estimates in this article, *we will also often make the following smallness assumption on the ADM mass*:

$$M\le\varepsilon. \tag{15.0.3}$$

15.1. ***Preliminary (weak) pointwise decay estimates.*** In this section, we provide some preliminary pointwise decay estimates that are essentially a consequence of the weighted Klainerman–Sobolev inequalities of Appendix B. Unlike the upgraded pointwise decay estimates of the next section, these estimates do not take into account the special structure of the reduced equations under the wave-coordinate condition.

We begin with a simple lemma concerning pointwise decay estimates for the Schwarzschild tail of the metric and its derivatives.

Lemma 15.1 (Decay estimates for $h^{(0)}$). *Let $h^{(0)}$ be as in* (1.2.1c), *and let I be any ∇-multi-index. Then the following pointwise estimate holds for* $(t,x)\in[0,\infty)\times\mathbb{R}^3$:

$$|\nabla^{I}h^{(0)}|\le CM(1+t+|q|)^{-(1+|I|)}, \tag{15.1.1a}$$

where M is the ADM mass.

Furthermore, if I is any ∇-multi-index and J is any $\mathfrak{Z}$-multi-index, then the following pointwise estimate holds for $(t,x)\in[0,\infty)\times\mathbb{R}^3$:

$$|\nabla^{I}\nabla_{\mathfrak{Z}}^{J}h^{(0)}|+|\nabla_{\mathfrak{Z}}^{J}\nabla^{I}h^{(0)}|\le CM(1+t+|q|)^{-(1+|I|)}. \tag{15.1.1b}$$

Remark 15.2. Since $H_{(0)\mu\nu}=-h_{\mu\nu}^{(0)}$ (where $H_{(0)}^{\mu\nu}$ is defined in (11.1.2)), the above estimates also hold if we replace $h^{(0)}$ with $H_{(0)}$.

Proof. The lemma follows from simple computations, the definition (4.2.1) of the cut-off function χ, the definition of $h^{(0)}$, and the definitions of the vector fields $Z\in\mathfrak{Z}$. □

Corollary 15.3 (Weak pointwise decay estimates; slight extension of [Lindblad and Rodnianski 2010, Corollary 9.4]). *Let $\ell\ge 10$ be an integer. Assume that the abstract initial data are asymptotically flat in the sense of* (1.0.4a)–(1.0.4f), *that the ADM mass smallness condition* (15.0.3) *holds, that the constraints* (4.1.1a)–(4.1.2b) *are satisfied, and that the initial data for the reduced system are constructed from the abstract initial data as described in Section 4.2. Let* $(g_{\mu\nu}\overset{\text{def}}{=}m_{\mu\nu}+h_{\mu\nu}^{(0)}+h_{\mu\nu}^{(1)},\mathcal{F}_{\mu\nu})$ *be the corresponding solution to the reduced system* (3.7.1a)–(3.7.1c) *existing on a slab* $(t,x)\in[0,T)\times\mathbb{R}^3$, *where $h^{(1)}$ is defined in* (1.2.1b). *In particular, by Proposition 4.2, the wave-coordinate condition* (3.1.1a) *holds for* $(t,x)\in[0,T)\times\mathbb{R}^3$. *Assume in addition that the pair* $(h^{(1)},\mathcal{F})$ *satisfies the energy bootstrap assumption* (15.0.1) *on the interval* $[0,T)$. *Then if ε is sufficiently small, the following pointwise estimates hold for* $(t,x)\in[0,T)\times\mathbb{R}^3$:

$$|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}||\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}| \le \begin{cases} C\varepsilon(1+t+|q|)^{-1}(1+t)^{\delta}(1+|q|)^{-1-\gamma} & \text{if } q>0,\\ C\varepsilon(1+t+|q|)^{-1}(1+t)^{\delta}(1+|q|)^{-1/2} & \text{if } q<0 \end{cases} \quad (|I|\le \ell-3), \tag{15.1.2a}$$

$$|\nabla_{\mathcal{Z}}^I h^{(1)}| \le \begin{cases} C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{-\gamma} & \text{if } q>0,\\ C\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1/2} & \text{if } q<0 \end{cases} \quad (|I|\le \ell-3), \tag{15.1.2b}$$

$$|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}| + (1+|q|)|\overline{\nabla}\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}| \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\gamma} & \text{if } q>0,\\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2} & \text{if } q<0 \end{cases} \quad (|I|\le \ell-4). \tag{15.1.2c}$$

In addition, the tensor field $H_{(1)}^{\mu\nu}$ *defined in* (11.1.2) *satisfies the same estimates as* $h_{\mu\nu}^{(1)}$. *Furthermore, if we make the substitution* $\gamma \to \delta$ *in the above inequalities, then the same estimates hold for the tensor fields* $h_{\mu\nu}^{(0)}$, $h_{\mu\nu} \overset{\text{def}}{=} h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}$, $H_{(0)\mu\nu} \overset{\text{def}}{=} -h_{\mu\nu}^{(0)}$, $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, *and* $H_{(1)}^{\mu\nu} \overset{\text{def}}{=} H^{\mu\nu} - H_{(0)}^{\mu\nu}$.

Proof. This corollary is a slight extension of Corollary 9.4 of [Lindblad and Rodnianski 2010], in which estimates for $h^{(0)} = -H_{(0)}$, $h^{(1)}$, and h were proved. The main idea in the proof is to use the weighted Klainerman–Sobolev estimates of Proposition B.1 under the assumption (15.0.2) together with the decay (1.0.4c)–(1.0.4f) of the initial data at spatial infinity and Lemma 15.1. The estimates for $\mathcal{F}$ follow in a straightforward fashion from the arguments of [Lindblad and Rodnianski 2010, Corollary 9.4] while the estimates for $H_{(1)}$ and H follow from those for $h^{(1)}$ and h together with (3.3.11a). □

In the next lemma, we use the weak decay estimates to derive pointwise estimates for the Schwarzschild tail term $\nabla_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}$ appearing on the right-hand side of (7.0.1).

Lemma 15.4 (Pointwise decay estimates for $\nabla_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}$ [Lindblad and Rodnianski 2010, Lemma 9.9]). *Let* $h^{(0)}$ *be the Schwarzschild part of* h *as defined in* (1.2.1c), *and assume the hypotheses/conclusions of Corollary 15.3. Let* I *be a* $\mathcal{Z}$*-multi-index subject to the restrictions stated below. Then if* ε *is sufficiently small, the following pointwise estimates hold for* $(t,x)\in[0,T)\times\mathbb{R}^3$, *where* M *is the ADM mass*:

$$|\nabla_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}| \le \begin{cases} CM\varepsilon(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta} & \text{if } q>0,\\ CM(1+t+|q|)^{-3} & \text{if } q<0 \end{cases} \quad (|I|\le \ell-3). \tag{15.1.3a}$$

Furthermore, the following pointwise estimates also hold for $(t,x)\in[0,T)\times\mathbb{R}^3$:

$$|\nabla_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}| \le CM \sum_{|J|\le|I|} (1+t+|q|)^{-3}|\nabla_{\mathcal{Z}}^J h^{(1)}| + \begin{cases} CM\varepsilon(1+t+|q|)^{-4} & \text{if } q>0,\\ CM(1+t+|q|)^{-3} & \text{if } q<0 \end{cases} \quad (|I|\le \ell). \tag{15.1.3b}$$

Proof. We first observe that $\widetilde{\Box}_g h^{(0)} = \Box_m h^{(0)} + H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(0)}$, where $\Box_m \overset{\text{def}}{=} (m^{-1})^{\kappa\lambda}\nabla_\kappa\nabla_\lambda$ is the Minkowski wave operator. From (15.1.1b), the definition of $h^{(0)}$, the Leibniz rule, and the fact that $\Box_m(1/r)=0$ for $r>0$, it follows that

$$|\nabla_{\mathcal{Z}}^I \Box_m h^{(0)}| \lesssim M(1+t+|q|)^{-3}\chi_0\Big(\frac{1}{2}\le\frac{r}{t}\le\frac{3}{4}\Big), \tag{15.1.4}$$

$$\big|\nabla_{\mathcal{Z}}^I\big(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(0)}\big)\big| \lesssim M(1+t+|q|)^{-3}\sum_{|J|\le|I|}|\nabla_{\mathcal{Z}}^J H|, \tag{15.1.5}$$

where $\chi_0\big(\frac{1}{2} \le z \le \frac{3}{4}\big)$ is the characteristic function of the interval $\big[\frac{1}{2}, \frac{3}{4}\big]$. Furthermore, using $H = -h^{(0)} - h^{(1)} + O^{\infty}(|h^{(0)} + h^{(1)}|^2)$, we deduce that

$$\sum_{|J|\le|I|} |\nabla_{\mathcal{Z}}^{J} H| \lesssim \varepsilon(1+t+|q|)^{-1} + \sum_{|J|\le|I|} |\nabla_{\mathcal{Z}}^{J} h^{(1)}|. \tag{15.1.6}$$

Using (15.1.5), (15.1.6), and the estimate (15.1.2b), we have that

$$\big|\nabla_{\mathcal{Z}}^{I}\big(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(0)}\big)\big| \lesssim \begin{cases} M\varepsilon(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta} & \text{if } q>0, \\ M\varepsilon(1+t+|q|)^{-4+\delta}(1+|q|)^{1/2} & \text{if } q<0 \end{cases} \quad (|I| \le \ell - 3) \tag{15.1.7}$$

and

$$\big|\nabla_{\mathcal{Z}}^{I}\big(H^{\kappa\lambda}\nabla_\kappa\nabla_\lambda h^{(0)}\big)\big| \lesssim M\varepsilon(1+t+|q|)^{-4} + M\varepsilon(1+t+|q|)^{-3} \sum_{|J|\le|I|} |\nabla_{\mathcal{Z}}^{J} h^{(1)}| \quad (|I| \le \ell). \tag{15.1.8}$$

Inequalities (15.1.3a) and (15.1.3b) now easily follow from the above estimates. □

15.2. *Initial upgraded pointwise decay estimates for $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{L}\mathcal{N}}$ and $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{T}\mathcal{T}}$.* In this section, we prove some upgraded pointwise decay estimates for the "favorable" components of the lower-order Lie derivatives of $\mathcal{F}$. Our estimates take into account the special structure revealed by our null decomposition of the electromagnetic equations of variations, a structure that was captured by Proposition 11.5 and that depends in part upon the wave-coordinate condition. We remark that in Section 15.3 some of these decay estimates will be further improved (hence our use of the terminology "initial upgraded" here).

Proposition 15.5 (Initial upgraded pointwise decay estimates for $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{L}\mathcal{N}}$ and $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{T}\mathcal{T}}$). *Assume the hypotheses/conclusions of Corollary 15.3. Then if ε is sufficiently small, the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$:*

$$|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{T}\mathcal{T}} \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{-\gamma-\delta} & \text{if } q>0, \\ C\varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{1/2-\delta} & \text{if } q<0 \end{cases} \quad (|I| \le \ell - 4). \tag{15.2.1}$$

Proof. Since $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|_{\mathcal{T}\mathcal{T}}| \approx |\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]| + |\rho[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]| + |\sigma[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$, it suffices to prove the desired decay estimates for $|\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$, $|\rho[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$, and $|\sigma[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$ separately. We provide proof for the null component $\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]$. The proofs for the components $\rho[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]$ and $\sigma[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]$ are similar, and we leave these details to the reader. Let $\mathcal{W} \overset{\text{def}}{=} \{(t,x) \mid |x| \ge 1 + t/2\} \cap \{(t,x) \mid |x| \le 2t - 1\}$ denote the "wave-zone" region. Then for $(t,x) \notin \mathcal{W}$, we have that $1 + |q| \approx (1+t+|q|)$. Using this fact, for $(t,x) \notin \mathcal{W}$, we can bound $|\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$ by the right-hand side of (15.2.1) by using the weak decay estimate (15.1.2a).

We now consider the case $(t,x) \in \mathcal{W}$. Let $f \overset{\text{def}}{=} r\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]$. Then from (11.1.13b), the fact that $r \approx (1+t+|q|) \approx (1+s+|q|)$ on $\mathcal{W}$, and the weak decay estimates of Corollary 15.3, it follows that (with ∂_q defined in Section 2.7)

$$|\partial_q f(t,x)| \lesssim \begin{cases} \varepsilon(1+s+|q|)^{-1+2\delta}(1+|q|)^{-1-\gamma-\delta} & \text{if } q>0, \\ \varepsilon(1+s+|q|)^{-1+2\delta}(1+|q|)^{-1/2-\delta} & \text{if } q<0. \end{cases} \tag{15.2.2}$$

Let $(\tau(q'), y(q'))$ be the q'-parametrized line segment of constant s and angular values that begins at (t,x) and terminates at the point (t_0, x_0) lying to the *past* of (t,x) and on the boundary of $\mathcal{W}$. Let q and s be the null coordinates corresponding to (t,x). Then the null coordinates corresponding to (t_0, x_0)

are $q_0 = \frac{s}{3} - \frac{2}{3}$ and $s_0 = s$. Integrating the inequality (15.2.2) along this line segment (i.e., integrating dq'), we have that

$$
\begin{aligned}
|f(t,x)| &\lesssim |f(t_0,x_0)| + \int_{q'=q}^{q'=s/3-2/3} \left. \begin{cases} \varepsilon(1+s+|q'|)^{-1+2\delta}(1+|q'|)^{-1-\gamma-\delta} & \text{if } q' > 0, \\ \varepsilon(1+s+|q'|)^{-1+\delta}(1+|q'|)^{-1/2-\delta} & \text{if } q' < 0 \end{cases} \right\} dq' \\
&\lesssim |f(t_0,x_0)| + \begin{cases} \varepsilon(1+s)^{-1+2\delta}(1+|q|)^{-\gamma-\delta} & \text{if } q > 0, \\ \varepsilon(1+s)^{-1+\delta}(1+|q|)^{1/2-\delta} & \text{if } q < 0. \end{cases} \qquad (15.2.3)
\end{aligned}
$$

From the facts that $r_0 \approx 1+|q_0| \approx 1+t_0+|q_0| \approx 1+s_0+|q_0| \approx 1+s$, together with the weak decay estimate (15.1.2a), it follows that

$$|f(t_0,x_0)| \lesssim \varepsilon(1+s)^{-1-\gamma+\delta}. \qquad (15.2.4)$$

Combining (15.2.3) and (15.2.4), and using the fact that $1+s \approx 1+t+|q|$, we deduce that $|\alpha[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}(t,x)]|$ is bounded from above by the right-hand side of (15.2.1). This completes our proof of (15.2.1) for the $\alpha[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]$ component. □

15.3. ***Upgraded pointwise decay estimates for $|\nabla_{\mathcal{Z}}^I h|$ and $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|$ and fully upgraded pointwise decay estimates for $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}}$ and $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}}$.*** In this section, we state two propositions that strengthen some of the pointwise decay estimates proved in Sections 15.1 and 15.2. Their proofs, which are provided in Sections 15.4 and 15.5, are based on a careful analysis of the special structure of the reduced equations and in particular rely upon the decompositions performed in Section 11, which in turn rely in part upon the wave-coordinate condition. These estimates play a central role in our derivation of the "strong" a priori energy estimate (16.1.8), which is the main step in the proof of our stability theorem.

Proposition 15.6 (Upgraded pointwise decay estimates for $\mathcal{F}$ and certain components of h, ∇h, and $\nabla_Z h$; extension of [Lindblad and Rodnianski 2010, Proposition 10.1]). *Assume the hypotheses/conclusions of Corollary 15.3. In particular, by Proposition 4.2, the wave-coordinate condition* (3.1.1a) *holds for* $(t,x) \in [0,T) \times \mathbb{R}^3$. *Then if ε is sufficiently small, for every vector field $Z \in \mathcal{Z}$, the following pointwise estimates hold for* $(t,x) \in [0,T) \times \mathbb{R}^3$:

$$|\nabla h|_{\mathcal{L}\mathcal{T}} + |\nabla \nabla_Z h|_{\mathcal{L}\mathcal{L}} \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{-\delta} & \textit{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-2+\delta}(1+|q|)^{1/2} & \textit{if } q < 0, \end{cases} \qquad (15.3.1a)$$

$$|h|_{\mathcal{L}\mathcal{T}} + |\nabla_Z h|_{\mathcal{L}\mathcal{L}} \le \begin{cases} C\varepsilon(1+t+|q|)^{-1} & \textit{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta} & \textit{if } q < 0, \end{cases} \qquad (15.3.1b)$$

$$|\nabla h|_{\mathcal{T}\mathcal{N}} \le C\varepsilon(1+t+|q|)^{-1}, \qquad (15.3.2a)$$

$$|\nabla h| \le C\varepsilon(1+t+|q|)^{-1}(1+\ln(1+t)), \qquad (15.3.2b)$$

$$|\mathcal{F}| \le C\varepsilon(1+t+|q|)^{-1}. \qquad (15.3.3)$$

Furthermore, the same estimates hold for the tensor fields $h_{\mu\nu}^{(0)}$, $h_{\mu\nu}^{(1)}$, $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$, $H_{(0)}^{\mu\nu}$, *and* $H_{(1)}^{\mu\nu}$.

Proposition 15.7 (Upgraded pointwise decay estimates for the lower-order derivatives of h and $\mathcal{F}$; extension of [Lindblad and Rodnianski 2010, Proposition 10.2]). *Under the assumptions of Proposition 15.6,* *let* $0 < \gamma' < \gamma - \delta$ *and* $0 < \delta < \mu' < \frac{1}{2}$ *be fixed constants. Let* I *be any* $\mathscr{Z}$*-multi-index subject to the restrictions stated below. Then there exist constants* M_k *and* C_k *depending on* γ', μ', *and* δ *such that, if* ε *is sufficiently small, then the following pointwise estimates hold for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}| + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}| \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1-\gamma'} & \text{if } q > 0, \\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 5), \tag{15.3.4a}$$

$$|\nabla_{\mathscr{Z}}^I h^{(1)}| \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-\gamma'} & \text{if } q > 0, \\ C_k\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 5), \tag{15.3.4b}$$

$$|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}| + (1+|q|)|\overline{\nabla}\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}| + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{T}\mathcal{T}} \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-2+M_k\varepsilon}(1+|q|)^{-\gamma'} & \text{if } q > 0, \\ C_k\varepsilon(1+t+|q|)^{-2+M_k\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \quad (|I| = k \leq \ell - 6). \tag{15.3.4c}$$

Furthermore, the same estimates hold for $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ *and* $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$ *if we replace* γ' *with* $M_k\varepsilon$.

15.4. ***Proof of Proposition 15.6.*** We only prove the estimates for $h_{\mu\nu}$ and $\mathscr{F}_{\mu\nu}$. The estimates for $h_{\mu\nu}^{(0)}$, $h_{\mu\nu}^{(1)}$, $H^{\mu\nu}$, $H_{(0)}^{\mu\nu}$, and $H_{(1)}^{\mu\nu}$ follow easily from those for $h_{\mu\nu}$, (3.3.11a), and Lemma 15.1.

15.4.1. *Proofs of* (15.3.1a) *and* (15.3.1b). We will argue as in Lemma 10.4 of [Lindblad and Rodnianski 2010]; we first provide a lemma that establishes a more general version of the desired estimates.

Lemma 15.8 (Pointwise estimates for $|\nabla\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{L}}$, $|\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{L}}$, $|\nabla\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{T}}$, and $|\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{T}}$ [Lindblad and Rodnianski 2010, Lemma 10.4]). *Under the hypotheses of Proposition 15.6, if* $k \leq \ell - 4$ *and* ε *is sufficiently small, then the following pointwise estimates hold for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$\sum_{|I| \leq k} |\nabla\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla\nabla_{\mathscr{Z}}^J h|_{\mathcal{L}\mathcal{T}}}_{\text{absent if } k=0} \lesssim \underbrace{\sum_{|K| \leq k-2} |\nabla\nabla_{\mathscr{Z}}^K h|}_{\text{absent if } k \leq 1} + \begin{cases} \varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{-2\delta} & \text{if } q > 0, \\ \varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{1/2-\delta} & \text{if } q < 0, \end{cases} \tag{15.4.1}$$

$$\sum_{|I| \leq k} |\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla_{\mathscr{Z}}^J h|_{\mathcal{L}\mathcal{T}}}_{\text{absent if } k=0} \lesssim \underbrace{\sum_{|K| \leq k-2} \int_{\varrho=|x|}^{\varrho=|x|+t} |\nabla\nabla_{\mathscr{Z}}^K h|(t+|q|-\varrho,\ \varrho x/|x|)\, d\varrho}_{\text{absent if } k \leq 1} + \begin{cases} \varepsilon(1+t+|q|)^{-1} & \text{if } q > 0, \\ \varepsilon(1+t+|q|)^{-1}(1+|q|)^{1/2+\delta} & \text{if } q < 0. \end{cases} \tag{15.4.2}$$

Furthermore, the same estimates hold for the tensor $H^{\mu\nu} \overset{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$.

Proof. By Proposition 11.1, we have that

$$\sum_{|I| \leq k} |\nabla\nabla_{\mathscr{Z}}^I h|_{\mathcal{L}\mathcal{L}} + \underbrace{\sum_{|J| \leq k-1} |\nabla\nabla_{\mathscr{Z}}^J h|_{\mathcal{L}\mathcal{T}}}_{\text{absent if } k=0} \lesssim \underbrace{\sum_{|K| \leq k-2} |\nabla\nabla_{\mathscr{Z}}^J h|}_{\text{absent if } k \leq 1} + \sum_{|J| \leq k} |\overline{\nabla}\nabla_{\mathscr{Z}}^J h| + \sum_{|I_1|+|I_2| \leq k} |\nabla_{\mathscr{Z}}^{I_1} h||\nabla\nabla_{\mathscr{Z}}^{I_2} h|. \tag{15.4.3}$$

By Corollary 15.3, we have that

$$\sum_{|J|\le k}|\overline{\nabla}\nabla_{\mathcal{Z}}^{J}h|+\sum_{|I_1|+|I_2|\le k}|\nabla_{\mathcal{Z}}^{I_1}h||\nabla\nabla_{\mathcal{Z}}^{I_2}h| \lesssim \begin{cases}\varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{-2\delta} & \text{if } q>0,\\ \varepsilon(1+t+|q|)^{-2+2\delta}(1+|q|)^{1/2-\delta} & \text{if } q<0\end{cases} \qquad (k\le \ell-4). \tag{15.4.4}$$

Combining (15.4.3) and (15.4.4), we deduce (15.4.1). Inequality (15.4.2) follows from integrating inequality (15.4.1) for $|\partial_q\nabla_{\mathcal{Z}}^{I}h|\lesssim|\nabla\nabla_{\mathcal{Z}}^{I}h|$, $q\overset{\text{def}}{=}|x|-t$, along the lines along which the angle $\omega\overset{\text{def}}{=}x/|x|$ and the null coordinate $s=|x|+t$ are constant (i.e., integrating dq) and using (15.1.2b) at $t=0$.

The proofs of the estimates for $H^{\mu\nu}$ follow from the estimates for $h_{\mu\nu}$, (3.3.11a), and Corollary 15.3. This concludes our proof of the lemma. □

Having proved the lemma, inequalities (15.3.1a) and (15.3.1b) now follow from inequalities (15.4.1) and (15.4.2) and the weak decay estimates of Corollary 15.3.

15.4.2. *Proof of* (15.3.3). Let $\mathcal{W}\overset{\text{def}}{=}\{(t,x)\mid |x|\ge 1+t/2\}\cap\{(t,x)\mid |x|\le 2t-1\}$ denote the "wave-zone" region. Note that $r\approx 1+t+|q|\approx 1+t+s$ for $(t,x)\in\mathcal{W}$. Now as in the proof of Proposition 15.5, inequality (15.3.3) follows from the weak decay estimates of Corollary 15.3 if $(t,x)\notin\mathcal{W}$. Furthermore, we have that $|\mathcal{F}|\approx|\underline{\alpha}[\mathcal{F}]|+|\alpha[\mathcal{F}]|+|\rho[\mathcal{F}]|+|\sigma[\mathcal{F}]|$, and by Proposition 15.5, inequality (15.3.3) has already been shown to hold for $|\alpha[\mathcal{F}]|+|\rho[\mathcal{F}]|+|\sigma[\mathcal{F}]|\approx|\mathcal{F}|_{\mathcal{L}\mathcal{N}}+|\mathcal{F}|_{\mathcal{T}\mathcal{T}}$.

It remains to prove the desired estimate for $|\underline{\alpha}[\mathcal{F}(t,x)]|$ under the assumption that $(t,x)\in\mathcal{W}$. To this end, we use (11.1.12), the weak decay estimates of Corollary 15.3, and Proposition 15.5 to deduce that if $(t,x)\in\mathcal{W}$ then

$$\left|\nabla_\Lambda(r\underline{\alpha}[\mathcal{F}])\right|\lesssim\varepsilon(1+t+|q|)^{-3/2+\delta}\left|r\underline{\alpha}[\mathcal{F}]\right|+\varepsilon(1+t+|q|)^{-2+3\delta}, \tag{15.4.5}$$

where $\Lambda\overset{\text{def}}{=}L+\frac{1}{4}h_{LL}\underline{L}$. Let $(\tau(\lambda),\,y(\lambda))$ be the integral curve[30] of the vector field Λ passing through the point $(t,x)=(\tau(\lambda_1),\,y(\lambda_1))\in\mathcal{W}$. By the already-proved smallness estimate (15.3.1b) for h_{LL}, every such integral curve must intersect the boundary of $\mathcal{W}$ at a point $(t_0,x_0)=(\tau(\lambda_0),\,y(\lambda_0))$ to the *past* of (t,x). Furthermore, by (15.3.1b) again, we have that $\frac{d\tau}{d\lambda}\approx 1$ along the integral curves, and for all $(\tau,y)\in\mathcal{W}$, we have that $|y|\approx\tau\approx 1+|\tau|\approx 1+|\tau|+||y|-\tau|$. We now set $f(\lambda)\overset{\text{def}}{=}\big||y(\lambda)|\underline{\alpha}[\mathcal{F}(\tau(\lambda),\,y(\lambda))]\big|$, integrate inequality (15.4.5) along the integral curve (which is contained in $\mathcal{W}$), use the assumption $0<\delta<\frac{1}{4}$, and change variables so that τ is the integration variable to obtain

$$\begin{aligned}\overbrace{\left|r\underline{\alpha}[\mathcal{F}](t,x)\right|}^{f(\lambda(t))}&\le\overbrace{\left|r_0\underline{\alpha}[\mathcal{F}(t_0,x_0)]\right|}^{f(\lambda_0)}+C\varepsilon\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}[1+\tau(\lambda)]^{-2+3\delta}\,d\lambda+C\varepsilon\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}[1+\tau(\lambda)]^{-3/2+\delta}f(\lambda)\,d\lambda\\&\le C\varepsilon+C\varepsilon\int_{\tau=t_0}^{\tau=t}(1+\tau)^{-2+3\delta}\,d\tau+C\varepsilon\int_{\tau=t_0}^{\tau=t}(1+\tau)^{-3/2+\delta}f(\lambda\circ\tau)\,d\tau\\&\le C\varepsilon+C\varepsilon\int_{\tau=t_0}^{\tau=t}(1+\tau)^{-3/2+\delta}f(\lambda\circ\tau)\,d\tau,\end{aligned} \tag{15.4.6}$$

[30]By integral curve, we mean the solution to the ODE system $\frac{d\tau}{d\lambda}=\Lambda^0(\tau,y)$ and $\frac{dy^j}{d\lambda}=\Lambda^j(\tau,y)$ $(j=1,2,3)$ passing through the point (t,x) at parameter value $\lambda=\lambda_1$.

where we have used (15.1.2a) to obtain the bound $|r_0\underline{\alpha}[\mathscr{F}(t_0, x_0)]| \le C\varepsilon$ for the point (t_0, x_0) lying on the boundary of $\mathscr{W}$. Applying Gronwall's lemma to (15.4.6), we deduce that

$$\left|r\underline{\alpha}[\mathscr{F}(t,x)]\right| \le C\varepsilon \exp\left(C\varepsilon \int_{\tau=t_0}^{\tau=t} (1+\tau)^{-3/2+\delta}\, d\tau\right) \le C\varepsilon, \tag{15.4.7}$$

from which it trivially follows that

$$\left|\underline{\alpha}[\mathscr{F}(t,x)]\right| \le C\varepsilon r^{-1} \le C\varepsilon(1+t+|q|)^{-1} \tag{15.4.8}$$

as desired.

15.4.3. *Proofs of* (15.3.2a) *and* (15.3.2b). In the next two lemmas, we will use the fact that the tensor field $h_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} - m_{\mu\nu}$ is a solution to the system

$$\widetilde{\square}_g h_{\mu\nu} = \mathfrak{H}_{\mu\nu}, \tag{15.4.9}$$

where the inhomogeneous term $\mathfrak{H}_{\mu\nu}$ is defined in (3.7.2a).

Lemma 15.9 (Pointwise estimates for the $\mathfrak{H}_{\mu\nu}$ inhomogeneities; extension of [Lindblad and Rodnianski 2010, Lemma 10.5]). *Suppose that the assumptions of Proposition 15.6 hold. Then if* ε *is sufficiently small*, *the following pointwise estimates hold for* $(t,x) \in [0,T) \times \mathbb{R}^3$:

$$|\mathfrak{H}|_{\mathcal{T}\mathcal{N}} \le C\varepsilon(1+t+|q|)^{-3/2+\delta}|\nabla h| + C\varepsilon(1+t+|q|)^{-5/2+\delta}, \tag{15.4.10a}$$

$$|\mathfrak{H}| \le C\varepsilon(1+t+|q|)^{-3/2+\delta}|\nabla h| + C|\nabla h|^2_{\mathcal{T}\mathcal{N}} + C\varepsilon^2(1+t+|q|)^{-2}. \tag{15.4.10b}$$

Proof. Lemma 15.9 follows from Proposition 11.3, Corollary 15.3, Proposition 15.5, the *already-proved* estimate (15.3.3), and the assumption $0 < \delta < \frac{1}{4}$. □

Lemma 15.10 (Integral inequalities for $|\nabla h|_{\mathcal{T}\mathcal{N}}$ and $|\nabla h|$; extension of [Lindblad and Rodnianski 2010, Lemma 10.6]). *Suppose that the assumptions of Proposition 15.6 hold. Then if* ε *is sufficiently small*, *the following integral inequalities hold for* $t \in [0,T)$:

$$(1+t)\||\nabla h|_{\mathcal{T}\mathcal{N}}(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon \int_0^t (1+\tau)^{-1/2+\delta}\|\nabla h(\tau,\cdot)\|_{L^\infty}\, d\tau, \tag{15.4.11a}$$

$$(1+t)\|\nabla h(t,\cdot)\|_{L^\infty} \le C\varepsilon + C\varepsilon^2 \ln(1+t) + C\varepsilon \int_0^t (1+\tau)^{-1/2+\delta}\|\nabla h(\tau,\cdot)\|_{L^\infty}\, d\tau + C\varepsilon \int_0^t (1+\tau)\||\nabla h|^2_{\mathcal{T}\mathcal{N}}(\tau,\cdot)\|_{L^\infty}\, d\tau. \tag{15.4.11b}$$

Proof. We first observe that (15.1.2b) and (15.3.1b) (the version for the tensor H) imply that the hypotheses of Lemma 13.2 and Corollary 13.3 hold. Therefore, using the lemma and the corollary with $\varpi(q) \overset{\text{def}}{=} 1$ and $\alpha \overset{\text{def}}{=} 0$, and noting that $h_{\mu\nu}$ satisfies the system (15.4.9), we have that

$$(1+t)|\nabla h|_{\mathcal{T}\mathcal{N}} \lesssim \sup_{0\le\tau\le t} \sum_{|I|\le 1} \|\nabla_{\mathcal{Z}}^I h(\tau,\cdot)\|_{L^\infty} + \int_{\tau=0}^t (1+\tau)\||\mathfrak{H}|_{\mathcal{T}\mathcal{N}}\|_{L^\infty(D_\tau)}\, d\tau + \sum_{|I|\le 2} \int_{\tau=0}^t (1+\tau)^{-1}\|\nabla_{\mathcal{Z}}^I h\|_{L^\infty(D_\tau)}\, d\tau. \tag{15.4.12}$$

Using (15.1.2b) (the version for the tensor h), we estimate the first and third terms on the right-hand side of (15.4.12) as follows:

$$\sup_{0\le\tau\le t}\sum_{|I|\le 1}\|\nabla_{\mathscr{Z}}^I h(\tau,\cdot)\|_{L^\infty}\le C\varepsilon, \tag{15.4.13}$$

$$\sum_{|I|\le 2}\int_{\tau=0}^{t}(1+\tau)^{-1}\|\nabla_{\mathscr{Z}}^I h\|_{L^\infty(D_\tau)}\,d\tau\le C\varepsilon\int_{\tau=0}^{\infty}(1+\tau)^{-3/2+\delta}\,d\tau\le C\varepsilon. \tag{15.4.14}$$

To estimate the second term, we use (15.4.10a) to conclude that for $x\in D_t$ we have that

$$(1+t)|\mathfrak{H}|_{\mathcal{T}\mathcal{N}}\le C\varepsilon(1+t)^{-1/2+\delta}|\nabla h|+C\varepsilon(1+t)^{-3/2+\delta}. \tag{15.4.15}$$

Inequality (15.4.11a) now follows from (15.4.12)–(15.4.15) and the fact that $C\varepsilon\int_0^t(1+\tau)^{-3/2+\delta}\,d\tau\le C\varepsilon$. Inequality (15.4.11b) can be obtained in a similar fashion by using (15.4.10b). □

To finish the proof of Proposition 15.6, we will apply the following Gronwall-type inequality:

Lemma 15.11 (Gronwall-type inequality; slight modification of [Lindblad and Rodnianski 2010, Lemma 10.7]). *Assume that the continuous functions $b(t)\ge 0$ and $c(t)\ge 0$ satisfy*

$$b(t)\le C\varepsilon+C\varepsilon\int_0^t(1+\tau)^{-1-a}c(\tau)\,d\tau, \tag{15.4.16a}$$

$$c(t)\le C\varepsilon\;+\;C\varepsilon^2\ln(1+t)+C\varepsilon\int_0^t(1+\tau)^{-1-a}c(\tau)\,d\tau+C\int_0^t(1+\tau)^{-1}b^2(\tau)\,d\tau \tag{15.4.16b}$$

for some positive constants a and C such that $\varepsilon<a/4C$ and $\varepsilon<2a/(1+4C^2)$. Then

$$b(t)\le 2C\varepsilon, \tag{15.4.17a}$$

$$c(t)\le 2C\varepsilon(1+a\ln(1+t)). \tag{15.4.17b}$$

Proof. We slightly modify the proof of [Lindblad and Rodnianski 2010, Lemma 10.7]. Let T be the largest time such that the bounds (15.4.17a)–(15.4.17b) hold. Then inserting these bounds into the inequalities (15.4.16a)–(15.4.16b) and using the bound (and the change of variables $z\stackrel{\text{def}}{=}a\ln(1+\tau)$)

$$\int_{\tau=0}^{\infty}(1+\tau)^{-1-a}(1+a\ln(1+\tau))\,d\tau=a^{-1}\int_{z=0}^{\infty}e^{-z}(1+z)\,dz=2a^{-1}, \tag{15.4.18}$$

we deduce that the following inequalities hold for $t\in[0,T]$:

$$b(t)\le C\varepsilon(1+4C\varepsilon a^{-1})<2C\varepsilon, \tag{15.4.19}$$

$$c(t)\le C\varepsilon(1+4C\varepsilon a^{-1}+(1+4C^2)\varepsilon\ln(1+t))<2C\varepsilon(1+a\ln(1+t)). \tag{15.4.20}$$

Since the above inequalities are a strict improvement of the assumed bounds (15.4.17a)–(15.4.17b), we thus conclude that $T=\infty$. □

To complete the proofs of (15.3.2a) and (15.3.2b), we apply Lemmas 15.10 and 15.11 with $b(t)\stackrel{\text{def}}{=}(1+t)\||\nabla h|_{\mathcal{T}\mathcal{N}}(t,\cdot)\|_{L^\infty}$ and $c(t)\stackrel{\text{def}}{=}(1+t)\|\nabla h(t,\cdot)\|_{L^\infty}$. This implies (15.3.2a) and (15.3.2b) with

$(1+t)$ in place of $(1+t+|q|)$. The additional decay in $|q|$ in (15.3.2a) and (15.3.2b) follows directly from (15.1.2a) (the version for the tensor h). $\square$

15.5. *Proof of Proposition 15.7.* We will prove the proposition using a series of inductive steps. We only prove the estimates for $h^{(1)}_{\mu\nu}$ and $\mathscr{F}_{\mu\nu}$. The estimates for $h_{\mu\nu}$ and $H^{\mu\nu}$ follow easily from those for $h^{(1)}_{\mu\nu}$, (3.3.11a), and Lemma 15.1. We first prove a technical lemma that will be used during the proof of the proposition.

Lemma 15.12 (Pointwise estimates for the $|\nabla^I_{\mathscr{Z}}\mathfrak{H}|$ inhomogeneities). *Suppose that the hypotheses of Proposition 15.6 hold, and let $\mathfrak{H}_{\mu\nu}$ be the inhomogeneous term on the right-hand side of the reduced equation* (3.7.1a). *Then if I is any $\mathscr{Z}$-multi-index with $|I| \leq \ell$, the following pointwise estimates hold for $(t,x) \in [0,T) \times \mathbb{R}^3$:*

$$|\nabla^I_{\mathscr{Z}}\mathfrak{H}| \leq C\varepsilon \sum_{|J|\leq|I|} (1+t+|q|)^{-1}\big(|\nabla\nabla^J_{\mathscr{Z}} h^{(1)}| + |\nabla^J_{\mathscr{Z}}\mathscr{F}|\big)$$
$$+ C \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |I_1|,|I_2|\leq|I|-1}} \big(|\nabla\nabla^{I_1}_{\mathscr{Z}} h^{(1)}| + |\mathscr{L}^{I_1}_{\mathscr{Z}}\mathscr{F}|\big)\big(|\nabla\nabla^{I_2}_{\mathscr{Z}} h^{(1)}| + |\mathscr{L}^{I_2}_{\mathscr{Z}}\mathscr{F}|\big) + C\varepsilon^2(1+t+|q|)^{-4}. \quad (15.5.1)$$

Proof. Lemma 15.12 follows from (11.1.5c), Lemma 15.1, the weak decay estimates of Corollary 15.3, (15.3.2a), (15.3.3), and the assumption that $0 < \delta < \frac{1}{4}$. We remark that the $C\varepsilon^2(1+t+|q|)^{-4}$ term arises from the estimate $|\nabla\nabla^{I_1}_{\mathscr{Z}} h^{(0)}||\nabla\nabla^{I_2}_{\mathscr{Z}} h^{(0)}| \leq C\varepsilon^2(1+t+|q|)^{-4}$. $\square$

We are now ready for the proof of Proposition 15.7. To prove (15.3.4a)–(15.3.4c), we will argue inductively, using the inequalities in the case $|I| \leq k$ to deduce that they hold in the case $|I| = k+1$. We also remark that the base case $k = 0$ is covered by our argument.

Induction Step 1: Upgraded pointwise decay estimates for $|\nabla^I_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{L}}$ for $|I| = k+1$ and $|\nabla^J_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{T}}$ for $|J| = k$. As a first step, we will use the wave-coordinate condition to upgrade the estimates for $|\nabla^I_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{L}}$ for $|I| = k+1$ and $|\nabla^J_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{T}}$ for $|J| = k$. To this end, we appeal to inequality (15.4.2), using inequality (15.3.4a) for h under the induction hypothesis to bound the integrand and thereby concluding that

$$\sum_{|I|=k+1} |\nabla^I_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{L}} + \sum_{|J|=k} |\nabla^J_{\mathscr{Z}} h|_{\mathcal{L}\mathcal{T}} \lesssim \begin{cases} \varepsilon(1+t+|q|)^{-1+M_{k-1}\varepsilon}(1+|q|)^{-M_{k-1}\varepsilon} & \text{if } q > 0, \\ \varepsilon(1+t+|q|)^{-1+M_{k-1}\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q < 0. \end{cases} \quad (15.5.2)$$

In the above estimates, the constant μ' is subject to the restrictions stated in the hypotheses of Proposition 15.7. Furthermore, since $H^{\mu\nu} = -h^{\mu\nu} + O^\infty(|h|^2)$, (15.1.2b) implies that the same estimates hold for the tensor H.

Induction Step 2: Upgraded pointwise decay estimates for $|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|$ and $|I| = k+1$. Let $\mathscr{W} \overset{\text{def}}{=} \{(t,x) \mid |x| \geq 1+t/2\} \cap \{(t,x) \mid |x| \leq 2t-1\}$ denote the "wave-zone" region. Then for $(t,x) \notin \mathscr{W}$, we have that $1+|q| \approx 1+t+|q|$. Using this fact, we see that for $(t,x) \notin \mathscr{W}$ the weak decay estimate (15.1.2a) implies that inequality (15.3.4a) holds for $|\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}|$ in the case $|I| = k+1$. Furthermore, by Proposition 15.5, the inequality (15.3.4a) holds for the null components $|\alpha[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]|$, $|\rho[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]|$, and $|\sigma[\mathscr{L}^I_{\mathscr{Z}}\mathscr{F}]|$ when $|I| = k+1$.

It remains to consider $|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}(t,x)]|$ in the case $(t,x)\in\mathcal{W}$. Note that $r\approx 1+t+|q|\approx 1+t+s$ for $(t,x)\in\mathcal{W}$. We will make use of the weight $\varpi(q)$ defined in (13.1.1). From (11.1.13a), Corollary 15.3 (the version for the tensor field h), Proposition 15.5, (15.3.1b), (15.3.3), the induction hypothesis, and (15.5.2), it follows that

$$\sum_{|I|\leq k+1}\left|\nabla_{\Lambda}\big(r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]\big)\right|\leq C\varepsilon(1+t+|q|)^{-1}\sum_{|I|\leq k+1}\left|r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]\right| + C\varepsilon(1+t+|q|)^{-(1+a)}+C\varepsilon^{2}(1+t+|q|)^{-1+C\varepsilon}, \quad (15.5.3)$$

where $0<a<\min\{\mu'-\delta,\gamma-\delta-\gamma'\}$ is a fixed constant and $\Lambda\stackrel{\text{def}}{=}L+\frac{1}{4}h_{LL}\underline{L}$. Note the importance of the independent estimate (15.3.1b) for bounding the second, fourth, and fifth sums on the right-hand side of (11.1.13a) and of the independent estimate (15.5.2) (in the case $|I|=k+1$) for bounding the third sum on the right-hand side of (11.1.13a).

Let $(\tau(\lambda),y(\lambda))$ be the integral curve (as defined in Section 15.4.2) of the vector field Λ passing through the point $(t,x)=(\tau(\lambda_1),y(\lambda_1))\in\mathcal{W}$. By the inequality (15.3.1b) for h_{LL}, every such integral curve must intersect the boundary of $\mathcal{W}$ at a point $(t_0,x_0)=(\tau(\lambda_0),y(\lambda_0))$ lying to the *past* of (t,x). Using (15.3.1b) again, we have that $\frac{dt}{d\lambda}\approx 1$ along the integral curves, and in the entire region $\mathcal{W}$, we have that $|y|\approx\tau\approx 1+|\tau|\approx 1+|\tau|+||y|-\tau|$. We define $f(\lambda)\stackrel{\text{def}}{=}\sum_{|I|\leq k+1}\big||y(\lambda)|\varpi(q(\lambda))\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}(\tau(\lambda),y(\lambda))]\big|$, where $q(\lambda)\stackrel{\text{def}}{=}|y(\lambda)|-\tau(\lambda)$. Note that $f(\lambda_1)=\sum_{|I|\leq k+1}|r\varpi(q)\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$, where $q\stackrel{\text{def}}{=}q(\lambda_1)=|x|-t$ while the weak decay estimate (15.1.2a) implies that $f(\lambda_0)\leq C\varepsilon$. Integrating inequality (15.5.3) and changing variables so that τ is the integration variable, we have that

$$\begin{aligned}\underbrace{f(\lambda_1)}_{f(\lambda(t))}&\leq f(\lambda_0)+C\varepsilon\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}[1+\tau(\lambda)]^{-1}f(\lambda)\,d\lambda\\&\quad+C\varepsilon\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}[1+\tau(\lambda)]^{-(1+a)}\,d\lambda+C\varepsilon^{2}\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}[1+\tau(\lambda)]^{-1+C\varepsilon}\,d\lambda\\&\leq C\varepsilon(1+t)^{C\varepsilon}+C\varepsilon\int_{\tau=t_0}^{\tau=t}(1+\tau)^{-1}f(\lambda\circ\tau)\,d\tau.\end{aligned}\quad (15.5.4)$$

Applying Gronwall's inequality to (15.5.4), we have that

$$\begin{aligned}f(\underbrace{\lambda\circ t}_{\lambda_1})&\leq C\varepsilon(1+t)^{C\varepsilon}\exp\bigg(C\varepsilon\int_{\tau=t_0}^{\tau=t}(1+\tau)^{-1}\,d\tau\bigg)\\&\leq C\varepsilon(1+t)^{2C\varepsilon},\end{aligned}\quad (15.5.5)$$

from which it easily follows that for $(t,x)\in\mathcal{W}$ we have that

$$\sum_{|I|\leq k+1}\left|\underline{\alpha}[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]\right|\leq C\varepsilon(1+t)^{-1+2C\varepsilon}\varpi^{-1}(q). \quad (15.5.6)$$

Combining (15.5.6) and the previous arguments covering $(t,x)\notin\mathcal{W}$ and the other null components of $\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}$, we have shown that the estimate (15.3.4a) holds for $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|$ in the case $|I|=k+1$.

Final Induction Step: Upgraded pointwise decay estimates for $|\nabla\nabla_{\mathscr{Z}}^I h|$ *and* $|\nabla_{\mathscr{Z}}^I h|$ $(|I| = k+1)$. Our first goal is to prove the following estimate in the case $|I| = k+1$:

$$|\widetilde{\square}_g \nabla_{\mathscr{Z}}^I h^{(1)}| \lesssim \varepsilon \sum_{|K|\leq|I|} (1+t+|q|)^{-1}|\nabla\nabla_{\mathscr{Z}}^K h^{(1)}| + \begin{cases}\varepsilon^2(1+t+|q|)^{-4+\delta}(1+|q|)^{-\delta} & \text{if } q>0,\\ \varepsilon(1+t+|q|)^{-3} & \text{if } q<0\end{cases}$$
$$+\begin{cases}\varepsilon^2(1+t+|q|)^{-2+2M_k\varepsilon}(1+|q|)^{-1-\gamma'} & \text{if } q>0,\\ \varepsilon^2(1+t+|q|)^{-2+2M_k\varepsilon}(1+|q|)^{-1/2+\mu'} & \text{if } q<0.\end{cases} \tag{15.5.7}$$

To prove (15.5.7), we first recall Corollary 11.7, which states that

$$|\widetilde{\square}_g \nabla_{\mathscr{Z}}^I h^{(1)}| \lesssim |\widehat{\nabla}_{\mathscr{Z}}^I \mathfrak{H}| + |\widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\square}_g h^{(0)}| + (1+t+|q|)^{-1} \sum_{|K|\leq|I|}\ \sum_{|J|+(|K|-1)_+\leq|I|} |\nabla\nabla_{\mathscr{Z}}^K h^{(1)}||\nabla_{\mathscr{Z}}^J H|$$
$$+ (1+|q|)^{-1} \sum_{|K|\leq|I|}\ \sum_{|J|+(|K|-1)_+\leq|I|} |\nabla\nabla_{\mathscr{Z}}^K h^{(1)}||\nabla_{\mathscr{Z}}^J H|_{\mathcal{L}\mathcal{L}} \tag{15.5.8}$$
$$+ (1+|q|)^{-1} \sum_{|K|\leq|I|}\ \sum_{|J'|+(|K|-1)_+\leq|I|-1} |\nabla\nabla_{\mathscr{Z}}^K h^{(1)}||\nabla_{\mathscr{Z}}^{J'} H|_{\mathcal{L}\mathcal{T}}$$
$$+ \underbrace{(1+|q|)^{-1} \sum_{|K|\leq|I|}\ \sum_{|J''|+(|K|-1)_+\leq|I|-2} |\nabla\nabla_{\mathscr{Z}}^K h^{(1)}||\nabla_{\mathscr{Z}}^{J''} H|}_{\text{absent if } |I|\leq 1 \text{ or } |K|=|I|}, \tag{15.5.9}$$

where $(|K|-1)_+ \overset{\text{def}}{=} 0$ if $|K|=0$ and $(|K|-1)_+ \overset{\text{def}}{=} |K|-1$ if $|K|\geq 1$. We first bound the terms from line (15.5.8) onwards, considering separately the cases $|K|<|I|$ and $|K|=|I|=k+1$. For $|K|<|I|=k+1$, we use (15.5.2) (for the tensor field H) and (15.3.4b) (for the tensor field H) under the induction hypotheses to conclude that

$$(1+|q|)^{-1} \sum_{\substack{|J|\leq k+1\\ |J'|\leq k\\ |J''|\leq k-1}} \left(|\nabla_{\mathscr{Z}}^J H|_{\mathcal{L}\mathcal{L}} + |\nabla_{\mathscr{Z}}^{J'} H|_{\mathcal{L}\mathcal{T}} + |\nabla_{\mathscr{Z}}^{J''} H|\right)$$
$$\lesssim \begin{cases}\varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1-M_k\varepsilon} & \text{if } q>0,\\ \varepsilon(1+t+|q|)^{-1+M_k\varepsilon}(1+|q|)^{-1/2+\mu'} & \text{if } q<0.\end{cases} \tag{15.5.10}$$

Also using (15.3.4a) under the induction hypotheses to bound $|\nabla\nabla_{\mathscr{Z}}^K h^{(1)}|$, we deduce that all of the terms from line (15.5.8) onwards in the case $|K|<|I|$ can be bounded by the last term on the right-hand side of (15.5.7).

We now consider the case $|K|=|I|=k+1$. Since $|J|\leq 1$ and $|J'|=0$ in this case, we can use (15.3.1b) (for the tensor field H) to deduce the bound

$$(1+|q|)^{-1} \sum_{|K|=|I|} \left(|\nabla\nabla_{\mathscr{Z}}^K h^{(1)}|\left(\sum_{|J|+(|K|-1)_+\leq|I|} |\nabla_{\mathscr{Z}}^J H|_{\mathcal{L}\mathcal{L}} + \sum_{|J'|+(|K|-1)_+\leq|I|-1} |\nabla_{\mathscr{Z}}^{J'} H|_{\mathcal{L}\mathcal{T}}\right)\right)$$
$$\lesssim \varepsilon \sum_{|K|=|I|} (1+t+|q|)^{-1}|\nabla\nabla_{\mathscr{Z}}^K h^{(1)}|. \tag{15.5.11}$$

Thus, all of the terms from line (15.5.8) onwards in the case $|K|=|I|=k+1$ can be bounded by the first term on the right-hand side of (15.5.7).

With the help of Corollary 15.3 (the version for the tensor field H), the

$$(1+t+|q|)^{-1}\sum_{|K|\le|I|}\ \sum_{|J|+(|K|-1)_+\le|I|}|\nabla_{\mathcal{Z}}^{J}H||\nabla\nabla_{\mathcal{Z}}^{K}h^{(1)}|$$

sum on the right-hand side of (15.5.9) can be bounded by the first sum on the right-hand side of (15.5.7).

For the $|\widehat{\nabla}_{\mathcal{Z}}^{I}\widetilde{\Box}_g h^{(0)}|$ term from the right-hand side of (15.5.9), we simply use Lemma 15.4, which shows that $|\widehat{\nabla}_{\mathcal{Z}}^{I}\widetilde{\Box}_g h^{(0)}|$ is bounded by the next-to-last term on the right-hand side of (15.5.7).

To bound the $|\nabla_{\mathcal{Z}}^{I}\mathfrak{H}|$ term from the right-hand side of (15.5.9), we apply Lemma 15.12. Using the already-proved upgraded estimates for $|\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|$ ($|I|\le k+1$), we see that the first and third sums from the right-hand side of (15.5.1) are bounded by the right-hand side of (15.5.7). The second sum

$$\sum_{\substack{|J|+|K|\le|I|\\|J|\le|K|<|I|}}\big(|\nabla\nabla_{\mathcal{Z}}^{J}h^{(1)}|+|\mathcal{L}_{\mathcal{Z}}^{J}\mathcal{F}|\big)\big(|\nabla\nabla_{\mathcal{Z}}^{K}h^{(1)}|+|\mathcal{L}_{\mathcal{Z}}^{K}\mathcal{F}|\big)$$

from the right-hand side of (15.5.1) can be bounded by the last term on the right-hand side of (15.5.7) by using the induction hypotheses since $|J|\le|K|\le k$. This completes the proof of (15.5.7) in the case of $|I|=k+1$.

To obtain the desired upgraded pointwise estimate for $|\nabla\nabla_{\mathcal{Z}}^{I}h^{(1)}|$, we will estimate the quantity

$$n_{k+1}(t)\stackrel{\text{def}}{=}(1+t)\sum_{|I|\le k+1}\big\|\varpi(q)\nabla\nabla_{\mathcal{Z}}^{I}h^{(1)}(t,\cdot)\big\|_{L^\infty},\tag{15.5.12}$$

where $\varpi(q)$ is the weight defined in (13.1.1). Our goal is to use Lemma 13.2 with $\phi\stackrel{\text{def}}{=}\nabla_{\mathcal{Z}}^{I}h_{\mu\nu}^{(1)}$ to obtain an integral inequality for $n_{k+1}(t)$ that is amenable to Gronwall's inequality. We begin by estimating the terms on the right-hand side of (13.2.8). First, with $a\stackrel{\text{def}}{=}\min(\mu'-\delta,\gamma-\delta-\gamma')>0$, by the weak decay estimate (15.1.2b), we have that

$$\varpi(q)|\nabla_{\mathcal{Z}}^{I}h^{(1)}|\lesssim\left\{\begin{matrix}\varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1+\gamma'-\gamma} & \text{if } q>0,\\ \varepsilon(1+t+|q|)^{-1+\delta}(1+|q|)^{1-\mu'} & \text{if } q<0\end{matrix}\right\}\lesssim\varepsilon(1+t)^{-a}\quad(|I|\le\ell-3).\tag{15.5.13}$$

This will serve as a suitable bound for estimating the first and fourth sums on the right-hand side of (13.2.8).

Next, using (15.5.7) and the definition (15.5.12), we deduce the following pointwise estimate:

$$\varpi(q)|\widetilde{\Box}_g\nabla_{\mathcal{Z}}^{I}h^{(1)}|\lesssim(1+t)^{-2}\big(\varepsilon n_{k+1}+\varepsilon^2(1+t)^{2M_k\varepsilon}+\varepsilon(1+t)^{-1/2-\mu'}\big).\tag{15.5.14}$$

This will serve as a suitable bound for estimating the third sum on the right-hand side of (13.2.8).

We now apply Lemma 13.2, using (15.5.13), (15.5.14), and the assumption $k+1 \leq \ell - 5$ to deduce that

$$
\begin{aligned}
n_{k+1}(t) \leq C \sup_{0 \leq \tau \leq t} \sum_{|I| \leq k+2} & \left\| \varpi(q) \nabla_{\mathcal{Z}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty} \\
& + C \sum_{|I| \leq k+1} \int_0^t \varepsilon \left\| \varpi(q) \nabla \nabla_{\mathcal{Z}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty} d\tau \\
& + C \sum_{|I| \leq k+1} \int_0^t (1+\tau) \left\| \varpi(q) |\widetilde{\Box}_g \nabla_{\mathcal{Z}}^I h^{(1)}|(\tau, \cdot) \right\|_{L^\infty(D_\tau)} d\tau \\
& + C \sum_{|I| \leq k+3} \int_0^t (1+\tau)^{-1} \left\| \varpi(q) \nabla_{\mathcal{Z}}^I h^{(1)}(\tau, \cdot) \right\|_{L^\infty(D_\tau)} d\tau \\
\leq C\varepsilon(1+t)^{-a} + C & \int_0^t (1+\tau)^{-1} \varepsilon n_{k+1}(\tau)\, d\tau \\
& + C \int_0^t (1+\tau)^{-1} \left\{ \varepsilon^2 (1+\tau)^{C\varepsilon} + \varepsilon (1+\tau)^{-1/2-\mu'} + \varepsilon (1+\tau)^{-a} \right\} d\tau \\
\leq C\varepsilon + C\varepsilon(1+t)^{C\varepsilon} & + C\varepsilon \int_0^t (1+\tau)^{-1} n_{k+1}(\tau)\, d\tau. \qquad (15.5.15)
\end{aligned}
$$

From (15.5.15) and Gronwall's inequality, we conclude that $n_{k+1}(t) \leq 2C\varepsilon(1+t)^{2C\varepsilon}$, which proves (15.3.4a) in the case $|I| = k+1$. As in our proof of Lemma 15.8, the estimate (15.3.4b) follows from integrating the bound for $|\partial_q \nabla_{\mathcal{Z}}^I h^{(1)}|$ implied by (15.3.4a) along the line $\omega \overset{\text{def}}{=} x/|x| = \text{constant}$ and $t + |x| = \text{constant}$, from the hyperplane $t = 0$, and using (15.1.2b) at $t = 0$. This closes the induction argument. We have completed the proof of Proposition 15.7 with the exception of showing that inequality (15.3.4c) holds for $|\overline{\nabla} \nabla_{\mathcal{Z}}^I h^{(1)}|$, $|\overline{\nabla} \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|$, $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}}$, and $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}}$, where $|I| \leq \ell - 6$. In the next paragraph, we address these inequalities using an argument that is not part of the induction process.

Upgraded pointwise decay estimates for $|\overline{\nabla} \nabla_{\mathcal{Z}}^I h^{(1)}|$, $|\overline{\nabla} \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|$, $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}}$, *and* $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}}$ $(|I| \leq \ell - 6)$. We first note that inequality (15.3.4c) for $|\overline{\nabla} \nabla_{\mathcal{Z}}^I h^{(1)}|$ and $|\overline{\nabla} \mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|$ follows from Lemma 6.16, (6.5.22), (15.3.4a), and (15.3.4b).

We now focus on proving the estimate (15.3.4c) for $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}}$ and $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}}$ in (15.3.4c); all of the other estimates of Proposition 15.7 have already been proved. Recall that $|\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}} \approx |\alpha[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]| + |\rho[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]| + |\sigma[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]|$. We will prove the desired estimate for $|\alpha[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]|$ in detail; the proofs for $|\rho[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]|$ and $|\sigma[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}]|$ are similar.

Our proof mirrors the proof of Proposition 15.5 except that we now are able to use the already-proved upgraded estimates of Proposition 15.7 in place of the weak decay estimates of Corollary 15.3. We will use the notation defined in the proof of Proposition 15.5. With the help of the upgraded pointwise decay estimates (15.3.4a) and (15.3.4b) (including the versions for the tensor field $h = h^{(0)} + h^{(1)}$), inequality (15.2.2) for $f(t,x) \overset{\text{def}}{=} r\alpha[\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}(t,x)]$ can be upgraded to

$$
|\partial_q f(t,x)| \leq \begin{cases} C_k \varepsilon (1+s)^{-1+C\varepsilon} (1+|q|)^{-1-\gamma'} & \text{if } q > 0, \\ C_k \varepsilon (1+s)^{-1+C\varepsilon} (1+|q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \qquad (|I| \leq \ell - 6). \tag{15.5.16}
$$

Arguing as in the proof of Proposition 15.5, and using in particular (15.2.4), we deduce from (15.5.16) that

$$\left|r\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}(t,x)]\right| \leq C\varepsilon(1+s)^{-1-\overbrace{(\gamma-\delta)}^{>0}} + \begin{cases} C_k\varepsilon(1+s)^{-1+C\varepsilon}(1+|q'|)^{-\gamma'} & \text{if } q'>0, \\ C_k\varepsilon(1+s)^{-1+C\varepsilon}(1+|q'|)^{1/2+\mu'} & \text{if } q'<0 \end{cases} \quad (|I|\leq \ell-6), \quad (15.5.17)$$

from which it easily follows that

$$\left|\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}(t,x)]\right| \leq \begin{cases} C_k\varepsilon(1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{-\gamma'} & \text{if } q>0, \\ C_k\varepsilon(1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q<0 \end{cases} \quad (|I|\leq \ell-6). \quad (15.5.18)$$

We have thus obtained the desired bound (15.3.4c) for $|\alpha[\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}]|$. □

16. Global existence and stability

In this section, we prove our main stability results. We separate our results into two theorems. The main conclusions are proved in Theorem 16.3, which is an easy consequence of Theorem 16.1. Theorem 16.1, which concerns the reduced equations (3.7.1a)–(3.7.1c), contains the crux of our bootstrap argument. In this theorem, we make certain assumptions concerning the smallness of the abstract initial data and various pointwise decay estimates for the solution on a local interval of existence $[0, T)$. We then use these assumptions to derive a "strong" a priori estimate for the energy $\mathcal{E}_{\ell;\gamma;\mu}(t)$ of the reduced solution on the same interval $[0, T)$. Furthermore, in Section 15, the pointwise decay assumptions of Theorem 16.1 were shown to be *automatic consequences* of the smallness assumptions on the data and the "weak" bootstrap assumption (15.0.1) for $\mathcal{E}_{\ell;\gamma;\mu}(t)$ as long as $\ell \geq 10$. Consequently, in our proof of Theorem 16.3, we will be able to appeal to the continuation principle of Proposition 14.1 to conclude that the solution to the reduced equation exists globally in time. Furthermore, this line of reasoning leads to an estimate on the size of $\mathcal{E}_{\ell;\gamma;\mu}(t)$, which can be used to deduce various decay estimates for the global solution. The wave-coordinate condition plays a central role in many of the estimates in this section.

16.1. ***Statement of the strong-a priori-energy-estimate theorem and proof of the global stability theorem.*** We begin by recalling that the norm $E_{\ell;\gamma}(0) \geq 0$ of the abstract initial data is

$$E^2_{\ell;\gamma}(0) \overset{\text{def}}{=} \|\nabla \underline{\mathring{h}}^{(1)}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{K}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{\mathfrak{D}}\|^2_{H^{\ell}_{1/2+\gamma}} + \|\mathring{\mathfrak{B}}\|^2_{H^{\ell}_{1/2+\gamma}}. \quad (16.1.1)$$

We furthermore recall that the energy $\mathcal{E}_{\ell;\gamma;\mu}(t) \geq 0$ of the reduced solution is

$$\mathcal{E}^2_{\ell;\gamma;\mu}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t} \sum_{|I|\leq\ell} \int_{\Sigma_\tau} \left(|\nabla\nabla_{\mathcal{Z}}^{I} h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^{I}\mathcal{F}|^2\right) w(q)\, d^3x. \quad (16.1.2)$$

In the above expressions, the weight function $w(q)$ and its derivative $w'(q)$ are

$$w = w(q) \overset{\text{def}}{=} \begin{cases} 1 + (1+|q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1+|q|)^{-2\mu} & \text{if } q < 0, \end{cases} \tag{16.1.3a}$$

$$w'(q) = \begin{cases} (1+2\gamma)(1+|q|)^{2\gamma} & \text{if } q > 0, \\ 2\mu(1+|q|)^{-2\mu-1} & \text{if } q < 0. \end{cases} \tag{16.1.3b}$$

The constants μ and γ are subject to the restrictions summarized in Section 2.14. The spacetime metric is split into the three pieces

$$g_{\mu\nu} = m_{\mu\nu} + h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}, \tag{16.1.4a}$$

$$h^{(0)}_{\mu\nu} = \chi\left(\frac{r}{t}\right)\chi(r)\frac{2M}{r}\delta_{\mu\nu}, \tag{16.1.4b}$$

where the cut-off function χ is defined in (4.2.1). Furthermore, by Proposition 10.4, if ε is sufficiently small and $E_{\ell;\gamma}(0) + M \le \varepsilon$, then the initial energy for the reduced solution satisfies

$$\mathcal{E}_{\ell;\gamma;\mu}(0) \lesssim E_{\ell;\gamma}(0) + M \lesssim \varepsilon. \tag{16.1.5}$$

We now state our technical theorem concerning the derivation of a "strong" a priori energy estimate. The proof will be provided in Section 16.2.

Theorem 16.1 (Derivation of a strong a priori energy estimate). *Let* $(g_{\mu\nu} \overset{\text{def}}{=} m_{\mu\nu} + \overbrace{h^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu}}^{h_{\mu\nu}}, \mathcal{F}_{\mu\nu})$ *be a local-in-time solution of the reduced equations* (3.7.1a)–(3.7.1c) *satisfying the wave-coordinate condition* (3.1.1a) *for* $(t, x) \in [0, T) \times \mathbb{R}^3$. *Let* $\ell \ge 0$ *be an integer. Suppose also that, for some constants* μ' *and* γ *satisfying* $0 < \mu' < \frac{1}{2}$ *and* $0 < \gamma < \frac{1}{2}$, *for all vector fields* $Z \in \mathcal{Z}$, *for all* $\mathcal{Z}$*-multi-indices* I *subject to the restrictions stated below, and for the sets* $\mathcal{L} = \{L\}$, $\mathcal{T} = \{L, e_1, e_2\}$, *and* $\mathcal{N} = \{\underline{L}, L, e_1, e_2\}$, *the following pointwise decay estimates hold for* $(t, x) \in [0, T) \times \mathbb{R}^3$:

$$(1+|q|)^{-1}|h|_{\mathcal{L}\mathcal{T}} + (1+|q|)^{-1}|\nabla_Z h|_{\mathcal{L}\mathcal{L}} + |\nabla h|_{\mathcal{T}\mathcal{N}} + |\mathcal{F}| \le C\varepsilon(1+t+|q|)^{-1}, \tag{16.1.6a}$$

$$\begin{aligned}&(1+|q|)^{-1}|\nabla_{\mathcal{Z}}^I h| + |\nabla\nabla_{\mathcal{Z}}^I h| + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}| \\ &\qquad \le \begin{cases} C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1-C\varepsilon} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'} & \text{if } q < 0 \end{cases} \qquad (|I| \le \lfloor \ell/2 \rfloor),\end{aligned} \tag{16.1.6b}$$

$$\begin{aligned}&|\overline{\nabla}\nabla_{\mathcal{Z}}^I h| + (1+|q|)|\overline{\nabla}\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}| + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I \mathcal{F}|_{\mathcal{T}\mathcal{T}} \\ &\qquad \le \begin{cases} C\varepsilon(1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{-C\varepsilon} & \text{if } q > 0, \\ C\varepsilon(1+t+|q|)^{-2+C\varepsilon}(1+|q|)^{1/2+\mu'} & \text{if } q < 0 \end{cases} \qquad (|I| \le \lfloor \ell/2 \rfloor).\end{aligned} \tag{16.1.6c}$$

In addition, assume that the following smallness conditions on the abstract initial data and ADM mass hold:

$$E_{\ell;\gamma}(0) + M \le \mathring{\varepsilon}. \tag{16.1.7}$$

Then for any constant μ *satisfying* $0 < \mu < \frac{1}{2} - \mu'$, *there exist positive constants* ε_ℓ, c_ℓ, *and* $\tilde{c}_\ell$ *depending on* ℓ, μ, μ', *and* γ *such that, if* $\mathring{\varepsilon} \le \varepsilon \le \varepsilon_\ell$, *then the following energy inequality holds for* $t \in [0, T)$:

$$\mathcal{E}_{\ell;\gamma;\mu}(t) \le c_\ell(\mathring{\varepsilon} + \varepsilon^{3/2})(1+t)^{\tilde{c}_\ell \varepsilon}. \tag{16.1.8}$$

Remark 16.2. By Lemma 15.1, the decompositions $h = h^{(0)} + h^{(1)}$ and $H = H_{(0)} + H_{(1)}$ (where $H^{\mu\nu} \stackrel{\text{def}}{=} (g^{-1})^{\mu\nu} - (m^{-1})^{\mu\nu}$), and the fact that $H_{(1)}^{\mu\nu} = -h^{(1)\mu\nu} + O^{\infty}(|h^{(0)} + h^{(1)}|^2)$, it follows that the estimates stated in the assumptions of the theorem also hold if we replace h with $h^{(0)}$, $H_{(0)}$, $h^{(1)}$, or $H_{(1)}$.

We now state and (using the results of Theorem 16.1) prove our main global stability theorem.

Theorem 16.3 (Global stability of the Minkowski spacetime solution). *Let* $(\mathring{\underline{g}}_{jk} = \delta_{jk} + \mathring{\underline{h}}_{jk}^{(0)} + \mathring{\underline{h}}_{jk}^{(1)}, \mathring{K}_{jk}, \mathring{\mathfrak{D}}_j, \mathring{\mathfrak{B}}_j)$ $(j, k = 1, 2, 3)$ *be abstract initial data on the manifold* $\mathbb{R}^3$ *for the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c) *that satisfy the constraints* (4.1.1a)–(4.1.2b), *and let* $(g_{\mu\nu}|_{t=0} = m_{\mu\nu} + h_{\mu\nu}^{(0)}|_{t=0} + h_{\mu\nu}^{(1)}|_{t=0}, \partial_t g_{\mu\nu}|_{t=0} = \partial_t h_{\mu\nu}^{(0)}|_{t=0} + \partial_t h_{\mu\nu}^{(1)}|_{t=0}, \mathscr{F}_{\mu\nu}|_{t=0})$ $(\mu, \nu = 0, 1, 2, 3)$ *be the corresponding initial data for the reduced system* (3.7.1a)–(3.7.1c) *as defined in Section 4.2. Assume that the abstract initial data are asymptotically flat in the sense that* (1.0.4a)–(1.0.4f) *hold. Let* $\ell \geq 10$ *be an integer, and let* $0 < \gamma < \frac{1}{2}$ *be a fixed constant. Let* $E_{\ell;\gamma}(0)$ *be the norm of the abstract data given in* (16.1.1), *and let* M *be the ADM mass corresponding to the abstract data. Then there exists a constant* $\varepsilon_\ell > 0$ *depending on* γ *and* ℓ *such that, if* $\varepsilon \leq \varepsilon_\ell$ *and if*

$$E_{\ell;\gamma}(0) + M \leq \varepsilon, \tag{16.1.9}$$

then the reduced data launch a unique, classical solution $(g_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$ *that exists for* $(t, x) \in (-\infty \times \infty) \times \mathbb{R}^3$. *The solution satisfies* both[31] *the reduced system* (3.7.1a)–(3.7.1c) *and the Einstein-nonlinear electromagnetic system* (1.0.1a)–(1.0.1c), *and the spacetime* $(\mathbb{R}^{1+3}, g_{\mu\nu})$ *is geodesically complete. In addition, the coordinates* (t, x) *form a global system of wave coordinates. Furthermore, there exists a constant* $0 < \upmu < \frac{1}{2}$ *(see Remark 1.2), and constants* $c_\ell > 0$ *and* $\tilde{c}_\ell > 0$ *depending on* γ *and* ℓ, *such that the solution's energy* (16.1.2) *satisfies the following bound for all* $t \in (-\infty, \infty)$:

$$\mathscr{E}_{\ell;\gamma;\upmu}(t) \leq c_\ell \varepsilon (1 + |t|)^{\tilde{c}_\ell \varepsilon}. \tag{16.1.10}$$

In addition, there exists a constant $C_\ell > 0$ *depending on* γ *and* ℓ *such that the following pointwise decay estimates hold for all* $(t, x) \in (-\infty, \infty) \times \mathbb{R}^3$:

$$\begin{aligned}
&(1+|t|+|q|)^{1-\tilde{c}_\ell\varepsilon}(1+|q|)^{-3/2}|h^{(1)}|_{\mathcal{L}\mathcal{T}} + (1+|t|+|q|)^{1-\tilde{c}_\ell\varepsilon}(1+|q|)^{-3/2}|\nabla_Z h^{(1)}|_{\mathcal{L}\mathcal{L}} \\
&\qquad + (1+|t|+|q|)^{1-\tilde{c}_\ell\varepsilon}(1+|q|)^{-1/2}|\nabla h^{(1)}|_{\mathcal{L}\mathcal{T}} + (1+|t|+|q|)^{1-\tilde{c}_\ell\varepsilon}(1+|q|)^{-1/2}|\nabla\nabla_Z h^{(1)}|_{\mathcal{L}\mathcal{L}} \\
&\qquad + |\nabla h^{(1)}|_{\mathcal{T}\mathcal{N}} + \{1+\ln(1+|t|)\}^{-1}|\nabla h^{(1)}| + |\mathscr{F}| \\
&\qquad\qquad \leq C_\ell \varepsilon (1+|t|+|q|)^{-1},
\end{aligned} \tag{16.1.11a}$$

$$\begin{aligned}
&(1+|q|)^{-1}|\nabla_{\mathscr{Z}}^I h^{(1)}| + |\nabla\nabla_{\mathscr{Z}}^I h^{(1)}| + |\mathcal{L}_{\mathscr{Z}}^I \mathscr{F}| \\
&\qquad \leq \begin{cases} C_\ell\varepsilon(1+|t|+|q|)^{-1+\tilde{c}_\ell\varepsilon}(1+|q|)^{-1-\gamma} & \text{if } q > 0, \\ C_\ell\varepsilon(1+|t|+|q|)^{-1+\tilde{c}_\ell\varepsilon}(1+|q|)^{-1/2} & \text{if } q < 0 \end{cases} \qquad (|I| \leq \ell - 3),
\end{aligned} \tag{16.1.11b}$$

$$\begin{aligned}
&|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}| + (1+|q|)|\overline{\nabla}\mathcal{L}_{\mathscr{Z}}^I \mathscr{F}| + |\mathcal{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{T}\mathcal{T}} \\
&\qquad \leq \begin{cases} C_\ell\varepsilon(1+|t|+|q|)^{-2+\tilde{c}_\ell\varepsilon}(1+|q|)^{-\gamma} & \text{if } q > 0, \\ C_\ell\varepsilon(1+|t|+|q|)^{-2+\tilde{c}_\ell\varepsilon}(1+|q|)^{1/2} & \text{if } q < 0 \end{cases} \qquad (|I| \leq \ell - 4).
\end{aligned} \tag{16.1.11c}$$

[31] Of course, we technically mean here that the pair $(h_{\mu\nu}^{(1)}, \mathscr{F}_{\mu\nu})$ is a solution to the version (3.7.1a)–(3.7.1c) of the reduced equations while the pair $(g_{\mu\nu}, \mathscr{F}_{\mu\nu})$ is a solution to (1.0.1a)–(1.0.1c).

Remark 16.4. Some of the $(1+|q|)$-decay estimates in inequalities (16.1.11a)–(16.1.11c) are not optimal and can be improved with additional work. For example, in [Lindblad and Rodnianski 2010, Section 16], with the help of the fundamental solution of the Minkowski wave operator $\Box_m$, the $(1+|q|)$-decay estimates (16.1.11b)–(16.1.11c) for the tensor field $h^{(1)}$ are strengthened by a power of $\frac{1}{2}$ in the interior region $\{q < 0\}$.

Remark 16.5. Proposition 4.2 shows that the wave-coordinate condition (3.1.1a) holds in the domain of classical existence of the solution to the reduced equations; this is why the reduced solution also satisfies the Einstein-nonlinear electromagnetic equations (1.0.1a)–(1.0.1c).

Remark 16.6. A global stability result for the reduced equations under the wave-coordinate assumption, without regard for the abstract initial data, can be deduced from the smallness of $\mathcal{E}_{\ell;\gamma;\mu}(0)+|M|$ (we could even allow for negative M!) together with the assumption $\liminf_{|x|\to\infty}|h^{(1)}(0,x)|=0$; this latter assumption, which is needed to deduce the inequalities (15.1.2b) at $t=0$, is automatically implied by the assumptions of Theorem 16.3.

Proof. We only discuss the region of spacetime in which $t \geq 0$; the argument for $t \leq 0$ is similar. We define $E_{\ell;\gamma}(0)+M \overset{\text{def}}{=} \mathring{\varepsilon}$. By Proposition 14.1, we can choose constants γ', μ, μ', and δ subject to the restrictions described in Section 2.14 (in particular, these constants depend on γ) and a constant $A_\ell > 0$ such that, if $\varepsilon \overset{\text{def}}{=} A_\ell\mathring{\varepsilon}$, A_ℓ is sufficiently large, and $\mathring{\varepsilon}$ is sufficiently small, then there exists a nontrivial spacetime slab $[0,T)\times\mathbb{R}^3$ upon which the solution to the reduced equations exists and satisfies the energy bound $\mathcal{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^\delta$ for $t\in[0,T)$. We then define

$$T_* \overset{\text{def}}{=} \sup\bigl\{T \bigm| \text{the solution exists classically and remains in the regime of hyperbolicity of the reduced equations, and } \mathcal{E}_{\ell;\gamma;\mu}(t) \leq \varepsilon(1+t)^\delta \text{ for } t\in[0,T)\bigr\}.$$

Note that, under the above assumptions, we have that $T_* > 0$.

We now observe that the main energy bootstrap assumption (15.0.1) is satisfied on $[0,T_*)$. Thus, if ε is sufficiently small, then by Propositions 15.6 and 15.7, all of the hypotheses of Theorem 16.1 are necessarily satisfied on $[0,T_*)$. Here, we are using the fact that $\lfloor \ell/2\rfloor \leq \ell-5$, which holds if $\ell \geq 10$. Consequently, the conclusion of that theorem (i.e., estimate (16.1.8)) allows us to deduce that the following energy estimate holds for $t\in[0,T_*)$:

$$\mathcal{E}_{\ell;\gamma;\mu}(t) \leq c_\ell(\mathring{\varepsilon}+\varepsilon^{3/2})(1+t)^{\tilde{c}_\ell\varepsilon} = c_\ell\Bigl(\frac{\varepsilon}{A_\ell}+\varepsilon^{3/2}\Bigr)(1+t)^{\tilde{c}_\ell\varepsilon}. \tag{16.1.12}$$

Now if $A_\ell > 3c_\ell$ and $\mathring{\varepsilon}$ is sufficiently small, then (16.1.12) implies that

$$\mathcal{E}_{\ell;\gamma;\mu}(t) < \tfrac{1}{2}A_\ell\mathring{\varepsilon}(1+t)^{A_\ell\tilde{c}_\ell\mathring{\varepsilon}} = \tfrac{1}{2}\varepsilon(1+t)^{\tilde{c}_\ell\varepsilon}, \tag{16.1.13}$$

which is a strict improvement over the bootstrap assumption (15.0.1). Thus, by (16.1.13), the weighted Klainerman–Sobolev inequality (B.4) (which, together with (6.5.22) and the smallness of $\mathcal{E}_{\ell;\gamma;\mu}(t)$, implies that the solution remains within the regime of hyperbolicity of the reduced equations), the continuation principle of Proposition 14.1, and the continuity of $\mathcal{E}_{\ell;\gamma;\mu}(t)$, it follows that, if A_ℓ is sufficiently large and $\mathring{\varepsilon}$ is sufficiently small, then $T_* = \infty$. Furthermore, under these assumptions, it is an obvious consequence of this reasoning that (16.1.13) holds for $t \in [0, \infty)$. After renaming the constants in (16.1.13), we arrive at (16.1.10).

The inequalities (16.1.11b) follow as in the proof of Corollary 15.3 but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1). Similarly, the inequalities (16.1.11a) follow as in our proof of Proposition 15.6 but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1). The inequalities (16.1.11c) for $|\overline{\nabla}\nabla_{\mathscr{Z}}^{I} h^{(1)}|$ and $|\overline{\nabla}\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}|$ follow from Lemma 6.16, (6.5.22), and (16.1.11b). The inequalities (16.1.11c) for $|\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}|_{\mathscr{L}\mathscr{N}}$ and $|\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}|_{\mathscr{T}\mathscr{T}}$ follow as in our proof of (15.2.1) but with the strong energy estimate (16.1.10) in place of the energy bootstrap assumption (15.0.1).

Based on these pointwise decay estimates, the geodesic completeness of the spacetime $(\mathbb{R}^{1+3}, g_{\mu\nu} \stackrel{\text{def}}{=} m_{\mu\nu} + h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)})$ follows as in [Lindblad and Rodnianski 2005, Section 16; Loizelet 2008, Section 9]. □

16.2. ***The main argument in the proof of Theorem 16.1.*** Our goal is to use *only* the assumptions of Theorem 16.1 to deduce (for all sufficiently small nonnegative ε and for sufficiently large fixed constants c_ℓ and $\tilde{c}_\ell$) the "strong" a priori energy estimate (16.1.8), which we restate for convenience:

$$\mathscr{E}_{\ell;\gamma;\mu}(t) \le c_\ell(\mathring{\mathscr{E}} + \varepsilon^{3/2})(1+t)^{\tilde{c}_\ell \varepsilon}. \tag{16.2.1}$$

The proof of (16.2.1) is based on a hierarchy of Gronwall-amenable inequalities for $\mathscr{E}_{k;\gamma;\mu}(t)$ $(0 \le k \le \ell)$. We derive this hierarchy by carefully analyzing the integrals of Proposition 12.3 involving the inhomogeneous terms $\mathfrak{H}_{\mu\nu}^{(1;I)}$ and $\mathfrak{F}_{(I)}^{\nu}$. We recall that the structure of these inhomogeneous terms is captured by Propositions 7.1 and 8.1, which state that $\nabla_{\mathscr{Z}}^{I} h_{\mu\nu}^{(1)}$ and $\mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}_{\mu\nu}$ are solutions to the following system of equations:

$$\widetilde{\square}_g \nabla_{\mathscr{Z}}^{I} h_{\mu\nu}^{(1)} = \mathfrak{H}_{\mu\nu}^{(1;I)} \qquad (\mu, \nu = 0, 1, 2, 3), \tag{16.2.2a}$$

$$\nabla_\lambda \mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}_{\mu\nu} + \nabla_\mu \mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}_{\nu\lambda} + \nabla_\nu \mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}_{\lambda\mu} = 0 \qquad (\lambda, \mu, \nu = 0, 1, 2, 3), \tag{16.2.2b}$$

$$N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}_{\kappa\lambda} = \mathfrak{F}_{(I)}^{\nu} \qquad (\nu = 0, 1, 2, 3). \tag{16.2.2c}$$

Most of the work goes into obtaining suitable estimates for the integrals involving $\mathfrak{H}_{\mu\nu}^{(1;I)}$ and $\mathfrak{F}_{(I)}^{\nu}$. In order to avoid impeding the flow of the proof, we prove most of the desired inequalities later in this section after the main argument. For the main part of the argument, we simply quote Corollaries 16.12 and 16.18, which are the key estimates that allow us to apply a suitable version of Gronwall's inequality. We will then return to the proofs of Corollaries 16.12 and 16.18, which follow from a large collection of lemmas, each of which involves the analysis of one of the constituent pieces of the integrals involving $\mathfrak{H}_{\mu\nu}^{(1;I)}$ and $\mathfrak{F}_{(I)}^{\nu}$.

We now proceed to the main argument. We first note that the hypotheses of Proposition 12.3 are implied by the hypotheses of Theorem 16.1. Therefore, we can use Proposition 12.3 (with $\dot{\mathscr{F}} \stackrel{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^{I}\mathscr{F}$ in the proposition) and Corollaries 16.12 and 16.18 to deduce that

$$\begin{aligned}
&\sum_{|I|\leq k}\int_{\Sigma_t}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x + \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\big(|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau\\
&\quad\leq C\sum_{|I|\leq k}\int_{\Sigma_0}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x + C\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau\\
&\qquad + C\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\Big(|\mathfrak{H}^{(1;I)}||\nabla\nabla_{\mathcal{Z}}^I h^{(1)}| + \big|(\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{0\nu})\mathfrak{F}^{\nu}_{(I)}\big|\Big)w(q)\,d^3x\,d\tau\\
&\quad\leq C\sum_{|I|\leq k}\int_{\Sigma_0}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x + CM\sum_{|I|\leq k}\int_0^t\left((1+\tau)^{-3/2}\sqrt{\int_{\Sigma_\tau}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|^2 w(q)\,d^3x}\right)d\tau\\
&\qquad + C\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau\\
&\qquad + C\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\big(|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau \qquad (16.2.3)\\
&\qquad + \underbrace{C\varepsilon\sum_{|J|\leq k-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^J h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^J\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau}_{\text{absent if } k=0} + C\varepsilon^3.
\end{aligned}$$

Recalling the definition (where the dependence on μ and γ is through $w(q)$)

$$\mathcal{E}^2_{k;\gamma;\mu}(t) \overset{\text{def}}{=} \sup_{0\leq\tau\leq t}\sum_{|I|\leq k}\int_{\Sigma_\tau}\big(|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2\big)w(q)\,d^3x$$

and introducing the quantity $\mathcal{S}_{k;\gamma;\mu}(t)\geq 0$, which is defined by

$$\mathcal{S}^2_{k;\gamma;\mu}(t) \overset{\text{def}}{=} \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\big(|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{L}\mathcal{N}} + |\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau, \qquad (16.2.4)$$

it therefore follows from the final inequality of (16.2.3) that

$$\begin{aligned}
\mathcal{E}^2_{k;\gamma;\mu}(t) + \mathcal{S}^2_{k;\gamma;\mu}(t) &\leq C\mathcal{E}^2_{k;\gamma;\mu}(0) + CM\int_0^t(1+\tau)^{-3/2}\mathcal{E}_{k;\gamma;\mu}(\tau)\,d\tau + C\varepsilon\int_0^t(1+\tau)^{-1}\mathcal{E}^2_{k;\gamma;\mu}(\tau)\,d\tau\\
&\quad + \underbrace{C\varepsilon\mathcal{S}^2_{k;\gamma;\mu}(t)}_{\text{absorb into left-hand side}} + C\varepsilon\int_0^t(1+\tau)^{-1+C\varepsilon}\mathcal{E}^2_{k-1;\gamma;\mu}(\tau)\,d\tau + C\varepsilon^3. \qquad (16.2.5)
\end{aligned}$$

For ε sufficiently small, we may absorb the $C\varepsilon\mathcal{S}^2_{k;\gamma;\mu}(t)$ term from (16.2.5) into the left-hand side at the expense of increasing all of the constants. We can similarly absorb the term $CM\int_0^t(1+\tau)^{-3/2}\mathcal{E}_{k;\gamma;\mu}(\tau)\,d\tau$ by using the inequality $CM\int_0^t(1+\tau)^{-3/2}\mathcal{E}_{k;\gamma;\mu}(\tau)\,d\tau \leq \frac{1}{2}\mathcal{E}^2_{k;\gamma;\mu}(t) + C^2M^2$, which follows from the algebraic estimate $CM\mathcal{E}_{k;\gamma;\mu}(\tau)\leq\frac{1}{4}\mathcal{E}^2_{k;\gamma;\mu}(\tau)+C^2M^2$, the integral inequality $\int_0^t(1+\tau)^{-3/2}\,d\tau\leq 2$, and the fact that $\mathcal{E}^2_{k;\gamma;\mu}(\tau)$ is increasing. If we also use the fact that $\mathcal{E}^2_{k;\gamma;\mu}(0)\leq C(E^2_{\ell;\gamma}(0)+M^2)\leq C\mathring{\varepsilon}^2$ (i.e, Proposition 10.4) and the inequality $M\leq\mathring{\varepsilon}$, then we arrive at the following inequality, valid for all small ε:

$$\mathscr{E}^2_{k;\gamma;\mu}(t) + \mathscr{S}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3) + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau + \underbrace{C\varepsilon \int_0^t (1+\tau)^{-1+C\varepsilon}\mathscr{E}^2_{k-1;\gamma;\mu}(\tau)\,d\tau}_{\text{absent if } k=0}. \quad (16.2.6)$$

For $k = 0$, (16.2.6) implies that

$$\mathscr{E}^2_{0;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3) + c_0\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{0;\gamma;\mu}(\tau)\,d\tau. \quad (16.2.7)$$

From (16.2.7) and Gronwall's inequality, we deduce that

$$\mathscr{E}^2_{0;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_0\varepsilon}. \quad (16.2.8)$$

Using (16.2.6) and the base case (16.2.8), we will argue inductively to derive the following estimate for $k \ge 1$:

$$\mathscr{E}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_k\varepsilon}. \quad (16.2.9)$$

Assuming that (16.2.9) holds for the case $k-1$, we insert inequality (16.2.9) for $\mathscr{E}^2_{k-1;\gamma;\mu}(t)$ into the right-hand side of (16.2.6) and deduce that

$$\begin{aligned} \mathscr{E}^2_{k;\gamma;\mu}(t) + \mathscr{S}^2_{k;\gamma;\mu}(t) &\le C(\mathring{\varepsilon}^2 + \varepsilon^3) + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau + C\varepsilon(\mathring{\varepsilon}^2 + \varepsilon^3)\int_0^t (1+\tau)^{-1+C\varepsilon}\,d\tau \\ &\le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{C\varepsilon} + C\varepsilon \int_0^t (1+\tau)^{-1}\mathscr{E}^2_{k;\gamma;\mu}(\tau)\,d\tau. \end{aligned} \quad (16.2.10)$$

Finally, from (16.2.10) and Gronwall's lemma, we conclude that, if ε is sufficiently small, then

$$\mathscr{E}^2_{k;\gamma;\mu}(t) \le C(\mathring{\varepsilon}^2 + \varepsilon^3)(1+t)^{c_k\varepsilon}. \quad (16.2.11)$$

We have therefore closed the induction and shown (16.1.8). This concludes the proof of Theorem 16.1.

16.3. ***Integral inequalities for the*** $\nabla^I_{\mathscr{Z}} h^{(1)}_{\mu\nu}$ ***inhomogeneities.*** In this section, we analyze the integrals in Proposition 12.3 corresponding to the inhomogeneous terms $\mathfrak{H}^{(1;I)}_{\mu\nu}$ in (16.2.2a). The main goal is to arrive at Corollary 16.12. The main point is that right-hand sides of the inequalities in the corollary can be bounded in terms of time integrals of the energies $\mathscr{E}_{k;\gamma;\mu}(t)$ (this was carried out in inequality (16.2.5)). As opposed to the estimates proved in Section 16.4, most of the estimates proved in this section are a straightforward generalization of the ones proved in [Lindblad and Rodnianski 2010]; i.e., the estimates involve a similar analysis but with additional terms arising from the presence of the $\mathscr{F}$ terms appearing on the right-hand side of the reduced equation (3.7.1a). The additional terms result in the presence of the $\mathscr{L}^J_{\mathscr{Z}}\mathscr{F}$ component of the first term on the right-hand side of inequality (16.3.2) and the $\mathscr{L}^{J'}_{\mathscr{Z}}\mathscr{F}$ component of the next-to-last term of the same inequality. The most important aspect of our analysis is showing that these additional terms respectively appear with the factors $\varepsilon(1+t)^{-1}$ and $\varepsilon(1+t)^{-1+C\varepsilon}$.

We begin with a lemma that follows easily from algebraic estimates of the form $|ab| \lesssim a^2 + b^2$:

Lemma 16.7 (Arithmetic-geometric mean inequality). *Let*

$$\mathfrak{H}^{(1;I)}_{\mu\nu} = \widehat{\nabla}^I_{\mathscr{Z}}\mathfrak{H}_{\mu\nu} - \widehat{\nabla}^I_{\mathscr{Z}}\widetilde{\Box} h^{(0)}_{\mu\nu} - (\widehat{\nabla}^I_{\mathscr{Z}}\widetilde{\Box}_g h^{(1)}_{\mu\nu} - \widetilde{\Box}_g \nabla^I_{\mathscr{Z}} h^{(1)}_{\mu\nu})$$

be the inhomogeneous term on the right-hand side of (7.0.1). *Then the following algebraic inequality holds*:

$$|\mathfrak{H}^{(1;I)}||\nabla\nabla_{\mathscr{Z}}^I h^{(1)}| \le \varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{Z}}^I \mathfrak{H}|^2 + \varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\Box}_g h_{\mu\nu}^{(1)} - \widetilde{\Box}_g \nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}|^2 + \varepsilon(1+t)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 + |\widehat{\nabla}_{\mathscr{Z}}^I \widetilde{\Box}_g h^{(0)}||\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|. \quad (16.3.1)$$

□

The next lemma provides a preliminary pointwise estimate for the $|\widehat{\nabla}_{\mathscr{Z}}^I \mathfrak{H}|$ term on the right-hand side of (16.3.1).

Lemma 16.8 (Pointwise estimates for the $|\nabla_{\mathscr{Z}}^I \mathfrak{H}|$ inhomogeneities; extension of [Lindblad and Rodnianski 2010, Lemma 11.2]). *Under the assumptions of Theorem 16.1, if I is any $\mathscr{Z}$-multi-index with $|I| \le \ell$ and if ε is sufficiently small, then the following pointwise estimates hold for $(t, x) \in [0, T) \times \mathbb{R}^3$*:

$$|\nabla_{\mathscr{Z}}^I \mathfrak{H}| \lesssim \varepsilon \sum_{|J|\le|I|} (1+t)^{-1} \left| \begin{pmatrix} \nabla\nabla_{\mathscr{Z}}^J h^{(1)} \\ \mathcal{L}_{\mathscr{Z}}^J \mathcal{F} \end{pmatrix} \right| + \varepsilon \sum_{|J|\le|I|} (1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'}|\overline{\nabla}\nabla_{\mathscr{Z}}^J h^{(1)}| + \varepsilon^2 \sum_{|J|\le|I|} (1+t+|q|)^{-1}(1+|q|)^{-1}|\nabla_{\mathscr{Z}}^J h^{(1)}| + \underbrace{\varepsilon \sum_{|J'|\le|I|-1} (1+t)^{-1+C\varepsilon} \left| \begin{pmatrix} \nabla\nabla_{\mathscr{Z}}^{J'} h^{(1)} \\ \mathcal{L}_{\mathscr{Z}}^{J'} \mathcal{F} \end{pmatrix} \right|}_{\textit{absent if } |I|=0} + \varepsilon^2(1+t+|q|)^{-4}. \quad (16.3.2)$$

Proof. By Proposition 11.3, we have that

$$|\nabla_{\mathscr{Z}}^I \mathfrak{H}| \lesssim |(\mathrm{i})| + |(\mathrm{ii})| + |(\mathrm{iii})|, \quad (16.3.3)$$

where

$$|(\mathrm{i})| = \sum_{|J|+|K|\le|I|} |\nabla\nabla_{\mathscr{Z}}^J h|_{\mathcal{T}\mathcal{N}}|\nabla\nabla_{\mathscr{Z}}^K h|_{\mathcal{T}\mathcal{N}} + |\overline{\nabla}\nabla_{\mathscr{Z}}^J h||\nabla\nabla_{\mathscr{Z}}^K h| + \underbrace{\sum_{|J''|+|K''|\le|I|-2} |\nabla\nabla_{\mathscr{Z}}^{J''} h||\nabla\nabla_{\mathscr{Z}}^{K''} h|}_{\text{absent if } |I|\le 1}, \quad (16.3.4)$$

$$|(\mathrm{ii})| = \sum_{|J|+|K|\le|I|} |\mathcal{L}_{\mathscr{Z}}^J \mathcal{F}||\mathcal{L}_{\mathscr{Z}}^K \mathcal{F}|, \quad (16.3.5)$$

$$|(\mathrm{iii})| = \sum_{|I_1|+|I_2|+|I_3|\le|I|} |\nabla_{\mathscr{Z}}^{I_1} h||\nabla\nabla_{\mathscr{Z}}^{I_2} h||\nabla\nabla_{\mathscr{Z}}^{I_3} h| + \sum_{|I_1|+|I_2|+|I_3|\le|I|} |\nabla_{\mathscr{Z}}^{I_1} h||\mathcal{L}_{\mathscr{Z}}^{I_2} \mathcal{F}||\mathcal{L}_{\mathscr{Z}}^{I_3} \mathcal{F}| + \sum_{|I_1|+|I_2|+|I_3|\le|I|} |\mathcal{L}_{\mathscr{Z}}^{I_1} \mathcal{F}||\mathcal{L}_{\mathscr{Z}}^{I_2} \mathcal{F}||\mathcal{L}_{\mathscr{Z}}^{I_3} \mathcal{F}|. \quad (16.3.6)$$

The desired bound for $|(\mathrm{i})|$ was proved in Lemma 11.2 of [Lindblad and Rodnianski 2010] by using the decomposition $h = h^{(1)} + h^{(0)}$ and by combining Lemma 15.1 and inequalities (16.1.6a)–(16.1.6c). The term $|(\mathrm{ii})|$ is the main contribution to $|\nabla_{\mathscr{Z}}^I \mathfrak{H}|$ arising from the presence of nonzero electromagnetic fields. To bound $|(\mathrm{ii})|$ by the right-hand side of (16.3.2), we consider the cases $(|J| = \ell,\ |K| = 0)$, $(|J| = 0,\ |K| = \ell)$, $(|J| \le \ell - 1,\ |K| \le \lfloor \ell/2 \rfloor)$, and $(|J| \le \lfloor \ell/2 \rfloor,\ |K| \le \ell - 1)$; clearly this exhausts all possible cases. In the first two cases, we use (16.1.6a) to achieve the desired bound while in the last two we use (16.1.6b). The cubic terms from case (iii) can be similarly bounded by using the decomposition $h = h^{(1)} + h^{(0)}$ and by combining Lemma 15.1 and inequality (16.1.6b). □

Using the previous lemma, we now derive the desired integral inequalities corresponding to the $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathcal{Z}}^I \mathfrak{H}|^2$ term on the right-hand side of (16.3.1).

Lemma 16.9 (Integral estimates for $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathcal{Z}}^I \mathfrak{H}|^2 w(q)$; extension of [Lindblad and Rodnianski 2010, Lemma 11.3]). *Under the assumptions of Theorem 16.1, if I is any $\mathcal{Z}$-multi-index with $|I| \leq \ell$ and if ε is sufficiently small, then the following integral estimate holds for $t \in [0, T)$:*

$$\begin{aligned}\varepsilon^{-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)|\widehat{\nabla}_{\mathcal{Z}}^I\mathfrak{H}|^2 w(q)\,d^3x\,d\tau &\\ \lesssim \varepsilon \sum_{|J|\leq|I|}\int_0^t\int_{\Sigma_\tau}\left((1+\tau)^{-1}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^J h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^J\mathcal{F}\end{pmatrix}\right|^2 w(q) + |\overline{\nabla}\nabla_{\mathcal{Z}}^J h^{(1)}|^2 w'(q)\right) d^3x\,d\tau &\\ + \underbrace{\varepsilon\sum_{|J'|\leq|I|-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^{J'} h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^{J'}\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau}_{\text{absent if } |I|=0} + \varepsilon^3. &\quad (16.3.7)\end{aligned}$$

Proof. After squaring both sides of (16.3.2), multiplying by $\varepsilon^{-1}(1+t)w(q)$, using the inequality $(1+|q|)^{-1}(1+q_-)^{-2\mu}w(q) \lesssim w'(q)$ (i.e., inequality (12.1.2)) and the fact that $\mu+\mu' < \frac{1}{2}$, and integrating, we see that the only terms that are not manifestly bounded by the right-hand side of (16.3.7) are

$$\varepsilon^3\sum_{|J|\leq|I|}\int_0^t\int_{\Sigma_t}(1+\tau)^{-1}(1+|q|)^{-2}|\nabla_{\mathcal{Z}}^J h^{(1)}|^2 w(q)\,d^3x\,d\tau. \quad (16.3.8)$$

The desired bound for these terms can be achieved with the help of the Hardy inequalities of Proposition C.1, which imply that

$$\int_{\Sigma_t}(1+\tau)^{-1}(1+|q|)^{-2}|\nabla_{\mathcal{Z}}^J h^{(1)}|^2 w(q)\,d^3x \lesssim \int_{\Sigma_t}(1+\tau)^{-1}|\nabla\nabla_{\mathcal{Z}}^J h^{(1)}|^2 w(q)\,d^3x. \quad (16.3.9)$$

This concludes the proof. □

We now derive the desired integral inequalities corresponding to the $|\widehat{\nabla}_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}||\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|$ term on the right-hand side of (16.3.1).

Lemma 16.10 (Integral estimates for $|\widehat{\nabla}_{\mathcal{Z}}^I \widetilde{\Box}_g h^{(0)}||\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|w(q)$ [Lindblad and Rodnianski 2010, Lemma 11.4]). *Let M be the ADM mass. Under the assumptions of Theorem 16.1, if I is a $\mathcal{Z}$-multi-index satisfying $|I| \leq \ell$ and if ε is sufficiently small, then the following integral inequality holds for $t \in [0, T)$:*

$$\int_0^t\int_{\Sigma_\tau}|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h^{(0)}||\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|w(q)\,d^3x\,d\tau \lesssim M\sum_{|J|\le|I|}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-2}|\nabla\nabla_{\mathscr{Z}}^J h^{(1)}|^2 w(q)\,d^3x\,d\tau + M\sum_{|J|\le|I|}\int_0^t\left((1+\tau)^{-3/2}\sqrt{\int_{\Sigma_\tau}|\nabla\nabla_{\mathscr{Z}}^J h^{(1)}|^2 w(q)\,d^3x}\right)d\tau. \tag{16.3.10}$$

Proof. We first use the Cauchy–Schwarz inequality for integrals to obtain

$$\int_0^t\int_{\Sigma_\tau}|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h^{(0)}||\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|w(q)\,d^3x\,d\tau \le \int_0^t\left[\left(\int_{\Sigma_\tau}|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h^{(0)}|^2 w(q)\,d^3x\right)^{1/2}\times\left(\int_{\Sigma_\tau}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\right)^{1/2}\right]d\tau. \tag{16.3.11}$$

Furthermore, under the present assumptions, the previous proof of inequality (15.1.3b) remains valid. Thus, from (15.1.3b) and the Hardy inequalities of Proposition C.1, it follows that

$$\int_{\Sigma_t}|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h^{(0)}|^2 w(q)\,d^3x \lesssim M^2(1+t)^{-3}+M^2(1+t)^{-4}\sum_{|J|\le|I|}\int_{\Sigma_t}|\nabla\nabla_{\mathscr{Z}}^J h^{(1)}|^2 w(q)\,d^3x. \tag{16.3.12}$$

The estimate (16.3.10) now follows from (16.3.11), (16.3.12), and the inequalities $\sqrt{|a|+|b|}\lesssim\sqrt{|a|}+\sqrt{|b|}$ and $|ab|\lesssim a^2+b^2$. □

The following integral estimate for the commutator term $\varepsilon^{-1}(1+t)|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h_{\mu\nu}^{(1)}-\widetilde{\Box}_g(\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)})|^2$ on the right-hand side of (16.3.1) was proved in [Lindblad and Rodnianski 2010]. Its lengthy proof is similar to our proof of Lemma 16.17 below, and we do not repeat it here.

Lemma 16.11 (Integral estimates for $\varepsilon^{-1}|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h_{\mu\nu}^{(1)}-\widetilde{\Box}_g\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}|^2 w(q)$ [Lindblad and Rodnianski 2010, Lemma 11.5]). *Under the assumptions of Theorem 16.1, if I is a $\mathscr{Z}$-multi-index satisfying $1\le|I|\le\ell$ and if ε is sufficiently small, then the following integral inequality holds for $t\in[0,T)$:*

$$\varepsilon^{-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)\big|\widehat{\nabla}_{\mathscr{Z}}^I\widetilde{\Box}_g h_{\mu\nu}^{(1)}-\widetilde{\Box}_g\nabla_{\mathscr{Z}}^I h_{\mu\nu}^{(1)}\big|^2 w(q)\,d^3x\,d\tau \lesssim \varepsilon\sum_{|J|\le|I|}\int_0^t\int_{\Sigma_\tau}\big((1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^J h^{(1)}|^2 w(q)+|\overline{\nabla}\nabla_{\mathscr{Z}}^J h^{(1)}|^2 w'(q)\big)\,d^3x\,d\tau + \varepsilon\sum_{|J'|\le|I|-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}|\nabla\nabla_{\mathscr{Z}}^{J'}h^{(1)}|^2 w(q)\,d^3x\,d\tau+\varepsilon^3. \tag{16.3.13}$$

Combining Lemmas 16.7, 16.9, 16.10, and 16.11, we arrive at the following corollary:

Corollary 16.12 (Estimates for the energy integrals corresponding to the $h^{(1)}$ inhomogeneities). *Under the assumptions of Theorem 16.1, if $0\le k\le\ell$ and ε is sufficiently small, then the following integral inequality holds for $t\in[0,T)$:*

$$\begin{aligned}\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}|\mathfrak{H}^{(1;I)}||\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|\,d^3x\,d\tau \lesssim M\sum_{|I|\leq k}\int_0^t\left((1+\tau)^{-3/2}\sqrt{\int_{\Sigma_\tau}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|^2 w(q)\,d^3x}\right)d\tau \\ +\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^I h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^I\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau \\ +\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 w'(q)\,d^3x\,d\tau \\ +\underbrace{\varepsilon\sum_{|J|\leq k-1}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}\left|\begin{pmatrix}\nabla\nabla_{\mathcal{Z}}^J h^{(1)}\\ \mathcal{L}_{\mathcal{Z}}^J\mathcal{F}\end{pmatrix}\right|^2 w(q)\,d^3x\,d\tau}_{\text{absent if } k=0} \\ +\varepsilon^3. \qquad (16.3.14)\end{aligned}$$

This completes our analysis of the integral inequalities for the $h_{\mu\nu}^{(1)}$ inhomogeneities.

16.4. ***Integral inequalities for the $\mathcal{L}_{\mathcal{Z}}^I\mathcal{F}_{\mu\nu}$ inhomogeneities.*** In this section, we estimate the integrals corresponding to the inhomogeneous terms in the $\mathcal{L}_{\mathcal{Z}}^I$-commuted electromagnetic equations. More precisely, we analyze the integrals in Proposition 12.3 corresponding to the inhomogeneous terms $\mathfrak{F}_{(I)}^{\nu}$ in (16.2.2c). The main goal is to arrive at Corollary 16.18. As was the case for Corollary 16.12, the main point is that right-hand sides of the inequalities in Corollary 16.18 can be bounded in terms of time integrals of the energies $\mathcal{E}_{k;\gamma;\mu}(t)$ (this was carried out in inequality (16.2.5)).

We begin with the following lemma, which provides pointwise estimates for the wave-coordinate-controlled quantities $|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|_{\mathcal{L}\mathcal{L}}$ and $|\nabla\nabla_{\mathcal{Z}}^J h^{(1)}|_{\mathcal{L}\mathcal{T}}$ for $|I|\leq\ell$ and $|J|\leq\ell-1$. These pointwise estimates will be used to help to derive suitable integral estimates later in this section.

Lemma 16.13 (Pointwise estimates for $\sum_{|I|\leq k}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|_{\mathcal{L}\mathcal{L}}+\sum_{|J|\leq k-1}|\nabla\nabla_{\mathcal{Z}}^J h^{(1)}|_{\mathcal{L}\mathcal{T}}$). *Under the assumptions of Theorem 16.1, if $0\leq k\leq\ell$ and ε is sufficiently small, then the following pointwise inequality holds for $(t,x)\in[0,T)\times\mathbb{R}^3$:*

$$\begin{aligned}\sum_{|I|\leq k}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}|_{\mathcal{L}\mathcal{L}}+\overbrace{\sum_{|J|\leq k-1}|\nabla\nabla_{\mathcal{Z}}^J h^{(1)}|_{\mathcal{L}\mathcal{T}}}^{\text{absent if } k=0} \\ \lesssim\sum_{|I|\leq k}|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|+\varepsilon(1+t+|q|)^{-2}\chi_0(1/2\leq r/t\leq 3/4)+\varepsilon^2(1+t+|q|)^{-3} \\ +\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{1/2+\mu'}|\nabla\nabla_{\mathcal{Z}}^I h^{(1)}| \\ +\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-1/2+\mu'}|\nabla_{\mathcal{Z}}^I h^{(1)}|+\overbrace{\sum_{|J'|\leq k-2}|\nabla\nabla_{\mathcal{Z}}^{J'} h^{(1)}|}^{\text{absent if } k\leq 1}, \qquad (16.4.1)\end{aligned}$$

where $\chi_0\left(\frac{1}{2}\leq z\leq\frac{3}{4}\right)$ is the characteristic function of the interval $\left[\frac{1}{2},\frac{3}{4}\right]$.

Proof. Lemma 16.13 follows from Lemma 11.2 (for the tensor field $h_{\mu\nu}^{(1)}$) and the pointwise decay assumptions (16.1.6b) for $h_{\mu\nu}^{(1)}$. □

In the next lemma, we derive pointwise estimates for the term $|(\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu|$. This term appears in the second spacetime integral on the right-hand side of (12.2.1), which is our basic energy inequality for the Faraday tensor and its Lie derivatives. The pointwise estimates are preliminary estimates that will be used in the subsequent lemma to estimate the corresponding spacetime integral.

Lemma 16.14 (Pointwise estimates for $|(\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu|$). *Let* $\mathfrak{F}_{(I)}^\nu = \widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu + [N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda})]$ *be the inhomogeneous term* (8.1.2b) *in the equations of variation* (8.1.1b) *satisfied by* $\dot{\mathscr{F}} \stackrel{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}$. *Under the assumptions of Theorem 16.1, if* $0 \le k \le \ell$ *and* ε *is sufficiently small, then the following pointwise inequality holds for* $(t,x) \in [0,T)\times\mathbb{R}^3$:

$$\begin{aligned}\sum_{|I|\le k}|(\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu| \lesssim \varepsilon \sum_{|I|\le k}&(1+t+|q|)^{-1}\big(|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|^2 + |\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2\big)\\ &+\varepsilon\sum_{|I|\le k}(1+|q|)^{-1}(1+q_-)^{-2\upmu}|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|^2\\ &+\varepsilon\sum_{|I|\le |k|}(1+|q|)^{-1}(1+q_-)^{-2\upmu}\big(|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{L}\mathcal{N}}^2 + |\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|_{\mathcal{T}\mathcal{T}}^2\big)\\ &+\varepsilon\sum_{|I|\le k}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|. \end{aligned}\tag{16.4.2}$$

Proof. From (11.1.11a) with $X_\nu \stackrel{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{0\nu}$, the pointwise decay assumptions of Theorem 16.1, the decomposition $h = h^{(0)} + h^{(1)}$, and the $h^{(0)}$-decay estimates of Lemma 15.1, it follows that

$$\begin{aligned}\sum_{|I|\le k}|(\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^\nu| \lesssim \sum_{\substack{|I|\le k\\ |I_1|+|I_2|\le|I|}}&|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}||\overline{\nabla}\nabla_{\mathscr{Z}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|\\ &+\sum_{\substack{|I|\le k\\ |I_1|+|I_2|\le|I|}}|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}||\nabla\nabla_{\mathscr{Z}}^{I_1} h^{(1)}|\big(|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}} + |\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big)\\ &+\sum_{\substack{|I|\le k\\ |I_1|+|I_2|+|I_3|\le|I|}}|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}||\nabla\nabla_{\mathscr{Z}}^{I_1} h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|\\ &+\sum_{\substack{|I|\le k\\ |I_1|+|I_2|+|I_3|\le|I|}}|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}||\nabla_{\mathscr{Z}}^{I_1} h^{(1)}||\nabla\nabla_{\mathscr{Z}}^{I_2} h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|\\ &+\varepsilon\sum_{|I|\le k}(1+t+|q|)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2\\ &+\varepsilon\sum_{|I|\le k}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2\\ &+\varepsilon\sum_{|I|\le k}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I \mathscr{F}|^2. \end{aligned}\tag{16.4.3}$$

Inequality (16.4.2) now follows from the assumptions of Theorem 16.1, (16.4.3), and repeated application of algebraic inequalities of the form $|ab| \lesssim \varsigma a^2 + \varsigma^{-1}b^2$. As an example, we consider the term $|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}}$ in the case that $|I_1|\le|I|\le\lfloor\ell/2\rfloor$ (such an inequality must be satisfied by either $|I_1|$ or $|I_2|$). Then with the help of (16.1.6b) and the fact that $\upmu+\upmu'<\frac{1}{2}$, it follows that, if ε is sufficiently small, then

$$|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}} \lesssim \varepsilon(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2+\varepsilon^{-1}(1+t+|q|)|\nabla\nabla_{\mathscr{Z}}^{I_1}h^{(1)}|^2|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}}^2$$
$$\lesssim \varepsilon(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2+\varepsilon(1+|q|)^{-1}(1+q_-)^{-2\upmu}|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}}^2. \qquad (16.4.4)$$

We now observe that the right-hand side of the above inequality is manifestly bounded by the right-hand side of (16.4.2). □

We now use the pointwise estimates of the previous lemma to estimate part of the second spacetime integral on the right-hand side of (12.2.1). These estimates are easier than the corresponding estimates involving the commutator term $N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})$, which are derived in Lemma 16.17.

Lemma 16.15 (Integral estimates for $|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I\mathfrak{F}^\nu|w(q)$). *Under the assumptions of Lemma 16.14, if $0\le k\le\ell$ and ε is sufficiently small, then the following integral inequality holds for $t\in[0,T)$:*

$$\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\big|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})\widehat{\mathscr{L}}_{\mathscr{Z}}^I\mathfrak{F}^\nu\big|w(q)\,d^3x\,d\tau$$
$$\lesssim \varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2w(q)\,d^3x\,d\tau+\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2w(q)\,d^3x\,d\tau$$
$$+\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|^2w'(q)\,d^3x\,d\tau$$
$$+\varepsilon\sum_{|I|\le k}\int_0^t\int_{\Sigma_\tau}\big(|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|_{\mathcal{L}\mathcal{N}}^2+|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|_{\mathcal{T}\mathcal{T}}^2\big)w'(q)\,d^3x\,d\tau. \qquad (16.4.5)$$

Proof. Inequality (16.4.5) follows from multiplying inequality (16.4.2) by $w(q)$, integrating $\int_0^t\int_{\Sigma_\tau}d^3x\,d\tau$, using the fact that $(1+|q|)^{-1}(1+q_-)^{-2\upmu}w(q)\lesssim w'(q)$, and using the Hardy estimate (16.3.9) to bound the integral corresponding to the last sum on the right-hand side of (16.4.2) by the second sum on the right-hand side of (16.4.5). □

The next lemma is a companion to Lemma 16.14. In the lemma, we derive pointwise estimates for the term $|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})]|$. This term appears in the second spacetime integral on the right-hand side of (12.2.1), which is our basic energy inequality for the Faraday tensor and its Lie derivatives. As before, these pointwise estimates are preliminary estimates that will be used in the subsequent lemma to estimate the corresponding spacetime integral.

Lemma 16.16 (Pointwise estimates for $|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})]|$). *Let $N^{\#\mu\nu\kappa\lambda}\cdot\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})$ be the inhomogeneous commutator term* (8.1.3b) *in the equations of variation* (8.1.1b) *satisfied by $\dot{\mathscr{F}}_{\mu\nu}\overset{\text{def}}{=}\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\mu\nu}$. Under the assumptions of Theorem 16.1, if $1\le k\le\ell$ and ε is sufficiently small, then the following pointwise inequality holds for $(t,x)\in[0,T)\times\mathbb{R}^3$:*

$$\sum_{|I|\leq k}\big|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})\big)\big|$$
$$\lesssim \varepsilon\sum_{|I|\leq|k|}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2+\varepsilon\sum_{|I|\leq|k|}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2$$
$$+\varepsilon\sum_{|I|\leq|k|}(1+|q|)^{-1}(1+q_-)^{-2\mu}\big(|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|_{\mathcal{L}\mathcal{N}}^2+|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|_{\mathcal{T}\mathcal{T}}^2\big)$$
$$+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu}|\nabla_{\mathscr{Z}}^I h^{(1)}|_{\mathcal{L}\mathcal{L}}^2$$
$$+\varepsilon\sum_{|J|\leq k-1}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu}|\nabla_{\mathscr{Z}}^J h^{(1)}|_{\mathcal{L}\mathcal{T}}^2$$
$$+\underbrace{\varepsilon\sum_{|J'|\leq k-2}(1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^{J'}h^{(1)}|^2}_{\textit{absent if } k=1}$$
$$+\varepsilon\sum_{|J|\leq k-1}(1+t+|q|)^{-1+C\varepsilon}|\mathscr{L}_{\mathscr{Z}}^J\mathscr{F}|^2. \tag{16.4.6}$$

Proof. From inequality (11.1.11b) with $X_\nu \stackrel{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu}$, the pointwise decay assumptions of Theorem 16.1, together with the decomposition $h=h^{(0)}+h^{(1)}$ and the $h^{(0)}$ decay estimates of Lemma 15.1, it follows that

$$\sum_{|I|\leq k}\big|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})\big)\big|$$
$$\lesssim \sum_{\substack{|I|\leq k,\ |I'|\leq k\\ |J|\leq 1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I'}h^{(1)}|_{\mathcal{L}\mathcal{L}}|\mathscr{L}_{\mathscr{Z}}^J\mathscr{F}|+\sum_{\substack{|I|\leq k,\ |I'|\leq k\\ |J|\leq 1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^J h^{(1)}|_{\mathcal{L}\mathcal{L}}|\mathscr{L}_{\mathscr{Z}}^{I'}\mathscr{F}|$$
$$+\sum_{|I|\leq k}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2|h|_{\mathcal{L}\mathcal{T}}+\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k+1\\ |I_1|,|I_2|\leq k}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}|\big(|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{L}\mathcal{N}}+|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|_{\mathcal{T}\mathcal{T}}\big)$$
$$+\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k+1\\ |I_1|,|I_2|\leq k}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|_{\mathcal{L}\mathcal{N}}|\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|$$
$$+\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k+1\\ |I_1|,|I_2|\leq k}}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|$$
$$+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2+\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2$$
$$+\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k+1\\ |I_1|\leq k-1,\ |I_2|\leq k-1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}|_{\mathcal{L}\mathcal{L}}|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|$$
$$+\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k\\ |I_1|\leq k-1,\ |I_2|\leq k-1}}(1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}|_{\mathcal{L}\mathcal{T}}|\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|$$

$$+ \underbrace{\sum_{\substack{|I|\leq k\\ |I_1|+|I_2|\leq k-1\\ |I_1|\leq k-2,\ |I_2|\leq k-1}} (1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}|}_{\text{absent if } k=1}$$
$$+ \sum_{\substack{|I|\leq k\\ |I_1|+|I_2|+|I_3|\leq k+1\\ |I_1|,|I_2|,|I_3|\leq k}} (1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\nabla_{\mathscr{Z}}^{I_2}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|$$
$$+ \sum_{\substack{|I|\leq k\\ |I_1|+|I_2|+|I_3|\leq k+1\\ |I_1|,|I_2|,|I_3|\leq k}} (1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\nabla_{\mathscr{Z}}^{I_1}h^{(1)}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|$$
$$+ \sum_{\substack{|I|\leq k\\ |I_1|+|I_2|+|I_3|\leq k+1\\ |I_1|,|I_2|,|I_3|\leq k}} (1+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_1}\mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_2}\mathscr{F}||\mathscr{L}_{\mathscr{Z}}^{I_3}\mathscr{F}|. \tag{16.4.7}$$

We remark that the $\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2$ and $\varepsilon\sum_{|I|\leq k}(1+t+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2$ sums on the right-hand side of (16.4.7) account for all of the terms containing a factor $\nabla_{\mathscr{Z}}^J h^{(0)}$ for some J. Inequality (16.4.6) now follows from (16.4.7), the pointwise decay assumptions of Theorem 16.1 (including the implied estimates for $h^{(1)}$), and simple algebraic estimates of the form $|ab|\lesssim \varsigma a^2+\varsigma^{-1}b^2$ (as in (16.4.4)). □

The next lemma is a companion to Lemma 16.15. In the lemma, we use the pointwise estimates of the previous lemma to estimate the part of the second spacetime integral on the right-hand side of (12.2.1) that was not addressed by Lemma 16.15.

Lemma 16.17 (Integral estimates for $|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})[N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})]|$). *Under the assumptions of Lemma 16.14, if $1\leq k\leq \ell$ and ε is sufficiently small, then the following integral inequality holds for $t\in[0,T)$:*

$$\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\big|(\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{0\nu})\big(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}_{\kappa\lambda}-\widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu\mathscr{F}_{\kappa\lambda})\big)\big|w(q)\,d^3x\,d\tau$$
$$\lesssim \varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2 w(q)\,d^3x\,d\tau+\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau$$
$$+\varepsilon\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}\big(|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|^2+|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2_{\mathcal{L}\mathcal{N}}+|\mathscr{L}_{\mathscr{Z}}^I\mathscr{F}|^2_{\mathcal{T}\mathcal{T}}\big)w'(q)\,d^3x\,d\tau$$
$$+\underbrace{\varepsilon\sum_{|J'|\leq k-2}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1+C\varepsilon}|\nabla\nabla_{\mathscr{Z}}^{J'}h^{(1)}|^2 w(q)\,d^3x\,d\tau}_{\textit{absent if } k=1}$$
$$+\varepsilon\sum_{|J|\leq k-1}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1+C\varepsilon}|\mathscr{L}_{\mathscr{Z}}^J\mathscr{F}|^2 w(q)\,d^3x\,d\tau+\varepsilon^3. \tag{16.4.8}$$

Proof. We begin by multiplying both sides of (16.4.6) by $w(q)$ and integrating $\int_0^t \int_{\Sigma_\tau} d^3x\, d\tau$. The integrals corresponding to the first and last sums on the right-hand side of (16.4.6) are manifestly bounded by the first and next-to-last terms on the right-hand side of (16.4.8). Using also the fact that $(1+|q|)^{-1}(1+q_-)^{-2\mu} w(q) \lesssim w'(q)$, we deduce that the integral corresponding to the third sum on the right-hand side of (16.4.6) is bounded by the third sum on the right-hand side of (16.4.8).

To bound the integral corresponding to the second sum on the right-hand side of (16.4.6), we simply use the Hardy inequalities of Proposition C.1 to derive the inequality

$$\sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1}(1+|q|)^{-2} |\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\, d^3x\, d\tau \lesssim \sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1} |\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\, d^3x\, d\tau. \quad (16.4.9)$$

After multiplication by ε, we see that the right-hand side of the above inequality is manifestly bounded by the second sum on the right-hand side of (16.4.8). Using the same reasoning, we obtain the following bound for the integral corresponding to the next-to-last sum on the right-hand side of (16.4.6):

$$\sum_{|J'|\le k-2} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon}(1+|q|)^{-2} |\nabla_{\mathscr{Z}}^{J'} h^{(1)}|^2 w(q)\, d^3x\, d\tau \lesssim \sum_{|J'|\le k-2} \int_0^t \int_{\Sigma_\tau} (1+\tau+|q|)^{-1+C\varepsilon} |\nabla\nabla_{\mathscr{Z}}^{J'} h^{(1)}|^2 w(q)\, d^3x\, d\tau. \quad (16.4.10)$$

We then multiply (16.4.10) by ε and observe that the right-hand side of the resulting inequality is manifestly bounded by the right-hand side of (16.4.8).

To estimate the integrals corresponding to the fourth and fifth sums on the right-hand side of (16.4.6), we will make use of the weight $\widetilde{w}(q)$, which is defined by

$$\widetilde{w}(q) \overset{\text{def}}{=} \min\{w'(q),\, (1+t+|q|)^{-1+C\varepsilon} w(q)\}. \quad (16.4.11)$$

We note that by (12.1.2) the following inequality is satisfied:

$$\widetilde{w}(q) \lesssim (1+|q|)^{-1} w(q). \quad (16.4.12)$$

With the help of Lemma 16.13, (16.4.12), and the Hardy inequalities of Proposition C.1, we estimate the integral corresponding to the fourth sum on the right-hand side of (16.4.6) as follows:

$$\sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} (1+t+|q|)^{-1+C\varepsilon}(1+|q|)^{-(2+C\varepsilon)}(1+q_-)^{-2\mu} |\nabla_{\mathscr{Z}}^I h^{(1)}|^2_{\mathcal{L}\mathcal{L}} w(q)\, d^3x\, d\tau$$
$$\lesssim \sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} |\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2_{\mathcal{L}\mathcal{L}} \widetilde{w}(q)\, d^3x\, d\tau$$

$$\begin{aligned}
&\lesssim \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w'(q)\,d^3x\,d\tau \\
&\qquad + \varepsilon^2\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-4}\chi_0^2\Big(\frac{1}{2}<\frac{r}{t}<\frac{3}{4}\Big)w'(q)\,d^3x\,d\tau \\
&\qquad + \varepsilon^4\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-6}w'(q)\,d^3x\,d\tau \\
&\qquad + \varepsilon^2\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau \\
&\qquad + \varepsilon^2\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau \\
&\qquad + \underbrace{\sum_{|J'|\leq k-2}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1+C\varepsilon}|\nabla\nabla_{\mathscr{Z}}^{J'} h^{(1)}|^2 w(q)\,d^3x\,d\tau}_{\text{absent if } k=1} \\
&\lesssim \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}|\overline{\nabla}\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w'(q)\,d^3x\,d\tau \\
&\qquad + \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau \\
&\qquad + \underbrace{\sum_{|J'|\leq k-2}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1+C\varepsilon}|\nabla\nabla_{\mathscr{Z}}^{J'} h^{(1)}|^2 w(q)\,d^3x\,d\tau}_{\text{absent if } k=1} + \varepsilon^2, \qquad (16.4.13)
\end{aligned}$$

where to pass to the final inequality we have again used Proposition C.1 to estimate

$$\sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau+|q|)^{-1}(1+|q|)^{-2}|\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau \lesssim \sum_{|I|\leq k}\int_0^t\int_{\Sigma_\tau}(1+\tau)^{-1}|\nabla\nabla_{\mathscr{Z}}^I h^{(1)}|^2 w(q)\,d^3x\,d\tau.$$

After multiplying both sides of (16.4.13) by ε, we see that the resulting right-hand side is manifestly bounded by the right-hand side of (16.4.8) as desired. The integral corresponding to the fifth sum on the right-hand side of (16.4.6) can be bounded via the same reasoning. □

Combining Lemmas 16.15 and 16.17, we arrive at the following corollary:

Corollary 16.18 (Estimates for the energy integrals corresponding to the $\mathscr{F}$ inhomogeneities). *Let*

$$\mathfrak{F}_{(I)}^{\nu} = \widehat{\mathscr{L}}_{\mathscr{Z}}^I \mathfrak{F}^{\nu} + [N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\kappa\lambda} - \widehat{\mathscr{L}}_{\mathscr{Z}}^I(N^{\#\mu\nu\kappa\lambda}\nabla_\mu \mathscr{F}_{\kappa\lambda})]$$

be the inhomogeneous term (8.1.3b) *in the equations of variation* (8.1.1b) *satisfied by* $\dot{\mathscr{F}}_{\mu\nu} \overset{\text{def}}{=} \mathscr{L}_{\mathscr{Z}}^I \mathscr{F}_{\mu\nu}$. *Under the assumptions of Theorem 16.1, if* $0 \leq k \leq \ell$ *and* ε *is sufficiently small, then the following integral* *inequality holds for* $t \in [0, T)$:

$$\sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} |(\mathcal{L}_{\mathcal{Z}}^I \mathscr{F}_{0\nu}) \mathfrak{F}_{(I)}^\nu| w(q)\, d^3x\, d\tau$$
$$\lesssim \varepsilon \sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1} \left| \begin{pmatrix} \nabla\nabla_{\mathcal{Z}}^I h^{(1)} \\ \mathcal{L}_{\mathcal{Z}}^I \mathscr{F} \end{pmatrix} \right|^2 w(q)\, d^3x\, d\tau$$
$$+ \varepsilon \sum_{|I|\le k} \int_0^t \int_{\Sigma_\tau} \big(|\overline{\nabla}\nabla_{\mathcal{Z}}^I h^{(1)}|^2 + |\mathcal{L}_{\mathcal{Z}}^I \mathscr{F}|_{\mathcal{L}\mathcal{N}}^2 + |\mathcal{L}_{\mathcal{Z}}^I \mathscr{F}|_{\mathcal{T}\mathcal{T}}^2 \big) w'(q)\, d^3x\, d\tau$$
$$+ \underbrace{\varepsilon \sum_{|J|\le k-1} \int_0^t \int_{\Sigma_\tau} (1+\tau)^{-1+C\varepsilon} \left| \begin{pmatrix} \nabla\nabla_{\mathcal{Z}}^J h^{(1)} \\ \mathcal{L}_{\mathcal{Z}}^J \mathscr{F} \end{pmatrix} \right|^2 w(q)\, d^3x\, d\tau}_{\textit{absent if } k=0} + \varepsilon^3. \quad (16.4.14)$$

Appendix A: Weighted Sobolev–Moser inequalities

The propositions and corollaries stated in this section were used in Section 10 to relate the smallness condition on the abstract initial data to a smallness condition on the initial energy of the corresponding solution to the reduced equations. The lemmas we state are slight extensions of Lemmas 2.4 and 2.5 of [Choquet-Bruhat and Christodoulou 1981] while the corollaries are easy (and nonoptimal) consequences of the lemmas. Throughout the appendix, we use the abbreviations

$$C_\eta^\ell \stackrel{\text{def}}{=} C_\eta^\ell(\mathbb{R}^3), \quad H_\eta^\ell \stackrel{\text{def}}{=} H_\eta^\ell(\mathbb{R}^3),$$

and so on (see Definitions 10.1 and 10.2). Furthermore, (x^1, x^2, x^3) denotes the standard Euclidean coordinate system on $\mathbb{R}^3$ and $|x| \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

Proposition A.1 (Weighted Sobolev embedding [Choquet-Bruhat and Christodoulou 1981, Lemma 2.4]). *Let ℓ and ℓ' be integers, and let η and η' be real numbers subject to the constraints $\ell' < \ell - \frac{3}{2}$ and $\eta' < \eta + \frac{3}{2}$. Assume that $v \in H_\eta^\ell$. Then $v \in C_{\eta'}^{\ell'}$, and*

$$\|v\|_{C_{\eta'}^{\ell'}} \lesssim \|v\|_{H_\eta^\ell}. \quad (A.1)$$

Proposition A.2 (Weighted Sobolev multiplication properties [Choquet-Bruhat and Christodoulou 1981, Lemma 2.5]). *Let $\ell_1, \dots, \ell_p \ge 0$ be integers, and let $\eta_1, \dots, \eta_p$ be real numbers. Suppose that $v_j \in H_{\eta_j}^{\ell_j}$ for $j = 1, \dots, p$. Assume that the integer ℓ satisfies $0 \le \ell \le \min\{\ell_1, \dots, \ell_p\}$ and $\ell \le \sum_{j=1}^p \ell_j - (p-1)\frac{3}{2}$ and that $\eta < \sum_{j=1}^p \eta_j + (p-1)\frac{3}{2}$. Then*

$$\prod_{j=1}^p v_j \in H_\eta^\ell, \quad (A.2)$$

and the multiplication map

$$H_{\eta_1}^{\ell_1} \times \cdots \times H_{\eta_p}^{\ell_p} \to H_\ell^\eta, \qquad (v_1, \dots, v_p) \to \prod_{j=1}^p v_j \quad (A.3)$$

is continuous.

Corollary A.3. *Let $\ell \geq 2$ be an integer, and let $\eta \geq 0$. Assume that $v_j \in H^\ell_\eta$ for $j = 1, \dots, p$ and that $I_1, \dots, I_p$ are $\underline{\nabla}$-multi-indices satisfying $\sum_{j=1}^p |I_j| \leq \ell$. Then*

$$(1 + |x|^2)^{(\eta + \sum_{j=1}^p |I_j|)/2} \prod_{i=1}^p \underline{\nabla}^{I_i} v_i \in L^2 \tag{A.4}$$

and

$$\left\| (1 + |x|^2)^{(\eta + \sum_{j=1}^p |I_j|)/2} \prod_{i=1}^p \underline{\nabla}^{I_i} v_i \right\|_{L^2} \lesssim \prod_{i=1}^p \|v_i\|_{H^\ell_\eta}. \tag{A.5}$$

Corollary A.4. *Let $\ell \geq 2$ be an integer, let $\mathfrak{K}$ be a compact set, and let $F(\cdot) \in C^\ell(\mathfrak{K})$ be a function. Assume that v_1 is a function on $\mathbb{R}^3$ such that $v_1(\mathbb{R}^3) \subset \mathfrak{K}$. Furthermore, assume that $\underline{\nabla} v_1, v_2 \in H^\ell_\eta$. Then $(F \circ v_1) v_2 \in H^\ell_\eta$, and*

$$\|(F \circ v_1) v_2\|_{H^\ell_\eta} \lesssim \|v_2\|_{H^\ell_\eta} |F|_{\mathfrak{K}} + \|(1 + |x|) v_2\|_{L^\infty} \|\underline{\nabla} v_1\|_{H^{\ell-1}_\eta} \sum_{j=1}^\ell |F^{(j)}|_{\mathfrak{K}} \|v_1\|^{j-1}_{L^\infty}, \tag{A.6}$$

where $F^{(j)}$ denotes the array of all j-th order partial derivatives of F with respect to its arguments and $|F^{(j)}|_{\mathfrak{K}} \overset{\text{def}}{=} \sup_{v \in \mathfrak{K}} |F^{(j)}(v)|$.

Appendix B: Weighted Klainerman–Sobolev inequalities

In this section, we recall the weighted Klainerman–Sobolev inequalities that were proved in [Lindblad and Rodnianski 2010]. Throughout this section, the weight function $w(q)$ is defined by

$$w \overset{\text{def}}{=} w(q) \overset{\text{def}}{=} \begin{cases} 1 + (1 + |q|)^{1+2\gamma} & \text{if } q > 0, \\ 1 + (1 + |q|)^{-2\mu} & \text{if } q < 0. \end{cases} \tag{B.1}$$

In this section, we assume that γ and μ are fixed constants satisfying $0 < \gamma < 1$ and $0 < \mu < \frac{1}{2}$. It easily follows from (B.1) that

$$w' \overset{\text{def}}{=} w'(q) = \begin{cases} (1 + 2\gamma)(1 + |q|)^{2\gamma} & \text{if } q > 0, \\ 2\mu(1 + |q|)^{-1-2\mu} & \text{if } q < 0, \end{cases} \tag{B.2}$$

and

$$w' \leq 4(1 + |q|)^{-1} w \leq 16\mu^{-1}(1 + q_-)^{2\mu} w'. \tag{B.3}$$

Proposition B.1 (Weighted Klainerman–Sobolev inequality [Lindblad and Rodnianski 2010, Proposition 14.1]). *There exists a $C > 0$ such that, for all $\phi(t, \cdot) \in C_0^\infty(\mathbb{R}^3)$, the following inequality holds*:

$$(1 + t + |x|)[(1 + |q|) w(q)]^{1/2} |\phi(t, x)| \leq C \sum_{|I| \leq 3} \left\| w^{1/2} \nabla^I_{\mathscr{Z}} \phi(t, \cdot) \right\|_{L^2}, \qquad q \overset{\text{def}}{=} |x| - t. \tag{B.4}$$

Furthermore,

$$(1 + t + |x|)[(1 + |q|) w(q)]^{1/2} |\nabla \phi(t, x)| \leq C \sum_{|I| \leq 3} \left\| w^{1/2} \nabla \nabla^I_{\mathscr{Z}} \phi(t, \cdot) \right\|_{L^2}, \qquad q \overset{\text{def}}{=} |x| - t. \tag{B.5}$$

Proof. Equation (B.4) was proved in the paper cited; (B.5) follows from Lemma 6.11 and (B.4). □

Appendix C: Hardy-type inequalities

In this section, we recall the weighted Hardy-type inequalities proved in [Lindblad and Rodnianski 2010].

Proposition C.1 (Hardy inequalities [Lindblad and Rodnianski 2010, Corollary 13.3]). *Let $\gamma > 0$ and $\mu > 0$, $q \stackrel{\text{def}}{=} |x| - t$, and let $w(q)$ and $w'(q)$ be as defined in* (B.1) *and* (B.2), *respectively. Then for any $-1 \leq a \leq 1$, there exists a $C > 0$ such that, for all $\phi \in C_0^\infty(\mathbb{R}^3)$, we have the integral inequality*

$$\int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}(1+|q|)^{-2}|\phi|^2 w(q)\, d^3x \leq C \int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}|\partial_r \phi|^2 w(q)\, d^3x, \tag{C.1}$$

where $\partial_r = \omega^b \partial_b$, $\omega^j \stackrel{\text{def}}{=} x^j/r$, denotes the radial vector field.

If in addition $a < 2\min\{\gamma, \mu\}$, then with

$$\widetilde{w}(q) \stackrel{\text{def}}{=} \min\{w'(q), (1+t+|q|)^{-1+a} w(q)\}, \tag{C.2}$$

there exists a constant $C > 0$ such that the integral inequality

$$\int_{\mathbb{R}^3} (1+t+|q|)^{-1+a}(1+|q|)^{-(a+2)}(1+q_-)^{-2\mu}|\phi|^2 w(q)\, d^3x \leq C \int_{\mathbb{R}^3} |\partial_r \phi|^2 \widetilde{w}(q)\, d^3x, \tag{C.3}$$

holds, where $q_- \stackrel{\text{def}}{=} |q|$ if $q \leq 0$ and $q_- = 0$ if $q > 0$.

Corollary C.2. *Assume the hypotheses of Proposition C.1, and let $P_{\mu\nu}$ be a type-$\binom{0}{2}$ tensor field. Let $\mathcal{V}$ and $\mathcal{W}$ be any two of the subsets of null frame-field vectors defined in* (5.1.12). *Then the same conclusions of Proposition C.1 hold if we replace $|\phi|$ and $|\partial_r \phi|$ with the contraction seminorms $|P|_{\mathcal{V}\mathcal{W}}$ and $|\nabla P|_{\mathcal{V}\mathcal{W}}$, respectively, where the contraction seminorms are defined in Definition 5.8.*

Proof. Let $\not{m}$ be the first fundamental form of the $S_{r,t}$ defined in (5.1.4b), and recall that the tensor $\not{m}_\mu^{\ \nu}$ projects m-orthogonally onto the $S_{r,t}$. Since $\partial_r = \frac{1}{2}(L - \underline{L})$, it follows from (5.1.9a), (5.1.9b), and (5.1.10) that

$$\partial_r(\underline{L}^\kappa \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa \underline{L}^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda}, \tag{C.4}$$

$$\partial_r(\underline{L}^\kappa L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa L^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda}, \tag{C.5}$$

$$\partial_r(L^\kappa \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa \underline{L}^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda}, \tag{C.6}$$

$$\partial_r(L^\kappa L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa L^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{C.7}$$

$$\partial_r(\not{m}_\mu^{\ \kappa} \underline{L}^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\not{m}_\mu^{\ \kappa} \underline{L}^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{C.8}$$

$$\partial_r(\underline{L}^\kappa \not{m}_\mu^{\ \lambda} P_{\kappa\lambda}) = \tfrac{1}{2}\underline{L}^\kappa \not{m}_\mu^{\ \lambda} (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{C.9}$$

$$\partial_r(\not{m}_\mu^{\ \kappa} L^\lambda P_{\kappa\lambda}) = \tfrac{1}{2}\not{m}_\mu^{\ \kappa} L^\lambda (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{C.10}$$

$$\partial_r(L^\kappa \not{m}_\mu^{\ \lambda} P_{\kappa\lambda}) = \tfrac{1}{2}L^\kappa \not{m}_\mu^{\ \lambda} (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \qquad (\mu = 0, 1, 2, 3), \tag{C.11}$$

$$\partial_r(\not{m}_\mu^{\ \kappa} \not{m}_\nu^{\ \lambda} P_{\kappa\lambda}) = \tfrac{1}{2}\not{m}_\mu^{\ \kappa} \not{m}_\nu^{\ \lambda} (\nabla_L - \nabla_{\underline{L}}) P_{\kappa\lambda} \quad (\mu, \nu = 0, 1, 2, 3). \tag{C.12}$$

That is to say, ∂_r commutes with the null decomposition of P. The conclusion of the corollary now easily follows from applying the proposition with ϕ equal to the scalar-valued functions $\underline{L}^\kappa \underline{L}^\lambda P_{\kappa\lambda}$, $\underline{L}^\kappa L^\lambda P_{\kappa\lambda}, \ldots,$ $\not{m}_\mu^{\ \kappa} \not{m}_\nu^{\ \lambda} P_{\kappa\lambda}$, respectively. □

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JARED SPECK: jspeck@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Room 2-163, Cambridge, MA 02139-4307, United States
http://math.mit.edu/~jspeck/

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