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A classical result by Cheng in 1976, improved later by Besson and Nadirashvili, says that the multiplicities of the eigenvalues of the Schrödinger operator $(-\Delta_g + \nu)$, where ν is C^∞ -smooth, on a compact Riemannian surface M are bounded in terms of the eigenvalue index and the genus of M . We prove that these multiplicity bounds hold for an L^p -potential ν , where $p > 1$. We also discuss similar multiplicity bounds for Laplace eigenvalues on singular Riemannian surfaces.

1. Introduction and statements of results

Multiplicity bounds. Let M be a connected compact surface. For a Riemannian metric g and C^∞ -smooth function ν on M we denote by

$$\lambda_0(g, \nu) < \lambda_1(g, \nu) \leq \dots \leq \lambda_k(g, \nu) \leq \dots$$

the eigenvalues of the Schrödinger operator $(-\Delta_g + \nu)$. If M has a nonempty boundary, we assume that the Dirichlet boundary condition is imposed.

The following theorem is an improved version of the statement originally discovered by Cheng [1976]. It is due to Besson [1980] for closed orientable surfaces and to Nadirashvili [1987] for general closed surfaces; multiplicity bounds for general boundary value problems were obtained in [Karpukhin et al. 2013].

Theorem 1.1. *Let (M, g) be a smooth compact surface, possibly with boundary. Then, for any C^∞ -smooth function ν on M , the multiplicity $m_k(g, \nu)$ of an eigenvalue $\lambda_k(g, \nu)$ satisfies the inequality*

$$m_k(g, \nu) \leq 2(2 - \chi - l) + 2k + 1, \quad k = 1, 2, \dots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

Above, we assume that $l = 0$ for closed surfaces. Note that even the fact that eigenvalue multiplicities on Riemannian surfaces are bounded is by no means trivial, and as is known [Colin de Verdière 1986; 1987], fails in higher dimensions, unless some specific hypotheses on a Riemannian metric or a potential are imposed. The key ingredient in the proof of Theorem 1.1 is the so-called *Cheng’s structure theorem* [Cheng 1976]: for any solution u to the Schrödinger equation with a smooth potential and any interior point $p \in M$ there exists a neighbourhood of p and its diffeomorphism onto a ball in \mathbb{R}^2 centred at the origin that maps the nodal set of u onto the nodal set of a homogeneous harmonic polynomial. This statement is

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based on a local approximation of solutions by harmonic homogeneous polynomials [Bers 1955] and, in particular, implies that the nodal set of a solution u is locally homeomorphic to its tangent cone. The latter property of nodal sets does not hold in higher dimensions [Bérard and Meyer 1982]. The structure theorem holds for sufficiently smooth solutions to the Schrödinger equation — see the Appendix — and consequently the multiplicity bounds in Theorem 1.1 hold for Hölder continuous potentials. Based on Cheng’s structure theorem, the multiplicity bounds for various eigenvalues problems have been extensively studied in the literature. We refer to the papers [Colin de Verdière 1987; Hoffmann-Ostenhof et al. 1999a; Hoffmann-Ostenhof et al. 1999b; Karpukhin et al. 2013] and references there for the details.

The purpose of this paper is to show that the multiplicity bounds continue to hold for rather weak potentials when no similar structure theorem for nodal sets is available. For a given real number $\delta \in (0, 2)$, we consider the class $K^{2,\delta}(M)$, introduced in [Aizenman and Simon 1982; Simon 1982], which is formed by absolutely integrable potentials v such that

$$\sup_{x \in M} \int_{B(x,r)} |x - y|^{-\delta} |v(y)| d\text{Vol}_g(y) \rightarrow 0 \quad \text{as } r \rightarrow 0, \quad (1-1)$$

where the absolute value $|x - y|$ above denotes the distance between x and y in the background metric g . It is a straightforward consequence of the Hölder inequality that any L^p -integrable function with $p > 1$ belongs to $K^{2,\delta}$ for some positive δ . However, unlike the traditional L^p -hypothesis, the potentials from $K^{2,\delta}(M)$ include certain physically important cases [Aizenman and Simon 1982; Simon 1982].

The hypothesis that $v \in K^{2,\delta}(M)$ implies that the measures $d\mu^\pm = v^\pm d\text{Vol}_g$, where v^+ and v^- are the positive and negative parts of v , are δ -uniform:

$$\mu^\pm(B(x, r)) \leq Cr^\delta \quad \text{for any } r > 0 \text{ and } x \in M$$

and some constant C . By the results of Maz’ja [1985] (see also [Kokarev 2014]) for such measures μ^\pm the Sobolev space $W^{1,2}(M, \text{Vol}_g)$ embeds compactly into $L_2(M, \mu^\pm)$. By standard perturbation theory [Kato 1976] (see also [Maz’ja 1985; Simon 1982]) we then conclude that the spectrum of the Schrödinger operator $(-\Delta_g + v)$ is discrete, bounded from below, and all eigenvalues have finite multiplicities. Our main result says that they satisfy the same multiplicity bounds.

Theorem 1.2. *Let (M, g) be a smooth compact surface, possibly with boundary. Then, for any absolutely integrable potential v from $K^{2,\delta}(M)$, where $\delta \in (0, 2)$, the multiplicity $m_k(g, v)$ of an eigenvalue $\lambda_k(g, v)$ satisfies the inequality*

$$m_k(g, v) \leq 2(2 - \chi - l) + 2k + 1, \quad k = 1, 2, \dots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

For the first eigenvalue $\lambda_1(g, v)$ the above multiplicity bound is sharp when M is homeomorphic to a sphere \mathbb{S}^2 or a projective plane $\mathbb{R}P^2$. When a potential v is smooth, there is an extensive literature [Colin de Verdière 1987; Nadirashvili 1987; Sévenec 2002] (and references therein) devoted to sharper multiplicity bounds for the first eigenvalue. In addition, in [Hoffmann-Ostenhof et al. 1999a; 1999b] the authors show that when M is a sphere or a disk the multiplicity bounds in Theorem 1.1 can be improved to

$m_k(g, \nu) \leq 2k - 1$ for $k \geq 2$. We have made no effort to improve our results in these directions. However, it is worth mentioning that the main topological result in [Sévenec 2002] does yield a sharper multiplicity bound for $\lambda_1(g, \nu)$ for some closed surfaces when a potential ν belongs to the space $K^{2,\delta}(M)$. More precisely, if M is a closed surface whose Euler–Poincaré number χ is negative, Theorem 5 of [Sévenec 2002] implies that $m_1(g, \nu) \leq 5 - \chi$ for any potential $\nu \in K^{2,\delta}(M)$. By the results in [Colin de Verdière 1987] this bound is sharp for $\mathbb{T}^2 \# \mathbb{T}^2$ and $\#n \mathbb{R}P^2$, where $n = 3, 4, 5$.

The multiplicity bounds in Theorem 1.1 also hold for eigenvalue problems on singular Riemannian surfaces; we discuss them in detail in Section 5. The proof of Theorem 1.2 is based on the delicate study of the nodal sets of Schrödinger eigenfunctions that we describe below.

Nodal sets of eigenfunctions. Let u be a solution to the eigenvalue problem

$$(-\Delta_g + \nu)u = \lambda u \quad \text{on } M, \quad (1-2)$$

where $\nu \in K^{2,\delta}(M)$ and, if $\partial M \neq \emptyset$, the Dirichlet boundary hypothesis is assumed. Recall that by results in [Simon 1982] such an eigenfunction u is Hölder continuous. We denote by $\mathcal{N}(u)$ its nodal set $u^{-1}(0)$.

By the results in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995] combined with the strong unique continuation property [Sawyer 1984; Chanillo and Sawyer 1990], in appropriate local coordinates around an interior point $x_0 \in M$ a nontrivial solution u has the form

$$u(x) = P_N(x - x_0) + O(|x - x_0|^{N+\delta}) \quad \text{for } x \in U,$$

where P_N is a nontrivial homogeneous harmonic polynomial on the Euclidean plane. We refer to Section 2 for a precise statement. The degree of this approximating homogeneous harmonic polynomial defines the so-called *vanishing order* $\text{ord}_x u$ for any interior point $x \in M$. Each point $x \in \mathcal{N}(u)$ has vanishing order at least one, and we define $\mathcal{N}^2(u)$ as the set of points x whose vanishing order $\text{ord}_x u$ is at least two.

The proof of Theorem 1.2 is based on the following key result.

Theorem 1.3. *Let (M, g) be a compact Riemannian surface, possibly with boundary, and let u be a nontrivial eigenfunction for the Schrödinger eigenvalue problem (1-2) with $\nu \in K^{2,\delta}(M)$, where $\delta \in (0, 2)$. Then the set $\mathcal{N}^2(u)$ is finite, and the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components. Moreover, for any $x \in \mathcal{N}^2(u)$, the number of connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ incident to x is an even integer that is at least $2 \text{ord}_x u$.*

The theorem says that the nodal set $\mathcal{N}(u)$ can be viewed as a graph: the vertices are points from $\mathcal{N}^2(u)$, and the edges are connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. This graph structure assigns to each $x \in \mathcal{N}^2(u)$ its degree $\deg x$, that is, the number of edges incident to x . If there is an edge that starts and ends at the same point, then it counts twice. The last statement of Theorem 1.3 says that $\deg x \geq 2 \text{ord}_x u$ for any $x \in \mathcal{N}^2(u)$. When the potential ν is smooth, Theorem 1.3 is a direct consequence of Cheng’s structure theorem and, in this case, the degree $\deg x$ is precisely $2 \text{ord}_x u$.

The proof of Theorem 1.3 uses essentially Courant’s nodal domain theorem, and is based on topological arguments, which are in turn built on the results in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995]. More precisely, one of the key ingredients is the study of prime ends

of nodal domains, which leads to a construction of neighbourhoods of $x \in \mathcal{N}(u)$ where a solution also has a finite number of nodal domains. Our method uses the properties of solutions in the interior of M only; it largely disregards their behaviour at the boundary. Consequently, the main results (Theorems 1.2 and 1.3) hold for rather general boundary value problems as long as Courant's nodal domain theorem holds; cf. [Karpukhin et al. 2013, Section 6]. The statement of Theorem 1.3 continues to hold for general solutions to the Schrödinger equation $(-\Delta + V)u = 0$ that have a finite number of nodal domains. Without the latter hypothesis for arbitrary L^p -potentials, it is unknown even whether the Hausdorff dimension of $\mathcal{N}^2(u)$ equals zero or not.

The paper is organised in the following way. In Section 2 we collect the background material on the strong unique continuation property, regularity of nodal sets, and recall the approximation results from [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995]. Here we also derive a number of consequences of these results that describe qualitative properties of nodal sets; they are used often in our sequel arguments. In the next section we recall the notion of Carathéodory's prime end and show that prime ends of nodal domains have the simplest possible structure: their impression always consists of a single point. In Section 4 we prove Theorems 1.2 and 1.3. In the last section we discuss multiplicity bounds for eigenvalue problems on surfaces with measures. We show that Laplace eigenvalue problems on singular Riemannian surfaces, such as Alexandrov surfaces of bounded integral curvature, can be viewed as particular instances of such problems. The paper also has an Appendix where we give details on Cheng's structure theorem for the reader's convenience.

2. Preliminaries

Background material. We start by collecting background material on solutions of the Schrödinger equation, which is used throughout the paper. From now on we assume that a potential V belongs to the space $K^{2,\delta}(M)$, where $\delta \in (0, 1)$. The superscript 2 in the notation for this function space refers to the dimension of M . Note that the space $K^{2,\delta}(M)$ is contained in the so-called *Kato space* formed by absolutely integrable functions V such that

$$\sup_{x \in M} \int_{B(x,r)} \ln\left(\frac{1}{|x-y|}\right) |v(y)| d\text{Vol}_g(y) \rightarrow 0 \quad \text{as } r \rightarrow 0;$$

see [Simon 1982]. Consider the Schrödinger equation

$$(-\Delta_g + V)u = 0 \quad \text{on } M, \tag{2-1}$$

understood in the distributional sense. As was mentioned above, by the results in [Simon 1982] its solutions are Hölder continuous. They also enjoy the following *strong unique continuation property*.

Proposition 2.1. *Let (M, g) be a smooth connected compact Riemannian surface, possibly with boundary, and let $x_0 \in M$ be an interior point. Let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$, such that*

$$u(x) = O(|x - x_0|^\ell) \quad \text{for any } \ell > 0.$$

Then u vanishes identically on M .

Proposition 2.1 is a consequence of the results in [Sawyer 1984], where the author proves that a solution u of the Schrödinger equation with the potential V from the Kato space $K^2(M)$ satisfies the unique continuation property: if u vanishes on a nonempty open subset, then it vanishes identically. As was pointed out in [Hoffmann-Ostenhof et al. 1995; Chanillo and Sawyer 1990], the argument in [Sawyer 1984] actually yields the strong unique continuation property.

The following fundamental statement is a combination of the main result in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992] with Proposition 2.1.

Proposition 2.2. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. For any interior point $x_0 \in M$, there exist a coordinate chart U around x_0 and a nontrivial homogeneous harmonic polynomial P_N of degree $N \geq 0$ on the Euclidean plane such that*

$$u(x) = P_N(x - x_0) + O(|x - x_0|^{N+\delta'}), \quad \text{where } x \in U,$$

for any $0 < \delta' < \delta$.

The proposition says that for any point $x \in M$ there is a well-defined *vanishing order* $\text{ord}_x u$ of a solution u at x , understood as the degree of the harmonic polynomial P_N . For a positive integer ℓ we define the set

$$\mathcal{N}^\ell(u) = \{x \in \text{Int } M \mid \text{ord}_x u \geq \ell\}.$$

Clearly, the nodal set $\mathcal{N}(u) = u^{-1}(0)$ is precisely the set $\mathcal{N}^1(u)$. Recall that a connected component of $M \setminus \mathcal{N}(u)$ is called a *nodal domain* of u . The combination of the Harnack inequality in [Aizenman and Simon 1982; Simon 1982] and the unique continuation property implies that a nontrivial solution u has different signs on adjacent nodal domains. Moreover, every point $x \in \mathcal{N}(u)$ belongs to the closure of at least two nodal domains.

Now suppose that u is an eigenfunction; that is, a solution to the eigenvalue problem (1-2). The following version of a classical statement is used in the sequel.

Courant's nodal domain theorem. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and $v \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then each nontrivial eigenfunction u corresponding to the eigenvalue $\lambda_k(g, v)$ of eigenvalue problem (1-2) has at most $k+1$ nodal domains.*

The proof follows standard arguments; see [Courant and Hilbert 1953]. It uses variational characterisation of eigenvalues $\lambda_k(g, v)$, the unique continuation property, Proposition 2.1, and the continuity of eigenfunctions up to the boundary. The latter can be deduced, for example, from the interior regularity [Simon 1982] by straightening the boundary locally and reflecting across it in an appropriate way.

Qualitative properties of nodal sets. Let u be a solution of the Schrödinger equation (2-1). If u is C^1 -smooth, then the implicit function theorem implies that the complement

$$\mathcal{N}^1(u) \setminus \mathcal{N}^2(u) \tag{2-2}$$

is a collection of C^1 -smooth arcs. The following celebrated nodal set regularity theorem due to [Hoffmann-Ostenhof et al. 1995] says that the latter holds under rather weak assumptions on a potential, when a solution u is not necessarily C^1 -smooth.

Proposition 2.3. *Let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then any point x in the complement (2-2) has a neighbourhood $U \subset M$ such that the set $\mathcal{N}^1(u) \cap U$ is the graph of a $C^{1,\delta}$ -smooth function with nonvanishing gradient. Further, if a potential V is $C^{k,\alpha}$ -smooth, then such a point x has a neighbourhood U such that $\mathcal{N}^1(u) \cap U$ is the graph of a $C^{k+3,\alpha}$ -smooth function with nonvanishing gradient.*

Below by *nodal edges* we mean the connected components of $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$. By Proposition 2.3 they are diffeomorphic to intervals of the real line, and their ends belong to the set $\mathcal{N}^2(u)$. We say that a nodal edge is *incident to* $x \in \mathcal{N}^2(u)$ if its closure contains x . A nodal edge is called a *nodal loop* if it is incident to one point $x \in \mathcal{N}^2(u)$ only. In other words, such a nodal edge starts and ends at the same point x .

The important consequence of Proposition 2.3 is the statement that nodal edges cannot accumulate to another nodal edge. We use this fact to describe a nodal set structure around an isolated point $x \in \mathcal{N}^2(u)$.

Corollary 2.4. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$. Then the number of nodal edges incident to x that are not nodal loops is finite. Moreover, any sequence of nodal loops incident to x has to contract to x .*

Proof. Let B be a neighbourhood of x whose closure does not contain any points in $\mathcal{N}^2(u)$. We view B as a unit ball in \mathbb{R}^2 centred at the origin $x = 0$. Suppose that there is an infinite number of nodal edges incident to x that are not nodal loops. Denote by Γ_i the connected components of the intersections of these nodal edges with the ball B whose closures $\bar{\Gamma}_i$ contain x . By Proposition 2.3, each $\bar{\Gamma}_i$ consists of a piece of a C^1 -smooth nodal arc and the origin x . They form a sequence of compact subsets of \bar{B} , and hence contain a subsequence that converges to a compact subset $\bar{\Gamma}_0 \subset \bar{B}$ in the Hausdorff distance. Clearly, the subset $\bar{\Gamma}_0$ belongs to the nodal set $\mathcal{N}(u)$ and contains the origin $x = 0$. Since the subsets $\bar{\Gamma}_i$ contain points on the boundary ∂B , then so does $\bar{\Gamma}_0$; in particular, the limit subset $\bar{\Gamma}_0$ does not coincide with x . Since the origin x is the only higher-order nodal point in \bar{B} , then $\bar{\Gamma}_0 \setminus \{x\}$ is the union of pieces of C^1 -smooth nodal edges. Without loss of generality, we may assume that the sequence $\bar{\Gamma}_i$ converges to a subset $\bar{\Gamma}_0$ such that $\bar{\Gamma}_0 \setminus \{x\}$ is a piece of a nodal edge. Now to get a contradiction we may either appeal to Proposition 2.3 directly, or argue in the following fashion. Let $x_i \in \bar{\Gamma}_i \cap \partial B$ be a sequence of points that converges to a point $x_0 \in \bar{\Gamma}_0 \cap \partial B$. We consider the two cases.

Case 1: *the complement $\bar{\Gamma}_0 \setminus \{x\}$ belongs to a nodal edge that intersects ∂B at x_0 transversally.* By Proposition 2.2, it is straightforward to see that the tangent line to Γ_0 at x_0 is precisely the kernel of an approximating linear function P_1 at x_0 . Since Γ_0 intersects ∂B at x_0 transversally, we conclude that the sequence $P_1((x_i - x_0)/|x_i - x_0|)$ is bounded away from zero for all sufficiently large i . On the other hand, by Proposition 2.2 we obtain $P_1(x_i - x_0) = O(|x_i - x_0|^{1+\delta})$, and arrive at a contradiction.

Case 2: *the complement $\bar{\Gamma}_0 \setminus \{x\}$ belongs to a nodal edge that is tangent to ∂B at x_0 .* Then there exists a sufficiently small ball B_0 centred at x_0 such that Γ_0 intersects ∂B_0 transversally. Choosing a sequence of

points $x'_i \in \Gamma_i \cap \partial B_0$ that converges to a point $x'_0 \in \Gamma_0 \cap B_0$, and arguing in a fashion similar to in Case 1, we again arrive at a contradiction.

Now we demonstrate the last statement of the lemma. Suppose that there is a sequence of nodal loops incident to x that do not contract to x . Choosing a subsequence and a sufficiently small neighbourhood B of x , we may assume that each nodal loop intersects with ∂B . Then the argument above shows that this sequence has to be finite. \square

We proceed with another statement on local properties of the nodal set near an isolated point $x \in \mathcal{N}^2(u)$.

Corollary 2.5. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$. Then there exists a neighbourhood B of x , viewed as a ball in the Euclidean plane, such that the zeroes of u on ∂B are precisely the intersections of the connected components of $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$ incident to x with ∂B .*

Proof. First, since x is isolated in $\mathcal{N}^2(u)$, one can choose a neighbourhood B such that it does not contain other points from $\mathcal{N}^2(u)$. Thus, for a proof of the lemma it is sufficient to show that the point x is not a limit point of the nodal edges that are not incident to x . This can be demonstrated following an argument similar to the one used in the proof of Corollary 2.4. \square

Let $x \in \mathcal{N}^2(u)$ be a point isolated in $\mathcal{N}^2(u)$ such that the number of nodal edges incident to x is finite. The number of these nodal edges, where nodal loops are counted twice, is a characteristic of a point x , called the *degree* and denoted by $\deg x$. It is closely related to the vanishing order $\text{ord}_x u$. More precisely, if a solution u is sufficiently smooth, then by Cheng's structure theorem [1976], it equals $2 \text{ord}_x u$. The following lemma describes its relationship to $\text{ord}_x u$ under rather weak regularity assumptions on u .

Lemma 2.6. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$ such that the degree $\deg x$ is finite. Then $\deg x$ is an even integer that is at least $2 \text{ord}_x u$.*

Proof. Denote by N the vanishing order $\text{ord}_x u$, that is, the degree of an approximating homogeneous harmonic polynomial $P_N(y - x)$; see Proposition 2.2. Choose a sufficiently small neighbourhood B of x such that it does not contain other points from $\mathcal{N}^2(u)$ and does not contain nodal loops. We identify B with a unit ball in the Euclidean plane such that the point x corresponds to the origin. By $B_\lambda \subset B$ we mean a neighbourhood that corresponds to a ball of radius λ , where $0 < \lambda < 1$. Consider the rescaled function

$$u_\lambda(y) = \lambda^{-N} u(\lambda \cdot y)$$

defined on the unit circle $S = \{y : |y| = 1\}$. Proposition 2.2 implies that $u_\lambda(y)$ converges uniformly to the homogeneous harmonic polynomial $P_N(y)$ as $\lambda \rightarrow 0$, when y ranges over the unit circle S . As is known, $P_N(y)$ changes sign on S precisely $2N$ times, and hence the corresponding zeroes are stable under the perturbation of $P_N(y)$. Thus, we conclude that for all sufficiently small $\lambda > 0$ the zeroes of u_λ lie in small pairwise nonintersecting neighbourhoods $U_i \subset S$, where $i = 1, \dots, 2N$, of the zeroes of $P_N(y)$,

and each U_i contains at least one zero of u_λ . Choosing a sufficiently small $\lambda > 0$, by Corollary 2.5 we may assume that the zeroes of u_λ correspond to the intersections of nodal edges incident to x with ∂B_λ . Further, the intersections of the nodal edges incident to x with B_λ lie in the cones

$$C_i(\lambda) = \{t \cdot \lambda U_i : 0 < t < 1\}, \quad \text{where } i = 1, \dots, 2N.$$

Since the cones $C_i(\lambda)$ are pairwise nonintersecting and each of them contains at least one connected piece of a nodal edge incident to x , we conclude that $\deg x$ is at least $2N$.

Now we claim that each cone $C_i(\lambda)$ contains an odd number of nodal edge pieces incident to x , and hence the degree $\deg x$ is an even integer. Indeed, the solution u has different signs on the connected components of $B_\lambda \setminus \cup C_i(\lambda)$ adjacent to the same cone; they coincide with the signs of u_λ and the approximating homogeneous harmonic polynomial P_N . Since u also has different signs on adjacent nodal domains, each nodal edge piece incident to x contributes to the change of sign, and the claim follows in a straightforward fashion. □

Properties of the vanishing order. The proof of Proposition 2.3 is based on the following improvement of Proposition 2.2 due to [Hoffmann-Ostenhof et al. 1995], which is important for our considerations in the sequel. Below we denote by B a coordinate chart viewed as a ball in the Euclidean plane, and by $B_{1/2}$ the ball of half the radius of B .

Proposition 2.7. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let B be a coordinate chart in the interior of M viewed as a ball in the Euclidean plane. Then for a sufficiently small B and any $\ell \geq 1$ there exists a constant $C > 0$ such that for any point $y \in \mathcal{N}^\ell(u) \cap B_{1/2}$ there exists a degree ℓ homogeneous harmonic polynomial P_ℓ^y such that*

$$|u(x) - P_\ell^y(x - y)| \leq C \left(\sup_B |u| \right) |x - y|^{\ell + \delta} \quad \text{for any } x \in B,$$

and the polynomials P_ℓ^y satisfy $|P_\ell^y(\bar{x})| \leq C_* (\sup_B |u|)$ for any $|\bar{x}| = 1$, where the constants C and C_* do not depend on a solution u .

Note that the harmonic polynomials P_ℓ^y above either vanish identically or coincide with approximating harmonic polynomials at y from Proposition 2.2. The main estimate of Proposition 2.7 is stated in [Hoffmann-Ostenhof et al. 1995, Theorem 1]. The bound for the values of the harmonic polynomials on the unit circle follows from the proof, and is explained explicitly in [Hoffmann-Ostenhof et al. 1995, p. 1256].

We proceed with studying the vanishing order $\text{ord}_x u$ as a function of $x \in M$. The following lemma is a straightforward consequence of Proposition 2.7. We include a proof for completeness of exposition.

Lemma 2.8. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then the function $\text{ord}_x u$ is upper-semicontinuous in the interior of M ; that is, for any sequence x_i converging to an interior point $x \in M$, one has the inequality $\limsup \text{ord}_{x_i}(u) \leq \text{ord}_x u$.*

Proof. It is sufficient to show that if x_i belong to $\mathcal{N}^\ell(u)$, then so does the limit point x . Without loss of generality, we may assume that the points x_i lie in a coordinate chart B that is identified with a unit ball in \mathbb{R}^2 centred at the origin $x = 0$, and $x_i \rightarrow 0$ as $i \rightarrow +\infty$. In addition, to simplify the notation, we assume that $\sup|u|$ on B equals 1. Let P_ℓ^i be a degree ℓ homogeneous harmonic polynomial corresponding to x_i from Proposition 2.7. Representing u as the sum of $u - P_\ell^i$ and P_ℓ^i , we obtain

$$|u(x)| \leq |u(x) - P_\ell^i(x - x_i)| + |P_\ell^i(x - x_i)| \leq C|x - x_i|^{\ell+\delta} + C_*|x - x_i|^\ell \quad \text{for any } x \in B,$$

where the second inequality for all sufficiently large i follows from Proposition 2.7. Passing to the limit as $i \rightarrow +\infty$, we get

$$|u(x)| \leq C'|x|^\ell \quad \text{for any } x \in B,$$

and conclude that the vanishing order at the origin is at least ℓ . □

Our last lemma says that the vanishing order $\text{ord}_x u$ is strictly upper-semicontinuous on $\mathcal{N}^2(u)$.

Lemma 2.9. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then for any sequence $x_i \in \mathcal{N}^2(u)$ converging to an interior point $x \in M$ we have $\limsup \text{ord}_{x_i}(u) < \text{ord}_x u$.*

Proof. As in the proof of Lemma 2.8, we assume that the points x_i belong to a coordinate chart B , viewed as a unit ball in \mathbb{R}^2 centred at the origin $x = 0$, and $x_i \rightarrow 0$ as $i \rightarrow +\infty$. We also suppose that $\sup|u|$ on B equals 1. First, by Lemma 2.8 we conclude that the upper limit $\limsup \text{ord}_{x_i}(u)$ is finite; we denote it by N . After a selection of a subsequence, we may assume that the vanishing order $\text{ord}_{x_i}(u)$ equals N for each x_i . By Lemma 2.8 it remains to show that the vanishing order $\text{ord}_x u$ at the origin x cannot be equal to N .

Suppose the contrary: the order of u at the origin equals $N \geq 2$. Let P_N be an approximating homogeneous harmonic polynomial for u at the origin. By Proposition 2.7, for a sufficiently large index i we have

$$\begin{aligned} |P_N(x) - P_N^i(x - x_i)| &\leq |u(x) - P_N(x)| + |u(x) - P_N^i(x - x_i)| \\ &\leq C(|x|^{N+\delta} + |x - x_i|^{N+\delta}) \quad \text{for any } x \in B, \end{aligned} \tag{2-3}$$

where P_N^i is an approximating homogeneous harmonic polynomial at x_i . Denote by λ_i the absolute value $|x_i|$, and by \bar{x}_i the point $\lambda_i^{-1}x_i$ on the unit circle. Setting $x = \lambda_i\bar{x}$ in inequality (2-3) and using the homogeneity of the left-hand side, we obtain

$$|P_N(\bar{x}) - P_N^i(\bar{x} - \bar{x}_i)| \leq (1 + 2^{N+\delta})C\lambda_i^\delta \quad \text{for any } |\bar{x}| = 1. \tag{2-4}$$

Without loss of generality, we may assume that the sequence \bar{x}_i converges to a point \bar{x}_0 , $|\bar{x}_0| = 1$. Setting \bar{x} to be equal to \bar{x}_i in inequality (2-4) and passing to the limit as $i \rightarrow +\infty$, we see that \bar{x}_0 is a zero of P_N . Recall that the nodal set of P_N consists of n straight lines passing through the origin; the vanishing order of the origin equals N , and any other nodal point, such as \bar{x}_0 , has vanishing order 1. On the other hand, by Proposition 2.7 the polynomials P_N^i are uniformly bounded on the unit circle and, since in polar coordinates they have the form

$$a_i r^N \cos(N\theta) + b_i r^N \sin(N\theta),$$

we conclude that, after a selection of a subsequence, they converge either to zero or to a harmonic homogeneous polynomial P_N^0 of degree N . If the former case occurs, then after passing to the limit in inequality (2-4) we see that $P_N(x)$ vanishes, and arrive at a contradiction. Now assume that the harmonic polynomials P_N^i converge to a nontrivial harmonic polynomial P_N^0 . Then the polynomials $P_N^i(\bar{x} - \bar{x}_i)$ converge uniformly to $P_N^0(\bar{x} - \bar{x}_0)$ and, passing to the limit in inequality (2-4), we conclude that $P_N(\bar{x})$ coincides identically with $P_N^0(\bar{x} - \bar{x}_0)$. Now, since $N \geq 2$, it is straightforward to arrive at a contradiction. The polynomial $P_N(\bar{x})$ has precisely $2N$ zeroes as \bar{x} ranges over the unit circle, while the polynomial $P_N^0(\bar{x} - \bar{x}_0)$ has at most $N + 1$. \square

Corollary 2.10. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then the set $\mathcal{N}^2(u)$ is totally disconnected: every nonempty connected subset is a single point. Moreover, the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ is open and dense in the nodal set.*

Proof. Suppose the contrary to the first statement. Then there exists a nonempty connected subset $C \subset \mathcal{N}^2(u)$ that is not a single point. Since any point $x \in C$ is the limit of a nontrivial sequence in C , by Lemma 2.9 we conclude that $C \subset \mathcal{N}^\ell(u)$ for any $\ell \geq 2$. Hence, the solution u vanishes to an infinite order at C and, by the strong unique continuation, Proposition 2.1, vanishes identically. This contradiction demonstrates the first statement.

By Lemma 2.8 the set $\mathcal{N}^2(u)$ is closed, and for a proof of the second statement of the corollary it remains to show that the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ is dense. Suppose the contrary. Then for some point $p \in \mathcal{N}(u)$ there exists a ball $B_\varepsilon(p)$ such that $C = B_\varepsilon(p) \cap \mathcal{N}(u)$ is contained in $\mathcal{N}^2(u)$. By the Harnack inequality [Aizenman and Simon 1982; Simon 1982] no point in the nodal set can be isolated, and we conclude that any $x \in C$ is the limit of a nontrivial sequence in C . Now we arrive at a contradiction in a fashion similar to the one above. \square

3. Prime ends of nodal domains

Now we study the nodal set $\mathcal{N}(u)$ from the point of view of the topology of nodal domains. More precisely, we describe the structure of prime ends of nodal domains. The notion of prime end goes back to Carathéodory [1913], who used it to describe the behaviour of conformal maps on the boundaries of simply connected domains. Later his theory was extended to general open subsets in manifolds [Epstein 1981]. However, main applications seem to be restricted to 2-dimensional problems [Milnor 2006]. We start by recalling the necessary definitions, following [Epstein 1981] closely.

Let $\Omega \subset M$ be a connected open subset, where we view M as the interior of a compact Riemannian surface. For a subdomain $D \subset \Omega$, we denote by ∂D the interior boundary:

$$\partial D = \Omega \cap \bar{D} \cap (\bar{\Omega} \setminus \bar{D}).$$

Definition 3.1. A *chain* in Ω is a sequence $\{D_i\}$, $i = 1, 2, \dots$, of open connected subsets of Ω such that

- ∂D_i is connected and nonempty for each i , and
- $\bar{D}_{i+1} \cap \Omega \subset D_i$ for each i .

Two chains $\{D_i\}$ and $\{D'_i\}$ are called *equivalent* if for any i there exists $j > i$ such that $D'_j \subset D_i$ and $D_j \subset D'_i$.

Definition 3.2. A chain in Ω is called a *topological chain* if there exists a point $p \in M$ such that

- the diameter of $(p \cup \partial D_i)$ tends to zero as $i \rightarrow +\infty$, and
- $\text{dist}(p, \partial D_i) > 0$ for each i .

The point p above is called the *principal point* of $\{D_i\}$. A *prime point* of Ω is the equivalence class of a topological chain.

Clearly, for a given topological chain the principal point $p \in \bar{\Omega}$ is unique. Note also that the above definitions do not depend on a metric on M . The set of all prime points of Ω is denoted by $\hat{\Omega}$. It is made into a topological space by taking the sets \hat{U} , formed by prime points represented by chains $\{D_i\}$ such that each D_i lies in an open subset $U \subset \Omega$, as a topological basis. There is a natural embedding $\omega : \Omega \rightarrow \hat{\Omega}$, defined by sending a point $x \in \Omega$ to the equivalence class of a sequence of concentric balls centred at x whose diameters tend to zero. As is shown in [Epstein 1981, Section 2], the map ω embeds Ω homeomorphically onto an open subset in $\hat{\Omega}$. A *prime end* of Ω is a prime point which is not in $\omega(\Omega)$. A *principal point* of a prime end is any principal point of any representative topological chain.

Although a given topological chain has only one principal point, a prime end may have many. The simplest example is given by considering a domain whose boundary has an oscillating behaviour similar to the graph of $\sin(1/x)$. The collection of all principal points is a subset of the *impression* $\bigcap \bar{D}_i$ of a prime end. The latter does not depend on a representative topological chain, and is a compact connected subset of the boundary $\partial\Omega$. Note also that a given point $x \in \partial\Omega$ can be a principal point of many different prime ends. We refer to [Epstein 1981; Milnor 2006] for examples and other details.

The following statement, proved in [Epstein 1981, Section 6], shows that prime ends give a useful compactification (the so-called *Carathéodory compactification*) of open subdomains.

Proposition 3.1. *Let (M, g) be a Riemannian surface, viewed as the interior of a compact surface, and let $\Omega \subset M$ be a connected open subset such that the first homology group $H_1(\Omega, \mathbb{Q})$ is finite-dimensional. Then there is a homeomorphism of $\hat{\Omega}$ onto a compact surface with boundary that maps the set of prime ends onto its boundary.*

We proceed with studying properties of nodal sets. The following lemma says that all prime ends of nodal domains have the simplest possible structure: any of them has only one principal point that coincides with its impression.

Lemma 3.2. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary. Let u be a nontrivial solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$, and let Ω be its nodal domain. Then for any prime end $[D_i]$ of Ω its impression $\bigcap \bar{D}_i$ consists of a single point. In particular, any prime end has only one principal point.*

Proof. First, the statement holds for any prime end that has a principal point x in the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Indeed, then the point x belongs to a nodal edge, which is the image of a C^1 -smooth regular path; see Proposition 2.3. By the implicit function theorem we can view a small nodal arc containing x as

a line segment in \mathbb{R}^2 . Then it is straightforward to see that any chain that has x as a principal point is equivalent to a chain that consists of concentric semidisks centred at x whose diameters converge to zero. Its impression consists of the point x only.

Now suppose that a given prime end has a principal point $x \in \mathcal{N}^2(u)$. Then we claim that its impression I does not have any points in $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Suppose the contrary. Then, since the impression I of a prime end is connected, we conclude that I contains a nontrivial arc C that belongs to some nodal edge; that is, C is a connected subset of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ that is not a single point, and $\text{dist}(x, C) > 0$. Let $\{D_i\}$ be a representative topological chain whose principal point is x , and let E_i be the set $\partial D_i \setminus I$, where ∂D_i is the boundary of D_i viewed as a subset of M . First, it is straightforward to see that for any $y \in C \subset I$ the distance $\text{dist}(y, E_i)$ converges to zero as $i \rightarrow +\infty$. For otherwise there is a neighbourhood U of y in \bar{D}_i such that $U \subset \bar{D}_i$ for any i . More precisely, viewing C around y as a straight segment in \mathbb{R}^2 , we may choose U to be diffeomorphic to a semidisk $B_\varepsilon^+(y)$, assuming that $\text{dist}(y, E_i) \geq 2\varepsilon$. Then we obtain the inclusions $U \subset I \subset \partial\Omega$, which are impossible. Thus, we see that any point $y \in C$ is the limit of a sequence $y_i \in \bar{E}_i$. Indeed, one can take as y_i a point at which the distance $\text{dist}(y, E_i)$ is attained. This implies that there is a sequence $C_i \subset \bar{E}_i$ of subsets that converges to a nodal arc C in the Hausdorff distance. Clearly, the sets $E_i \setminus (\partial D_i \cap \Omega)$ lie in the nodal set $\mathcal{N}(u)$, and since the interior boundaries $\partial D_i \cap \Omega$ converge to the point x , we conclude that for a sufficiently large i the subset C_i lies in the nodal set. Further, since the set $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ is open in the nodal set (see Lemma 2.8), we see that each C_i lies in $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Thus, without loss of generality, we may assume that the C_i are arcs of nodal edges. Combining the latter with Proposition 2.3, or following the argument in the proof of Corollary 2.4, we arrive at a contradiction.

Thus, the impression I does not have points in the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$, and is contained in $\mathcal{N}^2(u)$. By Corollary 2.10 the set $\mathcal{N}^2(u)$ is totally disconnected, and since the impression I is connected, it has to coincide with the point x . \square

Corollary 3.3. *Under the hypotheses of Lemma 3.2, the following statements hold:*

- (i) *Any point $x \in \partial\Omega$ is accessible; that is, it can be joined with any interior point in Ω by a continuous path $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = x$ and the image $\gamma(0, 1]$ lies in Ω .*
- (ii) *For any point $x \in \partial\Omega$ and any sufficiently small neighbourhood U of x there are only finitely many connected components U_1, \dots, U_k of $\Omega \cap U$ such that $x \in \bar{U}_i$, and the union $\bigcup \bar{U}_i$ is a neighbourhood of x in $\bar{\Omega}$.*
- (iii) *The boundary $\partial\Omega$ is locally connected.*

Proof. We derive the statements using the results in [Epstein 1981], which apply to open domains $\Omega \subset M$ whose first homology group $H_1(\Omega, \mathbb{Q})$ is finite-dimensional. Note that all statements are local, and hold trivially for the boundary points $x \in \mathcal{N}(u) \setminus \mathcal{N}^2(u)$. To prove the corollary for the boundary points $x \in \mathcal{N}^2(u)$ we may assume, after cutting Ω along smooth simple closed paths, that Ω has zero genus. Moreover, after cutting along paths joining points from $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ on different boundary components of Ω , we may assume that Ω is simply connected, and the results in [Epstein 1981] apply. Specifically, the first statement is a consequence of our Lemma 3.2 and Theorems 7.4 and 8.2 in that reference. The second

statement follows from Lemma 3.2 and [Epstein 1981, Theorem 8.2], and the third from Lemma 3.2 and [Epstein 1981, Theorem 8.3]. \square

4. The proofs

Proof of Theorem 1.3. Let (M, g) be a compact Riemannian surface, and u be a solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. First, we intend to generalise Theorem 1.3 to certain subdomains $\Omega \subset M$.

Definition 4.1. A connected open subset $\Omega \subset M$ is called a *proper subdomain* with respect to a solution u if its boundary consists of finitely many connected components and the solution u has finitely many nodal domains in Ω ; that is, the number of connected components of $\Omega \setminus \mathcal{N}(u)$ is finite.

If u is an eigenfunction, then by Courant's nodal domain theorem the surface M itself is a proper subdomain with respect to u . However, for our method it is also important to consider proper subdomains whose closures are contained in the interior of M . The hypothesis on the finite number of boundary components guarantees that such a domain Ω has finite topology and, by Proposition 3.1, is homeomorphic to the interior of a compact surface with boundary. The second hypothesis in Definition 4.1 mimics an important property of eigenfunctions, and is essential for our arguments in the sequel. Below we denote by $\mathcal{N}_\Omega(u)$ and $\mathcal{N}_\Omega^\ell(u)$ the sets $\mathcal{N}(u) \cap \Omega$ and $\mathcal{N}^\ell(u) \cap \Omega$ respectively.

Theorem 1.3 is a consequence of the following more general result.

Theorem 4.1. *Let (M, g) be a compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then for any proper subdomain $\Omega \subset M$ with respect to u the set $\mathcal{N}_\Omega^2(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components. Moreover, for any $x \in \mathcal{N}_\Omega^2(u)$ the number of connected components of $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ incident to x (if one connected component starts and ends at x , then it counts twice) is an even integer that is at least $2 \operatorname{ord}_x u$.*

The proof of Theorem 4.1 is based on the two lemmas below. The first lemma shows that proper neighbourhoods form a topological basis at any point $x \in \Omega$. Its proof relies on the topological consequences of our study of prime ends in Section 3.

Lemma 4.1. *Under the hypotheses of Theorem 4.1, for any point $x \in \mathcal{N}_\Omega(x)$ and any sufficiently small ball $B_\varepsilon(x)$ centred at x there exists a proper subdomain $U_\varepsilon(x)$ with respect to u such that $x \in U_\varepsilon(x) \subset B_\varepsilon(x)$.*

Proof. Let $x \in \mathcal{N}(u)$ be an interior nodal point in Ω , and $\Omega_1, \dots, \Omega_m$ be a collection of all nodal domains whose closure contains x . By Corollary 3.3 for any sufficiently small open ball $B_\varepsilon(x) \subset \Omega$ there are only finitely many connected components Ω_i^j , $j = 1, \dots, r_i$, of the intersection $B_\varepsilon(x) \cap \Omega_i$ whose closure contains x . Moreover, the union $F_i = \bigcup_j \bar{\Omega}_i^j$ is a neighbourhood of x in $\bar{\Omega}_i$. Thus, we conclude that the set $U_\varepsilon(x) = \operatorname{Int}(\bigcup F_i)$ contains x . Clearly, the connected components of the complement $U_\varepsilon(x) \setminus \mathcal{N}(u)$ are precisely the domains Ω_i^j , and it remains to show that $U_\varepsilon(x)$ has finitely many boundary components. Choosing $\varepsilon > 0$ such that the metric ball $B_\varepsilon(x)$ is homeomorphic to a ball in \mathbb{R}^2 , it is straightforward to see that any boundary component of $U_\varepsilon(x)$ that lies in $B_\varepsilon(x)$ bounds a union of nodal domains. Since the

number of nodal domains is finite, then choosing $\varepsilon > 0$ even smaller we conclude that $U_\varepsilon(x)$ is simply connected, and hence its boundary is connected. Thus, the neighbourhood $U_\varepsilon(x)$ is indeed a proper subdomain with respect to a solution u . \square

The second lemma says that if the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points, then it is necessarily finite, and the nodal set has the structure of a finite graph with the vertex set $\mathcal{N}_\Omega^2(u)$.

Lemma 4.2. *Under the hypotheses of Theorem 4.1, suppose that the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points. Then the set $\mathcal{N}_\Omega^2(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}_\Omega^2(u)$ has finitely many connected components.*

The proof of the last lemma appears at the end of the section. Now we proceed with the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.2 for a proof of the theorem it is sufficient to show that the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points in Ω . The second statement of the theorem is a direct consequence of Lemma 2.6. First, we consider the case of proper subdomains $\Omega \subset M$ whose closures are contained in the interior of M , $\bar{\Omega} \subset M$. Given such a subdomain Ω , it is straightforward to see that the *maximal vanishing order* $\ell = \max\{\text{ord}_x u\}$, where $x \in \Omega$, is finite. Indeed, otherwise there exists a point $p \in \bar{\Omega}$ that is the limit of points $x_i \in \Omega$ such that $\text{ord}_{x_i}(u) \rightarrow +\infty$ as $i \rightarrow +\infty$. Then, by Lemma 2.8, the solution u vanishes to an infinite order at p , and the strong unique continuation, Proposition 2.1, implies that u vanishes identically.

Let $\Omega \subset M$ be a proper subdomain whose closure is contained in the interior of M . We prove that the set $\mathcal{N}_\Omega^2(u)$ is finite by induction on the maximal vanishing order ℓ . Clearly, the statement holds for all solutions u and proper subdomains Ω such that the maximal vanishing order equals 2. Indeed, in this case by Lemma 2.9 the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points and, by Lemma 4.2, is finite. Now we perform an induction step. Suppose that the set $\mathcal{N}_\Omega^2(u)$ is finite for all solutions u to the Schrödinger equation (2-1) on M and all proper subdomains Ω whose closure is contained in the interior of M that satisfy

$$\max\{\text{ord}_x u : x \in \Omega\} \leq \ell - 1.$$

Now let u be a solution on M and Ω a proper subdomain such that the maximal vanishing order equals ℓ ,

$$\max\{\text{ord}_x u : x \in \Omega\} = \ell.$$

By Lemma 2.9 the set $\mathcal{N}_\Omega^\ell(u)$ consists of isolated points in Ω . Pick a point $p \in \mathcal{N}_\Omega^\ell(u)$. By Lemma 4.1, it has a neighbourhood U that is a proper subdomain such that $\bar{U} \subset \Omega$. Then the neighbourhood U may contain only finitely many points p_1, p_2, \dots, p_m whose vanishing order equals ℓ . Since the domain $U_0 = U \setminus \{p_1, \dots, p_m\}$ is proper with respect to u , the induction hypothesis implies that the set $\mathcal{N}^2(u) \cap U_0$ is finite. Hence, so is the set $\mathcal{N}^2(u) \cap U$. Thus, we conclude that $\mathcal{N}_\Omega^2(u)$ consists of isolated points in Ω and, by Lemma 4.2, is finite.

The statement that the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points in Ω for an arbitrary proper subdomain $\Omega \subset M$ follows directly from the case considered above together with Lemma 4.1. \square

Proof of Theorem 1.2. Now we show how Theorem 1.3 implies the multiplicity bounds. We give an argument following the strategy described in [Karpukhin et al. 2013, Section 6]. It relies on two lemmas that appear below. The first lemma gives a lower bound for the number of nodal domains via the vanishing order of points $x \in \mathcal{N}^2(u)$.

Lemma 4.3. *Under the hypotheses of Theorem 1.2, for any nontrivial eigenfunction u of an eigenvalue $\lambda_k(g, v)$ the number of its nodal domains is at least $\sum(\text{ord}_x u - 1) + \chi + l$, where the sum is taken over all points in $\mathcal{N}^2(u)$ and χ and l stand for the Euler–Poincaré number and the number of boundary components of M respectively.*

Before giving a proof we introduce some notation that is useful in the sequel. First, by Theorem 4.1, the nodal set $\mathcal{N}(u)$ of any eigenfunction u on M can be viewed as a finite graph, called a *nodal graph*. Its vertices are points in $\mathcal{N}^2(u)$ and the edges are connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Below we denote by \bar{M} a closed surface, viewed as the image of M under collapsing its boundary components to points, and by $\bar{\chi}$ its Euler–Poincaré number. Let $\bar{\mathcal{N}}(u)$ be the corresponding image of a nodal graph $\mathcal{N}(u)$, called the *reduced nodal graph*. Its edges are the same nodal arcs, and there are two types of vertices: vertices that correspond to the boundary components that contain limit points of nodal lines, called *boundary component vertices*, and genuine vertices that correspond to the points in $\mathcal{N}^2(u)$, called *interior vertices*. By *faces* of the graph $\bar{\mathcal{N}}(u)$ we mean the connected components of the complement $\bar{M} \setminus \bar{\mathcal{N}}(u)$. Clearly, they can be identified with the nodal domains of an eigenfunction u .

Proof of Lemma 4.3. Let $\bar{\mathcal{N}}(u)$ be a reduced nodal graph in \bar{M} . By Theorem 4.1 it is a finite graph, and let v, e and f be the number of its vertices, edges and faces respectively. We also denote by r the number of boundary component vertices in $\bar{\mathcal{N}}(u)$. Recall that the number of edges satisfies the relation $2e = \sum \text{deg } x$, where the sum is taken over all vertices. Since an eigenfunction u has different signs on adjacent nodal domains, the degree of each boundary component vertex is at least two, and we obtain

$$e \geq r + \frac{1}{2} \sum \text{deg } x \geq r + \sum \text{ord}_x u,$$

where the sum is taken over all interior vertices $x \in \mathcal{N}^2(u)$. The second inequality above follows from the relation $\text{deg } x \geq 2 \text{ord}_x u$; see Theorem 4.1. Viewing the number of vertices v as the sum $r + \sum 1$, where the sum is again taken over $x \in \mathcal{N}^2(u)$, by the Euler inequality [Giblin 2010, p. 207] we have

$$f \geq e - v + \bar{\chi} \geq \sum (\text{ord}_x u - 1) + \bar{\chi},$$

where $\bar{\chi} = \chi + l$ is the Euler–Poincaré number of \bar{M} . Since f is precisely the number of nodal domains, we are done. □

We proceed with the second lemma. In the case when the potential of a Schrödinger equation is smooth it is due to [Nadirashvili 1987]; see also [Karpukhin et al. 2013]. The proof relies essentially on Proposition 2.2.

Lemma 4.4. *Let (M, g) be a compact Riemannian surface, possibly with boundary, and let u_1, \dots, u_{2n} be a collection of nontrivial linearly independent solutions to the Schrödinger equation (2-1) with a*

potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then for a given interior point $x \in M$ there exists a nontrivial linear combination $u = \sum \alpha_i u_i$ whose vanishing order $\text{ord}_x u$ at the point x is at least n .

Proof. Let V be a linear space spanned by the functions u_1, \dots, u_{2n} , and V_i its subspace formed by solutions $u \in V$ whose vanishing order at x is at least i , $\text{ord}_x u \geq i$. Clearly, the subspaces V_i form a nested sequence, $V_{i+1} \subset V_i$. The lemma claims that V_n is nontrivial. Suppose the contrary, that the subspace V_n is trivial. Then, it is straightforward to see that the dimension of V satisfies the inequality

$$\dim V \leq 1 + \sum_{i=1}^{n-1} \dim(V_i/V_{i+1});$$

the equality occurs if the space V does not coincide with V_1 . By Proposition 2.2, the factor space V_i/V_{i+1} can be identified with a subspace of homogeneous harmonic polynomials on \mathbb{R}^2 of degree i . When the degree i is at least 1, the space of such polynomials has dimension two, and we obtain

$$\dim V \leq 1 + 2(n-1) = 2n-1.$$

Thus, we arrive at a contradiction with the hypotheses of the lemma. \square

Now we finish the proof of Theorem 1.2. Suppose the contrary to its statement. Then there exist at least $2(2-\chi-l)+2k+2$ linearly independent eigenfunctions corresponding to the eigenvalue $\lambda_k(\mu, g)$. Pick an interior point $x \in M$. By Lemma 4.4 there exists a new eigenfunction u whose vanishing order at the point x is at least $2-\chi-l+k+1$. Now Lemma 4.3 implies that the number of the nodal domains of u is at least $k+2$. Thus, we arrive at a contradiction with Courant's nodal domains theorem. \square

Proof of Lemma 4.2. Since the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points, we can view the nodal set $\mathcal{N}_\Omega(u)$ as a graph: the vertices are points in $\mathcal{N}_\Omega^2(u)$, and the edges are connected components of $\mathcal{N}_\Omega(u) \setminus \mathcal{N}_\Omega^2(u)$. Recall that the *degree* $\deg x$ of a vertex $x \in \mathcal{N}_\Omega^2(u)$ is defined as the number of edges incident to x ; if one edge starts and ends at x , then it counts twice. The following lemma says that the degree of each vertex has to be finite.

Lemma 4.5. *Under the hypotheses of Theorem 4.1, suppose that the set $\mathcal{N}_\Omega^2(u)$ consists of isolated points. Then the degree $\deg x$ of any point $x \in \mathcal{N}_\Omega^2(u)$ is finite.*

Proof. By Corollary 2.4 it is sufficient to show that the number of nodal loops that start and end at a given point $x \in \mathcal{N}_\Omega^2(u)$ is finite. Suppose the contrary, that the number of such nodal loops is infinite. Let $\bar{\Omega}$ be a compactification of Ω , obtained by adding one point for each boundary component. By Proposition 3.1 it is homeomorphic to a closed surface, and we denote by $\bar{\chi}$ its Euler–Poincaré number. Let Γ be a subgraph in the nodal graph formed by one vertex x and $m+2-\bar{\chi}$ nodal loops that start and end at x , where m is the number of nodal domains of u in Ω . Denote by $v=1$, $e=m+2-\bar{\chi}$ and f the number of vertices, edges and faces of Γ respectively. Here by the faces of Γ we mean the connected components of $\bar{\Omega} \setminus \Gamma$. Clearly, they are unions of nodal domains, and $f \leq m$. On the other hand, viewing Γ as a graph in $\bar{\Omega}$, by Euler's inequality [Giblin 2010, p. 207], we obtain

$$f \geq e - v + \bar{\chi} = m + 1.$$

This contradiction demonstrates the lemma. □

Now we prove the statement of Lemma 4.2: the set $\mathcal{N}_\Omega^2(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}_\Omega^2(u)$ has finitely many connected components. The argument below is based on the results in Section 2, and is close in the spirit to the one in [Karpukhin et al. 2013, Section 3].

Let $\bar{\Omega}$ be a closed surface obtained by collapsing boundary components of Ω to points. By $\bar{\mathcal{N}}_\Omega$ we denote the *reduced nodal graph* in $\bar{\Omega}$, defined in the proof of Theorem 1.2. Recall that its edges are the same nodal edges, and there are two types of vertices: vertices that correspond to the boundary components of Ω that contain limit points of nodal lines, called *boundary component vertices*, and genuine vertices that correspond to the points in $\mathcal{N}_\Omega^2(u)$, called *interior vertices*. For a proof of the lemma it is sufficient to show that $\bar{\mathcal{N}}_\Omega(u)$ is a finite graph. Our strategy is to show that

- (i) each boundary component vertex has a finite degree, and
- (ii) the number of interior vertices is finite in Ω .

We are going to construct new graphs in $\bar{\Omega}$ by *resolving interior vertices* in the following fashion. Let $x \in \mathcal{N}_\Omega^2(u)$ be an interior vertex. By Lemma 4.5 its degree is finite, and by Lemma 2.6 it is an even integer $2n$. Let B be a small disk centred at x that does not contain other vertices. By Corollary 2.5 we may assume that nodal edges nonincident to x lie in the complement $\Omega \setminus B$. Moreover, since the degree is finite, we may also assume that each nodal loop incident to x intersects ∂B in at least two points. Consider the intersections of nodal edges with B , and let Γ_i , where $i = 0, \dots, 2n - 1$, be their connected components incident to x . Pick points $y_i \in \bar{\Gamma}_i \cap \partial B$, one for each $i = 0, \dots, 2n - 1$. By the resolution of a vertex x we mean a new graph obtained by removing sub-arcs between x and y_i in each nodal edge incident to x and rounding them off by nonintersecting arcs in B joining the points y_{2j} and y_{2j+1} . If there was an edge that starts and ends at x , then such a procedure may make it into a loop. We remove all such loops, if they occur. A new graph, obtained by the resolution of one vertex, has one vertex less and at most as many faces as the original graph.

Proof of (i). Suppose the contrary. Let us resolve all interior vertices in $\bar{\mathcal{N}}_\Omega(u)$ in the way described above. The result is a graph Γ whose only vertices are boundary component vertices in $\bar{\mathcal{N}}_\Omega(u)$; let v be their number. Moreover, it has at most as many faces as $\bar{\mathcal{N}}_\Omega(u)$ — that is, no more than the number of nodal domains. Since there is a boundary component vertex in $\bar{\mathcal{N}}_\Omega(u)$ whose degree is infinite, the same vertex has an infinite degree in Γ . Let us remove all edges in Γ except for at least $v + m + 1 - \bar{\chi}$ of them, where m is the number of nodal domains and $\bar{\chi}$ is the Euler–Poincaré number of $\bar{\Omega}$. The result is a finite graph; it has precisely v vertices, and we denote by e and f the number of its edges and faces respectively. By Euler’s inequality, we obtain

$$f \geq e - v + \bar{\chi} = m + 1.$$

On the other hand, since removing an edge does not increase the number of faces, we have $f \leq m$. Thus, we arrive at a contradiction.

Proof of (ii). Suppose the contrary, and let v be the number of boundary component vertices in $\bar{\mathcal{N}}_\Omega(u)$. Let us resolve all interior vertices except for $v + m + 1 - \bar{\chi}$ of them. The result is a finite graph; we

denote by v' , e' and f' the number of its vertices, edges and faces respectively. Clearly, we have

$$v' \leq 2v + m + 1 - \bar{\chi} \quad \text{and} \quad e' \geq 2(v + m + 1 - \bar{\chi}),$$

where in the second inequality we used Lemma 2.6, saying that the degree of each vertex $x \in \mathcal{N}_{\Omega}^2(u)$ is at least 4. Combining these inequalities with the Euler inequality, we obtain

$$f' \geq e' - v' + \bar{\chi} \geq m + 1.$$

On the other hand, we have $f' \leq m$. Thus, we arrive at a contradiction. \square

5. Eigenvalue problems on singular Riemannian surfaces

Eigenvalue problems on surfaces with measures. The purpose of this section is to discuss multiplicity bounds on singular Riemannian surfaces. We start with recalling a useful general setting of eigenvalue problems on surfaces with measures, following [Kokarev 2014].

Let (M, g) be a compact Riemannian surface, possibly with boundary, and let μ be a finite absolutely continuous (with respect to $d\text{Vol}_g$) Radon measure on M that satisfies the *decay condition*

$$\mu(B(x, r)) \leq Cr^\delta, \quad \text{for any } r > 0 \text{ and } x \in M, \quad (5-1)$$

and some constants C and $\delta > 0$. Denote by $L_2^1(M, \text{Vol}_g)$ the space formed by distributions whose derivatives are in $L_2(M, \text{Vol}_g)$. Then by the results of Maz'ja [1985] (see also [Kokarev 2014]) the embedding

$$L_2(M, \mu) \cap L_2^1(M, \text{Vol}_g) \subset L_2(M, \mu)$$

is compact, the Dirichlet form $\int |\nabla u|^2 d\text{Vol}_g$ is closable in $L_2(M, \mu)$, and its spectrum is discrete. We denote by

$$\lambda_0(g, \mu) < \lambda_1(g, \mu) \leq \dots \leq \lambda_k(g, \mu) \leq \dots$$

the corresponding eigenvalues, and by $m_k(g, \mu)$ their multiplicities. As above, we always suppose that the Dirichlet boundary hypothesis is imposed if the boundary of M is nonempty. The eigenfunctions corresponding to an eigenvalue $\lambda_k(g, \mu)$ are distributional solutions to the Schrödinger equation

$$-\Delta_g u = \lambda_k(g, \mu) \mu u \quad \text{on } M. \quad (5-2)$$

The latter fact ensures that the analysis in Sections 2–4 carries over to yield the following result.

Theorem 5.1. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, endowed with a finite absolutely continuous Radon measure μ that satisfies hypothesis (5-1). Then the multiplicity $m_k(g, \mu)$ of a Laplace eigenvalue $\lambda_k(g, \mu)$ satisfies the inequality*

$$m_k(g, \mu) \leq 2(2 - \chi - l) + 2k + 1 \quad \text{for any } k = 1, 2, \dots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

Proof. First, we claim that the decay hypothesis (5-1) on the measure μ implies that its density belongs to the space $K^{2,\delta'}(M)$ for some $0 < \delta' < \delta$. Indeed, by Fubini's theorem and the change of variable formula, we obtain

$$\begin{aligned} \int_{B(x,r)} |x - y|^{-\delta'} d\mu &= \int_{r^{-\delta'}}^{+\infty} \mu\{y : |x - y|^{-\delta'} \geq t\} dt = \int_{r^{-\delta'}}^{+\infty} \mu(B(x, t^{-1/\delta'})) dt \\ &= \delta' \int_0^r s^{-\delta'-1} \mu(B(x, s)) ds \leq C \delta' \int_0^r s^{\delta-\delta'-1} ds. \end{aligned}$$

Second, using a variational characterisation of eigenvalues $\lambda_k(g, \mu)$, it is also straightforward to check that the standard proof of Courant's nodal domains theorem carries over for eigenfunctions u which satisfy (5-2). Hence, Theorem 4.1 applies, and then the argument in the proof of Theorem 1.2 carries over directly to yield the multiplicity bounds. □

Note that, since the Dirichlet energy is conformally invariant, if the measure μ is the volume measure of a metric h conformal to g then the quantities $\lambda_k(g, \mu)$ are precisely the Laplace eigenvalues of a metric h . More generally, the eigenvalue problems on surfaces with singular metrics can be also often viewed as particular instances of the setting of eigenvalues on measures. Below we discuss this point of view in more detail.

Let (M, g) be a Riemannian surface and h be a Riemannian metric of finite volume defined on the set $M \setminus S$, where S is a closed nowhere dense subset of zero measure. Here the set S plays the role of a singular set of h on M . Denote by μ the volume measure of the metric h . In the literature, e.g., [Cheeger 1983], the Dirichlet spectrum of a singular metric h is normally defined as the spectrum of the Dirichlet form

$$u \mapsto \int_{M \setminus S} |\nabla u|^2 d\text{Vol}_h \tag{5-3}$$

defined on the space $\mathcal{C} \subset L_2(M, \mu)$ of smooth compactly supported functions in $M \setminus S$. Suppose that the set S has zero Dirichlet capacity, the metric h is conformal on $M \setminus S$ to the metric g , and its volume measure μ satisfies the decay hypothesis (5-1). Then, it is straightforward to see that the spectrum of h is discrete and coincides with the set of eigenvalues $\lambda_k(g, \mu)$ defined above. Moreover, the construction makes sense even if a metric h is not smooth on $M \setminus S$ as long as the Dirichlet form (5-3) is well-defined. Theorem 5.1 gives multiplicity bounds for such eigenvalue problems. We end by discussing two examples: metrics with conical singularities and, more generally, Alexandrov surfaces of bounded integral curvature.

Example 1: metrics with conical singularities. Let M be a closed smooth surface and h be a metric on M with a number of conical singularities. Recall that a point $p \in M$ is called a *conical singularity* of order $\alpha > -1$ (or angle $2\pi(\alpha + 1)$) if in an appropriate local complex coordinate the metric h has the form $|z|^{2\alpha} \rho(z) |dz|^2$, where $\rho(z) > 0$. In other words, near p the metric is conformal to the Euclidean cone of total angle $2\pi(\alpha + 1)$. As is known, such a metric h is conformal to a genuine Riemannian metric g on M away from the singularities. If a surface M has a nonempty boundary, we do not exclude an infinite number of conical singularities accumulating to the boundary, and suppose that the volume measure Vol_h satisfies the decay hypothesis (5-1). For a surface with a finite number of conical singularities the

hypothesis on the volume measure is always satisfied. The Dirichlet integral with respect to the metric h is defined as an improper integral; by the conformal invariance, it satisfies the relation

$$\int_M |\nabla u|_h^2 d\text{Vol}_h = \int_M |\nabla u|_g^2 d\text{Vol}_g$$

for any smooth function u . Thus, we conclude that the Laplace eigenvalues and their multiplicities of a metric h coincide with the quantities $\lambda_k(g, \text{Vol}_h)$ and $m_k(g, \text{Vol}_h)$, defined above, and Theorem 5.1 yields the multiplicity bounds. Note that if a metric h has only a finite number of conical singularities, then the multiplicity bounds can also be obtained from arguments in [Karpukhin et al. 2013].

Example II: Alexandrov surfaces of bounded integral curvature. The most significant class of surfaces, illustrating our approach, is formed by the so-called Alexandrov surfaces of bounded integral curvature. Below we recall this notion and give a brief outline of its relevance to our setting; more details and references on the subject can be found in the surveys [Reshetnyak 1993; Troyanov 2009]. Eigenvalue problems on Alexandrov surfaces of bounded integral curvature are treated in detail in [Kokarev \geq 2014].

Definition. A metric space (M, d) , where M is a compact smooth surface, is called an *Alexandrov surface of bounded integral curvature* if:

- (i) the topology induced by d coincides with the original surface topology on M ;
- (ii) the metric space (M, d) is a *geodesic length space*; that is, any two points x and $y \in M$ can be joined by a path whose length is $d(x, y)$;
- (iii) the metric d is a C^0 -limit of distances of smooth Riemannian metrics g_n on M whose integral curvatures are bounded; that is,

$$\sup_n \int_M |K_{g_n}| d\text{Vol}_{g_n} < +\infty,$$

where K_{g_n} stands for the Gauss curvature of a metric g_n .

This is a large class of singular surfaces that contains, for example, all polyhedral surfaces as well as surfaces with conical singularities and their limits under the integral curvature bound. The hypothesis (iii) implies that after a selection of a subsequence the signed measures $K_{g_n} d\text{Vol}_{g_n}$ converge weakly to a measure ω on M . By the result of Alexandrov [Alexandrov and Zalgaller 1967], the measure ω is an intrinsic characteristic of (M, g) ; it does not depend on an approximating sequence of Riemannian metrics g_n , and is called the *curvature measure* of an Alexandrov surface. As an example, consider the surface of a unit cube in \mathbb{R}^3 . The metric on it is defined as the infimum of Euclidean lengths of all paths that lie on the surface of the cube and join two given points. As is known [Reshetnyak 1993; Troyanov 2009], its curvature measure is $\sum (\pi/2) \delta_p$, where δ_p is the Dirac mass and the sum runs over all vertices p of the cube.

Recall that a point $x \in M$ is called a *cusp* if $\omega(x) = 2\pi$. By the results of [Reshetnyak 1960; Huber 1960], any Alexandrov surface of bounded integral curvature and without cusps can be regarded as being “conformally equivalent” to a smooth Riemannian metric on a background compact surface. This means

that the distance function on such a surface has the form

$$d(x, y) = \inf_{\gamma} \left\{ \int_0^1 e^{u(\gamma(t))} |\dot{\gamma}(t)|_g dt \right\}$$

for some function u and a smooth background Riemannian metric g ; the infimum above is taken over smooth paths γ joining x and y . The conformal factor e^u here can be very singular, and is an L^{2p} -function, where $p > 1$. More precisely, the function u is the difference of weakly subharmonic functions [Reshetnyak 1960; 1993], and the set

$$S = \{x \in M : e^u(x) = 0\}$$

has zero capacity in M [Hayman and Kennedy 1976, Theorem 5.9].

Thus, an Alexandrov surface without cusps can be viewed as a surface with a ‘‘Riemannian metric’’ $h = e^u g$ on $M \setminus S$, whose distance function is precisely the original metric d . This ‘‘Riemannian metric’’ yields the *Alexandrov volume measure* $d\mu_h = e^{2u} d\text{Vol}_g$, which is another intrinsic characteristic of (M, d) ; it can be also defined via approximations by Riemannian metrics. More precisely, Alexandrov and Zalgaller [1967] show that if g_n is a sequence of Riemannian metrics that satisfy the hypothesis (iii) in the definition of an Alexandrov surface, then its volume measures Vol_{g_n} converge weakly to μ_h .

Since the set S has zero capacity, by conformal invariance it is straightforward to conclude that the relation

$$\int_{M \setminus S} |\nabla u|_h^2 d\mu_h = \int_M |\nabla u|_g^2 d\text{Vol}_g$$

holds for any smooth function u . Thus, the eigenvalues $\lambda_k(g, \mu_h)$ of the Dirichlet form $\int |\nabla u|^2 d\text{Vol}_g$ in $L_2(M, \mu_h)$ are indeed natural versions of Laplace eigenvalues on an Alexandrov surface without cusps. Since e^{2u} is an L^p -function, where $p > 1$, we conclude that the Alexandrov volume measure μ_h satisfies the decay hypothesis (5-1). In particular, the multiplicities $m_k(g, \mu_h)$ are finite and satisfy the inequalities in Theorem 5.1.

Appendix: Cheng’s structure theorem

The purpose of this section is to give details on Cheng’s structure theorem, discussed in Section 1. It is based on the following lemma.

Lemma A.1. *Let u be a $C^{1,1}$ -smooth function defined in a neighbourhood of the origin in \mathbb{R}^n that satisfies the relation*

$$u(x) = P_N(x) + O(|x|^{N+\delta}) \quad \text{as } x \rightarrow 0, \tag{A-1}$$

where P_N is a homogeneous polynomial of order N such that $|\nabla P_N(x)| \geq C|x|^{N-1}$. Then there exists a neighbourhood U of the origin and a Lipschitz homeomorphism Φ of it that preserves the origin and such that $u(x) = P_N(\Phi(x))$ for any $x \in U$. Moreover, if u is C^2 -smooth, then Φ is a C^1 -diffeomorphism.

Proof outline. The second term on the right-hand side can be viewed as the product $\alpha(x)|x|^{N+\delta'-1}$, where $0 < \delta' < \delta$ and $\alpha(x)$ is a function that is C^1 -smooth away from the origin and behaves like $O(|x|^{1+\delta-\delta'})$

as $x \rightarrow 0$. It is then straightforward to see that α is C^1 -smooth in a neighbourhood of the origin and, differentiating relation (A-1), we obtain

$$\nabla u(x) = \nabla P_N(x) + O(|x|^{N+\delta'-1}) \quad \text{as } x \rightarrow 0.$$

Given the last relation, if u is C^2 -smooth, the existence of the C^1 -diffeomorphism Φ follows from the argument in the proof of [Cheng 1976, Lemma 2.4]. This argument also works when u is $C^{1,1}$ -smooth, and in this case it yields a local Lipschitz homeomorphism Φ such that $u(x) = P_N(\Phi(x))$. \square

In dimension two any homogeneous harmonic polynomial of degree $N \geq 1$ satisfies the hypothesis $|\nabla P_N(x)| \geq C|x|^{N-1}$ and, combining the lemma above with Proposition 2.2, we obtain the following improved version of Cheng's result.

Cheng's structure theorem. *Let u be a $C^{1,1}$ -smooth solution of the Schrödinger equation*

$$(-\Delta + V)u = 0 \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{A-2}$$

where $V \in K^{2,\delta}(\Omega)$. Then for any nodal point $p \in \mathcal{N}(u)$ there is a neighbourhood U and a Lipschitz homeomorphism Φ of U onto a neighbourhood of the origin such that $u(x) = P_N(\Phi(x))$ for any $x \in U$, where P_N is an approximating homogeneous harmonic polynomial at p . Moreover, if u is C^2 -smooth, then Φ is a C^1 -diffeomorphism.

Cheng [1976] also states similar results in arbitrary dimension. However, in dimension $n > 2$ there are homogeneous harmonic polynomials for which the hypothesis $|\nabla P_N(x)| \geq C|x|^{N-1}$ fails, and thus Lemma A.1 cannot be used. As is shown in [Bérard and Meyer 1982, Appendix E], the latter hypothesis is necessary for the conclusion of Lemma A.1 to hold.

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