

# ANALYSIS & PDE

Volume 7

No. 6

2014

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WITH VARYING POLES**

## ON THE EIGENVALUES OF AHARONOV–BOHM OPERATORS WITH VARYING POLES

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We consider a magnetic operator of Aharonov–Bohm type with Dirichlet boundary conditions in a planar domain. We analyze the behavior of its eigenvalues as the singular pole moves in the domain. For any value of the circulation we prove that the  $k$ -th magnetic eigenvalue converges to the  $k$ -th eigenvalue of the Laplacian as the pole approaches the boundary. We show that the magnetic eigenvalues depend in a smooth way on the position of the pole, as long as they remain simple. In case of half-integer circulation, we show that the rate of convergence depends on the number of nodal lines of the corresponding magnetic eigenfunction. In addition, we provide several numerical simulations both on the circular sector and on the square, which find a perfect theoretical justification within our main results, together with the ones by the first author and Helffer in *Exp. Math.* **20**:3 (2011), 304–322.

### 1. Introduction

Let  $\Omega \subset \mathbb{R}^2$  be an open, simply connected, bounded set. For  $a = (a_1, a_2)$  varying in  $\Omega$ , we consider the magnetic Schrödinger operator

$$(i\nabla + A_a)^2 = -\Delta + i\nabla \cdot A_a + 2iA_a \cdot \nabla + |A_a|^2$$

acting on functions with zero boundary conditions on  $\partial\Omega$ , where  $A_a$  is a magnetic potential of Aharonov–Bohm type, singular at the point  $a$ . More specifically, the magnetic potential has the form

$$A_a(x) = \alpha \left( -\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right) + \nabla\chi \quad (1-1)$$

where  $x = (x_1, x_2) \in \Omega \setminus \{a\}$ ,  $\alpha \in (0, 1)$  is a fixed constant and  $\chi \in C^\infty(\bar{\Omega})$ . Since the regular part  $\chi$  does not play a significant role, throughout the paper we will suppose without loss of generality that  $\chi \equiv 0$ .

The magnetic field associated to this potential is a  $2\pi\alpha$ -multiple of the Dirac delta at  $a$ , orthogonal to the plane. A quantum particle moving in  $\Omega \setminus \{a\}$  will be affected by the magnetic potential, although it remains in a region where the magnetic field is zero (Aharonov–Bohm effect [1959]). We can think of the particle as being affected by the nontrivial topology of the set  $\Omega \setminus \{a\}$ .

We are interested in studying the behavior of the spectrum of the operator  $(i\nabla + A_a)^2$  as  $a$  moves in the domain and when it approaches its boundary. By standard spectral theory, the spectrum of such an

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Noris and Terracini are partially supported by the PRIN2009 grant “Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations”. Nys is a Research Fellow of the Belgian Fonds de la Recherche Scientifique (FNRS). Bonnaillie-Noël is supported by the ANR (Agence Nationale de la Recherche), project OPTIFORM no. ANR-12-BS01-0007-02.

MSC2010: 35J10, 35P20, 35Q40, 35Q60, 35J75.

Keywords: magnetic Schrödinger operators, eigenvalues, nodal domains.

operator consists of a diverging sequence of real positive eigenvalues (see Section 2). We will denote by  $\lambda_j^a$ ,  $j \in \mathbb{N} = \{1, 2, \dots\}$ , the eigenvalues counted with their multiplicity (see (2-3)) and by  $\varphi_j^a$  the corresponding eigenfunctions, normalized in the  $L^2(\Omega)$ -norm. We shall focus our attention on the extremal and critical points of the maps  $a \mapsto \lambda_j^a$ .

One motivation for our study is that, in the case of half-integer circulation, critical positions of the moving pole can be related to optimal partition problems. The link between spectral minimal partitions and nodal domains of eigenfunctions has been investigated in full detail in [Helffer 2010; Helffer and Hoffmann-Ostenhof 2010; 2013; Helffer et al. 2009; 2010a; 2010b]. By the results in [Helffer et al. 2009] in two dimensions, the boundary of a minimal partition is the union of finitely many regular arcs, meeting at some multiple intersection points dividing the angle in an equal fashion. If the multiplicity of the clustering domains is even, then the partition is nodal, that is to say it is the nodal set of an eigenfunction. On the other hand, the results in [Bonnaillie-Noël and Helffer 2011; Bonnaillie-Noël et al. 2009; 2010; Helffer and Hoffmann-Ostenhof 2013; Noris and Terracini 2010] suggest that the minimal partitions featuring a clustering point of odd multiplicity should be related to the nodal domains of eigenfunctions of Aharonov–Bohm Hamiltonians which corresponds to a critical value of the eigenfunction with respect to the moving pole.

Our first result states the continuity of the magnetic eigenvalues with respect to the position of the singularity, up to the boundary.

**Theorem 1.1.** *For every  $j \in \mathbb{N}$ , the function  $a \in \Omega \mapsto \lambda_j^a \in \mathbb{R}$  admits a continuous extension on  $\bar{\Omega}$ . More precisely, as  $a \rightarrow \partial\Omega$ , we have that  $\lambda_j^a$  converges to  $\lambda_j$ , the  $j$ -th eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$ .*

We remark that this holds for every  $\alpha \in (0, 1)$ ,  $\alpha$  being the circulation of the magnetic potential introduced in (1-1). As an immediate consequence of this result, we have that this map, being constant on  $\partial\Omega$ , always admits an interior extremal point.

**Corollary 1.2.** *For every  $j \in \mathbb{N}$ , the function  $a \in \Omega \mapsto \lambda_j^a \in \mathbb{R}$  has an extremal point in  $\Omega$ .*

Heuristically, we can interpret the previous theorem thinking at a magnetic potential  $A_b$ , singular at  $b \in \partial\Omega$ . The domain  $\Omega \setminus \{b\}$  coincides with  $\Omega$ , so that it has a trivial topology. For this reason, the magnetic potential is not experienced by a particle moving in  $\Omega$  and the operator acting on the particle is simply the Laplacian.

This result was first conjectured in the case  $j = 1$  in [Noris and Terracini 2010], where it was applied to show that the function  $a \mapsto \lambda_1^a$  has a global interior maximum, where it is not differentiable, corresponding to an eigenfunction of multiplicity exactly two. Numerical simulations in [Bonnaillie-Noël and Helffer 2011] supported the conjecture for every  $j$ . During the completion of this work, we became aware that the continuity of the eigenvalues with respect to multiple moving poles has been obtained independently in [Léna 2014].

We remark that the continuous extension up to the boundary is a nontrivial issue because the nature of the operator changes as  $a$  approaches  $\partial\Omega$ . This fact can be seen in the more specific case when  $\alpha = \frac{1}{2}$ , which is equivalent to the standard Laplacian on the double covering (see [Helffer et al. 1999; 2000; Noris and Terracini 2010]). We go then from a problem on a fixed domain with a varying operator (which

depends on the singularity  $a$ ) to a problem with a fixed operator (the Laplacian) and a varying domain (for the convergence of the eigenvalues of elliptic operators on varying domains, we refer to [Arendt and Daners 2007; Daners 2003]). In this second case, the singularity is transferred from the operator into the domain. Indeed, when  $a$  approaches the boundary, the double covering develops a corner at the origin. In particular, Theorem 7.1 in [Helffer et al. 2010a] cannot be applied in our case since there is no convergence in capacity of the domains.

In the light of the previous corollary it is natural to study additional properties of the extremal points. Our aim is to establish a relation between the nodal properties of  $\varphi_j^b$  and the vanishing order of  $|\lambda_j^a - \lambda_j^b|$  as  $a \rightarrow b$ . First of all we will need some additional regularity, which is guaranteed by the following theorem in the case of simple eigenvalues and regular domain.

**Theorem 1.3.** *Let  $b \in \Omega$ . If  $\lambda_j^b$  is simple then, for every  $j \in \mathbb{N}$ , the map  $a \in \Omega \mapsto \lambda_j^a$  is locally of class  $C^\infty$  in a neighborhood of  $b$ .*

In order to examine the link with the nodal set of eigenfunctions, we shall focus on the case  $\alpha = \frac{1}{2}$ . In this case, it was proved in [Helffer et al. 1999; 2000; Noris and Terracini 2010] (see also Proposition 2.4 below) that the eigenfunctions have an odd number of nodal lines ending at the pole  $a$  and an even number of nodal lines meeting at zeros different from  $a$ . We say that an eigenfunction has a zero of order  $k/2$  at a point if it has  $k$  nodal lines meeting at such point. More precisely, we give the following definition.

**Definition 1.4** (zero of order  $k/2$ ). Let  $f : \Omega \rightarrow \mathbb{C}$ ,  $b \in \Omega$  and  $k \in \mathbb{N}$ .

- (i) If  $k$  is even, we say that  $f$  has a zero of order  $k/2$  at  $b$  if it is of class at least  $C^{k/2}$  in a neighborhood of  $b$  and  $f(b) = \dots = D^{k/2-1} f(b) = 0$ , while  $D^{k/2} f(b) \neq 0$ .
- (ii) If  $k$  is odd, we say that  $f$  has a zero of order  $k/2$  at  $b$  if  $f(x^2)$  has a zero of order  $k$  at  $b$  (here  $x^2$  is the complex square).

**Theorem 1.5** [Noris and Terracini 2010, Theorem 1.1]. *Suppose that  $\alpha = \frac{1}{2}$ . Fix any  $j \in \mathbb{N}$ . If  $\varphi_j^b$  has a zero of order  $\frac{1}{2}$  at  $b \in \Omega$  then either  $\lambda_j^b$  is not simple, or  $b$  is not an extremal point of the map  $a \mapsto \lambda_j^a$ .*

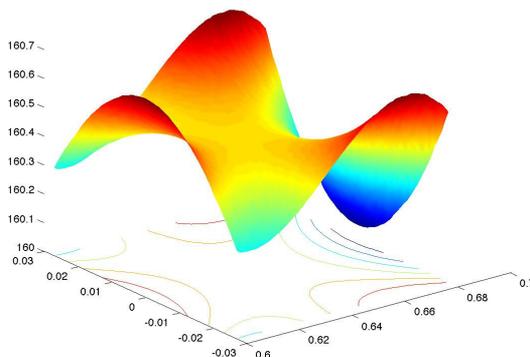
**Remark 1.6.** By joining this result with Corollary 1.2, we find that there is at least one extremal interior point (for the  $j$ -th eigenvalue) enjoying an alternative between degeneracy of the corresponding eigenvalue and the presence of a triple (or multiple) point nodal configuration for the corresponding eigenfunction.

Under the assumption that  $\lambda_j^b$  is simple, we prove here that the converse of Theorem 1.5 also holds. In addition, we show that the number of nodal lines of  $\varphi_j^b$  at  $b$  determines the order of vanishing of  $|\lambda_j^b - \lambda_j^a|$  as  $a \rightarrow b$ .

**Theorem 1.7.** *Suppose that  $\alpha = \frac{1}{2}$ . Fix any  $j \in \mathbb{N}$ . If  $\lambda_j^b$  is simple and  $\varphi_j^b$  has a zero of order  $k/2$  at  $b \in \Omega$ , with  $k \geq 3$  odd, then*

$$|\lambda_j^a - \lambda_j^b| \leq C|a - b|^{(k+1)/2} \quad \text{as } a \rightarrow b, \tag{1-2}$$

for a constant  $C > 0$  independent of  $a$ .



**Figure 1.**  $a \mapsto \lambda_3^a$ ,  $a \in \left\{ \left( \frac{m}{1000}, \frac{n}{1000} \right), 600 \leq m \leq 680, 0 \leq n \leq 30 \right\}$ .

In conclusion, in the case of half-integer circulation we have [Figure 1](#), which completes [Corollary 1.2](#).

**Corollary 1.8.** *Suppose that  $\alpha = \frac{1}{2}$ . Fix any  $j \in \mathbb{N}$ . If  $b \in \Omega$  is an extremal point of  $a \mapsto \lambda_j^a$  then either  $\lambda_j^b$  is not simple, or  $\varphi_j^b$  has a zero of order  $k/2$  at  $b$ ,  $k \geq 3$  odd. In this second case, the first  $(k - 1)/2$  terms of the Taylor expansion of  $\lambda_j^a$  at  $b$  cancel.*

**Remark 1.9.** When the order of the zero of the eigenfunction is at least  $\frac{3}{2}$ , the corresponding nodal set determines a regular partition of the domain, in the sense of [\[Helffer et al. 2009\]](#), where such a notion has been introduced and linked with the properties of boundaries of spectral minimal partitions. It is interesting to connect the variational properties of the partition with the characterization of the pole  $a$  as a critical point of the map  $a \mapsto \lambda_j^a$ . To this aim we performed a number of numerical computations. Rather surprisingly, the configurations of the triple (or multiple) point almost never appear at the maximum or minimum values of the eigenvalues, which are almost always nondifferentiability points, thus corresponding to degenerate eigenvalues. In the case of the angular sector, we observe in particular that any triple point configuration corresponds to a degenerate saddle point as illustrated in [Figure 1](#) (see also [Figures 7](#), [top](#), and [4](#)).

In [\[Noris et al.  \$\geq\$  2014\]](#) we intend to extend [Theorem 1.7](#) to the case  $b \in \partial\Omega$ . In this case we know from [Theorem 1.1](#) that  $\lambda_j^a$  converges to  $\lambda_j$  as  $a \rightarrow b \in \partial\Omega$  and we aim to estimate the rate of convergence depending on the number of nodal lines of  $\varphi_j$  at  $b$ , motivated by the numerical simulations in [\[Bonnaillie-Noël and Helffer 2011\]](#).

We would like to mention that the relation between the presence of a magnetic field and the number of nodal lines of the eigenfunctions, as well as the consequences on the behavior of the eigenvalues, have been recently studied in different contexts, giving rise to surprising conclusions. In [\[Berkolaiko 2013; Colin de Verdière 2013\]](#) the authors consider a magnetic Schrödinger operator on graphs and study the behavior of its eigenvalues as the circulation of the magnetic field varies. In particular, they consider an arbitrary number of singular poles, having circulation close to 0. They prove that the simple eigenvalues of the Laplacian (zero circulation) are critical values of the function  $\alpha \mapsto \lambda_j(\alpha)$ , which associates to the circulation  $\alpha$  the corresponding eigenvalue. In addition, they show that the number of nodal lines of the Laplacian eigenfunctions depends on the Morse index of  $\lambda_j(0)$ .

The paper is organized as follows. In [Section 2](#), we define the functional space  $\mathcal{D}_{A_a}^{1,2}(\Omega)$ , which is the more suitable space to consider our problem. We also recall a Hardy-type inequality and a theorem about the regularity of the eigenfunctions  $\varphi_j^a$ . Finally, in the case of a half-integer circulation, we recall the equivalence between the problem we consider and the standard Laplacian equation on the double covering. The first part of [Theorem 1.1](#), concerning the interior continuity of the eigenvalues  $\lambda_j^a$  is proved in [Section 3](#) and the second part concerning the extension to the boundary is studied in [Section 4](#). In [Section 5](#), we prove [Theorem 1.3](#). [Section 6](#) contains the proof of [Theorem 1.7](#). Finally, [Section 7](#) illustrates these results in the case of the angular sector of aperture  $\pi/4$  and the square.

### 2. Preliminaries

We will work in the functional space  $\mathcal{D}_{A_a}^{1,2}(\Omega)$ , which is defined as the completion of  $C_0^\infty(\Omega \setminus \{a\})$  with respect to the norm

$$\|u\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)} := \|(i\nabla + A_a)u\|_{L^2(\Omega)}.$$

As proved in [[Noris and Terracini 2010](#), Lemma 2.1], for example, we have an equivalent characterization

$$\mathcal{D}_{A_a}^{1,2}(\Omega) = \left\{ u \in H_0^1(\Omega), \frac{u}{|x-a|} \in L^2(\Omega) \right\},$$

and moreover we have that  $\mathcal{D}_{A_a}^{1,2}(\Omega)$  is continuously embedded in  $H_0^1(\Omega)$ : there exists a constant  $C > 0$  such that for every  $u \in \mathcal{D}_{A_a}^{1,2}(\Omega)$  we have

$$\|u\|_{H_0^1(\Omega)} \leq C \|u\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}. \tag{2-1}$$

This is proved by making use of a Hardy-type inequality by Laptev and Weidl [[1999](#)]. Such an inequality also holds for functions with nonzero boundary trace, as shown in [[Melgaard et al. 2004](#), Lemma 7.4] (see also [[Melgaard et al. 2005](#)]). More precisely, given  $D \subset \Omega$  simply connected and with smooth boundary, there exists a constant  $C > 0$  such that for every  $u \in \mathcal{D}_{A_a}^{1,2}(\Omega)$

$$\left\| \frac{u}{|x-a|} \right\|_{L^2(D)} \leq C \|(i\nabla + A_a)u\|_{L^2(D)}. \tag{2-2}$$

As a reference on Aharonov–Bohm operators we cite [[Rozenblum and Melgaard 2005](#)]. As a consequence of the continuous embedding, we have the following.

**Lemma 2.1.** *Let  $\text{Im}$  be the compact immersion of  $\mathcal{D}_{A_a}^{1,2}(\Omega)$  into  $(\mathcal{D}_{A_a}^{1,2}(\Omega))'$ . Then, the operator*

$$((i\nabla + A_a)^2)^{-1} \circ \text{Im} : \mathcal{D}_{A_a}^{1,2}(\Omega) \rightarrow \mathcal{D}_{A_a}^{1,2}(\Omega)$$

*is compact.*

As  $((i\nabla + A_a)^2)^{-1}$  is also self-adjoint and positive, we deduce that the spectrum of  $(i\nabla + A_a)^2$  consists of a diverging sequence of real positive eigenvalues, having finite multiplicity. They also admit the

variational characterization

$$\lambda_j^a = \inf_{\substack{W_j \subset \mathcal{D}_{A_a}^{1,2}(\Omega) \\ \dim W_j = j}} \sup_{\Phi \in W_j} \frac{\|\Phi\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}^2}{\|\Phi\|_{L^2(\Omega)}^2}. \tag{2-3}$$

Recall that  $A_a$  has the form (1-1) if and only if it satisfies

$$\nabla \times A_a = 0 \quad \text{in } \Omega \setminus \{a\} \quad \text{and} \quad \frac{1}{2\pi} \oint_{\sigma} A_a \cdot dx = \alpha \tag{2-4}$$

for every closed path  $\sigma$  which winds once around  $a$ . The value of the circulation strongly affects the behavior of the eigenfunctions, starting from their regularity, as the following lemma shows.

**Lemma 2.2** [Felli et al. 2011, Section 7]. *If  $A_a$  has the form (1-1) then  $\varphi_j^a \in C^{0,\alpha}(\Omega)$ , where  $\alpha$  is precisely the circulation of  $A_a$ .*

If the circulations of two magnetic potentials differ by an integer, the corresponding operators are equivalent under a gauge transformation, so that they have the same spectrum (see [Helffer et al. 1999, Theorem 1.1] and [Noris and Terracini 2010, Lemma 3.2]). For this reason, we can set  $\chi = 0$  in (2-4) and we can consider  $\alpha$  in the interval  $(0, 1)$  without losing generality. In the same papers it is shown that, when the circulations differ by a value  $\frac{1}{2}$ , one operator is equivalent to the other one composed with the complex square root. In particular, in case of half-integer circulation the operator is equivalent to the standard Laplacian in the double covering.

**Lemma 2.3** [Helffer et al. 1999, Lemma 3.3]. *Suppose that  $A_a$  has the form (2-4) with  $\alpha = \frac{1}{2}$  (and  $\chi = 0$ ). Then, with  $\theta$  being the angle of the polar coordinates, the function*

$$e^{-i\theta(y)} \varphi_j^a(y^2 + a) \quad \text{defined in } \{y \in \mathbb{C} : y^2 + a \in \Omega\}$$

*is real-valued and solves the following equation on its domain:*

$$-\Delta(e^{-i\theta(y)} \varphi_j^a(y^2 + a)) = 4\lambda_j^a |y|^2 e^{-i\theta(y)} \varphi_j^a(y^2 + a).$$

As a consequence, we have that, in the case of half-integer circulation,  $\varphi_j^a$  behaves, up to a complex phase, as an elliptic eigenfunction far from the singular point  $a$ . The behavior near  $a$  is, up to a complex phase, that of the square root of an elliptic eigenfunction. We summarize the known properties that we will need in the following proposition. The proofs can be found in [Felli et al. 2011, Theorem 1.3], [Helffer et al. 1999, Theorem 2.1] and [Noris and Terracini 2010, Theorem 1.5] (see also [Hartman and Wintner 1953]).

**Proposition 2.4.** *Let  $\alpha = \frac{1}{2}$ . There exists an odd integer  $k \geq 1$  such that  $\varphi_j^a$  has a zero of order  $k/2$  at  $a$ . Moreover, we have near  $a$  the asymptotic expansion*

$$\varphi_j^a(|x - a|, \theta_a) = e^{i\alpha\theta_a} \frac{|x - a|^{k/2}}{k} (c_k \cos(k\alpha\theta_a) + d_k \sin(k\alpha\theta_a)) + g(|x - a|, \theta_a),$$

where  $x - a = |x - a|e^{i\theta_a}$ ,  $c_k^2 + d_k^2 \neq 0$  and the remainder  $g$  satisfies

$$\lim_{r \rightarrow 0} \frac{\|g(r, \cdot)\|_{C^1(\partial D_r(a))}}{r^{k/2}} = 0,$$

where  $D_r(a)$  is the disk centered at  $a$  of radius  $r$ . In addition, there is a positive radius  $R$  such that  $(\varphi_j^a)^{-1}(\{0\}) \cap D_R(a)$  consists of  $k$  arcs of class  $C^\infty$ . If  $k \geq 3$  then the tangent lines to the arcs at the point  $a$  divide the disk into  $k$  equal sectors.

### 3. Continuity of the eigenvalues with respect to the pole in the interior of the domain

In this section we prove the first part of [Theorem 1.1](#), that is the continuity of the function  $a \mapsto \lambda_j^a$  when the pole  $a$  belongs to the interior of the domain.

**Lemma 3.1.** *Given  $a, b \in \Omega$  there exists a radial cut-off function  $\eta_a : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\eta_a(x) = 0$  for  $|x - a| < 2|b - a|$ ,  $\eta_a(x) = 1$  for  $|x - a| \geq \sqrt{2|b - a|}$ , and moreover*

$$\int_{\mathbb{R}^2} (|\nabla \eta_a|^2 + (1 - \eta_a^2)) dx \rightarrow 0 \quad \text{as } a \rightarrow b.$$

*Proof.* Given any  $0 < \varepsilon < 1$  we set

$$\eta(x) = \begin{cases} 0, & 0 \leq |x| \leq \varepsilon, \\ \frac{\log \varepsilon - \log |x|}{\log \varepsilon - \log \sqrt{\varepsilon}}, & \varepsilon \leq |x| \leq \sqrt{\varepsilon}, \\ 1, & x \geq \sqrt{\varepsilon}. \end{cases} \tag{3-1}$$

With  $\varepsilon = 2|b - a|$  and  $\eta_a(x) = \eta(x - a)$ , an explicit calculation shows that the properties are satisfied.  $\square$

**Lemma 3.2.** *Given  $a, b \in \Omega$  there exist  $\theta_a$  and  $\theta_b$  such that  $\theta_a - \theta_b \in C^\infty(\Omega \setminus \{ta + (1 - t)b, t \in [0, 1]\})$  and moreover in this set we have*

$$\alpha \nabla(\theta_a - \theta_b) = A_a - A_b.$$

*Proof.* Let  $a = (a_1, a_2)$  and  $b = (b_1, b_2)$ . Suppose that  $a_1 < b_1$ ; the other cases can be treated in a similar way. We shall provide a suitable branch of the polar angle centered at  $a$ , which is discontinuous on the half-line starting at  $a$  and passing through  $b$ . To this aim we consider the branch of the arctangent given by

$$\arctan : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We set

$$\theta_a = \begin{cases} \arctan \frac{x_2 - a_2}{x_1 - a_1}, & x_1 > a_1, x_2 \geq \frac{b_2 - a_2}{b_1 - a_1}x_1 + \frac{a_2b_1 - b_2a_1}{b_1 - a_1}, \\ \pi/2, & x_1 = a_1, x_2 > a_2, \\ \pi + \arctan \frac{x_2 - a_2}{x_1 - a_1}, & x_1 < a_1, \\ 3\pi/2, & x_1 = a_1, x_2 < a_2, \\ 2\pi + \arctan \frac{x_2 - a_2}{x_1 - a_1}, & x_1 > a_1, x_2 < \frac{b_2 - a_2}{b_1 - a_1}x_1 + \frac{a_2b_1 - b_2a_1}{b_1 - a_1}. \end{cases}$$

With this definition  $\theta_a$  is regular except on the half-line

$$x_2 = \frac{b_2 - a_2}{b_1 - a_1}x_1 + \frac{a_2b_1 - b_2a_1}{b_1 - a_1}, \quad x_1 > a_1,$$

and an explicit calculation shows that  $\alpha \nabla \theta_a = A_a$  in the set where it is regular. The definition of  $\theta_b$  is analogous: we keep the same half-line, whereas we replace  $(a_1, a_2)$  with  $(b_1, b_2)$  in the definition of the function. One can verify that  $\theta_a - \theta_b$  is regular except for the segment from  $a$  to  $b$ . □

Recall that in the following  $\varphi_j^a$  is an eigenfunction associated to  $\lambda_j^a$ , normalized in the  $L^2$ -norm. Moreover, we can assume that the eigenfunctions are orthogonal.

**Lemma 3.3.** *Given  $a, b \in \Omega$ , let  $\eta_a$  be defined as in Lemma 3.1 and let  $\theta_a, \theta_b$  be defined as in Lemma 3.2. Fix an integer  $k \geq 1$  and set, for  $j = 1, \dots, k$ ,*

$$\tilde{\varphi}_j = e^{i\alpha(\theta_a - \theta_b)} \eta_a \varphi_j^b.$$

Then  $\tilde{\varphi}_j \in \mathcal{D}_{A_a}^{1,2}(\Omega)$  and moreover, for every  $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ ,

$$(1 - \varepsilon_a) \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 \leq \left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|_{L^2(\Omega)}^2 \leq k \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2,$$

where  $\varepsilon_a \rightarrow 0$  as  $a \rightarrow b$ .

*Proof.* Let us prove first that  $\tilde{\varphi}_j \in \mathcal{D}_{A_a}^{1,2}(\Omega)$ . By Lemmas 3.1 and 3.2 we have that  $\theta_a - \theta_b \in C^\infty(\text{supp}\{\eta_a\})$ , so that  $\tilde{\varphi}_j \in H_0^1(\Omega)$ . Moreover  $\tilde{\varphi}_j(x) = 0$  if  $|x - a| < 2|b - a|$ , hence  $\tilde{\varphi}_j/|x - a| \in L^2(\Omega)$ . Concerning the inequalities, we compute on one hand

$$\left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|_{L^2(\Omega)}^2 \leq k \sum_{j=1}^k \alpha_j^2 \|\eta_a \varphi_j^b\|_{L^2(\Omega)}^2 \leq k \sum_{j=1}^k \alpha_j^2 = k \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2,$$

where we used the inequality  $\sum_{i,j=1}^k \alpha_i \alpha_j \leq k \sum_{j=1}^k \alpha_j^2$  and the fact that the eigenfunctions are orthogonal and normalized in the  $L^2(\Omega)$ -norm. On the other hand we compute

$$\left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 - \left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|_{L^2(\Omega)}^2 = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega} (1 - \eta_a^2) \varphi_i^b \bar{\varphi}_j^b dx.$$

Thanks to the regularity result proved by Felli, Ferrero and Terracini (see Lemma 2.2), we have that  $\varphi_i^b$  are bounded in  $L^\infty(\Omega)$ . Therefore the last quantity is bounded by

$$Ck \sum_{j=1}^k \alpha_j^2 \int_{\Omega} (1 - \eta_a^2) dx = Ck \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 \int_{\Omega} (1 - \eta_a^2) dx$$

and the conclusion follows from Lemma 3.1. □

We have all the tools to prove the first part of Theorem 1.1. We will use some ideas from [Helffer et al. 2010a, Theorem 7.1].

**Theorem 3.4.** *For every  $k \in \mathbb{N}$  the function  $a \in \Omega \mapsto \lambda_k^a \in \mathbb{R}$  is continuous.*

*Proof.* Step 1: First we prove that

$$\limsup_{a \rightarrow b} \lambda_k^a \leq \lambda_k^b.$$

To this aim it will be sufficient to exhibit a  $k$ -dimensional space  $E_k \subset \mathcal{D}_{A_a}^{1,2}(\Omega)$  with the property that

$$\|\Phi\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}^2 \leq (\lambda_k^b + \varepsilon'_a) \|\Phi\|_{L^2(\Omega)}^2 \quad \text{for every } \Phi \in E_k, \tag{3-2}$$

with  $\varepsilon'_a \rightarrow 0$  as  $a \rightarrow b$ . Let  $\text{span}\{\varphi_1^b, \dots, \varphi_k^b\}$  be any spectral space attached to  $\lambda_1^b, \dots, \lambda_k^b$ . Then we define

$$E_k := \text{span}\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_k\} \quad \text{with } \tilde{\varphi}_j = e^{i\alpha(\theta_a - \theta_b)} \eta_a \varphi_j^b.$$

We know from [Lemma 3.3](#) that  $E_k \subset \mathcal{D}_{A_a}^{1,2}(\Omega)$ . Moreover, it is immediate to see that  $\dim E_k = k$ . Let us now verify (3-2) with  $\Phi = \sum_{j=1}^k \alpha_j \tilde{\varphi}_j, \alpha_j \in \mathbb{R}$ . We compute

$$\|\Phi\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}^2 = \int_{\Omega} \left| \sum_{j=1}^k \alpha_j (i\nabla + A_b)(\eta_a \varphi_j^b) \right|^2 dx = \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j (i\nabla + A_b)^2 (\eta_a \varphi_i^b) (\eta_a \bar{\varphi}_j^b) dx, \tag{3-3}$$

where we have used the equality

$$(i\nabla + A_a)\tilde{\varphi}_j = e^{i\alpha(\theta_a - \theta_b)} (i\nabla + A_b)(\eta_a \varphi_j^b)$$

and integration by parts. Next notice that

$$(i\nabla + A_b)(\eta_a \varphi_i^b) = \eta_a (i\nabla + A_b)\varphi_i^b + i\varphi_i^b \nabla \eta_a,$$

so that

$$(i\nabla + A_b)^2 (\eta_a \varphi_i^b) = \eta_a (i\nabla + A_b)^2 \varphi_i^b + 2i(i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a - \varphi_i^b \Delta \eta_a.$$

By replacing in (3-3), we obtain

$$\begin{aligned} \|\Phi\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}^2 &= \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j (\lambda_i^b \varphi_i^b \eta_a + 2i(i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a - \varphi_i^b \Delta \eta_a) \bar{\varphi}_j^b \eta_a dx \\ &\leq \lambda_k^b \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 + \beta_a, \end{aligned} \tag{3-4}$$

where

$$\beta_a = \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j \{ \lambda_i^b (\eta_a^2 - 1) \varphi_i^b \bar{\varphi}_j^b + 2i \bar{\varphi}_j^b \eta_a (i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a - \varphi_i^b \bar{\varphi}_j^b \eta_a \Delta \eta_a \} dx. \tag{3-5}$$

We need to estimate  $\beta_a$ . From [Lemma 2.2](#) we deduce the existence of a constant  $C > 0$  such that  $\|\varphi_j^b\|_{L^\infty(\Omega)} \leq C$  for every  $j = 1, \dots, k$ . Hence

$$\left| \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j \lambda_i^b (\eta_a^2 - 1) \varphi_i^b \bar{\varphi}_j^b dx \right| \leq C \sum_{j=1}^k \alpha_j^2 \int_{\Omega} (1 - \eta_a^2) dx.$$

Using the fact that  $\|\varphi_j^b\|_{H_0^1(\Omega)}^2 \leq C \|\varphi_j^b\|_{\mathbb{D}_{A_b}^{1,2}(\Omega)}^2 = C\lambda_j^b$  (see (2-1)), we have

$$\left| \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j \bar{\varphi}_j^b \eta_a \nabla \varphi_i^b \cdot \nabla \eta_a \, dx \right| \leq C \sum_{j=1}^k \alpha_j^2 \left( \int_{\Omega} |\nabla \eta_a|^2 \, dx \right)^{1/2}.$$

Next we apply the Hardy inequality (2-2) to obtain

$$\begin{aligned} \left| \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j \varphi_i^b \bar{\varphi}_j^b \eta_a A_b \cdot \nabla \eta_a \, dx \right| &\leq C \sum_{j=1}^k \alpha_j^2 \int_{\Omega} |\varphi_j^b A_b \cdot \nabla \eta_a| \, dx \\ &\leq C \sum_{j=1}^k \alpha_j^2 \left\| \frac{\varphi_j^b}{x-b} \right\|_{L^2(\Omega)} \| (x-b) A_b \|_{L^\infty(\Omega)} \|\nabla \eta_a\|_{L^2(\Omega)} \\ &\leq C \sum_{j=1}^k \alpha_j^2 \|\nabla \eta_a\|_{L^2(\Omega)}. \end{aligned}$$

Concerning the last term in (3-5), similar estimates give

$$\begin{aligned} \left| \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j \varphi_i^b \bar{\varphi}_j^b \eta_a \Delta \eta_a \, dx \right| &= \left| \int_{\Omega} \sum_{i,j=1}^k \alpha_i \alpha_j (|\nabla \eta_a|^2 \varphi_i^b \bar{\varphi}_j^b + \eta_a \nabla \eta_a \cdot \nabla (\varphi_i^b \bar{\varphi}_j^b)) \, dx \right| \\ &\leq C \sum_{j=1}^k \alpha_j^2 \left( \int_{\Omega} |\nabla \eta_a|^2 \, dx \right)^{1/2}. \end{aligned}$$

In conclusion, we have obtained

$$|\beta_a| \leq C \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 \left\{ \int_{\Omega} (1 - \eta_a^2) \, dx + \left( \int_{\Omega} |\nabla \eta_a|^2 \, dx \right)^{1/2} \right\} = \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|_{L^2(\Omega)}^2 \varepsilon_a'',$$

with  $\varepsilon_a'' \rightarrow 0$  as  $a \rightarrow b$  by Lemma 3.1. By inserting the last estimate into (3-4) and then using Lemma 3.3 we obtain (3-2) with  $\varepsilon_a' = (\varepsilon_a'' + \lambda_k^b \varepsilon_a) / (1 - \varepsilon_a)$ .

Step 2: We now want to prove the second inequality,  $\liminf_{a \rightarrow b} \lambda_k^a \geq \lambda_k^b$ . From relation (2-1) and Step 1 we deduce

$$\|\varphi_j^a\|_{H_0^1(\Omega)}^2 \leq C \|\varphi_j^a\|_{\mathbb{D}_{A_a}^{1,2}(\Omega)}^2 \leq C\lambda_j^b.$$

Hence there exists  $\varphi_j^* \in H_0^1(\Omega)$  such that (up to subsequences)  $\varphi_j^a \rightharpoonup \varphi_j^*$  weakly in  $H_0^1(\Omega)$  and  $\varphi_j^a \rightarrow \varphi_j^*$  strongly in  $L^2(\Omega)$ , as  $a \rightarrow b$ . In particular we have

$$\int_{\Omega} |\varphi_j^*|^2 \, dx = 1 \quad \text{and} \quad q \int_{\Omega} \varphi_i^* \varphi_j^* \, dx = 0 \quad \text{if } i \neq j. \tag{3-6}$$

Moreover, Fatou’s lemma, relation (2-2) and Step 1 provide

$$\|\varphi_j^* / |x - b|\|_{L^2(\Omega)} \leq \liminf_{a \rightarrow b} \|\varphi_j^a / |x - a|\|_{L^2(\Omega)} \leq C \liminf_{a \rightarrow b} \|\varphi_j^a\|_{\mathbb{D}_{A_a}^{1,2}(\Omega)} = C \liminf_{a \rightarrow b} \sqrt{\lambda_j^a} \leq C \sqrt{\lambda_j^b},$$

so we deduce that  $\varphi_j^* \in \mathcal{D}_{A_b}^{1,2}(\Omega)$ .

Given a test function  $\phi \in C_0^\infty(\Omega \setminus \{b\})$ , consider  $a$  sufficiently close to  $b$  so that  $a \notin \text{supp}\{\phi\}$ . We have

$$\begin{aligned} \int_{\Omega} \lambda_j^a \varphi_j^a \bar{\phi} \, dx &= \int_{\Omega} \varphi_j^a \overline{(i\nabla + A_a)^2 \phi} \, dx \\ &= \int_{\Omega} \{-\Delta \varphi_j^a \bar{\phi} + \varphi_j^a \overline{[i\nabla \cdot A_a \phi + 2i A_a \cdot \nabla \phi + |A_a|^2 \phi]}\} \, dx \\ &= \int_{\Omega} \{(i\nabla + A_b)^2 \varphi_j^a \bar{\phi} - i\nabla \cdot (A_a + A_b) \varphi_j^a \bar{\phi} - 2i(A_a \cdot \nabla \bar{\phi} \varphi_j^a + A_b \cdot \nabla \varphi_j^a \bar{\phi}) \\ &\qquad\qquad\qquad + (|A_a|^2 - |A_b|^2) \varphi_j^a \bar{\phi}\} \, dx \\ &= \int_{\Omega} \{(i\nabla + A_b)^2 \varphi_j^a \bar{\phi} - i\nabla \cdot (A_a - A_b) \varphi_j^a \bar{\phi} - 2i \varphi_j^a (A_a - A_b) \cdot \nabla \bar{\phi} + (|A_a|^2 - |A_b|^2) \varphi_j^a \bar{\phi}\} \, dx, \end{aligned}$$

where in the last step we used the identity

$$-2i \int_{\Omega} A_b \cdot \nabla \varphi_j^a \bar{\phi} \, dx = 2i \int_{\Omega} (\nabla \cdot A_b \varphi_j^a \bar{\phi} + A_b \varphi_j^a \nabla \bar{\phi}) \, dx.$$

Since  $a, b \notin \text{supp}\{\phi\}$  then  $A_a \rightarrow A_b$  in  $C^\infty(\text{supp}\{\phi\})$ . Hence for a suitable subsequence we can pass to the limit in the previous expression obtaining

$$\int_{\Omega} (i\nabla + A_b)^2 \varphi_j^* \bar{\phi} \, dx = \int_{\Omega} \lambda_j^* \varphi_j^* \bar{\phi} \, dx \quad \text{for every } \phi \in C_0^\infty(\Omega \setminus \{b\}),$$

where  $\lambda_j^\infty := \liminf_{a \rightarrow b} \lambda_j^a$ . By density, the same is valid for  $\phi \in \mathcal{D}_{A_b}^{1,2}(\Omega)$ . As a consequence of the last equation and of (3-6), the functions  $\varphi_j^*$  are orthogonal in  $\mathcal{D}_{A_b}^{1,2}(\Omega)$  and hence

$$\begin{aligned} \lambda_k^b &= \inf_{\substack{W_k \subset \mathcal{D}_{A_b}^{1,2}(\Omega) \\ \dim W_k = k}} \sup_{\Phi \in W_k} \frac{\int_{\Omega} |(i\nabla + A_b)\Phi|^2 \, dx}{\int_{\Omega} |\Phi|^2 \, dx} \leq \sup_{(\alpha_1, \dots, \alpha_k) \neq 0} \frac{\int_{\Omega} |(i\nabla + A_b)(\sum_{j=1}^k \alpha_j \varphi_j^*)|^2 \, dx}{\int_{\Omega} |\sum_{j=1}^k \alpha_j \varphi_j^*|^2 \, dx} \\ &= \sup_{(\alpha_1, \dots, \alpha_k) \neq 0} \frac{\sum_{j=1}^k \alpha_j^2 \lambda_j^\infty}{\sum_{j=1}^k \alpha_j^2} \leq \lambda_k^\infty = \liminf_{a \rightarrow b} \lambda_k^a. \end{aligned}$$

This concludes Step 2 and the proof of the theorem. □

#### 4. Continuity of the eigenvalues with respect to the pole up to the boundary of the domain

In this section we prove the second part of [Theorem 1.1](#), that is the continuous extension up to the boundary of the domain. We will denote by  $\varphi_j$  an eigenfunction associated to  $\lambda_j$ , the  $j$ -th eigenvalue of the Laplacian in  $H_0^1(\Omega)$ . As usual, we suppose that the eigenfunctions are normalized in  $L^2$  and orthogonal. The following two lemmas can be proved exactly as the corresponding ones in [Section 3](#).

**Lemma 4.1.** *Given  $a \in \Omega$  and  $b \in \partial\Omega$  there exist  $\theta_a$  and  $\theta_b$  such that  $\theta_a \in C^\infty(\Omega \setminus \{ta + (1-t)b, t \in [0, 1]\})$ ,  $\theta_b \in C^\infty(\Omega)$ , and moreover in the respective sets of regularity the following holds:*

$$\alpha \nabla \theta_a = A_a, \quad \alpha \nabla \theta_b = A_b.$$

**Lemma 4.2.** *Given  $a \in \Omega$  and  $b \in \partial\Omega$ , let  $\eta_a$  be as defined in Lemma 3.1 and let  $\theta_a$  be as defined in Lemma 3.2. Set, for  $j = 1, \dots, k$ ,*

$$\tilde{\varphi}_j = e^{i\alpha\theta_a} \eta_a \varphi_j.$$

*Then, for every  $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ ,*

$$(1 - \varepsilon_a) \left\| \sum_{j=1}^k \alpha_j \varphi_j \right\|_{L^2(\Omega)}^2 \leq \left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|_{L^2(\Omega)}^2 \leq k \left\| \sum_{j=1}^k \alpha_j \varphi_j \right\|_{L^2(\Omega)}^2,$$

where  $\varepsilon_a \rightarrow 0$  as  $a \rightarrow b$ .

**Theorem 4.3.** *Suppose that  $a \in \Omega$  converges to  $b \in \partial\Omega$ . Then for every  $k \in \mathbb{N}$  we have that  $\lambda_k^a$  converges to  $\lambda_k$ .*

*Proof.* Following the scheme of the proof of Theorem 3.4 we proceed in two steps.

Step 1: First we show that

$$\limsup_{a \rightarrow b} \lambda_k^a \leq \lambda_k. \tag{4-1}$$

Since the proof is very similar to the one of Step 1 in Theorem 3.4 we will only point out the main differences. We define

$$E_k := \left\{ \Phi = \sum_{j=1}^k \alpha_j \tilde{\varphi}_j, \alpha_j \in \mathbb{R} \right\} \quad \text{with } \tilde{\varphi}_j = e^{i\alpha\theta_a} \eta_a \varphi_j.$$

We can verify the equality

$$(i\nabla + A_a)(e^{i\alpha\theta_a} \eta_a \varphi_j) = i e^{i\alpha\theta_a} \nabla(\eta_a \varphi_j),$$

so that we have

$$\|\Phi\|_{\mathfrak{D}_{A_a}^{1,2}(\Omega)}^2 = \int_{\Omega} \left| \sum_{j=1}^k \alpha_j \nabla(\eta_a \varphi_j) \right|^2 dx \leq \lambda_k \left\| \sum_{j=1}^k \alpha_j \varphi_j \right\|_{L^2(\Omega)}^2 + \beta_a,$$

with

$$\beta_a = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega} (|\nabla \eta_a|^2 \varphi_i \varphi_j + 2\eta_a \nabla \eta_a \cdot \nabla \varphi_j \varphi_i + (\eta_a^2 - 1) \nabla \varphi_i \cdot \nabla \varphi_j) dx.$$

Proceeding similarly to the proof of Theorem 3.4 we can estimate

$$|\beta_a| \leq \varepsilon_a'' \left\| \sum_{j=1}^k \alpha_j \varphi_j \right\|_{L^2(\Omega)}^2,$$

with  $\varepsilon_a'' \rightarrow 0$  as  $a \rightarrow b$ . In conclusion, using Lemma 4.2, we have obtained

$$\|\Phi\|_{\mathfrak{D}_{A_a}^{1,2}(\Omega)}^2 \leq \left( \lambda_k + \frac{\varepsilon_a'' + \lambda_k \varepsilon_a}{1 - \varepsilon_a} \right) \|\Phi\|_{L^2(\Omega)}^2 \quad \text{for every } \Phi \in E_k,$$

with  $\varepsilon_a, \varepsilon_a'' \rightarrow 0$  as  $a \rightarrow b$ . Therefore (4-1) is proved.

Step 2: We will now prove the second inequality

$$\liminf_{a \rightarrow b} \lambda_k^a \geq \lambda_k.$$

Given a test function  $\phi \in C_0^\infty(\Omega)$ , for  $a$  sufficiently close to  $b$  we have that

$$\{ta + (1 - t)b, t \in [0, 1]\} \subset \Omega \setminus \text{supp}\{\phi\}.$$

Then  $\phi \in \mathcal{D}_{A_a}^{1,2}(\Omega)$  and [Lemma 4.1](#) implies that  $e^{i\alpha\theta_a}\phi \in C_0^\infty(\Omega)$ . For this reason we can compute:

$$\int_{\Omega} \nabla(e^{-i\alpha\theta_b}\varphi_j^a) \cdot \nabla\bar{\phi} \, dx = \int_{\Omega} e^{-i\alpha\theta_b}\varphi_j^a (\overline{-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a})}) \, dx. \tag{4-2}$$

Since

$$-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a}) = (i\nabla + A_a)^2\phi - 2iA_a \cdot \nabla\phi - i\nabla \cdot A_a\phi - |A_a|^2\phi,$$

the right-hand side in (4-2) can be rewritten as

$$\int_{\Omega} ((i\nabla + A_a)^2(e^{-i\alpha\theta_b}\varphi_j^a)\bar{\phi} + e^{-i\alpha\theta_b}\varphi_j^a(2iA_a \cdot \nabla\bar{\phi} + i\nabla \cdot A_a\bar{\phi} - |A_a|^2\bar{\phi})) \, dx.$$

At this point notice that

$$(i\nabla + A_a)^2(e^{-i\alpha\theta_b}\varphi_j^a) = e^{-i\alpha\theta_b}((i\nabla + A_a)^2\varphi_j^a + i\nabla \cdot A_b\varphi_j^a + 2iA_b \cdot \nabla\varphi_j^a + |A_b|^2\varphi_j^a + 2A_a \cdot A_b\varphi_j^a).$$

By inserting this information in (4-2) we obtain

$$\int_{\Omega} \nabla(e^{-i\alpha\theta_b}\varphi_j^a) \cdot \nabla\bar{\phi} \, dx = \lambda_j^a \int_{\Omega} e^{-i\alpha\theta_b}\varphi_j^a\bar{\phi} \, dx + \beta_a, \tag{4-3}$$

with

$$\begin{aligned} \beta_a = \int_{\Omega} e^{-i\alpha\theta_b}\bar{\phi}(i\nabla \cdot A_b\varphi_j^a + 2iA_b \cdot \nabla\varphi_j^a + |A_b|^2\varphi_j^a + 2A_a \cdot A_b\varphi_j^a) \, dx \\ + \int_{\Omega} e^{-i\alpha\theta_b}\varphi_j^a(2iA_a \cdot \nabla\bar{\phi} + i\nabla \cdot A_a\bar{\phi} - |A_a|^2\bar{\phi}) \, dx. \end{aligned}$$

Integration by parts leads to

$$\beta_a = \int_{\Omega} e^{-i\alpha\theta_b}\varphi_j^a(-\bar{\phi}|A_a - A_b|^2 + 2i\nabla\bar{\phi} \cdot (A_a - A_b) + i\bar{\phi}\nabla \cdot (A_a - A_b)) \, dx,$$

so that  $|\beta_a| \rightarrow 0$  as  $a \rightarrow b$ , since  $A_a \rightarrow A_b$  in  $C^\infty(\text{supp}\{\phi\})$ . Therefore we can pass to the limit in (4-3) to obtain

$$\int_{\Omega} \nabla\varphi_j^* \cdot \nabla\bar{\phi} \, dx = \lambda_j^\infty \int_{\Omega} \varphi_j^*\bar{\phi} \, dx \quad \text{for every } \phi \in C_0^\infty(\Omega),$$

where  $\varphi_j^*$  is the weak limit of a suitable subsequence of  $e^{-i\alpha\theta_b}\varphi_j^a$  (given by Step 1) and  $\lambda_j^\infty := \liminf_{a \rightarrow b} \lambda_j^a$ . The conclusion of the proof is as in [Theorem 3.4](#). □

**Remark 4.4.** As a consequence of [Theorem 4.3](#) we obtain that  $e^{-i\alpha\theta_a}\varphi_j^a \rightarrow \varphi_j$  in  $H_0^1(\Omega)$  as  $a \rightarrow b \in \partial\Omega$ . Indeed, an inspection of the previous proof provides the weak convergence  $e^{-i\alpha\theta_a}\varphi_j^a \rightharpoonup \varphi_j$  in  $H_0^1(\Omega)$  and the convergence of the norms

$$\|e^{-i\alpha\theta_a}\varphi_j^a\|_{H_0^1(\Omega)}^2 = \|\varphi_j^a\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)}^2 = \lambda_j^a \rightarrow \lambda_j = \|\varphi_j\|_{H_0^1(\Omega)}^2,$$

as  $a \rightarrow b \in \partial\Omega$ , for every  $j \in \mathbb{N}$ .

### 5. Differentiability of the simple eigenvalues with respect to the pole

In this section we prove [Theorem 1.3](#). We omit the subscript in the notation of the eigenvalues and eigenfunctions; with this notation,  $\lambda^a$  is any eigenvalue of  $(i\nabla + A_a)^2$  and  $\varphi^a$  is an associated eigenfunction.

*Proof of Theorem 1.3.* Let  $b \in \Omega$  be such that  $\lambda^b$  is simple, as in the assumptions of the theorem. For  $R$  such that  $B_{2R}(b) \subset \Omega$ , let  $\xi$  be a cut-off function satisfying  $\xi \in C^\infty(\Omega)$ ,  $0 \leq \xi \leq 1$ ,  $\xi(x) = 1$  for  $x \in B_R(b)$  and  $\xi(x) = 0$  for  $x \in \Omega \setminus B_{2R}(b)$ . For every  $a \in B_R(b)$  we define the transformation

$$\Phi_a : \Omega \rightarrow \Omega, \quad \Phi_a(x) = \xi(x)(x - b + a) + (1 - \xi(x))x.$$

Then  $\varphi^a \circ \Phi_a \in \mathcal{D}_{A_b}^{1,2}(\Omega)$  and satisfies, for every  $a \in B_R(b)$ ,

$$(i\nabla + A_b)^2(\varphi^a \circ \Phi_a) + \mathcal{L}(\varphi^a \circ \Phi_a) = \lambda^a \varphi^a \circ \Phi_a \tag{5-1}$$

and

$$\int_{\Omega} |\varphi^a \circ \Phi_a|^2 |\det(\Phi'_a)| dx = 1, \tag{5-2}$$

where  $\mathcal{L}$  is a second-order operator of the form

$$\mathcal{L}v = - \sum_{i,j=1}^2 a^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^2 b^i(x) \frac{\partial v}{\partial x_i} + c(x)v,$$

with  $a^{ij}, b^i, c \in C^\infty(\Omega, \mathbb{C})$  vanishing in  $B_R(b)$  and outside of  $B_{2R}(b)$ . Notice that

$$\Phi'_a(x) = I + \nabla\xi(x) \otimes (a - b)$$

is a small perturbation of the identity whenever  $|b - a|$  is sufficiently small, so that the operator in the left-hand side of (5-1) is elliptic (see for example [\[Brezis 2011, Lemma 9.8\]](#)).

To prove the differentiability, we will use the implicit function theorem in Banach spaces. To this aim, we define the operator

$$F : B_R(b) \times \mathcal{D}_{A_b}^{1,2}(\Omega) \times \mathbb{R} \rightarrow (\mathcal{D}_{A_b}^{1,2}(\Omega))' \times \mathbb{R},$$

$$(a, v, \lambda) \mapsto \left( (i\nabla + A_b)^2 v + \mathcal{L}v - \lambda v, \int_{\Omega} |v|^2 |\det(\Phi'_a)| dx - 1 \right). \tag{5-3}$$

Notice that  $F$  is of class  $C^\infty$  by the ellipticity of the operator, provided that  $R$  is sufficiently small, and that  $F(a, \varphi^a \circ \Phi_a, \lambda^a) = 0$  for every  $a \in B_R(b)$ , as we saw in (5-1), (5-2). In particular we have  $F(b, \varphi^b, \lambda^b) = 0$ , since  $\Phi_b$  is the identity. We now have to verify that  $d_{(v,\lambda)} F(b, \varphi^b, \lambda^b)$ , the differential of  $F$  with respect

to the variables  $(v, \lambda)$  evaluated at the point  $(b, \varphi^b, \lambda^b)$ , belongs to  $\text{Inv}(\mathcal{D}_{A_b}^{1,2}(\Omega) \times \mathbb{R}, (\mathcal{D}_{A_b}^{1,2}(\Omega))' \times \mathbb{R})$ . The differential is given by

$$d_{(v,\lambda)}F(b, \varphi^b, \lambda^b) = \begin{pmatrix} (i\nabla + A_b)^2 - \lambda^b \text{Im} & -\varphi^b \\ 2 \int_{\Omega} \bar{\varphi}^b dx & 0 \end{pmatrix},$$

where  $\text{Im}$  is the compact immersion of  $\mathcal{D}_{A_b}^{1,2}(\Omega)$  in  $(\mathcal{D}_{A_b}^{1,2}(\Omega))'$ , which was introduced in [Lemma 2.1](#).

Let us first prove that it is injective. To this aim we have to show that, if  $(w, s) \in \mathcal{D}_{A_b}^{1,2}(\Omega) \times \mathbb{R}$  is such that

$$(i\nabla + A_b)^2 w - \lambda^b w = s\varphi^b, \tag{5-4}$$

$$2 \int_{\Omega} \bar{\varphi}^b w dx = 0, \tag{5-5}$$

then  $(w, s) = (0, 0)$ . Relations (5-5) and (5-2) (with  $a = b$  and  $\Phi_b$  the identity) imply that

$$w \neq k\varphi^b \quad \text{for all } k \neq 0. \tag{5-6}$$

By testing (5-4) by  $\varphi^b$  we obtain

$$s = \int_{\Omega} ((i\nabla + A_b)w \cdot \overline{(i\nabla + A_b)\varphi^b} - \lambda^b w \overline{\varphi^b}) dx.$$

On the other hand, testing by  $w$  the equation satisfied by  $\varphi^b$ , we see that  $s = 0$ , so that (5-4) becomes

$$(i\nabla + A_b)^2 w = \lambda^b w.$$

The assumption that  $\lambda^b$  is simple, together with (5-6), implies  $w = 0$ . This concludes the proof of the injectivity.

For the surjectivity, we have to show that for all  $(f, r) \in (\mathcal{D}_{A_b}^{1,2}(\Omega))' \times \mathbb{R}$  there exists  $(w, s) \in \mathcal{D}_{A_b}^{1,2}(\Omega) \times \mathbb{R}$  which verifies the following equalities

$$(i\nabla + A_b)^2 w - \lambda^b w = f + s\varphi^b, \tag{5-7}$$

$$2 \int_{\Omega} \bar{\varphi}^b w dx = r. \tag{5-8}$$

We recall that the operator  $(i\nabla + A_b)^2 - \lambda^b \text{Im} : \mathcal{D}_{A_b}^{1,2}(\Omega) \rightarrow (\mathcal{D}_{A_b}^{1,2}(\Omega))'$  is Fredholm of index 0. This is a standard fact, which can be proved for example by noticing that this operator is isomorphic to  $\text{Id} - \lambda^b((i\nabla + A_b)^2)^{-1}(\text{Im})$  through the Riesz isomorphism and because the operator  $(i\nabla + A_b)^2$  is invertible. This is Fredholm of index 0 because it has the form identity minus compact, the compactness coming from [Lemma 2.1](#). Therefore we have (through Riesz isomorphism)

$$\text{Range}((i\nabla + A_b)^2 - \lambda^b \text{Im}) = (\text{Ker}((i\nabla + A_b)^2 - \lambda^b \text{Im}))^\perp = (\text{span}\{\varphi^b\})^\perp, \tag{5-9}$$

where we used the assumption that  $\lambda^b$  is simple in the last equality. As a consequence, we obtain from (5-7) an expression for  $s$ :

$$s = - \int_{\Omega} f \bar{\varphi}^b dx.$$

Next we can decompose  $w$  in  $w_0 + w_1$  such that  $w_0 \in \text{Ker}((i\nabla + A_b)^2 - \lambda^b \text{Im})$  and  $w_1$  is in the orthogonal space. Condition (5-7) becomes

$$(i\nabla + A_b)^2 w_1 - \lambda^b w_1 = f - \varphi^b \int_{\Omega} f \bar{\varphi}^b dx \tag{5-10}$$

and (5-9) ensures the existence of a solution  $w_1$ . Given such  $w_1$ , condition (5-8) determines  $w_0$  as follows:

$$w_0 = \left( - \int_{\Omega} \bar{\varphi}^b w_1 dx + \frac{r}{2} \right) \varphi^b,$$

so that the surjectivity is also proved.

We conclude that the implicit function theorem applies, so that the maps  $a \in \Omega \mapsto \lambda^a \in \mathbb{R}$  and  $a \in \Omega \mapsto \varphi^a \circ \Phi_a \in \mathcal{D}_{A_b}^{1,2}(\Omega)$  are of class  $C^\infty$  locally in a neighborhood of  $b$ .  $\square$

By combining the previous result with a standard lemma of local inversion we deduce the following fact, which we will need in the next section.

**Corollary 5.1.** *Let  $b \in \Omega$ . If  $\lambda^b$  is simple then the map  $\Psi : \Omega \times \mathcal{D}_{A_b}^{1,2}(\Omega) \times \mathbb{R} \rightarrow \mathbb{R} \times (\mathcal{D}_{A_b}^{1,2}(\Omega))' \times \mathbb{R}$  given by*

$$\Psi(a, v, \lambda) = (a, F(a, v, \lambda)),$$

*with  $F$  defined in (5-3), is locally invertible in a neighborhood of  $(b, \varphi^b, \lambda^b)$ , with inverse  $\Psi^{-1}$  of class  $C^\infty$ .*

*Proof.* We saw in the proof of Theorem 1.3 that, if  $\lambda^b$  is simple, then  $d_{(v,\lambda)} F(b, \varphi^b, \lambda^b)$  is invertible. It is sufficient to apply Lemma 2.1 in Chapter 2 of the book of Ambrosetti and Prodi [1993].  $\square$

### 6. Vanishing of the derivative at a multiple zero

In this section we prove Theorem 1.7. Recall that here  $\alpha = \frac{1}{2}$ . We will need the following preliminary results.

**Lemma 6.1.** *Let  $\lambda > 0$  and let  $D_r = D_r(0) \subset \mathbb{R}^2$ . Consider the following set of equations for  $r > 0$  small:*

$$\begin{cases} -\Delta u = \lambda u & \text{in } D_r, \\ u = r^{k/2} f + g(r, \cdot) & \text{on } \partial D_r, \end{cases} \tag{6-1}$$

*where  $f, g(r, \cdot) \in H^1(\partial D_r)$  and  $g$  satisfies*

$$\lim_{r \rightarrow 0} \frac{\|g(r, \cdot)\|_{H^1(\partial D_r)}}{r^{k/2}} = 0 \tag{6-2}$$

*for some integer  $k \geq 3$ . Then for  $r$  sufficiently small there exists a unique solution to (6-1), which moreover satisfies*

$$\|u\|_{L^2(D_r)} \leq Cr^{(k+2)/2} \quad \text{and} \quad \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_r)} \leq Cr^{(k-1)/2},$$

*where  $C > 0$  is independent of  $r$ .*

*Proof.* Let  $z_1$  solve

$$\begin{cases} -\Delta z_1 = 0 & \text{in } D_1, \\ z_1 = f + r^{-k/2}g(r, \cdot) & \text{on } \partial D_1. \end{cases}$$

Since the quadratic form

$$\int_{D_1} (|\nabla v|^2 - \lambda r^2 v^2) dx \tag{6-3}$$

is coercive for  $v \in H_0^1(D_1)$  for  $r$  sufficiently small, there exists a unique solution  $z_2$  to the equation

$$\begin{cases} -\Delta z_2 - \lambda r^2 z_2 = \lambda r^2 z_1 & \text{in } D_1, \\ z_2 = 0 & \text{on } \partial D_1. \end{cases} \tag{6-4}$$

Then  $u(x) = r^{k/2}(z_1(x/r) + z_2(x/r))$  is the unique solution to (6-1). In order to obtain the desired bounds on  $u$  we will estimate separately  $z_1$  and  $z_2$ . Assumption (6-2) implies

$$\|z_1\|_{H^1(D_1)} = \|f + r^{-k/2}g(r, \cdot)\|_{H^{1/2}(\partial D_1)} \leq C\|f\|_{H^1(\partial D_1)}, \tag{6-5}$$

for  $r$  sufficiently small. We compare the function  $z_1$  to its limit function when  $r \rightarrow 0$ , which is the harmonic extension of  $f$  in  $D_1$ , which we will denote  $w$ . Then we have

$$\begin{cases} -\Delta(z_1 - w) = 0 & \text{in } D_1, \\ z_1 - w = r^{-k/2}g(r, \cdot) & \text{on } \partial D_1, \end{cases}$$

and hence (6-2) implies

$$\left\| \frac{\partial}{\partial \nu}(z_1 - w) \right\|_{L^2(\partial D_1)} \leq C\|z_1 - w\|_{H^1(\partial D_1)} = C \frac{\|g(r, \cdot)\|_{H^1(\partial D_1)}}{r^{k/2}} \rightarrow 0.$$

Then we estimate  $z_2$  as follows:

$$\|z_2\|_{L^2(D_1)}^2 \leq C \int_{D_1} |\nabla z_2|^2 dx \leq C \int_{D_1} (|\nabla z_2|^2 - \lambda r^2 z_2^2) dx \leq C\|\lambda r^2 z_1\|_{L^2(D_1)}\|z_2\|_{L^2(D_1)},$$

where we used the Poincaré inequality, the coercivity of the quadratic form (6-3) and the definition of  $z_2$  (6-4). Hence estimate (6-5) implies

$$\|z_2\|_{L^2(D_1)} \leq Cr^2\|f\|_{H^1(\partial D_1)} \rightarrow 0 \text{ as } r \rightarrow 0.$$

This and (6-5) provide, by a change of variables in the integral, the desired estimate on  $\|u\|_{L^2(D_r)}$ . Now, the standard bootstrap argument for elliptic equations applied to (6-4) provides

$$\|z_2\|_{H^2(D_1)} \leq C(\|\lambda r^2 z_1\|_{L^2(D_1)} + \|z_2\|_{L^2(D_1)}) \rightarrow 0,$$

and hence the trace embedding implies

$$\left\| \frac{\partial z_2}{\partial \nu} \right\|_{L^2(\partial D_1)} \leq C\|\nabla z_2\|_{H^1(D_1)} \leq C\|z_2\|_{H^2(D_1)} \rightarrow 0.$$

So, we have obtained that there exists  $C > 0$  independent of  $r$  such that

$$\left\| \frac{\partial}{\partial \nu}(z_1 + z_2) \right\|_{L^2(\partial D_1)} \leq C.$$

Finally, going back to the function  $u$ , we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_r)} = r^{(k-1)/2} \left\| \frac{\partial}{\partial \nu} (z_1 + z_2) \right\|_{L^2(\partial D_1)} \leq Cr^{(k-1)/2}$$

where we used the change of variable  $x = ry$ . □

**Lemma 6.2.** *Let  $\phi \in \mathfrak{D}_{A_a}^{1,2}(\Omega)$  ( $a \in \Omega$ ). Then*

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} \leq C \|\phi\|_{\mathfrak{D}_{A_a}^{1,2}(\Omega)} \tag{6-6}$$

where  $C$  only depends on  $\Omega$ .

*Proof.* Set  $\tilde{\phi}(y) = \phi(|a|y)$  defined for  $y \in \tilde{\Omega} = \{x/|a| : x \in \Omega\}$ . We apply this change of variables to the left-hand side in (6-6) and then use the trace embedding to obtain

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} = \|\tilde{\phi}\|_{L^2(\partial D_1)} \leq C \|\tilde{\phi}\|_{H^1(D_1)} \leq C \|\tilde{\phi}\|_{H^1(D_2)}.$$

We have that  $\tilde{\phi} \in H_{A_e}^1(\tilde{\Omega})$ , where  $e = a/|a|$ . Therefore we can apply relation (2-2) as follows:

$$\begin{aligned} \|\tilde{\phi}\|_{L^2(D_2)} &\leq \|y - e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y - e|} \right\|_{L^2(D_2)} \leq C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)}, \\ \|\nabla \tilde{\phi}\|_{L^2(D_2)} &\leq \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|A_a \tilde{\phi}\|_{L^2(D_2)} \\ &\leq \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|(y - e)A_e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y - e|} \right\|_{L^2(D_2)} \\ &\leq C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)}. \end{aligned}$$

We combine the previous inequalities obtaining

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} \leq C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} \leq C \|\phi\|_{\mathfrak{D}_{A_a}^{1,2}(\Omega)},$$

where in the last step we used the fact that the quadratic form is invariant under dilations. □

To simplify the notation we suppose without loss of generality that  $0 \in \Omega$  and we take  $b = 0$ . Moreover, we omit the subscript in the notation of the eigenvalues as we did in the previous section. As a first step in the proof of Theorem 1.7, we shall estimate  $|\lambda^a - \lambda^0|$  in the case when the pole  $a$  belongs to a nodal line of  $\varphi^0$  ending at 0. We make this restriction because all the constructions in the following proposition require that  $\varphi^0$  vanishes at  $a$ .

**Proposition 6.3.** *Suppose that  $\lambda^0$  is simple and that  $\varphi^0$  has a zero of order  $k/2$  at the origin, with  $k \geq 3$  odd. Denote by  $\Gamma$  a nodal line of  $\varphi^0$  with endpoint at 0 (which exists by Proposition 2.4) and take  $a \in \Gamma$ . Then there exists a constant  $C > 0$  independent of  $|a|$  such that*

$$|\lambda^a - \lambda^0| \leq C|a|^{k/2} \quad \text{as } |a| \rightarrow 0, \quad a \in \Gamma.$$

*Proof.* The idea of the proof is to construct a function  $u_a \in \mathcal{D}_{A_a}^{1,2}(\Omega)$  satisfying

$$(i\nabla + A_a)^2 u_a - \lambda^0 u_a = g_a, \quad \|u_a\|_{L^2(\Omega)} = 1 - \epsilon_a \tag{6-7}$$

with

$$\|g_a\|_{(\mathcal{D}_{A_a}^{1,2}(\Omega))'} \simeq |a|^{k/2} \quad \text{and} \quad |\epsilon_a| \simeq |a|^{(k+2)/2} \tag{6-8}$$

and then to apply [Corollary 5.1](#). For the construction of the function  $u_a$  we will heavily rely on the assumption  $a \in \Gamma$ .

Step 1: construction of  $u_a$ . We define it separately in  $D_{|a|} = D_{|a|}(0)$  and in its complement  $\Omega \setminus D_{|a|}$ , using the notation

$$\begin{cases} u_a = u_a^{\text{ext}} & \text{in } \Omega \setminus D_{|a|}, \\ u_a^{\text{int}} & \text{in } D_{|a|}. \end{cases} \tag{6-9}$$

Concerning the exterior function we set

$$u_a^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0)} \varphi^0, \tag{6-10}$$

where  $\theta_a, \theta_0$  are defined as in [Lemma 3.2](#) in such a way that  $\theta_a - \theta_0$  is regular in  $\Omega \setminus D_{|a|}$  (here  $\theta_0 = \theta$  is the angle in the usual polar coordinates, but we emphasize the position of the singularity in the notation). Therefore  $u_a^{\text{ext}}$  solves the magnetic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{\text{ext}} = \lambda^0 u_a^{\text{ext}} & \text{in } \Omega \setminus D_{|a|}, \\ u_a^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0)} \varphi^0 & \text{on } \partial D_{|a|}, \\ u_a^{\text{ext}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6-11}$$

For the definition of  $u_a^{\text{int}}$  we will first consider a related elliptic problem. Notice that, by our choice  $a \in \Gamma$ , we have that  $e^{-i\alpha\theta_0} \varphi^0$  is continuous on  $\partial D_{|a|}$ . Indeed,  $e^{-i\alpha\theta_0}$  restricted to  $\partial D_{|a|}$  is discontinuous only at the point  $a$ , where  $\varphi^0$  vanishes. Moreover, note that this boundary trace is at least  $H^1(\partial D_{|a|})$ . Indeed, the eigenfunction  $\varphi^0$  is  $C^\infty$  far from the singularity and  $e^{i\alpha\theta_0}$  is also regular except on the point  $a$ . Then, the boundary trace is differentiable almost everywhere.

This allows to apply [Lemma 6.1](#), thus providing the existence of a unique function  $\psi_a^{\text{int}}$ , a solution of the equation

$$\begin{cases} -\Delta \psi_a^{\text{int}} = \lambda^0 \psi_a^{\text{int}} & \text{in } D_{|a|}, \\ \psi_a^{\text{int}} = e^{-i\alpha\theta_0} \varphi^0 & \text{on } \partial D_{|a|}. \end{cases} \tag{6-12}$$

Then we complete our construction of  $u_a$  by setting

$$u_a^{\text{int}} = e^{i\alpha\theta_a} \psi_a^{\text{int}}, \tag{6-13}$$

which is well-defined since  $\theta_a$  is regular in  $D_{|a|}$ . Note that  $u_a^{\text{int}}$  solves the elliptic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{\text{int}} = \lambda^0 u_a^{\text{int}} & \text{in } D_{|a|}, \\ u_a^{\text{int}} = u_a^{\text{ext}} & \text{on } \partial D_{|a|}. \end{cases} \tag{6-14}$$

Step 2: estimate of the normal derivative of  $u_a^{\text{int}}$  along  $\partial D_{|a|}$ . By assumption,  $\varphi^0$  has a zero of order  $k/2$  at the origin, with  $k \geq 3$  odd. Hence by [Proposition 2.4](#) the following asymptotic expansion holds on

$\partial D_{|a|}$  as  $|a| \rightarrow 0$ :

$$e^{-i\alpha\theta_0}\varphi^0(|a|, \theta_0) = \frac{|a|^{k/2}}{k} [c_k \cos(k\alpha\theta_0) + d_k \sin(k\alpha\theta_0)] + g(|a|, \theta_0), \tag{6-15}$$

with

$$\lim_{|a| \rightarrow 0} \frac{\|g(|a|, \cdot)\|_{C^1(\partial D_{|a|})}}{|a|^{k/2}} = 0. \tag{6-16}$$

Hence Lemma 6.1 applies to  $\psi_a^{\text{int}}$  given in (6-12), giving a constant  $C$  independent of  $|a|$  such that

$$\|\psi_a^{\text{int}}\|_{L^2(D_{|a|})} \leq C|a|^{(k+2)/2} \quad \text{and} \quad \left\| \frac{\partial \psi_a^{\text{int}}}{\partial \nu} \right\|_{L^2(\partial D_{|a|})} \leq C|a|^{(k-1)/2}. \tag{6-17}$$

Finally, differentiating (6-13) we see that

$$(i\nabla + A_a)u_a^{\text{int}} = ie^{i\alpha\theta_a}\nabla\psi_a^{\text{int}},$$

so that, integrating, we obtain the  $L^2$ -estimate for the magnetic normal derivative of  $u_a^{\text{int}}$  along  $\partial D_{|a|}$

$$\|(i\nabla + A_a)u_a^{\text{int}} \cdot \nu\|_{L^2(\partial D_{|a|})} \leq C|a|^{(k-1)/2}. \tag{6-18}$$

Step 3: estimate of the normal derivative of  $u_a^{\text{ext}}$  along  $\partial D_{|a|}$ . We differentiate (6-10) to obtain

$$(i\nabla + A_a)u_a^{\text{ext}} = A_0u_a^{\text{ext}} + ie^{i\alpha(\theta_a - \theta_0)}\nabla\varphi^0. \tag{6-19}$$

On the other hand, the following holds a.e.:

$$\nabla\varphi^0 = iA_0\varphi^0 + e^{i\alpha\theta_0}\nabla(e^{-i\alpha\theta_0}\varphi^0),$$

so that

$$ie^{i\alpha(\theta_a - \theta_0)}\nabla\varphi^0 = -A_0u_a^{\text{ext}} + ie^{i\alpha\theta_a}\nabla(e^{-i\alpha\theta_0}\varphi^0).$$

Combining the last equality with (6-19) we obtain a.e.

$$(i\nabla + A_a)u_a^{\text{ext}} = ie^{i\alpha\theta_a}\nabla(e^{-i\alpha\theta_0}\varphi^0)$$

and hence  $|(i\nabla + A_a)u_a^{\text{ext}}| \leq C|a|^{k/2-1}$  on  $\partial D_{|a|}$  a.e., for some  $C$  not depending on  $|a|$ , by (6-15) and (6-16). Integrating on  $\partial D_{|a|}$  we arrive at the same estimate as for  $u_a^{\text{int}}$ , that is

$$\|(i\nabla + A_a)u_a^{\text{ext}} \cdot \nu\|_{L^2(\partial D_{|a|})} \leq C|a|^{(k-1)/2}. \tag{6-20}$$

Step 4: proof of (6-8). We test (6-11) with a test function  $\phi \in \mathcal{D}_{A_a}^{1,2}(\Omega)$  and apply the formula of integration by parts to obtain

$$\int_{\Omega \setminus D_{|a|}} \{(i\nabla + A_a)u_a^{\text{ext}} \overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{\text{ext}} \bar{\phi}\} dx = i \int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{\text{ext}} \cdot \nu \bar{\phi} d\sigma.$$

Similarly, (6-14) provides

$$\int_{D_{|a|}} \{(i\nabla + A_a)u_a^{\text{int}} \overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{\text{int}} \bar{\phi}\} dx = -i \int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{\text{int}} \cdot \nu \bar{\phi} d\sigma.$$

Then, we test the equation in (6-7) with  $\phi$ , we integrate by parts and we replace the previous equalities to get

$$\int_{\Omega} g_a \bar{\phi} dx = i \int_{\partial D_{|a|}} (i \nabla + A_a)(u_a^{\text{ext}} - u_a^{\text{int}}) \cdot \nu \bar{\phi} d\sigma.$$

To the previous expression we apply first the Hölder inequality and then the estimates obtained in the previous steps (6-18) and (6-20) to obtain

$$\begin{aligned} \left| \int_{\Omega} g_a \bar{\phi} dx \right| &\leq \| (i \nabla + A_a) u_a^{\text{int}} \cdot \nu \|_{L^2(\partial D_{|a|})} \|\phi\|_{L^2(\partial D_{|a|})} + \| (i \nabla + A_a) u_a^{\text{ext}} \cdot \nu \|_{L^2(\partial D_{|a|})} \|\phi\|_{L^2(\partial D_{|a|})} \\ &\leq C |a|^{(k-1)/2} \|\phi\|_{L^2(\partial D_{|a|})}. \end{aligned}$$

Finally, Lemma 6.2 provides the desired estimate on  $g_a$ . Then we estimate  $\epsilon_a$  as follows. Since  $\|u_a^{\text{ext}}\|_{L^2(\Omega \setminus D_{|a|})} = \|\varphi^0\|_{L^2(\Omega \setminus D_{|a|})}$  we have

$$\left| \|u_a\|_{L^2(\Omega)} - 1 \right| = \left| \|u_a^{\text{int}}\|_{L^2(D_{|a|})}^2 - \|\varphi^0\|_{L^2(D_{|a|})}^2 \right| \leq C |a|^{k+2}, \tag{6-21}$$

where in the last inequality we used the fact that  $\|\varphi^0\|_{L^2(D_{|a|})}^2 \leq C |a|^{k+2}$  by (6-15) and (6-16), and that  $\|u_a^{\text{int}}\|_{L^2(D_{|a|})}^2 = \|\psi_a^{\text{int}}\|_{L^2(D_{|a|})}^2 \leq C |a|^{k+2}$ , by (6-17).

Step 5: local inversion theorem. To conclude the proof we apply Corollary 5.1. Let  $\Psi$  be the function defined therein (recall that here  $b = 0$ ). The construction that we did in the previous steps ensures that

$$\begin{aligned} \Psi(a, \varphi^a \circ \Phi_a, \lambda^a) &= (a, 0, 0), \\ \Psi(a, u_a \circ \Phi_a, \lambda^0) &= (a, g_a \circ \Phi_a, \epsilon_a), \end{aligned}$$

with  $g_a, \epsilon_a$  satisfying (6-8). We proved in Theorem 3.4 that

$$|\lambda^a - \lambda^0| + \|\varphi^a \circ \Phi_a - \varphi^0\|_{\mathfrak{D}_{\lambda^0}^{1,2}(\Omega)} \rightarrow 0$$

as  $|a| \rightarrow 0$ . Moreover, it is not difficult to see that

$$\|u_a \circ \Phi_a - \varphi^0\|_{\mathfrak{D}_{\lambda^0}^{1,2}(\Omega)} \rightarrow 0$$

as  $|a| \rightarrow 0$ . Hence the points  $(a, \varphi^a \circ \Phi_a, \lambda^a)$  and  $(a, u_a \circ \Phi_a, \lambda^0)$  are approaching  $(0, \varphi^0, \lambda^0)$  in the space  $\Omega \times \mathfrak{D}_{\lambda^0}^{1,2}(\Omega) \times \mathbb{R}$  as  $|a| \rightarrow 0$ . Since  $\Psi$  admits an inverse of class  $C^\infty$  in a neighborhood of  $(0, \varphi^0, \lambda^0)$  (recall that  $\lambda^0$  is simple), we deduce that

$$\|(\varphi^a - u_a) \circ \Phi_a\|_{\mathfrak{D}_{\lambda^0}^{1,2}(\Omega)} + |\lambda^a - \lambda^0| \leq C (\|g_a\|_{\mathfrak{D}_{\lambda^a}^{1,2}(\Omega)} + |\epsilon_a|) \leq C |a|^{k/2},$$

for some constant  $C$  independent of  $a$ , which concludes the proof of the proposition. □

At this point we have proved the desired property only for pole  $a$  belonging to the nodal lines of  $\varphi^0$ . We would like to extend this result to all  $a$  sufficiently close to 0. We will proceed in the following way. Thanks to Theorem 1.3, we can consider the Taylor expansion of the function  $a \mapsto \lambda^a$  in a neighborhood of 0. Then Proposition 6.3 provides  $k$  vanishing conditions, corresponding to the  $k$  nodal lines of  $\varphi^0$ .

Finally, we will use these conditions to show that in fact the first terms of the polynomial are identically zero. Let us begin with a lemma on the existence and the form of the Taylor expansion.

**Lemma 6.4.** *If  $\lambda^0$  is simple then for  $a \in \Omega$  sufficiently close to 0 and for all  $H \in \mathbb{N}$*

$$\lambda^a - \lambda^0 = \sum_{h=1}^H |a|^h P_h(\vartheta(a)) + o(|a|^H), \tag{6-22}$$

where  $a = |a|(\cos \vartheta(a), \sin \vartheta(a))$  and

$$P_h(\vartheta) = \sum_{j=0}^h \beta_{j,h} \cos^j \vartheta \sin^{h-j} \vartheta \tag{6-23}$$

for some  $\beta_{j,h} \in \mathbb{R}$  not depending on  $|a|$ .

*Proof.* Since  $\lambda^0$  is simple,  $\lambda^a$  is also simple for  $a$  sufficiently close to 0. Then we proved in [Theorem 1.3](#) that  $\lambda_j^a$  is  $C^\infty$  in the variable  $a$ . As a consequence, we can consider the first terms of the Taylor expansion, with Peano rest, of  $\lambda_j^a$

$$\lambda^a - \lambda^0 = \sum_{h=1}^H \sum_{j=0}^h \frac{1}{j!(h-j)!} \frac{\partial^h \lambda^a}{\partial^j a_1 \partial^{h-j} a_2} \Big|_{a=0} a_1^j a_2^{h-j} + o(|a|^H),$$

where  $a = (a_1, a_2)$ . Setting

$$\beta_{j,h} = \frac{1}{j!(h-j)!} \frac{\partial^h \lambda^a}{\partial^j a_1 \partial^{h-j} a_2} \Big|_{a=0}$$

and  $a_1 = |a| \cos \vartheta(a)$ ,  $a_2 = |a| \sin \vartheta(a)$ , the thesis follows. □

The following lemma tells us that on the  $k$  nodal lines of  $\varphi^0$ , the first low-order polynomials cancel.

**Lemma 6.5.** *Suppose that  $\lambda^0$  is simple and that  $\varphi^0$  has a zero of order  $k/2$  at 0, with  $k \geq 3$  odd. Then there exists an angle  $\tilde{\vartheta} \in [0, 2\pi)$  and non-negative quantities  $\varepsilon_0, \dots, \varepsilon_{k-1}$  arbitrarily small such that*

$$P_h\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0 \quad \text{for all integers } l \in [0, k-1], h \in [1, (k-1)/2],$$

where  $P_h$  is defined in (6-23).

*Proof.* We know from [Proposition 2.4](#) that  $\varphi^0$  has  $k$  nodal lines with endpoint at 0, which we denote  $\Gamma_l$ ,  $l = 0, \dots, k-1$ . Take points  $a_l \in \Gamma_l$ ,  $l = 0, \dots, k-1$ , satisfying  $|a_0| = \dots = |a_{k-1}|$  and denote

$$a_l = |a_l|(\cos \vartheta(a_l), \sin \vartheta(a_l)).$$

First we claim that  $P_h(\vartheta(a_l)) = 0$  for all integers  $l \in [0, k-1]$ ,  $h \in [1, (k-1)/2]$ .

Indeed, suppose by contradiction that this is not the case for some  $l, h$  belonging to the intervals defined above. Then for such  $l, h$  the following holds by [Lemma 6.4](#):

$$\lambda^{a_l} - \lambda^0 = C|a_l|^h + o(|a_l|^h) \quad \text{for some } C \neq 0.$$

On the other hand we proved in Proposition 6.3 that there exists  $C > 0$  independent of  $a$  such that, for every  $l = 0, \dots, k - 1$ , we have

$$|\lambda^{a_l} - \lambda^0| \leq C|a_l|^{k/2} \quad \text{as } |a_l| \rightarrow 0.$$

This contradicts the last estimate because  $h \leq (k - 1)/2$ , so that the claim is proved.

Finally setting  $\tilde{\vartheta} := \vartheta(a_0)$ , Proposition 2.4 implies

$$\vartheta(a_l) = \tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l, \quad l = 1, \dots, k - 1, \quad \text{with } \varepsilon_l \rightarrow 0 \text{ as } |a_l| \rightarrow 0. \quad \square$$

The next lemma extends this previous property to all  $a$  close to 0.

**Lemma 6.6.** Fix  $k \geq 3$  odd. For any integer  $h \in [1, (k - 1)/2]$  consider any polynomial of the form

$$P_h(\vartheta) = \sum_{j=0}^h \beta_{j,h} \cos^j \vartheta \sin^{h-j} \vartheta, \tag{6-24}$$

with  $\beta_{j,h} \in \mathbb{R}$ . Suppose that there exist  $\tilde{\vartheta} \in [0, 2\pi)$  and  $\varepsilon_0, \dots, \varepsilon_{k-1}$  satisfying  $0 \leq \varepsilon_l \leq \frac{\pi}{4k}$  such that

$$P_h\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0 \quad \text{for every integer } l \in [0, k - 1].$$

Then  $P_h \equiv 0$ .

*Proof.* We prove the result by induction on  $h$ .

Step 1: Let  $h = 1$ ; then

$$P_1(\vartheta) = \beta_0 \sin \vartheta + \beta_1 \cos \vartheta$$

and the following conditions hold for  $l = 0, \dots, k - 1$ :

$$\beta_0 \sin\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) + \beta_1 \cos\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0. \tag{6-25}$$

In the case that for every  $l = 0, \dots, k - 1$  we have

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) \neq 0 \quad \text{and} \quad \cos\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) \neq 0,$$

system (6-25) has two unknowns  $\beta_0, \beta_1$  and  $k \geq 3$  linearly independent equations. Hence in this case  $\beta_0 = \beta_1 = 0$  and  $P_1 \equiv 0$ . In the case that there exists  $l$  such that

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0$$

then of course  $\cos(\tilde{\vartheta} + 2\pi l/k + \varepsilon_l) \neq 0$ , which implies  $\beta_1 = 0$ . We claim that in this case

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\right) \neq 0 \tag{6-26}$$

for every integer  $l' \in [0, k - 1]$  different from  $l$ . To prove the claim we proceed by contradiction. We can suppose without loss of generality that

$$\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l = 0 \quad \text{and} \quad \tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'} = \pi.$$

Then

$$l = -\frac{k}{2\pi}(\tilde{\vartheta} + \varepsilon_l) \quad \text{and} \quad l' = \frac{k}{2\pi}(\pi - \tilde{\vartheta} - \varepsilon_{l'})$$

so that

$$l' - l = \frac{k}{2} + k \frac{\varepsilon_l - \varepsilon_{l'}}{2\pi}.$$

The assumption  $0 \leq \varepsilon_l \leq \pi/(4k)$  implies

$$\frac{k}{2} - \frac{1}{4} \leq l' - l \leq \frac{k}{2} + \frac{1}{4}.$$

Since  $k \geq 3$  is an odd integer, the last estimate provides  $l' - l \notin \mathbb{N}$ , which is a contradiction. Therefore we have proved (6-26). Now consider any of the equations in (6-25) for  $l' \neq l$ . Inserting the information  $\beta_1 = 0$  and (6-26) we get  $\beta_0 = 0$  and hence  $P_1 \equiv 0$ . In the case that one of the cosines vanishes one can proceed in the same way, so we have proved the basis of the induction.

Step 2: Suppose that the statement is true for some  $h \leq (k - 3)/2$  and let us prove it for  $h + 1$ . The following conditions hold for  $l = 0, \dots, k - 1$ :

$$\sum_{j=0}^{h+1} \beta_j \cos^j\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) \sin^{h+1-j}\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0. \tag{6-27}$$

We can proceed similarly to Step 1. If none of the sines, cosines vanish then we have a system with  $h + 2 \leq (k + 1)/2$  unknowns and  $k$  linearly independent equations, hence  $P_{h+1} \equiv 0$ . Otherwise suppose that there exists  $l$  such that

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) = 0.$$

Then we saw in Step 1 that

$$\cos\left(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\right) \neq 0 \quad \text{and} \quad \sin\left(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\right) \neq 0$$

for every integer  $l' \in [0, k - 1]$  different from  $l$ . By rewriting  $P_{h+1}$  in the form

$$P_{h+1}(\vartheta) = \sin \vartheta P_h(\vartheta) + \beta_{h+1} \cos^{h+1} \vartheta,$$

with  $P_h$  as in (6-24), we deduce both that  $\beta_{h+1} = 0$  and that

$$P_h\left(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\right) = 0$$

for every  $l' \in [0, k - 1]$  different from  $l$ . These are  $k - 1$  conditions for a polynomial of order  $h \leq (k - 3)/2$ , so the induction hypothesis implies  $P_h \equiv 0$  and in turn  $P_{h+1} \equiv 0$ . □

*End of the proof of Theorem 1.7.* Take any  $a \in \Omega$  sufficiently close to 0, then by Lemma 6.4

$$\lambda^a - \lambda^0 = \sum_{h=1}^H |a|^h P_h(\vartheta(a)) + o(|a|^H).$$

By combining Lemmas 6.5 and 6.6 we obtain that  $P_h \equiv 0$  for every  $h \in [1, (k - 1)/2]$ , therefore  $|\lambda^a - \lambda^0| \leq C|a|^{(k+1)/2}$  for some constant  $C$  independent of  $a$ . □

### 7. Numerical illustration

Let us now illustrate some results of this paper using the Finite Element Library [Martin 2010] with isoparametric  $\mathbb{P}_6$  Lagrangian elements. We will restrict our attention to the case of half-integer circulation  $\alpha = \frac{1}{2}$ .

The numerical method we used here was presented in detail in [Bonnaillie-Noël and Helffer 2011]. Given a domain  $\Omega$  and a point  $a \in \Omega$ , to compute the eigenvalues  $\lambda_j^a$  of the Aharonov–Bohm operator  $(i\nabla + A_a)^2$  on  $\Omega$ , we compute those of the Dirichlet Laplacian on the double covering  $\Omega_a^{\mathcal{R}}$  of  $\Omega \setminus \{a\}$ , denoted by  $\mu_j^{\mathcal{R}}$ . This spectrum of the Laplacian on  $\Omega_a^{\mathcal{R}}$  is decomposed in two disjoint parts:

- the spectrum of the Dirichlet Laplacian on  $\Omega$ ,  $\lambda_j$ ,
- the spectrum of the magnetic Schrödinger operator  $(i\nabla + A_a)^2$ ,  $\lambda_j^a$ .

Thus we have

$$\{\mu_j^{\mathcal{R}}\}_{j \geq 1} = \{\lambda_j^a\}_{j \geq 1} \sqcup \{\lambda_j\}_{j \geq 1}.$$

Therefore by computing the spectrum of the Dirichlet Laplacian on  $\Omega$  and, for every  $a \in \Omega$ , that on the double covering  $\Omega_a^{\mathcal{R}}$ , we deduce the spectrum of the Aharonov–Bohm operator  $(i\nabla + A_a)^2$  on  $\Omega$ . This method avoids dealing with the singularity of the magnetic potential and furthermore allows us to work with real-valued functions. We have only to compute the spectrum of the Dirichlet Laplacian, which is quite standard. The only effort to be done is to mesh a double covering domain.

Let us now present the computations for the angular sector of aperture  $\pi/4$ :

$$\Sigma_{\pi/4} = \left\{ (x_1, x_2) \in \mathbb{R}^2, x_1 > 0, |x_2| < x_1 \tan \frac{\pi}{8}, x_1^2 + x_2^2 < 1 \right\}.$$

An analysis of the spectral minimal partitions of angular sectors can be found in [Bonnaillie-Noël and Léna 2014]. By symmetry, it is enough to compute the spectrum for  $a$  in the half-domain. We take a discretization grid of step  $1/N$  with  $N = 100$  or  $N = 1000$ :

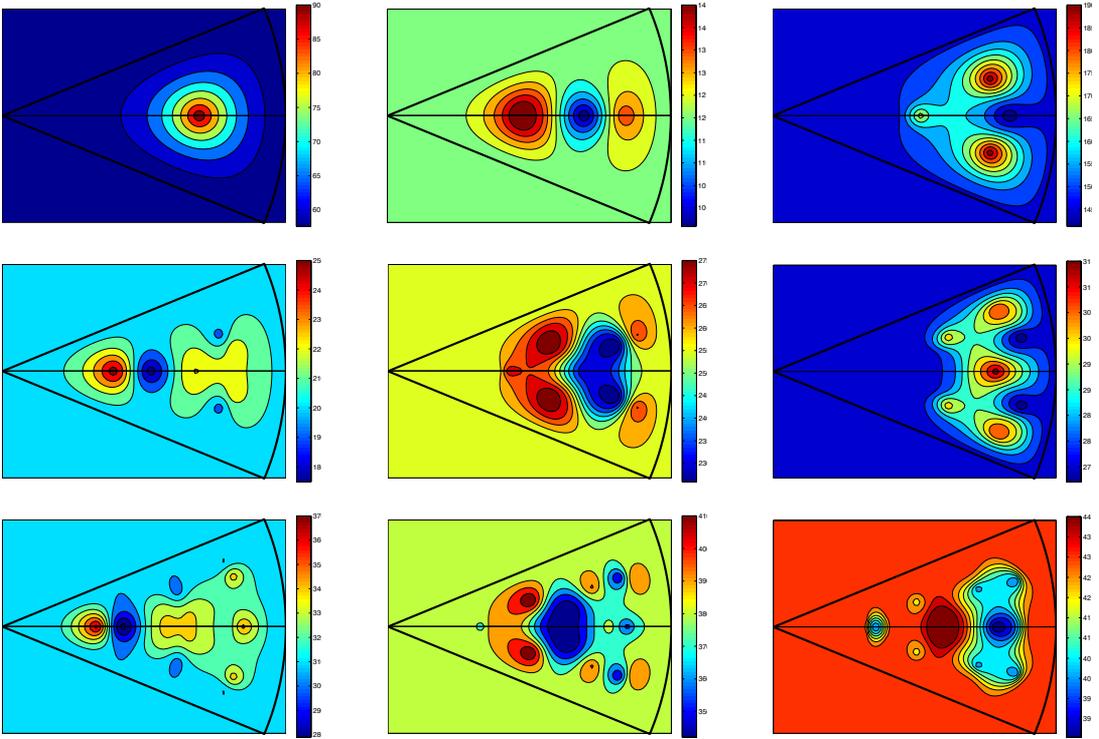
$$a \in \Pi_N := \left\{ \left( \frac{m}{N}, \frac{n}{N} \right), 0 < m < N, 0 < \frac{|n|}{m} < \tan \frac{\pi}{8}, \frac{m^2 + n^2}{N^2} < 1 \right\}.$$

Figure 2 gives the first nine eigenvalues  $\lambda_j^a$  for  $a \in \Pi_{100}$ . In these figures, the angular sector is represented by a dark thick line. Outside the angular sector are represented the eigenvalues  $\lambda_j$  of the Dirichlet Laplacian on  $\Sigma_{\pi/4}$  (which do not depend on  $a$ ). We observe the convergence proved in Theorem 1.1:

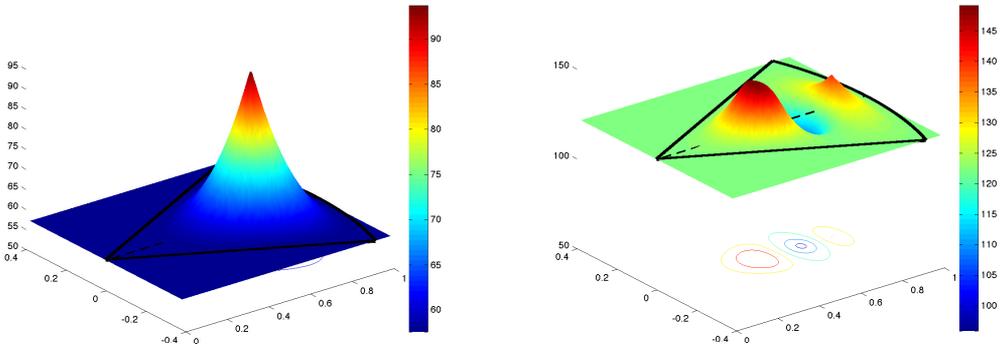
$$\text{for all } j \geq 1, \quad \lambda_j^a \rightarrow \lambda_j \quad \text{as } a \rightarrow \partial \Sigma_{\pi/4}.$$

In Figure 3, we provide the three-dimensional representation of the first two parts of Figure 2.

Let us now deal more accurately with the singular points on the symmetry axis. Numerically, we take a discretization step equal to  $\frac{1}{1000}$  and consider  $a \in \left\{ \left( \frac{m}{1000}, 0 \right), 1 \leq m \leq 1000 \right\}$ . Figure 4 gives the first nine eigenvalues of the Aharonov–Bohm operator  $(i\nabla + A_a)^2$  in  $\Sigma_{\pi/4}$ . Here we can identify the points  $a$  belonging to the symmetry axis such that  $\lambda_j^a$  is not simple. If we look for example at the first and second eigenvalues, we see that they are not simple respectively for one and three values of  $a$  on the symmetry axis. At such values, the function  $a \mapsto \lambda_j^a$ ,  $j = 1, 2$ , is not differentiable, as can be seen in Figure 3.



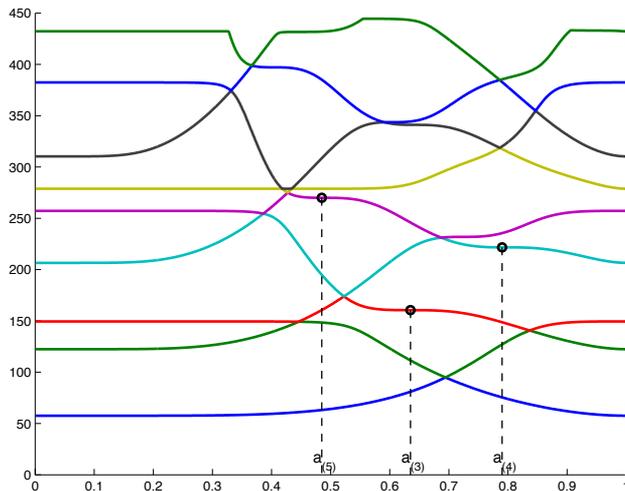
**Figure 2.** First nine eigenvalues of  $(i\nabla + A_a)^2$  in  $\Sigma_{\pi/4}$ ,  $a \in \Pi_{100}$ . Each graph depicts the level curves of  $a \mapsto \lambda_j^a$ , for  $j = 1, 2, 3$  (top),  $j = 4, 5, 6$  (middle) and  $j = 7, 8, 9$  (bottom).



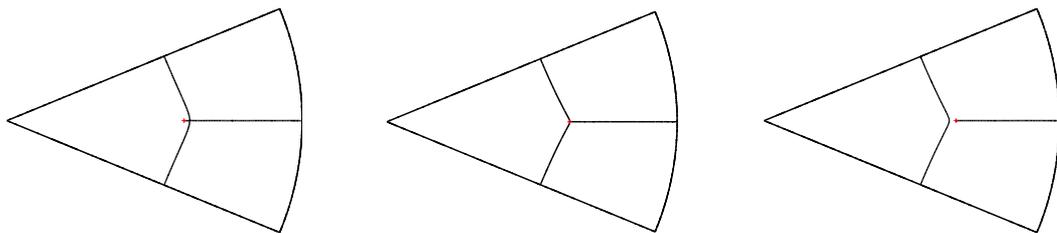
**Figure 3.** Three-dimensional representation of the first two panels of Figure 2:  $a \mapsto \lambda_1^a$  (left) and  $a \mapsto \lambda_2^a$  (right),  $a \in \Pi_{100}$ .

Figure 3 illustrates Theorem 1.3 for a domain with a piecewise- $C^\infty$  boundary: we see that the function  $a \mapsto \lambda_j^a$ ,  $j = 1, 2$ , is regular except at the points where the eigenvalue  $\lambda_j^a$  is not simple.

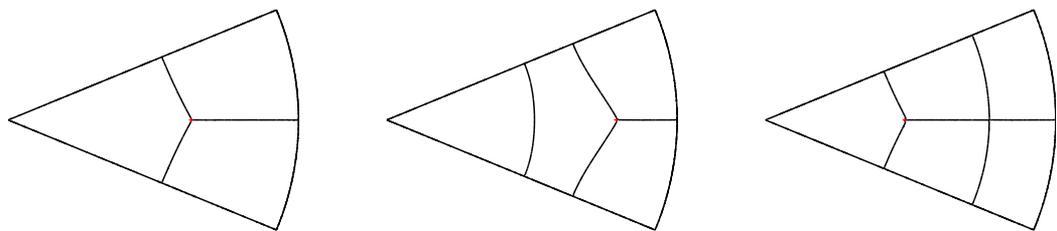
Going back to Figure 4, we see that the only critical points of  $\lambda_j^a$  which correspond to simple eigenvalues are inflexion points. As an example, we have analyzed the inflexion points for  $\lambda_3^a, \lambda_4^a, \lambda_5^a$  when  $a = (a_1, 0)$



**Figure 4.**  $a \mapsto \lambda_j^a$ ,  $a \in \{(\frac{m}{1000}, 0), 0 < m < 1000\}$ ,  $1 \leq j \leq 9$ .

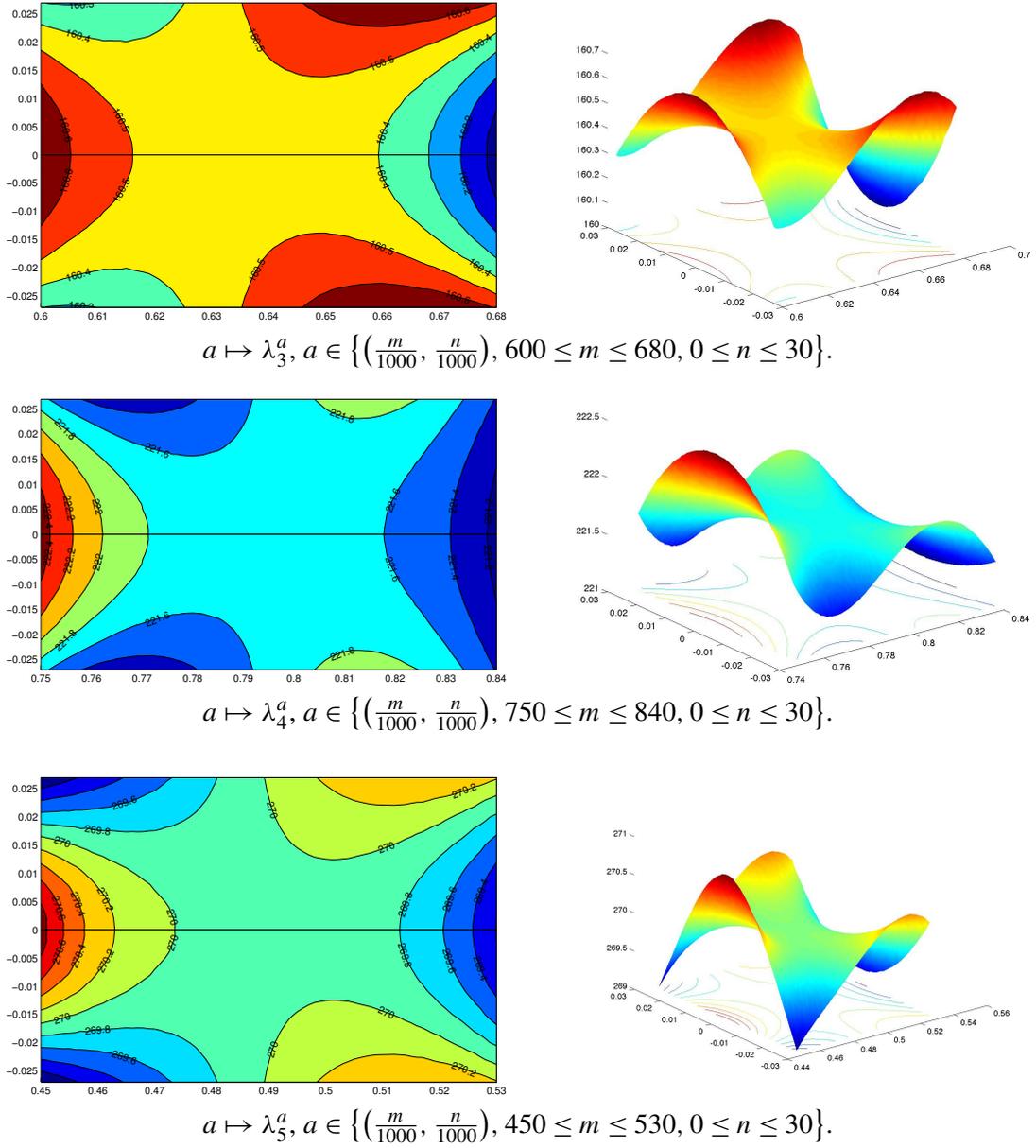


**Figure 5.** Nodal lines of an eigenfunction associated with  $\lambda_3^a$ ,  $a = (a_1, 0)$ ,  $a_1 = 0.6, 0.63, 0.65$ .



**Figure 6.** Nodal lines of an eigenfunction associated with  $\lambda_j^{a_{(j)}}$ ,  $j = 3, 4, 5$ .

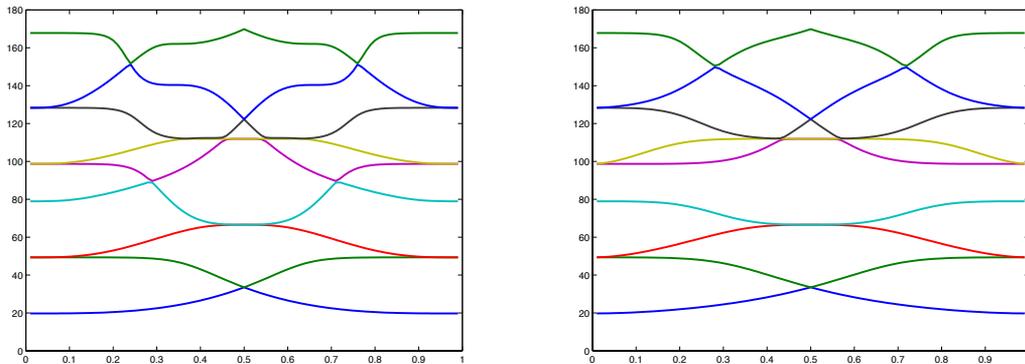
with  $a_1 \in (0.6, 0.7)$ ,  $a_1 \in (0.75, 0.85)$  and  $a_1 \in (0.45, 0.55)$  respectively. We will denote these points by  $a_{(j)}$ ,  $j = 3, 4, 5$ . Figure 5 gives the nodal lines for three different points  $a = (a_1, 0)$  on the symmetry axis  $y = 0$  with  $a_1 = 0.6, 0.63$  and  $0.65$ . This illustrates the emergence of a triple point when the pole is moved along the line  $y = 0$ . In Figure 6, we have plotted the nodal lines of the eigenfunctions  $\varphi_j^{a_{(j)}}$  associated with  $\lambda_j^{a_{(j)}}$ ,  $j = 3, 4, 5$ . We observe that each  $\varphi_j^{a_{(j)}}$  has a zero of order  $\frac{3}{2}$  at  $a_{(j)}$ . Correspondingly, the derivative of  $\lambda_j^a$  at  $a_{(j)}$  vanishes in Figure 4, thus illustrating Theorem 1.7. In the three examples proposed here, also the second derivative of  $\lambda_j^a$  vanishes at  $a_{(j)}$ .



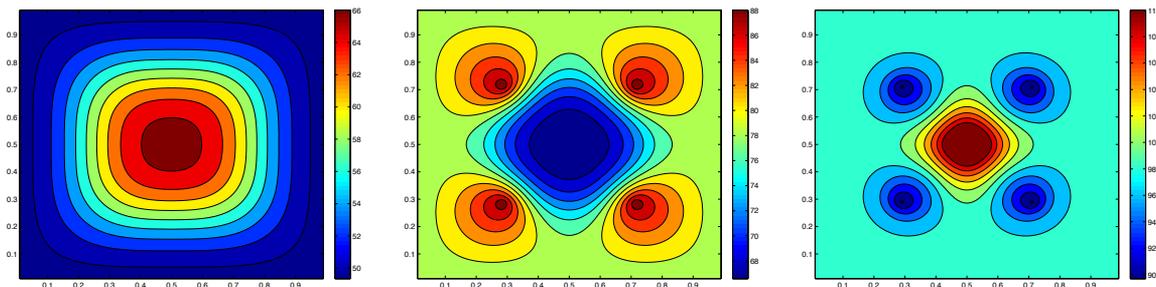
**Figure 7.**  $\lambda_j^a$  vs.  $a$  for  $a$  around the inflexion point  $a_{(j)}$ ,  $j = 3, 4, 5$ .

Let us now move a little the singular point around  $a_{(j)}$ . We use a discretization step of  $\frac{1}{1000}$ . Figure 7 represents the behavior of  $\lambda_j^a$  for  $a$  close to  $a_{(j)}$ . It indicates that these points are degenerated saddle points. The behavior of the function  $a \mapsto \lambda_j^a$ ,  $j = 3, 4, 5$ , around  $a_{(j)}$  is quite similar to that of the function  $(t, x) \mapsto t(t^2 - x^2)$  around the origin  $(0, 0)$ .

We remark that computing the first twelve eigenvalues of  $(i\nabla + A_a)^2$  on  $\Sigma_{\pi/4}$ , we have never found an eigenfunction for which five or more nodal lines end at a singular point  $a$ .



**Figure 8.**  $a \mapsto \lambda_j^a$  for  $a$  along the diagonal (left) or perpendicular bisector (right) of a square ( $1 \leq j \leq 9$ ).



**Figure 9.** Eigenvalues of  $(i\nabla + A_a)^2$  in  $[0, 1] \times [0, 1]$ ,  $a \in \Pi_{50}$ .



**Figure 10.** Nodal lines of an eigenfunction associated with  $\lambda_j^a$ ,  $j = 3, 4$ ,  $a = (\frac{1}{2}, \frac{1}{2})$ .

As we have already remarked, all the local maxima and minima of  $\lambda_j^a$  in Figure 4 correspond to nonsimple eigenvalues. Plotting the nodal lines of the corresponding eigenfunctions, we have found that they all have a zero of order  $\frac{1}{2}$  at  $a$ , i.e., one nodal line ending at  $a$ . Nonetheless, this is not a general fact: in performing the same analysis in the case  $\Omega$  is a square  $[0, 1] \times [0, 1]$ , we have found that the third and fourth eigenfunctions have a zero of order  $\frac{3}{2}$  at the center  $a = (\frac{1}{2}, \frac{1}{2})$ , see Figure 10, which is in this case a maximum of  $a \mapsto \lambda_3^a$  and a minimum of  $a \mapsto \lambda_4^a$ ; see Figures 8, 9. We observe in Figure 8 that the first and second derivatives of  $\lambda_3^a$  and of  $\lambda_4^a$  seem to vanish at the center  $a = (\frac{1}{2}, \frac{1}{2})$ .

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Received 3 Oct 2013. Revised 22 Feb 2014. Accepted 1 Apr 2014.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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APDE peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

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Volume 7 No. 6 2014

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