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SHARP CONSTANT FOR A k-PLANE TRANSFORM INEQUALITY

ALEXIS DROUOT

The k-plane transform $\mathcal{R}_k$ acting on test functions on $\mathbb{R}^d$ satisfies a dilation-invariant $L^p \to L^q$ inequality for some exponents p, q. We will make explicit some extremizers and the value of the best constant for any value of k and d, solving the endpoint case of a conjecture of Baernstein and Loss. This extends their own result for $k = 2$ and Christ's result for $k = d - 1$.

1. Introduction

Let us choose $d \geq 2$, $1 \leq k \leq d-1$ and denote by $\mathcal{G}_k$ the set of all k-planes in $\mathbb{R}^d$, meaning affine subspaces in $\mathbb{R}^d$ with dimension k. We define the k-plane transform of a continuous function with compact support $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathcal{R}_k f(\Pi) = \int_\Pi f \, d\lambda_\Pi,$$

where $\Pi \in \mathcal{G}_k$ and the measure λ_Π is the surface Lebesgue measure on Π. The operator $\mathcal{R}_k$ is known as the Radon transform for $k = d-1$ and as the X-ray transform for $k = 1$. It is known since the works of Oberlin and Stein [1982], Drury [1984] and Christ [1984] that $\mathcal{R}_k$ can be extended from $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$ to $L^{d+1}(\mathcal{G}_k, \sigma_k)$ where σ_k is a measure defined as follows. Let us denote by $\mathcal{M}_k$ the submanifold of $\mathcal{G}_k$ of all k-planes containing 0. The Lebesgue measure on $\mathbb{R}^d$ induces a natural measure on $\mathcal{M}_k$: there exists a unique probability measure μ_k on $\mathcal{M}_k$ invariant in the sense that if Ω is an orthogonal map and P is a subset of $\mathcal{M}_k$, then $\mu_k(P) = \mu_k(\Omega P)$. The construction of this measure can be found in [Mattila 1995]. This induces a measure σ_k on $\mathcal{G}_k$ such that

$$\sigma_k(A) = \int_{\Pi \in \mathcal{M}_k} \lambda_{\Pi^\perp}(\{x \in \Pi^\perp : x + \Pi \in A\}) \, d\mu_k(\Pi), \tag{1-1}$$

where $\lambda_{\Pi^\perp}$ denotes the Lebesgue surface measure on the $(d-k)$-plane $\Pi^\perp$. Equation (1-1) defines a measure on $\mathcal{G}_k$ invariant under translations and rotations in the following sense: if Ω is an orthogonal map, P is a subset of $\mathcal{G}_k$, and $x \in \mathbb{R}^d$, then $\sigma_k(P) = \sigma_k(\Omega P + x)$.

The $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$ to $L^{d+1}(\mathcal{G}_k, \sigma_k)$-boundedness of $\mathcal{R}_k$ leads to the inequality

$$\|\mathcal{R}_k f\|_{L^{d+1}(\mathcal{G}_k, \sigma_k)} \leq A(k,d) \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)} \tag{1-2}$$

for a certain constant $A(k,d)$, chosen to be optimal, that is,

$$A(k,d) = \sup\bigl\{\|\mathcal{R}_k f\|_{L^{d+1}(\mathcal{G}_k, \sigma_k)} : \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)} = 1\bigr\}. \tag{1-3}$$

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Functions realizing the supremum in (1-3) are called extremizers of (1-2).

Here are some standard questions about this inequality:

(1) What is the best constant?

(2) What are the extremizers?

(3) Is any extremizing sequence relatively compact, modulo the group of symmetries?

(4) What can we say about functions satisfying $\|\mathcal{R}_k f\|_{d+1} \geq c\|f\|_{\frac{d+1}{k+1}}$?

Some of the answers are already known for some values of k. Baernstein and Loss [1997] solved the first question for the special case $k=2$, and formulated a conjecture about the form of extremizers for a larger class of $L^p \to L^q$ inequalities. Christ solved their conjecture and answered all the above questions with the three papers [Christ 2011a; 2011b; 2011c] for the case $k=d-1$.

By a quite different approach, we will give here a proof of Baernstein and Loss' conjecture *for any values of* k, d in the inequality (1-2). Note that this concerns only the endpoint case of their general conjecture. The value of the extremizers provides the explicit value of the best constant in (1-2). In a subsequent paper [Drouot 2013] we give a positive answer to the third question in the radial case, which is much easier than the general case.

Main result. Our main result is the following theorem:

Theorem 1.1. *The constant $A(k,d)$ in (1-2) is given by*

$$A(k,d) = \left[2^{k-d}\frac{|S^k|^d}{|S^d|^k}\right]^{\frac{1}{d+1}},$$

and some extremizers are given by

$$h(x) = \left[\frac{C}{1+|Lx|^2}\right]^{\frac{k+1}{2}}, \tag{1-4}$$

where L is any invertible affine map on $\mathbb{R}^d$, and C is any positive constant.

To find the best constant in the k-plane inequality (1-2) we will use the method of competing symmetries introduced in [Carlen and Loss 1990]. We will need the existence of an additional symmetry $\mathcal{S}$ of (1-2) that changes the level sets of functions — this could be seen as a problem but it actually gives very helpful information on the structure of the inequality. The choice of this symmetry is the generalization of a symmetry found in [Christ 2011c] in the special case of the Radon transform.

Nevertheless, the approach followed by Carlen and Loss led them to the values of *all* extremizers, using some additional work for the equality case in the rearrangement inequality. This does not work for us, and so we do not prove that the extremizers are unique modulo the invertible affine maps. However, we prove in Section 4 that if all extremizers are of the form $F \circ L$ with F radial and L an invertible map, then all extremizers are of the form (1-4). Using this result, Flock [2013] proved the following theorem:

Theorem 1.2. *All extremizers of (1-2) are of the form (1-4).*

For the rest of the paper, let us note the following:

- Let A and B be positive functions. We will say that $A \lesssim B$ when there exists a universal constant C, which depends only on the dimension d and on the integer k, such that $A \le CB$. $A \gtrsim B$ means $B \lesssim A$, and $A \sim B$ will be used when $A \lesssim B$ and $B \lesssim A$.
- A radial function will be considered throughout the paper either as a function on $\mathbb{R}^d$ or as a function of the Euclidean norm, depending on the context.
- $|E|$ denotes the Lebesgue measure of a set E, except in the case of a sphere.
- $d(0, \Pi)$ denotes the Euclidean distance between 0 and a k-plane Π, that is,
$$d(0, \Pi) = \inf_{y \in \Pi} |y|.$$
- $|S^{m-1}|$ denotes the Lebesgue surface measure of the Euclidean sphere of $\mathbb{R}^m$.
- e_d is the vector $(0, \dots, 0, 1)$.
- For a vector x in $\mathbb{R}^d$, we will write $x = (x', x'')$ with $x' \in \mathbb{R}^{d-1}$ and $x'' \in \mathbb{R}$.
- $\|f\|_p$ denotes the L^p-norm of f with respect to a contextual measure.
- $\mathbb{R}^+$ is the set $(0, \infty)$.

2. Preliminaries

In this section we introduce some standard notions which will be useful for what follows. We will talk about the theory of radial, nonincreasing rearrangements of a function and about the special form of the k-plane transform for radial functions.

Let us consider a measure μ on $\mathbb{R}^d$ and a measurable subset E of $\mathbb{R}^d$. E^* denotes the unique closed ball centered at the origin such that $\mu(E^*) = \mu(E)$. Now for a measurable function f from $\mathbb{R}^d$ to $[0, \infty]$, and $t \ge 0$, let us denote
$$E_f(t) = \{x \in \mathbb{R}^d : |f(x)| \ge t\}.$$

Then we have the following proposition:

Proposition 2.1. *Let f be a measurable function from $\mathbb{R}^d$ to $\mathbb{R} \cup \{\pm\infty\}$. There exists a unique function f^* from $\mathbb{R}^d$ to $[0, \infty]$ such that*
$$E_{|f|}(t)^* = E_{f^*}(t). \tag{2-1}$$
Moreover, f^* is radial, and nonincreasing as a function of the norm. Furthermore, for all nonnegative functions $g, h \in L^p$ with $1 \le p \le \infty$, we have:

(i) $\|g\|_p = \|g^*\|_p$,

(ii) $\|g^* - h^*\|_p \le \|g - h\|_p$,

(iii) *if $g \le h$, then $g^* \le h^*$,*

(iv) *for all $\lambda \ge 0$, $\lambda g^* = (\lambda g)^*$.*

Points (i) to (iv) show that the nonlinear operator $f \mapsto f^*$ is actually a properly contractive operator (see Section 3). The map f^* is called the symmetric rearrangement of f (with respect to the measure μ).

We are now applying this theory to the k-plane transform. Christ [1984] proved that the k-plane transform satisfies the rearrangement inequality

$$\|\mathcal{R}g\|_q \le \|\mathcal{R}(g^*)\|_q. \tag{2-2}$$

That way, we can look for extremizers in the class of radial, nonincreasing functions. It obviously makes the study much easier, passing from functions on $\mathbb{R}^d$ to nonincreasing functions on $[0,\infty)$.

The geometric origin of the k-plane transform leads us to introduce the operator $\mathcal{T}$ defined on continuous, compactly supported functions on $\mathbb{R}^+$ as

$$\mathcal{T} f(r) = \int_0^\infty f\big(\sqrt{s^2+r^2}\big)s^{k-1}\,ds.$$

Then we have the following:

Lemma 2.2. *For all radial, continuous, compactly supported functions f on $\mathbb{R}^d$ and $\Pi \in \mathcal{G}$ such that $d(0,\Pi)=r$, we have*

$$\mathcal{R}f(\Pi) = |S^{k-1}|\cdot \mathcal{T} f(r). \tag{2-3}$$

For a proof, see, for instance, [Baernstein and Loss 1997]. The equation (2-3) shows that $\mathcal{T}$ is *almost* the k-plane transform. $\mathcal{T}$ acts on some Lebesgue spaces that we need to explicitly define, using the correspondence (2-3). Its domain is of course the space $L^p(\mathbb{R}^+, r^{d-1}\,dr)$. On the other hand, we have

$$\|\mathcal{R}f\|_q^q = \int_{\mathcal{G}} |\mathcal{R}f(\Pi)|^q\,d\sigma(\Pi) = |S^{k-1}|^q|S^{d-k-1}|\int_{r=0}^\infty |\mathcal{T} f(r)|^q r^{d-k-1}\,dr,$$

where the last line is obtained thanks to the formula (1.1) in [Baernstein and Loss 1997]. This shows that $\mathcal{T}$ maps $L^p(\mathbb{R}^+, r^{d-1}dr)$ to $L^q(\mathbb{R}^+, r^{d-k-1}dr)$.

3. Best constant and value of extremizers for the k-plane inequality

Here we want to prove the following:

Theorem 3.1. *An extremizer for the inequality (1-2) is given by*

$$f(x) = \left[\frac{1}{1+|x|^2}\right]^{\frac{k+1}{2}}. \tag{3-1}$$

As a matter of fact, since any invertible affine map is a symmetry of the inequality (1-2), this theorem is equivalent to Theorem 1.1.

Let us explain the process of the proof before the details. Our purpose here is to introduce two operators $V, \mathcal{S}$ acting on L^p, such that V and $\mathcal{S}$ preserve the L^p-norm *and*

$$\|\mathcal{R}f\|_q = \|\mathcal{R}\mathcal{S}f\|_q, \quad \|\mathcal{R}f\|_q \le \|\mathcal{R}Vf\|_q. \tag{3-2}$$

This means that V and $\mathcal{S}$ globally increase the functional $f \mapsto \|\mathcal{R}f\|_q/\|f\|_p$. Now using additional properties of $\mathcal{S}$ and V, we will apply a theorem from [Carlen and Loss 1990] to show that for any choice of $f \in L^p$ with norm 1, the sequence $(V\mathcal{S})^n f$ converges to an explicit function h that does not depend on f. Using (3-2), h must be an extremizer, and h is explicitly known.

In practice, the operator V will be the symmetric rearrangement $f \mapsto f^*$, and $\mathcal{S}$ will be a symmetry of the inequality. The operator $\mathcal{S}$ is special in a certain sense: it does not preserve the class of radial functions. Thus, if we were able to construct an extremizer such that $\mathcal{S}h = h$ and $Vh = h$, the explicit value of h could be determined. A way to construct such an extremizer is described in the next section. But we can already note that an extremizer satisfying this condition must satisfy $(V\mathcal{S})^n h = h$ for all n; this way, considering the sequence $(V\mathcal{S})^n f$ is probably a good idea.

Competing operators. As we said, we are following the approach introduced in [Carlen and Loss 1990]. We might also refer to the book [Bianchini et al. 2011]. First, we sum up the general results stated Chapter II, §3.4 of this book: let $\mathcal{B}$ be a Banach space of real valued functions, with norm $\|\cdot\|$. Let $\mathcal{B}^+$ be the cone of nonnegative functions, and assume that $\mathcal{B}^+$ is closed. Let us introduce some definitions:

Definition 3.2. An operator A on $\mathcal{B}$ is called properly contractive provided that:

(i) A is norm-preserving on $\mathcal{B}^+$, i.e., $\|Af\| = \|f\|$ for all $f \in \mathcal{B}^+$.

(ii) A is contractive on $\mathcal{B}^+$, i.e., for all $f, g \in \mathcal{B}^+$, $\|Af - Ag\| \le \|f - g\|$.

(iii) A is order-preserving on $\mathcal{B}^+$, i.e., for all $f, g \in \mathcal{B}^+$, $f \le g \Longrightarrow Af \le Ag$.

(iv) A is homogeneous of degree one on $\mathcal{B}^+$, i.e., for all $f \in \mathcal{B}^+, \lambda \ge 0$, $A(\lambda f) = \lambda Af$.

Note that we do not need A to be linear. Some examples of such operators are for instance the radial nonincreasing rearrangement $f \mapsto f^*$ or any linear isometry on $\mathcal{B}$.

Definition 3.3. Given a pair of properly contractive operators $\mathcal{S}$ and V, it is said that $\mathcal{S}$ competes with V if, for $f \in \mathcal{B}^+$,

$$f \in R(V) \cap \mathcal{S}R(V) \Longrightarrow \mathcal{S}f = f.$$

Here R denotes the range.

Theorem 3.4. *Suppose that $\mathcal{S}$ and V are both properly contractive, that $V^2 = V$ and that $\mathcal{S}$ competes with V. Suppose further that there is a dense set $\widetilde{\mathcal{B}} \subset \mathcal{B}^+$ and sets K_N satisfying $\bigcup_N K_N = \widetilde{\mathcal{B}}$ and for all integers N, $\mathcal{S}K_N \subset K_N$, $VK_N \subset K_N$, and VK_N is relatively compact in $\mathcal{B}$. Finally, suppose that there exists a function $h \in \mathcal{B}^+$ with $\mathcal{S}h = Vh = h$ and such that, for all $f \in \mathcal{B}^+$,*

$$\|Vf - h\| = \|f - h\| \Longrightarrow Vf = f. \tag{3-3}$$

Then, for any $f \in \mathcal{B}^+$,

$$Tf \equiv \lim_{n\to\infty} (V\mathcal{S})^n f$$

exists. Moreover, $\mathcal{S}T = T$ and $VT = T$.

An additional symmetry. Now we come back to the work of Christ. Using a correspondence between a convolution operator that he studied in [Christ 2011a; 2011b; 2012], and the Radon transform, he proved in [Christ 2011c] the existence of an additional symmetry for the Radon transform inequality. It is defined as

$$\mathcal{I} f(u,s) = \frac{1}{|s|^d} f\Big(\frac{u}{s}, \frac{1}{s}\Big).$$

It then satisfies $\|\mathcal{I} f\|_{\frac{d+1}{d}} = \|f\|_{\frac{d+1}{d}}$ and $\|\mathcal{R}_{d-1}\mathcal{I} f\|_{d+1} = \|\mathcal{R}_{d-1} f\|_{d+1}$. Fortunately, it happens that this symmetry, slightly modified, also works for the $L^p \to L^q$ inequality related to the k-plane transform.

Lemma 3.5. *Let $\mathcal{S}$ be the operator defined on L^p as*

$$\mathcal{S} f(u,s) = \frac{1}{|s|^{k+1}} f\Big(\frac{u}{s}, \frac{1}{s}\Big),$$

where $(u,s) \in \mathbb{R}^{d-1} \times (\mathbb{R} - \{0\})$. Then $\mathcal{S}$ is an isometry of L^p and satisfies the identity

$$\|\mathcal{R}\mathcal{S} f\|_q = \|\mathcal{R} f\|_q \tag{3-4}$$

for any nonnegative function f.

Proof. Let us check first that $\mathcal{S}$ is an isometry of L^p. Let us call

$$\Phi(x) = \left(\frac{x'}{x''}, \frac{1}{x''}\right)$$

for $x = (x', x'') \in \mathbb{R}^{d-1} \times (\mathbb{R} - \{0\})$. Then its Jacobian determinant is

$$J\Phi(x) = \frac{1}{|x''|^{d+1}},$$

which shows that $\|\mathcal{S} f\|_p = \|f\|_p$. Then we just have to prove (3-4). The proof is just calculation. Denote the unique k-plane containing the linearly independent points $x_0, \dots, x_k \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ by $\Pi(x_0, \dots, x_k)$ and let $\widetilde{\mathcal{R}} f$ be

$$\widetilde{\mathcal{R}} f(x_0, \dots, x_k) = \int_{\mathbb{R}^k} f\big(x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_k(x_k - x_0)\big)\, d\lambda_1 \dots d\lambda_k.$$

Thus we have the correspondence

$$V(x_0, \dots, x_k) \cdot \widetilde{\mathcal{R}} f(x_0, \dots, x_k) = \mathcal{R} f(\Pi(x_0, \dots, x_k)), \tag{3-5}$$

where $V(x_0, \dots, x_k)$ is the volume of the k-simplex $(x_0, \dots, x_k)$.

Lemma 3.6. *For all $f \in C_0^\infty$, for all $x_0, \dots, x_k \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, linearly independent and such that $\Phi(x_0), \dots, \Phi(x_k)$ exist and are linearly independent,*

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \dots, x_k) = \frac{(\widetilde{\mathcal{R}} f)(\Phi(x_0), \dots, \Phi(x_k))}{|x_0'' \cdots x_k''|}.$$

Proof. Let us call $\alpha = x_0'' + \lambda_1(x_1'' - x_0'') + \cdots + \lambda_k(x_k'' - x_0'')$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. Thus

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} \frac{1}{|\alpha|^{k+1}} f\left(\frac{x_0' + \lambda_1(x_1' - x_0') + \cdots + \lambda_k(x_k' - x_0') + e_d}{\alpha}\right) d\lambda. \tag{3-6}$$

Let us make the change of variables

$$\lambda_1' = \alpha^{-1}\lambda_1, \quad \ldots, \quad \lambda_{k-1}' = \alpha^{-1}\lambda_{k-1}, \quad \lambda_k' = \alpha^{-1}. \tag{3-7}$$

Then

$$d\lambda' = \frac{|x_k'' - x_0''|}{|\alpha|^{k+1}}\, d\lambda. \tag{3-8}$$

A proof of this formula is given in the Appendix. The equation (3-6) becomes

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} f\left(y_k + \lambda_k'(x_0' + e_d - x_0'' y_k) + \sum_{i=1}^{k-1} \lambda_i'(x_i' - x_0' - (x_i'' - x_0'')y_k)\right) \frac{d\lambda'}{|x_k'' - x_0''|},$$

where

$$y_i = \frac{x_i' - x_0'}{x_i'' - x_0''}.$$

This formula is somehow important: it shows that we are still integrating f over a k-plane. Which one? When we computed $\widetilde{\mathcal{R}}\mathcal{S} f(x_0, \ldots, x_k)$, we were interested only in the values of f on $\Phi(\Pi(x_0, \ldots, x_k))$. That way it is simple to guess that $\widetilde{\mathcal{R}}\mathcal{S} f(x_0, \ldots, x_k)$ is closely related to $\Pi(\Phi(x_0), \ldots, \Phi(x_k))$. And indeed, we just have to check that any of the points x_j can be written as

$$x_j = y_k + \lambda_k'(x_0' + e_d - y_k) + \sum_{i=1}^{k-1} \lambda_i'(x_i' - x_0' - (x_i'' - x_0'')y_k) \tag{3-9}$$

for a suitable choice of λ'. Indeed, taking $\lambda = e_j$ and λ' given by (3-7) for this choice of λ, we get the equality (3-9). Let us now make the other change of variables

$$\lambda_1' = \frac{\mu_1}{x_1'' - x_0''}, \quad \ldots, \quad \lambda_{k-1}' = \frac{\mu_{k-1}}{x_{k-1}'' - x_0''}, \quad \lambda_k' = \frac{\mu_k}{x_0''}.$$

We finally get

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} f\left(y_k' + \mu_k(\Phi(x_0) - y_k') + \sum_{i=1}^{k-1} \mu_i(y_i' - y_k')\right) \frac{d\mu}{|x_0''| \prod_{i=1}^{k-1} |x_i'' - x_0''|}.$$

Let us come back to Equation (3-5), the correspondence between $\mathcal{R}$ and $\widetilde{\mathcal{R}}$. We want to find a relation between $(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \ldots, x_k)$ and $(\widetilde{\mathcal{R}} f)(\Phi(x_0), \ldots, \Phi(x_k))$. The above algebra tells us that this is equivalent to finding a relation between the two volumes

$$V(\Phi(x_0), y_1, \ldots, y_k) \quad \text{and} \quad V(\Phi(x_0), \Phi(x_1), \ldots, \Phi(x_k)).$$

Lemma 3.7. *$V(\Phi(x_0), y_1, \dots, y_k)$ and $V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k))$ are related through*

$$\frac{V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k))}{V(\Phi(x_0), y_1, \dots, y_k)} = \prod_{i=1}^{k} \left| \frac{x_0''}{x_i''} - 1 \right|.$$

Proof. A direct calculation shows

$$\frac{x_i''}{x_i'' - x_0''}[\Phi(x_i) - \Phi(x_0)] = \frac{x_0'' x_i' + x_0'' e_d - x_i'' x_0' - x_i'' e_d}{x_0''(x_i'' - x_0'')}$$

and on the other hand, by definition of y_i and $\Phi(x_0)$,

$$y_i - \Phi(x_0) = \frac{x_0'' x_i' + x_0'' e_d - x_i'' x_0' - x_i'' e_d}{x_0''(x_i'' - x_0'')},$$

which proves the equality

$$\Phi(x_i) - \Phi(x_0) = \left(1 - \frac{x_0''}{x_i''}\right)[y_i - \Phi(x_0)].$$

Thus, using that

$$V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k)) = V(0, \Phi(x_1) - \Phi(x_0), \dots, \Phi(x_k) - \Phi(x_0)),$$

Lemma 3.7 is proved. □

Let us go back to the proof of Lemma 3.6. Using the correspondence described in (3-5) and the previous lemma, we finally get the equality

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \dots, x_k) = \frac{(\widetilde{\mathcal{R}} f)(\Phi(x_0), \dots, \Phi(x_k))}{|x_0'' \cdots x_k''|}. \qquad \square$$

At last, let us return to the proof of Lemma 3.5. Since the set of bad points $x_0, \dots, x_k$ (we mean points which do not satisfy the natural assumptions of Lemma 3.6) has null Lebesgue measure in $(\mathbb{R}^d)^{k+1}$, we do not consider them. Let us use Drury's formula [1984]:

$$\|\mathcal{R} f\|_q^q = \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \, f(x_0) \cdots f(x_k) \cdot \widetilde{\mathcal{R}} f(x_0, \dots, x_k)^{d-k}. \tag{3-10}$$

Now all that remains to be done is an easy change of variable $z_i = \Phi(x_i)$. Indeed,

$$\begin{aligned} &\|\mathcal{R}\mathcal{S} f\|_q^q \\ &= \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \frac{1}{|x_0''|^{k+1}} f(\Phi(x_0)) \cdots \frac{1}{|x_k''|^{k+1}} f(\Phi(x_k)) \cdot (\widetilde{\mathcal{R}}\mathcal{S} f(x_0, \dots, x_k))^{d-k} \\ &= \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \frac{1}{|x_0''|^{d+1}} f(\Phi(x_0)) \cdots \frac{1}{|x_k''|^{d+1}} f(\Phi(x_k)) \cdot (\widetilde{\mathcal{R}} f(\Phi(x_0), \dots, \Phi(x_k)))^{d-k} \\ &= \int_{(\mathbb{R}^d)^{k+1}} dz_0 \dots dz_k \, f(z_0) \cdots f(z_k) \cdot \widetilde{\mathcal{R}} f(z_0, \dots, z_k)^{d-k} = \|\mathcal{R} f\|_q^q. \end{aligned}$$

This completes the proof. □

It is a good time to prove a claim we made earlier: affine maps are symmetries.

Lemma 3.8. *Let $f \in L^p$ and L be an invertible affine map. Then*

$$\frac{\|\mathcal{R}(f \circ L)\|_q}{\|f \circ L\|_p} = \frac{\|\mathcal{R} f\|_q}{\|f\|_p}.$$

Proof. The proof is a direct consequence of the correspondence formula (3-5) and of Drury's formula (3-10). Indeed, let L be an invertible affine map; then

$$\widetilde{\mathcal{R}}(f \circ L)(x_0, \dots, x_k) = \widetilde{\mathcal{R}} f(Lx_0, \dots, Lx_k),$$

and with the change of variable $z_i = Lx_i$ in Drury's formula we get

$$\|\mathcal{R}(f \circ L)\|_q = |\det(L)|^{-\frac{1}{p}} \|\mathcal{R} f\|_q,$$

which ends the proof. □

Our goal is now to apply the general Theorem 3.4 about competing symmetries. The operator $\mathcal{S}$ and the rearrangement operator $V : f \mapsto f^*$ increase the L^q-norm of the k-plane transform, and preserve the norm of L^p-functions.

Proposition 3.9. *The operators V and $\mathcal{S}$ satisfy the assumptions of Theorem 3.4, with the Banach space $\mathcal{B} = L^p$.*

Proof. $\mathcal{S}$ and V are both properly contractive operators. Let us check that $\mathcal{S}$ competes with V: choose $f, g \in L^p$, radial, nonincreasing, such that $f = \mathcal{S} g$. Then

$$f(u,s) = \frac{1}{|s|^{k+1}} g\Big(\frac{u}{s}, \frac{1}{s}\Big), \tag{3-11}$$

and, specializing to $s = 1$, we get $f(u,1) = g(u,1)$. Since both f and g are radial, $f(x) = g(x)$ for all $|x| \geq 1$. Let us choose $s < 1$. Specializing (3-11) to $u = 0$, we get

$$f(0,s) = \frac{1}{|s|^{k+1}} g\Big(0, \frac{1}{s}\Big) = \frac{1}{|s|^{k+1}} f\Big(0, \frac{1}{s}\Big).$$

But

$$f\Big(0, \frac{1}{s}\Big) = |s|^{k+1} g(0,s),$$

which shows that $f(0,s) = g(0,s)$. Now again, since both f and g are radial, $f = g$ and $f = \mathcal{S} f$.

We now have to check that $\mathcal{S}$ and V satisfy the assumptions of Theorem 3.4. We follow the arguments of Carlen in [Bianchini et al. 2011]. Let us define

$$h(x) = \left[\frac{1}{1 + |x|^2}\right]^{\frac{k+1}{2}}.$$

Then $\mathcal{S} h = h$, $Vh = h$, and so with

$$K_N = \{f \in L^p : 0 \leq f \leq Nh\}$$

it is straightforward to check that $VK_N \subset K_N$ and $\mathcal{S} K_N \subset K_N$. Moreover VK_N is a compact subset of L^p. Indeed, let us consider a sequence $f_n \in VK_N$. Then f_n is radial, nonincreasing, and since h lies in L^∞ the sequence f_n is bounded in L^∞. Thus, by Helly's principle, f_n admits a subsequence that converges almost everywhere. But since $0 \le f_n \le Nh$, the dominated convergence theorem shows that this subsequence also converges in L^p, which implies that VK_N is relatively compact. At last, $\widetilde{L}^p = \bigcup_N K_N$ is a dense subset of nonnegative elements of L^p (since nonnegative, continuous, compactly supported functions are dense in L^p).

The hardest part is to prove the assumption (3-3). Fortunately, since h is strictly nonincreasing, it has already been done in [Carlen and Loss 1990]. □

We now close this subsection with the final key lemma for the explicit value of extremizers:

Lemma 3.10. *Let $h \in L^p$ such that $Vh = \mathcal{S}h = h$. Then there exists a constant C such that*

$$h(x) = C\left[\frac{1}{1+|x|^2}\right]^{\frac{k+1}{2}}.$$

Proof. Since h satisfies $\mathcal{S}h = Vh = h$, then h is equal to its own rearrangement and so is defined on (at least) $\mathbb{R}^d - \{0\}$. Moreover, $\mathcal{S}h$ must be radial. This leads to

$$\mathcal{S}h\big(u, \sqrt{1+|u|^2}\big) = \left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} h\left(\frac{u}{\sqrt{1+|u|^2}}, \frac{1}{\sqrt{1+|u|^2}}\right) = \left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} h(e_d),$$

using that h is radial. But, since $h = \mathcal{S}h$ is also radial,

$$\mathcal{S}h\big(u, \sqrt{1+|u|^2}\big) = \mathcal{S}h\big(0, \sqrt{1+2|u|^2}\big) = h\big(0, \sqrt{1+2|u|^2}\big).$$

Thus, we get the equality

$$h(x) = h(0,|x|) = \left[\frac{2}{1+|x|^2}\right]^{\frac{k+1}{2}} h(e_d) \tag{3-12}$$

for all $x \in \mathbb{R}^d$ such that $|x| \ge 1$. For $|x| < 1$, the equality $\mathcal{S}h = h$ shows that (3-12) is also right, which proves the lemma. □

Proof of the main theorem. Now we have all the material that we need to prove Theorem 3.1. Let $f_0 \ge 0$ be any function with L^p-norm equal to 1. Let us define the limit

$$h_0 = Tf_0 = \lim_{n\to\infty} (V\mathcal{S})^n f_0.$$

Using that $\mathcal{R}$ is bounded from $L^p \to L^q$, and equations (2-2), (3-4),

$$\|\mathcal{R}h_0\|_q = \lim_{n\to\infty} \|\mathcal{R}(V\mathcal{S})^n f_0\|_q \ge \|\mathcal{R}f_0\|_q. \tag{3-13}$$

Moreover, by Theorem 3.4, $Vh_0 = \mathcal{S}h_0 = h_0$, so h satisfies the assumptions of Lemma 3.10. We then get

$$h_0(x) = h_0(e_d)\left[\frac{2}{1+|x|^2}\right]^{\frac{k+1}{2}}.$$

Because of normalization and positivity of f_0, $h_0(e_d)$ can take only one value. It then follows from (3-13) that h_0 maximizes the norms of $\mathcal{R} f_0$, and thus it is an extremizer.

Value of the best constant. Here we compute the value of the best constant. We use the correspondence (2-3) described in the previous section, and only think about $\mathcal{T}$ and its related measurable spaces instead of $\mathcal{R}$. Let h be the radial extremizer

$$h(r) = \left[\frac{1}{1+r^2}\right]^{\frac{k+1}{2}}.$$

A family of integrals will be useful to compute its L^p-norm and the L^q-norm of $\mathcal{T}h$. These integrals are defined as

$$\int_0^\infty \frac{t^m}{(1+t^2)^{\frac{n}{2}}}\,dt.$$

A calculation shows that

$$\int_0^\infty \frac{t^m}{(1+t^2)^{\frac{n}{2}}}\,dt = \frac{\Gamma\left(\frac{1}{2}(m+1)\right)\Gamma\left(\frac{1}{2}(n-m-1)\right)}{\Gamma\left(\frac{1}{2}n\right)},$$

where Γ is the standard Euler gamma function. Then

$$\|h\|_p^p = \int_0^\infty \frac{r^{d-1}\,dr}{(1+r^2)^{\frac{d+1}{2}}} = \frac{\Gamma\left(\frac{1}{2}d\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}(d+1)\right)}.$$

Moreover,

$$\mathcal{T}h(r) = \frac{1}{\sqrt{1+r^2}} \int_0^\infty \frac{u^{k-1}\,du}{(1+|u|^2)^{\frac{k+1}{2}}},$$

and this leads to

$$\|\mathcal{T}h\|_q^q = \left(\frac{\Gamma(k-1)\Gamma(k+1)}{\Gamma(2k)}\right)^{d+1} \frac{\Gamma(d-k-1)\Gamma(d+1)}{\Gamma(2d-k)}.$$

The use of the fundamental relation

$$\tfrac{1}{2}|S^{n-1}|\Gamma\left(\tfrac{1}{2}n\right) = \pi^{\frac{n}{2}}$$

leads to

$$A(k,d) = \frac{\|\mathcal{R}h\|_q}{\|h\|_p} = \pi^{\frac{d-k}{2(d+1)}} \cdot \Gamma\left(\tfrac{1}{2}(d+1)\right)^{\frac{k}{d+1}} \cdot \Gamma\left(\tfrac{1}{2}(k+1)\right)^{-\frac{d}{d+1}} = \left[2^{k-d}\frac{|S^k|^d}{|S^d|^k}\right]^{\frac{1}{d+1}}.$$

4. The question of uniqueness

We shall discuss here the question of the uniqueness of extremizers of (1-2). For the sake of simplicity, we will assume $d \geq 3$. This is not a restricting assumption: indeed, for the case $d = 2$, the only k-plane transform is the Radon transform, and this has been thoroughly studied in [Christ 2011c].

The uniqueness problem for the Radon transform was solved in the same reference. The main tool for the proof is the following:

Theorem 4.1. *Let $k = d-1$, and let f be a nonnegative extremizer. Then there exist a radial, nonincreasing, nonnegative extremizer F and an invertible affine map L such that $f = F \circ L$.*

Then it turned out that the work was almost all done. Christ characterized all the extremizers using the uniqueness Theorem 4.1 two times, in a certain sense. His approach is very interesting because the question of uniqueness is curiously intertwined with the question of existence. Here we want to develop a different approach, for an arbitrary $1 \le k \le d-1$, *assuming* that a result similar to Theorem 4.1 is true. More accurately, we want to prove the following:

Theorem 4.2. *Let $1 \le k \le d-1$. Assume that any extremizer for the k-plane transform inequality (1-2) can be written $F \circ L$ with F a radial, nonincreasing extremizer and L an affine map. Then any nonincreasing radial extremizer is of the form*

$$x \mapsto \left(\frac{1}{a+b|x|}\right)^{\frac{k+1}{2}}. \tag{4-1}$$

As we mentioned in the introduction, the *ad hoc* assumption in this theorem was proved to be true by Flock [2013], inducing the complete characterization of extremizers.

One of the main tools here will be the use of the symmetry $\mathcal{S}$ combined with the fact that an extremizer is a radial function composed with an affine map. Thus we will use again the competing symmetry theory. From now we will assume that k is such that any extremizer for (1-2) can be written $f \circ L$ with f radial and L an affine map. Our main lemma follows; it shows that radial extremizers enjoy additional symmetries.

Lemma 4.3. *Let f be a radial, nonincreasing extremizer for (1-2). Then there exists a real number $\mu > 0$ such that*

$$(V\mathcal{S})^2 f(r) = \mu^{\frac{d}{p}} f(\mu r).$$

Proof. Since f is a radial, nonincreasing extremizer, then f is not the (almost everywhere) null function: there exists $\lambda_0 > 0$ such that $f(\lambda_0 e_d) \neq 0$. Because of dilation-invariance, we can assume $\lambda_0 = 1$.

$\mathcal{S} f$ is also an extremizer. It follows that there exist $F : \mathbb{R}^+ \to \mathbb{R}$, nonincreasing, a linear invertible map L and a vector $x_0 \in \mathbb{R}^d$ such that

$$\mathcal{S} f(x) = F(|x_0 + Lx|). \tag{4-2}$$

Computing $\mathcal{S} f\big(u, \sqrt{|u|^2+1}\big)$, we get

$$f(e_d)\left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} = F\Big(|x_0 + Lu + \sqrt{1+|u|^2}Le_d|\Big) \tag{4-3}$$

for all $u \in \mathbb{R}^{d-1} \times \{0\}$. Let $C = f(e_d) \neq 0$, and $I \subset \mathbb{R}^+$ the interval made of points that can be written $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ for some $u \in \mathbb{R}^{d-1} \times \{0\}$. We claim that the map F is strictly decreasing on I. Indeed, let us assume that there exists $0 < \alpha < \beta$ such that F is constant on $[\alpha, \beta]$. Pick $u \in \mathbb{R}^{d-1} \times \{0\}$ such that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d| \in (\alpha, \beta)$. For t close to 1, $|x_0 + Ltu + \sqrt{1+t^2|u|^2}Le_d| \in (\alpha, \beta)$, and thus for t close to 1 the map

$$t \mapsto F\big(|x_0 + Lu + \sqrt{1+|u|^2}Le_d|\big)$$

is constant. Because of (4-3), this is a contradiction.

The function F is then injective on I. Formula (4-3) shows that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ must be a function of $|u|^2$ only. To conclude the proof, we require the following lemma:

Lemma 4.4. *Let L be an invertible linear map such that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ depends only on $|u|$. Then $L(\mathbb{R}^{d-1}\times\{0\}) \subset (\operatorname{span}(Le_d))^\perp$, and $L|_{\mathbb{R}^{d-1}\times\{0\}}$ preserves the norm, modulo a multiplicative constant. Moreover, there exists $s_0 \in \mathbb{R}^d$ such that $x_0 = s_0 Le_d$.*

Proof. Let us choose $u = r\theta \in S^{d-2}\times\{0\}$. Then

$$\big|x_0 + rL\theta + \sqrt{1+r^2}Le_d\big|^2 = r^2|L\theta|^2 + \big|\sqrt{1+r^2}Le_d + x_0\big|^2 + 2r\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$$

depends only on r, and so does $r^2|L\theta|^2 + 2r\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$. As a consequence, $|L\theta|$ is a constant and $\langle L\theta, \sqrt{2}Le_d + x_0\rangle$ is a constant. Here we must assume $d \geq 3$, so the sphere S^{d-2} contains an infinity of points.

The condition that $|L\theta|$ is constant holds only if $L|_{\mathbb{R}^{d-1}\times\{0\}}$ preserves the norm, modulo a multiplicative constant. Thus the quantity

$$\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$$

must depend only on r. Specializing at θ and $-\theta$, for all r, $\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle = 0$. But since L is invertible, the space spanned by $L\theta$ has dimension $d-1$. Thus the space spanned by the vectors $\sqrt{1+r^2}Le_d + x_0$ for $r \geq 0$ has dimension 1, which proves that there exists s_0 such that $s_0 Le_d = x_0$. ☐

Composing with an isometry, we can assume that $L(\mathbb{R}^{d-1}\times\{0\}) \subset \mathbb{R}^{d-1}\times\{0\}$. Moreover, $|Lu|$ depends only on $|u|$, which implies that L restricted to $\mathbb{R}^{d-1}\times\{0\}$ must be a multiple of an isometry. We then deduce that there exist $a > 0, b > 0, s_0$ such that $|L(u+se_d) + x_0|^2 = a^2|u|^2 + b^2(s+s_0)^2$, *for all* $(u,s) \in \mathbb{R}^{d-1}\times\mathbb{R}$. Thus we get the fundamental relation between f and F:

$$\mathcal{S}f(u+se_d) = F\big(\sqrt{a^2|u|^2 + b^2(s+s_0)^2}\big).$$

Now, changing F to $G = F(\sqrt{ab}\,\cdot)$, G remains nonincreasing, and we get

$$\mathcal{S}f(u+se_d) = F\big(\sqrt{a^2|u|^2 + b^2(s+s_0)^2}\big) = G\left(\sqrt{\frac{a}{b}|u|^2 + \frac{b}{a}(s+s_0)^2}\right),$$

reducing the number of unknown parameters in our system. Thus, we have accomplished the first step in our identification program: we know how the operator $\mathcal{S}$ acts on radial extremizers. Now we have to understand how V acts on functions g whose form is

$$g : u + se_d \mapsto G\left(\sqrt{c|u|^2 + \frac{1}{c}(s+s_0)^2}\right).$$

First, we can assume that $s_0 = 0$: indeed, $g(\cdot - s_0 e_d)^* = g^*$. Moreover, G is decreasing and so the level sets of g are ellipsoids $c|u|^2 + c^{-1}s^2 \leq R^2$. The corresponding rearranged sets are balls of radius R', with R' satisfying the relation

$$R'^d = \frac{R^{d-1}}{c^{\frac{d-1}{2}}} c^{\frac{1}{2}} R = \frac{R^d}{c^{\frac{d-2}{2}}}.$$

Thus

$$Vg(se_d) = G(c^{\frac{d-2}{2d}} s) = \frac{1}{(c^{\frac{d-1}{d}} s)^{k+1}} f\left(\frac{e_d}{c^{\frac{d-1}{d}} s}\right),$$

coming back to the relation defining G, and using that f is radial. And then

$$Vg(se_d) = \frac{1}{(c^{\frac{d-1}{d}} s)^{k+1}} f\left(\frac{e_d}{c^{\frac{d-1}{d}} s}\right).$$

This characterizes the action of the operator $V\mathcal{S}$ on radial extremizers. More simply, calling $\lambda = c^{\frac{d-1}{d}}$, we have

$$V\mathcal{S} f(x) = \frac{1}{\lambda^{k+1}|x|^{k+1}} f\left(\frac{e_d}{\lambda|x|}\right).$$

Let us use again the competing symmetry theory: to construct an explicit extremizer of (1-2) we used iterations of $V\mathcal{S}$, applied to *any* function. Let us choose f_0 a radial extremizer. Then $V\mathcal{S} f_0$ is still a radial extremizer, and we know that there exists λ such that

$$V\mathcal{S} f_0(r) = \left(\frac{1}{\lambda r}\right)^{k+1} f_0\left(\frac{1}{\lambda r}\right).$$

Let us do that again: there exists λ' such that

$$(V\mathcal{S})^2 f_0(r) = \left(\frac{1}{\lambda' r}\right)^{k+1} (V\mathcal{S} f_0)\left(\frac{1}{\lambda' r}\right) = \left(\frac{1}{\lambda' r}\frac{\lambda' r}{\lambda}\right)^{k+1} f_0\left(\frac{\lambda' r}{\lambda}\right) = \frac{1}{\lambda^{k+1}} f_0\left(\frac{\lambda' r}{\lambda}\right).$$

Since the operator $V\mathcal{S}$ preserves the norm, we must have $\lambda\lambda'^d = 1$. Using the parameter μ such that $\lambda' = \mu\lambda$, we conclude the proof of Lemma 4.3. □

That proves that the operator $V\mathcal{S}$ acts on radial, nonincreasing extremizers as a dilation. Now let us consider $f_n = (V\mathcal{S})^{2n} f_0$. For each n, there exists μ_n such that

$$(V\mathcal{S})^{2n} f_0(r) = (\mu_n)^{\frac{d}{p}} f_0(\mu_n r).$$

But the sequence f_n converges in L^p to the extremizer h described in Theorem 3.1. Thus it converges weakly to a nonzero function, which is possible if and only if μ_n converges to a nonzero value. That ends the proof of Theorem 4.2: every nonnegative radial extremizer can be written

$$x \mapsto \left[\frac{1}{a + b|x|^2}\right]^{\frac{k+1}{2}}$$

with $a, b > 0$.

Appendix

Here we prove the jacobian formula (3-8). Define

$$\Psi(\lambda_1, \dots \lambda_k) = (\alpha^{-1}\lambda_1, \dots, \alpha^{-1}\lambda_{k-1}, \lambda_k).$$

We want to compute $J\psi(\lambda) = |\det(\nabla\psi)(\lambda_1, \dots, \lambda_k)|$. Note first that

$$\frac{\partial \alpha^{-1}}{\partial \lambda_i} = \alpha^{-2}(x_i'' - x_0'').$$

Thus

$$J\psi(\lambda) = \begin{vmatrix} -\alpha^{-2}(x_1''-x_0'')\lambda_1 + \alpha^{-1} & \dots & -\alpha^{-2}(x_1''-x_0'')\lambda_{k-1} & -\alpha^{-2}(x_1''-x_0'') \\ \vdots & \ddots & \vdots & \vdots \\ -\alpha^{-2}(x_{k-1}''-x_0'')\lambda_1 & \dots & -\alpha^{-2}(x_{k-1}''-x_0'')\lambda_{k-1} + \alpha^{-1} & -\alpha^{-2}(x_{k-1}''-x_0'') \\ -\alpha^{-2}(x_k''-x_0'')\lambda_1 & \dots & -\alpha^{-2}(x_k''-x_0'')\lambda_{k-1} & -\alpha^{-2}(x_k''-x_0'') \end{vmatrix}$$

$$= |\alpha|^{-k-1}|x_k''-x_0''| \begin{vmatrix} y_1\lambda_1 + 1 & \dots & y_1\lambda_{k-1} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1 & y_{k-1} \\ \lambda_1 & \dots & \lambda_{k-1} & 1 \end{vmatrix},$$

where $y_i = -\alpha^{-1}(x_i''-x_0'')$. We claim that the determinant appearing in the last line is always equal to 1. Indeed, consider the polynomial

$$P(z) = \det \begin{pmatrix} y_1\lambda_1 + 1 & \dots & y_1\lambda_{k-1} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1 & y_{k-1} \\ \lambda_1 & \dots & \lambda_{k-1} & z \end{pmatrix}.$$

It is of degree 1 in z. Moreover, we have

$$P'(1) = \det \begin{pmatrix} y_1\lambda_1 + 1 & \dots & y_1\lambda_{k-1} \\ \vdots & \ddots & \vdots \\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1 \end{pmatrix} = 1 + \langle y, \lambda\rangle, \tag{A-1}$$

$$P(2) = 2 + \langle y, \lambda\rangle. \tag{A-2}$$

Here $\langle y, \lambda\rangle = \sum_{i=1}^{k-1} \lambda_i y_i$. The formulas (A-1), (A-2) both come from the following lemma:

Lemma A.1. *If $u, v \in \mathbb{R}^p$, then*

$$\det(\mathbb{1} + u^t v) = 1 + \langle u, v\rangle.$$

Proof. The matrix $u^t v$ is of rank one. As a consequence, its only eigenvalue is its trace $\langle u, v\rangle$. The characteristic polynomial of $-u^t v$ is then

$$\det(z\mathbb{1} + u^t v) = z^{p-1}(z + \langle u, v\rangle).$$

Evaluating this at $z = 1$ proves the lemma. □

Applying this lemma to $u=\lambda$, $v=y$ leads to (A-1), and $u=(\lambda,1)$, $v=(y,1)$ leads to (A-2). Thus

$$P(z)=(1+\langle\lambda,y\rangle)z-\langle\lambda,y\rangle.$$

Evaluate this at $z=1$ to get the asserted claim, and then

$$J\psi(\lambda)=|\alpha|^{-k-1}|x_k''-x_0''|.$$

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WELL-POSEDNESS OF THE STOKES–CORIOLIS SYSTEM IN THE HALF-SPACE OVER A ROUGH SURFACE

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This paper is devoted to the well-posedness of the stationary 3D Stokes–Coriolis system set in a half-space with rough bottom and Dirichlet data which does not decrease at space infinity. Our system is a linearized version of the Ekman boundary layer system. We look for a solution of infinite energy in a space of Sobolev regularity. Following an idea of Gérard-Varet and Masmoudi, the general strategy is to reduce the problem to a bumpy channel bounded in the vertical direction thanks to a transparent boundary condition involving a Dirichlet to Neumann operator. Our analysis emphasizes some strong singularities of the Stokes–Coriolis operator at low tangential frequencies. One of the main features of our work lies in the definition of a Dirichlet to Neumann operator for the Stokes–Coriolis system with data in the Kato space $H^{1/2}_{\mathrm{uloc}}$.

1. Introduction

The goal of the present paper is to prove the existence and uniqueness of solutions to the Stokes–Coriolis system

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u|_\Gamma = u_0, \end{cases} \tag{1-1}$$

where

$$\Omega := \{x \in \mathbb{R}^3 : x_3 > \omega(x_h)\}, \quad \Gamma = \partial\Omega = \{x \in \mathbb{R}^3 : x_3 = \omega(x_h)\}$$

and $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded function.

When ω has some structural properties, such as periodicity, existence and uniqueness of solutions are easy to prove: our aim is to prove well-posedness when the function ω is arbitrary, say $\omega \in W^{1,\infty}(\mathbb{R}^2)$, and when the boundary data u_0 is not square integrable. More precisely, we wish to work with u_0 in a space of infinite energy of Sobolev regularity, such as Kato spaces. We refer to the end of this introduction for a definition of these uniformly locally Sobolev spaces L^2_{uloc}, H^s_{uloc}.

The interest for such function spaces to study fluid systems goes back to [Lemarié-Rieusset 1999; 2002], in which existence is proved for weak solutions of the Navier–Stokes equations in $\mathbb{R}^3$ with initial data in L^2_{uloc}. These works fall into the analysis of fluid flows with infinite energy, which is a field of intense research. Without being exhaustive, let us mention that:

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Keywords: Stokes–Coriolis system, Ekman boundary layer, rough boundaries, Dirichlet to Neumann operator, Saint-Venant estimate, Kato spaces.

- Cannon and Knightly [1970], Giga, Inui, and Matsui [Giga et al. 1999], Solonnikov [2003], Bae and Jin [2012] (local solutions), and Giga, Matsui, and Sawada [Giga et al. 2001] (global solutions) studied the nonstationary Navier–Stokes system in the whole space or in the half-space with initial data in L^∞ or in BUC (bounded uniformly continuous).
- Basson [2006] and Maekawa and Terasawa [2006] studied local solutions of the nonstationary Navier–Stokes system in the whole space with initial data in L^p_{uloc} spaces.
- Giga and Miyakawa [1989], Taylor [1992] (global solutions), and Kato [1992] studied local solutions to the nonstationary Navier–Stokes system, and Gala [2005] studied global solutions to a quasigeostrophic equation with initial data in Morrey spaces.
- Gallagher and Planchon [2002] studied the nonstationary Navier–Stokes system in $\mathbb{R}^2$ with initial data in the homogeneous Besov space $\dot{B}^{2/r-1}_{r,q}$.
- Giga et al. [2007] studied the nonstationary Ekman system in $\mathbb{R}^3_+$ with initial data in the Besov space $\dot{B}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ for $2 < p < \infty$; see also [Giga et al. 2006] (local solutions) and [Giga et al. 2008] (global solutions) on the Navier–Stokes–Coriolis system in $\mathbb{R}^3$, and [Yoneda 2009] for initial data spaces containing almost-periodic functions.
- Konieczny and Yoneda [2011] studied the stationary Navier–Stokes system in Fourier–Besov spaces.
- David Gérard-Varet and Nader Masmoudi [2010] studied the 2D Stokes system in the half-plane above a rough surface with $H^{1/2}_{\mathrm{uloc}}$ boundary data.
- Alazard, Burq, and Zuily [Alazard et al. 2013] studied the Cauchy problem for gravity water waves with data in H^s_{uloc}; in particular, they studied the Dirichlet to Neumann operator associated with the Laplacian in a domain $\Omega = \{(x, y) \in \mathbb{R}^{d+1} : \eta^*(x) < y < \eta(x)\}$, with $H^{1/2}_{\mathrm{uloc}}$ boundary data.

Despite this huge literature on initial value problems in fluid mechanics in spaces of infinite energy, we are not aware of any work concerning stationary systems and nonhomogeneous boundary value problems in $\mathbb{R}^3_+$. Let us emphasize that the derivation of energy estimates in stationary and time dependent settings are rather different: indeed, in a time dependent setting, boundedness of the solution at time t follows from boundedness of the initial data and of the associated semigroup. In a stationary setting and in a domain with a boundary, to the best of our knowledge, the only way to derive estimates without assuming any structure on the function ω is based on the arguments of Ladyzhenskaya and Solonnikov [1980] (see also [Gérard-Varet and Masmoudi 2010] for the Stokes system in a bumped half-plane).

In the present case, our motivation comes from the asymptotic analysis of highly rotating fluids near a rough boundary. Indeed, consider the system

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + \dfrac{1}{\varepsilon} e_3 \times u^\varepsilon + \nabla p^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \operatorname{div} u^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ u^\varepsilon|_{\Gamma^\varepsilon} = 0, \\ u^\varepsilon|_{x_3=1} = (V_h, 0), \end{cases} \tag{1-2}$$

where

$$\Omega^\varepsilon := \{x \in \mathbb{R}^3 : \varepsilon\omega(x_h/\varepsilon) < x_3 < 1\} \quad \text{and} \quad \Gamma^\varepsilon := \partial\Omega^\varepsilon \setminus \{x_3 = 1\}.$$

Then it is expected that u^ε is the sum of a two-dimensional interior flow $(u^{\text{int}}(x_h), 0)$ balancing the rotation with the pressure term and a boundary layer flow $u^{\text{BL}}(x/\varepsilon; x_h)$, located in the vicinity of the lower boundary. In this case, the equation satisfied by u^{BL} is precisely (1-1), with $u_0(y_h; x_h) = -(u^{\text{int}}(x_h), 0)$. Notice that x_h is the macroscopic variable and is a parameter in the equation on u^{BL}. The fact that the Dirichlet boundary condition is constant with respect to the fast variable y_h is the original motivation for study of the well-posedness of (1-1) in spaces of infinite energy, such as the Kato spaces H^s_{uloc}.

The system (1-2) models large-scale geophysical fluid flows in the linear regime. In order to get a physical insight into the physics of rotating fluids, we refer to the books [Greenspan 1980] (rotating fluids in general, including an extensive study of the linear regime) and [Pedlosky 1987] (focus on geophysical fluids). Ekman [1905] analyzed the effect of the interplay between viscous forces and the Coriolis acceleration on geophysical fluid flows.

For further remarks on the system (1-2), we refer to Section 7 in the book [Chemin et al. 2006] by Chemin, Desjardins, Gallagher, and Grenier, and to [Chemin et al. 2002], where a model with anisotropic viscosity is studied and an asymptotic expansion for u^ε is obtained.

Studying (1-1) with an arbitrary function ω is more realistic from a physical point of view, and also allows us to bring to light some bad behaviors of the system at low horizontal frequencies, which are masked in a periodic setting.

Our main result is the following.

Theorem 1. *Let $\omega \in W^{1,\infty}(\mathbb{R}^2)$, and let $u_{0,h} \in H^2_{\text{uloc}}(\mathbb{R}^2)^2$, $u_{0,3} \in H^1_{\text{uloc}}(\mathbb{R}^2)$. Assume that there exists $U_h \in H^{1/2}_{\text{uloc}}(\mathbb{R}^2)^2$ such that*

$$u_{0,3} - \nabla_h\omega \cdot u_{0,h} = \nabla_h \cdot U_h. \tag{1-3}$$

Then there exists a unique solution u of (1-1) such that

$$\sup_{l\in\mathbb{Z}^2} \|u\|_{H^1(((l+[0,1]^2)\times(-1,a))\cap\Omega)} < \infty \quad \text{for all } a > 0,$$

$$\sup_{l\in\mathbb{Z}^2} \sum_{\substack{\alpha\in\mathbb{N}^3\\ |\alpha|=q}} \int_1^\infty \int_{l+[0,1]^2} |\nabla^\alpha u|^2 < \infty$$

for some integer q sufficiently large, which does not depend on ω or u_0 (say $q \geq 4$).

Remark 1.1. • Assumption (1-3) is a compatibility condition, which stems from singularities at low horizontal frequencies in the system. When the bottom is flat, it merely becomes $u_{0,3} = \nabla_h \cdot U_h$. Notice that this condition only bears on the normal component of the velocity at the boundary: in particular, if $u_0 \cdot n|_\Gamma = 0$, then (1-3) is satisfied. We also stress that (1-3) is satisfied in the framework of highly rotating fluids near a rough boundary, since in this case $u_{0,3} = 0$ and $u_{0,h}$ is constant with respect to the microscopic variable.

• The singularities at low horizontal frequencies also account for the possible lack of integrability of the gradient far from the rough boundary: we were not able to prove that

$$\sup_{l\in\mathbb{Z}^2}\int_1^\infty\int_{l+[0,1]^2}|\nabla u|^2<\infty,$$

although this estimate is true for the Stokes system. In fact, looking closely at our proof, it seems that nontrivial cancellations should occur for such a result to hold in the Stokes–Coriolis case.

• Concerning the regularity assumptions on ω and u_0, it is classical to assume Lipschitz regularity on the boundary. The regularity required on u_0, however, may not be optimal, and stems in the present context from an explicit lifting of the boundary condition. It is possible that the regularity could be lowered if a different type of lifting were used, in the spirit of [Alazard et al. 2013, Proposition 4.3]. Let us stress as well that if ω is constant, then $H^{1/2}_{\mathrm{uloc}}$ regularity is enough (cf. Corollary 2.17).

The same tools can be used to prove a similar result for the Stokes system in three dimensions (we recall that [Gérard-Varet and Masmoudi 2010] is concerned with the Stokes system in two dimensions). In fact, the treatment of the Stokes system is easier, because the associated kernel is homogeneous and has no singularity at low frequencies. The results proved in Section 2 can be obtained thanks to the Green function associated with the Stokes system in three dimensions; see [Galdi 1994]. On the other hand, the arguments of Sections 3 and 4 can be transposed as such to the Stokes system in three dimensions. The main novelties of these sections, which rely on careful energy estimates, are concerned with the higher dimensional space rather than with the presence of the rotation term (except for Lemma 3.2).

The statement of Theorem 1 is very close to one of the main results of the paper [Gérard-Varet and Masmoudi 2010], namely, the well-posedness of the Stokes system in a bumped half-plane with boundary data in $H^{1/2}_{\mathrm{uloc}}(\mathbb{R})$. Of course, it shares the main difficulties of [Gérard-Varet and Masmoudi 2010]: spaces of functions of infinite energy, lack of a Poincaré inequality, irrelevancy of scalar tools (Harnack inequality, maximum principle) which do not apply to systems. But two additional problems are encountered when studying (1-1):

(1) Equation (1-1) is set in three dimensions, whereas the study of [Gérard-Varet and Masmoudi 2010] took place in two dimensions. This complicates the derivation of energy estimates. Indeed, the latter are based on the truncation method by Ladyzhenskaya and Solonnikov [1980], which consists more or less in multiplying (1-1) by $\chi_k u$, where $\chi_k\in\mathscr{C}_0^\infty(\mathbb{R}^{d-1})$ is a cut-off function in the horizontal variables such that $\operatorname{Supp}\chi_k\subset B_{k+1}$ and $\chi_k\equiv 1$ on B_k for $k\in\mathbb{N}$. If $d=2$, the size of the support of $\nabla\chi_k$ is bounded, while it is unbounded when $d=3$. This has a direct impact on the treatment of some commutator terms.

(2) Somewhat more importantly, the kernel associated with the Stokes–Coriolis operator has a more complicated expression than the one associated with the Stokes operator (see [Galdi 1994, Chapter IV] for the computation of the Green function associated to the Stokes system in the half-space). In the case of the Stokes–Coriolis operator, the kernel is not homogeneous, which prompts us to distinguish between high and low horizontal frequencies throughout the paper. Moreover, it exhibits strong singularities at low horizontal frequencies, which have repercussions on the whole proof and account for assumption (1-3).

The proof of Theorem 1 follows the same general scheme used in [Gérard-Varet and Masmoudi 2010] (this scheme has also been successfully applied in [Dalibard and Gérard-Varet 2011] in the case of a Navier slip boundary condition on the rough bottom): we first perform a thorough analysis of the Stokes–Coriolis system in $\mathbb{R}^3_+$, and we define the associated Dirichlet to Neumann operator for boundary data in $H^{1/2}_{\text{uloc}}$. In particular, we derive a representation formula for solutions of the Stokes–Coriolis system in $\mathbb{R}^3_+$, based on a decomposition of the kernel which distinguishes high and low frequencies, and singular/regular terms. We also prove a similar representation formula for the Dirichlet to Neumann operator. Then we derive an equivalent system to (1-1), set in a domain which is bounded in x_3 and in which a transparent boundary condition is prescribed on the upper boundary. These two preliminary steps are performed in Section 2. We then work with the equivalent system, for which we derive energy estimates in H^1_{uloc}; this allows us to prove existence in Section 3. Eventually, we prove uniqueness in Section 4. The appendices gathers several technical lemmas used throughout the paper.

Notation. We will be working with spaces of uniformly locally integrable functions, called Kato spaces, whose definition we now recall; see [Kato 1975]. Let $\vartheta \in \mathscr{C}^\infty_0(\mathbb{R}^d)$ be such that $\operatorname{Supp}\vartheta \subset [-1, 1]^d$, $\vartheta \equiv 1$ on $[-1/4, 1/4]^d$, and

$$\sum_{k\in\mathbb{Z}^d} \tau_k\vartheta(x) = 1 \quad \text{for all } x \in \mathbb{R}^d, \tag{1-4}$$

where τ_k is the translation operator defined by $\tau_k f(x) = f(x-k)$.

Then, for $s \geq 0$, $p \in [1, \infty)$,

$$L^p_{\text{uloc}}(\mathbb{R}^d) := \Big\{u \in L^p_{\text{loc}}(\mathbb{R}^d) : \sup_{k\in\mathbb{Z}^d} \|(\tau_k\vartheta)u\|_{L^p(\mathbb{R}^d)} < \infty\Big\},$$
$$H^s_{\text{uloc}}(\mathbb{R}^d) := \Big\{u \in H^s_{\text{loc}}(\mathbb{R}^d) : \sup_{k\in\mathbb{Z}^d} \|(\tau_k\vartheta)u\|_{H^s(\mathbb{R}^d)} < \infty\Big\}.$$

The space H^s_{uloc} is independent of the choice of the function ϑ; see [Alazard et al. 2013, Lemma 3.1].

We will also work in the domain $\Omega^b := \{x \in \mathbb{R}^3 : \omega(x_h) < x_3 < 0\}$, assuming that ω takes values in $(-1, 0)$. With a slight abuse of notation, we will write

$$\|u\|_{L^p_{\text{uloc}}(\Omega^b)} := \sup_{k\in\mathbb{Z}^2} \|(\tau_k\vartheta)u\|_{L^p(\Omega^b)},$$
$$\|u\|_{H^s_{\text{uloc}}(\Omega^b)} := \sup_{k\in\mathbb{Z}^2} \|(\tau_k\vartheta)u\|_{H^s(\Omega^b)},$$

where the function ϑ belongs to $\mathscr{C}^\infty_0(\mathbb{R}^2)$ and satisfies (1-4), $\operatorname{Supp}\vartheta \subset [-1, 1]^2$, $\vartheta \equiv 1$ on $[-1/4, 1/4]^2$, and $H^s_{\text{uloc}}(\Omega^b) = \{u \in H^s_{\text{loc}}(\Omega^b) : \|u\|_{H^s_{\text{uloc}}(\Omega^b)} < \infty\}$, $L^p_{\text{uloc}}(\Omega^b) = \{u \in L^p_{\text{loc}}(\Omega^b) : \|u\|_{L^p_{\text{uloc}}(\Omega^b)} < \infty\}$.

Throughout the proof, we will often use the notation $|\nabla^q u|$, where $q \in \mathbb{N}$, for the quantity

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|=q}} |\nabla^\alpha u|,$$

where $d = 2$ or 3, depending on the context.

2. Presentation of a reduced system and main tools

Following an idea of Gérard-Varet and Masmoudi [2010], the first step is to transform (1-1) so as to work in a domain bounded in the vertical direction (rather than a half-space). This allows us eventually to use Poincaré inequalities, which are paramount in the proof. To that end, we introduce an artificial flat boundary above the rough surface Γ, and we replace the Stokes–Coriolis system in the half-space above the artificial boundary by a transparent boundary condition, expressed in terms of a Dirichlet to Neumann operator.

In the rest of the article, without loss of generality, we assume that $\sup \omega =: \alpha < 0$ and $\inf \omega \geq -1$, and we place the artificial boundary at $x_3 = 0$. We set

$$\Omega^b := \{x \in \mathbb{R}^3 : \omega(x_h) < x_3 < 0\},$$
$$\Sigma := \{x_3 = 0\}.$$

The Stokes–Coriolis system differs in several aspects from the Stokes system; in the present paper, the most crucial differences are the lack of an explicit Green function, and the bad behavior of the system at low horizontal frequencies. The main steps of the proof are as follows:

(1) Prove existence and uniqueness of a solution of the Stokes–Coriolis system in a half-space with boundary data in $H^{1/2}(\mathbb{R}^2)$.

(2) Extend this well-posedness result to boundary data in $H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$.

(3) Define the Dirichlet to Neumann operator for functions in $H^{1/2}(\mathbb{R}^2)$, and extend it to functions in $H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$.

(4) Define an equivalent problem in Ω^b, with a transparent boundary condition at Σ, and prove the equivalence between the problem in Ω^b and the one in Ω.

(5) Prove existence and uniqueness of solutions of the equivalent problem.

Items (1)–(4) will be proved in the current section, and (5) in Sections 3 and 4.

2A. ***The Stokes–Coriolis system in a half-space.*** The first step is to study the properties of the Stokes–Coriolis system in $\mathbb{R}^3_+$, namely,

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p = 0 & \text{in } \mathbb{R}^3_+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3_+, \\ u|_{x_3=0} = v_0. \end{cases} \tag{2-1}$$

In order to prove the result of Theorem 1, we have to prove the existence and uniqueness of a solution u of the Stokes–Coriolis system in $H^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ such that, for some $q \in \mathbb{N}$ sufficiently large,

$$\sup_{l \in \mathbb{Z}^2} \int_{l+(0,1)^2} \int_1^\infty |\nabla^q u|^2 < \infty.$$

However, the Green function for the Stokes–Coriolis is far from being explicit, and its Fourier transform, for instance, is much less well-behaved than that of the Stokes system (which is merely the Poisson kernel). Therefore such a result is not so easy to prove. In particular, because of the singularities of the Fourier transform of the Green function at low frequencies, we are not able to prove that

$$\sup_{l\in\mathbb{Z}^2}\int_{l+(0,1)^2}\int_1^\infty |\nabla u|^2<\infty.$$

- We start by solving the system when $v_0\in H^{1/2}(\mathbb{R}^2)$.

Proposition 2.1. *Let $v_0\in H^{1/2}(\mathbb{R}^2)^3$ be such that*

$$\int_{\mathbb{R}^2}\frac{1}{|\xi|}|\hat v_{0,3}(\xi)|^2\,d\xi<\infty. \tag{2-2}$$

Then the system (2-1) admits a unique solution $u\in H^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ such that

$$\int_{\mathbb{R}^3_+}|\nabla u|^2<\infty.$$

Remark 2.2. The condition (2-2) stems from a singularity at low frequencies of the Stokes–Coriolis system, which we will encounter several times in the proof. Notice that (2-2) is satisfied in particular when $v_{0,3}=\nabla_h\cdot V_h$ for some $V_h\in H^{1/2}(\mathbb{R}^2)^2$, which is sufficient for further purposes.

Proof. Uniqueness. Consider a solution whose gradient is in $L^2(\mathbb{R}^3_+)$ and with zero boundary data on $x_3=0$. Then, using the Poincaré inequality, we infer that

$$\int_0^a\int_{\mathbb{R}^2}|u|^2\le C_a\int_0^a\int_{\mathbb{R}^2}|\nabla u|^2<\infty,$$

and therefore we can take the Fourier transform of u in the horizontal variables. Denoting by $\xi\in\mathbb{R}^2$ the Fourier variable associated with x_h, we get

$$\begin{cases}(|\xi|^2-\partial_3^2)\hat u_h+\hat u_h^\perp+i\xi\hat p=0,\\ (|\xi|^2-\partial_3^2)\hat u_3+\partial_3\hat p=0,\\ i\xi\cdot\hat u_h+\partial_3\hat u_3=0,\end{cases} \tag{2-3}$$

and

$$\hat u|_{x_3=0}=0.$$

Eliminating the pressure, we obtain

$$(|\xi|^2-\partial_3^2)^2\hat u_3-i\partial_3\xi^\perp\cdot\hat u_h=0.$$

Taking the scalar product of the first equation in (2-3) with $(\xi^\perp,0)$ and using the divergence-free condition, we are led to

$$(|\xi|^2-\partial_3^2)^3\hat u_3-\partial_3^2\hat u_3=0. \tag{2-4}$$

Notice that the solutions of this equation have a slightly different nature when $\xi\neq0$ or when $\xi=0$ (if $\xi=0$, the associated characteristic polynomial has a multiple root at zero). Therefore, as in [Gérard-Varet and Masmoudi 2010], we introduce a function $\varphi=\varphi(\xi)\in\mathscr{C}_0^\infty(\mathbb{R}^2)$ such that the support of φ does not contain zero. Then $\varphi\hat u_3$ satisfies the same equation as $\hat u_3$, and vanishes in a neighborhood of $\xi=0$.

For $\xi \neq 0$, the solutions of (2-4) are linear combinations of $\exp(-\lambda_k x_3)$ (with coefficients depending on ξ), where $(\lambda_k)_{1\le k\le 6}$ are the complex valued solutions of the equation

$$(\lambda^2 - |\xi|^2)^3 + \lambda^2 = 0. \tag{2-5}$$

Notice that none of the roots of this equation are purely imaginary, and that if λ is a solution of (2-5), so are $-\lambda$, $\bar\lambda$ and $-\bar\lambda$. Additionally, (2-5) has exactly one real-valued positive solution. Therefore, without loss of generality, we assume that λ_1, λ_2, λ_3 have strictly positive real part, while λ_4, λ_5, λ_6 have strictly negative real part, and $\lambda_1 \in \mathbb{R}$, $\bar\lambda_2 = \lambda_3$ with $\Im(\lambda_2) > 0$, $\Im(\lambda_3) < 0$.

On the other hand, the integrability condition on the gradient becomes

$$\int_{\mathbb{R}^3_+} (|\xi|^2 |\hat u(\xi, x_3)|^2 + |\partial_3 \hat u(\xi, x_3)|^2)\, d\xi\, dx_3 < \infty.$$

We infer immediately that $\varphi \hat u_3$ is a linear combination of $\exp(-\lambda_k x_3)$ for $1 \le k \le 3$: there exist

$$A_k : \mathbb{R}^2 \to \mathbb{C}^3 \quad \text{for } k = 1, 2, 3$$

such that

$$\varphi(\xi)\hat u_3(\xi, x_3) = \sum_{k=1}^{3} A_k(\xi) \exp(-\lambda_k(\xi) x_3).$$

Going back to (2-3), we also infer that

$$\begin{aligned} \varphi(\xi)\xi \cdot \hat u_h(\xi, x_3) &= -i \sum_{k=1}^{3} \lambda_k(\xi) A_k(\xi) \exp(-\lambda_k(\xi) x_3), \\ \varphi(\xi)\xi^\perp \cdot \hat u_h(\xi, x_3) &= i \sum_{k=1}^{3} \frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} A_k(\xi) \exp(-\lambda_k(\xi) x_3). \end{aligned} \tag{2-6}$$

Notice that, by (2-5),

$$\frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} = \frac{\lambda_k}{|\xi|^2 - \lambda_k^2} \quad \text{for } k = 1, 2, 3.$$

Thus the boundary condition $\hat u|_{x_3=0} = 0$ becomes

$$M(\xi) \begin{pmatrix} A_1(\xi) \\ A_2(\xi) \\ A_3(\xi) \end{pmatrix} = 0,$$

where

$$M := \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \dfrac{(|\xi|^2 - \lambda_1^2)^2}{\lambda_1} & \dfrac{(|\xi|^2 - \lambda_2^2)^2}{\lambda_2} & \dfrac{(|\xi|^2 - \lambda_3^2)^2}{\lambda_3} \end{pmatrix}.$$

Lemma 2.3. $$\det M = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(|\xi| + \lambda_1 + \lambda_2 + \lambda_3).$$

Since the proof of the result is a mere calculation, we have postponed it to Appendix A. It is then clear that M is invertible for all $\xi \neq 0$: indeed, it is easily checked that all the roots of (2-5) are simple, and we recall that λ_1, λ_2, λ_3 have positive real part.

We conclude that $A_1 = A_2 = A_3 = 0$, and thus $\varphi(\xi)\hat{u}(\xi, x_3) = 0$ for all $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ supported far from $\xi = 0$. Since $\hat{u} \in L^2(\mathbb{R}^2 \times (0, a))^3$ for all $a > 0$, we infer that $\hat{u} = 0$.

Existence. Now, given $v_0 \in H^{1/2}(\mathbb{R}^2)$, we define u through its Fourier transform in the horizontal variable. It is enough to define the Fourier transform for $\xi \neq 0$, since it is square integrable in ξ. Following the calculations above, we define coefficients A_1, A_2, A_3 by the equation

$$M(\xi)\begin{pmatrix} A_1(\xi) \\ A_2(\xi) \\ A_3(\xi) \end{pmatrix} = \begin{pmatrix} \hat{v}_{0,3} \\ i\xi \cdot \hat{v}_{0,h} \\ -i\xi^\perp \cdot \hat{v}_{0,h} \end{pmatrix} \quad \text{for all } \xi \neq 0. \tag{2-7}$$

As stated in Lemma 2.3, the matrix M is invertible, so that A_1, A_2, A_3 are well defined. We then set

$$\begin{aligned} \hat{u}_3(\xi, x_3) &:= \sum_{k=1}^{3} A_k(\xi) \exp(-\lambda_k(\xi) x_3), \\ \hat{u}_h(\xi, x_3) &:= \frac{i}{|\xi|^2} \sum_{k=1}^{3} A_k(\xi) \left(-\lambda_k(\xi)\xi + \frac{(|\xi|^2 - \lambda_k^2)^2}{\lambda_k} \xi^\perp \right) \exp(-\lambda_k(\xi) x_3). \end{aligned} \tag{2-8}$$

We have to check that the corresponding solution is sufficiently integrable, namely,

$$\begin{aligned} &\int_{\mathbb{R}^3_+} \left(|\xi|^2 |\hat{u}_h(\xi, x_3)|^2 + |\partial_3 \hat{u}_h(\xi, x_3)|^2 \right) d\xi \, dx_3 < \infty, \\ &\int_{\mathbb{R}^3_+} \left(|\xi|^2 |\hat{u}_3(\xi, x_3)|^2 + |\partial_3 \hat{u}_3(\xi, x_3)|^2 \right) d\xi \, dx_3 < \infty. \end{aligned} \tag{2-9}$$

Notice that by construction, $\partial_3 \hat{u}_3 = -i\xi \cdot \hat{u}_h$ (divergence-free condition), so that we only have to check three conditions.

To that end, we need to investigate the behavior of λ_k, A_k for ξ close to zero and for $\xi \to \infty$. We gather the results in the following lemma, whose proof is once again postponed to Appendix A:

Lemma 2.4. • *As $\xi \to \infty$, we have*

$$\begin{aligned} \lambda_1 &= |\xi| - \tfrac{1}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}), \\ \lambda_2 &= |\xi| - \frac{j^2}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}), \\ \lambda_3 &= |\xi| - \frac{j}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}), \end{aligned}$$

where $j = \exp(2i\pi/3)$, so that

$$\begin{pmatrix} A_1(\xi) \\ A_2(\xi) \\ A_3(\xi) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix} \begin{pmatrix} \hat{v}_{0,3} \\ -2|\xi|^{1/3}(i\xi \cdot \hat{v}_{0,h} - |\xi|\hat{v}_{0,3}) + O(|\hat{v}_0|) \\ -|\xi|^{-1/3} i\xi^\perp \cdot \hat{v}_{0,h} + O(|\hat{v}_0|) \end{pmatrix}. \tag{2-10}$$

- *As* $\xi \to 0$, *we have*

$$\lambda_1 = |\xi|^3 + O(|\xi|^7),$$
$$\lambda_2 = e^{i\pi/4} + O(|\xi|^2),$$
$$\lambda_3 = e^{-i\pi/4} + O(|\xi|^2).$$

As a consequence, for ξ *close to zero,*

$$\begin{aligned} A_1(\xi) &= \hat{v}_{0,3}(\xi) - \tfrac{1}{2}\sqrt{2}(i\xi\cdot\hat{v}_{0,h} + i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3}) + O(|\xi|^2|\hat{v}_0(\xi)|),\\ A_2(\xi) &= \tfrac{1}{2}(e^{-i\pi/4}i\xi\cdot\hat{v}_{0,h} + e^{i\pi/4}(i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3})) + O(|\xi|^2|\hat{v}_0(\xi)|),\\ A_3(\xi) &= \tfrac{1}{2}(e^{i\pi/4}i\xi\cdot\hat{v}_{0,h} + e^{-i\pi/4}(i\xi^\perp\hat{v}_{0,h} + |\xi|\hat{v}_{0,3})) + O(|\xi|^2|\hat{v}_0(\xi)|). \end{aligned} \tag{2-11}$$

- *For all* $a \geq 1$, *there exists a constant* $C_a > 0$ *such that*

$$a^{-1} \leq |\xi| \leq a \quad \Longrightarrow \quad \begin{cases} |\lambda_k(\xi)| + |\Re(\lambda_k(\xi))|^{-1} \leq C_a, \\ |A(\xi)| \leq C_a|\hat{v}_0(\xi)|. \end{cases}$$

We then decompose each integral in (2-9) into three pieces, one on $\{|\xi| > a\}$, one on $\{|\xi| < a^{-1}\}$, and the last one on $\{|\xi| \in (a^{-1}, a)\}$. All the integrals on $\{a^{-1} \leq |\xi| \leq a\}$ are bounded by

$$C_a \int_{a^{-1}<|\xi|<a} |\hat{v}_0(\xi)|^2\, d\xi \leq C_a\|v_0\|^2_{H^{1/2}(\mathbb{R}^2)}.$$

We thus focus on the two other pieces. We only treat the term

$$\int_{\mathbb{R}^3_+} |\xi|^2|\hat{u}_3(\xi, x_3)|^2\, d\xi\, dx_3,$$

since the two other terms can be evaluated using similar arguments.

$\triangleright$ On the set $\{|\xi| > a\}$, the difficulty comes from the fact that the contributions of the three exponentials compensate one another; hence a rough estimate is not possible. To simplify the calculations, we set

$$\begin{aligned} B_1 &= A_1 + A_2 + A_3,\\ B_2 &= A_1 + j^2A_2 + jA_3,\\ B_3 &= A_1 + jA_2 + j^2A_3, \end{aligned} \tag{2-12}$$

so that

$$\begin{pmatrix} A_1\\ A_2\\ A_3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1\\ 1 & j & j^2\\ 1 & j^2 & j \end{pmatrix}\begin{pmatrix} B_1\\ B_2\\ B_3 \end{pmatrix}.$$

Hence we have $A_k = (B_1 + \alpha_k B_2 + \alpha_k^2 B_3)/3$, where $\alpha_1 = 1$, $\alpha_2 = j$, $\alpha_3 = j^2$. Notice that $\alpha_k^3 = 1$ and $\sum_k \alpha_k = 0$. According to Lemma 2.4,

$$\begin{aligned} B_1 &= \hat{v}_{0,3},\\ B_2 &= -2|\xi|^{\frac{1}{3}}(i\xi\cdot\hat{v}_{0,h} - |\xi|\hat{v}_{0,3}) + O(|\hat{v}_0|),\\ B_3 &= -|\xi|^{-\frac{1}{3}}i\xi^\perp\cdot\hat{v}_{0,h} + O(|\hat{v}_0|). \end{aligned}$$

For all $\xi \in \mathbb{R}^2$, $|\xi| > a$, we have

$$|\xi|^2 \int_0^\infty |\hat{u}_3(\xi, x_3)|^2 \, dx_3 = |\xi|^2 \sum_{1\le k,l\le 3} A_k \bar{A}_l \frac{1}{\lambda_k + \bar{\lambda}_l}.$$

Using the asymptotic expansions in Lemma 2.4, we infer that

$$\frac{1}{\lambda_k + \bar{\lambda}_l} = \frac{1}{2|\xi|}\left(1 + \frac{\alpha_k^2 + \bar{\alpha}_l^2}{2}|\xi|^{-4/3} + O(|\xi|^{-8/3})\right).$$

Therefore, we obtain, for $|\xi| \gg 1$,

$$\begin{aligned} |\xi|^2 \sum_{1\le k,l\le 3} A_k \bar{A}_l \frac{1}{\lambda_k + \bar{\lambda}_l} &= \frac{|\xi|}{2} \sum_{1\le k,l\le 3} A_k \bar{A}_l \left(1 + \frac{\alpha_k^2 + \bar{\alpha}_l^2}{2}|\xi|^{-4/3} + O(|\xi|^{-8/3})\right) \\ &= \frac{|\xi|}{2}\big(|B_1|^2 + \tfrac{1}{2}(B_2\bar{B}_1 + \bar{B}_2 B_1)|\xi|^{-4/3} + O(|\hat{v}_0|^2)\big) \\ &= O(|\xi||\hat{v}_0|^2). \end{aligned}$$

Hence, since $v_0 \in H^{1/2}(\mathbb{R}^2)$, we deduce that

$$\int_{|\xi|>a} \int_0^\infty |\xi|^2 |\hat{u}_3|^2 \, dx_3 \, d\xi < +\infty.$$

$\triangleright$ On the set $|\xi| \le a$, we can use a crude estimate: we have

$$\int_{|\xi|\le a} \int_0^\infty |\xi|^2 |\hat{u}_3(\xi, x_3)|^2 \, dx_3 \, d\xi \le C \sum_{k=1}^3 \int_{|\xi|\le a} |\xi|^2 \frac{|A_k(\xi)|^2}{2\Re(\lambda_k(\xi))} \, d\xi.$$

Using the estimates of Lemma 2.4, we infer that

$$\begin{aligned} \int_{|\xi|\le a} \int_0^\infty |\xi|^2 |\hat{u}_3(\xi, x_3)|^2 \, dx_3 \, d\xi &\le C \int_{|\xi|\le a} |\xi|^2 \left((|\hat{v}_{0,3}(\xi)|^2 + |\xi|^2 |\hat{v}_{0,h}(\xi)|^2) \frac{1}{|\xi|^3} + |\xi|^2 |\hat{v}_0(\xi)|^2 \right) d\xi \\ &\le C \int_{|\xi|\le a} \left(\frac{|\hat{v}_{0,3}(\xi)|^2}{|\xi|} + |\xi| |\hat{v}_{0,h}(\xi)|^2 \right) d\xi < \infty, \end{aligned}$$

thanks to the assumption (2-2) on $\hat{v}_{0,3}$. In a similar way, we have

$$\begin{gathered} \int_{|\xi|\le a} \int_0^\infty |\xi|^2 |\hat{u}_h(\xi, x_3)|^2 \, dx_3 \, d\xi \le C \int_{|\xi|\le a} \left(\frac{|\hat{v}_{0,3}(\xi)|^2}{|\xi|} + |\xi| |\hat{v}_{0,h}(\xi)|^2 \right) d\xi, \\ \int_{|\xi|\le a} \int_0^\infty |\partial_3 \hat{u}_h(\xi, x_3)|^2 dx_3 \, d\xi \le C \int_{|\xi|\le a} |\hat{v}_0|^2 \, d\xi. \end{gathered}$$

Gathering all the terms, we deduce that

$$\int_{\mathbb{R}^3_+} (|\xi|^2 |\hat{u}(\xi, x_3)|^2 + |\partial_3 \hat{u}(\xi, x_3)|^2) \, d\xi \, dx_3 < \infty,$$

so that $\nabla u \in L^2(\mathbb{R}^3_+)$. □

Remark 2.5. Notice that thanks to the exponential decay in Fourier space, for all $p \in \mathbb{N}$ with $p \geq 2$, there exists a constant $C_p > 0$ such that

$$\int_1^\infty \int_{\mathbb{R}^2} |\nabla^p u|^2 \leq C_p \|v_0\|^2_{H^{1/2}}.$$

- We now extend the definition of a solution to boundary data in $H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$. We introduce the sets

$$\begin{aligned} \mathcal{H} &:= \{u \in H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2) : \exists U_h \in H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)^2,\ u = \nabla_h \cdot U_h\}, \\ \mathbb{K} &:= \{u \in H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)^3 : u_3 \in \mathcal{H}\}. \end{aligned} \tag{2-13}$$

In order to extend the definition of solutions to data which are only locally square integrable, we will first derive a representation formula for $v_0 \in H^{1/2}(\mathbb{R}^2)$. We will prove that the formula still makes sense when $v_0 \in \mathbb{K}$, and this will allow us to define a solution with boundary data in $\mathbb{K}$.

To that end, let us introduce some notation. According to the proof of Proposition 2.1, there exist $L_1, L_2, L_3 : \mathbb{R}^2 \to \mathcal{M}_3(\mathbb{C})$ and $q_1, q_2, q_3 : \mathbb{R}^2 \to \mathbb{C}^3$ such that

$$\begin{aligned} \hat{u}(\xi, x_3) &= \sum_{k=1}^3 L_k(\xi)\hat{v}_0(\xi)\exp(-\lambda_k(\xi)x_3), \\ \hat{p}(\xi, x_3) &= \sum_{k=1}^3 q_k(\xi) \cdot \hat{v}_0(\xi)\exp(-\lambda_k(\xi)x_3). \end{aligned} \tag{2-14}$$

For further reference, we state the following lemma:

Lemma 2.6. *For all $k \in \{1, 2, 3\}$ and all $\xi \in \mathbb{R}^2$, the following identities hold*:

$$(|\xi|^2 - \lambda_k^2)L_k + \begin{pmatrix} -L_{k,21} & -L_{k,22} & -L_{k,23} \\ L_{k,11} & L_{k,12} & L_{k,13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_1 q_{k,2} & i\xi_1 q_{k,3} \\ i\xi_2 q_{k,1} & i\xi_2 q_{k,2} & i\xi_2 q_{k,3} \\ -\lambda_k q_{k,1} & -\lambda_k q_{k,2} & -\lambda_k q_{k,3} \end{pmatrix} = 0$$

and, for $j = 1, 2, 3,\ k = 1, 2, 3$,

$$i\xi_1 L_{k,1j} + i\xi_2 L_{k,2j} - \lambda_k L_{k,3j} = 0.$$

Proof. Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ be such that $v_{0,3} = \nabla_h \cdot V_h$ for some $V_h \in H^{1/2}(\mathbb{R}^2)$. Then, according to Proposition 2.1, the couple (u, p) defined by (2-14) is a solution of (2-1). Therefore it satisfies (2-3). Plugging the definition (2-14) into (2-3), we infer that, for all $x_3 > 0$,

$$\int_{\mathbb{R}^2} \sum_{k=1}^3 \exp(-\lambda_k x_3)\mathcal{A}_k(\xi)\hat{v}_0(\xi)\, d\xi = 0, \tag{2-15}$$

where

$$\mathcal{A}_k := (|\xi|^2 - \lambda_k^2)L_k + \begin{pmatrix} -L_{k,21} & -L_{k,22} & -L_{k,23} \\ L_{k,11} & L_{k,12} & L_{k,13} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_1 q_{k,2} & i\xi_1 q_{k,3} \\ i\xi_2 q_{k,1} & i\xi_2 q_{k,2} & i\xi_2 q_{k,3} \\ -\lambda_k q_{k,1} & -\lambda_k q_{k,2} & -\lambda_k q_{k,3} \end{pmatrix}.$$

Since (2-15) holds for all v_0, we obtain

$$\sum_{k=1}^{3} \exp(-\lambda_k x_3)\mathscr{A}_k(\xi) = 0 \quad \text{for all } \xi \text{ and } x_3,$$

and since λ_1,λ_2, λ_3 are distinct for all $\xi \neq 0$, we deduce eventually that $\mathscr{A}_k(\xi) = 0$ for all ξ and k.

The second identity follows in a similar fashion from the divergence-free condition. □

Our goal is now to derive a representation formula for u, based on the formula satisfied by its Fourier transform, in such a way that the formula still makes sense when $v_0 \in \mathbb{K}$. The crucial part is to understand the action of the operators $\mathrm{Op}(L_k(\xi)\phi(\xi))$ on L^2_{uloc} functions, where $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. To that end, we will need to decompose $L_k(\xi)$ for ξ close to zero into several terms.

Lemma 2.4 provides asymptotic developments of L_1, L_2, L_3 and α_1, α_2, α_3 as $|\xi| \ll 1$ or $|\xi| \gg 1$. In particular, we have, for $|\xi| \ll 1$,

$$L_1(\xi) = \frac{\sqrt{2}}{2|\xi|}\begin{pmatrix} \xi_2(\xi_2-\xi_1) & -\xi_2(\xi_2+\xi_1) & -i\sqrt{2}\xi_2 \\ \xi_1(\xi_1-\xi_2) & \xi_1(\xi_2+\xi_1) & i\sqrt{2}\xi_1 \\ i|\xi|(\xi_2-\xi_1) & -i|\xi|(\xi_2+\xi_1) & \sqrt{2}|\xi| \end{pmatrix} + \big(O(|\xi|^2)\ \ O(|\xi|^2)\ \ O(|\xi|)\big), \tag{2-16}$$

$$L_2(\xi) = \frac{1}{2}\begin{pmatrix} 1 & i & \dfrac{2i(-\xi_1+\xi_2)}{|\xi|} \\ -i & 1 & \dfrac{-2i(\xi_1+\xi_2)}{|\xi|} \\ i(\xi_1 e^{-i\pi/4}-\xi_2 e^{i\pi/4}) & i(\xi_2 e^{-i\pi/4}+i\xi_1 e^{i\pi/4}) & e^{i\pi/4} \end{pmatrix} + \big(O(|\xi|^2)\ \ O(|\xi|^2)\ \ O(|\xi|)\big),$$

$$L_3(\xi) = \frac{1}{2}\begin{pmatrix} 1 & -i & \dfrac{2i(\xi_1+\xi_2)}{|\xi|} \\ i & 1 & \dfrac{-2i(\xi_1-\xi_2)}{|\xi|} \\ i(\xi_1 e^{i\pi/4}-\xi_2 e^{-i\pi/4}) & i(\xi_2 e^{i\pi/4}+i\xi_1 e^{-i\pi/4}) & e^{-i\pi/4} \end{pmatrix} + \big(O(|\xi|^2)\ \ O(|\xi|^2)\ \ O(|\xi|)\big).$$

The remainder terms are to be understood column-wise. Notice that the third column of L_k, that is, $L_k e_3$, always acts on $\hat{v}_{0,3} = i\xi \cdot \widehat{V}_h$. We thus introduce the following notation: for $k = 1, 2, 3$,

$$M_k := (L_k e_1 L_k e_2) \in \mathscr{M}_{3,2}(\mathbb{C}) \quad \text{and} \quad N_k := i L_k e_3\, {}^t\xi \in \mathscr{M}_{3,2}(\mathbb{C}).$$

M_k^1 (respectively N_k^1) denotes the 3×2 matrix whose coefficients are the nonpolynomial and homogeneous terms of order one in M_k (respectively N_k) for ξ close to zero. For instance,

$$M_1^1 := \frac{\sqrt{2}}{2|\xi|}\begin{pmatrix} \xi_2(\xi_2-\xi_1) & -\xi_2(\xi_2+\xi_1) \\ -\xi_1(\xi_2-\xi_1) & \xi_1(\xi_2+\xi_1) \\ 0 & 0 \end{pmatrix}, \qquad N_1^1 := \frac{i}{|\xi|}\begin{pmatrix} -\xi_2\xi_1 & \xi_2^2 \\ \xi_1^2 & \xi_1\xi_2 \\ 0 & 0 \end{pmatrix}.$$

We also set $M_k^{\text{rem}} = M_k - M_k^1$, $N_k^{\text{rem}} := N_k - N_k^1$ so that

$$\begin{aligned} &\text{for } \xi \text{ close to zero,} && M_1^{\text{rem}} = O(|\xi|),\\ &\text{for } k = 2, 3, && M_k^{\text{rem}} = O(1),\\ &\text{for all } k \in \{1, 2, 3\}, && N_k^{\text{rem}} = O(|\xi|). \end{aligned}$$

There are polynomial terms of order one in M_1^{rem} and N_k^{rem} (respectively of order 0 and 1 in M_k^{rem} for $k = 2, 3$) which account for the fact that the remainder terms are not $O(|\xi|^2)$. However, these polynomial terms do not introduce any singularity when they are differentiated, and thus, using the results of Appendix B, we get, for any integer $q \geq 1$,

$$|\nabla_\xi^q M_k^{\text{rem}}|,\ |\nabla_\xi^q N_k^{\text{rem}}| = O(|\xi|^{2-q} + 1) \quad \text{for } |\xi| \ll 1. \tag{2-17}$$

▷ Concerning the Fourier multipliers of order one M_k^1 and N_k^1, we will rely on the following lemma, which is proved in Appendix C:

Lemma 2.7. *There exists a constant C_I such that for all $i, j \in \{1, 2\}$, for any function $g \in \mathcal{S}(\mathbb{R}^2)$, for all $\zeta \in \mathscr{C}_0^\infty(\mathbb{R}^2)$, and for all $K > 0$,*

$$\begin{aligned} &\operatorname{Op}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\right)g(x)\\ &= C_I \int_{\mathbb{R}^2} dy \left[\frac{\delta_{i,j}}{|x-y|^3} - 3\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}\right] \times \{\rho*g(x) - \rho*g(y) - \nabla\rho*g(x)\cdot(x-y)\mathbf{1}_{|x-y|\leq K}\}, \end{aligned} \tag{2-18}$$

where $\rho := \mathcal{F}^{-1}\zeta \in \mathcal{S}(\mathbb{R}^2)$.

Definition 2.8. If L is a homogeneous, nonpolynomial function of order one in $\mathbb{R}^2$ of the form

$$L(\xi) = \sum_{1\leq i,j\leq 2} a_{ij}\frac{\xi_i\xi_j}{|\xi|},$$

then we define, for $\varphi \in W^{2,\infty}(\mathbb{R}^2)$,

$$\mathcal{I}[L]\varphi(x) := \sum_{1\leq i,j\leq 2} a_{ij}\int_{\mathbb{R}^2} dy\, \gamma_{ij}(x-y)\{\varphi(x) - \varphi(y) - \nabla\varphi(x)\cdot(x-y)\mathbf{1}_{|\mathbf{x}-\mathbf{y}|\leq\mathbf{K}}\},$$

where

$$\gamma_{i,j}(x) = C_I\left(\frac{\delta_{i,j}}{|x|^3} - 3\frac{x_ix_j}{|x|^5}\right).$$

Remark 2.9. The value of the number K in the formula (2-18) and in Definition 2.8 is irrelevant, since, for all $\varphi \in W^{2,\infty}(\mathbb{R}^2)$ and all $0 < K < K'$,

$$\int_{\mathbb{R}^2} dy\, \gamma_{ij}(x-y)\nabla\varphi(x)\cdot(x-y)\mathbf{1}_{\mathbf{K}<|\mathbf{x}-\mathbf{y}|\leq\mathbf{K}'} = 0$$

by symmetry arguments.

We then have the following bound:

Lemma 2.10. *Let $\varphi \in W^{2,\infty}(\mathbb{R}^2)$. Then, for all $1 \le i, j \le 2$,*

$$\left\| \mathcal{I}\left[\frac{\xi_i \xi_j}{|\xi|}\right]\varphi \right\|_{L^\infty(\mathbb{R}^2)} \le C\|\varphi\|_\infty^{1/2}\|\nabla^2\varphi\|_\infty^{1/2}.$$

Remark 2.11. We will often apply the above Lemma with $\varphi = \rho * g$, where $\rho \in \mathscr{C}^2(\mathbb{R}^2)$ is such that ρ and $\nabla^2\rho$ have bounded second order moments in L^2, and $g \in L^2_{\mathrm{uloc}}(\mathbb{R}^2)$. In this case, we have

$$\|\varphi\|_\infty \le C\|g\|_{L^2_{\mathrm{uloc}}}\|(1+|\cdot|^2)\rho\|_{L^2(\mathbb{R}^2)},$$
$$\|\nabla^2\varphi\|_\infty \le C\|g\|_{L^2_{\mathrm{uloc}}}\|(1+|\cdot|^2)\nabla^2\rho\|_{L^2(\mathbb{R}^2)}.$$

Indeed,

$$\begin{aligned}\|\rho * g\|_{L^\infty} &\le \sup_{x\in\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{1}{1+|x-y|^4}|g(y)|^2\,dy\right)^{\frac12}\left(\int_{\mathbb{R}^2}(1+|x-y|^4)|\rho(x-y)|^2\,dy\right)^{\frac12}\\ &\le C\|g\|_{L^2_{\mathrm{uloc}}}\|(1+|\cdot|^2)\rho\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

The L^∞ norm of $\nabla^2\varphi$ is estimated in exactly the same manner, simply replacing ρ by $\nabla^2\rho$.

Proof of Lemma 2.10. We split the integral in (2-18) into three parts:

$$\begin{aligned}\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\varphi(x) &= \int_{|x-y|\le K} dy\,\gamma_{ij}(x-y)\{\varphi(x)-\varphi(y)-\nabla\varphi(x)\cdot(x-y)\}\\ &\qquad + \int_{|x-y|\ge K} dy\,\gamma_{ij}(x-y)\varphi(x) - \int_{|x-y|\ge K} dy\,\gamma_{ij}(x-y)\varphi(y)\\ &= A(x)+B(x)+C(x).\end{aligned} \tag{2-19}$$

Concerning the first integral in (2-19), Taylor's formula implies

$$|A(x)| \le C\|\nabla^2\varphi\|_{L^\infty}\int_{|x-y|\le K}\frac{dy}{|x-y|} \le CK\|\nabla^2\varphi\|_{L^\infty}.$$

For the second and third integrals in (2-19),

$$|B(x)|+|C(x)| \le C\|\varphi\|_\infty\int_{|x-y|\ge K}\frac{dy}{|x-y|^3} \le CK^{-1}\|\varphi\|_\infty.$$

We infer that, for all $K>0$,

$$\left\|\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\varphi\right\|_\infty \le C(K\|\nabla^2\varphi\|_\infty + K^{-1}\|\varphi\|_\infty).$$

Optimizing in K (that is, choosing $K = \|\varphi\|_\infty^{1/2}/\|\nabla^2\varphi\|_\infty^{1/2}$), we obtain the desired inequality. □

▷ For the remainder terms M_k^{rem} and N_k^{rem} as well as the high-frequency terms, we will use the following estimates:

Lemma 2.12 (kernel estimates). *Let $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be such that $\phi(\xi) = 1$ for $|\xi| \le 1$. Define*

$$\varphi_{\mathrm{HF}}(x_h, x_3) := \mathscr{F}^{-1}\left(\sum_{k=1}^{3}(1-\phi)(\xi) L_k(\xi)\exp(-\lambda_k(\xi)x_3)\right),$$

$$\psi_1(x_h, x_3) := \mathscr{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi) M_k^{\mathrm{rem}}(\xi)\exp(-\lambda_k(\xi)x_3)\right),$$

$$\psi_2(x_h, x_3) := \mathscr{F}^{-1}\left(\sum_{k=1}^{3}\phi(\xi) N_k^{\mathrm{rem}}(\xi)\exp(-\lambda_k(\xi)x_3)\right).$$

Then the following estimates hold:

- *For all $q \in \mathbb{N}$, there exists $c_{0,q} > 0$ such that, for all $\alpha, \beta > c_{0,q}$, there exists $C_{\alpha,\beta,q} > 0$ such that*

$$|\nabla^q \varphi_{\mathrm{HF}}(x_h, x_3)| \le \frac{C_{\alpha,\beta,q}}{|x_h|^\alpha + |x_3|^\beta}.$$

- *For all $\alpha \in (0, 2/3)$ and all $q \in \mathbb{N}$, there exists $C_{\alpha,q} > 0$ such that*

$$|\nabla^q \psi_1(x_h, x_3)| \le \frac{C_{\alpha,q}}{|x_h|^{3+q} + |x_3|^{\alpha+q/3}}.$$

- *For all $\alpha \in (0, 2/3)$ and all $q \in \mathbb{N}$, there exists $C_{\alpha,q} > 0$ such that*

$$|\nabla^q \psi_2(x_h, x_3)| \le \frac{C_{\alpha,q}}{|x_h|^{3+q} + |x_3|^{\alpha+q/3}}.$$

Proof. • Let us first derive the estimate on φ_{HF} for $q = 0$. We seek to prove there exists $c_0 > 0$ such that

$$\text{for all } (\alpha, \beta) \in (c_0, \infty)^2, \quad \text{there exists } C_{\alpha,\beta} \text{ such that } |\varphi_{\mathrm{HF}}(x_h, x_3)| \le \frac{C_{\alpha,\beta}}{|x_h|^\alpha + |x_3|^\beta}. \tag{2-20}$$

To that end, it is enough to show that, for $\alpha \in \mathbb{N}^2$ and $\beta > 0$ with $|\alpha|, \beta \ge c_0$,

$$\sup_{x_3>0}\left(|x_3|^\beta \|\widehat{\varphi_{\mathrm{HF}}}(\,\cdot\,, x_3)\|_{L^1(\mathbb{R}^2)} + \|\nabla_\xi^\alpha \widehat{\varphi_{\mathrm{HF}}}(\,\cdot\,, x_3)\|_{L^1(\mathbb{R}^2)}\right) < \infty.$$

We recall that $\lambda_k(\xi) \sim |\xi|$ for $|\xi| \to \infty$. Moreover, using the estimates of Lemma 2.4, we infer that there exists $\gamma \in \mathbb{R}$ such that $L_k(\xi) = O(|\xi|^\gamma)$ for $|\xi| \gg 1$. Hence

$$\begin{aligned}
|x_3|^\beta |\hat{\varphi}_{\mathrm{HF}}(\xi, x_3)| &\le C|1-\phi(\xi)|\,|\xi|^\gamma \sum_{k=1}^{3}|x_3|^\beta \exp(-\Re(\lambda_k)x_3) \\
&\le C|1-\phi(\xi)|\,|\xi|^{\gamma-\beta} \sum_{k=1}^{3}|\Re(\lambda_k)x_3|^\beta \exp(-\Re(\lambda_k)x_3) \\
&\le C_\beta |\xi|^{\gamma-\beta}\mathbf{1}_{|\xi|\ge 1}.
\end{aligned}$$

Hence, for β large enough, for all $x_3 > 0$,

$$|x_3|^\beta \|\hat{\varphi}_{\mathrm{HF}}(\,\cdot\,, x_3)\|_{L^1(\mathbb{R}^2)} \le C_\beta.$$

In a similar fashion, for $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 1$, we have, as $|\xi| \to \infty$ (see Appendix B),

$$\nabla^\alpha L_k(\xi) = O(|\xi|^{\gamma-|\alpha|}),$$
$$\nabla^\alpha(\exp(-\lambda_k x_3)) = O((|\xi|^{1-|\alpha|}x_3 + |x_3|^{|\alpha|})\exp(-\Re(\lambda_k)x_3)) = O(|\xi|^{-|\alpha|}).$$

Moreover, we recall that $\nabla(1-\phi)$ is supported in a ring of the type $B_R \setminus B_1$ for some $R > 1$. As a consequence, we obtain, for all $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 1$,

$$|\nabla^\alpha \widehat{\varphi_{\mathrm{HF}}}(\xi, x_3)| \leq C_\alpha |\xi|^{\gamma-|\alpha|}\mathbf{1}_{|\xi|\geq 1},$$

so that

$$\|\nabla^\alpha \widehat{\varphi_{\mathrm{HF}}}(\,\cdot\,, x_3)\|_{L^1(\mathbb{R}^2)} \leq C_\alpha.$$

Thus φ_{HF} satisfies (2-20) for $q = 0$. For $q \geq 1$, the proof is the same, changing L_k into $|\xi|^{q_1}|\lambda_k|^{q_2}L_k$ with $q_1 + q_2 = q$.

• The estimates on ψ_1, ψ_2 are similar. The main difference lies in the degeneracy of λ_1 near zero. For instance, in order to derive an L^∞ bound on $|x_3|^{\alpha+q/3}\nabla^q\psi_1$, we look for an $L^\infty_{x_3}(L^1_\xi(\mathbb{R}^2))$ bound on $|x_3|^{\alpha+q/3}|\xi|^q\hat{\psi}_1(\xi, x_3)$. We have

$$\begin{aligned}\left| |x_3|^{\alpha+q/3}|\xi|^q\phi(\xi)\sum_{k=1}^{3} M_k^{\mathrm{rem}}\exp(-\lambda_k x_3)\right| &\leq C|x_3|^{\alpha+q/3}|\xi|^q\sum_{k=1}^{3}\exp(-\Re(\lambda_k)x_3)|M_k^{\mathrm{rem}}|\mathbf{1}_{|\xi|\leq \mathbf{R}}\\ &\leq C|\xi|^q\sum_{k=1}^{3}|\Re\lambda_k|^{-(\alpha+q/3)}|M_k^{\mathrm{rem}}|\mathbf{1}_{|\xi|\leq \mathbf{R}}\\ &\leq C|\xi|^q(|\xi|^{1-3\alpha-q}+1)\mathbf{1}_{|\xi|\leq \mathbf{R}}.\end{aligned}$$

The right-hand side is in L^1 provided $\alpha < 2/3$. We infer that

$$\left||x_3|^{\alpha+q/3}\nabla^q\psi_1(x)\right| \leq C_{\alpha,q} \quad \text{for all } x \text{ and all } \alpha \in (0, 2/3).$$

The other bound on ψ_1 is derived in a similar way, using the fact that

$$\nabla_\xi^q M_1^{\mathrm{rem}} = O(|\xi|^{2-q}+1)$$

for ξ in a neighborhood of zero. □

▷ We are now ready to state our representation formula:

Proposition 2.13 (representation formula). *Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ be such that $v_{0,3} = \nabla_h \cdot V_h$ for some $V_h \in H^{1/2}(\mathbb{R}^2)$, and let u be the solution of (2-1). For all $x \in \mathbb{R}^3$, let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be such that $\chi \equiv 1$ on $B(x_h, 1)$. Let $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be a cut-off function as in Lemma 2.12, and let φ_{HF}, ψ_1, ψ_2 be the associated kernels. For $k = 1, 2, 3$, set*

$$f_k(\,\cdot\,, x_3) := \mathscr{F}^{-1}(\phi(\xi)\exp(-\lambda_k x_3)).$$

Then

$$u(x)=\mathscr{F}^{-1}\Big(\sum_{k=1}^{3}L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\Big)(x)+\sum_{k=1}^{3}\mathscr{I}[M_k^1]f_k(\,\cdot\,,x_3)*((1-\chi)v_{0,h})(x)$$
$$+\sum_{k=1}^{3}\mathscr{I}[N_k^1]f_k(\,\cdot\,,x_3)*((1-\chi)V_h)(x)+\varphi_{\mathrm{HF}}*\begin{pmatrix}(1-\chi)v_{0,h}\\ \nabla\cdot((1-\chi)V_h)\end{pmatrix}(x)$$
$$+\psi_1*((1-\chi)v_{0,h})(x)+\psi_2*((1-\chi)V_h)(x).$$

As a consequence, for all $a>0$, there exists a constant C_a such that

$$\sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_0^a|u(x_h,x_3)|^2\,dx_3\,dx_h\le C_a(\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)}+\|V_h\|^2_{H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)}).$$

Moreover, there exists $q\in\mathbb{N}$ such that

$$\sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_1^\infty|\nabla^q u(x_h,x_3)|^2\,dx_3\,dx_h\le C(\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)}+\|V_h\|^2_{H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)}).$$

Remark 2.14. The integer q in the above proposition is explicit and does not depend on v_0. One can take $q=4$ for instance.

Proof. The proposition follows quite easily from the preceding lemmas. We have, according to Proposition 2.1,

$$u(x)=\mathscr{F}^{-1}\Big(\sum_{k=1}^{3}L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\Big)(x)$$
$$+\mathscr{F}^{-1}\Big(\sum_{k=1}^{3}L_k(\xi)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\Big)(x).$$

In the latter term, the cut-off function ϕ is introduced, writing simply $1=1-\phi+\phi$. We have, for the high-frequency term,

$$\mathscr{F}^{-1}\Big(\sum_{k=1}^{3}(1-\phi(\xi))L_k(\xi)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\Big)$$
$$=\mathscr{F}^{-1}\Big(\hat{\varphi}_{\mathrm{HF}}(\xi,x_3)\begin{pmatrix}\widehat{(1-\chi)v_{0,h}}(\xi)\\ \widehat{\nabla\cdot((1-\chi)V_h)}(\xi)\end{pmatrix}\Big)=\varphi_{\mathrm{HF}}(\,\cdot\,,x_3)*\begin{pmatrix}(1-\chi)v_{0,h}(\xi)\\ \nabla\cdot((1-\chi)V_h)(\xi)\end{pmatrix}$$

Notice that $\nabla_h\cdot((1-\chi)V_h)=(1-\chi)v_{0,3}-\nabla_h\chi\cdot V_h\in H^{1/2}(\mathbb{R}^2)$.

In the low-frequency terms, we distinguish between the horizontal and the vertical components of v_0. Let us deal with the vertical component, which is slightly more complicated. Since $v_{0,3} = \nabla_h \cdot V_h$, we have

$$\mathcal{F}^{-1}\left(\sum_{k=1}^{3} \phi(\xi) L_k(\xi) e_3 \widehat{\nabla_h \cdot ((1-\chi)V_h)}(\xi) \exp(-\lambda_k x_3)\right) \\ = \mathcal{F}^{-1}\left(\sum_{k=1}^{3} \phi(\xi) L_k(\xi) e_3 i\xi \cdot \widehat{(1-\chi)V_h}(\xi) \exp(-\lambda_k x_3)\right).$$

We recall that $N_k = i L_k e_3 {}^t\xi$, so that

$$L_k(\xi) e_3 i\xi \cdot \widehat{(1-\chi)V_h}(\xi) = N_k(\xi)\widehat{(1-\chi)V_h}(\xi).$$

Then, by definition of ψ_2 and f_k,

$$\begin{aligned}
&\mathcal{F}^{-1}\left(\sum_{k=1}^{3} \phi(\xi) N_k(\xi) \widehat{(1-\chi)V_h}(\xi) \exp(-\lambda_k x_3)\right) \\
&\quad= \mathcal{F}^{-1}\left(\sum_{k=1}^{3} \phi(\xi) N_k^1(\xi) \widehat{(1-\chi)V_h}(\xi) \exp(-\lambda_k x_3)\right) \\
&\qquad\qquad + \mathcal{F}^{-1}\left(\sum_{k=1}^{3} \phi(\xi) N_k^{\mathrm{rem}}(\xi) \widehat{(1-\chi)V_h}(\xi) \exp(-\lambda_k x_3)\right) \\
&\quad= \sum_{k=1}^{3} \mathcal{I}[N_k^1] f_k * ((1-\chi)\cdot V_h) + \mathcal{F}^{-1}(\hat{\psi}_2(\xi, x_3)\widehat{(1-\chi)\cdot V_h}(\xi)) \\
&\quad= \sum_{k=1}^{3} \mathcal{I}[N_k^1] f_k * ((1-\chi)\cdot V_h) + \psi_2 * ((1-\chi)\cdot V_h).
\end{aligned}$$

The representation formula follows.

There remains to bound every term occurring in the representation formula. In order to derive bounds on $(l + [0,1]^2) \times \mathbb{R}_+$ for some $l \in \mathbb{Z}^2$, we use the representation formula with a function $\chi_l \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $\chi_l \equiv 1$ on $l + [-1,2]^2$, and we assume that the derivatives of χ_l are bounded uniformly in l (take for instance $\chi_l = \chi(\cdot + l)$ for some $\chi \in \mathscr{C}_0^\infty$).

- According to Proposition 2.1, we have

$$\int_0^a \left\| \mathcal{F}^{-1}\left(\sum_{k=1}^{3} L_k(\xi) \begin{pmatrix} \widehat{\chi_l v_{0,h}}(\xi) \\ \nabla\cdot\widehat{(\chi_l V_h)} \end{pmatrix} \exp(-\lambda_k x_3)\right)\right\|_{L^2(\mathbb{R}^2)}^2 dx_3 \\ \le C_a(\|\chi_l v_{0,h}\|_{H^{1/2}}^2 + \|\nabla\chi_l \cdot V_h\|_{H^{1/2}}^2 + \|\chi_l v_{0,3}\|_{H^{1/2}(\mathbb{R}^2)}^2).$$

Using the formula

$$\|f\|_{H^{1/2}(\mathbb{R}^2)}^2 = \|f\|_{L^2}^2 + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^3}\, dx\, dy \quad \text{for all } f \in H^{1/2}(\mathbb{R}^2),$$

it can be easily proved that

$$\|\chi u\|_{H^{1/2}(\mathbb{R}^2)} \le C\|\chi\|_{W^{1,\infty}}\|u\|_{H^{1/2}(\mathbb{R}^2)} \tag{2-21}$$

for all $\chi \in W^{1,\infty}(\mathbb{R}^2)$ and for all $u \in H^{1/2}(\mathbb{R}^2)$, where the constant C only depends on the dimension. Therefore,

$$\|\chi_l v_{0,h}\|_{H^{1/2}} \le \sum_{k\in\mathbb{Z}^2} \|\chi_l \tau_k \vartheta v_{0,h}\|_{H^{1/2}} \le \sum_{\substack{k\in\mathbb{Z}^2\\ |k-l|\le 1+3\sqrt{2}}} \|\chi_l \tau_k \vartheta v_{0,h}\|_{H^{1/2}} \le C\|\chi_l\|_{W^{1,\infty}}\|v_{0,h}\|_{H^{1/2}_{\mathrm{uloc}}},$$

so that

$$\int_0^a \left\| \mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|^2_{L^2(\mathbb{R}^2)} dx_3 \le C_a(\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}} + \|V_h\|^2_{H^{1/2}_{\mathrm{uloc}}}).$$

Similarly,

$$\int_0^\infty \left\| \nabla\mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|^2_{L^2(\mathbb{R}^2)} dx_3 \le C(\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}} + \|V_h\|^2_{H^{1/2}_{\mathrm{uloc}}}).$$

Moreover, thanks to Remark 2.5, for any $q \ge 2$,

$$\int_1^\infty \left\| \nabla^q\mathcal{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi_l v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi_l V_h)}\end{pmatrix}\exp(-\lambda_k x_3)\right)\right\|^2_{L^2(\mathbb{R}^2)} dx_3 \le C_q(\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}} + \|V_h\|^2_{H^{1/2}_{\mathrm{uloc}}}).$$

• We now address the bounds of the terms involving the kernels φ_{HF}, ψ_1, ψ_2. According to Lemma 2.12, we have for instance, for all $x_3 > 0$, for all $x_h \in l + [0,1]^2$, for $\sigma \in \mathbb{N}^2$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \nabla^\sigma \varphi_{\mathrm{HF}}(y_h, x_3)\begin{pmatrix}(1-\chi_l)v_{0,h}\\ \nabla\cdot((1-\chi_l)V_h)\end{pmatrix}(x_h - y_h)\,dy_h\right| \\
&\qquad \le C_{\alpha,\beta,|\sigma|}\int_{|y_h|\ge 1} |v_0(x_h-y_h)|\frac{1}{|y_h|^\alpha + x_3^\beta}\,dy_h + C_{\alpha,\beta,|\sigma|}\int_{1\le|y_h|\le 2} |V_h(x_h-y_h)|\frac{1}{|y_h|^\alpha + x_3^\beta}\,dy_h \\
&\qquad \le C\|V_h\|_{L^2_{\mathrm{uloc}}}\frac{1}{1+x_3^\beta} + C\left(\int_{\mathbb{R}^2}\frac{|v_0(x_h-y_h)|^2}{1+|y_h|^\gamma}\,dy_h\right)^{\frac12}\left(\int_{|y_h|\ge 1}\frac{1+|y_h|^\gamma}{(|y_h|^\alpha + x_3^\beta)^2}\,dy_h\right)^{\frac12} \\
&\qquad \le C\|V_h\|_{L^2_{\mathrm{uloc}}}\frac{1}{1+x_3^\beta} + C\|v_0\|_{L^2_{\mathrm{uloc}}}\inf(1, x_3^{\beta((2+\gamma)/2\alpha-1)})
\end{aligned}$$

for all $\gamma > 2$ and for $\alpha, \beta > c_0$ and sufficiently large. In particular the $\dot{H}^q_{\mathrm{uloc}}$ bound follows. The local bounds in L^2_{uloc} near $x_3 = 0$ are immediate, since the right-hand side is uniformly bounded in x_3. The treatment of the terms with ψ_1, ψ_2 are analogous. Notice however that because of the slower decay of ψ_1, ψ_2 in x_3, we only have a uniform bound in $\dot{H}^q((l+[0,1]^2)\times(1,\infty))$ if q is large enough ($q \ge 2$ is sufficient).

• There remains to bound the terms involving $\mathcal{I}[M_k^1]$, $\mathcal{I}[N_k^1]$, using Lemma 2.7 and Remark 2.11. We have for instance, for all $x_3 > 0$,

$$\begin{aligned}\|\mathcal{I}[N_k^1] f_k * ((1-\chi_l)V_h)\|_{L^2(l+[0,1]^2)} \\ \le C\|V_h\|_{L^2_{\mathrm{uloc}}}\big(\|(1+|\cdot|^2) f_k(\cdot\,,x_3)\|_{L^2(\mathbb{R}^2)} + \|(1+|\cdot|^2)\nabla_h^2 f_k(\cdot\,,x_3)\|_{L^2(\mathbb{R}^2)}\big).\end{aligned}$$

Using the Plancherel formula, we infer

$$\begin{aligned}\|(1+|\cdot|^2) f_k(\cdot\,,x_3)\|_{L^2(\mathbb{R}^2)} &\le C\|\phi(\xi)\exp(-\lambda_k x_3)\|_{H^2(\mathbb{R}^2)} \\ &\le C\|\exp(-\lambda_k x_3)\|_{H^2(B_R)} + C\exp(-\mu x_3),\end{aligned}$$

where $R > 1$ is such that $\operatorname{Supp}\phi \subset B_R$ and μ is a positive constant depending only on ϕ. We have, for $k = 1, 2, 3$,

$$|\nabla^2 \exp(-\lambda_k x_3)| \le C(x_3|\nabla_\xi^2\lambda_k| + x_3^2|\nabla_\xi\lambda_k|^2)\exp(-\lambda_k x_3).$$

The asymptotic expansions in Lemma 2.4 together with the results of Appendix B imply that, for ξ in any neighborhood of zero,

$$\begin{aligned}&\nabla^2\lambda_1 = O(|\xi|), \quad \nabla\lambda_1 = O(|\xi|^2),\\ &\nabla^2\lambda_k = O(1), \quad \nabla\lambda_k = O(|\xi|) \quad \text{for } k = 2, 3.\end{aligned}$$

In particular, if $k = 2, 3$, since λ_k is bounded away from zero in a neighborhood of zero,

$$\int_0^\infty dx_3\|\exp(-\lambda_k x_3)\|_{H^2(B_R)}^2 < \infty.$$

On the other hand, the degeneracy of λ_1 near $\xi = 0$ prevents us from obtaining the same result. Notice however that

$$\int_0^a \|\exp(-\lambda_1 x_3)\|_{H^2(B_R)}^2 \le C_a$$

for all $a > 0$, and

$$\int_0^\infty \big\||\xi|^q\nabla^2\exp(-\lambda_1 x_3)\big\|_{L^2(B_R)}^2 < \infty$$

for $q \in \mathbb{N}$ large enough ($q \ge 4$). Hence the bound on $\nabla^q u$ follows. □

▷ The representation formula, together with its associated estimates, now allows us to extend the notion of solution to locally integrable boundary data. Before stating the corresponding result, let us prove a technical lemma about some nice properties of operators of the type $\mathcal{I}[\xi_i\xi_j/|\xi|]$, which we will use repeatedly.

Lemma 2.15. *Let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. Then, for all $g \in L^2_{\mathrm{uloc}}(\mathbb{R}^2)$ and all $\rho \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that $\nabla^\alpha\rho$ has bounded second order moments in L^2 for $0 \le \alpha \le 2$,*

$$\int_{\mathbb{R}^2}\varphi\mathcal{I}\Big[\frac{\xi_i\xi_j}{|\xi|}\Big]\rho * g = \int_{\mathbb{R}^2} g\mathcal{I}\Big[\frac{\xi_i\xi_j}{|\xi|}\Big]\check{\rho} * \varphi, \quad \int_{\mathbb{R}^2}\nabla\varphi\mathcal{I}\Big[\frac{\xi_i\xi_j}{|\xi|}\Big]\rho * g = -\int_{\mathbb{R}^2}\varphi\mathcal{I}\Big[\frac{\xi_i\xi_j}{|\xi|}\Big]\nabla\rho * g.$$

Remark 2.16. Notice that the second formula merely states that

$$\nabla\left(\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\rho * g\right) = \mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\nabla\rho * g$$

in the sense of distributions.

Proof. • The first formula is a consequence of Fubini's theorem: indeed,

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\rho * g \\ &= \int_{\mathbb{R}^6} dx\,dy\,dt\,\gamma_{ij}(x-y)g(t)\varphi(x)\times\{\rho(x-t)-\rho(y-t)-\nabla\rho(x-t)\cdot(x-y)\mathbf{1}_{|\mathbf{x}-\mathbf{y}|\le\mathbf{1}}\} \\ &\underset{y'=x+t-y}{=} \int_{\mathbb{R}^6} dx\,dy'\,dt\,\gamma_{ij}(y'-t)g(t)\varphi(x)\times\{\rho(x-t)-\rho(x-y')-\nabla\rho(x-t)\cdot(y'-t)\mathbf{1}_{|\mathbf{y}'-\mathbf{t}|\le\mathbf{1}}\}.\end{aligned}$$

Integrating with respect to x, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\rho * g &= \int_{\mathbb{R}^4} dy'\,dt\,\gamma_{ij}(y'-t)g(t)\{\varphi*\check{\rho}(t)-\varphi*\check{\rho}(y')-\varphi*\nabla\check{\rho}(t)\cdot(t-y')\mathbf{1}_{|\mathbf{y}'-\mathbf{t}|\le\mathbf{1}}\} \\ &= \int_{\mathbb{R}^2} dt\,g(t)\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\varphi*\check{\rho}.\end{aligned}$$

• The second formula is then easily deduced from the first: using the fact that $\nabla\check{\rho}(x) = -\nabla\rho(-x) = -\widecheck{\nabla\rho}(x)$, we infer

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla\varphi\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\rho * g &= \int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\check{\rho}*\nabla\varphi = \int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\nabla\check{\rho}*\varphi \\ &= -\int_{\mathbb{R}^2} g\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\widecheck{\nabla\rho}*\varphi = -\int_{\mathbb{R}^2} \varphi\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right]\nabla\rho * g.\end{aligned}$$

□

We are now ready to state the main result of this section:

Corollary 2.17. *Let $v_0 \in \mathbb{K}$ (recall that $\mathbb{K}$ is defined in (2-13).) Then there exists a unique solution u of (2-1) such that $u|_{x_3=0} = v_0$ and*

$$\begin{aligned} &\textit{for all } a>0, && \sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_0^a |u(x_h,x_3)|^2\,dx_3\,dx_h < \infty, \\ &\textit{there exists } q\in\mathbb{N}^*, && \sup_{k\in\mathbb{Z}^2}\int_{k+[0,1]^2}\int_1^\infty |\nabla^q u(x_h,x_3)|^2\,dx_3\,dx_h < \infty. \end{aligned} \tag{2-22}$$

Remark 2.18. As in Proposition 2.13, the integer q in the two results above is explicit and does not depend on v_0 (one can take $q = 4$ for instance).

Proof of Corollary 2.17.
Uniqueness. Let u be a solution of (2-1) satisfying (2-22) and such that $u|_{x_3=0} = 0$. We use the same type of proof as in Proposition 2.1; see also [Gérard-Varet and Masmoudi 2010]. Using a Poincaré inequality

near the boundary $x_3 = 0$, we have

$$\sup_{k\in\mathbb{Z}^2} \int_{k+[0,1]^2} \int_0^\infty |\nabla^q u(x_h, x_3)|^2 \, dx_3 \, dx_h < \infty.$$

Hence $u \in \mathscr{C}(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}^2))$ and we can take the Fourier transform of u with respect to the horizontal variable. The rest of the proof is identical to that of Proposition 2.1. The equations in (2-3) are meant in the sense of tempered distributions in x_h, and in the sense of distributions in x_3, which is enough to perform all calculations.

Existence. For all $x_h \in \mathbb{R}^2$, let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be such that $\chi \equiv 1$ on $B(x_h, 1)$. Then we set

$$\begin{aligned} u(x) = \mathscr{F}^{-1}\biggl(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix} \widehat{\chi v_{0,h}}(\xi) \\ \widehat{\nabla\cdot(\chi V_h)} \end{pmatrix} \exp(-\lambda_k x_3)\biggr)(x) + \sum_{k=1}^3 \mathscr{I}[M_k^1] f_k(\,\cdot\,, x_3) * ((1-\chi)v_{0,h})(x) \\ + \sum_{k=1}^3 \mathscr{I}[N_k^1] f_k(\,\cdot\,, x_3) * ((1-\chi)V_h)(x) + \varphi_{\mathrm{HF}} * \begin{pmatrix} (1-\chi)v_{0,h} \\ \nabla\cdot((1-\chi)V_h) \end{pmatrix}(x) \\ + \psi_1 * ((1-\chi)v_{0,h})(x) + \psi_2 * ((1-\chi)V_h)(x). \end{aligned} \tag{2-23}$$

We first claim that this formula does not depend on the choice of the function χ: indeed, let $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be such that $\chi_i \equiv 1$ on $B(x_h, 1)$. Then, since $\chi_1 - \chi_2 = 0$ on $B(x_h, 1)$ and $\chi_1 - \chi_2$ is compactly supported, we may write

$$\begin{aligned} &\sum_{k=1}^3 \mathscr{I}[M_k^1] f_k(\,\cdot\,, x_3) * ((\chi_1-\chi_2)v_{0,h}) + \psi_1 * ((\chi_1-\chi_2)v_{0,h}) \\ &\qquad = \mathscr{F}^{-1}\biggl(\sum_{k=1}^3 \phi(\xi) M_k \widehat{(\chi_1-\chi_2)v_{0,h}} \exp(-\lambda_k x_3)\biggr) \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=1}^3 \mathscr{I}[N_k^1] f_k(\,\cdot\,, x_3) * ((\chi_1-\chi_2)V_h) + \psi_2 * ((\chi_1-\chi_2)V_h) \\ &\qquad = \mathscr{F}^{-1}\biggl(\sum_{k=1}^3 \phi(\xi) N_k \widehat{(\chi_1-\chi_2)V_h} \exp(-\lambda_k x_3)\biggr) \\ &\qquad = \mathscr{F}^{-1}\biggl(\sum_{k=1}^3 \phi(\xi) L_k e_3 \mathscr{F}(\nabla\cdot(\chi_1-\chi_2)V_h) \exp(-\lambda_k x_3)\biggr). \end{aligned}$$

On the other hand,

$$\varphi_{\mathrm{HF}} * \begin{pmatrix} (\chi_1-\chi_2)v_{0,h} \\ \nabla\cdot((\chi_1-\chi_2)V_h) \end{pmatrix} = \mathscr{F}^{-1}\biggl(\sum_{k=1}^3 (1-\phi(\xi)) L_k \begin{pmatrix} \widehat{(\chi_1-\chi_2)v_{0,h}} \\ \widehat{\nabla\cdot((\chi_1-\chi_2)V_h)} \end{pmatrix} \exp(-\lambda_k x_3)\biggr).$$

Gathering all the terms, we find that the two definitions coincide. Moreover, u satisfies (2-22) (we refer to the proof of Proposition 2.13 for the derivation of such estimates: notice that the proof of Proposition 2.13 only uses local integrability properties of v_0).

It remains to prove that u is a solution of the Stokes system, which is not completely trivial due to the complexity of the representation formula. We start by deriving a duality formula: we claim that, for all $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^2)^3$ and all $x_3 > 0$,

$$\int_{\mathbb{R}^2} u(x_h, x_3)\cdot \eta(x_h)\,dx_h = \int_{\mathbb{R}^2} v_{0,h}(x_h)\cdot \mathscr{F}^{-1}\Big(\sum_{k=1}^3 (\overline{{}^tL_k}\hat\eta(\xi))_h \exp(-\bar\lambda_k x_3)\Big) - \int_{\mathbb{R}^2} V_h(x_h)\cdot \mathscr{F}^{-1}\Big(\sum_{k=1}^3 i\xi(\overline{{}^tL_k}\hat\eta(\xi))_3 \exp(-\bar\lambda_k x_3)\Big). \tag{2-24}$$

To that end, in (2-23), we may choose a function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ on the set

$$\{x \in \mathbb{R}^2 : d(x, \operatorname{Supp}\eta) \le 1\}.$$

We then transform every term in (2-23). We have, according to the Parseval formula,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathscr{F}^{-1}\left(\sum_{k=1}^3 L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\right)\cdot\eta \\
&\quad= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\sum_{k=1}^3 \overline{\hat\eta(\xi)}\cdot L_k(\xi)\begin{pmatrix}\widehat{\chi v_{0,h}}(\xi)\\ \widehat{\nabla\cdot(\chi V_h)}(\xi)\end{pmatrix}\exp(-\lambda_k x_3)\,d\xi \\
&\quad= \int_{\mathbb{R}^2}\chi v_{0,h}\mathscr{F}^{-1}\Big(\sum_{k=1}^3 (\overline{{}^tL_k}\hat\eta(\xi))_h\exp(-\bar\lambda_k x_3)\Big) - \int_{\mathbb{R}^2}\chi V_h\cdot\mathscr{F}^{-1}\Big(\sum_{k=1}^3 i\xi(\overline{{}^tL_k}\hat\eta(\xi))_3\exp(-\bar\lambda_k x_3)\Big).
\end{aligned}$$

Using standard convolution results, we have

$$\int_{\mathbb{R}^2}\psi_1 * ((1-\chi)v_{0,h})\eta = \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\,{}^t\check\psi_1 * \eta.$$

The terms with ψ_2 and φ_{HF} are transformed using identical computations. Concerning the term with $\mathscr{I}[M_k^1]$, we use Lemma 2.15, from which we infer that

$$\int_{\mathbb{R}^2}\mathscr{I}[M_k^1]f_k * ((1-\chi)v_{0,h})\eta = \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\mathscr{I}[{}^tM_k^1]\check f_k * \eta.$$

Notice also that, by the definition of M_k^1, $\check M_k^1 = M_k^1$. Therefore,

$$\int_{\mathbb{R}^2}\psi_1 * ((1-\chi)v_{0,h})\eta + \sum_{k=1}^3\int_{\mathbb{R}^2}\mathscr{I}[M_k^1]f_k * ((1-\chi)v_{0,h})\eta = \int_{\mathbb{R}^2}(1-\chi)v_{0,h}\cdot\mathscr{F}^{-1}\Big(\sum_{k=1}^3 {}^t\Big(\check L_k e_1\check L_k e_2\Big)\hat\eta\check\phi(\xi)\exp(-\check\lambda_k x_3)\Big)$$

and

$$\int_{\mathbb{R}^2}\psi_2 * ((1-\chi)V_h\eta + \sum_{k=1}^3\int_{\mathbb{R}^2}\mathscr{I}[N_k^1]f_k * ((1-\chi)V_h)\eta = \int_{\mathbb{R}^2}(1-\chi)V_h\cdot\mathscr{F}^{-1}\Big(\sum_{k=1}^3 \xi\,{}^t\Big(i\check L_k e_3\Big)\hat\eta\check\phi(\xi)\exp(-\check\lambda_k x_3)\Big).$$

Now we recall that if $v_0 \in H^{1/2}(\mathbb{R}^2) \cap \mathbb{K}$ is real-valued, so is the solution u of (2-1). Therefore, in Fourier space,

$$\overline{\hat{u}(\,\cdot\,, x_3)} = \check{\hat{u}}(\,\cdot\,, x_3) \quad \text{for all } x_3 > 0.$$

We infer in particular that

$$\sum_{k=1}^{3} \check{L}_k \exp(-\check{\lambda}_k x_3) = \sum_{k=1}^{3} \bar{L}_k \exp(-\bar{\lambda}_k x_3).$$

Gathering all the terms, we obtain (2-24).

Now let $\zeta \in \mathscr{C}_0^\infty(\mathbb{R}^2 \times (0,\infty))^3$ such that $\nabla \cdot \zeta = 0$, and $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^2 \times (0,\infty))$. We seek to prove that

$$\int_{\mathbb{R}^3_+} u(-\Delta\zeta - e_3 \times \zeta) = 0 \tag{2-25}$$

as well as

$$\int_{\mathbb{R}^3_+} u \cdot \nabla \eta = 0. \tag{2-26}$$

Using (2-24), we infer that

$$\begin{aligned}\int_{\mathbb{R}^3_+} & u\Bigl(-\Delta\zeta - e_3 \times \zeta\Bigr) \\ &= \int_0^\infty \int_{\mathbb{R}^2} v_{0,h} \mathscr{F}^{-1}\Bigl(\sum_{k=1}^{3} \overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi)\exp(-\bar{\lambda}_k x_3)\Bigr) + \int_0^\infty \int_{\mathbb{R}^2} V_h \mathscr{F}^{-1}\Bigl(\sum_{k=1}^{3} \overline{\mathcal{N}_k(\xi)}\hat{\zeta}(\xi)\exp(-\bar{\lambda}_k x_3)\Bigr),\end{aligned}$$

where

$$\mathcal{M}_k := (|\xi|^2 - \lambda_k^2)\,{}^tM_k + {}^tM_k \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{N}_k := (|\xi|^2 - \lambda_k^2)\,{}^tN_k + {}^tN_k \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to Lemma 2.6,

$$\mathcal{M}_k = \begin{pmatrix} i\xi_1 q_{k,1} & i\xi_2 q_{k,1} & -\lambda_k q_{k,1} \\ i\xi_1 q_{k,2} & i\xi_2 q_{k,2} & -\lambda_k q_{k,2} \end{pmatrix}$$

so that, since $i\xi \cdot \hat{\zeta}_h + \partial_3 \hat{\zeta}_3 = 0$,

$$\overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi, x_3) = (\partial_3\hat{\zeta}_3 - \bar{\lambda}_k \hat{\zeta}_3)\begin{pmatrix} \bar{q}_{k,1} \\ \bar{q}_{k,2} \end{pmatrix}.$$

Integrating in x_3, we find that

$$\int_0^\infty \overline{\mathcal{M}_k(\xi)}\hat{\zeta}(\xi, x_3)\exp(-\bar{\lambda}_k x_3)\,dx_3 = 0.$$

Similar arguments lead to

$$\int_0^\infty \int_{\mathbb{R}^2} V_h \mathscr{F}^{-1}\Bigl(\sum_{k=1}^{3} \overline{\mathcal{N}_k(\xi)}\hat{\zeta}(\xi, x_3)\exp(-\bar{\lambda}_k x_3)\Bigr) = 0$$

and to the divergence-free condition (2-26). $\square$

2B. ***The Dirichlet to Neumann operator for the Stokes–Coriolis system.*** We now define the Dirichlet to Neumann operator for the Stokes–Coriolis system with boundary data in $\mathbb{K}$. We start by deriving its expression for boundary data $v_0 \in H^{1/2}(\mathbb{R}^2)$ satisfying (2-2), for which we consider the unique solution u of (2-1) in $\dot{H}^1(\mathbb{R}^3_+)$. We recall that u is defined in Fourier space by (2-8). The corresponding pressure term is given by

$$\hat{p}(\xi, x_3) = \sum_{k=1}^{3} A_k(\xi) \frac{|\xi|^2 - \lambda_k(\xi)^2}{\lambda_k(\xi)} \exp(-\lambda_k(\xi) x_3).$$

The Dirichlet to Neumann operator is then defined by

$$\mathrm{DN}\, v_0 := -\partial_3 u|_{x_3=0} + p|_{x_3=0} e_3.$$

Consequently, in Fourier space, the Dirichlet to Neumann operator is given by

$$\widehat{\mathrm{DN}\, v_0}(\xi) = \sum_{k=1}^{3} A_k(\xi) \begin{pmatrix} (i/|\xi|^2)(-\lambda_k^2 \xi + (|\xi|^2 - \lambda_k^2)^2 \xi^\perp) \\ |\xi|^2/\lambda_k \end{pmatrix} =: M_{SC}(\xi) \hat{v}_0(\xi), \tag{2-27}$$

where $M_{SC} \in \mathcal{M}_{3,3}(\mathbb{C})$. Using the notations of the previous paragraph, we have

$$M_{SC} = \sum_{k=1}^{3} \lambda_k L_k + e_3\, {}^t q_k.$$

Let us first review a few useful properties of the Dirichlet to Neumann operator:

Proposition 2.19.
- *Behavior at large frequencies*: *when* $|\xi| \gg 1$,

$$M_{SC}(\xi) = \begin{pmatrix} |\xi| + \xi_1^2/|\xi| & \xi_1\xi_2/|\xi| & i\xi_1 \\ \xi_1\xi_2/|\xi| & |\xi| + \xi_2^2/|\xi| & i\xi_2 \\ -i\xi_1 & -i\xi_2 & 2|\xi| \end{pmatrix} + O(|\xi|^{1/3}).$$

- *Behavior at small frequencies*: *when* $|\xi| \ll 1$,

$$M_{SC}(\xi) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 & i(\xi_1+\xi_2)/|\xi| \\ 1 & 1 & i(\xi_2-\xi_1)/|\xi| \\ i(\xi_2-\xi_1)/|\xi| & -i(\xi_1+\xi_2)/|\xi| & \sqrt{2}/|\xi| - 1 \end{pmatrix} + O(|\xi|).$$

- *The horizontal part of the Dirichlet to Neumann operator*, *denoted by* DN_h, *maps* $H^{1/2}(\mathbb{R}^2)$ *into* $H^{-1/2}(\mathbb{R}^2)$.
- *Let* $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ *be such that* $\phi(\xi) = 1$ *for* $|\xi| \le 1$. *Then*

$$(1-\phi(D))\,\mathrm{DN}_3 : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2),$$
$$D\phi(D)\,\mathrm{DN}_3,\ |D|\phi(D)\,\mathrm{DN}_3 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2),$$

where, *classically*, $a(D)$ *denotes the operator defined in Fourier space by*

$$\widehat{a(D)u} = a(\xi)\hat{u}(\xi)$$

for $a \in \mathscr{C}(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2)$.

Remark 2.20. For $|\xi| \gg 1$, the Dirichlet to Neumann operator for the Stokes–Coriolis system has the same expression, at main order, as that of the Stokes system. This can be easily understood since, at large frequencies, the rotation term in the system (2-3) can be neglected in front of $|\xi|^2\hat{u}$, and therefore the system behaves roughly as the Stokes system.

Proof. The first two points follow from the expression (2-27) together with the asymptotic expansions in Lemma 2.4. Since they are lengthy but straightforward calculations, we postpone them to Appendix A.

The horizontal part of the Dirichlet to Neumann operator satisfies

$$|\widehat{\mathrm{DN}_h\, v_0}(\xi)| = O(|\xi|\,|\hat{v}_0(\xi)|) \quad \text{for } |\xi| \gg 1,$$
$$|\widehat{\mathrm{DN}_h\, v_0}(\xi)| = O(|\hat{v}_0(\xi)|) \qquad \text{for } |\xi| \ll 1.$$

Therefore, if $\int_{\mathbb{R}^2}(1+|\xi|^2)^{1/2}|\hat{v}_0(\xi)|^2\,d\xi < \infty$, we deduce that

$$\int_{\mathbb{R}^2}(1+|\xi|^2)^{-1/2}|\widehat{\mathrm{DN}_h\, v_0}(\xi)|^2\,d\xi < \infty.$$

Hence $\mathrm{DN}_h : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$.

In a similar way,

$$|\widehat{\mathrm{DN}_3\, v_0}(\xi)| = O(|\xi|\,|\hat{v}_0(\xi)|) \quad \text{for } |\xi| \gg 1,$$

so that if $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ is such that $\phi(\xi) = 1$ for ξ in a neighborhood of zero, there exists a constant C such that

$$|(1-\phi(\xi))\widehat{\mathrm{DN}_3\, v_0}(\xi)| \le C|\xi|\,|\hat{v}_0(\xi)| \quad \text{for all } \xi \in \mathbb{R}^2.$$

Therefore $(1-\phi(D))\,\mathrm{DN}_3 : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$.

The vertical part of the Dirichlet to Neumann operator, however, is singular at low frequencies. This is consistent with the singularity observed in $L_1(\xi)$ for ξ close to zero. More precisely, for ξ close to zero, we have

$$\widehat{\mathrm{DN}_3\, v_0}(\xi) = \frac{1}{|\xi|}\hat{v}_{0,3} + O(|\hat{v}_0(\xi)|).$$

Consequently, for all $\xi \in \mathbb{R}^2$,

$$|\xi\phi(\xi)\widehat{\mathrm{DN}_3\, v_0}(\xi)| \le C|\hat{v}_0(\xi)|. \qquad \square$$

Following [Gérard-Varet and Masmoudi 2010], we now extend the definition of the Dirichlet to Neumann operator to functions which are not square integrable in $\mathbb{R}^2$, but rather locally uniformly integrable. There are several differences with [Gérard-Varet and Masmoudi 2010]: First, the Fourier multiplier associated with DN is not homogeneous, even at the main order. Therefore its kernel (the inverse Fourier transform of the multiplier) is not homogeneous either, and, in general, does not have the same decay as the kernel of Stokes system. Moreover, the singular part of the Dirichlet to Neumann operator for low frequencies prevents us from defining DN on $H^{1/2}_{\mathrm{uloc}}$. Hence we will define DN on $\mathbb{K}$ only (see also Corollary 2.17).

Let us briefly recall the definition of the Dirichlet to Neumann operator for the Stokes system (see [Gérard-Varet and Masmoudi 2010]), which we denote by DN_S[1]. The Fourier multiplier of DN_S is

$$M_S(\xi) := \begin{pmatrix} |\xi| + \xi_1^2/|\xi| & \xi_1\xi_2/|\xi| & i\xi_1 \\ \xi_1\xi_2/|\xi| & |\xi| + \xi_2^2/|\xi| & i\xi_2 \\ -i\xi_1 & -i\xi_2 & 2|\xi| \end{pmatrix}.$$

The inverse Fourier transform of M_S in $\mathcal{S}'(\mathbb{R}^2)$ is homogeneous of order -3, and consists of two parts:

- The first is the inverse Fourier transform of coefficients equal to $i\xi_1$ or $i\xi_2$. This part is singular, and is the derivative of a Dirac mass at point $t = 0$.
- The second is the kernel denoted by K_S, which satisfies

$$|K_S(t)| \le \frac{C}{|t|^3}.$$

In particular, it is legitimate to say that

$$|\mathcal{F}^{-1} M_S(t)| \le \frac{C}{|t|^3} \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \setminus \{0\}).$$

Hence DN_S is defined on $H^{1/2}_{\mathrm{uloc}}$ in the following way: for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ be such that $\chi \equiv 1$ on the set $\{t \in \mathbb{R}^2 : d(t, \operatorname{Supp}\varphi) \le 1\}$. Then

$$\langle \mathrm{DN}_S\, u, \varphi\rangle_{\mathcal{D}',\mathcal{D}} := \langle \mathcal{F}^{-1}(M_S \widehat{\chi u}), \varphi\rangle_{H^{-1/2},H^{1/2}} + \int_{\mathbb{R}^2} K_S * ((1-\chi)u)\cdot\varphi.$$

The assumption on χ ensures that there is no singularity in the last integral, while the decay of K_S ensures its convergence. Notice also that the singular part (which is local in the physical space) is only present in the first term of the decomposition.

We wish to adopt a similar method here, but a few precautions must be taken because of the singularities at low frequencies, in the spirit of the representation formula (2-23). Hence, before defining the action of DN on $\mathbb{K}$, let us decompose the Fourier multiplier associated with DN. We have

$$M_{SC}(\xi) = M_S(\xi) + \phi(\xi)(M_{SC} - M_S)(\xi) + (1-\phi)(\xi)(M_{SC} - M_S)(\xi).$$

Concerning the third term, we have the following result, which is a straightforward consequence of Proposition 2.19 and Appendix B:

Lemma 2.21. *As $|\xi| \to \infty$, we have*

$$\nabla_\xi^\alpha (M_{SC} - M_S)(\xi) = O(|\xi|^{\frac{1}{3}-|\alpha|})$$

for $\alpha \in \mathbb{N}^2$, $0 \le |\alpha| \le 3$.

[1] Gérard-Varet and Masmoudi [2010] considered the Stokes system in $\mathbb{R}^2_+$ and not $\mathbb{R}^3_+$, but this part of their proof does not depend on the dimension.

We deduce from Lemma 2.21 that $\nabla^\alpha[(1-\phi(\xi))(M_{SC}-M_S)(\xi)]\in L^1(\mathbb{R}^2)$ for all $\alpha\in\mathbb{N}^2$ with $|\alpha|=3$, so that it follows from Lemma B.3 that there exists a constant $C>0$ such that

$$|\mathscr{F}^{-1}[(1-\phi(\xi))(M_{SC}-M_S)(\xi)](t)|\le\frac{C}{|t|^3}.$$

It remains to decompose $\phi(\xi)(M_{SC}-M_S)(\xi)$. As in Proposition 2.13, the multipliers which are homogeneous of order one near $\xi=0$ are treated separately. Note that since the last column and the last line of M_{SC} act on horizontal divergences (see Proposition 2.22), we are interested in multipliers homogeneous of order zero in $M_{SC,3i}$, $M_{SC,i3}$ for $i=1,2$, and homogeneous of order -1 in $M_{SC,33}$. In the following, we set

$$\bar M_h:=\frac{\sqrt2}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix},\qquad \bar M:=\begin{pmatrix}\bar M_h&0\\0&0\end{pmatrix},$$
$$V_1:=\frac{i\sqrt2}{2|\xi|}\begin{pmatrix}\xi_1+\xi_2\\\xi_1-\xi_2\end{pmatrix},\quad V_2:=\frac{i\sqrt2}{2|\xi|}\begin{pmatrix}-\xi_1+\xi_2\\-\xi_1-\xi_2\end{pmatrix}.$$

We decompose $M_{SC}-M_S$ near $\xi=0$ as

$$\phi(\xi)(M_{SC}-M_S)(\xi)=\bar M+\phi(\xi)\begin{pmatrix}M_1&V_1\\{}^tV_2&|\xi|^{-1}\end{pmatrix}-(1-\phi(\xi))\bar M+\phi(\xi)M^{\text{rem}},$$

where $M_1\in\mathscr{M}_2(\mathbb{C})$ only contains homogeneous and nonpolynomial terms of order one, and M^{rem}_{ij} contains either polynomial terms or remainder terms which are $o(|\xi|)$ for ξ close to zero if $1\le i,\,j\le2$. Looking closely at the expansions for λ_k in a neighborhood of zero (see (A-4)) and at the calculations in paragraph A.4.2, we infer that M^{rem}_{ij} contains either polynomial terms or remainder terms of order $O(|\xi|^2)$ if $1\le i,\,j\le2$. We emphasize that the precise expression of M^{rem} is not needed in the following, although it can be computed by pushing forward the expansions of Appendix A. In a similar fashion, $M^{\text{rem}}_{i,3}$ and $M^{\text{rem}}_{3,i}$ contain constant terms and remainder terms of order $O(|\xi|)$ for $i=1,2$ and $M^{\text{rem}}_{3,3}$ contains remainder terms of order $O(1)$. As a consequence, if we define the low-frequency kernels

$$K_i^{\text{rem}}:\mathbb{R}^2\to\mathscr{M}_2(\mathbb{C})\quad\text{for } 1\le i\le4$$

by

$$K_1^{\text{rem}}:=\mathscr{F}^{-1}\left(\phi\begin{pmatrix}M^{\text{rem}}_{11}&M^{\text{rem}}_{12}\\M^{\text{rem}}_{21}&M^{\text{rem}}_{22}\end{pmatrix}\right),$$
$$K_2^{\text{rem}}:=\mathscr{F}^{-1}\left(\phi\begin{pmatrix}M^{\text{rem}}_{13}\\M^{\text{rem}}_{23}\end{pmatrix}i\begin{pmatrix}\xi_1&\xi_2\end{pmatrix}\right),$$
$$K_3^{\text{rem}}:=\mathscr{F}^{-1}\left(-i\phi(\xi)\xi\begin{pmatrix}M^{\text{rem}}_{31}&M^{\text{rem}}_{32}\end{pmatrix}\right),$$
$$K_4^{\text{rem}}:=\mathscr{F}^{-1}\left(\phi(\xi)M^{\text{rem}}_{33}\begin{pmatrix}\xi_1^2&\xi_1\xi_2\\\xi_1\xi_2&\xi_2^2\end{pmatrix}\right),$$

we have, for $1 \le i \le 4$ (see Lemmas B.1 and B.5),

$$|K_i^{\mathrm{rem}}(x_h)| \le \frac{C}{|x_h|^3} \quad \text{for all } x_h \in \mathbb{R}^2.$$

We also denote by $M_{\mathrm{HF}}^{\mathrm{rem}}$ the kernel part of

$$\mathscr{F}^{-1}(-(1-\phi)\overline{M} + (1-\phi)(M_{SC} - M_S)),$$

which satisfies

$$|M_{\mathrm{HF}}^{\mathrm{rem}}(x_h)| \le \frac{C}{|x_h|^3} \quad \text{for all } x_h \in \mathbb{R}^2 \setminus \{0\}.$$

Notice that there is also a singular part in

$$\mathscr{F}^{-1}(-(1-\phi)\overline{M}),$$

which in fact corresponds to $\mathscr{F}^{-1}(-\overline{M})$, and which is therefore a Dirac mass at $x_h = 0$.

It remains to define the kernels homogeneous of order one besides M_1. We set

$$\begin{aligned} M_2 &:= V_1 i \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}, \\ M_3 &:= -i\xi\, {}^tV_2, \\ M_4 &:= \frac{1}{|\xi|} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}, \end{aligned}$$

so that M_1, M_2, M_3, M_4 are 2×2 real-valued matrices whose coefficients are linear combinations of $\xi_i\xi_j/|\xi|$. In the end, we will work with the following decomposition for the matrix M_{SC}, where the treatment of each of the terms has been explained above:

$$M_{SC} = M_S + \overline{M} + (1-\phi)(M_{SC} - M_S - \overline{M}) + \phi \begin{pmatrix} M_1 & V_1 \\ {}^tV_2 & |\xi|^{-1} \end{pmatrix} + \phi M^{\mathrm{rem}}.$$

We are now ready to extend the definition of the Dirichlet to Neumann operator to functions in $\mathbb{K}$: in the spirit of Proposition 2.13–Corollary 2.17, we derive a representation formula for functions in $\mathbb{K} \cap H^{1/2}(\mathbb{R}^2)^3$, which still makes sense for functions in $\mathbb{K}$:

Proposition 2.22. *Let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)^3$ such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ be such that $\chi \equiv 1$ on the set*

$$\{x \in \mathbb{R}^2 : d(x, \operatorname{Supp} \varphi \cup \operatorname{Supp} \Phi_h) \le 1\}.$$

Let $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^2_\xi)$ be such that $\phi(\xi) = 1$ if $|\xi| \le 1$, and let $\rho := \mathscr{F}^{-1}\phi$.

- *Let $v_0 \in H^{1/2}(\mathbb{R}^2)^3$ be such that $v_{0,3} = \nabla_h \cdot V_h$. Then*

$$
\begin{aligned}
&\langle \mathrm{DN}(v_0), \varphi\rangle_{\mathscr{D}',\mathscr{D}} \\
&\quad = \langle \mathrm{DN}_S(v_0), \varphi\rangle_{\mathscr{D}',\mathscr{D}} + \int_{\mathbb{R}^2} \varphi \cdot \bar{M} v_0 + \langle \mathscr{F}^{-1}((1-\phi)(M_{SC} - M_S - \bar{M})\widehat{\chi v_0}), \varphi\rangle_{H^{-1/2},H^{1/2}} \\
&\qquad + \int_{\mathbb{R}^2} \varphi \cdot M^{\mathrm{rem}}_{\mathrm{HF}} * ((1-\chi)v_0) + \left\langle \mathscr{F}^{-1}\left(\phi\left(M^{\mathrm{rem}} + \begin{pmatrix} M_1 & V_1 \\ {}^t V_2 & |\xi|^{-1}\end{pmatrix}\right)\begin{pmatrix}\widehat{\chi v_{0,h}} \\ i\xi \cdot \widehat{\chi V_h}\end{pmatrix}\right), \varphi \right\rangle_{H^{-1/2},H^{1/2}} \\
&\qquad + \int_{\mathbb{R}^2} \varphi_h \cdot \{\mathscr{I}[M_1](\rho * (1-\chi)v_{0,h}) + K_1^{\mathrm{rem}} * ((1-\chi)v_{0,h})\} \\
&\qquad + \int_{\mathbb{R}^2} \varphi_h \cdot \{\mathscr{I}[M_2](\rho * (1-\chi)V_h) + K_2^{\mathrm{rem}} * ((1-\chi)V_h)\} \\
&\qquad + \int_{\mathbb{R}^2} \Phi_h \cdot \{\mathscr{I}[M_3](\rho * (1-\chi)v_{0,h}) + K_3^{\mathrm{rem}} * ((1-\chi)v_{0,h})\} \\
&\qquad + \int_{\mathbb{R}^2} \Phi_h \cdot \{\mathscr{I}[M_4](\rho * (1-\chi)V_h) + K_4^{\mathrm{rem}} * ((1-\chi)V_h)\}.
\end{aligned}
$$

- *The above formula still makes sense when $v_0 \in \mathbb{K}$, which allows us to extend the definition of* DN *to* $\mathbb{K}$.

Remark 2.23. Notice that if $v_0 \in \mathbb{K}$ and $\varphi \in \mathbb{K}$ with $\varphi_3 = \nabla_h \cdot \Phi_h$, and if φ, Φ_h have compact support, then the right-hand side of the formula in Proposition 2.22 still makes sense. Therefore DN v_0 can be extended into a linear form on the set of functions in $\mathbb{K}$ with compact support. In this case, we will denote it by

$$\langle \mathrm{DN}(v_0), \varphi\rangle$$

without specifying the functional spaces.

The proof of the Proposition 2.22 is very close to those of Proposition 2.13 and Corollary 2.17, and therefore we leave it to the reader.

The goal is now to link the solution of the Stokes–Coriolis system in $\mathbb{R}^3_+$ with $v_0 \in \mathbb{K}$ and $\mathrm{DN}(v_0)$. This is done through the following lemma:

Lemma 2.24. *Let $v_0 \in \mathbb{K}$, and let u be the unique solution of (2-1) with $u|_{x_3=0} = v_0$, given by Corollary 2.17.*

Let $\varphi \in \mathscr{C}^\infty_0(\bar{\mathbb{R}}^3_+)^3$ be such that $\nabla \cdot \varphi = 0$. Then

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \langle \mathrm{DN}(v_0), \varphi|_{x_3=0}\rangle.$$

In particular, if $v_0 \in \mathbb{K}$ with $v_{0,3} = \nabla_h \cdot V_h$ and if v_0, V_h have compact support, then

$$\langle \mathrm{DN}(v_0), v_0\rangle \geq 0.$$

Remark 2.25. If $\varphi \in \mathscr{C}_0^\infty(\overline{\mathbb{R}^3_+})^3$ is such that $\nabla \cdot \varphi = 0$, then in particular

$$\begin{aligned}\varphi_{3|x_3=0}(x_h) &= -\int_0^\infty \partial_3 \varphi_3(x_h, z)\, dz \\ &= \int_0^\infty \nabla_h \cdot \varphi_h(x_h, z) = \nabla_h \cdot \Phi_h\end{aligned}$$

for $\Phi_h := \int_0^\infty \varphi_h(\,\cdot\,, z)\, dz \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. In particular $\varphi|_{x_3=0}$ is a suitable test function for Proposition 2.22.

Proof. The proof relies on two duality formulas in the spirit of (2-24), one for the Stokes–Coriolis system and the other for the Dirichlet to Neumann operator. We claim that if $v_0 \in \mathbb{K}$, then, on the one hand

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \int_{\mathbb{R}^2} v_0 \mathscr{F}^{-1}({}^t\overline{M}_{SC}(\xi) \hat{\varphi}|_{x_3=0}(\xi)), \tag{2-28}$$

and on the other hand, for any $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^2)^3$ such that $\eta_3 = \nabla_h \cdot \theta_h$ for some $\theta_h \in \mathscr{C}_0^\infty(\mathbb{R}^2)^2$,

$$\langle \mathrm{DN}(v_0), \eta \rangle_{\mathscr{D}', \mathscr{D}} = \int_{\mathbb{R}^2} v_0 \mathscr{F}^{-1}({}^t\overline{M}_{SC}(\xi) \hat{\eta}(\xi)). \tag{2-29}$$

Applying formula (2-29) with $\eta = \varphi|_{x_3=0}$ then yields the desired result. Once again, the proofs of (2-28) and (2-29) are close to that of (2-24). From (2-24), one has

$$\begin{aligned}\int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi &= -\int_{\mathbb{R}^3_+} u \cdot e_3 \times \varphi \\ &= -\int_{\mathbb{R}^2} v_0 \mathscr{F}^{-1}\Bigl(\int_0^\infty \sum_{k=1}^3 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k e_3 \times \hat{\varphi}\Bigr) \\ &= \int_{\mathbb{R}^2} v_0 \mathscr{F}^{-1}\left(\int_0^\infty \sum_{k=1}^3 \exp(-\bar{\lambda}_k x_3)\, {}^t\bar{L}_k \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \hat{\varphi}\right).\end{aligned}$$

Moreover, we deduce from the representation formula for u and from Lemma 2.15 a representation formula for ∇u:

$$\begin{aligned}\nabla u(x) = \mathscr{F}^{-1}&\Biggl(\sum_{k=1}^3 \exp(-\lambda_k x_3) L_k(\xi) \begin{pmatrix} \widehat{\chi v_{0,h}} \\ \widehat{\nabla \cdot (\chi V_h)} \end{pmatrix} \bigl(i\xi_1 \;\; i\xi_2 \;\; -\lambda_k\bigr)\Biggr)(x) \\ &+ \sum_{k=1}^3 \mathscr{I}[M_k^1] \nabla f_k(\,\cdot\,, x_3) * ((1-\chi) v_{0,h})(x) + \sum_{k=1}^3 \mathscr{I}[N_k^1] \nabla f_k(\,\cdot\,, x_3) * ((1-\chi) V_h)(x) \\ &+ \nabla \varphi_{\mathrm{HF}} * \begin{pmatrix} (1-\chi) v_{0,h}(\xi) \\ \nabla \cdot ((1-\chi) V_h) \end{pmatrix} + \nabla \psi_1 * ((1-\chi) v_{0,h})(x) + \nabla \psi_2 * ((1-\chi) V_h)(x).\end{aligned}$$

Then, proceeding exactly as in the proof of Corollary 2.17, we infer that

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}\Big(\sum_{k=1}^3 \int_0^\infty |\xi|^2 \exp(-\bar\lambda_k x_3)\, {}^t\bar L_k \hat\varphi(\xi, x_3)\, dx_3\Big) \\ - \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}\Big(\sum_{k=1}^3 \int_0^\infty \bar\lambda_k \exp(-\bar\lambda_k x_3)\, {}^t\bar L_k \partial_3 \hat\varphi(\xi, x_3)\, dx_3\Big).$$

Integrating by parts in x_3, we obtain

$$\int_0^\infty \exp(-\bar\lambda_k x_3)\, {}^t\bar L_k \partial_3 \hat\varphi(\xi, x_3)\, dx_3 = \bar\lambda_k \int_0^\infty \exp(-\bar\lambda_k x_3)\, {}^t\bar L_k \hat\varphi(\xi, x_3)\, dx_3 - {}^t\bar L_k \hat\varphi|_{x_3=0}(\xi).$$

Gathering the terms, we infer

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}\Big(\int_0^\infty \sum_{k=1}^3 \exp(-\bar\lambda_k x_3)\, {}^t\bar P_k \hat\varphi\Big) + \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}\Big(\sum_{k=1}^3 \bar\lambda_k\, {}^t\bar L_k \hat\varphi|_{x_3=0}\Big),$$

where

$$P_k := (|\xi|^2 - \lambda_k^2) L_k + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} L_k = -\begin{pmatrix} i\xi_1 \\ i\xi_2 \\ -\lambda_k \end{pmatrix} \begin{pmatrix} q_{k,1} & q_{k,2} & q_{k,3} \end{pmatrix}$$

according to Lemma 2.6. Therefore, since φ is divergence-free, we have

$${}^t\bar P_k \hat\varphi = (-\partial_3 \hat\varphi_3 + \bar\lambda_k \hat\varphi_3) \begin{pmatrix} \bar q_{k,1} \\ \bar q_{k,2} \\ \bar q_{k,3} \end{pmatrix},$$

so that eventually, after integrating by parts once more in x_3,

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^3_+} e_3 \times u \cdot \varphi = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}\left(\left[\sum_{k=1}^3 \bar\lambda_k\, {}^t\bar L_k + \begin{pmatrix} \bar q_{k,1} \\ \bar q_{k,2} \\ \bar q_{k,3} \end{pmatrix} {}^t e_3\right] \hat\varphi|_{x_3=0}\right) \\ = \int_{\mathbb{R}^2} v_0 \mathcal{F}^{-1}({}^t\overline{M}_{SC} \hat\varphi|_{x_3=0}).$$

The derivation of (2-29) is very similar to that of (2-24) and therefore we skip its proof. □

We conclude this section with some estimates on the Dirichlet to Neumann operator:

Lemma 2.26. *There exists a positive constant C such that the following property holds. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)^3$ be such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, and let $v_0 \in \mathbb{K}$ with $v_{0,3} = \nabla_h \cdot V_h$. Let $R \geq 1$ and $x_0 \in \mathbb{R}^2$ be such that*

$$\operatorname{Supp} \varphi \cup \operatorname{Supp} \Phi_h \subset B(x_0, R).$$

Then

$$|\langle \mathrm{DN}(v_0), \varphi \rangle_{\mathcal{D}', \mathcal{D}}| \leq C R (\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)})(\|v_0\|_{H^{1/2}_{\mathrm{uloc}}} + \|V_h\|_{H^{1/2}_{\mathrm{uloc}}}).$$

Moreover, if $v_0, V_h \in H^{1/2}(\mathbb{R}^2)$, then

$$|\langle \mathrm{DN}(v_0), \varphi \rangle_{\mathcal{D}', \mathcal{D}}| \leq C (\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)})(\|v_0\|_{H^{1/2}} + \|V_h\|_{H^{1/2}}).$$

Proof. The second inequality is classical and follows from the Fourier definition of the Dirichlet to Neumann operator. We therefore focus on the first inequality, for which we use the representation formula of Proposition 2.22.

We consider a truncation function χ such that $\chi \equiv 1$ on $B(x_0, R+1)$ and $\chi \equiv 0$ on $B(x_0, R+2)^c$, and such that $\|\nabla^\alpha \chi\|_\infty \leq C_\alpha$, with C_α independent of R, for all $\alpha \in \mathbb{N}$. We must evaluate three different types of term:

▹ Terms of the type

$$\int_{\mathbb{R}^2} K * ((1-\chi)v_0) \cdot \varphi,$$

where K is a matrix such that $|K(x)| \leq C|x|^{-3}$ for all $x \in \mathbb{R}^2$ (of course, we include in the present discussion all the variants involving V_h and Φ_h). These terms are bounded by

$$\begin{aligned} C \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|t|^3} |1-\chi(x-t)|\,|v_0(x-t)|\,|\varphi(x)|\,dx\,dt \\ \leq C \int_{\mathbb{R}^2} dx\, |\varphi(x)| \left(\int_{|t|\geq 1} \frac{|v_0(x-t)|^2}{|t|^3}\,dt \right)^{\frac12} \left(\int_{|t|\geq 1} \frac{1}{|t|^3}\,dt \right)^{\frac12} \\ \leq C \|v_0\|_{L^2_{\mathrm{uloc}}} \|\varphi\|_{L^1} \\ \leq CR \|v_0\|_{L^2_{\mathrm{uloc}}} \|\varphi\|_{L^2}. \end{aligned}$$

▹ Terms of the type

$$\int_{\mathbb{R}^2} \varphi_h \cdot \mathcal{I}[M]((1-\chi)v_{0,h}) * \rho,$$

where M is a 2×2 matrix whose coefficients are linear combinations of $\xi_i\xi_j/|\xi|$. Using Lemma 2.10 and Remark 2.11, these terms are bounded by

$$C\|\varphi\|_{L^1} \|v_0\|_{L^2_{\mathrm{uloc}}} \|(1+|\cdot|^2)\rho\|_{L^2}^{1/2} \|(1+|\cdot|^2)\nabla^2\rho\|_{L^2}^{1/2}.$$

Using Plancherel's theorem, we have (up to a factor 2π)

$$\begin{aligned} \|(1+|\cdot|^2)\rho\|_{L^2} &= \|(1-\Delta)\phi\|_{L^2(\mathbb{R}^2)} \leq C, \\ \|(1+|\cdot|^2)\nabla^2\rho\|_{L^2} &= \|(1-\Delta)|\cdot|^2\phi\|_{L^2(\mathbb{R}^2)} \leq C, \end{aligned}$$

so that eventually

$$\left| \int_{\mathbb{R}^2} \varphi_h \cdot \mathcal{I}[M]((1-\chi)v_{0,h}) * \rho \right| \leq C\|\varphi\|_{L^1} \|v_0\|_{L^2_{\mathrm{uloc}}} \leq CR \|v_0\|_{L^2_{\mathrm{uloc}}} \|\varphi\|_{L^2}.$$

▹ Terms of the type

$$\langle \mathcal{F}^{-1}(M(\xi)\widehat{\chi v_0}(\xi)), \varphi \rangle_{H^{-1/2},H^{1/2}} \quad \text{and} \quad \int_{\mathbb{R}^2} \varphi \cdot \bar{M} v_0,$$

where $M(\xi)$ is some kernel such that $\mathrm{Op}(M) : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$ and $\bar{M}$ is a constant matrix.

All these terms are bounded by

$$C\|\chi v_0\|_{H^{1/2}(\mathbb{R}^2)} \|\varphi\|_{H^{1/2}(\mathbb{R}^2)}.$$

In fact, the trickiest part of the lemma is proving that

$$\|\chi v_0\|_{H^{1/2}(\mathbb{R}^2)} \le C R \|v_0\|_{H^{1/2}_{\mathrm{uloc}}}. \tag{2-30}$$

To that end, we recall that

$$\|\chi v_0\|^2_{H^{1/2}(\mathbb{R}^2)} = \|\chi v_0\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|(\chi v_0)(x)-(\chi v_0)(y)|^2}{|x-y|^3}\,dx\,dy.$$

We consider a cut-off function ϑ satisfying (1-4), so that

$$\|\chi v_0\|^2_{L^2(\mathbb{R}^2)} \le \sum_{k\in\mathbb{Z}^2} \|(\tau_k\vartheta)\chi v_0\|^2_{L^2} \le \|\chi\|^2_\infty \sum_{\substack{k\in\mathbb{Z}^2\\ |k|\le CR}} \|(\tau_k\vartheta)v_0\|^2_{L^2} \le CR^2\|\chi\|^2_\infty \sup_k \|(\tau_k\vartheta)v_0\|^2_{L^2}.$$

Concerning the second term,

$$\begin{aligned}
&|\chi v_0(x)-\chi v_0(y)|^2\\
&\quad= \Big(\sum_{k\in\mathbb{Z}^2} \tau_k\vartheta(x)\chi(x)v_0(x) - \tau_k\vartheta(y)\chi(y)v_0(y)\Big)^2\\
&\quad= \sum_{\substack{k,l\in\mathbb{Z}^2\\ |k-l|\le 3}} [\tau_k\vartheta(x)\chi(x)v_0(x)-\tau_k\vartheta(y)\chi(y)v_0(y)][\tau_l\vartheta(x)\chi(x)v_0(x)-\tau_l\vartheta(y)\chi(y)v_0(y)]\\
&\qquad\qquad + \sum_{\substack{k,l\in\mathbb{Z}^2\\ |k-l|> 3}} [\tau_k\vartheta(x)\chi(x)v_0(x)-\tau_k\vartheta(y)\chi(y)v_0(y)][\tau_l\vartheta(x)\chi(x)v_0(x)-\tau_l\vartheta(y)\chi(y)v_0(y)].
\end{aligned}$$

Notice that, according to the assumptions on ϑ, if $|k-l|>3$, then $\tau_k\vartheta(x)\tau_l\vartheta(x)=0$ for all $x\in\mathbb{R}^2$. Moreover, if $\tau_k(x)\tau_l(y)\neq 0$, then $|x-y|\ge |k-l|-2$. Notice also that the first sum above contains $O(R^2)$ nonzero terms. Therefore, using the Cauchy–Schwartz inequality, we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|(\chi v_0)(x)-(\chi v_0)(y)|^2}{|x-y|^3}\,dx\,dy\\
&\qquad \le CR^2 \sup_{k\in\mathbb{Z}^2} \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|(\tau_k\vartheta\chi v_0)(x)-(\tau_k\vartheta\chi v_0)(y)|^2}{|x-y|^3}\,dx\,dy\\
&\qquad\qquad + \sum_{\substack{k,l\in\mathbb{Z}^2\\ |k-l|>3}} \frac{1}{(|k-l|-2)^3} \int_{\mathbb{R}^2\times\mathbb{R}^2} |\tau_k\vartheta(x)\chi(x)v_0(x)|\,|\tau_l\vartheta(y)\chi(y)v_0(y)|\,dx\,dy
\end{aligned}$$

Using (2-21), the first term is bounded by

$$CR^2\|\chi\|^2_{W^{1,\infty}}\|v_0\|^2_{H^{1/2}_{\mathrm{uloc}}},$$

while the second is bounded by $C\|v_0\|^2_{L^2_{\mathrm{uloc}}}$.

Gathering all the terms, we obtain (2-30). This concludes the proof. ☐

2C. ***Presentation of the new system.*** We now come to our main concern in this paper, which is proving the existence of weak solutions to the linear system of rotating fluids in the bumpy half-space (1-1). There are two features which make this problem particularly difficult. Firstly, the fact that the bottom is now bumpy rather than flat prevents us from using the Fourier transform in the tangential direction. Secondly, as the domain Ω is unbounded, it is not possible to rely on Poincaré type inequalities. We face this problem using an idea of [Gérard-Varet and Masmoudi 2010]. It consists in defining a problem equivalent to (1-1) yet posed in the bounded channel Ω^b, by the mean of a transparent boundary condition at the interface $\Sigma = \{x_3 = 0\}$, namely,

$$\begin{cases} -\Delta u + e_3 \times u + \nabla p = 0 & \text{in } \Omega^b, \\ \operatorname{div} u = 0 & \text{in } \Omega^b, \\ u|_\Gamma = u_0, & \\ -\partial_3 u + p e_3 = \mathrm{DN}(u|_{x_3=0}) & \text{on } \Sigma. \end{cases} \tag{2-31}$$

In the system above and throughout the rest of the paper, we assume without any loss of generality that $\sup \omega < 0$, $\inf \omega \geq -1$. Notice that thanks to assumption (1-3), we have

$$\begin{aligned} u_{3|x_3=0}(x_h) &= u_{0,3}(x_h) - \int_{\omega(x_h)}^0 \nabla_h \cdot u_h(x_h, z)\, dz \\ &= u_{0,3}(x_h) - \nabla_h \omega \cdot u_{0,h}(x_h) - \nabla_h \cdot \int_{\omega(x_h)}^0 u_h(x_h, z)\, dz \\ &= \nabla_h \cdot \left(U_h(x_h) - \int_{\omega(x_h)}^0 u_h(x_h, z)\, dz \right), \end{aligned}$$

so that $u_{3|x_3=0}$ satisfies the assumptions of Proposition 2.22.

Let us start by explaining the meaning of (2-31):

Definition 2.27. A function $u \in H^1_{\mathrm{uloc}}(\Omega^b)$ is a solution of (2-31) if it satisfies the bottom boundary condition $u|_\Gamma = u_0$ in the trace sense, and if, for all $\varphi \in \mathscr{C}_0^\infty(\overline{\Omega_b})$ such that $\nabla \cdot \varphi = 0$ and $\varphi|_\Gamma = 0$, we have

$$\int_{\Omega^b} (\nabla u \cdot \nabla \varphi + e_3 \times u \cdot \varphi) = -\langle \mathrm{DN}(u|_{x_3=0}), \varphi|_{x_3=0} \rangle_{\mathscr{D}', \mathscr{D}}.$$

Remark 2.28. Notice that if $\varphi \in \mathscr{C}_0^\infty(\overline{\Omega_b})$ is such that $\nabla \cdot \varphi = 0$ and $\varphi|_\Gamma = 0$, then

$$\varphi_{3|x_3=0} = \nabla_h \cdot \Phi_h, \quad \text{where } \Phi_h(x_h) := -\int_{\omega(x_h)}^0 \varphi_h(x_h, z)\, dz \in \mathscr{C}_0^\infty(\mathbb{R}^2).$$

Therefore φ is an admissible test function for Proposition 2.22.

We then have the following result, which is the Stokes–Coriolis equivalent of [Gérard-Varet and Masmoudi 2010, Proposition 9], and which follows easily from Lemma 2.24 and Corollary 2.17:

Proposition 2.29. *Let $u_0 \in L^2_{\mathrm{uloc}}(\mathbb{R}^2)$ satisfying (1-3), and assume that $\omega \in W^{1,\infty}(\mathbb{R}^2)$.*

- *Let* (u, p) *be a solution of* (1-1) *in* Ω *such that* $u \in H^1_{\mathrm{loc}}(\Omega)$ *and*

$$\text{for all } a > 0, \quad \sup_{l\in\mathbb{Z}^2} \int_{l+[0,1]^2} \int_{\omega(x_h)}^{a} (|u|^2 + |\nabla u|^2) < \infty,$$
$$\sup_{l\in\mathbb{Z}^2} \int_{l+[0,1]^2} \int_{1}^{\infty} |\nabla^q u|^2 < \infty$$

for some $q \in \mathbb{N}$, $q \geq 1$.

Then $u|_{\Omega^b}$ *is a solution of* (2-31), *and for* $x_3 > 0$, u *is given by* (2-23), *with* $v_0 := u|_{x_3=0} \in \mathbb{K}$.

- *Conversely, let* $u^- \in H^1_{\mathrm{uloc}}(\Omega^b)$ *be a solution of* (2-31), *and let* $v_0 := u^-|_{x_3=0} \in \mathbb{K}$. *Consider the function* $u^+ \in H^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ *defined by* (2-23). *Setting*

$$u(x) := \begin{cases} u^-(x) & \text{if } \omega(x_h) < x_3 < 0, \\ u^+(x) & \text{if } x_3 > 0, \end{cases}$$

the function $u \in H^1_{\mathrm{loc}}(\Omega)$ *is such that*

$$\text{for all } a > 0, \quad \sup_{l\in\mathbb{Z}^2} \int_{l+[0,1]^2} \int_{\omega(x_h)}^{a} (|u|^2 + |\nabla u|^2) < \infty,$$
$$\sup_{l\in\mathbb{Z}^2} \int_{l+[0,1]^2} \int_{1}^{\infty} |\nabla^q u|^2 < \infty$$

for some $q \in \mathbb{N}$ *sufficiently large, and is a solution of* (1-1).

As a consequence, we work with the system (2-31) from now on. In order to have a homogeneous Poincaré inequality in Ω^b, it is convenient to lift the boundary condition on Γ, so as to work with a homogeneous Dirichlet boundary condition. Therefore, we define $V = (V_h, V_3)$ by

$$V_h := u_{0,h}, \qquad V_3 := u_{0,3} - \nabla_h \cdot u_{0,h}(x_3 - \omega(x_h)).$$

Notice that $V|_{x_3=0} \in \mathbb{K}$ thanks to (1-3), and that V is divergence free. By definition, the function

$$\tilde{u} := u - V \mathbf{1}_{\mathbf{x}\in\Omega^{\mathbf{b}}}$$

is a solution of

$$\begin{cases} -\Delta\tilde{u} + e_3 \times \tilde{u} + \nabla\tilde{p} = f & \text{in } \Omega^b, \\ \operatorname{div}\tilde{u} = 0 & \text{in } \Omega^b, \\ \tilde{u}|_{\Gamma} = 0, \\ -\partial_3\tilde{u} + \tilde{p}e_3 = \mathrm{DN}(\tilde{u}|_{x_3=0^-}) + F & \text{on } \Sigma \times \{0\}, \end{cases} \tag{2-32}$$

where

$$f := \Delta V - e_3 \times V = \Delta_h V - e_3 \times V,$$
$$F := \mathrm{DN}(V|_{x_3=0}) + \partial_3 V|_{x_3=0}.$$

Notice that thanks to the regularity assumptions on u_0 and ω, we have, for all $l \in \mathbb{N}$ and for all $\varphi \in \mathscr{C}_0^\infty(\overline{\Omega^b})^3$ with $\operatorname{Supp}\varphi \subset ((-l, l)^2 \times (-1, 0)) \cap \overline{\Omega^b}$,

$$|\langle f, \varphi\rangle_{\mathscr{D}',\mathscr{D}}| \leq Cl(\|u_{0,h}\|_{H^2_{\mathrm{uloc}}} + \|u_{0,3}\|_{H^1_{\mathrm{uloc}}})\|\varphi\|_{H^1(\Omega^b)}, \tag{2-33}$$

where the constant C depends only on $\|\omega\|_{W^{1,\infty}}$. In a similar fashion, if $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)^3$ is such that $\varphi_3 = \nabla_h \cdot \Phi_h$ for some $\Phi_h \in \mathscr{C}_0^\infty(\mathbb{R}^2)^2$, and if $\operatorname{Supp}\varphi$, $\operatorname{Supp}\Phi_h \subset B(x_0, l)$, then, according to Lemma 2.26,

$$|\langle F, \varphi\rangle_{\mathscr{D}',\mathscr{D}}| \le Cl(\|u_{0,h}\|_{H^2_{\mathrm{uloc}}} + \|u_{0,3}\|_{H^1_{\mathrm{uloc}}} + \|U_h\|_{H^{1/2}_{\mathrm{uloc}}})(\|\varphi\|_{H^{1/2}(\mathbb{R}^2)} + \|\Phi_h\|_{H^{1/2}(\mathbb{R}^2)}). \tag{2-34}$$

2D. ***Strategy of the proof.*** From now on, we drop the ~'s in (2-32) so as to lighten the notation.

- In order to prove the existence of solutions of (2-32) in $H^1_{\mathrm{uloc}}(\Omega)$, we truncate horizontally the domain Ω, and we derive uniform estimates on the solutions of the Stokes–Coriolis system in the truncated domains. More precisely, we introduce, for all $n \in \mathbb{N}$, $k \in \mathbb{N}$,

$$\begin{aligned}
\Omega_n &:= \Omega^b \cap \{x \in \mathbb{R}^3 : |x_1| \le n,\ x_2 \le n\},\\
\Omega_{k,k+1} &:= \Omega_{k+1} \setminus \Omega_k,\\
\Sigma_n &:= \{(x_h, 0) \in \mathbb{R}^3 : |x_1| \le n,\ x_2 \le n\},\\
\Sigma_{k,k+1} &:= \Sigma_{k+1} \setminus \Sigma_k,\\
\Gamma_n &:= \Gamma \cap \{x \in \mathbb{R}^3 : |x_1| \le n,\ x_2 \le n\}.
\end{aligned}$$

We consider the Stokes–Coriolis system in Ω_n, with homogeneous boundary conditions on the lateral boundaries

$$\begin{cases}
-\Delta u_n + e_3 \times u_n + \nabla p_n = f, & x \in \Omega_n,\\
\nabla \cdot u_n = 0, & x \in \Omega_n,\\
u_n = 0, & x \in \Omega^b \setminus \Omega_n,\\
u_n = 0, & x \in \Gamma_n,\\
-\partial_3 u_n + p_n e_3|_{x_3=0} = \mathrm{DN}(u_n|_{x_3=0}) + F, & x \in \Sigma_n.
\end{cases} \tag{2-35}$$

Notice that the transparent boundary condition involving the Dirichlet to Neumann operator only makes sense if $u_n|_{x_3=0}$ is defined on the whole plane Σ (and not merely on Σ_n), due to the nonlocality of the operator DN. This accounts for the condition $u_n|_{\Omega^b\setminus\Omega_n} = 0$.

Taking u_n as a test function in (2-35), we get a first energy estimate on u_n

$$\begin{aligned}
\|\nabla u_n\|^2_{L^2(\Omega^b)} &= \underbrace{-\langle \mathrm{DN}(u_n|_{x_3=0}), u_n|_{x_3=0}\rangle}_{\le 0} - \langle F, u_n|_{x_3=0}\rangle + \langle f, u_n\rangle \\
&\le Cn\left(\|u_{n,h}|_{x_3=0}\|_{H^{1/2}(\Sigma_n)} + \left\| \int_{\omega(x_h)}^0 u_{n,h}(x_h, z')\,dz' \right\|_{H^{1/2}(\Sigma_n)}\right) + Cn\|u_n\|_{H^1(\Omega_n)}\\
&\le Cn\|u_n\|_{H^1(\Omega_n)},
\end{aligned} \tag{2-36}$$

where the constant C depends only on $\|u_0\|_{H^2_{\mathrm{uloc}}}$ and $\|\omega\|_{W^{1,\infty}}$. This implies, thanks to the Poincaré inequality,

$$E_n := \int_\Omega \nabla u_n \cdot \nabla u_n \le C_0 n^2. \tag{2-37}$$

The existence of u_n in $H^1(\Omega^b)$ follows. Uniqueness is a consequence of equality (2-36) with $F = 0$ and $f = 0$.

In order to prove the existence of u, we derive H^1_{uloc} estimates on u_n, uniform with respect to n. Then, passing to the limit in (2-35) and in the estimates, we deduce the existence of a solution of (2-32) in $H^1_{\mathrm{uloc}}(\Omega^b)$. In order to obtain H^1_{uloc} estimates on u_n, we follow the strategy in [Gérard-Varet and Masmoudi 2010], which is inspired by [Ladyženskaja and Solonnikov 1980]. We work with the energies

$$E_k := \int_{\Omega_k} \nabla u_n \cdot \nabla u_n. \tag{2-38}$$

The goal is to prove an inequality of the type

$$E_k \le C(k^2 + (E_{k+1} - E_k)) \quad \text{for all } k \in \{m, \dots, n\}, \tag{2-39}$$

where $m \in \mathbb{N}$ is a large, but fixed integer (independent of n) and C is a constant depending only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{\mathrm{uloc}}}$, $\|u_{0,3}\|_{H^1_{\mathrm{uloc}}}$, $\|U_h\|_{H^{1/2}_{\mathrm{uloc}}}$. Then, by backwards induction on k, we deduce that

$$E_k \le Ck^2 \quad \text{for all } k \in \{m, \dots, n\}$$

so that E_m in particular is bounded uniformly in n. Since the derivation of the energy estimates is invariant by translation in the horizontal variable, we infer that, for all $n \in \mathbb{N}$,

$$\sup_{c \in \mathscr{C}_m} \int_{(c\times(-1,0))\cap\Omega^b} |\nabla u_n|^2 \le C,$$

where

$$\mathscr{C}_m := \{c, \text{ square of edge of length } m \text{ contained in } \Sigma_n \text{ with vertices in } \mathbb{Z}^2\}. \tag{2-40}$$

Hence the uniform H^1_{uloc} bound on u_n is proved. As a consequence, by a diagonal argument, we can extract a subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ such that

$$\begin{aligned} u_{\psi(n)} &\rightharpoonup u && \text{weakly in } H^1(\Omega_k),\\ u_{\psi(n)}|_{x_3=0} &\rightharpoonup u|_{x_3=0} && \text{weakly in } H^{1/2}(\Sigma_k) \end{aligned}$$

for all $k \in \mathbb{N}$. Of course, u is a solution of the Stokes–Coriolis system in Ω^b, and $u \in H^1_{\mathrm{uloc}}(\Omega^b)$. Looking closely at the representation formula in Proposition 2.22, we infer that

$$\langle \mathrm{DN}\, u_{\psi(n)}|_{x_3=0}, \varphi\rangle_{\mathscr{D}',\mathscr{D}} \overset{n\to\infty}{\longrightarrow} \langle \mathrm{DN}\, u|_{x_3=0}, \varphi\rangle_{\mathscr{D}',\mathscr{D}}$$

for all admissible test functions φ. For instance,

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi M^{\mathrm{rem}}_{\mathrm{HF}} * (1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0}) &= \int_{\mathbb{R}^2} dx \int_{|t|\le k} dt\, \varphi(x) M^{\mathrm{rem}}_{\mathrm{HF}}(x-t)(1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0})(t)\\ &\quad + \int_{\mathbb{R}^2} dx \int_{|t|\ge k} dt\, \varphi(x) M^{\mathrm{rem}}_{\mathrm{HF}}(x-t)(1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0})(t). \end{aligned}$$

For all k, the first integral vanishes as $n \to \infty$ as a consequence of the weak convergence in $L^2(\Sigma_k)$. As for the second integral, let $R > 0$ be such that $\operatorname{Supp}\varphi \subset B_R$, and let $k \geq R+1$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} dx \int_{|t|\geq k} dt\, \varphi(x) M_{\mathrm{HF}}^{\mathrm{rem}}(x-t)((1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0})(t)) \\
&\quad \leq C \int_{\mathbb{R}^2} dx \int_{|t|\geq k} dt\, |\varphi(x)| \frac{1}{|x-t|^3} \big(|u_{\psi(n)}|_{x_3=0}(t)| + |u|_{x_3=0}(t)|\big) \\
&\qquad \leq C \int_{\mathbb{R}^2} dx\, |\varphi(x)| \bigg(\int_{|t|\geq k} \frac{1}{|x-t|^3}\, dt\bigg)^{\frac12} \bigg(\int_{|x-t|\geq 1} \frac{dt}{|x-t|^3} \big(\big|u|_{x_3=0}\big|^2 + |u_{\psi(n)}|_{x_3=0}|^2\big)\bigg)^{\frac12} \\
&\qquad \leq C(\|u|_{x_3=0}\|_{L^2_{\mathrm{uloc}}} + \sup_n \|u_n|_{x_3=0}\|_{L^2_{\mathrm{uloc}}}) \int_{\mathbb{R}^2} dx\, |\varphi(x)| \bigg(\int_{|t|\geq k} \frac{1}{|x-t|^3}\, dt\bigg)^{\frac12} \\
&\qquad \leq C(\|u|_{x_3=0}\|_{L^2_{\mathrm{uloc}}} + \sup_n \|u_n|_{x_3=0}\|_{L^2_{\mathrm{uloc}}}) \|\varphi\|_{L^1} (k-R)^{-\frac12}.
\end{aligned}$$

Hence the second integral vanishes as $k \to \infty$ uniformly in n. We infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \varphi M_{\mathrm{HF}}^{\mathrm{rem}} * ((1-\chi)(u_{\psi(n)}|_{x_3=0} - u|_{x_3=0})) = 0.$$

Therefore u is a solution of (2-32).

The final induction inequality will be much more complicated than (2-39), and the proof will also be more involved than that in [Gérard-Varet and Masmoudi 2010]. However, the general scheme will be very close to the one described above.

• Concerning uniqueness of solutions of (2-32), we use the same type of energy estimates as above. Once again, we give in the present paragraph a very rough idea of the computations, and we refer to Section 4 for all details. When $f = 0$ and $F = 0$, the energy estimates (2-39) become

$$E_k \leq C(E_{k+1} - E_k),$$

and therefore

$$E_k \leq r E_{k+1}$$

with $r := C/(1+C) \in (0, 1)$. Hence, by induction,

$$E_1 \leq r^{k-1} E_k \leq C r^{k-1} k^2$$

for all $k \geq 1$, since u is assumed to be bounded in $H^1_{\mathrm{uloc}}(\Omega^b)$. Letting $k \to \infty$, we deduce that $E_1 = 0$. Since all estimates are invariant by translation in x_h, we obtain that $u = 0$.

3. Estimates in the rough channel

This section is devoted to the proof of energy estimates of the type (2-39) for solutions of the system (2-35), which eventually lead to the existence of a solution of (2-32).

The goal is to prove that, for some $m \geq 1$ sufficiently large (but independent of n), E_m is bounded uniformly in n, which automatically implies the boundedness of u_n in $H^1_{\mathrm{uloc}}(\Omega^b)$. We reach this objective in two steps:

- We prove a Saint-Venant estimate: We claim that there exists a constant $C_1 > 0$ uniform in n such that, for all $m \in \mathbb{N} \setminus \{0\}$ and all $k \in \mathbb{N}$, $k \geq m$,

$$E_k \leq C_1 \left[k^2 + E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{3-1}$$

  The crucial fact is that C_1 depends only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{\mathrm{uloc}}}$, $\|u_{0,3}\|_{H^1_{\mathrm{uloc}}}$, $\|U_h\|_{H^{1/2}_{\mathrm{uloc}}}$, so that it is independent of n, k, and m.

- This estimate allows us to deduce the bound in $H^1_{\mathrm{uloc}}(\Omega)$ via a nontrivial induction argument.

Let us first explain the induction, assuming that (3-1) holds. The proof of (3-1) is postponed to Section 3B.

3A. ***Induction.*** We aim at deducing from (3-1) that there exists $m \in \mathbb{N} \setminus \{0\}$, $C > 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{\Omega_m} \nabla u_n \cdot \nabla u_n \leq C. \tag{3-2}$$

The proof of this uniform bound is divided into two points:

- Firstly, we deduce from (3-1), by downward induction on k, that there exist positive constants C_2, C_3, m_0, depending only on C_0 and C_1, appearing respectively in (2-37) and (3-1), such that, for all (k, m) such that $k \geq C_3 m$ and $m \geq m_0$,

$$E_k \leq C_2 \left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{3-3}$$

  Let us insist on the fact that C_2 and C_3 are independent of n, k, m. They will be adjusted in the course of the induction argument (see (3-8)).

- Secondly, we notice that (3-3) yields the bound we are looking for, choosing $k = \lfloor C_3 m \rfloor + 1$ and m large enough.

- We thus start with the proof of (3-3), assuming that (3-1) holds.

First, notice that thanks to (2-37), (3-3) is true for $k \geq n$ as soon as $C_2 \geq C_0$, remembering that $u_n = 0$ on $\Omega^b \setminus \Omega_n$. We then assume that (3-3) holds for $n, n-1, \dots, k+1$, where k is an integer such that $k \geq C_3 m$ (further conditions on C_2, C_3 will be derived at the end of the induction argument; see (3-7)).

We prove (3-3) at the rank k by contradiction. Assume that (3-3) does not hold at the rank k, so that

$$E_k > C_2 \left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j \geq m+k} \frac{E_{j+m} - E_j}{j} \right]. \tag{3-4}$$

Then the induction assumption implies

$$\begin{aligned} E_{k+m+1} - E_k &\le C_2\left[(k+m+1)^2 - k^2 + \frac{(k+m+1)^4 - k^4}{m^5} \sup_{j\ge k+m} \frac{E_{j+m}-E_j}{j}\right] \\ &\le C_2\left[2k(m+1) + (m+1)^2 + 80\frac{k^3}{m^4} \sup_{j\ge k+m} \frac{E_{j+m}-E_j}{j}\right]. \end{aligned} \tag{3-5}$$

Above, we have used the following inequality, which holds for all $k \ge m \ge 1$:

$$\begin{aligned} (k+m+1)^4 - k^4 &= 4k^3(m+1) + 6k^2(m+1)^2 + 4k(m+1)^3 + (m+1)^4 \\ &\le 8mk^3 + 6k^2 \times 4m^2 + 4k \times 8m^3 + 16m^4 \\ &\le 80mk^3. \end{aligned}$$

Using (3-4), (3-1), and (3-5), we get

$$\begin{aligned} C_2&\left[k^2 + m^3 + \frac{k^4}{m^5} \sup_{j\ge k+m} \frac{E_{j+m}-E_j}{j}\right] \\ &< E_k \le C_1\left[k^2 + 2C_2k(m+1) + C_2(m+1)^2 + \left(80C_2\frac{k^3}{m^4} + \frac{k^4}{m^5}\right) \sup_{j\ge k+m} \frac{E_{j+m}-E_j}{j}\right]. \end{aligned} \tag{3-6}$$

The constants $C_0, C_1 > 0$ are fixed and depend only on $\|\omega\|_{W^{1,\infty}}$ and $\|u_{0,h}\|_{H^2_{\text{uloc}}}$, $\|u_{0,3}\|_{H^1_{\text{uloc}}}$, $\|U_h\|_{H^{1/2}_{\text{uloc}}}$ (see (2-37) for the definition of C_0). We choose $m_0 > 1$, $C_2 > C_0$, and $C_3 \ge 1$ depending only on C_0 and C_1 so that

$$\begin{cases} k \ge C_3 m, \\ \text{and } m \ge m_0 \end{cases} \quad \text{implies} \quad \begin{cases} C_2(k^2+m^3) > C_1[k^2 + 2C_2k(m+1) + C_2(m+1)^2], \\ \text{and } C_2k^4/m^5 \ge C_1(80C_2k^3/m^4 + k^4/m^5). \end{cases} \tag{3-7}$$

One can easily check that it suffices to choose C_2, C_3, and m_0 so that

$$\begin{aligned} C_2 &> \max(2C_1, C_0), \\ (C_2 - C_1)C_3 &> 80C_1C_2, \\ \text{for all } m \ge m_0, \quad (C_2C_1 + C_1)(m+1)^2 &< m^3. \end{aligned} \tag{3-8}$$

Plugging (3-7) into (3-6), we reach a contradiction. Therefore (3-3) is true at the rank k. By induction, (3-3) is proved for all $m \ge m_0$ and for all $k \ge C_3 m$.

• It follows from (3-3), choosing $k = \lfloor C_3 m\rfloor + 1$, that there exists a constant $C > 0$, depending only on C_0, C_1, C_2, C_3, and therefore only on $\|\omega\|_{W^{1,\infty}}$ and on Sobolev–Kato norms on u_0 and U_h, such that, for all $m \ge m_0$,

$$E_{\lfloor m/2\rfloor} \le E_{\lfloor C_3 m\rfloor+1} \le C\left[m^3 + \frac{1}{m} \sup_{j\ge \lfloor C_3 m\rfloor+m+1} \frac{E_{j+m}-E_j}{j}\right]. \tag{3-9}$$

Let us now consider the set $\mathscr{C}_m$ defined by (2-40) for an even integer m. As $\mathscr{C}_m$ is finite, there exists a square c in $\mathscr{C}_m$ which maximizes

$$\{\|u_n\|_{H^1(\Omega_c)} : c \in \mathscr{C}_m\},$$

where $\Omega_c = \{x \in \Omega^b : x_h \in c\}$. We then shift u_n in such a manner that c is centered at 0. We call $\tilde{u}_n$ the shifted function. It is still compactly supported, but in Ω_{2n} instead of in Ω_n:

$$\int_{\Omega_{2n}} |\nabla \tilde{u}_n|^2 = \int_{\Omega_n} |\nabla u_n|^2 \quad \text{and} \quad \int_{\Omega_{m/2}} |\nabla \tilde{u}_n|^2 = \int_{\Omega_c} |\nabla u_n|^2.$$

Analogously to E_k, we define $\widetilde{E}_k$. Since the arguments leading to the derivation of energy estimates are invariant by horizontal translation, and all constants depend only on Sobolev norms on u_0, U_h, and ω, we infer that (3-9) still holds when E_k is replaced by $\widetilde{E}_k$. On the other hand, recall that $\widetilde{E}_{m/2}$ maximizes $\|\tilde{u}_n\|^2_{H^1(\Omega_c)}$ on the set of squares of edge length m. Moreover, in the set

$$\Sigma_{j+m} \setminus \Sigma_j \quad \text{for } j \geq 1,$$

there are at most $4(j+m)/m$ squares of edge length m. As a consequence, we have, for all $j \in \mathbb{N}^*$,

$$\widetilde{E}_{j+m} - \widetilde{E}_j \leq 4 \frac{j+m}{m} \widetilde{E}_{m/2},$$

so that (3-9) written for $\tilde{u}_n$ becomes

$$\begin{aligned} \widetilde{E}_{m/2} &\leq C \left[m^3 + \frac{1}{m^2} \left(\sup_{j \geq (C_3+1)m} 1 + \frac{m}{j} \right) \widetilde{E}_{m/2} \right] \\ &\leq C \left[m^3 + \frac{1}{m^2} \widetilde{E}_{m/2} \right]. \end{aligned}$$

This estimate being uniform in $m \in \mathbb{N}$ provided $m \geq m_0$, we can take m large enough and get

$$\widetilde{E}_{m/2} \leq C \frac{m^3}{1 - C(1/m^2)},$$

so that eventually there exists $m \in \mathbb{N}$ such that

$$\sup_{c \in \mathscr{C}_m} \|u_n\|^2_{H^1((c \times (-1,0) \cap \Omega^b))} \leq C \frac{m^3}{1 - C(1/m^2)}.$$

This means exactly that u_n is uniformly bounded in $H^1_{\mathrm{uloc}}(\Omega^b)$. Existence follows, as explained in Section 2D.

3B. *Saint-Venant estimate.* This part is devoted to the proof of (3-1). We carry out a Saint-Venant estimate on the system (2-35), focusing on having constants uniform in n as explained in Section 2D. The preparatory work of Sections 2A and 2B allows us to focus on very few issues. The main problem is the nonlocality of the Dirichlet to Neumann operator, which at first sight does not seem to be compatible with getting estimates independent of the size of the support of u_n.

Let $n \in \mathbb{N} \setminus \{0\}$ be fixed. Also let $\varphi \in \mathscr{C}_0^\infty(\Omega^b)$ such that

$$\nabla \cdot \varphi = 0, \quad \varphi = 0 \text{ on } \Omega^b \setminus \Omega_n, \quad \varphi|_{x_3 = \omega(x_h)} = 0. \tag{3-10}$$

Remark 2.28 states that such a function φ is an appropriate test function for (2-35). In the spirit of Definition 2.27, the weak formulation for (2-35) is

$$\int_{\Omega^b} \nabla u_n \cdot \nabla \varphi + \int_{\Omega^b} u_{n,h}^{\perp} \cdot \varphi_h = -\langle \mathrm{DN}(u_n|_{x_3=0^-}), \varphi|_{x_3=0^-}\rangle_{\mathscr{D}',\mathscr{D}} - \langle F, \varphi|_{x_3=0^-}\rangle_{\mathscr{D}',\mathscr{D}} + \langle f, \varphi\rangle_{\mathscr{D}',\mathscr{D}}. \quad (3\text{-}11)$$

Thanks to the representation formula for DN in Proposition 2.22, and to the estimates (2-33) for f and (2-34) for F, the weak formulation (3-11) still makes sense for $\varphi \in H^1(\Omega^b)$ satisfying (3-10).

In the sequel we drop the n subscripts. Note that all constants appearing in the inequalities below are uniform in n. However, one should be aware that E_k defined by (2-38) *depends on n.* Furthermore, we denote $u|_{x_3=0^-}$ by v_0.

In order to estimate E_k, we introduce a smooth cutoff function $\chi_k = \chi_k(y_h)$ supported in Σ_{k+1} and identically equal to 1 on Σ_k. We carry out energy estimates on the system (2-35). Remember that a test function has to meet the conditions (3-10). We therefore choose

$$\begin{aligned}\varphi = \begin{pmatrix}\varphi_h \\ \nabla\cdot\Phi_h\end{pmatrix} &:= \begin{pmatrix}\chi_k u_h \\ -\nabla_h\cdot(\chi_k \int_{\omega(x_h)}^{z} u_h(x_h, z')\,dz')\end{pmatrix} \in H^1(\Omega^b),\\ &= \chi_k u - \begin{pmatrix} 0 \\ \nabla_h \chi_k(x_h)\cdot \int_{\omega(x_h)}^{z} u_h(x_h,z')\,dz'\end{pmatrix},\end{aligned}$$

which can be readily checked to satisfy (3-10). Notice that this choice of test function is different from the one in [Gérard-Varet and Masmoudi 2010], which is merely $\chi_k u$. Aside from being a suitable test function for (2-35), the function φ has the advantage of being divergence free, so that there will be no need to estimate commutator terms stemming from the pressure.

Plugging φ into the weak formulation (3-11), we get

$$\begin{aligned}\int_\Omega \chi_k |\nabla u|^2 = -\int_\Omega \nabla u\cdot(\nabla\chi_k)u + \int_\Omega \nabla u_3 \cdot \nabla\Big(\nabla_h\chi_k(x_h)\cdot\int_{\omega(x_h)}^{z} u_h(x_h,z')\,dz'\Big)\\ -\langle \mathrm{DN}(v_0), \varphi|_{x_3=0^-}\rangle - \langle F, \varphi|_{x_3=0^-}\rangle + \langle f, \varphi\rangle. \quad (3\text{-}12)\end{aligned}$$

Before coming to the estimates, we state an easy bound on Φ_h and φ:

$$\|\Phi_h\|_{H^1(\Omega^b)} + \|\varphi\|_{H^1(\Omega^b)} + \|\Phi_h|_{x_3=0}\|_{H^{1/2}(\mathbb{R}^2)} + \|\varphi|_{x_3=0}\|_{H^{1/2}(\mathbb{R}^2)} \le C E_{k+1}^{1/2}. \quad (3\text{-}13)$$

As we have recourse to Lemma 2.26 to estimate some terms in (3-12), we use (3-13) repeatedly in the sequel, sometimes with slight changes.

We have to estimate each of the terms appearing in (3-12). The most difficult term is the one involving the Dirichlet to Neumann operator, because of the nonlocal feature: although v_0 is supported in Σ_n, $\mathrm{DN}(v_0)$ is not in general. However, each term in (3-12), except $-\langle \mathrm{DN}(v_0), \varphi|_{x_3=0^-}\rangle$, is local, and hence very easy to bound. Let us sketch the estimates of the local terms. For the first term, we simply use the Cauchy–Schwarz and Poincaré inequalities:

$$\Big|\int_\Omega \nabla u\cdot(\nabla\chi_k)u\Big| \le C\Big(\int_{\Omega_{k,k+1}} |\nabla u|^2\Big)^{\frac12}\Big(\int_{\Omega_{k,k+1}} |u|^2\Big)^{\frac12} \le C(E_{k+1}-E_k).$$

In the same fashion, using (3-13), we find that the second term is bounded by

$$\begin{aligned}
&\left|\int_\Omega \nabla u_3\cdot\nabla\left(\nabla_h\chi_k(x_h)\cdot\int_{\omega(x_h)}^z u_h(x_h,z')\,dz'\right)dx_h\,dz\right|\\
&\quad\le \int_\Omega |\nabla u_3||\nabla\nabla_h\chi_k(x_h)|\int_{\omega(x_h)}^z |u_h(x_h,z')|\,dz'\,dx_h\,dz\\
&\qquad+\int_\Omega |\nabla_h u_3||\nabla_h\chi_k(x_h)|\int_{\omega(x_h)}^z |\nabla_h u_h(x_h,z')|\,dz'\,dx_h\,dz+\int_\Omega |\partial_3 u_3\nabla_h\chi_k(x_h)\cdot u_h(x_h,z)|\,dx_h\,dz\\
&\quad\le C(E_{k+1}-E_k).
\end{aligned}$$

We finally bound the last two terms in (3-12) using (3-13), and (2-34) or (2-33):

$$\begin{aligned}
|\langle F,\varphi|_{x_3=0^-}\rangle| &\le C(k+1)\left[\|\chi_k u_h|_{x_3=0^-}\|_{H^{1/2}(\mathbb{R}^2)}+\left\|\nabla_h\cdot\left(\chi_k\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz'\right)\right\|_{H^{1/2}(\mathbb{R}^2)}\right]\\
&\le C(k+1)[E_{k+1}^{1/2}+(E_{k+1}-E_k)^{1/2}]\le C(k+1)E_{k+1}^{1/2},\\
|\langle f,\varphi\rangle| &\le (k+1)E_{k+1}^{1/2}.
\end{aligned}$$

The last term to handle is $-\langle \mathrm{DN}_h(v_0),\varphi|_{x_3=0^-}\rangle$. The issue of the nonlocality of the Dirichlet to Neumann operator is already present for the Stokes system. Again, we attempt to adapt the ideas of [Gérard-Varet and Masmoudi 2010]. In order to handle the large scales of $\mathrm{DN}(v_0)$, we are led to introduce the auxiliary parameter $m\in\mathbb{N}^*$, which appears in (3-1). We decompose v_0 into

$$\begin{aligned}
v_0=\begin{pmatrix}\chi_k v_{0,h}-\nabla_h\cdot(\chi_k\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix}+\begin{pmatrix}(\chi_{k+m}-\chi_k)v_{0,h}\\ -\nabla_h\cdot((\chi_{k+m}-\chi_k)\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix}\\
+\begin{pmatrix}(1-\chi_{k+m})v_{0,h}\\ -\nabla_h\cdot((1-\chi_{k+m})\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix}.
\end{aligned}$$

The truncations on the vertical component of v_0 are put inside the horizontal divergence in order to apply the Dirichlet to Neumann operator to functions in $\mathbb{K}$.

The term corresponding to the truncation of v_0 by χ_k, namely,

$$\begin{aligned}
&-\left\langle \mathrm{DN}\begin{pmatrix}\chi_k v_{0,h}\\ -\nabla_h\cdot(\chi_k\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix},\begin{pmatrix}\varphi_h|_{x_3=0^-}\\ \nabla_h\cdot\Phi_h|_{x_3=0^-}\end{pmatrix}\right\rangle\\
&\qquad=-\left\langle \mathrm{DN}\begin{pmatrix}\chi_k v_{0,h}\\ -\nabla_h\cdot(\chi_k\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix},\begin{pmatrix}\chi_k v_{0,h}\\ -\nabla_h\cdot(\chi_k\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix}\right\rangle,
\end{aligned}$$

is negative by positivity of the operator DN (see Lemma 2.24). For the term corresponding to the truncation by $\chi_{k+m}-\chi_k$, we resort to Lemma 2.26 and (3-13). This yields

$$\left|\left\langle \mathrm{DN}\begin{pmatrix}(\chi_{k+m}-\chi_k)v_{0,h}\\ -\nabla_h\cdot((\chi_{k+m}-\chi_k)\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz')\end{pmatrix},\begin{pmatrix}\varphi_h|_{x_3=0^-}\\ \nabla_h\cdot\Phi_h|_{x_3=0^-}\end{pmatrix}\right\rangle\right|\le C(E_{k+m+1}-E_k)^{\frac12}E_{k+1}^{1/2}.$$

However, the estimate of Lemma 2.26 is not refined enough to address the large scales independently of n. For the term

$$\left\langle \mathrm{DN}\begin{pmatrix} (1-\chi_{k+m})v_{0,h} \\ -\nabla_h\cdot\big((1-\chi_{k+m})\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz'\big)\end{pmatrix}, \begin{pmatrix}\varphi_h|_{x_3=0^-}\\ \nabla_h\cdot\Phi_h|_{x_3=0^-}\end{pmatrix}\right\rangle,$$

we must have a closer look at the representation formula given in Proposition 2.22. Let

$$\tilde v_0 := \begin{pmatrix} (1-\chi_{k+m})v_{0,h} \\ -\nabla_h\cdot\big((1-\chi_{k+m})\int_{\omega(x_h)}^0 u_h(x_h,z')\,dz'\big)\end{pmatrix} = \begin{pmatrix}(1-\chi_{k+m})v_{0,h}\\ -\nabla_h\cdot\tilde V_h\end{pmatrix}.$$

We take $\chi := \chi_{k+1}$ in the formula of Proposition 2.22. If $m\ge 2$, $\operatorname{Supp}\chi_{k+1}\cap\operatorname{Supp}(1-\chi_{k+m})=\varnothing$, so that the formula of Proposition 2.22 becomes[2]

$$\begin{aligned}\langle \mathrm{DN}\,\tilde v_0,\varphi\rangle &= \int_{\mathbb{R}^2}\varphi|_{x_3=0^-}\cdot K_S*\tilde v_0 + \int_{\mathbb{R}^2}\varphi|_{x_3=0^-}\cdot M^{\mathrm{rem}}_{\mathrm{HF}}*\tilde v_0\\ &\quad+\int_{\mathbb{R}^2}\varphi_{h|x_3=0^-}\cdot\{\mathcal{I}[M_1](\rho*\tilde v_{0,h})+K_1^{\mathrm{rem}}*\tilde v_{0,h}\}\\ &\quad+\int_{\mathbb{R}^2}\varphi_{h|x_3=0^-}\cdot\{\mathcal{I}[M_2](\rho*\tilde V_h)+K_2^{\mathrm{rem}}*\tilde V_h\}\\ &\quad+\int_{\mathbb{R}^2}\Phi_{h|x_3=0^-}\cdot\{\mathcal{I}[M_3](\rho*\tilde v_{0,h})+K_3^{\mathrm{rem}}*\tilde v_{0,h}\}\\ &\quad+\int_{\mathbb{R}^2}\Phi_{h|x_3=0^-}\cdot\{\mathcal{I}[M_4](\rho*\tilde V_h)+K_4^{\mathrm{rem}}*\tilde V_h\}.\end{aligned}$$

Thus we have two types of terms to estimate:

- On the one hand are the convolution terms with the kernels K_S, $M^{\mathrm{rem}}_{\mathrm{HF}}$, and K_i^{rem} for $1\le i\le 4$, which all decay like $1/|x_h|^3$.
- On the other hand are the terms involving $\mathcal{I}[M_i]$ for $1\le i\le 4$.

For the first ones, we rely on the following nontrivial estimate:

Lemma 3.1. *For all $k\ge m$,*

$$\left\|\tilde v_0 * \frac{1}{|\cdot|^3}\right\|_{L^2(\Sigma_{k+1})} \le C\frac{k^{3/2}}{m^2}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right)^{\frac12}. \tag{3-14}$$

This estimate still holds with $\tilde V_h$ in place of $\tilde v_0$.

For the second ones, we have recourse to:

Lemma 3.2. *For all $k\ge m$ and all $1\le i,j\le 2$,*

$$\left\|\mathcal{I}\left[\frac{\xi_i\xi_j}{|\xi|}\right](\rho*\tilde v_{0,h})\right\|_{L^2(\Sigma_{k+1})} \le C\frac{k^2}{m^{5/2}}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right)^{\frac12}. \tag{3-15}$$

[2] Here we use in a crucial (but hidden) way the fact that the zero-order terms at low frequencies are constant. Indeed, such terms are local, so that $\int_{\mathbb{R}^2}\varphi|_{x_3=0^-}\cdot\overline{M}\tilde v_0 = 0$.

This estimate still holds with $\tilde{V}_h$ in place of $v_{0,h}$.

We postpone the proofs of these two key lemmas to Section 3C. Applying repeatedly Lemmas 3.1 and Lemma 3.2 together with the estimates (3-13), we are finally led to the estimate

$$E_k \le C\left((k+1)E_{k+1}^{1/2} + (E_{k+1} - E_k) + E_{k+1}^{1/2}(E_{k+m+1} - E_k)^{1/2} + \frac{k^2}{m^{5/2}} E_{k+1}^{1/2}\left(\sup_{j\ge k+m} \frac{E_{j+m} - E_j}{j}\right)^{\frac{1}{2}}\right)$$

for all $k \ge m \ge 1$. Now, since E_k is increasing in k, we have

$$E_{k+1} \le E_k + (E_{k+m+1} - E_k).$$

Using Young's inequality, we infer that, for all $\nu > 0$, there exists a constant C_ν such that, for all $k \ge 1$,

$$E_k \le \nu E_k + C_\nu\left(k^2 + E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j\ge k+m} \frac{E_{j+m} - E_j}{j}\right).$$

Choosing $\nu < 1$, inequality (3-1) follows.

3C. *Proof of the key lemmas.* It remains to establish the estimates (3-14) and (3-15). The proofs are quite technical, but similar ideas and tools are used in both.

Proof of Lemma 3.1. We use an idea of Gérard-Varet and Masmoudi [2010] to treat the large scales: we decompose the set $\Sigma \setminus \Sigma_{k+m}$ as

$$\Sigma \setminus \Sigma_{k+m} = \bigcup_{j=1}^{\infty} \Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}.$$

On every set $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$, we bound the L^2 norm of $\tilde{v}_0$ by $E_{k+m(j+1)} - E_{k+mj}$. Let us stress here a technical difference with the work of Gérard-Varet and Masmoudi: since Σ has dimension two, the area of the set $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$ is of order $(k+mj)m$. In particular, we expect

$$E_{k+m(j+1)} - E_{k+mj} \sim (k+mj)m\|u\|^2_{H^1_{\mathrm{uloc}}}$$

to grow with j. Thus we work with the quantity

$$\sup_{j\ge k+m} \frac{E_{j+m} - E_j}{j},$$

which we expect to be bounded uniformly in n, k, rather than with $\sup_{j\ge k+m}(E_{j+m} - E_j)$.

Now, applying the Cauchy–Schwarz inequality yields, for $\eta > 0$,

$$\int_{\Sigma_{k+1}} dy \left(\int_{\mathbb{R}^2} \frac{1}{|y-t|^3} \tilde{v}_0(t)\, dt\right)^2 \le C \int_{\Sigma_{k+1}} dy \int_{\Sigma\setminus\Sigma_{k+m}} \frac{|t|}{|y-t|^{3+2\eta}}\, dt \int_{\Sigma\setminus\Sigma_{k+m}} \frac{|\tilde{v}_0(t)|^2}{|t|\,|y-t|^{3-2\eta}}\, dt.$$

The role of the division by the $|t|$ factor in the second integral is precisely to force the apparition of the quantities $(E_{j+m}-E_j)/j$. More precisely, for $y\in\Sigma_{k+1}$ and $m\geq 1$,

$$\begin{aligned}\int_{\Sigma\setminus\Sigma_{k+m}}\frac{|\tilde v_0(t)|^2}{|t|\,|y-t|^{3-2\eta}}\,dt&=\sum_{j=1}^{\infty}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}}\frac{|\tilde v_0(t)|^2}{|t|\,|y-t|^{3-2\eta}}\,dt\\&\leq C\sum_{j=1}^{\infty}(E_{k+m(j+1)}-E_{k+mj})\frac{1}{(k+mj)|mj+k-|y|_\infty|^{3-2\eta}}\\&\leq C\biggl(\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\biggr)\sum_{j=1}^{\infty}\frac{1}{|mj+k-|y|_\infty|^{3-2\eta}}\\&\leq C_\eta\frac{1}{m}\frac{1}{|m+k-|y|_\infty|^{2-2\eta}}\biggl(\sup_{j\geq k+m}\frac{E_{j+m}-E_j}{j}\biggr),\end{aligned}$$

where $|x|_\infty:=\max(|x_1|,|x_2|)$ for $x\in\mathbb{R}^2$. A simple rescaling yields

$$\begin{aligned}\int_{\Sigma_{k+1}}\int_{\Sigma\setminus\Sigma_{k+m}}&\frac{|t|}{|y-t|^{3+2\eta}|m+k-|y|_\infty|^{2-2\eta}}\,dt\,dy\\&=\int_{\Sigma_{1+1/k}}\int_{\Sigma\setminus\Sigma_{1+m/k}}\frac{|t|}{|y-t|^{3+2\eta}|1+m/k-|y|_\infty|^{2-2\eta}}\,dt\,dy.\end{aligned}$$

Let us assume that $k\geq m\geq 2$ and take $\eta\in\left]\frac12,1\right[$. We decompose $\Sigma\setminus\Sigma_{1+m/k}$ as $(\Sigma\setminus\Sigma_2)\cup(\Sigma_2\setminus\Sigma_{1+m/k})$. On the one hand, since $|t-y|\geq C|t-y|_\infty\geq C(|t|_\infty-|y|_\infty)\geq C(|t|_\infty-3/2)$,

$$\int_{\Sigma_{1+1/k}}\int_{\Sigma\setminus\Sigma_2}\frac{|t|}{|y-t|^{3+2\eta}|1+m/k-|y|_\infty|^{2-2\eta}}\,dt\,dy\leq C_\eta\int_{\Sigma_{1+1/k}}\frac{dy}{|1+m/k-|y|_\infty|^{2-2\eta}}.$$

Decomposing $\Sigma_{1+1/k}$ into elementary regions of the type $\Sigma_{r+dr}\setminus\Sigma_r$, on which $|y|_\infty\simeq r$, we infer that the right-hand side of the above inequality is bounded by

$$\begin{aligned}C\int_0^{1+1/k}\frac{r}{|1+m/k-r|^{2-2\eta}}\,dr&\leq C\int_0^{1+1/k}\frac{dr}{|r+(m-1)/k|^{2-2\eta}}\\&\leq C_\eta\biggl(\Bigl(1+\frac{m}{k}\Bigr)^{2\eta-1}-\Bigl(\frac{m-1}{k}\Bigr)^{2\eta-1}\biggr)\leq C_\eta.\end{aligned}$$

On the other hand, $y\in\Sigma_{1+1/k}$ implies $|1+m/k-|y|_\infty|\geq(m-1)/k$, so

$$\begin{aligned}\int_{\Sigma_{1+1/k}}\int_{\Sigma_2\setminus\Sigma_{1+m/k}}&\frac{|t|}{|y-t|^{3+2\eta}|1+m/k-|y|_\infty|^{2-2\eta}}\,dt\,dy\\&\leq C\Bigl(\frac{k}{m-1}\Bigr)^{2-2\eta}\int_{\Sigma_{1+1/k}}dy\int_{\Sigma_2\setminus\Sigma_{1+m/k}}\frac{dt}{|t-y|^{3+2\eta}}\\&\leq C\Bigl(\frac{k}{m-1}\Bigr)^{2-2\eta}\int_{X\in\mathbb{R}^2,\,(m-1)/k\leq|X|\leq C}\frac{dX}{|X|^{3+2\eta}}\leq C_\eta\Bigl(\frac{k}{m}\Bigr)^3.\end{aligned}$$

Gathering these bounds leads to (3-14). ∎

Proof of Lemma 3.2. As in the preceding proof, the overall strategy is to decompose

$$(1-\chi_{k+m})v_{0,h}=\sum_{j=1}^{\infty}(\chi_{k+m(j+1)}-\chi_{k+mj})v_{0,h}.$$

In the course of the proof, we introduce some auxiliary parameters, whose meanings we explain. We cannot use Lemma 2.10 as such, because we will need a much finer estimate. We therefore rely on the splitting (2-19) with $K := m/2$. An important property is the fact that $\rho := \mathcal{F}^{-1}\phi$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of rapidly decreasing functions.

As in the proof of Lemma 2.10, for $K = m/2$ and $x \in \Sigma_{k+1}$, we have

$$|A(x)| \le Cm\|\nabla^2\rho * ((1-\chi_{k+m}v_{0,h}))\|_{L^\infty(\Sigma_{k+1+m/2})},$$

and for all $\alpha > 0$ and all $y \in \Sigma_{k+1+m/2}$,

$$\begin{aligned}|\nabla^2\rho * (1-\chi_{k+m})v_{0,h}(y)| &\le \int_{\Sigma\setminus\Sigma_{k+m}} |\nabla^2\rho(y-t)|\,|v_{0,h}(t)|\,dt\\ &\le \left(\int_{\Sigma\setminus\Sigma_{k+m}} |\nabla^2\rho(y-t)|^2|t|^\alpha\,dt\right)^{\frac12}\left(\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha}\,dt\right)^{\frac12}.\end{aligned}$$

Yet, on the one hand, for $\alpha > 2$,

$$\begin{aligned}\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha}\,dt &= \sum_{j=1}^{\infty}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} \frac{|v_{0,h}(t)|^2}{|t|^\alpha}\,dt\\ &\le \left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right)\sum_{j=1}^{\infty}\frac{1}{(k+mj)^{\alpha-1}}\\ &\le C\frac{1}{m}\frac{1}{(k+m)^{\alpha-2}}\left(\sup_{j\ge k+m}\frac{E_{j+m}-E_j}{j}\right).\end{aligned}$$

On the other hand, $y \in \Sigma_{k+1+m/2}$ and $t \in \Sigma\setminus\Sigma_{k+m}$ implies $|y-t| \ge m/2-1$,

$$\begin{aligned}\int_{\Sigma\setminus\Sigma_{k+m}} |\nabla^2\rho(y-t)|^2|t|^\alpha\,dt &\le C\int_{\Sigma\setminus\Sigma_{k+m}} |\nabla^2\rho(y-t)|^2(|y-t|^\alpha+|y|^\alpha)\,dt\\ &\le C\left(\left(k+1+\frac{m}{2}\right)^\alpha\int_{|s|\ge m/2-1}|\nabla^2\rho(s)|^2+\int_{|s|\ge m/2-1}|\nabla^2\rho(s)|^2|s|^\alpha\right).\end{aligned}$$

Now, since $\rho \in \mathcal{S}(\mathbb{R}^2)$, for all $\beta > 0$, $\alpha > 0$, there exists a constant $C_{\alpha,\beta}$ such that

$$\int_{|s|\ge m/2-1}(1+|s|^\alpha)|\nabla^2\rho(s)|^2 \le C_\beta m^{-2\beta}.$$

The role of auxiliary parameter β is to "eat" the powers of k in order to get a Saint-Venant estimate for which the induction procedure of Section 3A works. Gathering the latter bounds, we obtain, for $k \geq m$,

$$\|A\|_{L^\infty(\Sigma_{k+1})} \leq C_\beta k m^{-\beta}\left(\sup_{j\geq k+m} \frac{E_{j+m}-E_j}{j}\right)^{\frac{1}{2}}. \tag{3-16}$$

The second term in (2-19) is even simpler to estimate. One ends up with

$$\|B\|_{L^\infty(\Sigma_{k+1})} \leq C_\beta k m^{-\beta}\left(\sup_{j\geq k+m} \frac{E_{j+m}-E_j}{j}\right)^{\frac{1}{2}}. \tag{3-17}$$

Therefore A and B satisfy the desired estimate, since

$$\|A\|_{L^2(\Sigma_{k+1})} \leq Ck\|A\|_{L^\infty(\Sigma_{k+1})}, \qquad \|B\|_{L^2(\Sigma_{k+1})} \leq Ck\|B\|_{L^\infty(\Sigma_{k+1})}.$$

The last integral in (2-19) is more intricate, because it is a convolution integral. Furthermore, $\rho * (1-\chi_{k+m})v_{0,h}(y)$ is no longer supported in $\Sigma \setminus \Sigma_{k+m}$. The idea is to "exchange" the variables y and t, that is, to replace the kernel $|x-y|^{-3}$ by $|x-t|^{-3}$. Indeed, we have, for all $x, y, t \in \mathbb{R}^2$,

$$\left|\frac{1}{|x-y|^3} - \frac{1}{|x-t|^3}\right| \leq \frac{C|y-t|}{|x-y|\,|x-t|^3} + \frac{C|y-t|}{|x-y|^3|x-t|}. \tag{3-18}$$

We decompose the integral term accordingly. We obtain, using the fast decay of ρ,

$$\begin{aligned}
&\int_{|x-y|\geq m/2} dy\, \frac{1}{|x-y|^3}|\rho * ((1-\chi_{k+m})v_{0,h})(y)| \\
&\quad\leq C\int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt\, \frac{1}{|x-t|^3}|\rho(y-t)|\,|v_{0,h}(t)| \\
&\qquad\qquad + C\int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt\, \frac{|y-t|}{|x-y|^3|x-t|}|\rho(y-t)|\,|v_{0,h}(t)| \\
&\qquad\qquad + C\int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt\, \frac{|y-t|}{|x-y|\,|x-t|^3}|\rho(y-t)|\,|v_{0,h}(t)| \\
&\quad\leq C\int_{\Sigma\setminus\Sigma_{k+m}} dt\, \frac{1}{|x-t|^3}|v_{0,h}(t)| + C\int_{|x-y|\geq m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} dt\, \frac{|y-t|}{|x-y|^3|x-t|}|\rho(y-t)|\,|v_{0,h}(t)|.
\end{aligned}$$

The first term on the right hand side above can be addressed thanks to Lemma 3.1. We focus on the second term. As above, we use the Cauchy–Schwarz inequality

$$\begin{aligned}
&\int_{\Sigma\setminus\Sigma_{k+m}} \frac{|y-t|\,|\rho(y-t)|}{|x-t|}|v_{0,h}(t)|\,dt \\
&\qquad\leq \sum_{j=1}^{\infty}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} \frac{|y-t|\,|\rho(y-t)|}{|x-t|}|v_{0,h}(t)|\,dt \\
&\qquad\leq \left(\sup_{j\geq k+m}\frac{E_{m+j}-E_j}{j}\right)^{\frac{1}{2}} \sum_{j=1}^{\infty}\frac{1}{k+mj-|x|_\infty}\left(\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2|\rho(y-t)|^2|t|\,dt\right)^{\frac{1}{2}}.
\end{aligned}$$

The idea is to use the fast decay of ρ so as to bound the integral over $\Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$. However, $\sum_{j=1}^{\infty} 1/(k+mj-|x|) = \infty$, so that we also need to recover some decay with respect to j in this integral. For $t \in \Sigma_{k+m(j+1)} \setminus \Sigma_{k+mj}$,

$$1 \le \frac{|t|-|x|_\infty}{k+mj-|x|_\infty} \le \frac{|t|}{k+mj-|x|_\infty},$$

so that, for all $\eta > 0$,

$$\begin{aligned}\int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} & |y-t|^2|\rho(y-t)|^2|t|\,dt \\ &\le \frac{1}{(k+mj-|x|_\infty)^{2\eta}} \int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2|\rho(y-t)|^2|t|^{1+2\eta}\,dt \\ &\le \frac{C}{(k+mj-|x|_\infty)^{2\eta}} \int_{\Sigma_{k+m(j+1)}\setminus\Sigma_{k+mj}} |y-t|^2(|y-t|^{1+2\eta}+|y|^{1+2\eta})|\rho(y-t)|^2\,dt \\ &\le \frac{C_\eta}{(k+mj-|x|_\infty)^{2\eta}}(1+|y-x|^{1+2\eta}+|x|^{1+2\eta})).\end{aligned}$$

Summing in j, we have, as before,

$$\sum_{j=1}^{\infty} \frac{1}{(k+mj-|x|_\infty)^{1+\eta}} \le \frac{C_\eta}{m(k+m-|x|_\infty)^\eta} \le \frac{C_\eta}{m^{1+\eta}}$$

so that, for $0<\eta<\frac{1}{2}$, one finally obtains, for $x \in \Sigma_{k+1}$,

$$\begin{aligned}\int_{|x-y|\ge m/2} dy \int_{\Sigma\setminus\Sigma_{k+m}} & \frac{|y-t|\,|\rho(y-t)|}{|x-y|^3|x-t|}|v_{0,h}(t)|\,dt \\ &\le Cm^{-1-\eta}\left(\sup_{j\ge k+m}\frac{E_{m+j}-E_j}{j}\right)^{\frac{1}{2}} \int_{|x-y|\ge m/2} [|x-y|^{-\frac{5}{2}+\eta}+|x|^{\frac{1}{2}+\eta}|x-y|^{-3}]\,dy \\ &\le Cm^{-\frac{3}{2}}\left[1+\left(\frac{k}{m}\right)^{\frac{1}{2}+\eta}\right]\left(\sup_{j\ge k+m}\frac{E_{k+j}-E_j}{j}\right)^{\frac{1}{2}}.\end{aligned}$$

Gathering all the terms, and again using the fact that

$$\|F\|_{L^2(\Sigma_{k+1})} \le Ck\|F\|_{L^\infty(\Sigma_{k+1})} \quad \text{for all } F \in L^\infty(\Sigma_{k+1}),$$

we infer that, for all $k \ge m$ and all $\eta > 0$,

$$\|C\|_{L^2(\Sigma_{k+1})} \le C_\eta \frac{k^{3/2+\eta}}{m^{2+\eta}}\left(\sup_{j\ge k+m}\frac{E_{k+j}-E_j}{j}\right)^{\frac{1}{2}}.$$

Choose $\eta = 1/2$; Lemma 3.2 is thus proved. □

4. Uniqueness

This section is devoted to the proof of uniqueness of solutions of (2-32). Therefore we consider the system (2-32) with $f=0$ and $F=0$, and we intend to prove that the solution u is identically zero.

Following the notations of the previous section, we set

$$E_k := \int_{\Omega_k} \nabla u \cdot \nabla u.$$

We can carry out the same estimates as those of Section 3B and get a constant $C_1 > 0$ such that, for all $m \in \mathbb{N}$ and all $k \geq m$,

$$E_k \leq C_1 \left(E_{k+m+1} - E_k + \frac{k^4}{m^5} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j} \right). \tag{4-1}$$

Let m be a positive even integer and $\varepsilon > 0$ be fixed. Analogously to Section 3A, the set $\mathscr{C}_m$ is defined by

$$\mathscr{C}_m := \{c, \text{ square with edge of length } m \text{ with vertices in } \mathbb{Z}^2\}.$$

Note that the situation is not quite the same as in Section 3A since this set is infinite. The values of $E_c := \int_{\Omega_c} |\nabla u|^2$ when $c \in \mathscr{C}_m$ are bounded by $Cm^2 \|u\|^2_{H^1_{\text{uloc}}(\Omega^b)}$, so the following supremum exists:

$$\mathscr{E}_m := \sup_{c \in \mathscr{C}_m} E_c < \infty,$$

but it may not be attained. Therefore, for $\varepsilon > 0$, we choose a square $c \in \mathscr{C}_m$ such that $\mathscr{E}_m - \varepsilon \leq E_c \leq \mathscr{E}_m$. As in Section 3A, up to a shift we can always assume that c is centered in 0.

From (4-1), we retrieve, for all $m, k \in \mathbb{N}$ with $k \geq m$,

$$E_k \leq \frac{C_1}{C_1+1} E_{k+m+1} + \frac{C_1}{C_1+1} \frac{k^4}{m^5} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j}.$$

Again, the conclusion $E_k = 0$ would be very easy to get if there were no second term in the right-hand side taking into account the large scales due to the nonlocal operator DN.

An induction argument then implies that, for all $r \in \mathbb{N}$,

$$E_k \leq \left(\frac{C_1}{C_1+1}\right)^r E_{k+r(m+1)} + \sum_{r'=0}^{r-1} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} \sup_{j \geq k+m} \frac{E_{j+m} - E_j}{j}. \tag{4-2}$$

Now, for $\kappa := \ln(C_1/(C_1+1)) < 0$ and $k \in \mathbb{N}$ large enough, the function $x \mapsto \exp(\kappa(x+1))(k+x(m+1))^4$ is decreasing on $(-1, \infty)$, so that

$$\begin{aligned}
\sum_{r'=0}^{r-1} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} &\leq \sum_{r'=0}^{\infty} \left(\frac{C_1}{C_1+1}\right)^{r'+1} \frac{(k+r'(m+1))^4}{m^5} \\
&\leq \frac{1}{m^5} \int_{-1}^{\infty} \exp(\kappa(x+1))(k+x(m+1))^4 \, dx \\
&\leq C \frac{k^5}{m^6} \int_{-(m+1)/k}^{\infty} \exp\left(\frac{\kappa k}{m+1} u\right)(1+u)^4 \, du, \\
&\leq C \frac{k^5}{m^6}
\end{aligned}$$

since $k/(m+1) \geq 1/2$ as soon as $k \geq m \geq 1$. Therefore, we conclude from (4-2) for $k = m$ that, for all $r \in \mathbb{N}$,

$$\begin{aligned}
\mathscr{E}_m - \varepsilon \leq E_m = E_c &\leq \left(\frac{C_1}{C_1+1}\right)^r E_{m+r(m+1)} + \frac{C}{m} \sup_{j\geq 2m} \frac{E_{j+m}-E_j}{j} \\
&\leq \left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 \|u\|^2_{H^1_{\mathrm{uloc}}} + 4\frac{C}{m} \sup_{j\geq 2m} \frac{j+m}{jm} \mathscr{E}_m \\
&\leq \left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 \|u\|^2_{H^1_{\mathrm{uloc}}} + \frac{C}{m^2} \mathscr{E}_m.
\end{aligned}$$

Since the constants are uniform in m, we have, for m sufficiently large and for all $\varepsilon > 0$,

$$\mathscr{E}_m \leq C\left[\left(\frac{C_1}{C_1+1}\right)^r (r+1)^2(m+1)^2 + \varepsilon\right],$$

which, letting $r \to \infty$ and $\varepsilon \to 0$, gives $\mathscr{E}_m = 0$. The latter holds for all m large enough, and thus we have $u = 0$.

Appendix A. Proof of Lemmas 2.3 and 2.4

This section is devoted to the proofs of Lemma 2.3, which gives a formula for the determinant of M, and Lemma 2.4, which contains the low and high frequency expansions of the main functions we work with, namely, λ_k and A_k. As A_1, A_2, A_3 can be expressed in terms of the eigenvalues λ_k solution to (2-5), it is essential to begin by stating some properties of the latter. Usual properties on the roots of polynomials entail that the eigenvalues satisfy

$$\begin{gathered}
\Re(\lambda_k) > 0 \quad \text{for } k = 1, 2, 3, \qquad \lambda_1 \in \,]0, \infty[, \qquad \lambda_2 = \overline{\lambda_3}, \\
-(\lambda_1\lambda_2\lambda_3)^2 = -|\xi|^6, \quad \lambda_1\lambda_2\lambda_3 = |\xi|^3, \quad (|\xi|^2-\lambda_1^2)(|\xi|^2-\lambda_2^2)(|\xi|^2-\lambda_3^2) = |\xi|^2, \\
\frac{(|\xi|^2-\lambda_k^2)^2}{\lambda_k} = \frac{\lambda_k}{|\xi|^2-\lambda_k^2},
\end{gathered} \tag{A-1}$$

and can be computed exactly:

$$\lambda_1^2(\xi) = |\xi|^2 + \left(\frac{-|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}} - \left(\frac{|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}}, \tag{A-2a}$$

$$\lambda_2^2(\xi) = |\xi|^2 + j\left(\frac{-|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}} - j^2\left(\frac{|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}}, \tag{A-2b}$$

$$\lambda_3^2(\xi) = |\xi|^2 + j^2\left(\frac{-|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}} - j\left(\frac{|\xi|^2 + (|\xi|^4+4/27)^{1/2}}{2}\right)^{\frac{1}{3}}. \tag{A-2c}$$

A.1. ***Expansion of the eigenvalues*** $\boldsymbol{\lambda_k}$**.** The expansions below follow directly from the exact formulas (A-2). In high frequencies, that is, for $|\xi| \gg 1$, we have

$$\lambda_1^2 = |\xi|^2(1 - |\xi|^{-\frac{4}{3}} + O(|\xi|^{-\frac{8}{3}})), \qquad \lambda_1 = |\xi| - \tfrac{1}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}), \tag{A-3a}$$

$$\lambda_2^2 = |\xi|^2(1 - j^2|\xi|^{-\frac{4}{3}} + O(|\xi|^{-\frac{8}{3}})), \quad \lambda_2 = |\xi| - \frac{j^2}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}), \tag{A-3b}$$

$$\lambda_3^2 = |\xi|^2(1 - j|\xi|^{-\frac{4}{3}} + O(|\xi|^{-\frac{8}{3}})), \qquad \lambda_3 = |\xi| - \frac{j}{2}|\xi|^{-\frac{1}{3}} + O(|\xi|^{-\frac{5}{3}}). \tag{A-3c}$$

In low frequencies, that is, for $|\xi| \ll 1$, we have

$$\begin{aligned}
\left(|\xi|^4 + \frac{4}{27}\right)^{\frac{1}{2}} &= \frac{2}{\sqrt{27}}\left[1 + \frac{27}{8}|\xi|^4 + O(|\xi|^8)\right],\\
\left(\frac{-|\xi|^2 + (|\xi|^4 + 4/27)^{1/2}}{2}\right)^{\frac{1}{3}} &= \frac{1}{\sqrt{3}} - \frac{1}{2}|\xi|^2 - \frac{\sqrt{3}}{8}|\xi|^4 + O(|\xi|^6),\\
\left(\frac{|\xi|^2 + (|\xi|^4 + 4/27)^{1/2}}{2}\right)^{\frac{1}{3}} &= \frac{1}{\sqrt{3}} + \frac{1}{2}|\xi|^2 - \frac{\sqrt{3}}{8}|\xi|^4 + O(|\xi|^6),
\end{aligned}$$

from which we deduce

$$\lambda_2^2 = i + \tfrac{3}{2}|\xi|^2 - \tfrac{3}{8}i|\xi|^4 + O(|\xi|^6), \qquad \lambda_2 = e^{i\pi/4}\big(1 - \tfrac{3}{4}i|\xi|^2 + \tfrac{3}{32}|\xi|^4 + O(|\xi|^6)\big), \tag{A-4a}$$

$$\lambda_3^2 = -i + \tfrac{3}{2}|\xi|^2 + \tfrac{3}{8}i|\xi|^4 + O(|\xi|^6), \quad \lambda_3 = e^{-i\pi/4}\big(1 + \tfrac{3}{4}i|\xi|^2 + \tfrac{3}{32}|\xi|^4 + O(|\xi|^6)\big). \tag{A-4b}$$

Since $\lambda_1\lambda_2\lambda_3 = |\xi|^3$, we infer that

$$\lambda_1 = |\xi|^3 + O(|\xi|^7).$$

A.2. ***Expansion of*** $\boldsymbol{A_1}$***,*** $\boldsymbol{A_2}$***, and*** $\boldsymbol{A_3}$**.** Let us recall that $A_k = A_k(\xi)$, $k = 1, \dots, 3$, solve the linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1\\ \lambda_1 & \lambda_2 & \lambda_3\\ (|\xi|^2 - \lambda_1^2)^2/\lambda_1 & (|\xi|^2 - \lambda_2^2)^2/\lambda_2 & (|\xi|^2 - \lambda_3^2)^2/\lambda_3 \end{pmatrix}}_{=:M(\xi)} \begin{pmatrix} A_1\\ A_2\\ A_3 \end{pmatrix} = \begin{pmatrix} \widehat{v_{0,3}}\\ i\xi \cdot \widehat{v_{0,h}}\\ -i\xi^\perp \cdot \widehat{v_{0,h}} \end{pmatrix}.$$

The exact computation of A_k is not necessary. For the record, note however that A_k can be written in the form of a quotient

$$A_k = \frac{P(\xi_1, \xi_2, \lambda_1, \lambda_2, \lambda_3)}{Q(|\xi|, \lambda_1, \lambda_2, \lambda_3)}, \tag{A-5}$$

where P is a polynomial with complex coefficients and

$$Q := \det(M) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(|\xi| + \lambda_1 + \lambda_2 + \lambda_3). \tag{A-6}$$

This formula for $\det(M)$ is shown using the relations (A-1):

$$\begin{aligned}
&\det(M)\\
&= \frac{\lambda_2^2(|\xi|^2-\lambda_3^2)^2-\lambda_3^2(|\xi|^2-\lambda_2^2)^2}{\lambda_2\lambda_3}-\frac{\lambda_1^2(|\xi|^2-\lambda_3^2)^2-\lambda_3^2(|\xi|^2-\lambda_1^2)^2}{\lambda_1\lambda_3}+\frac{\lambda_1^2(|\xi|^2-\lambda_2^2)^2-\lambda_2^2(|\xi|^2-\lambda_1^2)^2}{\lambda_1\lambda_2}\\
&= |\xi|(\lambda_1(\lambda_2^2-\lambda_3^2)-\lambda_2(\lambda_1^2-\lambda_3^2)+\lambda_3(\lambda_1^2-\lambda_2^2))+\lambda_2\lambda_3(\lambda_3^2-\lambda_2^2)-\lambda_1\lambda_3(\lambda_3^2-\lambda_1^2)+\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)\\
&= (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(|\xi|+\lambda_1+\lambda_2+\lambda_3).
\end{aligned}$$

This proves (A-6), and thus Lemma 2.3.

We now concentrate on the expansions of $M(\xi)$ for $|\xi|\gg 1$ and $|\xi|\ll 1$.

A.2.1. *High frequency expansion.* At high frequencies, it is convenient to work with the quantities B_1, B_2, B_3 introduced in (2-12). Indeed, inserting the expansions (A-3) into the system (2-7) yields

$$\begin{aligned}
B_1 &= \widehat{v_{0,3}},\\
|\xi|B_1-\tfrac{1}{2}|\xi|^{-\frac{1}{3}}B_2+O(|\xi|^{-\frac{5}{3}}|A|) &= i\xi\cdot\widehat{v_{0,h}},\\
|\xi|^{\frac{1}{3}}B_3+O(|\xi|^{-1}|A|) &= -i\xi^{\perp}\cdot\widehat{v_{0,h}}.
\end{aligned}$$

Of course A and B are of the same order, so that the above system becomes

$$\begin{aligned}
&B_1=\hat{v}_{0,3},\\
&B_2=2|\xi|^{\frac{1}{3}}(|\xi|\widehat{v_{0,3}}-i\xi\cdot\widehat{v_{0,h}})+O(|\xi|^{-\frac{4}{3}}|B|),\\
&B_3=-i|\xi|^{-\frac{1}{3}}\xi^{\perp}\cdot\widehat{v_{0,h}}+O(|\xi|^{-\frac{4}{3}}|B|).
\end{aligned}$$

We infer immediately that $|B|=O(|\xi|^{4/3}|\widehat{v_0}|)$, and therefore the result of Lemma 2.4 follows.

A.2.2. *Low frequency expansion.* At low frequencies, we invert M thanks to the adjugate matrix formula

$$M^{-1}(\xi)=\frac{1}{\det(M(\xi))}[\mathrm{Cof}(M(\xi))]^T.$$

We have

$$\frac{(|\xi|^2-\lambda_2^2)^2}{\lambda_2}=\frac{e^{i\pi}(1+O(|\xi|^2))}{e^{i\pi/4}(1+O(|\xi|^2))}=-e^{-i\pi/4}+O(|\xi|^2)=\overline{\frac{(|\xi|^2-\lambda_3^2)^2}{\lambda_3}}.$$

Hence,

$$M(\xi)=\begin{pmatrix}1 & 1 & 1\\ O(|\xi|^3) & e^{i\pi/4}+O(|\xi|^2) & e^{-i\pi/4}+O(|\xi|^2)\\ |\xi|+O(|\xi|^5) & -e^{-i\pi/4}+O(|\xi|^2) & -e^{i\pi/4}+O(|\xi|^2)\end{pmatrix}$$

and

$$\mathrm{Cof}(M)=\begin{pmatrix}-2i & |\xi|e^{-i\pi/4} & -|\xi|e^{i\pi/4}\\ \sqrt{2}i & -e^{i\pi/4}-|\xi| & e^{-i\pi/4}+|\xi|\\ -\sqrt{2}i & -e^{-i\pi/4} & e^{i\pi/4}\end{pmatrix}+O(|\xi|^2).$$

We deduce that

$$M^{-1}(\xi) = -\frac{1}{2i(1+(\sqrt{2}/2)|\xi|+O(|\xi|^2))}[\mathrm{Cof}(M(\xi))]^T$$
$$= \begin{pmatrix} 1-\frac{\sqrt{2}}{2}|\xi| & -\frac{\sqrt{2}}{2}\big[1-\frac{\sqrt{2}}{2}|\xi|\big] & +\frac{\sqrt{2}}{2}\big[1-\frac{\sqrt{2}}{2}|\xi|\big] \\ (e^{i\pi/4}/2)|\xi| & -(1/2i)\big[-e^{i\pi/4}-\big(1-\frac{\sqrt{2}}{2}e^{i\pi/4}\big)|\xi|\big] & -(e^{i\pi/4}/2)\big[1-\frac{\sqrt{2}}{2}|\xi|\big] \\ (e^{-i\pi/4}/2)|\xi| & -(1/2i)\big[e^{-i\pi/4}+\big(1-\frac{\sqrt{2}}{2}e^{-i\pi/4}\big)|\xi|\big] & -(e^{-i\pi/4}/2)\big[1-\frac{\sqrt{2}}{2}|\xi|\big] \end{pmatrix} + O(|\xi|^2).$$

Finally,

$$A_1 = \left(1-\frac{\sqrt{2}}{2}|\xi|\right)\widehat{v_{0,3}} - \frac{\sqrt{2}}{2}i(\xi+\xi^\perp)\cdot\widehat{v_{0,h}} + O(|\xi|^2|\widehat{v_0}|), \tag{A-7a}$$

$$A_2 = \frac{e^{i\pi/4}}{2}|\xi|\widehat{v_{0,3}} + \tfrac{1}{2}e^{i\pi/4}\xi\cdot\widehat{v_{0,h}} - \tfrac{1}{2}e^{-i\pi/4}\xi^\perp\cdot\widehat{v_{0,h}} + O(|\xi|^2|\widehat{v_0}|), \tag{A-7b}$$

$$A_3 = \frac{e^{-i\pi/4}}{2}|\xi|\widehat{v_{0,3}} - \tfrac{1}{2}e^{-i\pi/4}\xi\cdot\widehat{v_{0,h}} + \tfrac{1}{2}e^{i\pi/4}\xi^\perp\cdot\widehat{v_{0,h}} + O(|\xi|^2|\widehat{v_0}|). \tag{A-7c}$$

A.3. ***Low frequency expansion for L_1, L_2, and L_3.*** For the sake of completeness, we sketch the low-frequency expansion of L_1 in detail. We recall that

$$L_k(\xi)\widehat{v_0}(\xi) = \begin{pmatrix} (i/|\xi|^2)(-\lambda_k\xi + ((|\xi|^2-\lambda_k^2)^2/\lambda_k)\xi^\perp) \\ 1 \end{pmatrix} A_k(\xi)$$

Hence, for $|\xi| \ll 1$,

$$L_1(\xi) = \begin{pmatrix} (i/|\xi|)\xi^\perp + O(|\xi|^2) \\ 1 \end{pmatrix} \begin{pmatrix} -\frac{i\sqrt{2}}{2}(\xi_1-\xi_2) & -\frac{i\sqrt{2}}{2}(\xi_1+\xi_2) & 1-\frac{\sqrt{2}}{2}|\xi| \end{pmatrix} + O(|\xi|^2),$$

which yields (2-16). The calculations for L_2 and L_3 are completely analogous.

A.4. ***The Dirichlet to Neumann operator.*** Let us recall the expression of the operator DN in Fourier space:

$$\widehat{\mathrm{DN}(v^0)} = \sum_{k=1}^{3} \begin{pmatrix} (i/|\xi|^2)[(|\xi|^2-\lambda_k^2)^2\xi^\perp - \lambda_k^2\xi] \\ \lambda_k + (|\xi|^2-\lambda_k^2)/\lambda_k \end{pmatrix} A_k \tag{A-8}$$

$$= \begin{pmatrix} -i\widehat{v_3^0}(\xi)\xi \\ i\xi\cdot\widehat{v_h^0}(\xi) \end{pmatrix} + \sum_{k=1}^{3} \begin{pmatrix} (i/|\xi|^2)[(|\xi|^2-\lambda_k^2)^2\xi^\perp + (|\xi|^2-\lambda_k^2)\xi] \\ (|\xi|^2-\lambda_k^2)/\lambda_k \end{pmatrix} A_k. \tag{A-9}$$

A.4.1. *High frequency expansion.* Using the exact formula (A-9) for $\widehat{\mathrm{DN}\,v_0}$ together with the expansions (A-3) and (2-10), we get for the high frequencies

$$\widehat{\mathrm{DN}\,v_0} = \begin{pmatrix} -i\widehat{v_3^0}(\xi)\xi \\ i\xi\cdot\widehat{v_h^0}(\xi) \end{pmatrix} + \begin{pmatrix} (i/|\xi|^2)\big((|\xi|^{\frac{4}{3}}B_3 + O(|\xi|^{\frac{4}{3}}|\widehat{v_0}|))\xi^\perp + (|\xi|^{\frac{2}{3}}B_2 + O(|\xi|^{\frac{2}{3}}|\widehat{v_0}|))\xi\big) \\ |\xi|^{-\frac{1}{3}}B_2 + O(|\xi|^{-\frac{1}{3}}|\widehat{v_0}|) \end{pmatrix}$$

$$= \begin{pmatrix} |\xi|\widehat{v_h^0} + (\xi\cdot\widehat{v_h^0}/|\xi|)\xi + i\widehat{v_3^0}\xi \\ 2|\xi|\widehat{v_3^0} - i\xi\cdot\widehat{v_h^0} \end{pmatrix} + O(|\xi|^{\frac{1}{3}}|\widehat{v_0}|). \tag{A-10}$$

A.4.2. *Low frequency expansion.* For $|\xi| \ll 1$, using (A-8), (A-4), and (A-7) leads to

$$\begin{aligned}\widehat{\mathrm{DN}_h\, v_0} &= \frac{i}{2|\xi|^2}\sum_{\pm}(-\xi^\perp \mp i\xi + O(|\xi|^3))(e^{\pm i\pi/4}|\xi|\widehat{v_{0,3}} \pm e^{\pm i\pi/4}\xi\cdot\widehat{v_{0,h}} \mp e^{\mp i\pi/4}\xi^\perp\cdot\widehat{v_{0,h}} + O(|\xi|^2|\widehat{v_0}|))\\ &= \frac{\sqrt{2}i}{2}\frac{\xi-\xi^\perp}{|\xi|}\widehat{v_{0,3}} + \frac{\sqrt{2}}{2}(\widehat{v_{0,h}} + \widehat{v_{0,h}}^\perp) + O(|\xi||\widehat{v_0}|).\end{aligned}$$

For the vertical component of the operator DN, we have in low frequencies

$$\begin{aligned}\widehat{\mathrm{DN}_3\, v_0} &= i\xi\cdot\widehat{v_{0,h}} + \left(\frac{1}{|\xi|} + O(|\xi|)\right)A_1(\xi) - (e^{i\pi/4} + O(|\xi|^2))A_2(\xi) - (e^{-i\pi/4} + O(|\xi|^2))A_3(\xi)\\ &= \frac{\widehat{v_{0,3}}}{|\xi|} - \frac{\sqrt{2}}{2}\widehat{v_{0,3}} - \frac{\sqrt{2}i}{2}\frac{\xi\cdot\widehat{v_{0,h}} + \xi^\perp\cdot\widehat{v_{0,h}}}{|\xi|} + O(|\xi||\widehat{v_0}|).\end{aligned}$$

Appendix B. Lemmas for the remainder terms

The goal of this section is to prove that the various remainder terms encountered throughout the paper decay like $|x|^{-3}$. To that end, we introduce the algebra

$$E := \Big\{ f \in \mathscr{C}([0,\infty),\mathbb{R}) : \exists \mathscr{A} \subset \mathbb{R} \text{ finite},\ \exists r_0 > 0,\ f(r) = \sum_{\alpha\in\mathscr{A}} r^\alpha f_\alpha(r) \text{ for all } r \in [0, r_0),$$
$$\text{where, for all } \alpha \in \mathscr{A},\ f_\alpha : \mathbb{R} \to \mathbb{R} \text{ is analytic in } B(0, r_0) \Big\}. \qquad \text{(B-1)}$$

We then have the following result:

Lemma B.1. *Let $\varphi \in \mathscr{S}'(\mathbb{R}^2)$.*

- *Assume that* Supp $\hat{\varphi} \subset B(0,1)$, *and that $\hat{\varphi}(\xi) = f(|\xi|)$ for ξ in a neighborhood of zero, with $f \in E$ and $f(r) = O(r^\alpha)$ for some $\alpha > 1$. Then $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ and there exists a constant C such that*
$$|\varphi(x)| \le \frac{C}{|x|^3} \quad \text{for all } x \in \mathbb{R}^2.$$
- *Assume that* Supp $\hat{\varphi} \subset \mathbb{R}^2 \setminus B(0,1)$, *and that $\hat{\varphi}(\xi) = f(|\xi|^{-1})$ for $|\xi| > 1$, with $f \in E$ and $f(r) = O(r^\alpha)$ for some $\alpha > -1$. Then $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ and there exists a constant C such that*
$$|\varphi(x)| \le \frac{C}{|x|^3} \quad \text{for all } x \in \mathbb{R}^2.$$

We prove the Lemma in several steps: we first give some properties of the algebra E. We then compute the derivatives of order 3 of functions of the type $f(|\xi|)$ and $f(|\xi|^{-1})$. Eventually, we explain the link between the bounds in Fourier space and in the physical space.

Properties of the algebra E.

Lemma B.2. • *E is stable by differentiation.*

- *Let* $f \in E$ *with* $f(r) = \sum_{\alpha \in \mathscr{A}} r^\alpha f_\alpha(r)$, *and let* $\alpha_0 \in \mathbb{R}$. *Assume that*

$$f(r) = O(r^{\alpha_0})$$

for r *in a neighborhood of zero. Then*

$$\inf\{\alpha \in \mathscr{A} : f_\alpha(0) \neq 0\} \geq \alpha_0.$$

- *Let* $f \in E$, *and let* $\alpha_0 \in \mathbb{R}$ *such that*

$$f(r) = O(r^{\alpha_0})$$

for r *in a neighborhood of zero. Then*

$$f'(r) = O(r^{\alpha_0 - 1})$$

for $0 < r \ll 1$.

Proof. The first point simply follows from the chain rule and the fact that if f_α is analytic in $B(0, r_0)$, so is f'_α. Concerning the second point, notice that we can always choose the set $\mathscr{A}$ and the functions f_α so that

$$f(r) = r^{\alpha_1} f_{\alpha_1}(r) + \cdots + r^{\alpha_s} f_{\alpha_s}(r),$$

where $\alpha_1 < \cdots < \alpha_s$ and f_{α_i} is analytic in $B(0, r_0)$ with $f_{\alpha_i}(0) \neq 0$. Therefore

$$f(r) \sim r^{\alpha_1} f_{\alpha_1}(0), \quad \text{as } r \to 0,$$

so that $r^{\alpha_1} = O(r^{\alpha_0})$. It follows that $\alpha_1 \geq \alpha_0$. Using the same expansion, we also obtain

$$f'(r) = \sum_{i=1}^{s} \alpha_i r^{\alpha_i - 1} f_{\alpha_i}(r) + r^{\alpha_i} f'_{\alpha_i}(r) = O(r^{\alpha_1 - 1}).$$

Since $r^{\alpha_1} = O(r^{\alpha_0})$, we infer eventually that $f'(r) = O(r^{\alpha_0 - 1})$. □

Differentiation formulas. Now, since we wish to apply the preceding lemma to functions of the type $f(|\xi|)$, or $f(|\xi|^{-1})$, where $f \in E$, we need to have differentiation formulas for such functions. Tedious but easy computations yield, for $\varphi \in \mathscr{C}^3(\mathbb{R})$,

$$\partial^3_{\xi_i} f(|\xi|) = \left(3\frac{\xi_i^3}{|\xi|^5} - 3\frac{\xi_i}{|\xi|^3}\right) f'(|\xi|) + \left(3\frac{\xi_i}{|\xi|^2} - \frac{\xi_i^3}{|\xi|^4}\right) f''(|\xi|) + \frac{\xi_i^3}{|\xi|^3} f^{(3)}(|\xi|)$$
$$\partial^3_{\xi_i} f(|\xi|^{-1}) = \left(9\frac{\xi_i}{|\xi|^5} - 11\frac{\xi_i^3}{|\xi|^7}\right) f'(|\xi|^{-1}) + \left(3\frac{\xi_i}{|\xi|^6} - 7\frac{\xi_i^3}{|\xi|^8}\right) f''(|\xi|^{-1}) + \frac{\xi_i^3}{|\xi|^9} f^{(3)}(|\xi|^{-1}).$$

In particular, if $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is such that $\varphi(\xi) = f(|\xi|)$ for ξ in a neighborhood of zero, where $f \in E$ is such that $f(r) = O(r^\alpha)$ for r close to zero, we infer that

$$|\partial^3_{\xi_1}\varphi(\xi)| + |\partial^3_{\xi_2}\varphi(\xi)| = O(|\xi|^{\alpha - 3})$$

for $|\xi| \ll 1$. In a similar fashion, if $\varphi(\xi) = f(|\xi|^{-1})$ for ξ in a neighborhood of zero, where $f \in E$ is such that $f(r) = O(r^\alpha)$ for r close to zero, we infer that

$$|\partial_{\xi_1}^3 \varphi(\xi)| + |\partial_{\xi_2}^3 \varphi(\xi)| = O\{|\xi|^{-4}(|\xi|^{-1})^{-\alpha-1} + |\xi|^{-5}(|\xi|^{-1})^{-\alpha-2} + |\xi|^{-6}(|\xi|^{-1})^{-\alpha-3}\} = O(|\xi|^{\alpha-3}).$$

Moments of order 3 in the physical space.

Lemma B.3. *Let $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ be such that $\partial_{\xi_1}^3 \varphi, \partial_{\xi_2}^3 \varphi \in L^1(\mathbb{R}^2)$.*
Then

$$|\mathcal{F}^{-1}(\varphi)(x_h)| \le \frac{C}{|x_h|^3} \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \setminus \{0\}).$$

Proof. The proof follows from the formula

$$x_h^\alpha \mathcal{F}^{-1}(\varphi) = i \mathcal{F}^{-1}(\nabla_\xi^\alpha \varphi)$$

for all $\alpha \in \mathbb{N}^2$ such that $|\alpha| = 3$. When $\varphi \in \mathcal{S}(\mathbb{R}^2)$, the formula is a consequence of standard properties of the Fourier transform. It is then extended to $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ by duality. □

Remark B.4. Notice that constants or polynomials of order less that two satisfy the assumptions of the above lemma. In this case, the inverse Fourier transform is a distribution whose support is $\{0\}$ (Dirac mass or derivative of a Dirac mass). This is of course compatible with the result of Lemma B.3.

The result of Lemma B.1 then follows easily. It only remains to explain how we can apply it to the functions in the present paper. To that end, we first notice that, for all $k \in \{1, 2, 3\}$, λ_k is a function of $|\xi|$ only, say $\lambda_k = f_k(|\xi|)$. In a similar fashion,

$$L_k(\xi) = G_k^0(|\xi|) + \xi_1 G_k^1(|\xi|) + \xi_2 G_k^2(|\xi|).$$

We then claim the following result:

Lemma B.5.
- *For all $k \in \{1, 2, 3\}$, $j \in \{0, 1, 2\}$, the functions f_k, G_k^j, as well as*

$$r \mapsto f_k(r^{-1}), \quad r \mapsto G_k^j(r^{-1}) \tag{B-2}$$

  all belong to E.
- *For ξ in a neighborhood of zero,*

$$M_k^{\text{rem}} = P_k(\xi) + \sum_{1 \le i,j \le 2} \xi_i \xi_j a_k^{ij}(|\xi|) + \xi \cdot b_k(|\xi|),$$
$$N_k^{\text{rem}} = Q_k(\xi) + \sum_{1 \le i,j \le 2} \xi_i \xi_j c_k^{ij}(|\xi|) + \xi \cdot d_k(|\xi|),$$

  where P_k, Q_k are polynomials, and $a_k^{ij}, c_k^{i,j} \in E$ and $b_k, d_k \in E^2$ with $b_k(r), d_k(r) = O(r)$ for r close to zero.
- *There exists a function $m \in E$ such that*

$$(M_{SC} - M_S)(\xi) = m(|\xi|^{-1})$$

  for $|\xi| \gg 1$.

The lemma can be easily proved using the formulas (A-2) together with the Maclaurin series for functions of the type $x \mapsto (1+x)^s$ for $s \in \mathbb{R}$.

Appendix C. Fourier multipliers supported in low frequencies

This appendix is concerned with the proof of Lemma 2.7, which is a slight variant of a result by Droniou and Imbert [2006] on integral formulas for the fractional Laplacian. Notice that this corresponds to the operator $\mathcal{I}[|\xi|] = \mathcal{I}[(\xi_1^2+\xi_2^2)/|\xi|]$. We recall that $g \in \mathcal{S}(\mathbb{R}^2)$, $\zeta \in \mathscr{C}_0^\infty(\mathbb{R}^2)$, and $\rho := \mathcal{F}^{-1}\zeta \in \mathcal{S}(\mathbb{R}^2)$. Then, for all $x \in \mathbb{R}^2$,

$$\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|}\right) * \mathcal{F}^{-1}(\xi_i\xi_j\zeta(\xi)\hat{g}(\xi))(x).$$

As explained in [Droniou and Imbert 2006], the function $|\xi|^{-1}$ is locally integrable in $\mathbb{R}^2$ and therefore belongs to $\mathcal{S}'(\mathbb{R}^2)$. Its inverse Fourier transform is a radially symmetric distribution with homogeneity $-2+1=-1$. Hence there exists a constant C_I such that

$$\mathcal{F}^{-1}\left(\frac{1}{|\xi|}\right) = \frac{C_I}{|x|}.$$

We infer that

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) &= \frac{C_I}{|\cdot|} * \partial_{ij}(\rho * g)\\ &= C_I \int_{\mathbb{R}^2} \frac{1}{|x-y|}\partial_{ij}(\rho * g)(y)\,dy\\ &= C_I \int_{\mathbb{R}^2} \frac{1}{|y|}\partial_{ij}(\rho * g)(x+y)\,dy.\end{aligned}$$

The idea is to put the derivatives ∂_{ij} on the kernel $1/|y|$ through integrations by parts. As such, it is not possible to realize this idea. Indeed, $y \mapsto \partial_i(1/|y|)\partial_j(\rho * g)(x+y)$ is not integrable in the vicinity of 0. In order to compensate for this lack of integrability, we consider an even function $\theta \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $0 \le \theta \le 1$ and $\theta = 1$ on $B(0, K)$, and we introduce the auxiliary function

$$U_x(y) := \rho * g(x+y) - \rho * g(x) - \theta(y)(y\cdot\nabla)\rho * g(x),$$

which satisfies

$$|U_x(y)| \le C|y|^2, \quad |\nabla_y U_x(y)| \le C|y| \tag{C-1}$$

for y close to 0. Then, for all $y \in \mathbb{R}^2$,

$$\partial_{y_i}\partial_{y_j}U_x = \partial_{y_i}\partial_{y_j}\rho * g(x+y) - (\partial_{y_i}\partial_{y_j}\theta)(y\cdot\nabla)\rho * g(x) - (\partial_{y_j}\theta)\partial_{x_i}\rho * g(x) - (\partial_{y_i}\theta)\partial_{x_j}\rho * g(x),$$

where

$$y \mapsto -(\partial_{y_i}\partial_{y_j}\theta)(y\cdot\nabla)\rho * g(x) - (\partial_{y_j}\theta)\partial_{x_i}\rho * g(x) - (\partial_{y_i}\theta)\partial_{x_j}\rho * g(x)$$

is an odd function. Therefore, for all $\varepsilon > 0$,

$$\int_{\varepsilon<|y|<\varepsilon^{-1}} \frac{1}{|y|}\partial_{ij}(\rho * g)(x+y)\,dy = \int_{\varepsilon\le|y|\le 1/\varepsilon} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}U_x(y)\,dy.\sum$$

A first integration by parts yields

$$\begin{aligned}\int_{\varepsilon\le|y|\le 1/\varepsilon} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}\rho*g(x+y)\,dy &= \int_{\varepsilon\le|y|\le 1/\varepsilon} \frac{1}{|y|}\partial_{y_i}\partial_{y_j}U_x(y)\,dy \\ &= \int_{|y|=\varepsilon} \frac{1}{|y|}\partial_{y_j}U_x(y)n_i(y)\,dy + \int_{|y|=1/\varepsilon} \frac{1}{|y|}\partial_{y_j}U_x(y)n_i(y)\,dy + \int_{\varepsilon\le|y|\le 1/\varepsilon} \frac{y_i}{|y|^3}\partial_{y_j}U_x(y)\,dy.\end{aligned}$$

The first boundary integral vanishes as $\varepsilon\to 0$ because of (C-1), and the second thanks to the fast decay of $\rho*g\in\mathcal{S}(\mathbb{R}^2)$. Another integration by parts leads to

$$\begin{aligned}\int_{\varepsilon\le|y|\le 1/\varepsilon} \frac{y_i}{|y|^3}\partial_{y_j}U_x(y)\,dy &= \int_{|y|=\varepsilon} \frac{y_i}{|y|^3}U_x(y)n_j(y)\,dy + \int_{|y|=1/\varepsilon} \frac{y_i}{|y|^3}U_x(y)n_j(y)\,dy + \int_{\varepsilon\le|y|\le 1/\varepsilon}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)\,dy \\ &\xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^2}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)\,dy,\end{aligned}$$

where

$$\partial_{y_i}\partial_{y_j}\frac{1}{|y|} = -\frac{\delta_{ij}}{|y|^3} + 3\frac{y_iy_j}{|y|^5},\qquad \left|\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right| \le \frac{C}{|y|^3},$$

and the boundary terms vanish because of (C-1) and the fast decay of U_x. Therefore, for all $x\in\mathbb{R}^2$,

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) &= C_I\int_{\mathbb{R}^2}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)U_x(y)\,dy \\ &= C_I\int_{\mathbb{R}^2}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)[\rho*g(x+y)-\rho*g(x)-\theta(y)(y\cdot\nabla)\rho*g(x)]\,dy \\ &= C_I\int_{B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)[\rho*g(x+y)-\rho*g(x)-y\cdot\nabla\rho*g(x)]\,dy \\ &\qquad + C_I\int_{\mathbb{R}^2\setminus B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)[\rho*g(x+y)-\rho*g(x)]\,dy \\ &\qquad - C_I\int_{\mathbb{R}^2\setminus B(0,K)}\left(\partial_{y_i}\partial_{y_j}\frac{1}{|y|}\right)\theta(y)(y\cdot\nabla)\rho*g(x)\,dy.\end{aligned}$$

The last integral is zero as $y\mapsto\theta(y)(\partial_{y_i}\partial_{y_j}(1/|y|))y$ is odd. We then perform a last change of variables by setting $y'=x+y$, and we obtain

$$\begin{aligned}\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|}\zeta(\xi)\hat{g}(\xi)\right)(x) &= -\int_{|x-y'|\le K}\gamma_{ij}(x-y')\{\rho*g(y')-\rho*g(x)-(y'-x)\nabla\rho*g(x)\}\,dy' \\ &\qquad - \int_{|x-y'|\ge K}\gamma_{ij}(x-y')\{\rho*g(y')-\rho*g(x)\}\,dy'.\end{aligned}$$

This completes the proof of Lemma 2.7.

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OPTIMAL CONTROL OF SINGULAR FOURIER MULTIPLIERS BY MAXIMAL OPERATORS

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Dedicated to the memory of Adela Moyua, 1956–2013.

We control a broad class of singular (or "rough") Fourier multipliers by geometrically defined maximal operators via general weighted $L^2(\mathbb{R})$ norm inequalities. The multipliers involved are related to those of Coifman, Rubio de Francia and Semmes, satisfying certain weak Marcinkiewicz-type conditions that permit highly oscillatory factors of the form $e^{i|\xi|^\alpha}$ for both α positive and negative. The maximal functions that arise are of some independent interest, involving fractional averages associated with tangential approach regions (related to those of Nagel and Stein), and more novel "improper fractional averages" associated with "escape" regions. Some applications are given to the theory of L^p–L^q multipliers, oscillatory integrals and dispersive PDE, along with natural extensions to higher dimensions.

1. Introduction and statements of results

Given a Fourier multiplier m, with corresponding convolution operator T_m, there has been considerable interest in identifying, where possible, "geometrically defined" maximal operators $\mathcal{M}$ for which a weighted L^2-norm inequality of the form

$$\int_{\mathbb{R}^n} |T_m f|^2 w \leq \int_{\mathbb{R}^n} |f|^2 \mathcal{M} w \tag{1}$$

holds for all admissible input functions f and weight functions w. This very general Fourier multiplier problem was made particularly explicit in the 1970s in work of A. Córdoba and C. Fefferman [1976], following the emergence of fundamental connections between the theory of Fourier multipliers and elementary geometric notions such as curvature (see [Fefferman 1971; Córdoba 1977; Stein 1979], in particular). Such control of a multiplier m by a maximal operator $\mathcal{M}$, combined with an elementary duality argument, reveals that, for $p, q \geq 2$,

$$\|m\|_{p,q} := \|T_m\|_{L^p - L^q} \leq \|\mathcal{M}\|^{1/2}_{L^{(q/2)'} - L^{(p/2)'}}. \tag{2}$$

Thus it is of particular interest to identify an "optimal" maximal operator $\mathcal{M}$ for which (1) holds, in the sense that (2) permits optimal L^p–L^q bounds for $\mathcal{M}$ to be transferred to optimal bounds for T_m.

There are a variety of results of this nature, although often formulated in terms of the convolution kernel rather than the multiplier. For example, if T denotes a Calderón–Zygmund singular integral operator

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on $\mathbb{R}^n$, such as the Hilbert transform on the line, Córdoba and Fefferman [1976] (see also [Hunt et al. 1973]) showed that for each $s > 1$ there is a constant $C_s < \infty$ for which

$$\int_{\mathbb{R}^n} |Tf|^2 w \leq C_s \int_{\mathbb{R}^n} |f|^2 (Mw^s)^{1/s} \tag{3}$$

holds, where M denotes the classical Hardy–Littlewood maximal operator. This result extends to weighted L^p estimates for $1 < p < \infty$; see [Córdoba and Fefferman 1976]. The inequality (3) may be viewed as a consequence of the classical theory of Muckenhoupt A_p weights through the fundamental fact that if $(Mw^s)^{1/s} < \infty$ a.e. and $s > 1$ then $(Mw^s)^{1/s} \in A_1 \subset A_2$; see [Stein 1993] and the references there. Of course, for any fixed $s > 1$ the maximal operator $w \mapsto (Mw^s)^{1/s}$ in (3) is not optimal, since it fails to be L^p-bounded in the range $1 < p \leq s$, while T is bounded on L^p for all $1 < p < \infty$. More recently this was remedied by Wilson [1989], who showed that[1]

$$\int_{\mathbb{R}^n} |Tf|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M^3 w, \tag{4}$$

where $M^3 = M \circ M \circ M$ denotes the 3-fold composition of M with itself. As with (3), this useful result extends to weighted L^p norms for $1 < p < \infty$; see [Wilson 1989; Pérez 1994; Reguera and Thiele 2012]. There are numerous further results belonging to the considerable theory surrounding the A_p weights; see for example [García-Cuerva and Rubio de Francia 1985; Pérez 1995; Hytönen 2012; Lacey et al. 2014; Hytönen et al. 2013; Lerner 2013].

In the setting of *oscillatory integrals* the controlling maximal operators appear to acquire a much more interesting geometric nature, well beyond the scope of the classical A_p theory. This is illustrated well by a compelling and seemingly very deep conjecture concerning the classical *Bochner–Riesz multipliers*,

$$m_\delta(\xi) = (1 - |\xi|^2)_+^\delta,$$

where $\xi \in \mathbb{R}^n$ and $\delta \geq 0$. Of course, m_0 is simply the characteristic function of the unit ball in $\mathbb{R}^n$, allowing us to interpret m_δ for $\delta > 0$ as a certain regularisation of this characteristic function. The classical Bochner–Riesz conjecture concerns the range of exponents p for which m_δ is an L^p-multiplier. In the 1970s, A. Córdoba [1977] and E. M. Stein [1979] raised the possibility that a weighted inequality of the form (1) holds where $\mathcal{M}$ is some suitable variant of the Nikodym maximal operator

$$\mathcal{N}_\delta w(x) := \sup_{T \ni x} \frac{1}{|T|} \int_T w;$$

see also [Fefferman 1971; 1973]. Here the supremum is taken over all cylindrical tubes of eccentricity less than $1/\delta$ that contain the point x. This maximal operator $\mathcal{M}$ should be geometrically defined (very much like $\mathcal{N}_\delta$) and its known/conjectured bounds should be similar to those of $\mathcal{N}_\delta$, thus essentially implying the full Bochner–Riesz conjecture via (2).[2] Such a result is rather straightforward for $n = 1$ as it reduces to

[1] Throughout this paper we shall write $A \lesssim B$ if there exists a constant c such that $A \leq cB$. In particular, this constant will always be independent of the input function f and weight function w. The relations $A \gtrsim B$ and $A \sim B$ are defined similarly.

[2] Similar weighted inequalities relating the Fourier restriction and Kakeya conjectures have also received some attention in the literature; see [Bennett et al. 2006] for further discussion.

the aforementioned inequality for the Hilbert transform. In higher dimensions this question is far from having a satisfactory answer already for $n = 2$ (see [Bourgain 1991; Christ 1985; Carbery et al. 1992; Carbery and Soria 1997a; 1997b; Carbery and Seeger 2000; Bennett et al. 2006; Duoandikoetxea et al. 2008; Lee et al. 2012; Córdoba and Rogers 2014] for some related results). The associated convolution kernel

$$K_\delta(x) := \mathscr{F}^{-1} m_\delta(x) = \frac{c J_{n/2+\delta}(2\pi|x|)}{|x|^{n/2+\delta}} = c \frac{e^{2\pi i|\xi|} + e^{-2\pi i|\xi|} + o(1)}{|\xi|^{(n+1)/2+\delta}},$$

unlike the Hilbert kernel, is (for δ sufficiently small) very far from being Lebesgue integrable. Here J_λ denotes the Bessel function of order λ, making K_δ highly oscillatory.

In [Bennett and Harrison 2012], using arguments from [Bennett et al. 2006] in the setting of Fourier extension operators, we gave nontrivial examples of such "optimal" control of oscillatory kernels on the line by geometrically defined maximal operators. In particular, for integers $\ell \geq 3$, we showed that

$$\int_{\mathbb{R}} \left|e^{i(\cdot)^\ell} * f\right|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^2 \mathcal{M} M^4 w, \tag{5}$$

where

$$\mathcal{M} w(x) := \sup_{(y,r)\in\Gamma(x)} \frac{1}{r^{1/(\ell-1)}} \int_{y-r}^{y+r} w$$

and

$$\Gamma(x) = \left\{(y,r) : 0 < r \leq 1,\ |x-y| \leq r^{-1/(\ell-1)}\right\}. \tag{6}$$

The maximal operator $\mathcal{M}$ here may be interpreted as a fractional Hardy–Littlewood maximal operator associated with an approach region $\Gamma(x)$. This maximal operator is similar in spirit to those studied by Nagel and Stein [1984], although here tangential approach to infinite order is permitted. It is shown in [Bennett and Harrison 2012] that $\mathcal{M}$ has a sharp bound on $L^{(\ell/2)'}$, which may be reconciled via (5) with a sharp L^ℓ bound for convolution with e^{ix^ℓ}. We note in passing that the factors of the Hardy–Littlewood maximal operator appearing in (5) are of secondary importance as $\mathcal{M}$ and $M^2\mathcal{M}M^4$ share the same L^p–L^q mapping properties. This follows from the L^p-boundedness of M for $1 < p \leq \infty$.

In this paper we seek an understanding of the "map" $m \mapsto \mathcal{M}$, from Fourier multiplier to optimal controlling maximal operator, for which (1) holds. As we shall see, an inequality of the form (1) does indeed hold for a wide class of multipliers m and a surprisingly rich family of geometrically defined maximal operators $\mathcal{M}$. This class of multipliers is sufficiently singular to apply to a variety of highly oscillatory convolution kernels, placing (5) in a much broader context. The maximal operators turn out to be fractional Hardy–Littlewood maximal operators associated with a diverse family of approach and "escape" regions in the half-space. While such operators corresponding to approach regions have arisen before [Nagel and Stein 1984; Bennett et al. 2006; Bennett and Harrison 2012], those associated with "escape" regions appear to be quite novel, involving improper fractional averages.

As is well known, at least in one dimension, the *variation* of a multiplier can play a decisive role in determining its behaviour as an operator. For example, if a multiplier m is of bounded variation on the line, then it often satisfies the same norm inequalities as the Hilbert transform. This is a straightforward consequence of the elementary identity

$$T_m = \lim_{t\to-\infty} m(t)I + \frac{1}{2}\int_{\mathbb{R}} (I + iM_{-t}HM_t)\,dm(t). \tag{7}$$

Here I denotes the identity operator on $\mathbb{R}^d$, the modulation operator M_t is given by $M_t f(x) = e^{-2\pi ixt} f(x)$, and $dm(t)$ denotes the Lebesgue–Stieltjes measure (which we identify with $|m'(t)|\,dt$ throughout). In particular, combining this with (4) quickly leads to the inequality

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^3 w. \tag{8}$$

Invoking classical weighted Littlewood–Paley theory for dyadic decompositions of the line (see [Wilson 2007] and [Bennett and Harrison 2012] for further discussion) leads to the following weighted version of the Marcinkiewicz multiplier theorem (cf. [Kurtz 1980]).

Theorem 1. *If $m : \mathbb{R} \to \mathbb{C}$ is a bounded function which is uniformly of bounded variation on dyadic intervals, that is,*

$$\sup_{R>0} \int_{R\le|\xi|\le 2R} |m'(\xi)|\,d\xi < \infty, \tag{9}$$

then

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^7 w.$$

The control of m here by a power of the Hardy–Littlewood maximal operator is optimal in the sense that Theorem 1, combined with the Hardy–Littlewood maximal theorem, implies the classical Marcinkiewicz multiplier theorem via (2). It would seem unlikely that the particular power of M that features here is best possible; here and throughout this paper we do not concern ourselves with such finer points.

Our goal is to establish versions of Theorem 1 which apply to much more singular (or "rougher") multipliers. A natural class of singular multipliers on the line, defined in terms of the so-called "r-variation" was introduced by Coifman, Rubio de Francia and Semmes in [Coifman et al. 1988]. For a function m on an interval $[a, b]$ we define the r-variation of m to be the supremum of the quantity

$$\left(\sum_{j=0}^{N-1} |m(x_{j+1}) - m(x_j)|^r\right)^{1/r}$$

over all partitions $a = x_0 < x_1 < \cdots < x_N = b$ of $[a, b]$. We say that m is a V_r *multiplier* if it has uniformly bounded r-variation on each dyadic interval. (Of course, if $r = 1$ this class reduces to the classical Marcinkiewicz multipliers.) In [Coifman et al. 1988] it is shown that if m is a V_r multiplier then m is an $L^p(\mathbb{R})$ multiplier for $|1/p - 1/2| < 1/r$, considerably generalising the classical Marcinkiewicz multiplier theorem on the line. With the possible exception of the endpoint, this result is sharp, as may be seen from the specific multipliers

$$m_{\alpha,\beta}(\xi) := \frac{e^{i|\xi|^\alpha}}{(1+|\xi|^2)^{\beta/2}}, \quad \alpha, \beta \ge 0, \tag{10}$$

first studied by Hirschman [1959] (see [Stein 1970] for further discussion). Indeed $m_{\alpha,\beta}$ is a V_r multiplier if $\beta r = \alpha$, while being an L^p multiplier if and only if $\alpha|1/p - 1/2| \le \beta$; see [Hirschman 1959; Miyachi 1981]. The endpoint case $|1/p - 1/2| = 1/r$ remains open in general for V_r multipliers; see [Tao and Wright 2001] for further discussion and related results.

For the purposes of identifying *optimal* controlling maximal operators we will confine attention to a subclass of the V_r multipliers that retains some of the structure of the specific example (10). Before we describe this subclass let us discuss some motivating examples.

The multiplier corresponding to the convolution kernel e^{ix^ℓ} appearing in (5) coincides with the (generalised) Airy function

$$Ai^{(\ell)}(\xi) = \int_{-\infty}^{\infty} e^{i(x^\ell + x\xi)}\, dx = c_0 \frac{e^{ic_1|\xi|^{\ell/(\ell-1)}} + o(1)}{|\xi|^{(\ell-2)/(2(\ell-1))}}$$

as $|\xi| \to \infty$; here c_0 and c_1 are appropriate constants. As standard Airy function asymptotics reveal, the variation of this multiplier on dyadic intervals is unbounded. This multiplier, with its highly oscillatory behaviour as $|\xi| \to \infty$, belongs to a more general class of multipliers satisfying

$$m(\xi) = O\big(|\xi|^{-\beta}\big), \quad m'(\xi) = O\big(|\xi|^{-\beta+\alpha-1}\big) \tag{11}$$

as $|\xi| \to \infty$. Here $\alpha, \beta \ge 0$, and of course the specific multiplier in (10) is a model example. In addition to multipliers whose derivatives can have strong singularities at *infinity*, it is also natural to consider those which are singular at a *point*. In particular, we might hope to control multipliers satisfying (11) as $|\xi| \to 0$ for $\alpha, \beta \le 0$. Such singular multipliers, which were studied by Miyachi [1980; 1981], arise frequently in the study of oscillatory and oscillatory-singular integrals; see for example [Stein 1993; Miyachi 1981; Sjölin 1981; Chanillo et al. 1986]. See also [Miyachi 1980; 1981] for a general $L^p(\mathbb{R})$ (and Hardy space $H^p(\mathbb{R})$) multiplier theorem under the specific hypothesis (11). The following class of multipliers, which we denote $\mathscr{C}(\alpha, \beta)$, involves a Marcinkiewicz-type variation condition specifically designed to capture these Miyachi-type examples.

The class of multipliers. For each $\alpha, \beta \in \mathbb{R}$, let $\mathscr{C}(\alpha, \beta)$ be the class of functions $m : \mathbb{R} \to \mathbb{C}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^\alpha \ge 1\}, \tag{12}$$

$$\sup_\xi |\xi|^\beta |m(\xi)| < \infty, \tag{13}$$

$$\sup_{R^\alpha \ge 1} \sup_{\substack{I \subseteq [R,2R] \\ \ell(I) = R^{-\alpha} R}} R^\beta \int_{\pm I} |m'(\xi)|\, d\xi < \infty. \tag{14}$$

Here the supremum is taken over all subintervals I of $[R, 2R]$ of length $\ell(I) = R^{-\alpha} R$.

Remarks.

(i) The support condition (12) has no content for $\alpha = 0$. For $\alpha > 0$ and $\alpha < 0$ it reduces to, respectively, $\operatorname{supp}(m) \subseteq \{|\xi| \ge 1\}$ and $\operatorname{supp}(m) \subseteq \{|\xi| \le 1\}$. A similar interpretation applies to the outermost supremum in (14).

(ii) The case $\alpha = 0$ is of course somewhat degenerate. As is easily verified, the class $\mathscr{C}(\alpha, \beta)$ reduces to the classical Marcinkiewicz multipliers when $\alpha = \beta = 0$. Further, the fractional integration multiplier $\xi \mapsto |\xi|^{-\beta}$ lies in $\mathscr{C}(0, \beta)$.

(iii) The model behaviour of a multiplier in $\mathscr{C}(\alpha, \beta)$ in the nondegenerate case $\alpha \neq 0$ is that of the Miyachi multipliers (11) as $|\xi|^{\alpha} \to \infty$.

(iv) An elementary calculation reveals that if m lies in $\mathscr{C}(\alpha, \beta)$ then m is a V_r multiplier provided $\beta r = \alpha$. We also note that the additional structure of the class $\mathscr{C}(\alpha, \beta)$ yields L^p–L^q estimates for certain $q \neq p$ — see the forthcoming Corollary 5.

(v) An elementary change of variables argument reveals that a multiplier $m \in \mathscr{C}(\alpha, \beta)$ if and only if $\widetilde{m} \in \mathscr{C}(-\alpha, -\beta)$, where $\widetilde{m}(\xi) := m(1/\xi)$. The main point is that the diffeomorphism $\mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $\xi \mapsto 1/\xi$ preserves dyadic intervals and (essentially) any lattice structure within them.

(vi) Unlike the V_r multipliers, if $\alpha \neq 0$ the class $\mathscr{C}(\alpha, \beta)$ is not dilation-invariant due to the distinguished role of the unit scale $R = 1$. See the forthcoming Theorem 3 for a natural dilation-invariant formulation.

We now introduce the family of maximal operators that will control these multipliers via (1).

The controlling maximal operators. For $\alpha, \beta \in \mathbb{R}$ we define the maximal operator $\mathcal{M}_{\alpha,\beta}$ by

$$\mathcal{M}_{\alpha,\beta} f(x) = \sup_{(r,y)\in\Gamma_\alpha(x)} \frac{r^{2\beta}}{r} \int_{|y-z|\leq r} f(z)\, dz, \tag{15}$$

where

$$\Gamma_\alpha(x) = \bigl\{(r, y) : 0 < r^{\alpha} \leq 1 \text{ and } |y - x| \leq r^{1-\alpha}\bigr\}. \tag{16}$$

This family of maximal operators is of some independent interest. When $\alpha = 0$ the approach region $\Gamma_\alpha(x)$ is simply a cone with vertex x, and the associated maximal operator $\mathcal{M}_{\alpha,\beta}$ is equivalent to the classical fractional Hardy–Littlewood maximal operator

$$M_{2\beta} w(x) := \sup_{r>0} \frac{r^{2\beta}}{r} \int_{x-r}^{x+r} w. \tag{17}$$

When $0 < \alpha < 1$ the maximal operators $\mathcal{M}_{\alpha,\beta}$ have also been considered before and originate in work of Nagel and Stein [1984] on fractional maximal operators associated with more general nontangential approach regions. However, as we have already mentioned, the above definitions also permit $\alpha \geq 1$ and $\alpha < 0$, where one sees dramatic transitions in the nature of the region Γ_α. In particular if $\alpha \geq 1$ then the situation is similar to that in (6), where tangential approach to infinite order is permitted; see [Bennett et al. 2006] for the origins of such regions. Furthermore, for $\alpha < 0$ we have

$$\Gamma_\alpha(x) = \bigl\{(r, y) : r \geq 1 \text{ and } |y - x| \leq r^{1-\alpha}\bigr\},$$

which may be viewed as an "escape", rather than "approach", region. Notice also that if $\beta < 0$ we interpret $\mathcal{M}_{\alpha,\beta}$ as an *improper* fractional maximal operator.

The maximal operators $\mathcal{M}_{\alpha,\beta}$ are significant improvements on the controlling maximal operators $w \mapsto (Mw^s)^{1/s}$ that typically arise via classical A_p-weighted inequalities. Crudely estimating $\mathcal{M}_{\alpha,\beta} w$ pointwise using Hölder's inequality reveals that

$$\mathcal{M}_{\alpha,\beta} w \leq (Mw^s)^{1/s} \quad \text{when } 2s\beta = \alpha. \tag{18}$$

This allows the forthcoming Theorem 2 to be reconciled with certain A_p-weighted inequalities established by Chanillo, Kurtz and Sampson in [Chanillo et al. 1983; 1986]. In Section 2 we provide necessary and sufficient conditions for $\mathcal{M}_{\alpha,\beta}$ to be bounded from L^p to L^q. In particular, we see that $\mathcal{M}_{\alpha,\beta}$ is bounded on L^s when $2s\beta = \alpha$, a property that does not follow from (18).

The main result of this paper is the following.

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}$. If $m \in \mathscr{C}(\alpha, \beta)$ then*

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w. \tag{19}$$

It is interesting to contrast this result with the recent weighted variational Carleson theorem of Do and Lacey [2012]; see also [Oberlin et al. 2012; Lacey 2007].

As may be expected, the factors of the Hardy–Littlewood maximal operator M arising in Theorem 2 are of secondary importance, and to some extent occur for technical reasons. Since M is bounded on L^p for all $1 < p \leq \infty$, the maximal operators $M^6 \mathcal{M}_{\alpha,\beta} M^4$ and $\mathcal{M}_{\alpha,\beta}$ share the same L^p–L^q bounds. The forthcoming Theorem 4 clarifies the L^p–L^q behaviour of these operators.

It is perhaps helpful to make some further remarks about the nonsingular case $\alpha = 0$ of Theorem 2. As is immediately verified, the class of multipliers $\mathscr{C}(0, \beta)$ is precisely those satisfying the conditions

$$\sup_{\xi \in \mathbb{R}} |\xi|^\beta |m(\xi)| < \infty, \tag{20}$$

and

$$\sup_{R>0} R^\beta \int_{R \leq |\xi| \leq 2R} |m'(\xi)| \, d\xi < \infty. \tag{21}$$

For such "classical" multipliers, Theorem 2 reduces to the weighted inequality

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 M_{2\beta} M^4 w, \tag{22}$$

where $M_{2\beta}$ is the fractional Hardy–Littlewood maximal operator given by (17). When $\beta = 0$, the conditions (20) and (21) become those of the classical Marcinkiewicz multiplier theorem, and the resulting inequality (22) reduces — up to factors of M — to the classical Theorem 1. Noting that the multiplier $\xi \mapsto |\xi|^{-\beta}$ lies in $\mathscr{C}(0, \beta)$, again up to factors of M we recover the one-dimensional case of Pérez's [1995] result.

Of course the class $\mathscr{C}(\alpha, \beta)$ is neither scale-invariant nor facilitates quantification of the implicit constants in Theorem 2. Our arguments, along with elementary scaling considerations, reveal the following.

Theorem 3. *Let $\alpha, \beta \in \mathbb{R}$ and $\lambda, C > 0$. If $m : \mathbb{R} \to \mathbb{C}$ is such that*

$$\operatorname{supp}(m) \subseteq \{\xi : |\xi|^\alpha \geq \lambda^\alpha\}, \tag{23}$$

$$\sup_\xi |\xi|^\beta |m(\xi)| \leq C, \tag{24}$$

$$\sup_{R^\alpha \geq \lambda^\alpha} \sup_{\substack{I \subseteq [R,2R] \\ \ell(I)=(R/\lambda)^{-\alpha}R}} R^\beta \int_{\pm I} |m'(\xi)|\, d\xi \leq C, \tag{25}$$

then there exists an absolute constant $c > 0$ such that

$$\int_{\mathbb{R}} |T_m f|^2 w \leq cC^2 \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}^\lambda_{\alpha,\beta} M^4 w, \tag{26}$$

where

$$\mathcal{M}^\lambda_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma^\lambda_\alpha(x)} \frac{r^{2\beta}}{r} \int_{y-r}^{y+r} w \quad \textit{and} \quad \Gamma^\lambda_\alpha(x) = \big\{(y,r) : 0 < r^\alpha \leq \lambda^{-\alpha},\ |x-y| \leq \lambda^{-\alpha} r^{1-\alpha}\big\}.$$

The hypotheses of Theorem 3 are scale-invariant. More precisely, if m satisfies (23)–(25) with parameter $\lambda = \eta$, then $\eta^\beta m(\eta\,\cdot)$ satisfies (23)–(25) with parameter $\lambda = 1$.

Organisation of the paper. Our proof of Theorem 2 rests crucially on a certain Littlewood–Paley type square function estimate. This is presented in Section 3. Section 4 contains the proof of Theorem 2, Section 5 concerns extensions to higher dimensions, and finally Section 6 is devoted to the L^p–L^q boundedness properties of the maximal operators $\mathcal{M}_{\alpha,\beta}$. We begin by presenting some applications and interpretations of Theorem 2.

2. Applications and interpretations

Here we present three distinct applications (or interpretations) of Theorem 2.

2.1. *L^p–L^q multipliers.* Our first application of Theorem 2 is to the theory of L^p–L^q multipliers on the line. Such a multiplier theorem will follow from Theorem 2 via (2) once we have suitable bounds on the maximal operators $\mathcal{M}_{\alpha,\beta}$.

Theorem 4. *Let $1 < p \leq q \leq \infty$ and $\alpha, \beta \in \mathbb{R}$. If $\alpha > 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if*

$$\beta \geq \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{27}$$

If $\alpha = 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if

$$\beta = \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{28}$$

If $\alpha < 0$ then $\mathcal{M}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if and only if

$$\beta \leq \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{29}$$

Remarks. When $\alpha = 0$ Theorem 4 of course reduces to the well-known L^p–L^q boundedness properties of the classical fractional Hardy–Littlewood maximal operator in one dimension; see [Muckenhoupt and Wheeden 1974]. For $0 \leq \alpha < 1$ (the case of nontangential approach regions) and $p = q$, this result follows from the work of [Nagel and Stein 1984]. Certain particular cases of Theorem 4 in the region $\alpha > 1$ are established in [Bennett and Harrison 2012], following arguments in [Bennett et al. 2006]. Our proof, which extends further the arguments in [Bennett et al. 2006], follows by establishing a corresponding endpoint Hardy space result when $p = 1$; see Section 6.

Combining Theorems 2 and 4 yields the following unweighted Marcinkiewicz-type multiplier theorem.

Corollary 5. *Let $2 \leq p \leq q < \infty$, $\alpha, \beta \in \mathbb{R}$ and suppose $m \in \mathscr{C}(\alpha, \beta)$. If one of*

$$\begin{aligned} &\alpha > 0 \quad \textit{and} \quad \beta \geq \alpha\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{p} - \frac{1}{q}, \\ &\alpha = 0 \quad \textit{and} \quad \beta = \frac{1}{p} - \frac{1}{q}, \\ \textit{or} \quad &\alpha < 0 \quad \textit{and} \quad \beta \leq \alpha\Big(\frac{1}{2} - \frac{1}{p}\Big) + \frac{1}{p} - \frac{1}{q} \end{aligned}$$

holds, then m is an $L^p(\mathbb{R})$–$L^q(\mathbb{R})$ multiplier.

Remarks. Corollary 5, which modestly generalises a number of well-known results, is optimal subject to the (inevitable) constraint $p, q \geq 2$; see [Miyachi 1980; 1981]. However, as the examples in those papers also suggest, unless $p = q$, Corollary 5 is unlikely to lead to optimal results in the full range $1 \leq p, q \leq \infty$. If $\alpha \neq 0$ then, by duality and interpolation, we may conclude that m is an $L^p(\mathbb{R})$ multiplier for all $1 < p < \infty$ satisfying the familiar condition $|1/2 - 1/p| \leq \beta/\alpha$. This generalises the L^p (as opposed to H^p) multiplier results of [Miyachi 1980] in dimension $n = 1$. If $\alpha = 0$ then Corollary 5 reduces to the classical one-dimensional Marcinkiewicz multiplier theorem on setting $p = q$, since m is a Marcinkiewicz multiplier if and only if $m \in \mathscr{C}(0, 0)$. The special case $\alpha = 0$ also generalises the classical Hardy–Littlewood–Sobolev theorem on fractional integration since the multiplier $|\xi|^{-\beta}$ belongs to $\mathscr{C}(0, \beta)$.

2.2. ***Oscillatory convolution kernels on the line.*** The method of stationary phase permits Theorem 2 to be applied to a variety of explicit oscillatory convolution operators on the line. For example, for $a > 0$ with $a \neq 1$ and $1 - a/2 \leq b < 1$, consider the convolution kernel $K_{a,b} : \mathbb{R}\backslash\{0\} \to \mathbb{C}$ given by

$$K_{a,b}(x) = \frac{e^{i|x|^a}}{(1 + |x|)^b}.$$

The corresponding convolution operator is well understood on L^p, with

$$\|K_{a,b} * f\|_p \lesssim \|f\|_p \quad \Longleftrightarrow \quad p_0 \leq p \leq p_0', \tag{30}$$

where $p_0 = a/(a + b - 1)$; see [Sjölin 1981; Jurkat and Sampson 1981]. As we shall see, an application of Theorem 2 quickly leads to the following.

Theorem 6. *If $a > 0$ with $a \neq 1$ and $1 - a/2 \leq b < 1$, then*

$$\int_{\mathbb{R}} |K_{a,b} * f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w, \tag{31}$$

where $\alpha = \dfrac{a}{a-1}$ *and* $\beta = \dfrac{a/2+b-1}{a-1}$.

This theorem is optimal in the sense that it allows us to recover (30) (and indeed more general L^p–L^q estimates) from Theorem 4 via (2). Notice that if $0 < a < 1$ then $\alpha := a/(a-1) < 0$ and so the controlling maximal operator $\mathcal{M}_{\alpha,\beta}$ corresponds to an *escape* region. Similarly, if $a > 1$ then $\alpha > 0$ and so $\mathcal{M}_{\alpha,\beta}$ corresponds to an *approach* region. Theorem 6 may of course be viewed as a generalisation (modulo factors of M) of the inequality (5).

In order to deduce Theorem 6 from Theorem 2 we simply observe that, up to a couple of well-behaved "error" terms, the multiplier $\widehat{K}_{a,b}$ belongs to $\mathscr{C}(\alpha, \beta)$. Let us begin by handling the portion of $K_{a,b}$ in a neighbourhood of the origin (where the kernel lacks smoothness). Let $\eta \in C_c^\infty(\mathbb{R})$ be an even function satisfying $\eta(x) = 1$ for $|x| \leq 1$, and write $K_{a,b} = K_{a,b,0} + K_{a,b,\infty}$, where $K_{a,b,0} = \eta K_{a,b}$. Since $K_{a,b,0}$ is rapidly decreasing, by the Cauchy–Schwarz inequality we have

$$\int_{\mathbb{R}} |K_{a,b,0} * f|^2 w \leq \|K_{a,b,0}\|_1 \int_{\mathbb{R}} |f|^2 |K_{a,b,0}| * w \lesssim \int_{\mathbb{R}} |f|^2 M^1 w,$$

where

$$M^1 w(x) := \sup_{r \geq 1} \frac{1}{2r} \int_{x-r}^{x+r} w.$$

The claimed inequality (31) for the portion of the kernel $K_{a,b,0}$ now follows from the elementary pointwise bound

$$M^1 w \lesssim A M^1 w \leq \mathcal{M}_{\alpha,\beta} M^1 w \leq \mathcal{M}_{\alpha,\beta} M w \leq M^6 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where the averaging operator A is given by

$$A w(x) = \frac{1}{2} \int_{x-1}^{x+1} w.$$

It thus remains to prove (31) for the portion $K_{a,b,\infty}$. In order to force the support hypothesis (12) we introduce a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\xi) = 0$ when $|\xi|^\alpha \leq 1$ and $\psi(\xi) = 1$ when $|\xi|^\alpha \geq 2$. Writing $m_0 = (1-\psi)\widehat{K}_{a,b,\infty}$ and $m_1 = \psi \widehat{K}_{a,b,\infty}$, it suffices to show that

$$\int_{\mathbb{R}} |T_{m_j} f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta} M^4 w \tag{32}$$

for $j = 0, 1$. A standard stationary phase argument (see [Sjölin 1981] for explicit details) reveals that m_1 satisfies the Miyachi-type bounds (11) as $|\xi|^\alpha \to \infty$. Hence $m_1 \in \mathscr{C}(\alpha, \beta)$, yielding (32) for $j = 1$ by Theorem 2. The multiplier m_0 is less interesting, being the Fourier transform of a rapidly decreasing function (again, see [Sjölin 1981] for further details). Arguing as we did for the portion $K_{a,b,0}$ establishes (32) for $j = 0$, completing the proof.

For a more far-reaching discussion relating to the asymptotics of Fourier transforms of oscillatory kernels, see [Stein 1993], and what Stein refers to as the "duality of phases".

2.3. ***Spatial regularity of solutions of dispersive equations.*** Theorem 2 has an interesting interpretation in the context of spatial regularity of solutions to dispersive equations. For example, applying[3] Theorem 2 to the multiplier $m_{2,\beta}$ given by (10) yields

$$\int_{\mathbb{R}} \left|e^{i\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(I-\partial^2)^{\beta/2} f\right|^2 M^6 \mathcal{M}_{2,\beta} M^4 w$$

for all $\beta \geq 0$. Using the scale-invariant inequality (26) with $\lambda = t^{-1/2}$, a similar statement may be made for the operator $e^{it\partial^2}$, namely

$$\int_{\mathbb{R}} \left|e^{it\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(t^{-1}I-\partial^2)^{\beta/2} f\right|^2 M^6 \mathcal{M}_{2,\beta}^{t^{-1/2}} M^4 w,$$

with implicit constant independent of $t > 0$. It is perhaps more natural to rewrite this as

$$\int_{\mathbb{R}} \left|e^{it\partial^2} f\right|^2 w \lesssim \int_{\mathbb{R}} \left|(I-t\partial^2)^{\beta/2} f\right|^2 M^6 \mathfrak{M}_t M^4 w,$$

where

$$\mathfrak{M}_t w(x) := \sup_{(y,r)\in\Lambda_t(x)} r^{2\beta} \frac{1}{t^{1/2}r} \int_{y-t^{1/2}r}^{y+t^{1/2}r} w \quad \text{and} \quad \Lambda_t(x) = \left\{(y,r) : 0 < r \leq 1,\ |x-y| \leq t^{1/2}/r\right\},$$

so that the degeneracy as $t \to 0$ is more apparent. The resulting L^p multiplier theorem at $t = 1$ (see Corollary 5 in the case $q = p$) is the inequality

$$\left\|e^{i\partial^2} f\right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{W^{\beta,p}}$$

for $\beta \geq 2|1/2 - 1/p|$. Here $W^{\beta,p}$ denotes the classical inhomogeneous L^p Sobolev space. This optimal Sobolev inequality, which goes back to Miyachi [1981], describes the regularity loss in $L^p(\mathbb{R})$ for a solution to the Schrödinger equation with initial data in $L^p(\mathbb{R})$. Naturally this interpretation applies equally well to the wave, Airy and more general (pseudo)differential dispersive equations. Similar conclusions, at least for the Schrödinger equation, may be reached in higher dimensions using the results of Section 5; see also [Miyachi 1981].

3. Weighted inequalities for a lattice square function

In this section we present the forward and reverse weighted Littlewood–Paley square function estimates that underpin our proof of Theorem 2. We formulate our results in $\mathbb{R}^n$ in anticipation of higher-dimensional applications in Section 5.

Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\operatorname{supp}(\widehat{\Psi}) \subseteq [-1, 1]^n$ and

$$\sum_{k\in\mathbb{Z}^n} \widehat{\Psi}(\xi - k) = 1$$

for all $\xi \in \mathbb{R}^n$. Such a function may of course be constructed by defining $\widehat{\Psi} = \chi_{[-1/2,1/2]^n} * \phi$, for a function $\phi \in C_c^\infty(\mathbb{R}^n)$ of suitably small support and integral 1.

[3]Strictly speaking we are applying Theorem 2 to a portion of the multiplier supported away from the origin, and dealing with the portion near the origin by other (elementary) means. See Section 2.2 for further details.

For each $t \in (0,\infty)^n$ we define the $n\times n$ dilation matrix $\delta(t) := \mathrm{diag}(t_1,\dots,t_n)$, and the rectangular box $B(t) := \delta(t)^{-1}[-1,1]^n = [-1/t_1, 1/t_1]\times\cdots\times[-1/t_n, 1/t_n]$.

Now let $R' \in (0,\infty)^n$ and decompose $\mathbb{R}^n$ into a lattice of rectangles $\{\rho_k\}$ as follows. For each $k\in\mathbb{Z}^n$ let

$$\rho_k = \delta(R')\big(\{k\} + \big[-\tfrac{1}{2},\tfrac{1}{2}\big]^n\big),$$

making ρ_k the axis-parallel rectangular cell centred at $\delta(R')k = (R'_1k_1,\dots,R'_nk_n)$ with j-th side length R'_j. Defining $\Psi_k \in \mathcal{S}(\mathbb{R}^n)$ by

$$\widehat{\Psi}_k(\xi) = \widehat{\Psi}(\delta(R')^{-1}\xi - k),$$

we have

$$\sum_{k\in\mathbb{Z}^n}\widehat{\Psi}_k \equiv 1, \tag{33}$$

$$\mathrm{supp}(\widehat{\Psi}_k) \subseteq \tilde{\rho}_k,$$

for each $k\in\mathbb{Z}^n$. Here $\tilde{\rho}_k$ denotes the concentric double of ρ_k. Finally, let the operator S_k be given by $\widehat{S_kf} = \widehat{\Psi}_k\hat{f}$.

For the operators S_k we have the following essentially standard square function estimate. Very similar results may be found in several places in the literature, including [Córdoba 1982; Rubio de Francia 1985; Bennett et al. 2006].

Proposition 7.
$$\int_{\mathbb{R}^n}\sum_k |S_kf|^2 w \lesssim \int_{\mathbb{R}^n}|f|^2 M_S w \tag{34}$$

uniformly in R', where M_S denotes the strong maximal function.

A reverse weighted inequality, where the function f is controlled by the square function $\big(\sum_k |S_kf|^2\big)^{1/2}$, is rather more subtle, and is the main content of this section.

Theorem 8. *Suppose $R\in(0,\infty)^n$ is such that $R_j \geq R'_j$ for each $1\leq j\leq n$, and let ρ be an axis-parallel rectangle in $\mathbb{R}^n$ of j-th side length R_j. If $\mathrm{supp}(\hat{f})\subseteq\rho$ then*

$$\int_{\mathbb{R}^n}|f|^2 w \lesssim \int_{\mathbb{R}^n}\sum_k |S_kf|^2 M_S A_{R,R'} M_S w,$$

where the operator $A_{R,R'}$ is given by

$$A_{R,R'}w(x) = \sup_{y\in\{x\}+B(R')}\frac{1}{|B(R)|}\int_{\{y\}+B(R)} w.$$

Remark. As the following proof reveals, Theorem 8 continues to hold if the operators S_k are replaced by the genuine frequency-projection operators defined by $\widehat{S_kf} = \chi_{\rho_k}\hat{f}$.

Proof of Theorem 8. We begin by exploiting the Fourier support hypothesis on f to mollify the weight w. Let $\Phi\in\mathcal{S}(\mathbb{R}^n)$ be an even function satisfying $\widehat{\Phi} = 1$ on $[-1,1]^n$. Observe that if we define $\Phi_R\in\mathcal{S}(\mathbb{R}^n)$ by $\widehat{\Phi}_R(\xi) = \widehat{\Phi}(\delta(R)^{-1}\xi) = \widehat{\Phi}(\xi_1/R_1,\dots,\xi_n/R_n)$, then $f = f*(M_{\xi_\rho}\Phi_R)$. Here

$M_{\xi_\rho}\Phi_R(x)=e^{-2\pi i x\cdot\xi_\rho}\Phi_R(x)$ and ξ_ρ denotes the centre of ρ. A standard application of the Cauchy–Schwarz inequality and Fubini's theorem reveals that

$$\int_{\mathbb{R}^n}|f|^2w=\int_{\mathbb{R}^n}|f*(M_{\xi_\rho}\Phi_R)|^2w\le\|\Phi_R\|_1\int_{\mathbb{R}^n}|f|^2|\Phi_R|*w\lesssim\int_{\mathbb{R}^n}|f|^2w_1,\tag{35}$$

where $w_1:=|\Phi_R|*w$. The final inequality here follows since the functions Φ_R are normalised in L^1.

Now, by (33) we have

$$f=\sum_k S_kf.$$

This raises issues of orthogonality for the operators S_k on $L^2(w_1)$. Although the weight w_1 is smooth, in order for us to have any (almost) orthogonality we should expect to need an improved smoothness consistent with a mollification by $|\Phi_{R'}|$ rather than $|\Phi_R|$. We thus seek an efficient way of dominating w_1 by such an improved weight.[4] This ingredient, which is based on an argument in [Bennett et al. 2006], comes in two simple steps. First define the weight function w_2 by

$$w_2(x)=\sup_{y\in\{x\}+B(R')}w_1(y).$$

Certainly w_2 dominates w_1 pointwise, although w_2 will not in general be sufficiently smooth for our purposes. Let $\Theta\in\mathcal{S}(\mathbb{R}^n)$ be a nonnegative function whose Fourier transform is nonnegative and compactly supported, and let

$$w_3=\Theta_{R'}*w_2,$$

where $\Theta_{R'}$ is defined by $\widehat{\Theta}_{R'}(\xi)=\widehat{\Theta}(\delta(R')^{-1}\xi)=\widehat{\Theta}(\xi_1/R'_1,\dots,\xi_n/R'_n)$. By construction w_3 has Fourier support in $\{\xi:|\xi_j|\lesssim R'_j,\ 1\le j\le n\}$ and so by Parseval's theorem we have the desired almost orthogonality:

$$\langle S_kf,S_{k'}f\rangle_{L^2(w_3)}=0\quad\text{if }|k-k'|\gtrsim 1.\tag{36}$$

Despite its improved smoothness, this new weight w_3 continues to dominate w_1.

Lemma 9. $w_2\lesssim w_3$.

Proof. By dilating Θ by an absolute constant if necessary, we may assume that $\Theta\gtrsim 1$ on $[-1,1]^n$. Consequently,

$$w_3(0)\gtrsim\frac{1}{|B(R')|}\int_{B(R')}w_2(x)\,dx.$$

Now let $B_1,B_2,\dots,B_{2^n}$ be the intersections of $B(R')$ with the 2^n coordinate hyperoctants of $\mathbb{R}^n$. It will suffice to show that there exists $\ell\in\{1,2,\dots,2^n\}$ such that $w_2(x)\ge w_2(0)$ for all $x\in B_\ell$. To see this we suppose, for a contradiction, that there exist $x_\ell\in B_\ell$ such that $w_2(x_\ell)<w_2(0)$ for each $1\le\ell\le 2^n$. Thus, by the definition of w_2 we have

$$\sup_{x\in\{x_\ell\}+B(R')}w_1(x)<w_2(0)\quad\text{for }1\le\ell\le 2^n.$$

[4]This idea is somewhat reminiscent of the classical fact that if $(Mw^s)^{1/s}<\infty$ a.e. and $s>1$ then $w\le(Mw^s)^{1/s}\in A_1\subset A_2$; see the discussion following (3).

However, since
$$B(R') \subseteq \bigcup_{\ell=1}^{2^n} (\{x_\ell\} + B(R')),$$
$\sup_{x\in B(R')} w_1(x) < w_2(0)$, contradicting the definition of $w_2(0)$. □

Combining (35), Lemma 9 and the orthogonality property (36) we obtain
$$\int_{\mathbb{R}^n} |f|^2 w \lesssim \int_{\mathbb{R}^n} \sum_k |S_k f|^2 w_3. \tag{37}$$

In order to complete the proof of Theorem 8 it remains to show that $w_3(x) \lesssim M_S A_{R,R'} M_S w(x)$ uniformly in x and R, R'. Since $w_3(x) \lesssim M_S w_2(x)$ it suffices to show that $w_2(x) \lesssim A_{R,R'} M_S w(x)$. Further, by translation invariance, it is enough to deal with the case $x = 0$. To see this we define the maximal operator $M_S^{(R)}$ by
$$M_S^{(R)} w(y) = \sup_{r\geq 1} \frac{1}{|rB(R)|} \int_{\{y\}+rB(R)} w.$$

Notice that $M_S^{(R)} w \leq M_S w$. Using the rapid decay of Φ and elementary considerations we have
$$w_1(y) = |\Phi_R| * w(y) \lesssim M_S^{(R)} w(y) \lesssim \frac{1}{|B(R)|} \int_{\{y\}+B(R)} M_S^{(R)} w$$
and so
$$w_2(0) \lesssim \sup_{y\in B(R')} \frac{1}{|B(R)|} \int_{\{y\}+B(R)} M_S w = A_{R,R'} M_S w(0)$$
uniformly in R, R', as required. □

4. The proof of Theorem 2

The proof we present combines the essential ingredients of the standard proof of the Marcinkiewicz multiplier theorem (see [Stein 1970; Duoandikoetxea 2001], for example) and the square function estimates from Section 3.

By standard weighted Littlewood–Paley theory (see [Bennett and Harrison 2012] for further details) it suffices to prove that
$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^5 \mathcal{M}_{\alpha,\beta} M w \tag{38}$$
holds for functions f with Fourier support in the dyadic interval $\pm[R, 2R]$, with bounds uniform in $R^\alpha \geq 1$.

Suppose that $\operatorname{supp}(\hat{f}) \subseteq \pm[R, 2R]$ for some $R^\alpha \geq 1$. We begin by applying Theorem 8 with $n = 1$, $R' = R^{-\alpha} R$ and $\rho = \pm[R, 2R]$. For each $k \in \mathbb{Z}$ let ρ_k, $\tilde{\rho}_k$, Ψ_k and S_k be as in Section 3. By Theorem 8 we have
$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} \sum_k |S_k T_m f|^2 M A_{R,R'} M w \tag{39}$$
uniformly in $R^\alpha \geq 1$. Of course the case $R = 1$ (as with the case $\alpha = 0$) is somewhat degenerate here, although we note that the conclusion (39) does retain some content.

Next we invoke the standard representation formula

$$S_kT_mf(x) = m(a_k)S_kf(x) + \int_{\tilde{\rho}_k} U_\xi S_kf(x)m'(\xi)\,d\xi, \tag{40}$$

where $a_k = \inf \tilde{\rho}_k$ and U_ξ is defined by

$$\widehat{U_\xi f} = \chi_{[\xi,\infty)}\hat{f}. \tag{41}$$

In order to see (40), which is a minor variant of (7), we use the Fourier inversion formula to write

$$\begin{aligned}
S_kT_mf(x) &= \int_{\tilde{\rho}_k} e^{ix\xi}\widehat{\Psi}_k(\xi)m(\xi)\hat{f}(\xi)\,d\xi \\
&= -\int_{\tilde{\rho}_k} \frac{\partial}{\partial\xi}\left(\int_\xi^\infty \widehat{\Psi}_k(t)\hat{f}(t)e^{ixt}dt\right)m(\xi)\,d\xi \\
&= m(a_k)S_kf(x) + \int_{\tilde{\rho}_k}\left(\int_{\mathbb{R}} \chi_{[\xi,\infty)}(t)\widehat{\Psi}_k(t)\hat{f}(t)e^{ixt}\,dt\right)m'(\xi)\,d\xi \\
&= m(a_k)S_kf(x) + \int_{\tilde{\rho}_k} U_\xi S_kf(x)m'(\xi)\,d\xi.
\end{aligned}$$

Applying Minkowski's inequality to (40), we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}} |S_kT_mf|^2 MA_{R,R'}Mw\right)^{1/2} \\
&\qquad \le |m(a_k)|\left(\int_{\mathbb{R}} |S_kf|^2 MA_{R,R'}Mw\right)^{1/2} + \int_{\tilde{\rho}_k}\left(\int_{\mathbb{R}} |U_\xi S_kf|^2 MA_{R,R'}Mw\right)^{1/2}|m'(\xi)|\,d\xi.
\end{aligned}$$

Since $U_\xi = \frac{1}{2}(I + iM_{-\xi}HM_\xi)$, where $M_\xi f(x) := e^{-2\pi ix\xi}f(x)$ and H is the Hilbert transform, an application of (4) yields

$$\int_{\mathbb{R}} |U_\xi S_kf|^2 MA_{R,R'}Mw \lesssim \int_{\mathbb{R}} |S_kf|^2 M^4A_{R,R'}Mw$$

uniformly in ξ, k and R. Using this along with the hypotheses (13) and (14) yields

$$\int_{\mathbb{R}} |S_kT_mf|^2 MA_{R,R'}Mw \lesssim R^{-2\beta}\int_{\mathbb{R}} |S_kf|^2 M^4A_{R,R'}Mw$$

uniformly in k and R. Here we have used the fact that $|a_k| \sim R$. Thus by (39) and Proposition 7 we have

$$\int_{\mathbb{R}} |T_mf|^2 w \lesssim R^{-2\beta}\int_{\mathbb{R}} |f|^2 M^5A_{R,R'}Mw$$

uniformly in $R^\alpha \ge 1$. Inequality (38) now follows from the elementary observation that

$$R^{-2\beta}A_{R,R^{-\alpha}R}w(x) \lesssim \mathcal{M}_{\alpha,\beta}w(x)$$

uniformly in x and $R^\alpha \ge 1$.

5. Extensions to higher dimensions

Theorem 2 has a natural generalisation to higher dimensions. It should be pointed out that this generalisation, being of Marcinkiewicz type in formulation, is not motivated by multipliers of the form (11), but rather by tensor products of such one-dimensional multipliers. For the sake of simplicity we confine our attention to two dimensions. Just as with the classical Marcinkiewicz multiplier theorem, this is already typical of the general situation.

For $\alpha, \beta \in \mathbb{R}^2$ let $\mathscr{C}(\alpha, \beta)$ denote the class of functions $m : \mathbb{R}^2 \to \mathbb{C}$ for which

$$\operatorname{supp}(m) \subseteq \{\xi \in \mathbb{R}^2 : |\xi_1|^{\alpha_1} \geq 1,\ |\xi_2|^{\alpha_2} \geq 1\}, \tag{42}$$

$$\sup_{\xi_2} \sup_{\xi_1} |\xi_2|^{\beta_2} |\xi_1|^{\beta_1} |m(\xi_1, \xi_2)| < \infty, \tag{43}$$

$$\sup_{\xi_2} |\xi_2|^{\beta_2} \Biggl\{ \sup_{R_1^{\alpha_1} \geq 1} \sup_{\substack{I_1 \subseteq [R_1, 2R_1] \\ \ell(I_1) = R_1^{-\alpha_1} R_1}} R_1^{\beta_1} \int_{\pm I_1} \left| \frac{\partial m}{\partial \xi_1} \right| d\xi_1 \Biggr\} < \infty, \tag{44}$$

$$\sup_{\xi_1} |\xi_1|^{\beta_1} \Biggl\{ \sup_{R_2^{\alpha_2} \geq 1} \sup_{\substack{I_2 \subseteq [R_2, 2R_2] \\ \ell(I_2) = R_2^{-\alpha_2} R_2}} R_2^{\beta_2} \int_{\pm I_2} \left| \frac{\partial m}{\partial \xi_2} \right| d\xi_2 \Biggr\} < \infty, \tag{45}$$

$$\sup_{R_2^{\alpha_2} \geq 1} \sup_{\substack{I_2 \subseteq [R_2, 2R_2] \\ \ell(I_2) = R_2^{-\alpha_2} R_2}} \sup_{R_1^{\alpha_1} \geq 1} \sup_{\substack{I_1 \subseteq [R_1, 2R_1] \\ \ell(I_1) = R_1^{-\alpha_1} R_1}} R_2^{\beta_2} R_1^{\beta_1} \int_{\pm I_2} \int_{\pm I_1} \left| \frac{\partial^2 m}{\partial \xi_1 \partial \xi_2} \right| d\xi_1 d\xi_2 < \infty. \tag{46}$$

Although these conditions might appear rather complicated, it is straightforward to verify that the tensor product $\mathscr{C}(\alpha_1, \beta_1) \otimes \mathscr{C}(\alpha_2, \beta_2)$ is contained in $\mathscr{C}(\alpha, \beta)$, and that $\mathscr{C}(0, 0)$ is precisely the classical Marcinkiewicz multipliers on $\mathbb{R}^2$.

Theorem 10. *If $m \in \mathscr{C}(\alpha, \beta)$ then*

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} |f|^2 M_S^9 \mathcal{M}_{\alpha,\beta} M_S^7 w,$$

where

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(r_1, y_1) \in \Gamma_{\alpha_1}(x_1)} \sup_{(r_2, y_2) \in \Gamma_{\alpha_2}(x_2)} \frac{r_1^{2\beta_1}}{r_1} \frac{r_2^{2\beta_2}}{r_2} \int_{|y_1 - z_1| \leq r_1} \int_{|y_2 - z_2| \leq r_2} w(z)\, dz$$

and M_S denotes the strong maximal function.

Proof of Theorem 10. The proof we present is very similar to the one-dimensional case. By standard weighted Littlewood–Paley theory (again, see [Bennett and Harrison 2012] for details) it suffices to prove that

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} |f|^2 M_S^8 \mathcal{M}_{\alpha,\beta} M_S w \tag{47}$$

holds for functions f with Fourier support in $(\pm[R_1, 2R_1]) \times (\pm[R_2, 2R_2])$, with bounds uniform in $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$.

Assuming such a restriction we can apply Theorem 8 with $n = 2$, $R' = (R_1^{-\alpha_1} R_1, R_2^{-\alpha_2} R_2)$ and

$\rho = (\pm[R_1, 2R_1]) \times (\pm[R_2, 2R_2])$. For each $k \in \mathbb{Z}^2$ let ρ_k, $\tilde{\rho}_k$, Ψ_k and S_k be as in Section 3. By Theorem 8 we have

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim \int_{\mathbb{R}^2} \sum_{k\in\mathbb{Z}^2} |S_k T_m f|^2 M_S A_{R,R'} M_S w \tag{48}$$

uniformly in R.

In what follows $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ denote the coordinate projections $\pi_1 x = x_1$ and $\pi_2 x = x_2$, and for each k we define $a_k \in \mathbb{R}^2$ by $a_k = (\inf \pi_1 \tilde{\rho}_k, \inf \pi_2 \tilde{\rho}_k)$, making a_k the bottom-left vertex of the axis-parallel rectangle $\tilde{\rho}_k$.

Now, taking our cue again from the standard proof of the classical Marcinkiewicz multiplier theorem, we write

$$S_k T_m f(x) = m(a_k) S_k f(x) + \int_{\pi_1\tilde{\rho}_k} U^{(1)}_{\xi_1} S_k f(x) \frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\, d\xi_1 + \int_{\pi_2\tilde{\rho}_k} U^{(2)}_{\xi_2} S_k f(x) \frac{\partial m}{\partial \xi_2}(\pi_1 a_k, \xi_2)\, d\xi_2$$
$$+ \int_{\tilde{\rho}_k} U^{(2)}_{\xi_2} U^{(1)}_{\xi_1} S_k f(x) \frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \tag{49}$$

where $U^{(j)}_{\xi_j}$ denotes the operator U_{ξ_j}, defined in (41), acting in the j-th variable. Applying Minkowski's inequality we obtain

$$\left(\int_{\mathbb{R}^2} |S_k T_m f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \leq |m(a_k)| \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2}$$
$$+ \int_{\pi_1\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(1)}_{\xi_1} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\right| d\xi_1$$
$$+ \int_{\pi_2\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(2)}_{\xi_2} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_2}(\pi_1 a_k, \xi_2)\right| d\xi_2$$
$$+ \int_{\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |U^{(2)}_{\xi_2} U^{(1)}_{\xi_1} S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}(\xi_1, \xi_2)\right| d\xi.$$

We denote the summands on the right side by I, II, III, IV.

For I we use the facts that $|\pi_1 a_k| \sim R_1$ and $|\pi_2 a_k| \sim R_2$, along with (43) to obtain

$$I \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S A_{R,R'} M_S w\right)^{1/2}$$

uniformly in k. For II, following the proof of Theorem 2, we apply (4) in the first variable to obtain

$$II \lesssim \int_{\pi_1\tilde{\rho}_k} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S^4 A_{R,R'} M_S w\right)^{1/2} \left|\frac{\partial m}{\partial \xi_1}(\xi_1, \pi_2 a_k)\right| d\xi_1,$$

which by (44) yields

$$II \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S^4 A_{R,R'} M_S w\right)^{1/2}$$

uniformly in k. By (45) and symmetry, it follows that III satisfies the same bound. The final term IV is potentially the most interesting as it involves using a weighted bound on the double Hilbert transform. By a twofold application of (4), followed by (46), we obtain

$$IV \lesssim R_2^{-\beta_2} R_1^{-\beta_1} \left(\int_{\mathbb{R}^2} |S_k f|^2 M_S^7 A_{R,R'} M_S w \right)^{1/2}.$$

Thus, by (48) and Proposition 7 we have

$$\int_{\mathbb{R}^2} |T_m f|^2 w \lesssim R_2^{-2\beta_2} R_1^{-2\beta_1} \int_{\mathbb{R}^2} |S_k f|^2 M_S^8 A_{R,R'} M_S w$$

uniformly in $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$. Inequality (47) now follows on observing that

$$R_2^{-2\beta_2} R_1^{-2\beta_1} A_{R,R'} w(x) \lesssim \mathcal{M}_{\alpha,\beta} w(x)$$

uniformly in x and $R_1^{\alpha_1}, R_2^{\alpha_2} \geq 1$. □

Remarks. The above arguments raise certain basic questions about weighted inequalities for various multiparameter operators in harmonic analysis. For instance, for which powers $k \in \mathbb{N}$ do we have

$$\int_{\mathbb{R}^n} |Tf|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 M_S^k w$$

for classical product Calderón–Zygmund operators T on $\mathbb{R}^n$ with $n \geq 2$? As we have seen, crudely applying the one-dimensional result of Wilson [1989] separately in each variable allows us to take $k = 3n$. Reducing this power would of course lead to a reduction in the number of factors of M_S in the statement of Theorem 10.

As we have already discussed, since Theorem 10 involves Marcinkiewicz-type hypotheses it really belongs to the "multiparameter" theory of multipliers. It is conceivable that a variant may be obtained involving a Hörmander-type hypothesis on sublacunary annuli in $\mathbb{R}^n$; that is, involving hypotheses on quantities of the form

$$\int_{R_j \leq |\xi| < R_{j+1}} \left| \left(\frac{\partial}{\partial \xi} \right)^{\gamma} m(\xi) \right|^2 d\xi$$

for certain sublacunary sequences (R_j) and multi-indices γ. A very general result of this type (which might permit the radii (R_j) to accumulate away from zero) is likely to be difficult as it would naturally apply to the Bochner–Riesz multipliers. There are of course many other conditions that one might impose, from the above all the way down to the higher-dimensional analogue of the Miyachi condition (11) in [Miyachi 1980; 1981]; see also [Carbery 1985].

6. Proof of Theorem 4

In this section we give a proof of Theorem 4. Our argument is a generalisation of those in [Bennett et al. 2006; Bennett and Harrison 2012]; see also [Nagel and Stein 1984]. As the case $\alpha = 0$ reduces to the L^p–L^q boundedness of the classical fractional Hardy–Littlewood maximal function, we may assume that $\alpha \neq 0$.

The claimed necessity of the conditions (27), (28) and (29) follows from testing the putative L^p–L^q

bound for $\mathcal{M}_{\alpha,\beta}$ on the characteristic function $f_\nu = \chi_{[-\nu,\nu]}$. The necessary conditions follow by taking limits as both $\nu \to 0$ and $\nu \to \infty$. We leave these elementary calculations to the reader.

It will suffice to establish the L^p–L^q boundedness of $\mathcal{M}_{\alpha,\beta}$ for exponents $1 < p \le q \le \infty$ on the sharp line

$$\beta = \frac{\alpha}{2q} + \frac{1}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big). \tag{50}$$

As our proof of Theorem 4 rests on a Hardy space estimate, it is necessary to regularise the averaging in the definition of $\mathcal{M}_{\alpha,\beta}$. To this end let P be a nonnegative, compactly supported bump function which is positive on $[-1, 1]$, let $P_r(x) = r^{-1}P(x/r)$, and define the maximal operator $\tilde{\mathcal{M}}_{\alpha,\beta}$ by

$$\tilde{\mathcal{M}}_{\alpha,\beta} w(x) = \sup_{(y,r)\in\Gamma_\alpha(x)} r^{2\beta}|P_r * w(y)|.$$

Since $\mathcal{M}_{\alpha,\beta} w \lesssim \tilde{\mathcal{M}}_{\alpha,\beta} w$ pointwise uniformly, it suffices to prove that $\tilde{\mathcal{M}}_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ when (50) holds. Since $\tilde{\mathcal{M}}_{\alpha,0}$ is bounded on $L^\infty(\mathbb{R})$, and $\tilde{\mathcal{M}}_{\alpha,1/2}$ is bounded from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$, by analytic interpolation [Stein 1970] it suffices to prove that $\tilde{\mathcal{M}}_{\alpha,\alpha/2}$ is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. We establish this by showing that

$$\|\tilde{\mathcal{M}}_{\alpha,\alpha/2} a\|_1 \lesssim 1 \tag{51}$$

uniformly in H^1-atoms a. By translation invariance we may suppose that the support interval I of a is centred at the origin. Our estimates will be based on the standard and elementary pointwise bound

$$|P_r * a(x)| \lesssim \begin{cases} 1/|I| & \text{if } r \le |I|,\ |x| \le \frac{5}{2}|I|, \\ |I|/r^2 & \text{if } r \ge |I|,\ |x| \le \frac{5}{2}r, \\ 0 & \text{otherwise,} \end{cases}$$

which follows from the smoothness of P and the mean value zero property of a. As the nature of Γ_α is fundamentally different in the cases $\alpha < 0$, $0 < \alpha \le 1$ and $\alpha > 1$, we divide the analysis into three cases. For $\alpha < 0$ and $\alpha > 0$ the interesting situation is, respectively, when $|I| \gtrsim 1$ and $|I| \lesssim 1$.

Case 1: $\alpha < 0$. Elementary considerations reveal that if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2} a(x) \lesssim \begin{cases} |I| & \text{if } |x| \lesssim 1, \\ |I|/|x|^{-(2-\alpha)/(1-\alpha)} & \text{otherwise,} \end{cases}$$

and if $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2} a(x) \lesssim \begin{cases} |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I|\,|x|^{-(2-\alpha)/(1-\alpha)} & \text{otherwise.} \end{cases}$$

In both cases (51) follows by direct calculation.

Case 2: $0 < \alpha \le 1$. For technical reasons it is convenient to deal first with the particularly simple case $\alpha = 1$. If $|I| \gtrsim 1$ then arguing similarly we obtain

$$\tilde{\mathcal{M}}_{1,1/2} a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ 0 & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{1,1/2}a(x) \lesssim \begin{cases} 1 & \text{if } |x| \lesssim 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly in both cases (51) follows immediately.

Suppose now that $0 < \alpha < 1$. If $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ 0 & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I||x|^{-(2-\alpha)/(1-\alpha)} & \text{if } |I|^{1-\alpha} \lesssim |x| \lesssim 1, \\ 0 & \text{otherwise.} \end{cases}$$

Again, in both cases (51) follows directly.

Case 3: $\alpha > 1$. If $|I| \gtrsim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-1} & \text{if } |x| \lesssim |I|, \\ |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{otherwise,} \end{cases}$$

and if $|I| \lesssim 1$ then

$$\tilde{\mathcal{M}}_{\alpha,\alpha/2}a(x) \lesssim \begin{cases} |I|^{-(1-\alpha)} & \text{if } |x| \lesssim |I|^{1-\alpha}, \\ |I|^{-1}|x|^{\alpha/(1-\alpha)} & \text{if } |I|^{1-\alpha} \lesssim |x|. \end{cases}$$

Once again (51) follows.

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THE HARTREE EQUATION FOR INFINITELY MANY PARTICLES II: DISPERSION AND SCATTERING IN 2D

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We consider the nonlinear Hartree equation for an interacting gas containing infinitely many particles and we investigate the large-time stability of the stationary states of the form $f(-\Delta)$, describing a homogeneous quantum gas. Under suitable assumptions on the interaction potential and on the momentum distribution f, we prove that the stationary state is asymptotically stable in dimension 2. More precisely, for any initial datum which is a small perturbation of $f(-\Delta)$ in a Schatten space, the system weakly converges to the stationary state for large times.

1. Introduction

This article is the continuation of [Lewin and Sabin 2014], where we considered the nonlinear Hartree equation for infinitely many particles. However, the main result of the present article does not rely on that paper.

The Hartree equation can be written using the formalism of density matrices as

$$\begin{cases} i\partial_t\gamma = [-\Delta + w * \rho_\gamma, \gamma], \\ \gamma(0) = \gamma_0. \end{cases} \tag{1}$$

Here $\gamma(t)$ is the one-particle density matrix of the system, which is a bounded nonnegative self-adjoint operator on $L^2(\mathbb{R}^d)$ with $d \geqslant 1$, and $\rho_\gamma(t,x) = \gamma(t,x,x)$ is the density of particles in the system at time t. Also, w is the interaction potential between the particles, which we assume to be smooth and rapidly decaying at infinity.

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The starting point of [Lewin and Sabin 2014] was the observation that (1) has many stationary states. Indeed, if $f \in L^\infty(\mathbb{R}_+, \mathbb{R})$ is such that

$$\int_{\mathbb{R}^d} |f(|k|^2)|\, dk < +\infty,$$

then the operator

$$\gamma_f := f(-\Delta)$$

(the Fourier multiplier by $k \mapsto f(|k|^2)$) is a bounded self-adjoint operator which commutes with $-\Delta$ and whose density

$$\rho_{\gamma_f}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(|k|^2)\, dk \quad \text{for all } x \in \mathbb{R}^d$$

is constant. Hence, for $w \in L^1(\mathbb{R}^d)$, $w * \rho_{\gamma_f}$ is also constant, and $[w * \rho_{\gamma_f}, \gamma_f] = 0$. Therefore $\gamma(t) \equiv \gamma_f$ is a stationary solution to (1). The purpose of [Lewin and Sabin 2014] and of this article is to investigate the stability of these stationary states, under "local perturbations". We do not necessarily think of small perturbations in norm, but we typically think of $\gamma(0) - \gamma_f$ as being compact.

The simplest choice is $f \equiv 0$, which corresponds to the vacuum case. We are interested here in the case of $f \not\equiv 0$, describing an infinite, homogeneous gas containing infinitely many particles and with positive constant density $\rho_{\gamma_f} > 0$. Four important physical examples are:

- *Fermi gas at zero temperature*:
$$\gamma_f = \mathbb{1}(-\Delta \leqslant \mu), \quad \mu > 0; \tag{2}$$
- *Fermi gas at positive temperature* $T > 0$:
$$\gamma_f = \frac{1}{e^{(-\Delta-\mu)/T} + 1}, \quad \mu \in \mathbb{R}; \tag{3}$$
- *Bose gas at positive temperature* $T > 0$:
$$\gamma_f = \frac{1}{e^{(-\Delta-\mu)/T} - 1}, \quad \mu < 0; \tag{4}$$
- *Boltzmann gas at positive temperature* $T > 0$:
$$\gamma_f = e^{(\Delta+\mu)/T}, \quad \mu \in \mathbb{R}. \tag{5}$$

In the density matrix formalism, the number of particles in the system is given by $\operatorname{Tr}\gamma$. It is clear that $\operatorname{Tr}\gamma_f = +\infty$ in the previous examples, since γ_f is a translation-invariant (hence noncompact) operator. Because they contain infinitely many particles, these systems also have infinite energy. In [Lewin and Sabin 2014], we proved the existence of global solutions to (1) in the defocusing case $\widehat{w} \geqslant 0$, when the initial datum γ_0 has a finite *relative energy* counted with respect to the stationary states γ_f given in (2)–(5), in dimensions $d = 1, 2, 3$. We also proved the orbital stability of γ_f.

In this work, we are interested in the *asymptotic stability* of γ_f. As usual for Schrödinger equations, we cannot expect strong convergence in norm and we will instead prove that $\gamma(t) \rightharpoonup \gamma_f$ weakly as $t \to \pm\infty$, if the initial datum γ_0 is close enough to γ_f. Physically, this means that a small defect added to the translation-invariant state γ_f disappears for large times due to dispersive effects, and the system locally relaxes towards the homogeneous gas. More precisely, we are able to describe the exact behavior of $\gamma(t)$ for large times, by proving that

$$e^{-it\Delta}(\gamma(t)-\gamma_f)e^{it\Delta} \underset{t\to\pm\infty}{\longrightarrow} Q_\pm$$

strongly in a Schatten space (hence, for instance, for the operator norm). This nonlinear scattering result means that the perturbation $\gamma(t)-\gamma_f$ of the homogeneous gas evolves for large times as in the case of free particles:

$$\gamma(t)-\gamma_f \underset{t\to\pm\infty}{\simeq} e^{it\Delta}Q_\pm e^{-it\Delta} \underset{t\to\pm\infty}{\rightharpoonup} 0.$$

If $f\equiv 0$ and $\gamma_0=|u_0\rangle\langle u_0|$ is a rank-one orthogonal projection, then (1) reduces to the well-known Hartree equation for one function

$$\begin{cases} i\partial_t u=(-\Delta+w*|u|^2)u, \\ u(0)=u_0. \end{cases} \tag{6}$$

There is a large literature about scattering for the nonlinear equation (6), for instance [Ginibre and Velo 1980; 2000; Strauss 1981; Hayashi and Tsutsumi 1987; Mochizuki 1989; Nakanishi 1999]. The intuitive picture is that the nonlinear term is negligible for small u, since $(w*|u|^2)u$ is formally of order 3. It is important to realize that this intuition does not apply in the case $f\neq 0$ considered in this paper. Indeed the nonlinear term is not small and it behaves linearly with respect to the small parameter $\gamma-\gamma_f$:

$$[w*\rho_\gamma,\gamma]=[w*\rho_{\gamma-\gamma_f},\gamma]\simeq[w*\rho_{\gamma-\gamma_f},\gamma_f]\neq 0. \tag{7}$$

One of the main purposes of this paper is to rigorously study the linear response of the homogeneous Hartree gas γ_f (the last term in (7)), which is a very important object in the physical literature called the Lindhard function [Lindhard 1954; Giuliani and Vignale 2005, Chapter 4]. For a general f, our main result requires that the interaction potential w be small enough, in order to control the linear term. Under the natural assumption that f is strictly decreasing (as it is in the three physical examples (3)–(5)), the condition can be weakened in the defocusing case $\widehat{w}\geqslant 0$.

The paper is organized as follows. In the next section we state our main result and make several comments. In Section 3 we study the linear response in detail, before turning to the higher-order terms in the expansion of the wave operator in Section 4. Apart from the linear response, our method requires us to treat separately the next $d-1$ terms of this expansion, in spacial dimension d. Even if all the other estimates are valid in any dimension, in this paper we only deal with the second order in dimension $d=2$.

2. Main result

In the whole paper, we denote by $\mathcal{B}(\mathfrak{H})$ the space of bounded operators on the Hilbert space $\mathfrak{H}$. The corresponding operator norm is $\|A\|$. We use the notation $\mathfrak{S}^p(\mathfrak{H})$ for the Schatten space of all the compact operators A on $\mathfrak{H}$ such that $\operatorname{Tr}|A|^p<\infty$, with $|A|=\sqrt{A^*A}$, and use the norm $\|A\|_{\mathfrak{S}^p(\mathfrak{H})}:=(\operatorname{Tr}|A|^p)^{1/p}$. We refer to [Simon 1977] for the properties of Schatten spaces. The spaces $\mathfrak{S}^2(\mathfrak{H})$ and $\mathfrak{S}^1(\mathfrak{H})$ correspond to Hilbert–Schmidt and trace-class operators. We often use the shorthand notation $\mathcal{B}$ and $\mathfrak{S}^p$ when the Hilbert space $\mathfrak{H}$ is clear from the context.

Our main result is the following.

Theorem 1 (dispersion and scattering in 2D). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be such that*

$$\int_0^\infty \big(1 + r^{\frac{k}{2}}\big)|f^{(k)}(r)|\,dr < \infty \quad \text{for } k = 0, \dots, 4 \tag{8}$$

and $\gamma_f := f(-\Delta)$. Denote by $\check{g}$ the Fourier inverse on $\mathbb{R}^2$ of $g(k) = f(|k|^2)$. Let $w \in W^{1,1}(\mathbb{R}^2)$ be such that

$$\|\check{g}\|_{L^1(\mathbb{R}^2)}\|\widehat{w}\|_{L^\infty(\mathbb{R}^2)} < 4\pi \tag{9}$$

or, if $f' < 0$ a.e. on $\mathbb{R}_+$, such that

$$\max\big(\varepsilon_g \widehat{w}(0)_+, \|\check{g}\|_{L^1(\mathbb{R}^2)}\|(\widehat{w})_-\|_{L^\infty(\mathbb{R}^2)}\big) < 4\pi, \tag{10}$$

where $(\widehat{w})_-$ is the negative part of $\widehat{w}$ and $0 \leqslant \varepsilon_g \leqslant \|\check{g}\|_{L^1(\mathbb{R}^2)}$ is a constant depending only on g (defined later in Section 3).

Then, there exists a constant $\varepsilon_0 > 0$ (depending only on w and f) such that, for any $\gamma_0 \in \gamma_f + \mathfrak{S}^{4/3}$ with

$$\|\gamma_0 - \gamma_f\|_{\mathfrak{S}^{4/3}} \leqslant \varepsilon_0,$$

there exists a unique solution $\gamma \in \gamma_f + C_t^0(\mathbb{R}, \mathfrak{S}^2)$ to the Hartree equation (1) with initial datum γ_0, such that

$$\rho_\gamma - \rho_{\gamma_f} \in L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2).$$

Furthermore, $\gamma(t)$ scatters around γ_f at $t = \pm\infty$, in the sense that there exists $Q_\pm \in \mathfrak{S}^4$ such that

$$\lim_{t\to\pm\infty}\big\|e^{-it\Delta}(\gamma(t) - \gamma_f)e^{it\Delta} - Q_\pm\big\|_{\mathfrak{S}^4} = \lim_{t\to\pm\infty}\big\|\gamma(t) - \gamma_f - e^{it\Delta}Q_\pm e^{-it\Delta}\big\|_{\mathfrak{S}^4} = 0. \tag{11}$$

Before explaining our strategy to prove Theorem 1, we make some comments.

First, we notice that the gases at positive temperature, (3), (4) and (5), are all covered by the theorem with condition (10), since the corresponding f is smooth, strictly decreasing and exponentially decaying at infinity. Our result does not cover the Fermi gas at zero temperature (2), however. We show in Section 3 that its linear response is unbounded and it is a challenging task to better understand its dynamical stability.

The next remark concerns the assumption (9), which says that the interactions must be small or, equivalently, that the gas must contain few particles having a small momentum (if $\check{g} \geqslant 0$, then the condition can be written as $f(0)\|\widehat{w}\|_{L^\infty(\mathbb{R}^2)} < 2$ and hence $f(|k|^2)$ must be small for small k). Our method does not work without condition (9) if no other information on w and f is provided. However, under the natural additional assumption that f is strictly decreasing, we can replace condition (9) by the weaker condition (10). The latter says that the negative part of $\widehat{w}$ and the value at zero of the positive part should be small (with a better constant for the latter). We will explain later where condition (10) comes from, but we mention already that we are not able to deal with an arbitrary large potential $\widehat{w}$ in a neighborhood of the origin, even in the defocusing case. We also recall that the focusing or defocusing character of our equation is governed by the sign of $\hat{w}$ and not of w, as it is for (6). This is seen from the sign of the nonlinear term

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(x-y)\rho_{\gamma-\gamma_f}(x)\rho_{\gamma-\gamma_f}(y)\,dx\,dy = (2\pi)^{\frac{d}{2}}\int_{\mathbb{R}^d}\hat{w}(k)|\hat{\rho}_{\gamma-\gamma_f}(k)|^2\,dk,$$

which appears in the relative energy of the system [Lewin and Sabin 2014, Equations (9)–(10)].

Then, we note that in our previous article [Lewin and Sabin 2014] we proved the existence of global solutions under the assumption that the initial state γ_0 has a finite relative entropy with respect to γ_f (and for f being one of the physical examples (2)–(5)). By the Lieb–Thirring inequality [Frank et al. 2011; 2013; Lewin and Sabin 2014], this implies that $\rho_{\gamma(t)}-\rho_{\gamma_f}\in L^\infty_t(L^2_x)$. By interpolation, we therefore get that $\rho_{\gamma(t)}-\rho_{\gamma_f}\in L^p_t(L^2_x)$ for every $2\leqslant p\leqslant\infty$. This requires, of course, that the initial perturbation $\gamma_0-\gamma_f$ be small in $\mathfrak{S}^{4/3}$. Our method does not allow us to replace this condition by the fact that γ_0 has a small relative entropy with respect to γ_f.

Let us finally mention that our results hold for *small* initial data, where the smallness is not only qualitative (meaning that $\gamma_0-\gamma_f\in\mathfrak{S}^{4/3}$, for instance) but also quantitative, since we need that $\|\gamma_0-\gamma_f\|_{\mathfrak{S}^{4/3}}$ be small enough. This is a well-known restriction, coming from our method of proof, based on a fixed-point argument. The literature on nonlinear Schrödinger equations suggests that in order to remove this smallness assumption one would need some assumption on w like $\hat{w}\geqslant 0$, as well as some additional (almost) conservation laws [Cazenave 2003]. Our study of the linear response operator however indicates that the situation is involved and more information on the momentum distribution f is certainly also necessary.

We now explain our strategy for proving Theorem 1. The idea of the proof relies on a fixed-point argument, in the spirit of [Lewin and Sabin 2014, Section 5]. If we can prove that $\rho_\gamma-\rho_{\gamma_f}\in L^2_{t,x}(\mathbb{R}_+\times\mathbb{R}^2)$, then we deduce from [Yajima 1987; Frank et al. 2014] that there exists a family of unitary operators $U_V(t)\in C^0_t(\mathbb{R}_+,\mathcal{B})$ on $L^2(\mathbb{R}^2)$ such that

$$\gamma(t)=U_V(t)\gamma_0U_V(t)^*$$

for all $t\in\mathbb{R}_+$. We furthermore have

$$U_V(t)=e^{it\Delta}\mathcal{W}_V(t),$$

where $\mathcal{W}_V(t)$ is the wave operator. By iterating Duhamel's formula, we can expand the latter in a series as

$$\mathcal{W}_V(t)=1+\sum_{n\geqslant 1}\mathcal{W}_V^{(n)}(t), \tag{12}$$

with

$$\mathcal{W}_V^{(n)}(t):=(-i)^n\int_0^t dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\,e^{-it_n\Delta}V(t_n)e^{i(t_n-t_{n-1})\Delta}\cdots e^{i(t_2-t_1)\Delta}V(t_1)e^{it_1\Delta}.$$

The idea is to find a solution to the nonlinear equation

$$\rho_Q(t)=\rho\big[e^{it\Delta}\mathcal{W}_{w*\rho_Q}(t)(\gamma_f+Q_0)\mathcal{W}_{w*\rho_Q}(t)^*e^{-it\Delta}\big]-\rho_{\gamma_f}, \tag{13}$$

by a fixed-point argument on the variable $\rho_Q\in L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)$, where $Q:=\gamma-\gamma_f$ and $Q_0=\gamma_0-\gamma_f$.

Inserting the expansion (12) of the wave operator $\mathcal{W}_V$, the nonlinear equation (13) may be written as

$$\rho_Q = \rho\big[e^{it\Delta} Q_0 e^{-it\Delta}\big] - \mathcal{L}(\rho_Q) + \mathcal{R}(\rho_Q), \tag{14}$$

where $\mathcal{L}$ is linear and $\mathcal{R}(\rho_Q)$ contains higher-order terms. The sign convention for $\mathcal{L}$ is motivated by the stationary case [Frank et al. 2013]. The linear operator $\mathcal{L}$ can be written

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2,$$

where

$$\mathcal{L}_1(\rho_Q) = -\rho\big[e^{it\Delta}(\mathcal{W}^{(1)}_{w*\rho_Q}(t)\gamma_f + \gamma_f \mathcal{W}^{(1)}_{w*\rho_Q}(t)^*)e^{-it\Delta}\big]$$

and

$$\mathcal{L}_2(\rho_Q) = -\rho\big[e^{it\Delta}(\mathcal{W}^{(1)}_{w*\rho_Q}(t)Q_0 + Q_0 \mathcal{W}^{(1)}_{w*\rho_Q}(t)^*)e^{-it\Delta}\big].$$

Note that $\mathcal{L}_2$ depends on Q_0 and it can always be controlled by adding suitable assumptions on Q_0. On the other hand, the other linear operator $\mathcal{L}_1$ does not depend on the studied solution; it only depends on the functions w and f.

In Section 3, we study the linear operator $\mathcal{L}_1$ in detail, and we prove that it is a space-time Fourier multiplier of the form $\widehat{w}(k) m_f(\omega, k)$ where m_f is a famous function in the physics literature called the *Lindhard function* [Lindhard 1954; Mihaila 2011; Giuliani and Vignale 2005], which only depends on f and d. In particular, we investigate the question of when $\mathcal{L}_1$ is bounded on $L^p_{t,x}(\mathbb{R} \times \mathbb{R}^2)$, and we show this is the case when w and f are sufficiently smooth. For the Fermi sea (2), we prove that $\mathcal{L}_1$ is unbounded on $L^2_{t,x}$.

The next step is to invert the linear part by rewriting (14) in the form

$$\rho_Q = (1 + \mathcal{L})^{-1}\big(\rho\big[e^{it\Delta} Q_0 e^{-it\Delta}\big] + \mathcal{R}(\rho_Q)\big) \tag{15}$$

and applying a fixed-point method. In the time-independent case, a similar technique was used for the Dirac sea in [Hainzl et al. 2005]. In order to be able to invert the Fourier multiplier $\mathcal{L}_1$, we need that

$$\boxed{\min_{(\omega,k)\in\mathbb{R}\times\mathbb{R}^2} |\widehat{w}(k) m_f(\omega, k) + 1| > 0.} \tag{16}$$

Then $1 + \mathcal{L} = 1 + \mathcal{L}_1 + \mathcal{L}_2$ is invertible if Q_0 is small enough. In Section 3 we prove the simple estimate

$$|m_f(k, \omega)| \leqslant (4\pi)^{-1} \|\check{g}\|_{L^1(\mathbb{R}^2)}$$

and this leads to our condition (9). If f is strictly decreasing, then we are able to prove that the imaginary part of $m_f(k, \omega)$ is never 0 if $k \neq 0$ or $\omega \neq 0$. Since $m_f(\omega, k)$ has a fixed sign for $\omega = 0$ and $k = 0$, everything boils down to investigating the properties of m_f at $(\omega, k) = (0, 0)$. At this point m_f will usually not be continuous, and it can take both positive and negative values. We have

$$\limsup_{\substack{k\to 0\\ \omega\to 0}} \Re\, m_f(\omega, k) = (4\pi)^{-1} \|\check{g}\|_{L^1(\mathbb{R}^2)}$$

and we set

$$\liminf_{\substack{k\to 0\\ \omega\to 0}} \Re\, m_f(\omega, k) =: -(4\pi)^{-1} \varepsilon_g,$$

leading to our condition (10). It is well known in the physics literature that the imaginary part of the Lindhard function plays a crucial role in the dynamics of the homogeneous Fermi gas. In our rigorous analysis it is used to invert the linear response operator outside of the origin. The behavior of $m_f(\omega,k)$ for $(\omega,k)\to(0,0)$ is, however, involved and $1+\mathcal{L}_1$ is not invertible if $\widehat{w}(0)>\varepsilon_g/(4\pi)$ or $\widehat{w}(0)<-\|\check{g}\|_{L^1(\mathbb{R}^2)}/(4\pi)$.

For the Fermi gas at zero temperature (2) we will prove that the minimum in (16) is always zero, except when $\widehat{w}$ vanishes sufficiently quickly at the origin; this means that $1+\mathcal{L}_1$ is never invertible. It is an interesting open question to understand the asymptotic stability of the Fermi sea.

Once the linear response $\mathcal{L}$ has been inverted, it remains to study the zeroth-order term $\rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big]$ and the higher-order terms contained in $\mathcal{R}(\rho_Q)$. At this step we use a recent Strichartz estimate in Schatten spaces due to Frank, Lieb, Seiringer and the first author.

Theorem 2 (Strichartz estimate on wave operator [Frank et al. 2014, Theorem 3]). *Fix $d\geqslant 1$ and q such that $1+d/2\leqslant q<\infty$, and p such that $2/p+d/q=2$. Let also $0<\varepsilon<1/p$. There exists $C=C(d,p,\varepsilon)>0$ such that for any $V\in L^p_t(\mathbb{R},L^q_x(\mathbb{R}^d))$ and any $t\in\mathbb{R}$, we have the estimates*

$$\big\|\mathcal{W}^{(1)}_V(t)\big\|_{\mathfrak{S}^{2q}}\leqslant C\|V\|_{L^p_tL^q_x} \tag{17}$$

and

$$\big\|\mathcal{W}^{(n)}_V(t)\big\|_{\mathfrak{S}^{2\lceil q/n\rceil}}\leqslant \frac{C^n}{(n!)^{\frac{1}{p}-\varepsilon}}\|V\|^n_{L^p_tL^q_x}\quad \text{for all } n\geqslant 2. \tag{18}$$

The estimate (17) is the dual version of

$$\|\rho_{e^{it\Delta}Ae^{-it\Delta}}\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))}\leqslant C\|A\|_{\mathfrak{S}^{\frac{2q}{q+1}}} \tag{19}$$

for any (p,q) such that $2/p+d/q=d$ and $1\leqslant q\leqslant 1+2/d$; see [Frank et al. 2014, Theorem 1]. The estimate (19) is useful to deal with the first-order term involving Q_0 in (15), leading to the natural condition that $Q_0\in\mathfrak{S}^{4/3}$ in dimension $d=2$ with $p=q=2$.

In dimension d, it seems natural to prove that $\rho_Q\in L^{1+2/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)$. The estimate (18) turns out to be enough to deal with the terms of order at least $d+1$ but it does not seem to help for the terms of order d and lower, because the wave operators $\mathcal{W}^{(n)}_V$ with small n belong to a Schatten space with a too-large exponent. Apart from the linear response, we are therefore left with $d-1$ terms for which a more detailed computation is necessary. We are not able to do this in any dimension (the number of such terms grows with d), but we can deal with the second-order term in dimension $d=2$,

$$\rho\big[e^{it\Delta}\big(\mathcal{W}^{(2)}_{w*\rho_Q}(t)\gamma_f+\mathcal{W}^{(1)}_{w*\rho_Q}(t)\gamma_f\mathcal{W}^{(1)}_{w*\rho_Q}(t)^*+\gamma_f\mathcal{W}^{(2)}_{w*\rho_Q}(t)^*\big)e^{-it\Delta}\big],$$

which then finishes the proof of the theorem in this case. The second-order term is the topic of Section 5.

Even if our final result only covers the case $d=2$, we have several estimates in any dimension $d\geqslant 2$. With the results of this paper, only the terms of orders from 2 to d remain to be studied to obtain a result similar to Theorem 1 (with $\rho_\gamma-\rho_{\gamma_f}\in L^{1+2/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)$) in dimensions $d\geqslant 3$.

3. Linear response theory

3.1. ***Computation of the linear response operator.*** As we have explained before, we deal here with the linear response $\mathscr{L}_1$ associated with the homogeneous state γ_f. The first order in Duhamel's formula is defined by

$$Q_1(t) := -i\int_0^t e^{i(t-t')\Delta}[w * \rho_{Q(t')}, \gamma_f]e^{i(t'-t)\Delta}\,dt'.$$

We see that it is a linear expression in ρ_Q, and we compute its density as a function of ρ_Q.

Proposition 1 (uniform bound on $\mathscr{L}_1$). *Let $d \geqslant 1$, $f \in L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $\int_{\mathbb{R}^d}|f(k^2)|\,dk < +\infty$, and $w \in L^1(\mathbb{R}^d)$. The linear operator $\mathscr{L}_1$ defined for all $\varphi \in \mathscr{D}(\mathbb{R}_+\times\mathbb{R}^d)$ by*

$$\mathscr{L}_1(\varphi)(t) := -\rho[Q_1(t)] = \rho\left[i\int_0^t e^{i(t-t')\Delta}[w*\varphi(t'),\gamma_f]e^{i(t'-t)\Delta}\,dt'\right]$$

is a space-time Fourier multiplier by the kernel $K^{(1)} = K^{(1)}(\omega,k) = \widehat{w}(k)\,m_f(\omega,k)$, where

$$\boxed{[\mathscr{F}_\omega^{-1}m_f](t,k) := 2\,\mathbb{1}_{t\geqslant 0}\sqrt{2\pi}\,\sin(t|k|^2)\check{g}(2tk)} \tag{20}$$

(*we recall that $g(k) := f(k^2)$ and that $\check{g}$ is its Fourier inverse*). *This means that for all $\varphi \in \mathscr{D}(\mathbb{R}_+\times\mathbb{R}^d)$ we have*

$$\mathscr{F}_{t,x}[\mathscr{L}_1(\varphi)](\omega,k) = \widehat{w}(k)m_f(\omega,k)[\mathscr{F}_{t,x}\varphi](\omega,k) \quad \textit{for all } (\omega,k)\in\mathbb{R}\times\mathbb{R}^d,$$

where $\mathscr{F}_{t,x}$ is the space-time Fourier transform. Furthermore, if $\int_0^\infty |x|^{2-d}|\check{g}(x)|\,dx < \infty$, then $m_f \in L^\infty_{\omega,k}(\mathbb{R}\times\mathbb{R}^d)$ and we have the explicit estimates

$$\|m_f\|_{L^\infty_{\omega,k}} \leqslant \frac{1}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}\,dx\right) \tag{21}$$

and

$$\|\mathscr{L}_1\|_{L^2_{t,x}\to L^2_{t,x}} \leqslant \frac{\|\widehat{w}\|_{L^\infty}}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}\,dx\right). \tag{22}$$

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R}_+\times\mathbb{R}^d)$. In order to compute $\mathscr{L}(\varphi)$, we use the relation

$$\int_0^\infty \operatorname{Tr}[W(t,x)Q_1(t)]\,dt = \int_0^\infty\int_{\mathbb{R}^d} W(t,x)\rho_{Q_1}(t,x)\,dx\,dt,$$

valid for any function $W \in \mathscr{D}(\mathbb{R}_+\times\mathbb{R}^d)$. This leads to

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d} W(t,x)\rho_{Q_1}(t,x)\,dx\,dt\\ &\quad= \frac{-i}{(2\pi)^d}\int_0^\infty\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-2i(t-t')k\cdot\ell}\times\widehat{w}(t,-k)\widehat{V}(t',k)\big(g\big(\ell-\tfrac12 k\big)-g\big(\ell+\tfrac12 k\big)\big)\,d\ell\,dk\,dt'dt,\end{aligned}$$

where $g(k) := f(k^2)$ and $V = w*\varphi$. Computing the ℓ-integral gives

$$\int_{\mathbb{R}^d} e^{-2i(t-t')k\cdot\ell}\big(g\big(\ell-\tfrac12 k\big)-g\big(\ell+\tfrac12 k\big)\big)\,d\ell = -(2\pi)^{\frac d2}2i\,\sin((t-t')|k|^2)\check{g}(2(t-t')k).$$

Hence, using that $\widehat{V} = (2\pi)^{d/2}\widehat{w}\widehat{\varphi}$, we find that

$$\int_0^\infty\int_{\mathbb{R}^d} W(t,x)\rho_{Q_1}(t,x)\,dx\,dt$$
$$= -2\int_0^\infty\int_0^t\int_{\mathbb{R}^d} \sin((t-t')|k|^2)\check{g}(2(t-t')k)\widehat{w}(k)\widehat{w}(t,-k)\widehat{\varphi}(t',k)\,dk\,dt'dt.$$

Since g is radial, $\check{g}$ is also radial and we have

$$|m_f(\omega,k)| \leqslant 2\int_0^\infty \left|\sin(t|k|^2)\right||\check{g}(2t|k|)|\,dt \leqslant 2\int_0^\infty \frac{|\sin(t|k|)|}{|k|}|\check{g}(2t)|\,dt \leqslant \frac{1}{2}\int_0^\infty r|\check{g}(r)|\,dr. \qquad \square$$

We now make several remarks about the previous result.

First, the physical examples for g are

$$g(k) = \begin{cases} \mathbb{1}(|k|^2 \leqslant \mu), & \mu > 0, \\ e^{-(|k|^2-\mu)/T}, & T > 0,\ \mu \in \mathbb{R}, \\ \dfrac{1}{e^{(|k|^2-\mu)/T}+1}, & T > 0,\ \mu \in \mathbb{R}, \\ \dfrac{1}{e^{(|k|^2-\mu)/T}-1}, & T > 0,\ \mu < 0. \end{cases}$$

In the last three choices, g is a Schwartz function, hence $\check{g} \in L^1(\mathbb{R}^d)$. For the first choice of g (Fermi sea at zero temperature), we have $\check{g}(r) \sim r^{-1}\sin r$, which obviously does not satisfy $r\check{g}(r) \in L^1(0,+\infty)$.

Then, we remark that (21) is optimal without more assumptions on f. Indeed, for $\omega = 0$ and small k, we find

$$m_f(0,k) = 2\int_0^\infty \sin(t|k|^2)\check{g}(2tk)\,dt \underset{k\to 0}{\longrightarrow} \frac{1}{2}\int_0^\infty r\check{g}(r)\,dr = \frac{1}{2|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d}\frac{\check{g}(x)}{|x|^{d-2}}\,dx.$$

We conclude that (21) is optimal if $\check{g}$ has a constant sign (for instance if f is decreasing, as in the physical examples (3)–(5)). Similarly, (22) is optimal if both $\check{g}$ and w have a constant sign (then $|\widehat{w}(0)| = \|\widehat{w}\|_{L^\infty}$).

In general the function m_f is complex-valued and it is not an easy task to determine when $\widehat{w}(k)m_f(\omega,k)$ stays far from -1. Since the stationary linear response is real ($\Im m_f(0,k) \equiv 0$), the condition should at least involve the maximum or the minimum of m_f on the set $\{\omega = 0\}$, depending on the sign of $\widehat{w}$. Even if the function m_f is bounded on $\mathbb{R}\times\mathbb{R}^d$ by (21), it will usually not be continuous at the point $(0,0)$. Under the additional condition that f is strictly decreasing, we are able to prove that

$$\{\Im m_f(\omega,k) = 0\} = \{\omega = 0\} \cup \{k = 0\}$$

and this can be used to replace the assumption on $\widehat{w}$ by one on $(\widehat{w})_-$ and $\widehat{w}(0)_+$. In order to explain this, we first compute m_f in the case of a Fermi gas at zero temperature, $f(k^2) = \mathbb{1}(|k|^2 \leqslant \mu)$.

Proposition 2 (linear response at zero temperature). *Let $d \geqslant 1$ and $\mu > 0$. Then, for the Fermi sea at zero temperature $\gamma_f = \mathbb{1}(-\Delta \leqslant \mu)$, the corresponding Fourier multiplier $m_f(\omega,k) := m_d^{\mathrm{F}}(\mu,\omega,k)$ of* *the linear response operator in dimension d is given by*

$$m_1^{\mathrm{F}}(\mu,\omega,k)=\frac{1}{2\sqrt{2\pi}|k|}\log\left|\frac{(|k|^2+2|k|\sqrt{\mu})^2-\omega^2}{(|k|^2-2|k|\sqrt{\mu})^2-\omega^2}\right| \\ +i\frac{\pi}{4\sqrt{2\pi}|k|}\big(\mathbb{1}(|\omega-|k|^2|\leqslant 2\sqrt{\mu}|k|)-\mathbb{1}(|\omega+|k|^2|\leqslant 2\sqrt{\mu}|k|)\big) \quad (23)$$

for $d=1$; by

$$m_2^{\mathrm{F}}(\mu,\omega,k) \\ =\frac{\pi}{2\sqrt{2}}\left(2-\frac{\operatorname{sgn}(|k|^2+\omega)}{|k|^2}\big((|k|^2+\omega)^2-4\mu|k|^2\big)_+^{\frac12}-\frac{\operatorname{sgn}(|k|^2-\omega)}{|k|^2}\big((|k|^2-\omega)^2-4\mu|k|^2\big)_+^{\frac12}\right) \\ +i\frac{\pi}{2\sqrt{2}|k|^2}\Big(\big((|k|^2-\omega)^2-4\mu|k|^2\big)_-^{\frac12}-\big((|k|^2+\omega)^2-4\mu|k|^2\big)_-^{\frac12}\Big). \quad (24)$$

for $d=2$; and by

$$m_d^{\mathrm{F}}(\mu,\omega,k)=\frac{|\mathbb{S}^{d-2}|\mu^{\frac{d-1}{2}}}{(2\pi)^{\frac{d-1}{2}}}\int_0^1 m_1^{\mathrm{F}}\big(\mu(1-r^2),\omega,k\big)r^{d-2}\,dr \quad \textit{for } d\geqslant 2, \\ =\frac{|\mathbb{S}^{d-3}|\mu^{\frac{d-2}{2}}}{(2\pi)^{\frac{d-2}{2}}}\int_0^1 m_2^{\mathrm{F}}\big(\mu(1-r^2),\omega,k\big)r^{d-3}\,dr \quad \textit{for } d\geqslant 3. \quad (25)$$

The formula for m_d^{F} is well known in the physics literature [Lindhard 1954; Mihaila 2011; Giuliani and Vignale 2005, Chapter 4]. It is also possible to derive an explicit expression for $m_3^{\mathrm{F}}(\mu,\omega,k)$; see [Giuliani and Vignale 2005, Chapter 4]. We remark that $m_d^{\mathrm{F}}(\mu,0,k)$ coincides with the time-independent linear response computed in [Frank et al. 2013, Theorem 2.5].

From the formulas we see that the real part of m_d^{F} can have both signs. It is always positive for $\omega=0$ and it can take negative values for $\omega\neq 0$. For instance, in dimension $d=2$, on the curve $\omega=|k|^2+2\sqrt{\mu}|k|$ the imaginary part vanishes and we get

$$m_2^{\mathrm{F}}\big(\mu,|k|^2+2\sqrt{\mu}|k|,k\big)=\frac{\pi}{\sqrt{2}}\left(1-\sqrt{1+\frac{2\sqrt{\mu}}{|k|}}\right)\underset{k\to 0}{\longrightarrow}-\infty. \quad (26)$$

In particular, if $\hat{w}(k)/\sqrt{|k|}\to+\infty$ when $k\to 0$, then $\hat{w}(k)m_f(|k|^2+2\sqrt{\mu}|k|,k)\to-\infty$ when $|k|\to 0$. Since, on the other hand, $\hat{w}(k)m_f(|k|^2+2\sqrt{\mu}|k|,k)\to 0$ when $|k|\to\infty$, we conclude that the function must cross -1, so $(1+\mathscr{L}_1)^{-1}$ is not bounded.

An important feature of m_d^{F}, which we are going to use in the positive temperature case, is that the imaginary part $\Im m_d^{\mathrm{F}}(\mu,\omega,k)$ has a constant sign on $\{\omega>0\}$ and on $\{\omega<0\}$. Before we discuss this in detail, we provide the proof of the proposition.

Proof. A calculation shows that the Fourier inverse $\check{g}_1$ of the radial g in dimension $d=1$ is given by

$$\check{g}_1(\mu,x)=\sqrt{\frac{2}{\pi}}\frac{\sin(\sqrt{\mu}|x|)}{|x|}. \quad (27)$$

In dimension $d \geqslant 2$ we can write

$$\begin{aligned}
\check{g}_d(\mu,|x|) =& \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathbb{1}(|k|^2 \leqslant \mu) e^{ik\cdot x} \\
=& \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}} dk_1 \int_{\mathbb{R}^{d-1}} dk_\perp \mathbb{1}(|k_1|^2 \leqslant \mu - |k_\perp|^2) e^{ik_1|x|} \\
=& \frac{|\mathbb{S}^{d-2}|\mu^{\frac{d-1}{2}}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}} dk_1 \int_0^1 \mathbb{1}\big(|k_1|^2 \leqslant \mu(1-r^2)\big) e^{ik_1|x|} r^{d-2}\, dr \\
=& \frac{|\mathbb{S}^{d-2}|\mu^{\frac{d-1}{2}}}{(2\pi)^{\frac{d-1}{2}}}\int_0^1 \check{g}_1\big(\mu(1-r^2),|x|\big) r^{d-2}\, dr \\
=& \frac{2|\mathbb{S}^{d-2}|}{(2\pi)^{\frac{d}{2}}}\frac{\mu^{\frac{d-1}{2}}}{|x|}\int_0^1 \sin\big(\sqrt{\mu}|x|\sqrt{1-r^2}\big) r^{d-2}\, dr.
\end{aligned} \tag{28}$$

Similarly, we have in dimension $d \geqslant 3$

$$\check{g}_d(\mu,|x|) = \frac{|\mathbb{S}^{d-3}|\mu^{\frac{d-2}{2}}}{(2\pi)^{\frac{d-2}{2}}}\int_0^1 \check{g}_2\big(\mu(1-r^2),|x|\big) r^{d-3}\, dr. \tag{29}$$

Now we can compute the multiplier $m_d^{\mathrm{F}}(\mu,\omega,k)$ for $d = 1, 2$. We start with $d = 1$, for which we have

$$[\mathcal{F}_\omega^{-1} m_{f,1}](t,k) = 4\mathbb{1}_{t\geqslant 0}\frac{\sin(t|k|^2)\sin(2\sqrt{\mu}t|k|)}{2t|k|}.$$

It remains to compute the time Fourier transform. We use the formula, valid for any $a, b \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^\infty \frac{\sin(at)\sin(bt)}{t} e^{-it\omega}\, dt \\
&= \tfrac{1}{4}\log\left|\frac{(a+b)^2-\omega^2}{(a-b)^2-\omega^2}\right| + i\frac{\pi}{8}\big(\operatorname{sgn}(a-b-\omega) - \operatorname{sgn}(a+b-\omega) + \operatorname{sgn}(a+b+\omega) - \operatorname{sgn}(a-b+\omega)\big),
\end{aligned}$$

and obtain (23). To provide the more explicit expression in dimension 2, we use the formula

$$\forall a \in \mathbb{R}, \quad \frac{1}{a}\int_0^1 \log\frac{|a+2\sqrt{1-r^2}|}{|a-2\sqrt{1-r^2}|}\, dr = \frac{\pi}{2} - \frac{\pi}{2}\left(1-\frac{4}{a^2}\right)_+^{1/2},$$

which leads to the claimed form (24) of $m_2^{\mathrm{F}}(\mu,\omega,k)$. □

Now we will use the imaginary part of m_d^{F} to show that $1 + \mathcal{L}_1$ is invertible with bounded inverse when $\widehat{w} \geqslant 0$ with $\widehat{w}(0)$ not too large and f is strictly decreasing.

Corollary 1 ($1 + \mathcal{L}_1$ in the defocusing case). *Let $d \geqslant 1$ and $f \in L^\infty(\mathbb{R}_+,\mathbb{R})$ such that $f'(r) < 0$ for all $r > 0$ and $\int_0^\infty (r^{d/2-1}|f(r)| + |f'(r)|)\, dr < \infty$. Assume furthermore that $\int_{\mathbb{R}^d} |x|^{2-d}|\check{g}(x)|\, dx < \infty$ with $g(k) = f(|k|^2)$. If $w \in L^1(\mathbb{R}^d)$ is an even function such that*

$$\|(\widehat{w})_-\|_{L^\infty}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}\, dx\right) < 2|\mathbb{S}^{d-1}| \tag{30}$$

and such that

$$\varepsilon_g \widehat{w}(0)_+ < 2|\mathbb{S}^{d-1}|, \quad \textit{where } \varepsilon_g := -\liminf_{\substack{k\to 0\\ \omega\to 0}} \frac{\Re m_f(\omega,k)}{2|\mathbb{S}^{d-1}|}, \tag{31}$$

then we have

$$\min_{(\omega,k)\in\mathbb{R}\times\mathbb{R}^d} |\widehat{w}(k)m_f(\omega,k)-1| > 0$$

and $(1+\mathscr{L}_1)$ *is invertible on* $L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)$ *with bounded inverse.*

Proof. First we recall that m_f is uniformly bounded by (21). Therefore, we only have to look at the set

$$A = \left\{k\in\mathbb{R}^d : |\widehat{w}(k)| \geqslant \frac{1}{4|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}}\,dx\right)\right\}.$$

On the complement of A, we have $|\widehat{w}\, m_f + 1| \geqslant \frac12$. Since $\widehat{w}(k)\to 0$ when $|k|\to\infty$, A is a compact set. Next, from the integral formula

$$f(|k|^2) = -\int_0^\infty \mathbb{1}(|k|^2\leqslant s)\, f'(s)\,ds,$$

we infer that

$$m_f(\omega,k) = -\int_0^\infty m_d^{\mathrm{F}}(s,\omega,k)\, f'(s)\,ds.$$

This integral representation can be used to prove that m_f is continuous on $\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}$. In general, the function m_f is not continuous at $(0,0)$, however.

Since $m_d^{\mathrm{F}}(s,0,k)\geqslant 0$ for all k and $s\geqslant 0$, we conclude that $m_f(0,k)\geqslant 0$ and that

$$m_f(0,k)\widehat{w}(k) \geqslant -m_f(0,k)\widehat{w}(k)_- \geqslant -\|\widehat{w}_-\|_{L^\infty}\frac{1}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}\,dx\right),$$

due to (21). In particular,

$$|m_f(0,k)\widehat{w}(k)+1| \geqslant 1-\|\widehat{w}_-\|_{L^\infty}\frac{1}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}\,dx\right) > 0,$$

due to our assumption on $(\widehat{w})_-$. Similarly, we have $m_f(\omega,0)=0$ for all $\omega\neq 0$ and therefore $m_f(\omega,0)\widehat{w}(0)+1=1$ is invertible on $\{k=0,\omega\neq 0\}$.

Now we look at $k\neq 0$ and $\omega>0$ and we prove that the imaginary part of m_f never vanishes. We give the argument for $d=1$, as it is very similar for $d\geqslant 2$, using the integral representation (24). We have

$$\Im m_f(\omega,k) = -\frac{\pi}{4\sqrt{2\pi}|k|}\times\int_0^\infty \Big(\mathbb{1}\big((\omega-|k|^2)^2\leqslant 4s|k|^2\big) - \mathbb{1}\big((\omega+|k|^2)^2\leqslant 4s|k|^2\big)\Big) f'(s)\,ds.$$

The difference of the two Heaviside functions is always nonnegative for $\omega>0$. Furthermore, it is equal to 1 for all s in the interval

$$\frac{(\omega-|k|^2)^2}{4|k|^2}\leqslant s\leqslant\frac{(\omega+|k|^2)^2}{4|k|^2}.$$

Therefore we have

$$\Im m_f(\omega,k) \geqslant \frac{\pi}{4\sqrt{2\pi}|k|}\int_{\frac{(\omega-|k|^2)^2}{4|k|^2}}^{\frac{(\omega+|k|^2)^2}{4|k|^2}} |f'(s)|\,ds > 0$$

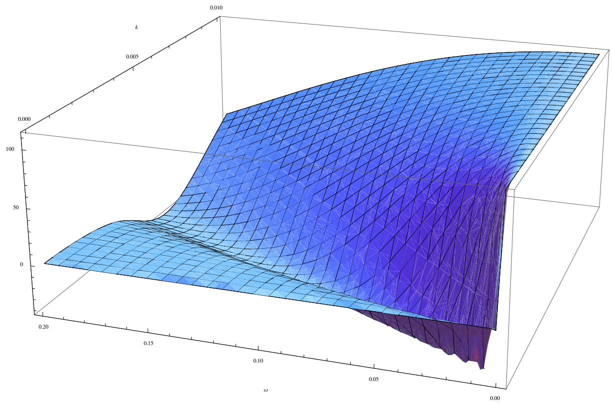

Figure 1. Plot of $\Re\, m_f(\omega, k)$ in the fermionic case (3) for $d = 2$, $T = 100$ and $\mu = 1$, in a neighborhood of $(\omega, k) = (0, 0)$.

for all $\omega > 0$ and $k \neq 0$. For $\omega < 0$, we can simply use that $\Im m_f(\omega, k) = -\Im m_f(-\omega, k)$, and this concludes the proof that the imaginary part does not vanish outside of $\{k = 0\} \cup \{\omega = 0\}$.

From the previous argument, we see that everything boils down to understanding the behavior of $\Re m_f$ in a neighborhood of $(0, 0)$. At this point the maximal value is $\frac{1}{2}\int_0^\infty r\check{g}(r)\,dr$ and the minimal value is $-\varepsilon_g 2|\mathbb{S}^{d-1}|$, by definition, hence the result follows. $\square$

We remark that

$$\Re\, m_f(\omega, k) \underset{\substack{k\to 0\\ \omega\to 0}}{\simeq} \frac{1}{2}\int_0^\infty t\,\check{g}(t)\cos\left(\frac{\omega}{2|k|}t\right)dt$$

and therefore we can express

$$-\varepsilon_g := \frac{1}{4|\mathbb{S}^{d-1}|}\min_{a\in\mathbb{R}}\int_0^\infty t\,\check{g}(t)\cos(at)\,dt.$$

In the three physical cases (3)–(5), the function f satisfies the assumptions of the corollary, and therefore $1 + \mathcal{L}_1$ is invertible with bounded inverse when w satisfies (30) and (31). Numerical computations show that ε_g is always positive, but usually smaller than the maximum, by a factor of between 2 and 10. As an illustration, we display the function $\Re m_f(\omega, k)$ for $T = 100$ and $\mu = 1$ in Figure 1.

3.2. ***Boundedness of the linear response in $L^p_{t,x}$.*** We have studied the boundedness of $\mathcal{L}_1$ from $L^2_{t,x}$ to $L^2_{t,x}$. This is useful in dimension $d = 2$, where the density ρ_Q naturally belongs to $L^2_{t,x}$. However, in other space dimensions, we would like to prove that ρ_Q belongs to $L^{1+2/d}_{t,x}$, and hence it makes sense to ask whether $\mathcal{L}_1$ is bounded from $L^p_{t,x}$ to $L^p_{t,x}$. This is the topic of this section. The study of Fourier multipliers acting on L^p is a classical subject in harmonic analysis. We use theorems of Stein and Marcinkiewicz to infer the required boundedness.

Proposition 3 (boundedness of the linear response on L^p). *Let $w \in L^1(\mathbb{R}^d)$ with $|x|^{d+2}w \in L^1(\mathbb{R}^d)$ be such that*

$$\left(\prod_{i\in I}|k_i|^2\partial_{k_i}\right)\widehat{w}(k) \in L^\infty_k(\mathbb{R}^d) \quad \textit{for all } I \subset \{1,\dots,d\}.$$

Let also $h : \mathbb{R}^d \to \mathbb{R}$ *be an even function such that for all* $\alpha \in \mathbb{N}^d$, $|\alpha| \leqslant d+3$,

$$\int_{\mathbb{R}^d} (1+|k|^{d+4})|\partial^\alpha h(k)|\,dk < +\infty \quad \text{and} \quad \Big(\prod_{i\in I} \partial_{k_i}\Big) h \in L^\infty_k(\mathbb{R}^d) \quad \text{for all } I \subset \{1,\dots,d\}.$$

Then the Fourier multiplier

$$\mathcal{F}_t\big\{\mathbb{1}(t \geqslant 0)\sin(t|k|^2)h(2tk)\big\}$$

defines a bounded operator from $L^p_{t,x}$ *to itself for every* $1 < p < \infty$.

The conditions on h are fulfilled if, for instance, h is a Schwartz function, hence they are fulfilled for our physical examples (3)–(5), where we take $h = \check{g}$.

Proof. We define

$$m_1(t,k) = \mathbb{1}(t \geqslant 1)\widehat{w}(k)\sin(t|k|^2)h(2tk),$$
$$m_2(t,k) = \mathbb{1}(0 \leqslant t \leqslant 1)\widehat{w}(k)\sin(t|k|^2)h(2tk),$$

and use a different criterion for these two multipliers.

To show that m_1 defines a bounded operator on L^p, we use the criterion of Stein [1970, II §2, Theorem 1]. We write $m_1(t,k) = \widehat{w}(k)\widetilde{m}_1(t,k)$. We first prove estimates on $\widetilde{m}_1$, which then imply that m_1 defines a bounded Fourier multiplier on L^p, by Stein's theorem. Computing the inverse Fourier transform of $\widetilde{m}_1$, one has

$$M_1(t,x) := [\mathcal{F}_k^{-1}\widetilde{m}_1](t,x) = \mathbb{1}(t \geqslant 1)(2\pi)^{-d/2}\int_{\mathbb{R}^d} \sin(t|k|^2)h(2tk)e^{ix\cdot k}\,dk.$$

Then, we have

$$\nabla_x M_1(t,x) = \mathbb{1}(t \geqslant 1)\frac{(2\pi)^{-d/2}}{(2t)^{d+1}} i \int_{\mathbb{R}^d} kh(k)\sin\frac{|k|^2}{4t}\,e^{i\frac{x\cdot k}{2t}}\,dk. \tag{32}$$

From this formula, we see that, for all (t,x),

$$t^{d+2}|\nabla_x M_1(t,x)| \leqslant C\int_{\mathbb{R}^d} |k|^3|h(k)|\,dk. \tag{33}$$

Next, let $1 \leqslant j \leqslant d$ and notice that

$$x_j^{d+2}e^{i\frac{x\cdot k}{2t}} = \frac{d^{d+2}}{dk_j^{d+2}}(2t)^{d+2}(-i)^{d+2}e^{i\frac{x\cdot k}{2t}},$$

and hence by an integration by parts we obtain

$$x_j^{d+2}\nabla_x M_1(t,x) = \mathbb{1}(t \geqslant 1)(2\pi)^{-d/2}2ti(-i)^{d+2}\int_{\mathbb{R}^d} \frac{d^{d+2}}{dk_j^{d+2}}\left[kh(k)\sin\frac{|k|^2}{4t}\right]e^{i\frac{x\cdot k}{2t}}\,dk.$$

When the k_j-derivative hits $\sin(|k|^2/4t)$ at least once, one gains a factor of (at least) $1/(4t)$, canceling the $2t$ before the integral; the only term that we have to prove is bounded in t is where all the k_j-derivatives hit the term $kh(k)$, which is

$$\mathbb{1}(t \geqslant 1)(2\pi)^{-\frac{d}{2}}2it(-i)^{d+2}\int_{\mathbb{R}^d} \frac{d^{d+2}}{dk_j^{d+2}}[kh(k)]\sin\frac{|k|^2}{4t}\,e^{i\frac{x\cdot k}{2t}}\,dk.$$

It is also bounded, since $|\sin(|k|^2/4t)| \leqslant |k|^2/4t$. We deduce that, for all (t,x),

$$|x|^{d+2}|\nabla_x M_1(t,x)| \leqslant C \sup_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leqslant d+2}} \int_{\mathbb{R}^d} (1+|k|^{d+2})|\partial^\alpha h(k)|\,dk. \tag{34}$$

For the time derivative we use the form

$$M_1(t,x) = \mathbb{1}(t \geqslant 1)(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} h(2tk)\sin(t|k|^2)\cos(x\cdot k)\,dk$$

to infer that

$$\begin{aligned}\partial_t M_1(t,x) &= 2\,\mathbb{1}(t\geqslant 1)(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} k\cdot\nabla_k h(2tk)\sin(t|k|^2)\cos(x\cdot k)\,dk \\ &\quad + \mathbb{1}(t\geqslant 1)(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |k|^2 h(2tk)\cos(t|k|^2)\cos(x\cdot k)\,dk \\ &= \frac{2\,\mathbb{1}(t\geqslant 1)}{(2t)^{d+1}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} k\cdot\nabla_k h(k)\sin\frac{|k|^2}{4t}\cos\frac{x\cdot k}{2t}\,dk \\ &\quad + \frac{\mathbb{1}(t\geqslant 1)}{(2t)^{d+2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |k|^2 h(k)\cos\frac{|k|^2}{4t}\cos\frac{x\cdot k}{2t}\,dk. \end{aligned}\tag{35}$$

By the same method as before, we infer

$$\|(t,x)\|^{d+2}|\partial_t M_1(t,x)| \leqslant C \sup_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leqslant d+3}} \int_{\mathbb{R}^d} (1+|k|^{d+4})|\partial^\alpha h(k)|\,dk. \tag{36}$$

Now let us go back to the multiplier m_1. We have

$$\mathcal{F}_x^{-1} m_1(t,x) = (2\pi)^{\frac{d}{2}}(w \star M_1(t,\cdot))(x),$$

and hence

$$\nabla_{t,x}\mathcal{F}_x^{-1} m_1(t,x) = (2\pi)^{\frac{d}{2}}(w \star \nabla_{t,x} M_1(t,\cdot))(x).$$

First we have

$$|t^{d+2}\nabla_{t,x}\mathcal{F}_x^{-1} m_1(t,x)| \leqslant C\|w\|_{L^1_x}\|t^{d+2}\nabla_{t,x}M_1(t,x)\|_{L^\infty_{t,x}},$$

which is finite thanks to (33), (34) and (36). Next,

$$|x|^{d+2}|\nabla_{t,x}\mathcal{F}_x^{-1} m_1(t,x)| \leqslant C\big\||\cdot|^{d+2}w\big\|_{L^1_x}\|\nabla_{t,x}M_1(t,x)\|_{L^\infty_{t,x}} + C\|w\|_{L^1_x}\big\||x|^{d+2}\nabla_{t,x}M_1(t,x)\big\|_{L^\infty_{t,x}}.$$

The second term is finite also from (33) and (34), while the first term is finite by the expressions (32) and (35). As a consequence, we can apply Stein's theorem to m_1 and we deduce that the corresponding operator is bounded on $L^p_{t,x}$ for all $1 < p < \infty$.

The multiplier m_2 is treated differently. We show that

$$m_2 \in L^1_t(\mathbb{R}, \mathcal{B}(L^p_x \to L^p_x)),$$

which is enough to show that m_2 defines a bounded operator on $L^p_{t,x}$. Indeed, for any $\varphi \in L^p_{t,x}$, define the Fourier multiplication operator T_{m_2} by

$$(T_{m_2}\varphi)(t,x) = \int_{\mathbb{R}} \mathcal{F}_x^{-1}\big[m_2(t-t',\cdot)(\mathcal{F}_x\varphi)(t',\cdot)\big](x)\,dt'.$$

Then, we have

$$\|T_{m_2}\varphi(t)\|_{L^p_x} \leqslant \int_{\mathbb{R}} \big\|\mathcal{F}_x^{-1}[m_2(t-t',\cdot)(\mathcal{F}_x\varphi)(t',\cdot)]\big\|_{L^p_x}\,dt' \leqslant \int_{\mathbb{R}} \|m_2(t-t')\|_{\mathcal{B}(L^p_x\to L^p_x)}\|\varphi(t')\|_{L^p_x}\,dt',$$

and hence

$$\|T_{m_2}\varphi\|_{L^p_{t,x}} \leqslant \|m_2\|_{L^1_t(\mathbb{R},\mathcal{B}(L^p_x\to L^p_x))}\|\varphi\|_{L^p_{t,x}}.$$

Hence, let us show that $\|m_2\|_{L^p_x\to L^p_x} \in L^1_t$. We estimate $\|m_2\|_{L^p_x\to L^p_x}$ by the Marcinkiewicz theorem [Grafakos 2008, Corollary 5.2.5]. Namely, we have to show that for indices $1 \leqslant i_1,\dots,i_\ell \leqslant d$ all different, we have

$$k_{i_1}\cdots k_{i_\ell}\partial_{k_{i_1}}\cdots\partial_{k_{i_\ell}} m_2(t,k) \in L^\infty_k,$$

and if so the Marcinkiewicz theorem tells us that

$$\|m_2(t)\|_{L^p_x\to L^p_x} \leqslant C \sup_{i_1,\dots,i_\ell} \|k_{i_1}\cdots k_{i_\ell}\partial_{k_{i_1}}\cdots\partial_{k_{i_\ell}} m_2(t,k)\|_{L^\infty_k}.$$

A direct computation shows that

$$|k_{i_1}\cdots k_{i_\ell}\partial_{k_{i_1}}\cdots\partial_{k_{i_\ell}} m_2(t,k)| \leqslant C\mathbb{1}_{0\leqslant t\leqslant 1} \sum_{I\subset\{i_1,\dots,i_\ell\}}\sum_{J\subset I} |k_{i_1}|^2\cdots|k_{i_\ell}|^2|\partial_I\hat{w}(k)|\,|(\partial_J g)(2tk)|,$$

hence

$$\|k_{i_1}\cdots k_{i_\ell}\partial_{k_{i_1}}\cdots\partial_{k_{i_\ell}} m_2(t,k)\|_{L^\infty_k} \leqslant C\mathbb{1}_{0\leqslant t\leqslant 1} \sup_{I,J\subset\{i_1,\dots,i_\ell\}} \big\||k_{i_1}|^2\cdots|k_{i_\ell}|^2|\partial_I\hat{w}(k)|\big\|_{L^\infty_k}\|\partial_J g\|_{L^\infty_k},$$

which is obviously an L^1_t–function. □

4. Higher-order terms

In this section, we explain how to treat the higher-order terms in (14). We recall the decomposition of the solution for all $t \geqslant 0$:

$$\rho_Q(t) = \rho\big[e^{it\Delta}\mathcal{W}_{w*\rho_Q}(t)(\gamma_f + Q_0)\mathcal{W}_{w*\rho_Q}(t)^* e^{-it\Delta}\big] - \rho_{\gamma_f}.$$

We first estimate the terms involving Q_0, in dimension 2.

Lemma 1. *Let $Q_0 \in \mathfrak{S}^{4/3}(L^2(\mathbb{R}^2))$ and $V \in L^2_{t,x}(\mathbb{R}_+\times\mathbb{R}^2)$. Then, we have the following estimate for all $n,m \geqslant 0$:*

$$\big\|\rho\big[e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^*e^{-it\Delta}\big]\big\|_{L^2_{t,x}(\mathbb{R}_+\times\mathbb{R}^2)} \leqslant C\|Q_0\|_{\mathfrak{S}^{4/3}}\frac{C^{n+m}\|V\|^{n+m}_{L^2_{t,x}}}{(n!)^{\frac14}(m!)^{\frac14}}$$

for some $C > 0$ independent of Q_0, n, m and V.

Proof. Defining $\mathcal{W}_V^{(0)}(t) := 1$, for $n, m \geqslant 0$ the density of

$$e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}$$

is estimated by duality in the following fashion. Let $U \in L^2_{t,x}(\mathbb{R}_+ \times \mathbb{R}^2)$. The starting point is the formula

$$\int_0^\infty \int_{\mathbb{R}^2} U(t,x)\rho\big[e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big](t,x)\,dx\,dt \\ = \int_0^\infty \operatorname{Tr}\big[U(t,x)e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big]\,dt.$$

By cyclicity of the trace, we have

$$\operatorname{Tr}\big[U(t,x)e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big] = \operatorname{Tr}\big[\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}U(t,x)e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\big].$$

A straightforward generalization of Theorem 2 shows that we have

$$\left\| \int_0^\infty \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}U(t,x)e^{it\Delta}\mathcal{W}_V^{(n)}(t)\,dt \right\|_{\mathfrak{S}^4} \leqslant \|U\|_{L^2_{t,x}} \frac{C^n\|V\|^n_{L^2_{t,x}}}{(n!)^{\frac14}} \frac{C^m\|V\|^m_{L^2_{t,x}}}{(m!)^{\frac14}},$$

and hence using that $Q_0 \in \mathfrak{S}^{4/3}$ and Hölder's inequality we infer that

$$\big\|\rho\big[e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big]\big\|_{L^2_{t,x}} \leqslant \|Q_0\|_{\mathfrak{S}^{4/3}} \frac{C^n\|V\|^n_{L^2_{t,x}}}{(n!)^{\frac14}} \frac{C^m\|V\|^m_{L^2_{t,x}}}{(m!)^{\frac14}}.$$

This concludes the proof of the lemma. □

When $d \geqslant 2$, the corresponding result is:

Lemma 2. *Let $d \geqslant 2$, $Q_0 \in \mathfrak{S}^{\frac{d+2}{d+1}}(L^2(\mathbb{R}^d))$, $1 < q \leqslant 1 + 2/d$ and p such that $2/p + d/q = d$. Let $V \in L^{p'}_t L^{q'}_x(\mathbb{R}_+ \times \mathbb{R}^d)$. Then, we have the following estimate for any $n, m \geqslant 0$:*

$$\big\|\rho\big[e^{it\Delta}\mathcal{W}_V^{(n)}(t)Q_0\mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big]\big\|_{L^p_t L^q_x(\mathbb{R}_+\times\mathbb{R}^d)} \leqslant C\|Q_0\|_{\mathfrak{S}^{\frac{d+2}{d+1}}} \frac{C^{n+m}\|V\|^{n+m}_{L^{p'}_t L^{q'}_x}}{(n!)^{\frac{1}{2q'}}(m!)^{\frac{1}{2q'}}}$$

for some $C > 0$ independent of Q_0, n, m and V.

The proof follows the same lines as for $d = 2$, and relies on the following estimate for any n, m:

$$\left\| \int_0^\infty \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}U(t,x)e^{it\Delta}\mathcal{W}_V^{(n)}(t)\,dt \right\|_{\mathfrak{S}^{d+2}} \leqslant \|U\|_{L^{p'}_t L^{q'}_x} \frac{C^n\|V\|^n_{L^{p'}_t L^{q'}_x}}{(n!)^{\frac{1}{2q'}}} \frac{C^m\|V\|^m_{L^{p'}_t L^{q'}_x}}{(m!)^{\frac{1}{2q'}}}.$$

We see that the terms involving Q_0 can be treated in any dimension, provided that Q_0 is in an adequate Schatten space. This is not the case for the terms involving γ_f, for which we can only deal with the higher orders.

Lemma 3. *Let $d \geqslant 1$, $g : \mathbb{R}^d \to \mathbb{R}$ such that $\check{g} \in L^1(\mathbb{R}^d)$, $1 < q \leqslant 1 + 2/d$ and p such that $2/p + d/q = d$. Let $V \in L_t^{p'} L_x^{q'}(\mathbb{R}_+ \times \mathbb{R}^d)$. Then, for all n, m such that $n + m + 1 \geqslant 2q'$, we have*

$$\|\rho[e^{it\Delta}\mathcal{W}_V^{(n)}(t)\gamma_f \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}]\|_{L_t^p L_x^q} \leqslant C\|\check{g}\|_{L^1} \frac{C^n \|V\|^n_{L_t^{p'} L_x^{q'}}}{(n!)^{\frac{1}{2q'}}} \frac{C^m \|V\|^m_{L_t^{p'} L_x^{q'}}}{(m!)^{\frac{1}{2q'}}},$$

where $\gamma_f = g(-i\nabla)$.

Proof. Again we argue by duality. Let $U \in L_t^{p'} L_x^{q'}$. Without loss of generality, we can assume that $U, V \geqslant 0$. Then, we evaluate

$$\begin{aligned}
&\int_0^\infty \mathrm{Tr}\big[U(t,x) e^{it\Delta}\mathcal{W}_V^{(n)}(t)\gamma_f \mathcal{W}_V^{(m)}(t)^* e^{-it\Delta}\big]\,dt \\
&= (-i)^n i^m \int_0^\infty dt \int_{0 \leqslant s_1 \leqslant \cdots \leqslant s_m \leqslant t} ds_1 \cdots ds_m \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant t} dt_1 \cdots dt_n \\
&\qquad \times \mathrm{Tr}\big[V(s_1, x - 2is_1\nabla) \cdots V(s_m, x - 2is_m\nabla) U(t, x - 2it\nabla) V(t_n, x - 2it_n\nabla) \cdots V(t_1, x - 2it_1\nabla)\gamma_f\big],
\end{aligned}$$

where we used the relation

$$e^{-it\Delta} W(t,x) e^{it\Delta} = W(t, x - 2it\nabla).$$

In the spirit of [Frank et al. 2014], we gather the terms using the cyclicity of the trace as

$$\begin{aligned}
&\mathrm{Tr}\big[V(s_1, x - 2is_1\nabla) \cdots V(s_m, x - 2is_m\nabla) U(t, x - 2it\nabla) \times V(t_n, x - 2it_n\nabla) \cdots V(t_1, x - 2it_1\nabla)\gamma_f\big] \\
&\quad = \mathrm{Tr}\big[V(s_1, x - 2is_1\nabla)^{\frac12} V(s_2, x - 2is_2\nabla)^{\frac12} \cdots V(s_m, x - 2is_m\nabla)^{\frac12} U(t, x - 2it\nabla)^{\frac12} \\
&\qquad \times U(t, x - 2it\nabla)^{\frac12} V(t_n, x - 2it_n\nabla)^{\frac12} \cdots V(t_1, x - 2it_1\nabla)^{\frac12}\gamma_f V(s_1, x - 2is_1\nabla)^{\frac12}\big].
\end{aligned} \tag{37}$$

The first ingredient to estimate this trace is [Frank et al. 2014, Lemma 1], which states that

$$\|\varphi_1(\alpha x - i\beta\nabla)\varphi_2(\gamma x - i\delta\nabla)\|_{\mathfrak{S}^r} \leqslant \frac{\|\varphi_1\|_{L^r(\mathbb{R}^d)}\|\varphi_2\|_{L^r(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{r}} |\alpha\delta - \beta\gamma|^{\frac{d}{r}}} \quad \text{for all } r \geqslant 2. \tag{38}$$

The second ingredient, to treat the term with γ_f, is a generalization of this inequality involving γ_f.

Lemma 4. *There exists a constant $C > 0$ such that for all $t, s \in \mathbb{R}$ we have*

$$\|\varphi_1(x + 2it\nabla) g(-i\nabla)\varphi_2(x + 2is\nabla)\|_{\mathfrak{S}^r} \leqslant \frac{\|\check{g}\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}} \frac{\|\varphi_1\|_{L^r(\mathbb{R}^d)}\|\varphi_2\|_{L^r(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{r}}|t - s|^{\frac{d}{r}}} \tag{39}$$

for all $r \geqslant 2$.

We remark that (39) reduces to (38) when $g = 1$ and $\check{g} = (2\pi)^{\frac{d}{2}}\delta_0$. We postpone the proof of this lemma, and use it to estimate (37) in the following way:

$$\begin{aligned}
&\big|\mathrm{Tr}\big[V(s_1, x - 2is_1\nabla) \cdots V(s_m, x - 2is_m\nabla) U(t, x - 2it\nabla) V(t_n, x - 2it_n\nabla) \cdots V(t_1, x - 2it_1\nabla)\gamma_f\big]\big| \\
&\qquad \leqslant C \frac{\|V(s_1)\|_{L^{q'}} \cdots \|V(s_m)\|_{L^{q'}} \|U(t)\|_{L^{q'}} \|V(t_n)\|_{L^{q'}} \|V(t_1)\|_{L^{q'}}}{|s_1 - t_1|^{\frac{d}{2q'}} \cdots |s_m - t|^{\frac{d}{2q'}} |t - t_n|^{\frac{d}{2q'}} \cdots |t_2 - t_1|^{\frac{d}{2q'}}}.
\end{aligned}$$

Here, we have used the condition $n+m+1 \geqslant 2q'$ to ensure that the operator inside the trace is trace-class by Hölder's inequality. From this point the proof is identical to the proof of [Frank et al. 2014, Theorem 3]. □

Proof of Lemma 4. The inequality is immediate if $r=\infty$. Hence, by complex interpolation, we only have to prove it for $r=2$. We have

$$\begin{aligned}\|\varphi_1(t,x+2it\nabla)g(-i\nabla)\varphi_2(s,x+2is\nabla)\|_{\mathfrak{S}^2}^2 &= \operatorname{Tr}\big[\varphi_1(x)^2 e^{i(t-s)\Delta} g(-i\nabla)\varphi_2(x)^2 e^{i(s-t)\Delta} g(-i\nabla)\big]\\ &= \frac{(2\pi)^{-2d}}{|t-s|^d}\iint \varphi_1(x)^2 \Big|\big(\check{g}*e^{-i\frac{|\cdot|^2}{4(t-s)}}\big)(x-y)\Big|^2 \varphi_2(y)^2\,dx\,dy\\ &\leqslant \frac{(2\pi)^{-2d}}{|t-s|^d}\|\check{g}\|_{L^1}^2\|\varphi_1\|_{L^2}^2\|\varphi_2\|_{L^2}^2. \qquad \square\end{aligned}$$

In dimension d, we want to prove that ρ_Q belongs to $L^{1+2/d}_{t,x}$, hence we consider $q=1+2/d$ and $q'=1+d/2$. The previous result estimates the terms of order $n+m+1\geqslant d+2$, that is, $n+m\geqslant d+1$. The case $n+m=1$ corresponds exactly to the linear response studied in the previous section. In dimension $d=2$, we see that we are still lacking the case $n+m=2$, which is what we call the second order. The next section is devoted to this order. We are not able to treat the terms with $1<n+m\leqslant d$ in other dimensions.

5. Second order in 2D

The study of the linear response is not enough to prove dispersion for the Hartree equation in 2D. We also have to estimate the second-order term, which we first compute explicitly in any dimension, and then study only in dimension 2.

5.1. ***Exact computation in any dimension.*** Define the second-order term in the Duhamel expansion of $Q(t)$,

$$Q_2(t) := (-i)^2\int_0^t ds\int_0^s dt_1\, e^{i(t-s)\Delta}[V(s), e^{i(s-t_1)\Delta}[V(t_1),\gamma_f]e^{i(t_1-s)\Delta}]e^{i(s-t)\Delta},$$

where we again used the notation $V=w*\rho_Q$. We explicitly compute its density. To do so, we let $W\in\mathfrak{D}(\mathbb{R}_+\times\mathbb{R}^d)$ and use the relation

$$\int_0^\infty\int_{\mathbb{R}^d} W(t,x)\rho_{Q_2}(t,x)\,dx\,dt = \int_0^\infty \operatorname{Tr}[W(t)Q_2(t)]\,dt.$$

For any $(p,q)\in\mathbb{R}^d\times\mathbb{R}^d$ we have

$$\begin{aligned}\widehat{Q}_2(t,p,q) = -\frac{1}{(2\pi)^d}\int_0^t ds\int_0^s dt_1\int_{\mathbb{R}^d} dq_1\, e^{i(t-s)(p^2-q^2)}&\\ \times\big[\widehat{V}(s,p-q_1)e^{i(s-t_1)(q_1^2-q^2)}\widehat{V}(t_1,q_1-q)(g(q)-g(q_1))&\\ -\widehat{V}(s,q_1-q)e^{i(s-t_1)(p^2-q_1^2)}\widehat{V}(t_1,p-q_1)(g(q_1)-g(p))\big]&.\end{aligned}$$

Using that

$$\operatorname{Tr}[W(t)Q_2(t)] = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \widehat{w}(t,q-p)\widehat{Q}_2(t,p,q)\,dp\,dq,$$

we arrive at the formula

$$\int_0^\infty\int_{\mathbb{R}^d} W(t,x)\rho_{Q_2}(t,x)\,dx\,dt$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d} dt\,ds\,dt_1 dk\,d\ell K^{(2)}(t-s,s-t_1;k,\ell)\widehat{w}(t,-k)\widehat{\rho}_Q(s,k-\ell)\widehat{\rho}_Q(t_1,\ell),$$

with

$$K^{(2)}(t,s;k,\ell) = \mathbb{1}_{t\geqslant 0}\mathbb{1}_{s\geqslant 0}\frac{4\widehat{w}(\ell)\widehat{w}(k-\ell)}{(2\pi)^{\frac{d}{2}}}\sin(tk\cdot(k-\ell))\sin(\ell\cdot(tk+s\ell))\check{g}(2(tk+s\ell)).$$

5.2. *Estimates in 2D.*

Proposition 4. *Assume that $g\in L^1(\mathbb{R}^2)$ is such that $|x|^a|\check{g}(x)|\in L^\infty(\mathbb{R}^2)$ for some $a>3$. Assume also that w is such that $(1+|k|^{1/2})|\widehat{w}(k)|\in L^\infty(\mathbb{R}^2)$. Then, if $\rho_Q\in L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)$, we have*

$$\|\rho_{Q_2}\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leqslant C\|(1+|\cdot|^2)^{\frac{a}{2}}\check{g}\|_{L^\infty}\|(1+|\cdot|^{\frac{1}{2}})\widehat{w}\|_{L^\infty}\|\rho_Q\|^2_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^2)}, \tag{40}$$

for some constant $C(g,w)$ depending only on g and w.

Proof. First, we have the estimate

$$\left|\int_{\mathbb{R}^3} G(t_1-t_2,t_2-t_3)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_1\,dt_2\,dt_3\right| \leqslant C\|G\|_{L^2L^1}\prod_{i=1}^3\|f_i\|_{L^2}$$

for any G, and hence

$$\left|\int_{\mathbb{R}^3} K^{(2)}(t_1-t_2,t_2-t_3;k,\ell)\widehat{w}(t_1,-k)\widehat{\rho}_Q(t_2,k-\ell)\widehat{\rho}_Q(t_3,\ell)\,dt_1\,dt_2\,dt_3\right|$$
$$\leqslant \left\|K^{(2)}(t,s;k,\ell)\right\|_{L^2_tL^1_s}\|\widehat{w}(\cdot,-k)\|_{L^2}\|\widehat{\rho}_Q(\cdot,k-\ell)\|_{L^2}\|\widehat{\rho}_Q(\cdot,\ell)\|_{L^2}.$$

Let us therefore estimate $\|K^{(2)}(t,s;k,\ell)\|_{L^2_tL^1_s}$. To do so, we use the bounds $|\sin(tk\cdot(k-\ell))|\leqslant 1$ and $|\sin(\ell\cdot(tk+s\ell))|\leqslant|\ell||tk+s\ell|$ to get

$$\|K^{(2)}(t,s;k,\ell)\|^2_{L^2_tL^1_s} \leqslant \frac{16\widehat{w}(\ell)^2\widehat{w}(k-\ell)^2}{(2\pi)^d}\ell^2\int_{\mathbb{R}} dt\left|\int_{\mathbb{R}} ds\,|tk+s\ell||\check{g}(2(tk+s\ell))|\right|^2.$$

We let

$$u = \ell s + t\frac{k\cdot\ell}{\ell} \quad\text{and}\quad v = \sqrt{k^2-\frac{(k\cdot\ell)^2}{\ell^2}}t$$

and notice that

$$|tk+s\ell| = \left(\ell^2\left(s+t\frac{k\cdot\ell}{\ell^2}\right)^2+\left(k^2-\frac{(k\cdot\ell)^2}{\ell^2}\right)t^2\right)^{\frac{1}{2}} = \sqrt{u^2+v^2}.$$

Since $\check{g}$ is a radial function, we find that

$$\ell^2 \int_{\mathbb{R}} dt \left| \int_{\mathbb{R}} ds \, |tk+s\ell| |\check{g}(2(tk+s\ell))| \right|^2 = \frac{|\ell|}{(k^2\ell^2-(k\cdot\ell)^2)^{\frac{1}{2}}} \int_{\mathbb{R}} dv \left| \int_{\mathbb{R}} du \sqrt{u^2+v^2} |\check{g}(2\sqrt{u^2+v^2})| \right|^2.$$

The double integral on the right is finite under some mild decay assumptions on $\check{g}$; for instance, it is finite if $|\check{g}(r)| \leqslant C(1+r^2)^{-a/2}$ for some $a>3$. Noticing that $(k^2\ell^2-(k\cdot\ell)^2)^{1/2} = |\det(k,\ell)|$, we thus have

$$\begin{aligned} &|\langle W, \rho_{Q_2}\rangle| \\ &\leqslant C \|(1+|\cdot|^2)^{\frac{a}{2}} \check{g}\|_{L^\infty} \int_{\mathbb{R}^{2d}} dk \, d\ell \frac{\|\widehat{w}(\cdot,-k)\|_{L^2} |\widehat{w}(k-\ell)| \|\widehat{\rho}_Q(\cdot,k-\ell)\|_{L^2} |\ell|^{\frac{1}{2}} |\widehat{w}(\ell)| \|\widehat{\rho}_Q(\cdot,\ell)\|_{L^2}}{|\det(k,\ell)|^{\frac{1}{2}}}. \end{aligned}$$

We prove the following inequality of Hardy–Littlewood–Sobolev type:

Lemma 5. *For any functions f, g, h we have*

$$\left| \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{f(k)g(k-\ell)h(\ell)}{|\det(k,\ell)|^{\frac{1}{2}}} \, dk \, d\ell \right| \leqslant C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \tag{41}$$

Proof. Since $\det(k,\ell) = k_1\ell_2 - k_2\ell_1$, we first fix $k_1 \neq 0$, $\ell_1 \neq 0$, $k_1 \neq \ell_1$ and estimate

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \frac{f(k_1,k_2)g(k_1-\ell_1,k_2-\ell_2)h(\ell_1,\ell_2)}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \right| \\ &\quad \leqslant \left(\int_{\mathbb{R}^2} \frac{|f(k_1,k_2)|^{\frac{3}{2}} |g(k_1-\ell_1,k_2-\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \right)^{\frac{1}{3}} \times \left(\int_{\mathbb{R}^2} \frac{|f(k_1,k_2)|^{\frac{3}{2}} |h(\ell_1,\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \right)^{\frac{1}{3}} \\ &\qquad \times \left(\int_{\mathbb{R}^2} \frac{|g(k_1-\ell_1,k_2-\ell_2)|^{\frac{3}{2}} |h(\ell_1,\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \right)^{\frac{1}{3}}. \end{aligned}$$

We then have

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{|f(k_1,k_2)|^{\frac{3}{2}} |g(k_1-\ell_1,k_2-\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \\ &\qquad = \int_{\mathbb{R}^2} \frac{|f(k_1,k_2)|^{\frac{3}{2}} |g(k_1-\ell_1,\ell_2)|^{\frac{3}{2}}}{|k_2(k_1-\ell_1)-\ell_2 k_1|^{\frac{1}{2}}} \, dk_2 d\ell_2 \\ &\qquad = \frac{1}{|k_1||k_1-\ell_1|} \int_{\mathbb{R}^2} \frac{|f(k_1,k_2/(k_1-\ell_1))|^{\frac{3}{2}} |g(k_1-\ell_1,\ell_2/k_1)|^{\frac{3}{2}}}{|k_2-\ell_2|^{\frac{1}{2}}} \, dk_2 d\ell_2 \\ &\qquad \leqslant \frac{C}{|k_1||k_1-\ell_1|} \|f(k_1,\cdot/(k_1-\ell_1))\|_{L^2}^{\frac{3}{2}} \|g(k_1-\ell_1,\cdot/k_1)\|_{L^2}^{\frac{3}{2}} \\ &\qquad \leqslant \frac{C}{|k_1|^{\frac{1}{4}}|k_1-\ell_1|^{\frac{1}{4}}} \|f(k_1,\cdot)\|_{L^2}^{\frac{3}{2}} \|g(k_1-\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}, \end{aligned}$$

and, in the same fashion,

$$\int_{\mathbb{R}^2} \frac{|f(k_1,k_2)|^{\frac{3}{2}}|h(\ell_1,\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}}\,dk_2d\ell_2 \leqslant \frac{C}{|k_1|^{\frac{1}{4}}|\ell_1|^{\frac{1}{4}}}\|f(k_1,\cdot)\|_{L^2}^{\frac{3}{2}}\|h(\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}},$$

$$\int_{\mathbb{R}^2} \frac{|g(k_1-\ell_1,k_2-\ell_2)|^{\frac{3}{2}}|h(\ell_1,\ell_2)|^{\frac{3}{2}}}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}}\,dk_2d\ell_2 \leqslant \frac{C}{|\ell_1|^{\frac{1}{4}}|k_1-\ell_1|^{\frac{1}{4}}}\|g(k_1-\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}\|h(\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}.$$

As a consequence, we have

$$\left|\int_{\mathbb{R}^2} \frac{f(k_1,k_2)g(k_1-\ell_1,k_2-\ell_2)h(\ell_1,\ell_2)}{|k_1\ell_2-k_2\ell_1|^{\frac{1}{2}}}\,dk_2d\ell_2\right| \leqslant C\frac{\|f(k_1,\cdot)\|_{L^2}\|g(k_1-\ell_1,\cdot)\|_{L^2}\|h(\ell_1,\cdot)\|_{L^2}}{|k_1|^{\frac{1}{6}}|\ell_1|^{\frac{1}{6}}|k_1-\ell_1|^{\frac{1}{6}}}.$$

We now need a multilinear Hardy–Littlewood–Sobolev-type inequality. Integrating over (k_1,ℓ_1) we find that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{f(k)g(k-\ell)h(\ell)}{|\det(k,\ell)|^{\frac{1}{2}}}\,dk\,d\ell\right| \\
&\leqslant C\int_{\mathbb{R}^2} \frac{\|f(k_1,\cdot)\|_{L^2}\|g(k_1-\ell_1,\cdot)\|_{L^2}\|h(\ell_1,\cdot)\|_{L^2}}{|k_1|^{\frac{1}{6}}|\ell_1|^{\frac{1}{6}}|k_1-\ell_1|^{\frac{1}{6}}}\,dk_1d\ell_1 \\
&\leqslant C\left(\int_{\mathbb{R}^2} \frac{\|g(k_1-\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}\|h(\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}}{|k_1|^{\frac{1}{2}}}\,dk_1d\ell_1\right)^{\frac{1}{3}}\times\left(\int_{\mathbb{R}^2} \frac{\|f(k_1,\cdot)\|_{L^2}^{\frac{3}{2}}\|g(k_1-\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}}{|\ell_1|^{\frac{1}{2}}}\,dk_1d\ell_1\right)^{\frac{1}{3}} \\
&\qquad\qquad\times\left(\int_{\mathbb{R}^2} \frac{\|f(k_1,\cdot)\|_{L^2}^{\frac{3}{2}}\|h(\ell_1,\cdot)\|_{L^2}^{\frac{3}{2}}}{|k_1-\ell_1|^{\frac{1}{2}}}\,dk_1d\ell_1\right)^{\frac{1}{3}} \\
&\leqslant C\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2},
\end{aligned}$$

where in the last line we have used the 2D Hardy–Littlewood–Sobolev inequality. □

From the lemma, we deduce that

$$|\langle W,\rho_{Q_2}\rangle| \leqslant C\left\|(1+|\cdot|^2)^{\frac{a}{2}}\check{g}\right\|_{L^\infty}\|(1+|\cdot|^{\frac{1}{2}})\hat{w}\|_{L^\infty}\|\rho_Q\|_{L^2_{t,x}}^2,$$

which ends the proof of the proposition. □

6. Proof of the main theorem

Proof of Theorem 1. Let $T>0$. Assume also that $\|Q_0\|_{\mathfrak{S}^{4/3}}\leqslant 1$. We solve the equation

$$\begin{aligned}
\rho_Q(t) &= \rho\big[e^{it\Delta}\mathcal{W}_{w*\rho_Q}(t)(\gamma_f+Q_0)\mathcal{W}_{w*\rho_Q}(t)^*e^{-it\Delta}\big]-\rho_{\gamma_f} \\
&= \rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big]-\mathcal{L}(\rho_Q)+\mathcal{R}(\rho_Q)
\end{aligned}$$

by a fixed-point argument. Here $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2$, where $\mathcal{L}_1$ was studied in Section 3 and

$$\mathcal{L}_2(\rho_Q)=-\rho\big[e^{it\Delta}(\mathcal{W}^{(1)}_{w*\rho_Q}(t)Q_0+Q_0\mathcal{W}^{(1)}_{w*\rho_Q}(t)^*)e^{-it\Delta}\big].$$

As explained in Proposition 1 and in Corollary 1, under the assumption (9) (or (10) when f is strictly decreasing), $(1+\mathscr{L}_1)$ is invertible with bounded inverse on $L^2_{t,x}$. The operator $1+\mathscr{L} = 1+\mathscr{L}_1+\mathscr{L}_2$ is invertible with bounded inverse when

$$\|\mathscr{L}_2\| < \frac{1}{\|(1+\mathscr{L}_1)^{-1}\|}.$$

By Lemma 1, we have

$$\|\mathscr{L}_2\| \leqslant C\|w\|_{L^1}\|Q_0\|_{\mathfrak{S}^{4/3}}$$

and therefore the condition can be expressed as

$$\|Q_0\|_{\mathfrak{S}^{4/3}} < \frac{1}{C\|w\|_{L^1}\|(1+\mathscr{L}_1)^{-1}\|}.$$

Then, we can write

$$\rho_Q(t) = (1+\mathscr{L})^{-1}\big(\rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big]+\mathscr{R}(\rho_Q)\big).$$

For any $\varphi \in L^2_{t,x}([0,T]\times\mathbb{R}^2)$, define

$$F(\varphi)(t) = \rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big]+\mathscr{R}(\varphi).$$

We apply the Banach fixed-point theorem to the map $(1+\mathscr{L})^{-1}F$. To do so, we expand F as

$$\begin{aligned}F(\varphi)(t) = \rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big] &+ \sum_{n+m\geqslant 2}\rho\big[e^{it\Delta}\mathscr{W}_{w*\varphi}(t)Q_0\mathscr{W}_{w*\varphi}(t)^*e^{-it\Delta}\big]\\ &+\sum_{n+m=2}\rho\big[e^{it\Delta}\mathscr{W}^{(n)}_{w*\varphi}(t)\gamma_f\mathscr{W}^{(m)}_{w*\varphi}(t)^*e^{-it\Delta}\big]+\sum_{n+m\geqslant 3}\rho\big[e^{it\Delta}\mathscr{W}^{(n)}_{w*\varphi}(t)\gamma_f\mathscr{W}^{(m)}_{w*\varphi}(t)^*e^{-it\Delta}\big].\end{aligned}$$

By the Strichartz estimate (19), we have

$$\big\|\rho\big[e^{it\Delta}Q_0e^{-it\Delta}\big]\big\|_{L^2_{t,x}} \leqslant C\|Q_0\|_{\mathfrak{S}^{4/3}}.$$

By Lemma 1, we have

$$\Big\|\sum_{n+m\geqslant 2}\rho\big[e^{it\Delta}\mathscr{W}_{w*\varphi}(t)Q_0\mathscr{W}_{w*\varphi}(t)^*e^{-it\Delta}\big]\Big\|_{L^2_{t,x}} \leqslant C\|Q_0\|_{\mathfrak{S}^{4/3}}\sum_{n+m\geqslant 2}\frac{C^{n+m}\|w*\varphi\|^{n+m}_{L^2_{t,x}}}{(n!)^{\frac14}(m!)^{\frac14}}.$$

By Proposition 4, we have

$$\Big\|\sum_{n+m=2}\rho\big[e^{it\Delta}\mathscr{W}^{(n)}_{w*\varphi}(t)\gamma_f\mathscr{W}^{(m)}_{w*\varphi}(t)^*e^{-it\Delta}\big]\Big\|_{L^2_{t,x}} \leqslant C\|(1+|\cdot|^2)^{\frac{a}{2}}\check{g}\|_{L^\infty}\|(1+|\cdot|^{\frac12})\hat{w}\|_{L^\infty}\|\varphi\|^2_{L^2_{t,x}}.$$

Finally, by Lemma 3 we have

$$\Big\|\sum_{n+m\geqslant 3}\rho\big[e^{it\Delta}\mathscr{W}^{(n)}_{w*\varphi}(t)\gamma_f\mathscr{W}^{(m)}_{w*\varphi}(t)^*e^{-it\Delta}\big]\Big\|_{L^2_{t,x}} \leqslant C\|\check{g}\|_{L^1}\sum_{n+m\geqslant 3}\frac{C^{n+m}\|w*\varphi\|^{n+m}_{L^2_{t,x}}}{(n!)^{\frac14}(m!)^{\frac14}}.$$

We deduce that, for all $\varphi \in L^2_{t,x}([0,T]\times\mathbb{R}^2)$, we have the estimate

$$\|(1+\mathscr{L})^{-1}F(\varphi)\|_{L^2_{t,x}} \leqslant C\|(1+\mathscr{L})^{-1}\|\big(\|Q_0\|_{\mathfrak{S}^{4/3}}+A\big(\|\varphi\|_{L^2_{t,x}}\big)\big),$$

where we used the notation

$$A(z) = C \sum_{n+m\geqslant 2} \frac{C^{n+m}(\|w\|_{L^1} z)^{n+m}}{(n!)^{\frac14}(m!)^{\frac14}} + C\big\|(1+|\cdot|^2)^{\frac{a}{2}}\check{g}\big\|_{L^\infty}\big\|(1+|\cdot|^{\frac12})\widehat{w}\big\|_{L^\infty} z^2 + C\|\check{g}\|_{L^1} \sum_{n+m\geqslant 3} \frac{C^{n+m}(\|w\|_{L^1} z)^{n+m}}{(n!)^{\frac14}(m!)^{\frac14}}.$$

We have $A(z) = O(z^2)$ as $z \to 0$. As a consequence, there exist $C_0, z_0 > 0$, depending only on $\|w\|_{L^1}$, $\|(1+|\cdot|^2)^{a/2}\check{g}\|_{L^\infty}\|(1+|\cdot|^{1/2})\widehat{w}\|_{L^\infty}$ and $\|\check{g}\|_{L^1}$, such that

$$|A(z)| \leqslant C_0 z^2$$

for all $|z| \leqslant z_0$. Choosing

$$R = \min\left(z_0, \frac{1}{2C_0\|(1+\mathscr{L})^{-1}\|}\right)$$

and

$$\|Q_0\|_{\mathfrak{S}^{4,3}} \leqslant \min\left(1, \frac{R}{2C\|(1+\mathscr{L})^{-1}\|}\right)$$

leads to the estimate

$$\|(1+\mathscr{L})^{-1}F(\varphi)\|_{L^2_{t,x}} \leqslant R$$

for all $\|\varphi\|_{L^2_{t,x}} \leqslant R$, independently of the maximal time $T > 0$. Similar estimates show that F is also a contraction on this ball, up to diminishing R if necessary. The Banach fixed-point theorem shows that there exists a solution for any time $T > 0$, with a uniform estimate with respect to T. Having built this solution $\varphi_0 \in L^2_{t,x}(\mathbb{R}_+ \times \mathbb{R}^2)$, we define the operator γ as

$$\gamma(t) = e^{it\Delta}\mathscr{W}_{w*\varphi_0}(t)(\gamma_f + Q_0)\mathscr{W}_{w*\varphi_0}(t)^* e^{-it\Delta}.$$

We have $\varphi_0 = \rho_\gamma - \rho_{\gamma_f}$ by definition.

From [Frank et al. 2014, Theorem 3], we know that $\mathscr{W}_{w*\varphi_0} - 1 \in C^0_t(\mathbb{R}_+, \mathfrak{S}^4)$ and that $\mathscr{W}_{w*\varphi_0} - 1$ admits a strong limit in $\mathfrak{S}^4$ when $t \to \infty$, which gives that $\gamma - \gamma_f \in C^0(\mathbb{R}_+, \mathfrak{S}^4)$ and our scattering result (11). Next, we remark that since $w \in W^{1,1}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$, we have $w * \varphi_0 \in L^2_t(L^\infty_x \cap L^2_x)$. From [Lewin and Sabin 2014, Lemma 7] and the fact that $g \in L^2(\mathbb{R}^2)$ (due to (8)), we deduce that $(\mathscr{W}_{w*\varphi_0}(t) - 1)\gamma_f \in C^0(\mathbb{R}_+, \mathfrak{S}^2)$. This now shows that $\gamma - \gamma_f \in C^0(\mathbb{R}_+, \mathfrak{S}^2)$. Of course, we can perform the same procedure for negative times and this finishes the proof of Theorem 1. □

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ON THE EIGENVALUES OF AHARONOV–BOHM OPERATORS WITH VARYING POLES

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We consider a magnetic operator of Aharonov–Bohm type with Dirichlet boundary conditions in a planar domain. We analyze the behavior of its eigenvalues as the singular pole moves in the domain. For any value of the circulation we prove that the k-th magnetic eigenvalue converges to the k-th eigenvalue of the Laplacian as the pole approaches the boundary. We show that the magnetic eigenvalues depend in a smooth way on the position of the pole, as long as they remain simple. In case of half-integer circulation, we show that the rate of convergence depends on the number of nodal lines of the corresponding magnetic eigenfunction. In addition, we provide several numerical simulations both on the circular sector and on the square, which find a perfect theoretical justification within our main results, together with the ones by the first author and Helffer in *Exp. Math.* **20**:3 (2011), 304–322.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open, simply connected, bounded set. For $a = (a_1, a_2)$ varying in Ω, we consider the magnetic Schrödinger operator

$$(i\nabla + A_a)^2 = -\Delta + i\nabla \cdot A_a + 2iA_a \cdot \nabla + |A_a|^2$$

acting on functions with zero boundary conditions on $\partial\Omega$, where A_a is a magnetic potential of Aharonov–Bohm type, singular at the point a. More specifically, the magnetic potential has the form

$$A_a(x) = \alpha\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right) + \nabla\chi \tag{1-1}$$

where $x = (x_1, x_2) \in \Omega \setminus \{a\}$, $\alpha \in (0, 1)$ is a fixed constant and $\chi \in C^\infty(\bar{\Omega})$. Since the regular part χ does not play a significant role, throughout the paper we will suppose without loss of generality that $\chi \equiv 0$.

The magnetic field associated to this potential is a $2\pi\alpha$-multiple of the Dirac delta at a, orthogonal to the plane. A quantum particle moving in $\Omega \setminus \{a\}$ will be affected by the magnetic potential, although it remains in a region where the magnetic field is zero (Aharonov–Bohm effect [1959]). We can think of the particle as being affected by the nontrivial topology of the set $\Omega \setminus \{a\}$.

We are interested in studying the behavior of the spectrum of the operator $(i\nabla + A_a)^2$ as a moves in the domain and when it approaches its boundary. By standard spectral theory, the spectrum of such an

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Keywords: magnetic Schrödinger operators, eigenvalues, nodal domains.

operator consists of a diverging sequence of real positive eigenvalues (see Section 2). We will denote by λ_j^a, $j \in \mathbb{N} = \{1, 2, \dots\}$, the eigenvalues counted with their multiplicity (see (2-3)) and by φ_j^a the corresponding eigenfunctions, normalized in the $L^2(\Omega)$-norm. We shall focus our attention on the extremal and critical points of the maps $a \mapsto \lambda_j^a$.

One motivation for our study is that, in the case of half-integer circulation, critical positions of the moving pole can be related to optimal partition problems. The link between spectral minimal partitions and nodal domains of eigenfunctions has been investigated in full detail in [Helffer 2010; Helffer and Hoffmann-Ostenhof 2010; 2013; Helffer et al. 2009; 2010a; 2010b]. By the results in [Helffer et al. 2009] in two dimensions, the boundary of a minimal partition is the union of finitely many regular arcs, meeting at some multiple intersection points dividing the angle in an equal fashion. If the multiplicity of the clustering domains is even, then the partition is nodal, that is to say it is the nodal set of an eigenfunction. On the other hand, the results in [Bonnaillie-Noël and Helffer 2011; Bonnaillie-Noël et al. 2009; 2010; Helffer and Hoffmann-Ostenhof 2013; Noris and Terracini 2010] suggest that the minimal partitions featuring a clustering point of odd multiplicity should be related to the nodal domains of eigenfunctions of Aharonov–Bohm Hamiltonians which corresponds to a critical value of the eigenfunction with respect to the moving pole.

Our first result states the continuity of the magnetic eigenvalues with respect to the position of the singularity, up to the boundary.

Theorem 1.1. *For every $j \in \mathbb{N}$, the function $a \in \Omega \mapsto \lambda_j^a \in \mathbb{R}$ admits a continuous extension on $\overline{\Omega}$. More precisely, as $a \to \partial\Omega$, we have that λ_j^a converges to λ_j, the j-th eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.*

We remark that this holds for every $\alpha \in (0, 1)$, α being the circulation of the magnetic potential introduced in (1-1). As an immediate consequence of this result, we have that this map, being constant on $\partial\Omega$, always admits an interior extremal point.

Corollary 1.2. *For every $j \in \mathbb{N}$, the function $a \in \Omega \mapsto \lambda_j^a \in \mathbb{R}$ has an extremal point in Ω.*

Heuristically, we can interpret the previous theorem thinking at a magnetic potential A_b, singular at $b \in \partial\Omega$. The domain $\Omega \setminus \{b\}$ coincides with Ω, so that it has a trivial topology. For this reason, the magnetic potential is not experienced by a particle moving in Ω and the operator acting on the particle is simply the Laplacian.

This result was first conjectured in the case $j = 1$ in [Noris and Terracini 2010], where it was applied to show that the function $a \mapsto \lambda_1^a$ has a global interior maximum, where it is not differentiable, corresponding to an eigenfunction of multiplicity exactly two. Numerical simulations in [Bonnaillie-Noël and Helffer 2011] supported the conjecture for every j. During the completion of this work, we became aware that the continuity of the eigenvalues with respect to multiple moving poles has been obtained independently in [Léna 2014].

We remark that the continuous extension up to the boundary is a nontrivial issue because the nature of the operator changes as a approaches $\partial\Omega$. This fact can be seen in the more specific case when $\alpha = \frac{1}{2}$, which is equivalent to the standard Laplacian on the double covering (see [Helffer et al. 1999; 2000; Noris and Terracini 2010]). We go then from a problem on a fixed domain with a varying operator (which depends on the singularity a) to a problem with a fixed operator (the Laplacian) and a varying domain (for the convergence of the eigenvalues of elliptic operators on varying domains, we refer to [Arendt and Daners 2007; Daners 2003]). In this second case, the singularity is transferred from the operator into the domain. Indeed, when a approaches the boundary, the double covering develops a corner at the origin. In particular, Theorem 7.1 in [Helffer et al. 2010a] cannot be applied in our case since there is no convergence in capacity of the domains.

In the light of the previous corollary it is natural to study additional properties of the extremal points. Our aim is to establish a relation between the nodal properties of φ_j^b and the vanishing order of $|\lambda_j^a - \lambda_j^b|$ as $a \to b$. First of all we will need some additional regularity, which is guaranteed by the following theorem in the case of simple eigenvalues and regular domain.

Theorem 1.3. *Let $b \in \Omega$. If λ_j^b is simple then, for every $j \in \mathbb{N}$, the map $a \in \Omega \mapsto \lambda_j^a$ is locally of class C^∞ in a neighborhood of b.*

In order to examine the link with the nodal set of eigenfunctions, we shall focus on the case $\alpha = \frac{1}{2}$. In this case, it was proved in [Helffer et al. 1999; 2000; Noris and Terracini 2010] (see also Proposition 2.4 below) that the eigenfunctions have an odd number of nodal lines ending at the pole a and an even number of nodal lines meeting at zeros different from a. We say that an eigenfunction has a zero of order $k/2$ at a point if it has k nodal lines meeting at such point. More precisely, we give the following definition.

Definition 1.4 (zero of order $k/2$). Let $f : \Omega \to \mathbb{C}$, $b \in \Omega$ and $k \in \mathbb{N}$.

(i) If k is even, we say that f has a zero of order $k/2$ at b if it is of class at least $C^{k/2}$ in a neighborhood of b and $f(b) = \cdots = D^{k/2-1} f(b) = 0$, while $D^{k/2} f(b) \neq 0$.

(ii) If k is odd, we say that f has a zero of order $k/2$ at b if $f(x^2)$ has a zero of order k at b (here x^2 is the complex square).

Theorem 1.5 [Noris and Terracini 2010, Theorem 1.1]. *Suppose that $\alpha = \frac{1}{2}$. Fix any $j \in \mathbb{N}$. If φ_j^b has a zero of order $\frac{1}{2}$ at $b \in \Omega$ then either λ_j^b is not simple, or b is not an extremal point of the map $a \mapsto \lambda_j^a$.*

Remark 1.6. By joining this result with Corollary 1.2, we find that there is at least one extremal interior point (for the j-th eigenvalue) enjoying an alternative between degeneracy of the corresponding eigenvalue and the presence of a triple (or multiple) point nodal configuration for the corresponding eigenfunction.

Under the assumption that λ_j^b is simple, we prove here that the converse of Theorem 1.5 also holds. In addition, we show that the number of nodal lines of φ_j^b at b determines the order of vanishing of $|\lambda_j^b - \lambda_j^a|$ as $a \to b$.

Theorem 1.7. *Suppose that $\alpha = \frac{1}{2}$. Fix any $j \in \mathbb{N}$. If λ_j^b is simple and φ_j^b has a zero of order $k/2$ at $b \in \Omega$, with $k \geq 3$ odd, then*

$$|\lambda_j^a - \lambda_j^b| \leq C|a-b|^{(k+1)/2} \quad \text{as } a \to b, \tag{1-2}$$

for a constant $C > 0$ independent of a.

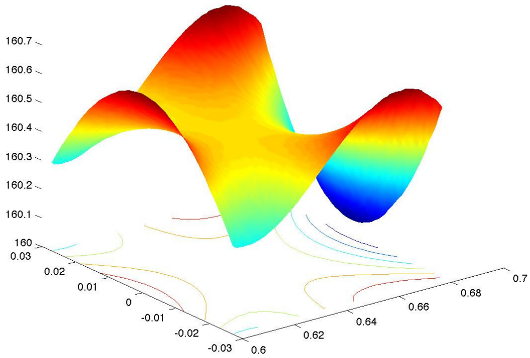


Figure 1. $a \mapsto \lambda_3^a$, $a \in \left\{\left(\frac{m}{1000}, \frac{n}{1000}\right), 600 \le m \le 680, 0 \le n \le 30\right\}$.

In conclusion, in the case of half-integer circulation we have Figure 1, which completes Corollary 1.2.

Corollary 1.8. *Suppose that* $\alpha = \frac{1}{2}$. *Fix any* $j \in \mathbb{N}$. *If* $b \in \Omega$ *is an extremal point of* $a \mapsto \lambda_j^a$ *then either* λ_j^b *is not simple*, *or* φ_j^b *has a zero of order* $k/2$ *at* b, $k \ge 3$ *odd. In this second case*, *the first* $(k-1)/2$ *terms of the Taylor expansion of* λ_j^a *at* b *cancel.*

Remark 1.9. When the order of the zero of the eigenfunction is at least $\frac{3}{2}$, the corresponding nodal set determines a regular partition of the domain, in the sense of [Helffer et al. 2009], where such a notion has been introduced and linked with the properties of boundaries of spectral minimal partitions. It is interesting to connect the variational properties of the partition with the characterization of the pole a as a critical point of the map $a \mapsto \lambda_j^a$. To this aim we performed a number of numerical computations. Rather surprisingly, the configurations of the triple (or multiple) point almost never appear at the maximum or minimum values of the eigenvalues, which are almost always nondifferentiability points, thus corresponding to degenerate eigenvalues. In the case of the angular sector, we observe in particular that any triple point configuration corresponds to a degenerate saddle point as illustrated in Figure 1 (see also Figures 7, top, and 4).

In [Noris et al. $\ge$ 2014] we intend to extend Theorem 1.7 to the case $b \in \partial\Omega$. In this case we know from Theorem 1.1 that λ_j^a converges to λ_j as $a \to b \in \partial\Omega$ and we aim to estimate the rate of convergence depending on the number of nodal lines of φ_j at b, motivated by the numerical simulations in [Bonnaillie-Noël and Helffer 2011].

We would like to mention that the relation between the presence of a magnetic field and the number of nodal lines of the eigenfunctions, as well as the consequences on the behavior of the eigenvalues, have been recently studied in different contexts, giving rise to surprising conclusions. In [Berkolaiko 2013; Colin de Verdière 2013] the authors consider a magnetic Schrödinger operator on graphs and study the behavior of its eigenvalues as the circulation of the magnetic field varies. In particular, they consider an arbitrary number of singular poles, having circulation close to 0. They prove that the simple eigenvalues of the Laplacian (zero circulation) are critical values of the function $\alpha \mapsto \lambda_j(\alpha)$, which associates to the circulation α the corresponding eigenvalue. In addition, they show that the number of nodal lines of the Laplacian eigenfunctions depends on the Morse index of $\lambda_j(0)$.

The paper is organized as follows. In Section 2, we define the functional space $\mathcal{D}^{1,2}_{A_a}(\Omega)$, which is the more suitable space to consider our problem. We also recall a Hardy-type inequality and a theorem about the regularity of the eigenfunctions φ^a_j. Finally, in the case of a half-integer circulation, we recall the equivalence between the problem we consider and the standard Laplacian equation on the double covering. The first part of Theorem 1.1, concerning the interior continuity of the eigenvalues λ^a_j is proved in Section 3 and the second part concerning the extension to the boundary is studied in Section 4. In Section 5, we prove Theorem 1.3. Section 6 contains the proof of Theorem 1.7. Finally, Section 7 illustrates these results in the case of the angular sector of aperture $\pi/4$ and the square.

2. Preliminaries

We will work in the functional space $\mathcal{D}^{1,2}_{A_a}(\Omega)$, which is defined as the completion of $C^\infty_0(\Omega\backslash\{a\})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} := \|(i\nabla + A_a)u\|_{L^2(\Omega)}.$$

As proved in [Noris and Terracini 2010, Lemma 2.1], for example, we have an equivalent characterization

$$\mathcal{D}^{1,2}_{A_a}(\Omega) = \left\{u \in H^1_0(\Omega),\ \frac{u}{|x-a|} \in L^2(\Omega)\right\},$$

and moreover we have that $\mathcal{D}^{1,2}_{A_a}(\Omega)$ is continuously embedded in $H^1_0(\Omega)$: there exists a constant $C > 0$ such that for every $u \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ we have

$$\|u\|_{H^1_0(\Omega)} \leq C\|u\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)}. \tag{2-1}$$

This is proved by making use of a Hardy-type inequality by Laptev and Weidl [1999]. Such an inequality also holds for functions with nonzero boundary trace, as shown in [Melgaard et al. 2004, Lemma 7.4] (see also [Melgaard et al. 2005]). More precisely, given $D \subset \Omega$ simply connected and with smooth boundary, there exists a constant $C > 0$ such that for every $u \in \mathcal{D}^{1,2}_{A_a}(\Omega)$

$$\left\|\frac{u}{|x-a|}\right\|_{L^2(D)} \leq C\|(i\nabla + A_a)u\|_{L^2(D)}. \tag{2-2}$$

As a reference on Aharonov–Bohm operators we cite [Rozenblum and Melgaard 2005]. As a consequence of the continuous embedding, we have the following.

Lemma 2.1. *Let* Im *be the compact immersion of* $\mathcal{D}^{1,2}_{A_a}(\Omega)$ *into* $(\mathcal{D}^{1,2}_{A_a}(\Omega))'$. *Then, the operator*

$$((i\nabla + A_a)^2)^{-1} \circ \mathrm{Im} : \mathcal{D}^{1,2}_{A_a}(\Omega) \to \mathcal{D}^{1,2}_{A_a}(\Omega)$$

is compact.

As $((i\nabla+A_a)^2)^{-1}$ is also self-adjoint and positive, we deduce that the spectrum of $(i\nabla+A_a)^2$ consists of a diverging sequence of real positive eigenvalues, having finite multiplicity. They also admit the variational characterization

$$\lambda_j^a = \inf_{\substack{W_j \subset \mathcal{D}^{1,2}_{A_a}(\Omega) \\ \dim W_j = j}} \sup_{\Phi \in W_j} \frac{\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)}}{\|\Phi\|^2_{L^2(\Omega)}}. \tag{2-3}$$

Recall that A_a has the form (1-1) if and only if it satisfies

$$\nabla \times A_a = 0 \quad \text{in } \Omega \setminus \{a\} \quad \text{and} \quad \frac{1}{2\pi} \oint_\sigma A_a \cdot dx = \alpha \tag{2-4}$$

for every closed path σ which winds once around a. The value of the circulation strongly affects the behavior of the eigenfunctions, starting from their regularity, as the following lemma shows.

Lemma 2.2 [Felli et al. 2011, Section 7]. *If A_a has the form (1-1) then $\varphi_j^a \in C^{0,\alpha}(\Omega)$, where α is precisely the circulation of A_a.*

If the circulations of two magnetic potentials differ by an integer, the corresponding operators are equivalent under a gauge transformation, so that they have the same spectrum (see [Helffer et al. 1999, Theorem 1.1] and [Noris and Terracini 2010, Lemma 3.2]). For this reason, we can set $\chi = 0$ in (2-4) and we can consider α in the interval $(0, 1)$ without losing generality. In the same papers it is shown that, when the circulations differ by a value $\frac{1}{2}$, one operator is equivalent to the other one composed with the complex square root. In particular, in case of half-integer circulation the operator is equivalent to the standard Laplacian in the double covering.

Lemma 2.3 [Helffer et al. 1999, Lemma 3.3]. *Suppose that A_a has the form (2-4) with $\alpha = \frac{1}{2}$ (and $\chi = 0$). Then, with θ being the angle of the polar coordinates, the function*

$$e^{-i\theta(y)} \varphi_j^a(y^2 + a) \quad \textit{defined in } \{y \in \mathbb{C} : y^2 + a \in \Omega\}$$

is real-valued and solves the following equation on its domain:

$$-\Delta(e^{-i\theta(y)} \varphi_j^a(y^2 + a)) = 4\lambda_j^a |y|^2 e^{-i\theta(y)} \varphi_j^a(y^2 + a).$$

As a consequence, we have that, in the case of half-integer circulation, φ_j^a behaves, up to a complex phase, as an elliptic eigenfunction far from the singular point a. The behavior near a is, up to a complex phase, that of the square root of an elliptic eigenfunction. We summarize the known properties that we will need in the following proposition. The proofs can be found in [Felli et al. 2011, Theorem 1.3], [Helffer et al. 1999, Theorem 2.1] and [Noris and Terracini 2010, Theorem 1.5] (see also [Hartman and Wintner 1953]).

Proposition 2.4. *Let $\alpha = \frac{1}{2}$. There exists an odd integer $k \geq 1$ such that φ_j^a has a zero of order $k/2$ at a. Moreover, we have near a the asymptotic expansion*

$$\varphi_j^a(|x-a|, \theta_a) = e^{i\alpha\theta_a} \frac{|x-a|^{k/2}}{k} \big(c_k \cos(k\alpha\theta_a) + d_k \sin(k\alpha\theta_a)\big) + g(|x-a|, \theta_a),$$

where $x-a=|x-a|e^{i\theta_a}$, $c_k^2+d_k^2\neq 0$ *and the remainder* g *satisfies*

$$\lim_{r\to 0}\frac{\|g(r,\cdot)\|_{C^1(\partial D_r(a))}}{r^{k/2}}=0,$$

where $D_r(a)$ *is the disk centered at* a *of radius* r. *In addition, there is a positive radius* R *such that* $(\varphi_j^a)^{-1}(\{0\})\cap D_R(a)$ *consists of* k *arcs of class* C^∞. *If* $k\geq 3$ *then the tangent lines to the arcs at the point* a *divide the disk into* k *equal sectors.*

3. Continuity of the eigenvalues with respect to the pole in the interior of the domain

In this section we prove the first part of Theorem 1.1, that is the continuity of the function $a\mapsto\lambda_j^a$ when the pole a belongs to the interior of the domain.

Lemma 3.1. *Given* $a,b\in\Omega$ *there exists a radial cut-off function* $\eta_a:\mathbb{R}^2\to\mathbb{R}$ *such that* $\eta_a(x)=0$ *for* $|x-a|<2|b-a|$, $\eta_a(x)=1$ *for* $|x-a|\geq\sqrt{2|b-a|}$, *and moreover*

$$\int_{\mathbb{R}^2}\big(|\nabla\eta_a|^2+(1-\eta_a^2)\big)\,dx\to 0\quad\text{ as } a\to b.$$

Proof. Given any $0<\varepsilon<1$ we set

$$\eta(x)=\begin{cases}0, & 0\leq|x|\leq\varepsilon,\\ \dfrac{\log\varepsilon-\log|x|}{\log\varepsilon-\log\sqrt{\varepsilon}}, & \varepsilon\leq|x|\leq\sqrt{\varepsilon},\\ 1, & x\geq\sqrt{\varepsilon}.\end{cases}\tag{3-1}$$

With $\varepsilon=2|b-a|$ and $\eta_a(x)=\eta(x-a)$, an explicit calculation shows that the properties are satisfied. □

Lemma 3.2. *Given* $a,b\in\Omega$ *there exist* θ_a *and* θ_b *such that* $\theta_a-\theta_b\in C^\infty(\Omega\setminus\{ta+(1-t)b,\ t\in[0,1]\})$ *and moreover in this set we have*

$$\alpha\nabla(\theta_a-\theta_b)=A_a-A_b.$$

Proof. Let $a=(a_1,a_2)$ and $b=(b_1,b_2)$. Suppose that $a_1<b_1$; the other cases can be treated in a similar way. We shall provide a suitable branch of the polar angle centered at a, which is discontinuous on the half-line starting at a and passing through b. To this aim we consider the branch of the arctangent given by

$$\arctan:\mathbb{R}\to\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right).$$

We set

$$\theta_a=\begin{cases}\arctan\dfrac{x_2-a_2}{x_1-a_1}, & x_1>a_1,\ x_2\geq\dfrac{b_2-a_2}{b_1-a_1}x_1+\dfrac{a_2b_1-b_2a_1}{b_1-a_1},\\ \pi/2, & x_1=a_1,\ x_2>a_2,\\ \pi+\arctan\dfrac{x_2-a_2}{x_1-a_1}, & x_1<a_1,\\ 3\pi/2, & x_1=a_1,\ x_2<a_2,\\ 2\pi+\arctan\dfrac{x_2-a_2}{x_1-a_1}, & x_1>a_1,\ x_2<\dfrac{b_2-a_2}{b_1-a_1}x_1+\dfrac{a_2b_1-b_2a_1}{b_1-a_1}.\end{cases}$$

With this definition θ_a is regular except on the half-line

$$x_2 = \frac{b_2 - a_2}{b_1 - a_1} x_1 + \frac{a_2 b_1 - b_2 a_1}{b_1 - a_1}, \qquad x_1 > a_1,$$

and an explicit calculation shows that $\alpha \nabla \theta_a = A_a$ in the set where it is regular. The definition of θ_b is analogous: we keep the same half-line, whereas we replace (a_1, a_2) with (b_1, b_2) in the definition of the function. One can verify that $\theta_a - \theta_b$ is regular except for the segment from a to b. □

Recall that in the following φ_j^a is an eigenfunction associated to λ_j^a, normalized in the L^2-norm. Moreover, we can assume that the eigenfunctions are orthogonal.

Lemma 3.3. *Given $a, b \in \Omega$, let η_a be defined as in Lemma 3.1 and let θ_a, θ_b be defined as in Lemma 3.2. Fix an integer $k \ge 1$ and set, for $j = 1, \dots, k$,*

$$\tilde{\varphi}_j = e^{i\alpha(\theta_a - \theta_b)} \eta_a \varphi_j^b.$$

Then $\tilde{\varphi}_j \in \mathfrak{D}^{1,2}_{A_a}(\Omega)$ and moreover, for every $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$,

$$(1 - \varepsilon_a) \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|^2_{L^2(\Omega)} \le \left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|^2_{L^2(\Omega)} \le k \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|^2_{L^2(\Omega)},$$

where $\varepsilon_a \to 0$ as $a \to b$.

Proof. Let us prove first that $\tilde{\varphi}_j \in \mathfrak{D}^{1,2}_{A_a}(\Omega)$. By Lemmas 3.1 and 3.2 we have that $\theta_a - \theta_b \in C^\infty(\operatorname{supp}\{\eta_a\})$, so that $\tilde{\varphi}_j \in H^1_0(\Omega)$. Moreover $\tilde{\varphi}_j(x) = 0$ if $|x - a| < 2|b - a|$, hence $\tilde{\varphi}_j / |x - a| \in L^2(\Omega)$. Concerning the inequalities, we compute on one hand

$$\left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|^2_{L^2(\Omega)} \le k \sum_{j=1}^k \alpha_j^2 \| \eta_a \varphi_j^b \|^2_{L^2(\Omega)} \le k \sum_{j=1}^k \alpha_j^2 = k \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|^2_{L^2(\Omega)},$$

where we used the inequality $\sum_{i,j=1}^k \alpha_i \alpha_j \le k \sum_{j=1}^k \alpha_j^2$ and the fact that the eigenfunctions are orthogonal and normalized in the $L^2(\Omega)$-norm. On the other hand we compute

$$\left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|^2_{L^2(\Omega)} - \left\| \sum_{j=1}^k \alpha_j \tilde{\varphi}_j \right\|^2_{L^2(\Omega)} = \sum_{i,j=1}^k \alpha_i \alpha_j \int_\Omega (1 - \eta_a^2) \varphi_i^b \bar{\varphi}_j^b \, dx.$$

Thanks to the regularity result proved by Felli, Ferrero and Terracini (see Lemma 2.2), we have that φ_i^b are bounded in $L^\infty(\Omega)$. Therefore the last quantity is bounded by

$$Ck \sum_{j=1}^k \alpha_j^2 \int_\Omega (1 - \eta_a^2) \, dx = Ck \left\| \sum_{j=1}^k \alpha_j \varphi_j^b \right\|^2_{L^2(\Omega)} \int_\Omega (1 - \eta_a^2) \, dx$$

and the conclusion follows from Lemma 3.1. □

We have all the tools to prove the first part of Theorem 1.1. We will use some ideas from [Helffer et al. 2010a, Theorem 7.1].

Theorem 3.4. *For every $k \in \mathbb{N}$ the function $a \in \Omega \mapsto \lambda_k^a \in \mathbb{R}$ is continuous.*

Proof. Step 1: First we prove that

$$\limsup_{a\to b} \lambda_k^a \le \lambda_k^b.$$

To this aim it will be sufficient to exhibit a k-dimensional space $E_k \subset \mathcal{D}^{1,2}_{A_a}(\Omega)$ with the property that

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \le (\lambda_k^b + \varepsilon'_a)\|\Phi\|^2_{L^2(\Omega)} \qquad \text{for every } \Phi \in E_k, \tag{3-2}$$

with $\varepsilon'_a \to 0$ as $a \to b$. Let $\operatorname{span}\{\varphi_1^b, \dots, \varphi_k^b\}$ be any spectral space attached to $\lambda_1^b, \dots, \lambda_k^b$. Then we define

$$E_k := \operatorname{span}\{\tilde\varphi_1, \dots, \tilde\varphi_k\} \qquad \text{with } \tilde\varphi_j = e^{i\alpha(\theta_a-\theta_b)}\eta_a\varphi_j^b.$$

We know from Lemma 3.3 that $E_k \subset \mathcal{D}^{1,2}_{A_a}(\Omega)$. Moreover, it is immediate to see that $\dim E_k = k$. Let us now verify (3-2) with $\Phi = \sum_{j=1}^k \alpha_j\tilde\varphi_j$, $\alpha_j \in \mathbb{R}$. We compute

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \int_\Omega \left|\sum_{j=1}^k \alpha_j(i\nabla + A_b)(\eta_a\varphi_j^b)\right|^2 dx = \int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j(i\nabla + A_b)^2(\eta_a\varphi_i^b)(\eta_a\bar\varphi_j^b)\,dx, \tag{3-3}$$

where we have used the equality

$$(i\nabla + A_a)\tilde\varphi_j = e^{i\alpha(\theta_a-\theta_b)}(i\nabla + A_b)(\eta_a\varphi_j^b)$$

and integration by parts. Next notice that

$$(i\nabla + A_b)(\eta_a\varphi_i^b) = \eta_a(i\nabla + A_b)\varphi_i^b + i\varphi_i^b\nabla\eta_a,$$

so that

$$(i\nabla + A_b)^2(\eta_a\varphi_i^b) = \eta_a(i\nabla + A_b)^2\varphi_i^b + 2i(i\nabla + A_b)\varphi_i^b\cdot\nabla\eta_a - \varphi_i^b\Delta\eta_a.$$

By replacing in (3-3), we obtain

$$\begin{aligned}\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} &= \int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j\big(\lambda_i^b\varphi_i^b\eta_a + 2i(i\nabla + A_b)\varphi_i^b\cdot\nabla\eta_a - \varphi_i^b\Delta\eta_a\big)\bar\varphi_j^b\eta_a\,dx \\ &\le \lambda_k^b\left\|\sum_{j=1}^k \alpha_j\varphi_j^b\right\|^2_{L^2(\Omega)} + \beta_a, \end{aligned}\tag{3-4}$$

where

$$\beta_a = \int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j\big\{\lambda_i^b(\eta_a^2-1)\varphi_i^b\bar\varphi_j^b + 2i\bar\varphi_j^b\eta_a(i\nabla + A_b)\varphi_i^b\cdot\nabla\eta_a - \varphi_i^b\bar\varphi_j^b\eta_a\Delta\eta_a\big\}\,dx. \tag{3-5}$$

We need to estimate β_a. From Lemma 2.2 we deduce the existence of a constant $C > 0$ such that $\|\varphi_j^b\|_{L^\infty(\Omega)} \le C$ for every $j = 1, \dots, k$. Hence

$$\left|\int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j\lambda_i^b(\eta_a^2-1)\varphi_i^b\bar\varphi_j^b\,dx\right| \le C\sum_{j=1}^k \alpha_j^2\int_\Omega (1-\eta_a^2)\,dx.$$

Using the fact that $\|\varphi_j^b\|^2_{H_0^1(\Omega)} \le C\|\varphi_j^b\|^2_{\mathcal{D}^{1,2}_{A_b}(\Omega)} = C\lambda_j^b$ (see (2-1)), we have

$$\left|\int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j\bar{\varphi}_j^b \eta_a \nabla\varphi_i^b\cdot\nabla\eta_a\,dx\right| \le C\sum_{j=1}^k \alpha_j^2\left(\int_\Omega |\nabla\eta_a|^2\,dx\right)^{1/2}.$$

Next we apply the Hardy inequality (2-2) to obtain

$$\begin{aligned}\left|\int_\Omega \sum_{i,j=1}^k \alpha_i\alpha_j\varphi_i^b\bar{\varphi}_j^b\eta_a A_b\cdot\nabla\eta_a\,dx\right| &\le C\sum_{j=1}^k\alpha_j^2\int_\Omega |\varphi_j^b A_b\cdot\nabla\eta_a|\,dx\\ &\le C\sum_{j=1}^k \alpha_j^2\left\|\frac{\varphi_j^b}{x-b}\right\|_{L^2(\Omega)}\|(x-b)A_b\|_{L^\infty(\Omega)}\|\nabla\eta_a\|_{L^2(\Omega)}\\ &\le C\sum_{j=1}^k\alpha_j^2\|\nabla\eta_a\|_{L^2(\Omega)}.\end{aligned}$$

Concerning the last term in (3-5), similar estimates give

$$\begin{aligned}\left|\int_\Omega\sum_{i,j=1}^k\alpha_i\alpha_j\varphi_i^b\bar{\varphi}_j^b\eta_a\Delta\eta_a\,dx\right| &= \left|\int_\Omega\sum_{i,j=1}^k\alpha_i\alpha_j\big(|\nabla\eta_a|^2\varphi_i^b\bar{\varphi}_j^b+\eta_a\nabla\eta_a\cdot\nabla(\varphi_i^b\bar{\varphi}_j^b)\big)\,dx\right|\\ &\le C\sum_{j=1}^k\alpha_j^2\left(\int_\Omega|\nabla\eta_a|^2\,dx\right)^{1/2}.\end{aligned}$$

In conclusion, we have obtained

$$|\beta_a| \le C\left\|\sum_{j=1}^k\alpha_j\varphi_j^b\right\|^2_{L^2(\Omega)}\left\{\int_\Omega(1-\eta_a^2)\,dx+\left(\int_\Omega|\nabla\eta_a|^2\,dx\right)^{1/2}\right\} = \left\|\sum_{j=1}^k\alpha_j\varphi_j^b\right\|^2_{L^2(\Omega)}\varepsilon_a'',$$

with $\varepsilon_a''\to 0$ as $a\to b$ by Lemma 3.1. By inserting the last estimate into (3-4) and then using Lemma 3.3 we obtain (3-2) with $\varepsilon_a' = (\varepsilon_a''+\lambda_k^b\varepsilon_a)/(1-\varepsilon_a)$.

Step 2: We now want to prove the second inequality, $\liminf_{a\to b}\lambda_k^a \ge \lambda_k^b$. From relation (2-1) and Step 1 we deduce

$$\|\varphi_j^a\|^2_{H_0^1(\Omega)} \le C\|\varphi_j^a\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \le C\lambda_j^b.$$

Hence there exists $\varphi_j^\star\in H_0^1(\Omega)$ such that (up to subsequences) $\varphi_j^a \rightharpoonup \varphi_j^\star$ weakly in $H_0^1(\Omega)$ and $\varphi_j^a\to\varphi_j^\star$ strongly in $L^2(\Omega)$, as $a\to b$. In particular we have

$$\int_\Omega|\varphi_j^\star|^2\,dx = 1 \qquad\text{and}\qquad q\int_\Omega\varphi_i^\star\varphi_j^\star\,dx = 0 \quad \text{if } i\ne j. \tag{3-6}$$

Moreover, Fatou's lemma, relation (2-2) and Step 1 provide

$$\|\varphi_j^\star/|x-b|\|_{L^2(\Omega)} \le \liminf_{a\to b}\|\varphi_j^a/|x-a|\|_{L^2(\Omega)} \le C\liminf_{a\to b}\|\varphi_j^a\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = C\liminf_{a\to b}\sqrt{\lambda_j^a}\le C\sqrt{\lambda_j^b},$$

so we deduce that $\varphi_j^\star \in \mathcal{D}^{1,2}_{A_b}(\Omega)$.

Given a test function $\phi \in C_0^\infty(\Omega\backslash\{b\})$, consider a sufficiently close to b so that $a \notin \operatorname{supp}\{\phi\}$. We have

$$\begin{aligned}
\int_\Omega \lambda_j^a \varphi_j^a \bar\phi\, dx &= \int_\Omega \varphi_j^a \overline{(i\nabla + A_a)^2\phi}\, dx \\
&= \int_\Omega \bigl\{-\Delta\varphi_j^a \bar\phi + \varphi_j^a [\overline{i\nabla\cdot A_a\phi + 2iA_a\cdot\nabla\phi + |A_a|^2\phi}]\bigr\}\, dx \\
&= \int_\Omega \bigl\{(i\nabla + A_b)^2\varphi_j^a\bar\phi - i\nabla\cdot(A_a+A_b)\varphi_j^a\bar\phi - 2i(A_a\cdot\nabla\bar\phi\varphi_j^a + A_b\cdot\nabla\varphi_j^a\bar\phi) \\
&\qquad\qquad + (|A_a|^2 - |A_b|^2)\varphi_j^a\bar\phi\bigr\}\, dx \\
&= \int_\Omega \bigl\{(i\nabla + A_b)^2\varphi_j^a\bar\phi - i\nabla\cdot(A_a - A_b)\varphi_j^a\bar\phi - 2i\varphi_j^a(A_a - A_b)\cdot\nabla\bar\phi + (|A_a|^2 - |A_b|^2)\varphi_j^a\bar\phi\bigr\}\, dx,
\end{aligned}$$

where in the last step we used the identity

$$-2i\int_\Omega A_b\cdot\nabla\varphi_j^a\bar\phi\, dx = 2i\int_\Omega (\nabla\cdot A_b\varphi_j^a\bar\phi + A_b\varphi_j^a\nabla\bar\phi)\, dx.$$

Since $a, b \notin \operatorname{supp}\{\phi\}$ then $A_a \to A_b$ in $C^\infty(\operatorname{supp}\{\phi\})$. Hence for a suitable subsequence we can pass to the limit in the previous expression obtaining

$$\int_\Omega (i\nabla + A_b)^2\varphi_j^\star\bar\phi\, dx = \int_\Omega \lambda_j^\star\varphi_j^\star\bar\phi\, dx \qquad \text{for every } \phi \in C_0^\infty(\Omega\backslash\{b\}),$$

where $\lambda_j^\infty := \liminf_{a\to b}\lambda_j^a$. By density, the same is valid for $\phi \in \mathcal{D}^{1,2}_{A_b}(\Omega)$. As a consequence of the last equation and of (3-6), the functions $\varphi_j^\star$ are orthogonal in $\mathcal{D}^{1,2}_{A_b}(\Omega)$ and hence

$$\begin{aligned}
\lambda_k^b = \inf_{\substack{W_k\subset\mathcal{D}^{1,2}_{A_b}(\Omega)\\ \dim W_k = k}} \sup_{\Phi\in W_k} \frac{\int_\Omega |(i\nabla + A_b)\Phi|^2\, dx}{\int_\Omega |\Phi|^2\, dx} &\le \sup_{(\alpha_1,\dots,\alpha_k)\neq 0} \frac{\int_\Omega \bigl|(i\nabla + A_b)\bigl(\sum_{j=1}^k \alpha_j\varphi_j^\star\bigr)\bigr|^2\, dx}{\int_\Omega \bigl|\sum_{j=1}^k \alpha_j\varphi_j^\star\bigr|^2\, dx} \\
&= \sup_{(\alpha_1,\dots,\alpha_k)\neq 0} \frac{\sum_{j=1}^k \alpha_j^2\lambda_j^\infty}{\sum_{j=1}^k \alpha_j^2} \le \lambda_k^\infty = \liminf_{a\to b}\lambda_k^a.
\end{aligned}$$

This concludes Step 2 and the proof of the theorem. □

4. Continuity of the eigenvalues with respect to the pole up to the boundary of the domain

In this section we prove the second part of Theorem 1.1, that is the continuous extension up to the boundary of the domain. We will denote by φ_j an eigenfunction associated to λ_j, the j-th eigenvalue of the Laplacian in $H_0^1(\Omega)$. As usual, we suppose that the eigenfunctions are normalized in L^2 and orthogonal. The following two lemmas can be proved exactly as the corresponding ones in Section 3.

Lemma 4.1. *Given $a \in \Omega$ and $b \in \partial\Omega$ there exist θ_a and θ_b such that $\theta_a \in C^\infty(\Omega\backslash\{ta + (1-t)b,\ t \in [0,1]\})$, $\theta_b \in C^\infty(\Omega)$, and moreover in the respective sets of regularity the following holds*:

$$\alpha\nabla\theta_a = A_a, \quad \alpha\nabla\theta_b = A_b.$$

Lemma 4.2. *Given $a \in \Omega$ and $b \in \partial\Omega$, let η_a be as defined in Lemma 3.1 and let θ_a be as defined in Lemma 3.2. Set, for $j = 1, \dots, k$,*

$$\tilde{\varphi}_j = e^{i\alpha\theta_a}\eta_a\varphi_j.$$

Then, for every $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$,

$$(1-\varepsilon_a)\left\|\sum_{j=1}^{k}\alpha_j\varphi_j\right\|^2_{L^2(\Omega)} \le \left\|\sum_{j=1}^{k}\alpha_j\tilde{\varphi}_j\right\|^2_{L^2(\Omega)} \le k\left\|\sum_{j=1}^{k}\alpha_j\varphi_j\right\|^2_{L^2(\Omega)},$$

where $\varepsilon_a \to 0$ as $a \to b$.

Theorem 4.3. *Suppose that $a \in \Omega$ converges to $b \in \partial\Omega$. Then for every $k \in \mathbb{N}$ we have that λ_k^a converges to λ_k.*

Proof. Following the scheme of the proof of Theorem 3.4 we proceed in two steps.

Step 1: First we show that

$$\limsup_{a\to b}\lambda_k^a \le \lambda_k. \tag{4-1}$$

Since the proof is very similar to the one of Step 1 in Theorem 3.4 we will only point out the main differences. We define

$$E_k := \left\{\Phi = \sum_{j=1}^{k}\alpha_j\tilde{\varphi}_j,\ \alpha_j \in \mathbb{R}\right\} \quad \text{with } \tilde{\varphi}_j = e^{i\alpha\theta_a}\eta_a\varphi_j.$$

We can verify the equality

$$(i\nabla + A_a)(e^{i\alpha\theta_a}\eta_a\varphi_j) = ie^{i\alpha\theta_a}\nabla(\eta_a\varphi_j),$$

so that we have

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \int_\Omega\left|\sum_{j=1}^{k}\alpha_j\nabla(\eta_a\varphi_j)\right|^2 dx \le \lambda_k\left\|\sum_{j=1}^{k}\alpha_j\varphi_j\right\|^2_{L^2(\Omega)} + \beta_a,$$

with

$$\beta_a = \sum_{i,j=1}^{k}\alpha_i\alpha_j\int_\Omega\left(|\nabla\eta_a|^2\varphi_i\varphi_j + 2\eta_a\nabla\eta_a\cdot\nabla\varphi_j\varphi_i + (\eta_a^2-1)\nabla\varphi_i\cdot\nabla\varphi_j\right)dx.$$

Proceeding similarly to the proof of Theorem 3.4 we can estimate

$$|\beta_a| \le \varepsilon_a''\left\|\sum_{j=1}^{k}\alpha_j\varphi_j\right\|^2_{L^2(\Omega)},$$

with $\varepsilon_a'' \to 0$ as $a \to b$. In conclusion, using Lemma 4.2, we have obtained

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \le \left(\lambda_k + \frac{\varepsilon_a'' + \lambda_k\varepsilon_a}{1-\varepsilon_a}\right)\|\Phi\|^2_{L^2(\Omega)} \qquad \text{for every } \Phi \in E_k,$$

with $\varepsilon_a, \varepsilon_a'' \to 0$ as $a \to b$. Therefore (4-1) is proved.

Step 2: We will now prove the second inequality

$$\liminf_{a\to b}\lambda_k^a \geq \lambda_k.$$

Given a test function $\phi \in C_0^\infty(\Omega)$, for a sufficiently close to b we have that

$$\{ta+(1-t)b,\ t\in[0,1]\}\subset \Omega\setminus \operatorname{supp}\{\phi\}.$$

Then $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ and Lemma 4.1 implies that $e^{i\alpha\theta_a}\phi \in C_0^\infty(\Omega)$. For this reason we can compute:

$$\int_\Omega \nabla(e^{-i\alpha\theta_b}\varphi_j^a)\cdot\nabla\bar{\phi}\,dx = \int_\Omega e^{-i\alpha\theta_b}\varphi_j^a\overline{(-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a}))}\,dx. \tag{4-2}$$

Since

$$-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a}) = (i\nabla+A_a)^2\phi - 2iA_a\cdot\nabla\phi - i\nabla\cdot A_a\phi - |A_a|^2\phi,$$

the right-hand side in (4-2) can be rewritten as

$$\int_\Omega \big((i\nabla+A_a)^2(e^{-i\alpha\theta_b}\varphi_j^a)\bar{\phi} + e^{-i\alpha\theta_b}\varphi_j^a(2iA_a\cdot\nabla\bar{\phi} + i\nabla\cdot A_a\bar{\phi} - |A_a|^2\bar{\phi})\big)\,dx.$$

At this point notice that

$$(i\nabla+A_a)^2(e^{-i\alpha\theta_b}\varphi_j^a) = e^{-i\alpha\theta_b}\big((i\nabla+A_a)^2\varphi_j^a + i\nabla\cdot A_b\varphi_j^a + 2iA_b\cdot\nabla\varphi_j^a + |A_b|^2\varphi_j^a + 2A_a\cdot A_b\varphi_j^a\big).$$

By inserting this information in (4-2) we obtain

$$\int_\Omega \nabla(e^{-i\alpha\theta_b}\varphi_j^a)\cdot\nabla\bar{\phi}\,dx = \lambda_j^a\int_\Omega e^{-i\alpha\theta_b}\varphi_j^a\bar{\phi}\,dx + \beta_a, \tag{4-3}$$

with

$$\beta_a = \int_\Omega e^{-i\alpha\theta_b}\bar{\phi}\big(i\nabla\cdot A_b\varphi_j^a + 2iA_b\cdot\nabla\varphi_j^a + |A_b|^2\varphi_j^a + 2A_a\cdot A_b\varphi_j^a\big)\,dx$$
$$+ \int_\Omega e^{-i\alpha\theta_b}\varphi_j^a\big(2iA_a\cdot\nabla\bar{\phi} + i\nabla\cdot A_a\bar{\phi} - |A_a|^2\bar{\phi}\big)\,dx.$$

Integration by parts leads to

$$\beta_a = \int_\Omega e^{-i\alpha\theta_b}\varphi_j^a\big(-\bar{\phi}|A_a-A_b|^2 + 2i\nabla\bar{\phi}\cdot(A_a-A_b) + i\bar{\phi}\nabla\cdot(A_a-A_b)\big)\,dx,$$

so that $|\beta_a|\to 0$ as $a\to b$, since $A_a\to A_b$ in $C^\infty(\operatorname{supp}\{\phi\})$. Therefore we can pass to the limit in (4-3) to obtain

$$\int_\Omega \nabla\varphi_j^\star\cdot\nabla\bar{\phi}\,dx = \lambda_j^\infty\int_\Omega \varphi_j^\star\bar{\phi}\,dx \qquad \text{for every } \phi\in C_0^\infty(\Omega),$$

where $\varphi_j^\star$ is the weak limit of a suitable subsequence of $e^{-i\alpha\theta_b}\varphi_j^a$ (given by Step 1) and $\lambda_j^\infty := \liminf_{a\to b}\lambda_j^a$. The conclusion of the proof is as in Theorem 3.4. □

Remark 4.4. As a consequence of Theorem 4.3 we obtain that $e^{-i\alpha\theta_a}\varphi_j^a \to \varphi_j$ in $H_0^1(\Omega)$ as $a \to b \in \partial\Omega$. Indeed, an inspection of the previous proof provides the weak convergence $e^{-i\alpha\theta_a}\varphi_j^a \rightharpoonup \varphi_j$ in $H_0^1(\Omega)$ and the convergence of the norms

$$\|e^{-i\alpha\theta_a}\varphi_j^a\|^2_{H_0^1(\Omega)} = \|\varphi_j^a\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \lambda_j^a \to \lambda_j = \|\varphi_j\|^2_{H_0^1(\Omega)},$$

as $a \to b \in \partial\Omega$, for every $j \in \mathbb{N}$.

5. Differentiability of the simple eigenvalues with respect to the pole

In this section we prove Theorem 1.3. We omit the subscript in the notation of the eigenvalues and eigenfunctions; with this notation, λ^a is any eigenvalue of $(i\nabla + A_a)^2$ and φ^a is an associated eigenfunction.

Proof of Theorem 1.3. Let $b \in \Omega$ be such that λ^b is simple, as in the assumptions of the theorem. For R such that $B_{2R}(b) \subset \Omega$, let ξ be a cut-off function satisfying $\xi \in C^\infty(\Omega)$, $0 \le \xi \le 1$, $\xi(x) = 1$ for $x \in B_R(b)$ and $\xi(x) = 0$ for $x \in \Omega \setminus B_{2R}(b)$. For every $a \in B_R(b)$ we define the transformation

$$\Phi_a : \Omega \to \Omega, \qquad \Phi_a(x) = \xi(x)(x - b + a) + (1 - \xi(x))x.$$

Then $\varphi^a \circ \Phi_a \in \mathcal{D}^{1,2}_{A_b}(\Omega)$ and satisfies, for every $a \in B_R(b)$,

$$(i\nabla + A_b)^2(\varphi^a \circ \Phi_a) + \mathcal{L}(\varphi^a \circ \Phi_a) = \lambda^a \varphi^a \circ \Phi_a \tag{5-1}$$

and

$$\int_\Omega |\varphi^a \circ \Phi_a|^2 |\det(\Phi_a')|\, dx = 1, \tag{5-2}$$

where $\mathcal{L}$ is a second-order operator of the form

$$\mathcal{L}v = -\sum_{i,j=1}^{2} a^{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b^i(x)\frac{\partial v}{\partial x_i} + c(x)v,$$

with $a^{ij}, b^i, c \in C^\infty(\Omega, \mathbb{C})$ vanishing in $B_R(b)$ and outside of $B_{2R}(b)$. Notice that

$$\Phi_a'(x) = I + \nabla\xi(x) \otimes (a - b)$$

is a small perturbation of the identity whenever $|b - a|$ is sufficiently small, so that the operator in the left-hand side of (5-1) is elliptic (see for example [Brezis 2011, Lemma 9.8]).

To prove the differentiability, we will use the implicit function theorem in Banach spaces. To this aim, we define the operator

$$\begin{aligned} F : B_R(b) \times \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{R} &\to (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R}, \\ (a, v, \lambda) &\mapsto \left((i\nabla + A_b)^2 v + \mathcal{L}v - \lambda v,\ \int_\Omega |v|^2 |\det(\Phi_a')|\, dx - 1 \right). \end{aligned} \tag{5-3}$$

Notice that F is of class C^∞ by the ellipticity of the operator, provided that R is sufficiently small, and that $F(a, \varphi^a \circ \Phi_a, \lambda^a) = 0$ for every $a \in B_R(b)$, as we saw in (5-1), (5-2). In particular we have $F(b, \varphi^b, \lambda^b) = 0$, since Φ_b is the identity. We now have to verify that $\mathrm{d}_{(v,\lambda)}F(b, \varphi^b, \lambda^b)$, the differential of F with respect to the variables (v, λ) evaluated at the point $(b, \varphi^b, \lambda^b)$, belongs to $\mathrm{Inv}(\mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{R}, (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R})$. The differential is given by

$$d_{(v,\lambda)} F(b, \varphi^b, \lambda^b) = \begin{pmatrix} (i\nabla + A_b)^2 - \lambda^b \mathrm{Im} & -\varphi^b \\ 2\int_\Omega \bar{\varphi}^b \, dx & 0 \end{pmatrix},$$

where Im is the compact immersion of $\mathcal{D}^{1,2}_{A_b}(\Omega)$ in $(\mathcal{D}^{1,2}_{A_b}(\Omega))'$, which was introduced in Lemma 2.1.

Let us first prove that it is injective. To this aim we have to show that, if $(w, s) \in \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{R}$ is such that

$$(i\nabla + A_b)^2 w - \lambda^b w = s\varphi^b, \tag{5-4}$$

$$2\int_\Omega \bar{\varphi}^b w \, dx = 0, \tag{5-5}$$

then $(w, s) = (0, 0)$. Relations (5-5) and (5-2) (with $a = b$ and Φ_b the identity) imply that

$$w \neq k\varphi^b \quad \text{for all } k \neq 0. \tag{5-6}$$

By testing (5-4) by φ^b we obtain

$$s = \int_\Omega \big((i\nabla + A_b) w \cdot \overline{(i\nabla + A_b)\varphi^b} - \lambda^b w \overline{\varphi^b} \big) \, dx.$$

On the other hand, testing by w the equation satisfied by φ^b, we see that $s = 0$, so that (5-4) becomes

$$(i\nabla + A_b)^2 w = \lambda^b w.$$

The assumption that λ^b is simple, together with (5-6), implies $w = 0$. This concludes the proof of the injectivity.

For the surjectivity, we have to show that for all $(f, r) \in (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R}$ there exists $(w, s) \in \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{R}$ which verifies the following equalities

$$(i\nabla + A_b)^2 w - \lambda^b w = f + s\varphi^b, \tag{5-7}$$

$$2\int_\Omega \bar{\varphi}^b w \, dx = r. \tag{5-8}$$

We recall that the operator $(i\nabla + A_b)^2 - \lambda^b \,\mathrm{Im} : \mathcal{D}^{1,2}_{A_b}(\Omega) \to (\mathcal{D}^{1,2}_{A_b}(\Omega))'$ is Fredholm of index 0. This is a standard fact, which can be proved for example by noticing that this operator is isomorphic to $\mathrm{Id} - \lambda^b((i\nabla + A_b)^2)^{-1}(\mathrm{Im})$ through the Riesz isomorphism and because the operator $(i\nabla + A_b)^2$ is invertible. This is Fredholm of index 0 because it has the form identity minus compact, the compactness coming from Lemma 2.1. Therefore we have (through Riesz isomorphism)

$$\mathrm{Range}((i\nabla + A_b)^2 - \lambda^b \,\mathrm{Im}) = (\mathrm{Ker}((i\nabla + A_b)^2 - \lambda^b \,\mathrm{Im}))^\perp = (\mathrm{span}\{\varphi^b\})^\perp, \tag{5-9}$$

where we used the assumption that λ^b is simple in the last equality. As a consequence, we obtain from (5-7) an expression for s:

$$s = -\int_\Omega f \overline{\varphi^b} \, dx.$$

Next we can decompose w in w_0+w_1 such that $w_0 \in \operatorname{Ker}((i\nabla+A_b)^2-\lambda^b \operatorname{Im})$ and w_1 is in the orthogonal space. Condition (5-7) becomes

$$(i\nabla + A_b)^2 w_1 - \lambda^b w_1 = f - \varphi^b \int_\Omega f\overline{\varphi^b}\,dx \tag{5-10}$$

and (5-9) ensures the existence of a solution w_1. Given such w_1, condition (5-8) determines w_0 as follows:

$$w_0 = \left(-\int_\Omega \bar{\varphi}^b w_1\,dx + \frac{r}{2}\right)\varphi^b,$$

so that the surjectivity is also proved.

We conclude that the implicit function theorem applies, so that the maps $a \in \Omega \mapsto \lambda^a \in \mathbb{R}$ and $a \in \Omega \mapsto \varphi^a \circ \Phi_a \in \mathcal{D}^{1,2}_{A_b}(\Omega)$ are of class C^∞ locally in a neighborhood of b. □

By combining the previous result with a standard lemma of local inversion we deduce the following fact, which we will need in the next section.

Corollary 5.1. *Let $b \in \Omega$. If λ^b is simple then the map $\Psi : \Omega \times \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{R} \to \mathbb{R} \times (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R}$ given by*

$$\Psi(a, v, \lambda) = (a, F(a, v, \lambda)),$$

with F defined in (5-3), is locally invertible in a neighborhood of $(b, \varphi^b, \lambda^b)$, with inverse Ψ^{-1} of class C^∞.

Proof. We saw in the proof of Theorem 1.3 that, if λ^b is simple, then $\mathrm{d}_{(v,\lambda)}F(b, \varphi^b, \lambda^b)$ is invertible. It is sufficient to apply Lemma 2.1 in Chapter 2 of the book of Ambrosetti and Prodi [1993]. □

6. Vanishing of the derivative at a multiple zero

In this section we prove Theorem 1.7. Recall that here $\alpha = \frac{1}{2}$. We will need the following preliminary results.

Lemma 6.1. *Let $\lambda > 0$ and let $D_r = D_r(0) \subset \mathbb{R}^2$. Consider the following set of equations for $r > 0$ small:*

$$\begin{cases} -\Delta u = \lambda u & \text{in } D_r, \\ u = r^{k/2} f + g(r,\cdot) & \text{on } \partial D_r, \end{cases} \tag{6-1}$$

where $f, g(r,\cdot) \in H^1(\partial D_r)$ and g satisfies

$$\lim_{r\to 0} \frac{\|g(r,\cdot)\|_{H^1(\partial D_r)}}{r^{k/2}} = 0 \tag{6-2}$$

for some integer $k \geq 3$. Then for r sufficiently small there exists a unique solution to (6-1), which moreover satisfies

$$\|u\|_{L^2(D_r)} \leq C r^{(k+2)/2} \quad \text{and} \quad \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial D_r)} \leq C r^{(k-1)/2},$$

where $C > 0$ is independent of r.

Proof. Let z_1 solve

$$\begin{cases} -\Delta z_1 = 0 & \text{in } D_1, \\ z_1 = f + r^{-k/2} g(r,\cdot) & \text{on } \partial D_1. \end{cases}$$

Since the quadratic form

$$\int_{D_1} (|\nabla v|^2 - \lambda r^2 v^2)\,dx \tag{6-3}$$

is coercive for $v \in H_0^1(D_1)$ for r sufficiently small, there exists a unique solution z_2 to the equation

$$\begin{cases} -\Delta z_2 - \lambda r^2 z_2 = \lambda r^2 z_1 & \text{in } D_1, \\ z_2 = 0 & \text{on } \partial D_1. \end{cases} \tag{6-4}$$

Then $u(x) = r^{k/2}(z_1(x/r) + z_2(x/r))$ is the unique solution to (6-1). In order to obtain the desired bounds on u we will estimate separately z_1 and z_2. Assumption (6-2) implies

$$\|z_1\|_{H^1(D_1)} = \|f + r^{-k/2} g(r,\cdot)\|_{H^{1/2}(\partial D_1)} \le C \|f\|_{H^1(\partial D_1)}, \tag{6-5}$$

for r sufficiently small. We compare the function z_1 to its limit function when $r \to 0$, which is the harmonic extension of f in D_1, which we will denote w. Then we have

$$\begin{cases} -\Delta(z_1 - w) = 0 & \text{in } D_1, \\ z_1 - w = r^{-k/2} g(r,\cdot) & \text{on } \partial D_1, \end{cases}$$

and hence (6-2) implies

$$\left\| \frac{\partial}{\partial \nu}(z_1 - w) \right\|_{L^2(\partial D_1)} \le C \|z_1 - w\|_{H^1(\partial D_1)} = C \frac{\|g(r,\cdot)\|_{H^1(\partial D_1)}}{r^{k/2}} \to 0.$$

Then we estimate z_2 as follows:

$$\|z_2\|^2_{L^2(D_1)} \le C \int_{D_1} |\nabla z_2|^2\,dx \le C \int_{D_1} (|\nabla z_2|^2 - \lambda r^2 z_2^2)\,dx \le C \|\lambda r^2 z_1\|_{L^2(D_1)} \|z_2\|_{L^2(D_1)},$$

where we used the Poincaré inequality, the coercivity of the quadratic form (6-3) and the definition of z_2 (6-4). Hence estimate (6-5) implies

$$\|z_2\|_{L^2(D_1)} \le C r^2 \|f\|_{H^1(\partial D_1)} \to 0 \quad \text{as } r \to 0.$$

This and (6-5) provide, by a change of variables in the integral, the desired estimate on $\|u\|_{L^2(D_r)}$. Now, the standard bootstrap argument for elliptic equations applied to (6-4) provides

$$\|z_2\|_{H^2(D_1)} \le C(\|\lambda r^2 z_1\|_{L^2(D_1)} + \|z_2\|_{L^2(D_1)}) \to 0,$$

and hence the trace embedding implies

$$\left\| \frac{\partial z_2}{\partial \nu} \right\|_{L^2(\partial D_1)} \le C \|\nabla z_2\|_{H^1(D_1)} \le C \|z_2\|_{H^2(D_1)} \to 0.$$

So, we have obtained that there exists $C > 0$ independent of r such that

$$\left\| \frac{\partial}{\partial \nu}(z_1 + z_2) \right\|_{L^2(\partial D_1)} \le C.$$

Finally, going back to the function u, we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_r)} = r^{(k-1)/2} \left\| \frac{\partial}{\partial \nu}(z_1+z_2) \right\|_{L^2(\partial D_1)} \le C r^{(k-1)/2}$$

where we used the change of variable $x = ry$. □

Lemma 6.2. *Let $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ $(a \in \Omega)$. Then*

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} \le C \|\phi\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \tag{6-6}$$

where C only depends on Ω.

Proof. Set $\tilde{\phi}(y) = \phi(|a|y)$ defined for $y \in \tilde{\Omega} = \{x/|a| : x \in \Omega\}$. We apply this change of variables to the left-hand side in (6-6) and then use the trace embedding to obtain

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} = \|\tilde{\phi}\|_{L^2(\partial D_1)} \le C \|\tilde{\phi}\|_{H^1(D_1)} \le C \|\tilde{\phi}\|_{H^1(D_2)}.$$

We have that $\tilde{\phi} \in H^1_{A_e}(\tilde{\Omega})$, where $e = a/|a|$. Therefore we can apply relation (2-2) as follows:

$$\|\tilde{\phi}\|_{L^2(D_2)} \le \|y-e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y-e|} \right\|_{L^2(D_2)} \le C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)},$$

$$\begin{aligned} \|\nabla\tilde{\phi}\|_{L^2(D_2)} &\le \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|A_a\tilde{\phi}\|_{L^2(D_2)} \\ &\le \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|(y-e)A_e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y-e|} \right\|_{L^2(D_2)} \\ &\le C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)}. \end{aligned}$$

We combine the previous inequalities obtaining

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} \le C \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} \le C \|\phi\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)},$$

where in the last step we used the fact that the quadratic form is invariant under dilations. □

To simplify the notation we suppose without loss of generality that $0 \in \Omega$ and we take $b = 0$. Moreover, we omit the subscript in the notation of the eigenvalues as we did in the previous section. As a first step in the proof of Theorem 1.7, we shall estimate $|\lambda^a - \lambda^0|$ in the case when the pole a belongs to a nodal line of φ^0 ending at 0. We make this restriction because all the constructions in the following proposition require that φ^0 vanishes at a.

Proposition 6.3. *Suppose that λ^0 is simple and that φ^0 has a zero of order $k/2$ at the origin, with $k \ge 3$ odd. Denote by Γ a nodal line of φ^0 with endpoint at 0 (which exists by Proposition 2.4) and take $a \in \Gamma$. Then there exists a constant $C > 0$ independent of $|a|$ such that*

$$|\lambda^a - \lambda^0| \le C|a|^{k/2} \quad \text{as } |a| \to 0,\ a \in \Gamma.$$

Proof. The idea of the proof is to construct a function $u_a \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ satisfying

$$(i\nabla + A_a)^2 u_a - \lambda^0 u_a = g_a, \qquad \|u_a\|_{L^2(\Omega)} = 1 - \epsilon_a \tag{6-7}$$

with

$$\|g_a\|_{(\mathcal{D}^{1,2}_{A_a}(\Omega))'} \simeq |a|^{k/2} \qquad \text{and} \qquad |\epsilon_a| \simeq |a|^{(k+2)/2} \tag{6-8}$$

and then to apply Corollary 5.1. For the construction of the function u_a we will heavily rely on the assumption $a \in \Gamma$.

<u>Step 1</u>: construction of u_a. We define it separately in $D_{|a|} = D_{|a|}(0)$ and in its complement $\Omega \setminus D_{|a|}$, using the notation

$$\begin{cases} u_a = u_a^{\text{ext}} & \text{in } \Omega \setminus D_{|a|}, \\ u_a^{\text{int}} & \text{in } D_{|a|}. \end{cases} \tag{6-9}$$

Concerning the exterior function we set

$$u_a^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0)} \varphi^0, \tag{6-10}$$

where θ_a, θ_0 are defined as in Lemma 3.2 in such a way that $\theta_a - \theta_0$ is regular in $\Omega \setminus D_{|a|}$ (here $\theta_0 = \theta$ is the angle in the usual polar coordinates, but we emphasize the position of the singularity in the notation). Therefore u_a^{ext} solves the magnetic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{\text{ext}} = \lambda^0 u_a^{\text{ext}} & \text{in } \Omega \setminus D_{|a|}, \\ u_a^{\text{ext}} = e^{i\alpha(\theta_a - \theta_0)} \varphi^0 & \text{on } \partial D_{|a|}, \\ u_a^{\text{ext}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6-11}$$

For the definition of u_a^{int} we will first consider a related elliptic problem. Notice that, by our choice $a \in \Gamma$, we have that $e^{-i\alpha\theta_0}\varphi^0$ is continuous on $\partial D_{|a|}$. Indeed, $e^{-i\alpha\theta_0}$ restricted to $\partial D_{|a|}$ is discontinuous only at the point a, where φ^0 vanishes. Moreover, note that this boundary trace is at least $H^1(\partial D_{|a|})$. Indeed, the eigenfunction φ^0 is C^∞ far from the singularity and $e^{i\alpha\theta_0}$ is also regular except on the point a. Then, the boundary trace is differentiable almost everywhere.

This allows to apply Lemma 6.1, thus providing the existence of a unique function ψ_a^{int}, a solution of the equation

$$\begin{cases} -\Delta \psi_a^{\text{int}} = \lambda^0 \psi_a^{\text{int}} & \text{in } D_{|a|}, \\ \psi_a^{\text{int}} = e^{-i\alpha\theta_0} \varphi^0 & \text{on } \partial D_{|a|}. \end{cases} \tag{6-12}$$

Then we complete our construction of u_a by setting

$$u_a^{\text{int}} = e^{i\alpha\theta_a} \psi_a^{\text{int}}, \tag{6-13}$$

which is well-defined since θ_a is regular in $D_{|a|}$. Note that u_a^{int} solves the elliptic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{\text{int}} = \lambda^0 u_a^{\text{int}} & \text{in } D_{|a|}, \\ u_a^{\text{int}} = u_a^{\text{ext}} & \text{on } \partial D_{|a|}. \end{cases} \tag{6-14}$$

<u>Step 2</u>: estimate of the normal derivative of u_a^{int} along $\partial D_{|a|}$. By assumption, φ^0 has a zero of order $k/2$ at the origin, with $k \geq 3$ odd. Hence by Proposition 2.4 the following asymptotic expansion holds on

$\partial D_{|a|}$ as $|a| \to 0$:

$$e^{-i\alpha\theta_0}\varphi^0(|a|,\theta_0) = \frac{|a|^{k/2}}{k}[c_k \cos(k\alpha\theta_0) + d_k \sin(k\alpha\theta_0)] + g(|a|,\theta_0), \tag{6-15}$$

with

$$\lim_{|a|\to 0} \frac{\|g(|a|,\cdot)\|_{C^1(\partial D_{|a|})}}{|a|^{k/2}} = 0. \tag{6-16}$$

Hence Lemma 6.1 applies to ψ_a^{int} given in (6-12), giving a constant C independent of $|a|$ such that

$$\|\psi_a^{\text{int}}\|_{L^2(D_{|a|})} \le C|a|^{(k+2)/2} \quad \text{and} \quad \left\| \frac{\partial \psi_a^{\text{int}}}{\partial \nu} \right\|_{L^2(\partial D_{|a|})} \le C|a|^{(k-1)/2}. \tag{6-17}$$

Finally, differentiating (6-13) we see that

$$(i\nabla + A_a)u_a^{\text{int}} = ie^{i\alpha\theta_a}\nabla\psi_a^{\text{int}},$$

so that, integrating, we obtain the L^2-estimate for the magnetic normal derivative of u_a^{int} along $\partial D_{|a|}$

$$\|(i\nabla + A_a)u_a^{\text{int}} \cdot \nu\|_{L^2(\partial D_{|a|})} \le C|a|^{(k-1)/2}. \tag{6-18}$$

Step 3: estimate of the normal derivative of u_a^{ext} along $\partial D_{|a|}$. We differentiate (6-10) to obtain

$$(i\nabla + A_a)u_a^{\text{ext}} = A_0 u_a^{\text{ext}} + ie^{i\alpha(\theta_a - \theta_0)}\nabla\varphi^0. \tag{6-19}$$

On the other hand, the following holds a.e.:

$$\nabla\varphi^0 = iA_0\varphi^0 + e^{i\alpha\theta_0}\nabla(e^{-i\alpha\theta_0}\varphi^0),$$

so that

$$ie^{i\alpha(\theta_a-\theta_0)}\nabla\varphi^0 = -A_0 u_a^{\text{ext}} + ie^{i\alpha\theta_a}\nabla(e^{-i\alpha\theta_0}\varphi^0).$$

Combining the last equality with (6-19) we obtain a.e.

$$(i\nabla + A_a)u_a^{\text{ext}} = ie^{i\alpha\theta_a}\nabla(e^{-i\alpha\theta_0}\varphi^0)$$

and hence $|(i\nabla + A_a)u_a^{\text{ext}}| \le C|a|^{k/2-1}$ on $\partial D_{|a|}$ a.e., for some C not depending on $|a|$, by (6-15) and (6-16). Integrating on $\partial D_{|a|}$ we arrive at the same estimate as for u_a^{int}, that is

$$\|(i\nabla + A_a)u_a^{\text{ext}} \cdot \nu\|_{L^2(\partial D_{|a|})} \le C|a|^{(k-1)/2}. \tag{6-20}$$

Step 4: proof of (6-8). We test (6-11) with a test function $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ and apply the formula of integration by parts to obtain

$$\int_{\Omega\setminus D_{|a|}} \{(i\nabla + A_a)u_a^{\text{ext}}\overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{\text{ext}}\bar{\phi}\}\,dx = i\int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{\text{ext}} \cdot \nu\bar{\phi}\,d\sigma.$$

Similarly, (6-14) provides

$$\int_{D_{|a|}} \{(i\nabla + A_a)u_a^{\text{int}}\overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{\text{int}}\bar{\phi}\}\,dx = -i\int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{\text{int}} \cdot \nu\bar{\phi}\,d\sigma.$$

Then, we test the equation in (6-7) with ϕ, we integrate by parts and we replace the previous equalities to get

$$\int_\Omega g_a \bar{\phi}\, dx = i \int_{\partial D_{|a|}} (i\nabla + A_a)(u_a^{\text{ext}} - u_a^{\text{int}}) \cdot \nu \bar{\phi}\, d\sigma.$$

To the previous expression we apply first the Hölder inequality and then the estimates obtained in the previous steps (6-18) and (6-20) to obtain

$$\begin{aligned}\left| \int_\Omega g_a \bar{\phi}\, dx \right| &\le \|(i\nabla + A_a)u_a^{\text{int}} \cdot \nu\|_{L^2(\partial D_{|a|})} \|\phi\|_{L^2(\partial D_{|a|})} + \|(i\nabla + A_a)u_a^{\text{ext}} \cdot \nu\|_{L^2(\partial D_{|a|})} \|\phi\|_{L^2(\partial D_{|a|})} \\ &\le C|a|^{(k-1)/2} \|\phi\|_{L^2(\partial D_{|a|})}.\end{aligned}$$

Finally, Lemma 6.2 provides the desired estimate on g_a. Then we estimate ϵ_a as follows. Since $\|u_a^{\text{ext}}\|_{L^2(\Omega \setminus D_{|a|})} = \|\varphi^0\|_{L^2(\Omega \setminus D_{|a|})}$ we have

$$\big| \|u_a\|_{L^2(\Omega)} - 1 \big| = \big| \|u_a^{\text{int}}\|^2_{L^2(D_{|a|})} - \|\varphi^0\|^2_{L^2(D_{|a|})} \big| \le C|a|^{k+2}, \tag{6-21}$$

where in the last inequality we used the fact that $\|\varphi^0\|^2_{L^2(D_{|a|})} \le C|a|^{k+2}$ by (6-15) and (6-16), and that $\|u_a^{\text{int}}\|^2_{L^2(D_{|a|})} = \|\psi_a^{\text{int}}\|^2_{L^2(D_{|a|})} \le C|a|^{k+2}$, by (6-17).

Step 5: local inversion theorem. To conclude the proof we apply Corollary 5.1. Let Ψ be the function defined therein (recall that here $b = 0$). The construction that we did in the previous steps ensures that

$$\begin{aligned}\Psi(a, \varphi^a \circ \Phi_a, \lambda^a) &= (a, 0, 0), \\ \Psi(a, u_a \circ \Phi_a, \lambda^0) &= (a, g_a \circ \Phi_a, \epsilon_a),\end{aligned}$$

with g_a, ϵ_a satisfying (6-8). We proved in Theorem 3.4 that

$$|\lambda^a - \lambda^0| + \|\varphi^a \circ \Phi_a - \varphi^0\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} \to 0$$

as $|a| \to 0$. Moreover, it is not difficult to see that

$$\|u_a \circ \Phi_a - \varphi^0\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} \to 0$$

as $|a| \to 0$. Hence the points $(a, \varphi^a \circ \Phi_a, \lambda^a)$ and $(a, u_a \circ \Phi_a, \lambda^0)$ are approaching $(0, \varphi^0, \lambda^0)$ in the space $\Omega \times \mathcal{D}^{1,2}_{A_0}(\Omega) \times \mathbb{R}$ as $|a| \to 0$. Since Ψ admits an inverse of class C^∞ in a neighborhood of $(0, \varphi^0, \lambda^0)$ (recall that λ^0 is simple), we deduce that

$$\|(\varphi^a - u_a) \circ \Phi_a\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} + |\lambda^a - \lambda^0| \le C(\|g_a\|_{(\mathcal{D}^{1,2}_{A_a}(\Omega))'} + |\epsilon_a|) \le C|a|^{k/2},$$

for some constant C independent of a, which concludes the proof of the proposition. □

At this point we have proved the desired property only for pole a belonging to the nodal lines of φ^0. We would like to extend this result to all a sufficiently close to 0. We will proceed in the following way. Thanks to Theorem 1.3, we can consider the Taylor expansion of the function $a \mapsto \lambda^a$ in a neighborhood of 0. Then Proposition 6.3 provides k vanishing conditions, corresponding to the k nodal lines of φ^0.

Finally, we will use these conditions to show that in fact the first terms of the polynomial are identically zero. Let us begin with a lemma on the existence and the form of the Taylor expansion.

Lemma 6.4. *If λ^0 is simple then for $a \in \Omega$ sufficiently close to 0 and for all $H \in \mathbb{N}$*

$$\lambda^a - \lambda^0 = \sum_{h=1}^{H} |a|^h P_h(\vartheta(a)) + o(|a|^H), \tag{6-22}$$

where $a = |a|(\cos\vartheta(a), \sin\vartheta(a))$ and

$$P_h(\vartheta) = \sum_{j=0}^{h} \beta_{j,h} \cos^j\vartheta \sin^{h-j}\vartheta \tag{6-23}$$

for some $\beta_{j,h} \in \mathbb{R}$ not depending on $|a|$.

Proof. Since λ^0 is simple, λ^a is also simple for a sufficiently close to 0. Then we proved in Theorem 1.3 that λ_j^a is C^∞ in the variable a. As a consequence, we can consider the first terms of the Taylor expansion, with Peano rest, of λ_j^a

$$\lambda^a - \lambda^0 = \sum_{h=1}^{H}\sum_{j=0}^{h} \frac{1}{j!(h-j)!} \frac{\partial^h \lambda^a}{\partial^j a_1 \partial^{h-j} a_2}\bigg|_{a=0} a_1^j a_2^{h-j} + o(|a|^H),$$

where $a = (a_1, a_2)$. Setting

$$\beta_{j,h} = \frac{1}{j!(h-j)!} \frac{\partial^h \lambda^a}{\partial^j a_1 \partial^{h-j} a_2}\bigg|_{a=0}$$

and $a_1 = |a|\cos\vartheta(a)$, $a_2 = |a|\sin\vartheta(a)$, the thesis follows. □

The following lemma tells us that on the k nodal lines of φ^0, the first low-order polynomials cancel.

Lemma 6.5. *Suppose that λ^0 is simple and that φ^0 has a zero of order $k/2$ at 0, with $k \geq 3$ odd. Then there exists an angle $\tilde{\vartheta} \in [0, 2\pi)$ and non-negative quantities $\varepsilon_0, \dots, \varepsilon_{k-1}$ arbitrarily small such that*

$$P_h\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0 \quad \textit{for all integers } l \in [0, k-1],\ h \in [1, (k-1)/2],$$

where P_h is defined in (6-23).

Proof. We know from Proposition 2.4 that φ^0 has k nodal lines with endpoint at 0, which we denote Γ_l, $l = 0, \dots, k-1$. Take points $a_l \in \Gamma_l$, $l = 0, \dots, k-1$, satisfying $|a_0| = \cdots = |a_{k-1}|$ and denote

$$a_l = |a_l|(\cos\vartheta(a_l), \sin\vartheta(a_l)).$$

First we claim that $P_h(\vartheta(a_l)) = 0$ for all integers $l \in [0, k-1]$, $h \in [1, (k-1)/2]$.

Indeed, suppose by contradiction that this is not the case for some l, h belonging to the intervals defined above. Then for such l, h the following holds by Lemma 6.4:

$$\lambda^{a_l} - \lambda^0 = C|a_l|^h + o(|a_l|^h) \quad \text{for some } C \neq 0.$$

On the other hand we proved in Proposition 6.3 that there exists $C > 0$ independent of a such that, for every $l = 0, \dots, k-1$, we have

$$|\lambda^{a_l} - \lambda^0| \le C|a_l|^{k/2} \quad \text{as } |a_l| \to 0.$$

This contradicts the last estimate because $h \le (k-1)/2$, so that the claim is proved.

Finally setting $\tilde{\vartheta} := \vartheta(a_0)$, Proposition 2.4 implies

$$\vartheta(a_l) = \tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l, \quad l = 1, \dots, k-1, \quad \text{with } \varepsilon_l \to 0 \text{ as } |a_l| \to 0. \qquad \square$$

The next lemma extends this previous property to all a close to 0.

Lemma 6.6. *Fix $k \ge 3$ odd. For any integer $h \in [1, (k-1)/2]$ consider any polynomial of the form*

$$P_h(\vartheta) = \sum_{j=0}^{h} \beta_{j,h} \cos^j \vartheta \sin^{h-j} \vartheta, \tag{6-24}$$

with $\beta_{j,h} \in \mathbb{R}$. Suppose that there exist $\tilde{\vartheta} \in [0, 2\pi)$ and $\varepsilon_0, \dots, \varepsilon_{k-1}$ satisfying $0 \le \varepsilon_l \le \dfrac{\pi}{4k}$ such that

$$P_h\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0 \quad \textit{for every integer } l \in [0, k-1].$$

Then $P_h \equiv 0$.

Proof. We prove the result by induction on h.

Step 1: Let $h = 1$; then

$$P_1(\vartheta) = \beta_0 \sin\vartheta + \beta_1 \cos\vartheta$$

and the following conditions hold for $l = 0, \dots, k-1$:

$$\beta_0 \sin\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) + \beta_1 \cos\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0. \tag{6-25}$$

In the case that for every $l = 0, \dots, k-1$ we have

$$\sin\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) \ne 0 \quad \text{and} \quad \cos\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) \ne 0,$$

system (6-25) has two unknowns β_0, β_1 and $k \ge 3$ linearly independent equations. Hence in this case $\beta_0 = \beta_1 = 0$ and $P_1 \equiv 0$. In the case that there exists l such that

$$\sin\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0$$

then of course $\cos(\tilde{\vartheta} + 2\pi l/k + \varepsilon_l) \ne 0$, which implies $\beta_1 = 0$. We claim that in this case

$$\sin\Big(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\Big) \ne 0 \tag{6-26}$$

for every integer $l' \in [0, k-1]$ different from l. To prove the claim we proceed by contradiction. We can suppose without loss of generality that

$$\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l = 0 \quad \text{and} \quad \tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'} = \pi.$$

Then
$$l = -\frac{k}{2\pi}(\tilde{\vartheta} + \varepsilon_l) \quad \text{and} \quad l' = \frac{k}{2\pi}(\pi - \tilde{\vartheta} - \varepsilon_{l'})$$
so that
$$l' - l = \frac{k}{2} + k\frac{\varepsilon_l - \varepsilon_{l'}}{2\pi}.$$
The assumption $0 \le \varepsilon_l \le \pi/(4k)$ implies
$$\frac{k}{2} - \frac{1}{4} \le l' - l \le \frac{k}{2} + \frac{1}{4}.$$
Since $k \ge 3$ is an odd integer, the last estimate provides $l' - l \notin \mathbb{N}$, which is a contradiction. Therefore we have proved (6-26). Now consider any of the equations in (6-25) for $l' \ne l$. Inserting the information $\beta_1 = 0$ and (6-26) we get $\beta_0 = 0$ and hence $P_1 \equiv 0$. In the case that one of the cosines vanishes one can proceed in the same way, so we have proved the basis of the induction.

Step 2: Suppose that the statement is true for some $h \le (k-3)/2$ and let us prove it for $h+1$. The following conditions hold for $l = 0, \dots, k-1$:
$$\sum_{j=0}^{h+1} \beta_j \cos^j\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) \sin^{h+1-j}\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0. \tag{6-27}$$
We can proceed similarly to Step 1. If none of the sines, cosines vanish then we have a system with $h+2 \le (k+1)/2$ unknowns and k linearly independent equations, hence $P_{h+1} \equiv 0$. Otherwise suppose that there exists l such that
$$\sin\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) = 0.$$
Then we saw in Step 1 that
$$\cos\Big(\tilde{\vartheta} + \frac{2\pi l}{k} + \varepsilon_l\Big) \ne 0 \quad \text{and} \quad \sin\Big(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\Big) \ne 0$$
for every integer $l' \in [0, k-1]$ different from l. By rewriting P_{h+1} in the form
$$P_{h+1}(\vartheta) = \sin\vartheta\, P_h(\vartheta) + \beta_{h+1}\cos^{h+1}\vartheta,$$
with P_h as in (6-24), we deduce both that $\beta_{h+1} = 0$ and that
$$P_h\Big(\tilde{\vartheta} + \frac{2\pi l'}{k} + \varepsilon_{l'}\Big) = 0$$
for every $l' \in [0, k-1]$ different from l. These are $k-1$ conditions for a polynomial of order $h \le (k-3)/2$, so the induction hypothesis implies $P_h \equiv 0$ and in turn $P_{h+1} \equiv 0$. □

End of the proof of Theorem 1.7. Take any $a \in \Omega$ sufficiently close to 0, then by Lemma 6.4
$$\lambda^a - \lambda^0 = \sum_{h=1}^{H} |a|^h P_h(\vartheta(a)) + o(|a|^H).$$
By combining Lemmas 6.5 and 6.6 we obtain that $P_h \equiv 0$ for every $h \in [1, (k-1)/2]$, therefore $|\lambda^a - \lambda^0| \le C|a|^{(k+1)/2}$ for some constant C independent of a. □

7. Numerical illustration

Let us now illustrate some results of this paper using the Finite Element Library [Martin 2010] with isoparametric $\mathbb{P}_6$ Lagrangian elements. We will restrict our attention to the case of half-integer circulation $\alpha = \frac{1}{2}$.

The numerical method we used here was presented in detail in [Bonnaillie-Noël and Helffer 2011]. Given a domain Ω and a point $a \in \Omega$, to compute the eigenvalues λ_j^a of the Aharonov–Bohm operator $(i\nabla + A_a)^2$ on Ω, we compute those of the Dirichlet Laplacian on the double covering $\Omega_a^{\mathcal{R}}$ of $\Omega \setminus \{a\}$, denoted by $\mu_j^{\mathcal{R}}$. This spectrum of the Laplacian on $\Omega_a^{\mathcal{R}}$ is decomposed in two disjoint parts:

- the spectrum of the Dirichlet Laplacian on Ω, λ_j,
- the spectrum of the magnetic Schrödinger operator $(i\nabla + A_a)^2$, λ_j^a.

Thus we have

$$\{\mu_j^{\mathcal{R}}\}_{j\geq 1} = \{\lambda_j^a\}_{j\geq 1} \sqcup \{\lambda_j\}_{j\geq 1}.$$

Therefore by computing the spectrum of the Dirichlet Laplacian on Ω and, for every $a \in \Omega$, that on the double covering $\Omega_a^{\mathcal{R}}$, we deduce the spectrum of the Aharonov–Bohm operator $(i\nabla + A_a)^2$ on Ω. This method avoids dealing with the singularity of the magnetic potential and furthermore allows us to work with real-valued functions. We have only to compute the spectrum of the Dirichlet Laplacian, which is quite standard. The only effort to be done is to mesh a double covering domain.

Let us now present the computations for the angular sector of aperture $\pi/4$:

$$\Sigma_{\pi/4} = \Big\{(x_1, x_2) \in \mathbb{R}^2, x_1 > 0, |x_2| < x_1 \tan\frac{\pi}{8}, x_1^2 + x_2^2 < 1\Big\}.$$

An analysis of the spectral minimal partitions of angular sectors can be found in [Bonnaillie-Noël and Léna 2014]. By symmetry, it is enough to compute the spectrum for a in the half-domain. We take a discretization grid of step $1/N$ with $N = 100$ or $N = 1000$:

$$a \in \Pi_N := \Big\{\Big(\frac{m}{N}, \frac{n}{N}\Big), 0 < m < N, 0 < \frac{|n|}{m} < \tan\frac{\pi}{8}, \frac{m^2+n^2}{N^2} < 1\Big\}.$$

Figure 2 gives the first nine eigenvalues λ_j^a for $a \in \Pi_{100}$. In these figures, the angular sector is represented by a dark thick line. Outside the angular sector are represented the eigenvalues λ_j of the Dirichlet Laplacian on $\Sigma_{\pi/4}$ (which do not depend on a). We observe the convergence proved in Theorem 1.1:

$$\text{for all } j \geq 1, \quad \lambda_j^a \to \lambda_j \quad \text{as} \quad a \to \partial\Sigma_{\pi/4}.$$

In Figure 3, we provide the three-dimensional representation of the first two parts of Figure 2.

Let us now deal more accurately with the singular points on the symmetry axis. Numerically, we take a discretization step equal to $\frac{1}{1000}$ and consider $a \in \big\{\big(\frac{m}{1000}, 0\big), 1 \leq m \leq 1000\big\}$. Figure 4 gives the first nine eigenvalues of the Aharonov–Bohm operator $(i\nabla + A_a)^2$ in $\Sigma_{\pi/4}$. Here we can identify the points a belonging to the symmetry axis such that λ_j^a is not simple. If we look for example at the first and second eigenvalues, we see that they are not simple respectively for one and three values of a on the symmetry axis. At such values, the function $a \mapsto \lambda_j^a$, $j = 1, 2$, is not differentiable, as can be seen in Figure 3.

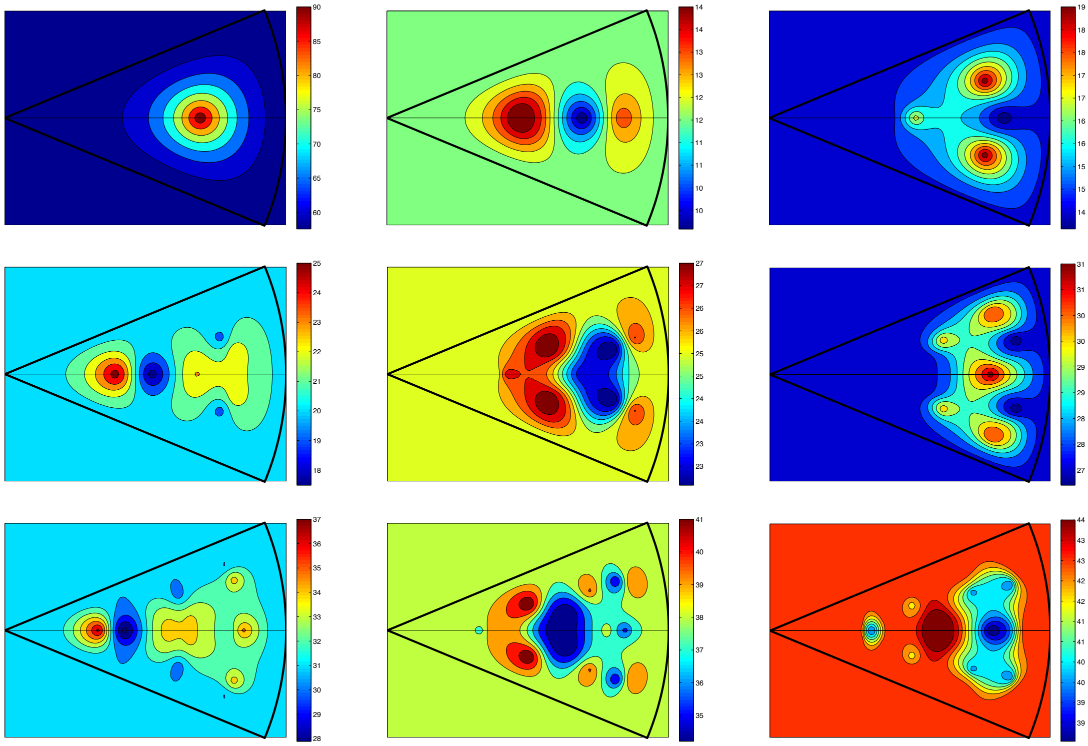


Figure 2. First nine eigenvalues of $(i\nabla + A_a)^2$ in $\Sigma_{\pi/4}$, $a \in \Pi_{100}$. Each graph depicts the level curves of $a \mapsto \lambda_j^a$, for $j = 1, 2, 3$ (top), $j = 4, 5, 6$ (middle) and $j = 7, 8, 9$ (bottom).

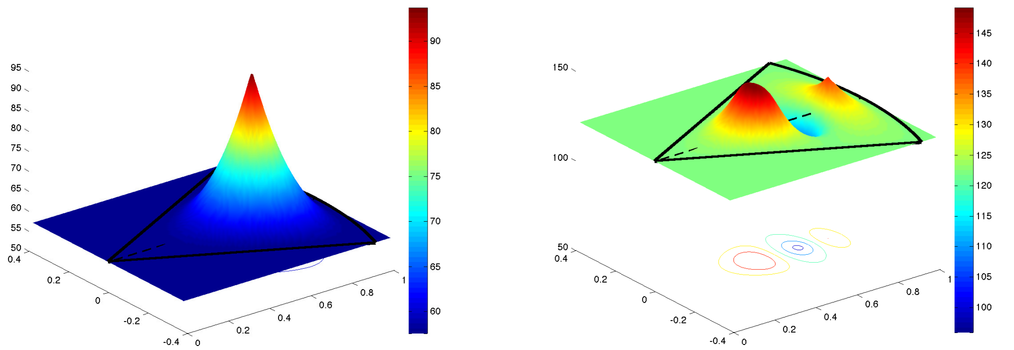


Figure 3. Three-dimensional representation of the first two panels of Figure 2: $a \mapsto \lambda_1^a$ (left) and $a \mapsto \lambda_2^a$ (right), $a \in \Pi_{100}$.

Figure 3 illustrates Theorem 1.3 for a domain with a piecewise-C^∞ boundary: we see that the function $a \mapsto \lambda_j^a$, $j = 1, 2$, is regular except at the points where the eigenvalue λ_j^a is not simple.

Going back to Figure 4, we see that the only critical points of λ_j^a which correspond to simple eigenvalues are inflexion points. As an example, we have analyzed the inflexion points for λ_3^a, λ_4^a, λ_5^a when $a = (a_1, 0)$

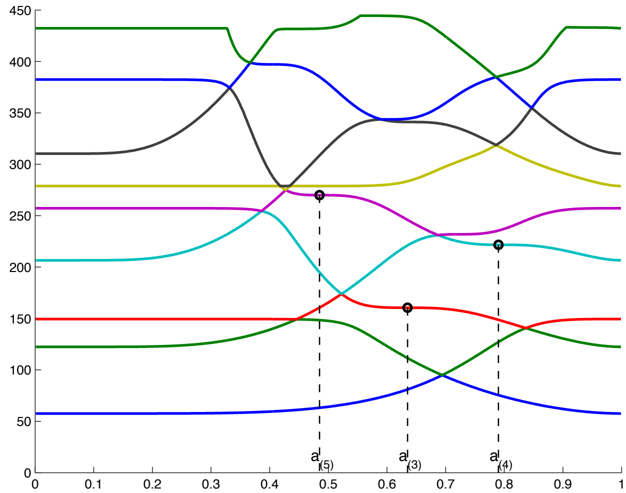


Figure 4. $a \mapsto \lambda_j^a$, $a \in \left\{\left(\frac{m}{1000}, 0\right), 0 < m < 1000\right\}$, $1 \le j \le 9$.

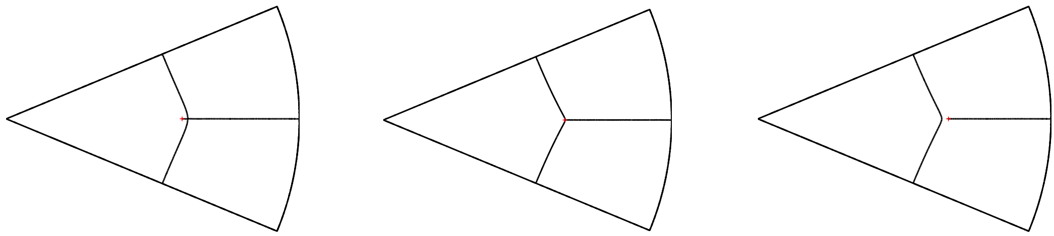


Figure 5. Nodal lines of an eigenfunction associated with λ_3^a, $a = (a_1, 0)$, $a_1 = 0.6, 0.63, 0.65$.

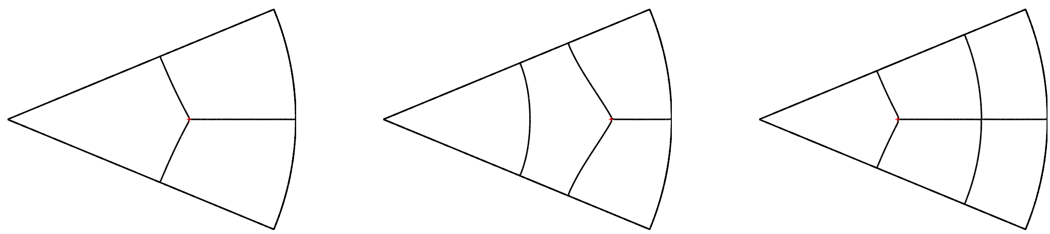
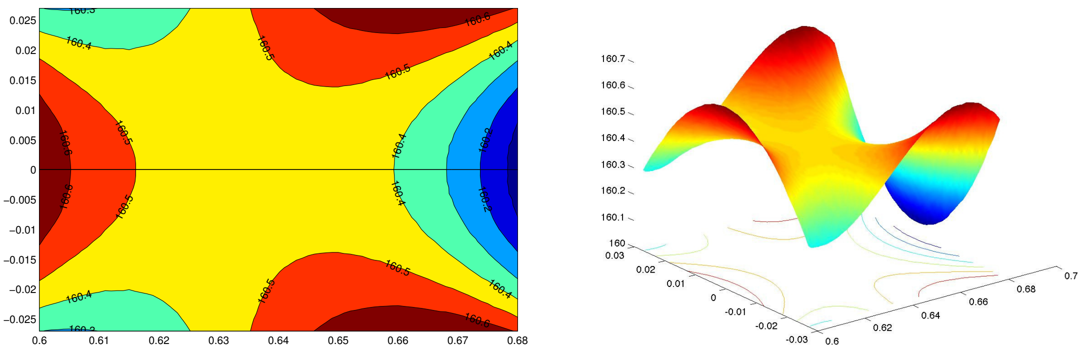
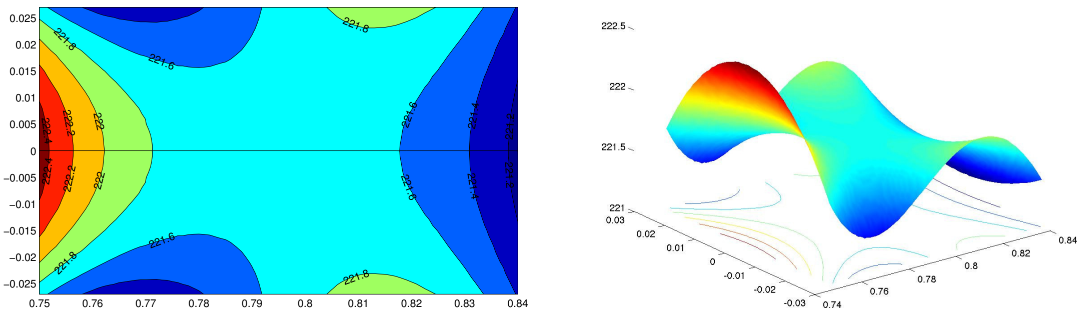


Figure 6. Nodal lines of an eigenfunction associated with $\lambda_j^{a(j)}$, $j = 3, 4, 5$.

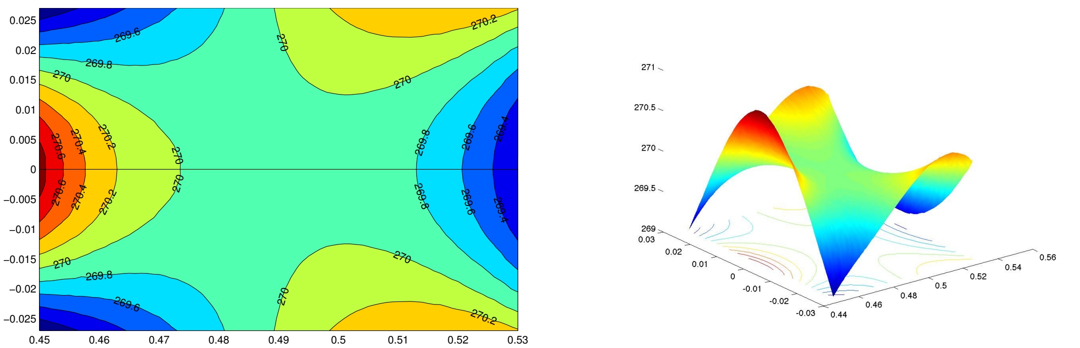
with $a_1 \in (0.6, 0.7)$, $a_1 \in (0.75, 0.85)$ and $a_1 \in (0.45, 0.55)$ respectively. We will denote these points by $a_{(j)}$, $j = 3, 4, 5$. Figure 5 gives the nodal lines for three different points $a = (a_1, 0)$ on the symmetry axis $y = 0$ with $a_1 = 0.6, 0.63$ and 0.65. This illustrates the emergence of a triple point when the pole is moved along the line $y = 0$. In Figure 6, we have plotted the nodal lines of the eigenfunctions $\varphi_j^{a(j)}$ associated with $\lambda_j^{a(j)}$, $j = 3, 4, 5$. We observe that each $\varphi_j^{a(j)}$ has a zero of order $\frac{3}{2}$ at $a_{(j)}$. Correspondingly, the derivative of λ_j^a at $a_{(j)}$ vanishes in Figure 4, thus illustrating Theorem 1.7. In the three examples proposed here, also the second derivative of λ_j^a vanishes at $a_{(j)}$.



$a \mapsto \lambda_3^a$, $a \in \left\{\left(\frac{m}{1000}, \frac{n}{1000}\right), 600 \leq m \leq 680, 0 \leq n \leq 30\right\}$.



$a \mapsto \lambda_4^a$, $a \in \left\{\left(\frac{m}{1000}, \frac{n}{1000}\right), 750 \leq m \leq 840, 0 \leq n \leq 30\right\}$.



$a \mapsto \lambda_5^a$, $a \in \left\{\left(\frac{m}{1000}, \frac{n}{1000}\right), 450 \leq m \leq 530, 0 \leq n \leq 30\right\}$.

Figure 7. λ_j^a vs. a for a around the inflexion point $a_{(j)}$, $j = 3, 4, 5$.

Let us now move a little the singular point around $a_{(j)}$. We use a discretization step of $\frac{1}{1000}$. Figure 7 represents the behavior of λ_j^a for a close to $a_{(j)}$. It indicates that these points are degenerated saddle points. The behavior of the function $a \mapsto \lambda_j^a$, $j = 3, 4, 5$, around $a_{(j)}$ is quite similar to that of the function $(t, x) \mapsto t(t^2 - x^2)$ around the origin $(0, 0)$.

We remark that computing the first twelve eigenvalues of $(i\nabla + A_a)^2$ on $\Sigma_{\pi/4}$, we have never found an eigenfunction for which five or more nodal lines end at a singular point a.

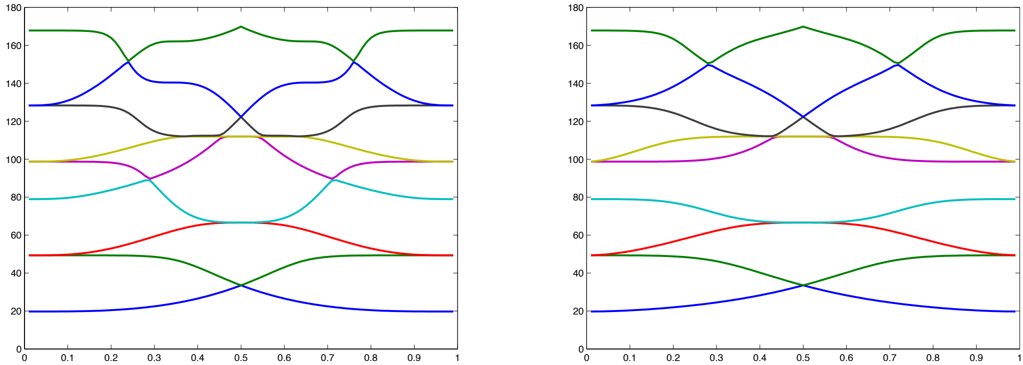


Figure 8. $a \mapsto \lambda_j^a$ for a along the diagonal (left) or perpendicular bisector (right) of a square ($1 \le j \le 9$).

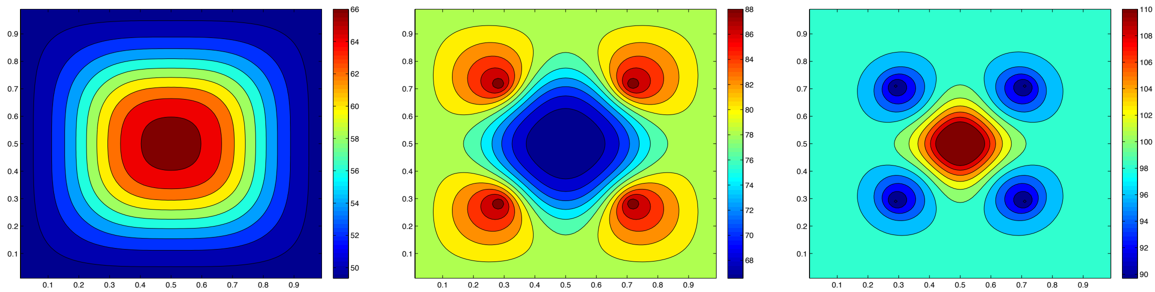


Figure 9. Eigenvalues of $(i\nabla + A_a)^2$ in $[0, 1] \times [0, 1]$, $a \in \Pi_{50}$.

Figure 10. Nodal lines of an eigenfunction associated with λ_j^a, $j = 3, 4$, $a = \left(\frac{1}{2}, \frac{1}{2}\right)$.

As we have already remarked, all the local maxima and minima of λ_j^a in Figure 4 correspond to nonsimple eigenvalues. Plotting the nodal lines of the corresponding eigenfunctions, we have found that they all have a zero of order $\frac{1}{2}$ at a, i.e., one nodal line ending at a. Nonetheless, this is not a general fact: in performing the same analysis in the case Ω is a square $[0, 1] \times [0, 1]$, we have found that the third and fourth eigenfunctions have a zero of order $\frac{3}{2}$ at the center $a = \left(\frac{1}{2}, \frac{1}{2}\right)$, see Figure 10, which is in this case a maximum of $a \mapsto \lambda_3^a$ and a minimum of $a \mapsto \lambda_4^a$; see Figures 8, 9. We observe in Figure 8 that the first and second derivatives of λ_3^a and of λ_4^a seem to vanish at the center $a = \left(\frac{1}{2}, \frac{1}{2}\right)$.

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ON MULTIPLICITY BOUNDS FOR SCHRÖDINGER EIGENVALUES ON RIEMANNIAN SURFACES

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A classical result by Cheng in 1976, improved later by Besson and Nadirashvili, says that the multiplicities of the eigenvalues of the Schrödinger operator $(-\Delta_g + \nu)$, where ν is C^∞-smooth, on a compact Riemannian surface M are bounded in terms of the eigenvalue index and the genus of M. We prove that these multiplicity bounds hold for an L^p-potential ν, where $p > 1$. We also discuss similar multiplicity bounds for Laplace eigenvalues on singular Riemannian surfaces.

1. Introduction and statements of results

Multiplicity bounds. Let M be a connected compact surface. For a Riemannian metric g and C^∞-smooth function ν on M we denote by

$$\lambda_0(g,\nu) < \lambda_1(g,\nu) \leqslant \cdots \leqslant \lambda_k(g,\nu) \leqslant \cdots$$

the eigenvalues of the Schrödinger operator $(-\Delta_g + \nu)$. If M has a nonempty boundary, we assume that the Dirichlet boundary condition is imposed.

The following theorem is an improved version of the statement originally discovered by Cheng [1976]. It is due to Besson [1980] for closed orientable surfaces and to Nadirashvili [1987] for general closed surfaces; multiplicity bounds for general boundary value problems were obtained in [Karpukhin et al. 2013].

Theorem 1.1. *Let (M, g) be a smooth compact surface, possibly with boundary. Then, for any C^∞-smooth function ν on M, the multiplicity $m_k(g,\nu)$ of an eigenvalue $\lambda_k(g,\nu)$ satisfies the inequality*

$$m_k(g,\nu) \leqslant 2(2-\chi-l) + 2k + 1, \quad k = 1, 2, \ldots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

Above, we assume that $l = 0$ for closed surfaces. Note that even the fact that eigenvalue multiplicities on Riemannian surfaces are bounded is by no means trivial, and as is known [Colin de Verdière 1986; 1987], fails in higher dimensions, unless some specific hypotheses on a Riemannian metric or a potential are imposed. The key ingredient in the proof of Theorem 1.1 is the so-called *Cheng's structure theorem* [Cheng 1976]: for any solution u to the Schrödinger equation with a smooth potential and any interior point $p \in M$ there exists a neighbourhood of p and its diffeomorphism onto a ball in $\mathbb{R}^2$ centred at the origin that maps the nodal set of u onto the nodal set of a homogeneous harmonic polynomial. This statement is

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based on a local approximation of solutions by harmonic homogeneous polynomials [Bers 1955] and, in particular, implies that the nodal set of a solution u is locally homeomorphic to its tangent cone. The latter property of nodal sets does not hold in higher dimensions [Bérard and Meyer 1982]. The structure theorem holds for sufficiently smooth solutions to the Schrödinger equation — see the Appendix — and consequently the multiplicity bounds in Theorem 1.1 hold for Hölder continuous potentials. Based on Cheng's structure theorem, the multiplicity bounds for various eigenvalues problems have been extensively studied in the literature. We refer to the papers [Colin de Verdière 1987; Hoffmann-Ostenhof et al. 1999a; Hoffmann-Ostenhof et al. 1999b; Karpukhin et al. 2013] and references there for the details.

The purpose of this paper is to show that the multiplicity bounds continue to hold for rather weak potentials when no similar structure theorem for nodal sets is available. For a given real number $\delta \in (0,2)$, we consider the class $K^{2,\delta}(M)$, introduced in [Aizenman and Simon 1982; Simon 1982], which is formed by absolutely integrable potentials v such that

$$\sup_{x\in M}\int_{B(x,r)}|x-y|^{-\delta}|v(y)|\,d\mathrm{Vol}_g(y)\to 0 \quad \text{as } r\to 0, \tag{1-1}$$

where the absolute value $|x-y|$ above denotes the distance between x and y in the background metric g. It is a straightforward consequence of the Hölder inequality that any L^p-integrable function with $p>1$ belongs to $K^{2,\delta}$ for some positive δ. However, unlike the traditional L^p-hypothesis, the potentials from $K^{2,\delta}(M)$ include certain physically important cases [Aizenman and Simon 1982; Simon 1982].

The hypothesis that $v\in K^{2,\delta}(M)$ implies that the measures $d\mu^\pm = v^\pm d\mathrm{Vol}_g$, where v^+ and v^- are the positive and negative parts of v, are δ-uniform:

$$\mu^\pm(B(x,r))\leqslant Cr^\delta \quad \text{for any } r>0 \text{ and } x\in M$$

and some constant C. By the results of Maz'ja [1985] (see also [Kokarev 2014]) for such measures $\mu^\pm$ the Sobolev space $W^{1,2}(M,\mathrm{Vol}_g)$ embeds compactly into $L_2(M,\mu^\pm)$. By standard perturbation theory [Kato 1976] (see also [Maz'ja 1985; Simon 1982]) we then conclude that the spectrum of the Schrödinger operator $(-\Delta_g+v)$ is discrete, bounded from below, and all eigenvalues have finite multiplicities. Our main result says that they satisfy the same multiplicity bounds.

Theorem 1.2. *Let (M,g) be a smooth compact surface, possibly with boundary. Then, for any absolutely integrable potential v from $K^{2,\delta}(M)$, where $\delta\in(0,2)$, the multiplicity $m_k(g,v)$ of an eigenvalue $\lambda_k(g,v)$ satisfies the inequality*

$$m_k(g,v)\leqslant 2(2-\chi-l)+2k+1, \quad k=1,2,\dots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

For the first eigenvalue $\lambda_1(g,v)$ the above multiplicity bound is sharp when M is homeomorphic to a sphere $\mathbb{S}^2$ or a projective plane $\mathbb{R}P^2$. When a potential v is smooth, there is an extensive literature [Colin de Verdière 1987; Nadirashvili 1987; Sévennec 2002] (and references therein) devoted to sharper multiplicity bounds for the first eigenvalue. In addition, in [Hoffmann-Ostenhof et al. 1999a; 1999b] the authors show that when M is a sphere or a disk the multiplicity bounds in Theorem 1.1 can be improved to

$m_k(g, \nu) \leqslant 2k-1$ for $k \geqslant 2$. We have made no effort to improve our results in these directions. However, it is worth mentioning that the main topological result in [Sévennec 2002] does yield a sharper multiplicity bound for $\lambda_1(g, \nu)$ for some closed surfaces when a potential v belongs to the space $K^{2,\delta}(M)$. More precisely, if M is a closed surface whose Euler–Poincaré number χ is negative, Theorem 5 of [Sévennec 2002] implies that $m_1(g, \nu) \leqslant 5-\chi$ for any potential $v \in K^{2,\delta}(M)$. By the results in [Colin de Verdière 1987] this bound is sharp for $\mathbb{T}^2 \# \mathbb{T}^2$ and $\# n\, \mathbb{R}P^2$, where $n = 3, 4, 5$.

The multiplicity bounds in Theorem 1.1 also hold for eigenvalue problems on singular Riemannian surfaces; we discuss them in detail in Section 5. The proof of Theorem 1.2 is based on the delicate study of the nodal sets of Schrödinger eigenfunctions that we describe below.

Nodal sets of eigenfunctions. Let u be a solution to the eigenvalue problem

$$(-\Delta_g + \nu)u = \lambda u \quad \text{on } M, \tag{1-2}$$

where $\nu \in K^{2,\delta}(M)$ and, if $\partial M \neq \varnothing$, the Dirichlet boundary hypothesis is assumed. Recall that by results in [Simon 1982] such an eigenfunction u is Hölder continuous. We denote by $\mathcal{N}(u)$ its nodal set $u^{-1}(0)$.

By the results in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995] combined with the strong unique continuation property [Sawyer 1984; Chanillo and Sawyer 1990], in appropriate local coordinates around an interior point $x_0 \in M$ a nontrivial solution u has the form

$$u(x) = P_N(x - x_0) + O(|x - x_0|^{N+\delta}) \quad \text{for } x \in U,$$

where P_N is a nontrivial homogeneous harmonic polynomial on the Euclidean plane. We refer to Section 2 for a precise statement. The degree of this approximating homogeneous harmonic polynomial defines the so-called *vanishing order* $\operatorname{ord}_x u$ for any interior point $x \in M$. Each point $x \in \mathcal{N}(u)$ has vanishing order at least one, and we define $\mathcal{N}^2(u)$ as the set of points x whose vanishing order $\operatorname{ord}_x u$ is at least two.

The proof of Theorem 1.2 is based on the following key result.

Theorem 1.3. *Let (M, g) be a compact Riemannian surface, possibly with boundary, and let u be a nontrivial eigenfunction for the Schrödinger eigenvalue problem (1-2) with $\nu \in K^{2,\delta}(M)$, where $\delta \in (0, 2)$. Then the set $\mathcal{N}^2(u)$ is finite, and the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components. Moreover, for any $x \in \mathcal{N}^2(u)$, the number of connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ incident to x is an even integer that is at least $2 \operatorname{ord}_x u$.*

The theorem says that the nodal set $\mathcal{N}(u)$ can be viewed as a graph: the vertices are points from $\mathcal{N}^2(u)$, and the edges are connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. This graph structure assigns to each $x \in \mathcal{N}^2(u)$ its degree $\deg x$, that is, the number of edges incident to x. If there is an edge that starts and ends at the same point, then it counts twice. The last statement of Theorem 1.3 says that $\deg x \geqslant 2 \operatorname{ord}_x u$ for any $x \in \mathcal{N}^2(u)$. When the potential ν is smooth, Theorem 1.3 is a direct consequence of Cheng's structure theorem and, in this case, the degree $\deg x$ is precisely $2 \operatorname{ord}_x u$.

The proof of Theorem 1.3 uses essentially Courant's nodal domain theorem, and is based on topological arguments, which are in turn built on the results in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995]. More precisely, one of the key ingredients is the study of prime ends of nodal domains, which leads to a construction of neighbourhoods of $x \in \mathcal{N}(u)$ where a solution also has a finite number of nodal domains. Our method uses the properties of solutions in the interior of M only; it largely disregards their behaviour at the boundary. Consequently, the main results (Theorems 1.2 and 1.3) hold for rather general boundary value problems as long as Courant's nodal domain theorem holds; cf. [Karpukhin et al. 2013, Section 6]. The statement of Theorem 1.3 continues to hold for general solutions to the Schrödinger equation $(-\Delta+V)u=0$ that have a finite number of nodal domains. Without the latter hypothesis for arbitrary L^p-potentials, it is unknown even whether the Hausdorff dimension of $\mathcal{N}^2(u)$ equals zero or not.

The paper is organised in the following way. In Section 2 we collect the background material on the strong unique continuation property, regularity of nodal sets, and recall the approximation results from [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992; Hoffmann-Ostenhof et al. 1995]. Here we also derive a number of consequences of these results that describe qualitative properties of nodal sets; they are used often in our sequel arguments. In the next section we recall the notion of Carathéodory's prime end and show that prime ends of nodal domains have the simplest possible structure: their impression always consists of a single point. In Section 4 we prove Theorems 1.2 and 1.3. In the last section we discuss multiplicity bounds for eigenvalue problems on surfaces with measures. We show that Laplace eigenvalue problems on singular Riemannian surfaces, such as Alexandrov surfaces of bounded integral curvature, can be viewed as particular instances of such problems. The paper also has an Appendix where we give details on Cheng's structure theorem for the reader's convenience.

2. Preliminaries

Background material. We start by collecting background material on solutions of the Schrödinger equation, which is used throughout the paper. From now on we assume that a potential V belongs to the space $K^{2,\delta}(M)$, where $\delta \in (0,1)$. The superscript 2 in the notation for this function space refers to the dimension of M. Note that the space $K^{2,\delta}(M)$ is contained in the so-called *Kato space* formed by absolutely integrable functions V such that

$$\sup_{x\in M}\int_{B(x,r)} \ln\Big(\frac{1}{|x-y|}\Big)|v(y)|\,d\mathrm{Vol}_g(y)\to 0 \quad \text{as } r\to 0;$$

see [Simon 1982]. Consider the Schrödinger equation

$$(-\Delta_g+V)u=0 \quad \text{on } M, \tag{2-1}$$

understood in the distributional sense. As was mentioned above, by the results in [Simon 1982] its solutions are Hölder continuous. They also enjoy the following *strong unique continuation property*.

Proposition 2.1. *Let (M,g) be a smooth connected compact Riemannian surface, possibly with boundary, and let $x_0\in M$ be an interior point. Let u be a nontrivial solution of the Schrödinger equation (2-1) with $V\in K^{2,\delta}(M)$, where $0<\delta<1$, such that*

$$u(x)=O(|x-x_0|^{\ell}) \quad \textit{for any } \ell>0.$$

Then u vanishes identically on M.

Proposition 2.1 is a consequence of the results in [Sawyer 1984], where the author proves that a solution u of the Schrödinger equation with the potential V from the Kato space $K^2(M)$ satisfies the unique continuation property: if u vanishes on a nonempty open subset, then it vanishes identically. As was pointed out in [Hoffmann-Ostenhof et al. 1995; Chanillo and Sawyer 1990], the argument in [Sawyer 1984] actually yields the strong unique continuation property.

The following fundamental statement is a combination of the main result in [Hoffmann-Ostenhof and Hoffmann-Ostenhof 1992] with Proposition 2.1.

Proposition 2.2. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. For any interior point $x_0 \in M$, there exist a coordinate chart U around x_0 and a nontrivial homogeneous harmonic polynomial P_N of degree $N \geqslant 0$ on the Euclidean plane such that*

$$u(x) = P_N(x - x_0) + O(|x - x_0|^{N+\delta'}), \quad \text{where } x \in U,$$

for any $0 < \delta' < \delta$.

The proposition says that for any point $x \in M$ there is a well-defined *vanishing order* $\operatorname{ord}_x u$ of a solution u at x, understood as the degree of the harmonic polynomial P_N. For a positive integer ℓ we define the set

$$\mathcal{N}^\ell(u) = \{x \in \operatorname{Int} M \mid \operatorname{ord}_x u \geqslant \ell\}.$$

Clearly, the nodal set $\mathcal{N}(u) = u^{-1}(0)$ is precisely the set $\mathcal{N}^1(u)$. Recall that a connected component of $M \setminus \mathcal{N}(u)$ is called a *nodal domain* of u. The combination of the Harnack inequality in [Aizenman and Simon 1982; Simon 1982] and the unique continuation property implies that a nontrivial solution u has different signs on adjacent nodal domains. Moreover, every point $x \in \mathcal{N}(u)$ belongs to the closure of at least two nodal domains.

Now suppose that u is an eigenfunction; that is, a solution to the eigenvalue problem (1-2). The following version of a classical statement is used in the sequel.

Courant's nodal domain theorem. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and $\nu \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then each nontrivial eigenfunction u corresponding to the eigenvalue $\lambda_k(g, \nu)$ of eigenvalue problem (1-2) has at most $k+1$ nodal domains.*

The proof follows standard arguments; see [Courant and Hilbert 1953]. It uses variational characterisation of eigenvalues $\lambda_k(g, \nu)$, the unique continuation property, Proposition 2.1, and the continuity of eigenfunctions up to the boundary. The latter can be deduced, for example, from the interior regularity [Simon 1982] by straightening the boundary locally and reflecting across it in an appropriate way.

Qualitative properties of nodal sets. Let u be a solution of the Schrödinger equation (2-1). If u is C^1-smooth, then the implicit function theorem implies that the complement

$$\mathcal{N}^1(u) \setminus \mathcal{N}^2(u) \tag{2-2}$$

is a collection of C^1-smooth arcs. The following celebrated nodal set regularity theorem due to [Hoffmann-Ostenhof et al. 1995] says that the latter holds under rather weak assumptions on a potential, when a solution u is not necessarily C^1-smooth.

Proposition 2.3. *Let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then any point x in the complement (2-2) has a neighbourhood $U \subset M$ such that the set $\mathcal{N}^1(u) \cap U$ is the graph of a $C^{1,\delta}$-smooth function with nonvanishing gradient. Further, if a potential V is $C^{k,\alpha}$-smooth, then such a point x has a neighbourhood U such that $\mathcal{N}^1(u) \cap U$ is the graph of a $C^{k+3,\alpha}$-smooth function with nonvanishing gradient.*

Below by *nodal edges* we mean the connected components of $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$. By Proposition 2.3 they are diffeomorphic to intervals of the real line, and their ends belong to the set $\mathcal{N}^2(u)$. We say that a nodal edge is *incident* to $x \in \mathcal{N}^2(u)$ if its closure contains x. A nodal edge is called a *nodal loop* if it is incident to one point $x \in \mathcal{N}^2(u)$ only. In other words, such a nodal edge starts and ends at the same point x.

The important consequence of Proposition 2.3 is the statement that nodal edges cannot accumulate to another nodal edge. We use this fact to describe a nodal set structure around an isolated point $x \in \mathcal{N}^2(u)$.

Corollary 2.4. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$. Then the number of nodal edges incident to x that are not nodal loops is finite. Moreover, any sequence of nodal loops incident to x has to contract to x.*

Proof. Let B be a neighbourhood of x whose closure does not contain any points in $\mathcal{N}^2(u)$. We view B as a unit ball in $\mathbb{R}^2$ centred at the origin $x = 0$. Suppose that there is an infinite number of nodal edges incident to x that are not nodal loops. Denote by Γ_i the connected components of the intersections of these nodal edges with the ball B whose closures $\bar{\Gamma}_i$ contain x. By Proposition 2.3, each $\bar{\Gamma}_i$ consists of a piece of a C^1-smooth nodal arc and the origin x. They form a sequence of compact subsets of $\bar{B}$, and hence contain a subsequence that converges to a compact subset $\bar{\Gamma}_0 \subset \bar{B}$ in the Hausdorff distance. Clearly, the subset $\bar{\Gamma}_0$ belongs to the nodal set $\mathcal{N}(u)$ and contains the origin $x = 0$. Since the subsets $\bar{\Gamma}_i$ contain points on the boundary ∂B, then so does $\bar{\Gamma}_0$; in particular, the limit subset $\bar{\Gamma}_0$ does not coincide with x. Since the origin x is the only higher-order nodal point in $\bar{B}$, then $\bar{\Gamma}_0 \setminus \{x\}$ is the union of pieces of C^1-smooth nodal edges. Without loss of generality, we may assume that the sequence $\bar{\Gamma}_i$ converges to a subset $\bar{\Gamma}_0$ such that $\bar{\Gamma}_0 \setminus \{x\}$ is a piece of a nodal edge. Now to get a contradiction we may either appeal to Proposition 2.3 directly, or argue in the following fashion. Let $x_i \in \bar{\Gamma}_i \cap \partial B$ be a sequence of points that converges to a point $x_0 \in \bar{\Gamma}_0 \cap \partial B$. We consider the two cases.

Case 1: *the complement $\bar{\Gamma}_0 \setminus \{x\}$ belongs to a nodal edge that intersects ∂B at x_0 transversally.* By Proposition 2.2, it is straightforward to see that the tangent line to Γ_0 at x_0 is precisely the kernel of an approximating linear function P_1 at x_0. Since Γ_0 intersects ∂B at x_0 transversally, we conclude that the sequence $P_1((x_i - x_0)/|x_i - x_0|)$ is bounded away from zero for all sufficiently large i. On the other hand, by Proposition 2.2 we obtain $P_1(x_i - x_0) = O(|x_i - x_0|^{1+\delta})$, and arrive at a contradiction.

Case 2: *the complement $\bar{\Gamma}_0 \setminus \{x\}$ belongs to a nodal edge that is tangent to ∂B at x_0.* Then there exists a sufficiently small ball B_0 centred at x_0 such that Γ_0 intersects ∂B_0 transversally. Choosing a sequence of points $x_i' \in \Gamma_i \cap \partial B_0$ that converges to a point $x_0' \in \Gamma_0 \cap B_0$, and arguing in a fashion similar to in Case 1, we again arrive at a contradiction.

Now we demonstrate the last statement of the lemma. Suppose that there is a sequence of nodal loops incident to x that do not contract to x. Choosing a subsequence and a sufficiently small neighbourhood B of x, we may assume that each nodal loop intersects with ∂B. Then the argument above shows that this sequence has to be finite. □

We proceed with another statement on local properties of the nodal set near an isolated point $x \in \mathcal{N}^2(u)$.

Corollary 2.5. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$. Then there exists a neighbourhood B of x, viewed as a ball in the Euclidean plane, such that the zeroes of u on ∂B are precisely the intersections of the connected components of $\mathcal{N}^1(u) \setminus \mathcal{N}^2(u)$ incident to x with ∂B.*

Proof. First, since x is isolated in $\mathcal{N}^2(u)$, one can choose a neighbourhood B such that it does not contain other points from $\mathcal{N}^2(u)$. Thus, for a proof of the lemma it is sufficient to show that the point x is not a limit point of the nodal edges that are not incident to x. This can be demonstrated following an argument similar to the one used in the proof of Corollary 2.4. □

Let $x \in \mathcal{N}^2(u)$ be a point isolated in $\mathcal{N}^2(u)$ such that the number of nodal edges incident to x is finite. The number of these nodal edges, where nodal loops are counted twice, is a characteristic of a point x, called the *degree* and denoted by $\deg x$. It is closely related to the vanishing order $\operatorname{ord}_x u$. More precisely, if a solution u is sufficiently smooth, then by Cheng's structure theorem [1976], it equals $2 \operatorname{ord}_x u$. The following lemma describes its relationship to $\operatorname{ord}_x u$ under rather weak regularity assumptions on u.

Lemma 2.6. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let $x \in \mathcal{N}^2(u)$ be an isolated point in $\mathcal{N}^2(u)$ such that the degree $\deg x$ is finite. Then $\deg x$ is an even integer that is at least $2 \operatorname{ord}_x u$.*

Proof. Denote by N the vanishing order $\operatorname{ord}_x u$, that is, the degree of an approximating homogeneous harmonic polynomial $P_N(y - x)$; see Proposition 2.2. Choose a sufficiently small neighbourhood B of x such that it does not contain other points from $\mathcal{N}^2(u)$ and does not contain nodal loops. We identify B with a unit ball in the Euclidean plane such that the point x corresponds to the origin. By $B_\lambda \subset B$ we mean a neighbourhood that corresponds to a ball of radius λ, where $0 < \lambda < 1$. Consider the rescaled function

$$u_\lambda(y) = \lambda^{-N} u(\lambda \cdot y)$$

defined on the unit circle $S = \{y : |y| = 1\}$. Proposition 2.2 implies that $u_\lambda(y)$ converges uniformly to the homogeneous harmonic polynomial $P_N(y)$ as $\lambda \to 0$, when y ranges over the unit circle S. As is known, $P_N(y)$ changes sign on S precisely $2N$ times, and hence the corresponding zeroes are stable under the perturbation of $P_N(y)$. Thus, we conclude that for all sufficiently small $\lambda > 0$ the zeroes of u_λ lie in small pairwise nonintersecting neighbourhoods $U_i \subset S$, where $i = 1, \ldots, 2N$, of the zeroes of $P_N(y)$, and each U_i contains at least one zero of u_λ. Choosing a sufficiently small $\lambda > 0$, by Corollary 2.5 we may assume that the zeroes of u_λ correspond to the intersections of nodal edges incident to x with ∂B_λ. Further, the intersections of the nodal edges incident to x with B_λ lie in the cones

$$C_i(\lambda) = \{t \cdot \lambda U_i : 0 < t < 1\}, \quad \text{where } i = 1, \dots, 2N.$$

Since the cones $C_i(\lambda)$ are pairwise nonintersecting and each of them contains at least one connected piece of a nodal edge incident to x, we conclude that $\deg x$ is at least $2N$.

Now we claim that each cone $C_i(\lambda)$ contains an odd number of nodal edge pieces incident to x, and hence the degree $\deg x$ is an even integer. Indeed, the solution u has different signs on the connected components of $B_\lambda \setminus \cup C_i(\lambda)$ adjacent to the same cone; they coincide with the signs of u_λ and the approximating homogeneous harmonic polynomial P_N. Since u also has different signs on adjacent nodal domains, each nodal edge piece incident to x contributes to the change of sign, and the claim follows in a straightforward fashion. □

Properties of the vanishing order. The proof of Proposition 2.3 is based on the following improvement of Proposition 2.2 due to [Hoffmann-Ostenhof et al. 1995], which is important for our considerations in the sequel. Below we denote by B a coordinate chart viewed as a ball in the Euclidean plane, and by $B_{1/2}$ the ball of half the radius of B.

Proposition 2.7. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Let B be a coordinate chart in the interior of M viewed as a ball in the Euclidean plane. Then for a sufficiently small B and any $\ell \geqslant 1$ there exists a constant $C > 0$ such that for any point $y \in \mathcal{N}^\ell(u) \cap B_{1/2}$ there exists a degree ℓ homogeneous harmonic polynomial P_ℓ^y such that*

$$|u(x) - P_\ell^y(x-y)| \leqslant C\big(\sup_B |u|\big)|x-y|^{\ell+\delta} \quad \text{for any } x \in B,$$

and the polynomials P_ℓ^y satisfy $|P_\ell^y(\bar{x})| \leqslant C_*(\sup_B |u|)$ for any $|\bar{x}| = 1$, where the constants C and C_* do not depend on a solution u.

Note that the harmonic polynomials P_ℓ^y above either vanish identically or coincide with approximating harmonic polynomials at y from Proposition 2.2. The main estimate of Proposition 2.7 is stated in [Hoffmann-Ostenhof et al. 1995, Theorem 1]. The bound for the values of the harmonic polynomials on the unit circle follows from the proof, and is explained explicitly in [Hoffmann-Ostenhof et al. 1995, p. 1256].

We proceed with studying the vanishing order $\operatorname{ord}_x u$ as a function of $x \in M$. The following lemma is a straightforward consequence of Proposition 2.7. We include a proof for completeness of exposition.

Lemma 2.8. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then the function $\operatorname{ord}_x u$ is upper-semicontinuous in the interior of M; that is, for any sequence x_i converging to an interior point $x \in M$, one has the inequality $\limsup \operatorname{ord}_{x_i}(u) \leqslant \operatorname{ord}_x u$.*

Proof. It is sufficient to show that if x_i belong to $\mathcal{N}^\ell(u)$, then so does the limit point x. Without loss of generality, we may assume that the points x_i lie in a coordinate chart B that is identified with a unit ball in $\mathbb{R}^2$ centred at the origin $x = 0$, and $x_i \to 0$ as $i \to +\infty$. In addition, to simplify the notation, we assume that $\sup|u|$ on B equals 1. Let P^i_ℓ be a degree ℓ homogeneous harmonic polynomial corresponding to x_i from Proposition 2.7. Representing u as the sum of $u - P^i_\ell$ and P^i_ℓ, we obtain

$$|u(x)| \leqslant |u(x) - P^i_\ell(x - x_i)| + |P^i_\ell(x - x_i)| \leqslant C|x - x_i|^{\ell+\delta} + C_*|x - x_i|^\ell \quad \text{for any } x \in B,$$

where the second inequality for all sufficiently large i follows from Proposition 2.7. Passing to the limit as $i \to +\infty$, we get

$$|u(x)| \leqslant C'|x|^\ell \quad \text{for any } x \in B,$$

and conclude that the vanishing order at the origin is at least ℓ. □

Our last lemma says that the vanishing order $\operatorname{ord}_x u$ is strictly upper-semicontinuous on $\mathcal{N}^2(u)$.

Lemma 2.9. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then for any sequence $x_i \in \mathcal{N}^2(u)$ converging to an interior point $x \in M$ we have* $\limsup \operatorname{ord}_{x_i}(u) < \operatorname{ord}_x u$.

Proof. As in the proof of Lemma 2.8, we assume that the points x_i belong to a coordinate chart B, viewed as a unit ball in $\mathbb{R}^2$ centred at the origin $x = 0$, and $x_i \to 0$ as $i \to +\infty$. We also suppose that $\sup|u|$ on B equals 1. First, by Lemma 2.8 we conclude that the upper limit $\limsup \operatorname{ord}_{x_i}(u)$ is finite; we denote it by N. After a selection of a subsequence, we may assume that the vanishing order $\operatorname{ord}_{x_i}(u)$ equals N for each x_i. By Lemma 2.8 it remains to show that the vanishing order $\operatorname{ord}_x u$ at the origin x cannot be equal to N.

Suppose the contrary: the order of u at the origin equals $N \geqslant 2$. Let P_N be an approximating homogeneous harmonic polynomial for u at the origin. By Proposition 2.7, for a sufficiently large index i we have

$$\begin{aligned} |P_N(x) - P^i_N(x - x_i)| &\leqslant |u(x) - P_N(x)| + |u(x) - P^i_N(x - x_i)| \\ &\leqslant C(|x|^{N+\delta} + |x - x_i|^{N+\delta}) \quad \text{for any } x \in B, \end{aligned} \tag{2-3}$$

where P^i_N is an approximating homogeneous harmonic polynomial at x_i. Denote by λ_i the absolute value $|x_i|$, and by $\bar{x}_i$ the point $\lambda_i^{-1}x_i$ on the unit circle. Setting $x = \lambda_i\bar{x}$ in inequality (2-3) and using the homogeneity of the left-hand side, we obtain

$$|P_N(\bar{x}) - P^i_N(\bar{x} - \bar{x}_i)| \leqslant (1 + 2^{N+\delta})C\lambda_i^\delta \quad \text{for any } |\bar{x}| = 1. \tag{2-4}$$

Without loss of generality, we may assume that the sequence $\bar{x}_i$ converges to a point $\bar{x}_0$, $|\bar{x}_0| = 1$. Setting $\bar{x}$ to be equal to $\bar{x}_i$ in inequality (2-4) and passing to the limit as $i \to +\infty$, we see that $\bar{x}_0$ is a zero of P_N. Recall that the nodal set of P_N consists of n straight lines passing through the origin; the vanishing order of the origin equals N, and any other nodal point, such as $\bar{x}_0$, has vanishing order 1. On the other hand, by Proposition 2.7 the polynomials P^i_N are uniformly bounded on the unit circle and, since in polar coordinates they have the form

$$a_i r^N \cos(N\theta) + b_i r^N \sin(N\theta),$$

we conclude that, after a selection of a subsequence, they converge either to zero or to a harmonic homogeneous polynomial P_N^0 of degree N. If the former case occurs, then after passing to the limit in inequality (2-4) we see that $P_N(x)$ vanishes, and arrive at a contradiction. Now assume that the harmonic polynomials P_N^i converge to a nontrivial harmonic polynomial P_N^0. Then the polynomials $P_N^i(\bar{x}-\bar{x}_i)$ converge uniformly to $P_N^0(\bar{x}-\bar{x}_0)$ and, passing to the limit in inequality (2-4), we conclude that $P_N(\bar{x})$ coincides identically with $P_N^0(\bar{x}-\bar{x}_0)$. Now, since $N\geqslant 2$, it is straightforward to arrive at a contradiction. The polynomial $P_N(\bar{x})$ has precisely $2N$ zeroes as $\bar{x}$ ranges over the unit circle, while the polynomial $P_N^0(\bar{x}-\bar{x}_0)$ has at most $N+1$. □

Corollary 2.10. *Let (M,g) be a smooth compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution of the Schrödinger equation (2-1) with $V\in K^{2,\delta}(M)$, where $0<\delta<1$. Then the set $\mathcal{N}^2(u)$ is totally disconnected: every nonempty connected subset is a single point. Moreover, the complement $\mathcal{N}(u)\setminus\mathcal{N}^2(u)$ is open and dense in the nodal set.*

Proof. Suppose the contrary to the first statement. Then there exists a nonempty connected subset $C\subset\mathcal{N}^2(u)$ that is not a single point. Since any point $x\in C$ is the limit of a nontrivial sequence in C, by Lemma 2.9 we conclude that $C\subset\mathcal{N}^\ell(u)$ for any $\ell\geqslant 2$. Hence, the solution u vanishes to an infinite order at C and, by the strong unique continuation, Proposition 2.1, vanishes identically. This contradiction demonstrates the first statement.

By Lemma 2.8 the set $\mathcal{N}^2(u)$ is closed, and for a proof of the second statement of the corollary it remains to show that the complement $\mathcal{N}(u)\setminus\mathcal{N}^2(u)$ is dense. Suppose the contrary. Then for some point $p\in\mathcal{N}(u)$ there exists a ball $B_\varepsilon(p)$ such that $C=B_\varepsilon(p)\cap\mathcal{N}(u)$ is contained in $\mathcal{N}^2(u)$. By the Harnack inequality [Aizenman and Simon 1982; Simon 1982] no point in the nodal set can be isolated, and we conclude that any $x\in C$ is the limit of a nontrivial sequence in C. Now we arrive at a contradiction in a fashion similar to the one above. □

3. Prime ends of nodal domains

Now we study the nodal set $\mathcal{N}(u)$ from the point of view of the topology of nodal domains. More precisely, we describe the structure of prime ends of nodal domains. The notion of prime end goes back to Carathéodory [1913], who used it to describe the behaviour of conformal maps on the boundaries of simply connected domains. Later his theory was extended to general open subsets in manifolds [Epstein 1981]. However, main applications seem to be restricted to 2-dimensional problems [Milnor 2006]. We start by recalling the necessary definitions, following [Epstein 1981] closely.

Let $\Omega\subset M$ be a connected open subset, where we view M as the interior of a compact Riemannian surface. For a subdomain $D\subset\Omega$, we denote by ∂D the interior boundary:

$$\partial D=\Omega\cap\bar{D}\cap(\overline{\Omega\setminus D}).$$

Definition 3.1. A *chain* in Ω is a sequence $\{D_i\}$, $i=1,2,\dots$, of open connected subsets of Ω such that

- ∂D_i is connected and nonempty for each i, and
- $\bar{D}_{i+1}\cap\Omega\subset D_i$ for each i.

Two chains $\{D_i\}$ and $\{D_i'\}$ are called *equivalent* if for any i there exists $j > i$ such that $D_j' \subset D_i$ and $D_j \subset D_i'$.

Definition 3.2. A chain in Ω is called a *topological chain* if there exists a point $p \in M$ such that

- the diameter of $(p \cup \partial D_i)$ tends to zero as $i \to +\infty$, and
- $\operatorname{dist}(p, \partial D_i) > 0$ for each i.

The point p above is called the *principal point* of $\{D_i\}$. A *prime point* of Ω is the equivalence class of a topological chain.

Clearly, for a given topological chain the principal point $p \in \bar{\Omega}$ is unique. Note also that the above definitions do not depend on a metric on M. The set of all prime points of Ω is denoted by $\widehat{\Omega}$. It is made into a topological space by taking the sets $\widehat{U}$, formed by prime points represented by chains $\{D_i\}$ such that each D_i lies in an open subset $U \subset \Omega$, as a topological basis. There is a natural embedding $\omega : \Omega \to \widehat{\Omega}$, defined by sending a point $x \in \Omega$ to the equivalence class of a sequence of concentric balls centred at x whose diameters tend to zero. As is shown in [Epstein 1981, Section 2], the map ω embeds Ω homeomorphically onto an open subset in $\widehat{\Omega}$. A *prime end* of Ω is a prime point which is not in $\omega(\Omega)$. A *principal point* of a prime end is any principal point of any representative topological chain.

Although a given topological chain has only one principal point, a prime end may have many. The simplest example is given by considering a domain whose boundary has an oscillating behaviour similar to the graph of $\sin(1/x)$. The collection of all principal points is a subset of the *impression* $\bigcap \bar{D}_i$ of a prime end. The latter does not depend on a representative topological chain, and is a compact connected subset of the boundary $\partial\Omega$. Note also that a given point $x \in \partial\Omega$ can be a principal point of many different prime ends. We refer to [Epstein 1981; Milnor 2006] for examples and other details.

The following statement, proved in [Epstein 1981, Section 6], shows that prime ends give a useful compactification (the so-called *Carathéodory compactification*) of open subdomains.

Proposition 3.1. *Let (M, g) be a Riemannian surface, viewed as the interior of a compact surface, and let $\Omega \subset M$ be a connected open subset such that the first homology group $H_1(\Omega, \mathbb{Q})$ is finite-dimensional. Then there is a homeomorphism of $\widehat{\Omega}$ onto a compact surface with boundary that maps the set of prime ends onto its boundary.*

We proceed with studying properties of nodal sets. The following lemma says that all prime ends of nodal domains have the simplest possible structure: any of them has only one principal point that coincides with its impression.

Lemma 3.2. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary. Let u be a nontrivial solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$, and let Ω be its nodal domain. Then for any prime end $[D_i]$ of Ω its impression $\bigcap \bar{D}_i$ consists of a single point. In particular, any prime end has only one principal point.*

Proof. First, the statement holds for any prime end that has a principal point x in the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Indeed, then the point x belongs to a nodal edge, which is the image of a C^1-smooth regular path; see Proposition 2.3. By the implicit function theorem we can view a small nodal arc containing x as a line segment in $\mathbb{R}^2$. Then it is straightforward to see that any chain that has x as a principal point is equivalent to a chain that consists of concentric semidisks centred at x whose diameters converge to zero. Its impression consists of the point x only.

Now suppose that a given prime end has a principal point $x \in \mathcal{N}^2(u)$. Then we claim that its impression I does not have any points in $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Suppose the contrary. Then, since the impression I of a prime end is connected, we conclude that I contains a nontrivial arc C that belongs to some nodal edge; that is, C is a connected subset of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ that is not a single point, and $\mathrm{dist}(x, C) > 0$. Let $\{D_i\}$ be a representative topological chain whose principal point is x, and let E_i be the set $\partial D_i \setminus I$, where ∂D_i is the boundary of D_i viewed as a subset of M. First, it is straightforward to see that for any $y \in C \subset I$ the distance $\mathrm{dist}(y, E_i)$ converges to zero as $i \to +\infty$. For otherwise there is a neighbourhood U of y in $\bar{D}_i$ such that $U \subset \bar{D}_i$ for any i. More precisely, viewing C around y as a straight segment in $\mathbb{R}^2$, we may choose U to be diffeomorphic to a semidisk $B^+_\varepsilon(y)$, assuming that $\mathrm{dist}(y, E_i) \geqslant 2\varepsilon$. Then we obtain the inclusions $U \subset I \subset \partial\Omega$, which are impossible. Thus, we see that any point $y \in C$ is the limit of a sequence $y_i \in \bar{E}_i$. Indeed, one can take as y_i a point at which the distance $\mathrm{dist}(y, E_i)$ is attained. This implies that there is a sequence $C_i \subset \bar{E}_i$ of subsets that converges to a nodal arc C in the Hausdorff distance. Clearly, the sets $E_i \setminus (\partial D_i \cap \Omega)$ lie in the nodal set $\mathcal{N}(u)$, and since the interior boundaries $\partial D_i \cap \Omega$ converge to the point x, we conclude that for a sufficiently large i the subset C_i lies in the nodal set. Further, since the set $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ is open in the nodal set (see Lemma 2.8), we see that each C_i lies in $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Thus, without loss of generality, we may assume that the C_i are arcs of nodal edges. Combining the latter with Proposition 2.3, or following the argument in the proof of Corollary 2.4, we arrive at a contradiction.

Thus, the impression I does not have points in the complement $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$, and is contained in $\mathcal{N}^2(u)$. By Corollary 2.10 the set $\mathcal{N}^2(u)$ is totally disconnected, and since the impression I is connected, it has to coincide with the point x. □

Corollary 3.3. *Under the hypotheses of Lemma 3.2, the following statements hold*:

(i) *Any point $x \in \partial\Omega$ is accessible*; *that is, it can be joined with any interior point in Ω by a continuous path $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and the image $\gamma(0, 1]$ lies in Ω.*

(ii) *For any point $x \in \partial\Omega$ and any sufficiently small neighbourhood U of x there are only finitely many connected components $U_1, \dots, U_k$ of $\Omega \cap U$ such that $x \in \overline{U}_i$, and the union $\bigcup \overline{U}_i$ is a neighbourhood of x in $\bar{\Omega}$.*

(iii) *The boundary $\partial\Omega$ is locally connected.*

Proof. We derive the statements using the results in [Epstein 1981], which apply to open domains $\Omega \subset M$ whose first homology group $H_1(\Omega, \mathbb{Q})$ is finite-dimensional. Note that all statements are local, and hold trivially for the boundary points $x \in \mathcal{N}(u) \setminus \mathcal{N}^2(u)$. To prove the corollary for the boundary points $x \in \mathcal{N}^2(u)$ we may assume, after cutting Ω along smooth simple closed paths, that Ω has zero genus. Moreover, after cutting along paths joining points from $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$ on different boundary components of Ω, we may assume that Ω is simply connected, and the results in [Epstein 1981] apply. Specifically, the first statement is a consequence of our Lemma 3.2 and Theorems 7.4 and 8.2 in that reference. The second statement follows from Lemma 3.2 and [Epstein 1981, Theorem 8.2], and the third from Lemma 3.2 and [Epstein 1981, Theorem 8.3]. □

4. The proofs

Proof of Theorem 1.3. Let (M, g) be a compact Riemannian surface, and u be a solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. First, we intend to generalise Theorem 1.3 to certain subdomains $\Omega \subset M$.

Definition 4.1. A connected open subset $\Omega \subset M$ is called a *proper subdomain* with respect to a solution u if its boundary consists of finitely many connected components and the solution u has finitely many nodal domains in Ω; that is, the number of connected components of $\Omega \setminus \mathcal{N}(u)$ is finite.

If u is an eigenfunction, then by Courant's nodal domain theorem the surface M itself is a proper subdomain with respect to u. However, for our method it is also important to consider proper subdomains whose closures are contained in the interior of M. The hypothesis on the finite number of boundary components guarantees that such a domain Ω has finite topology and, by Proposition 3.1, is homeomorphic to the interior of a compact surface with boundary. The second hypothesis in Definition 4.1 mimics an important property of eigenfunctions, and is essential for our arguments in the sequel. Below we denote by $\mathcal{N}_\Omega(u)$ and $\mathcal{N}^\ell_\Omega(u)$ the sets $\mathcal{N}(u) \cap \Omega$ and $\mathcal{N}^\ell(u) \cap \Omega$ respectively.

Theorem 1.3 is a consequence of the following more general result.

Theorem 4.1. *Let (M, g) be a compact Riemannian surface, possibly with boundary, and let u be a nontrivial solution to the Schrödinger equation (2-1) with a potential $V \in K^{2,\delta}(M)$, where $0 < \delta < 1$. Then for any proper subdomain $\Omega \subset M$ with respect to u the set $\mathcal{N}^2_\Omega(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components. Moreover, for any $x \in \mathcal{N}^2_\Omega(u)$ the number of connected components of $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ incident to x (if one connected component starts and ends at x, then it counts twice) is an even integer that is at least $2 \operatorname{ord}_x u$.*

The proof of Theorem 4.1 is based on the two lemmas below. The first lemma shows that proper neighbourhoods form a topological basis at any point $x \in \Omega$. Its proof relies on the topological consequences of our study of prime ends in Section 3.

Lemma 4.1. *Under the hypotheses of Theorem 4.1, for any point $x \in \mathcal{N}_\Omega(x)$ and any sufficiently small ball $B_\varepsilon(x)$ centred at x there exists a proper subdomain $U_\varepsilon(x)$ with respect to u such that $x \in U_\varepsilon(x) \subset B_\varepsilon(x)$.*

Proof. Let $x \in \mathcal{N}(u)$ be an interior nodal point in Ω, and $\Omega_1, \dots, \Omega_m$ be a collection of all nodal domains whose closure contains x. By Corollary 3.3 for any sufficiently small open ball $B_\varepsilon(x) \subset \Omega$ there are only finitely many connected components Ω_i^j, $j = 1, \dots, r_i$, of the intersection $B_\varepsilon(x) \cap \Omega_i$ whose closure contains x. Moreover, the union $F_i = \bigcup_j \overline{\Omega}_i^j$ is a neighbourhood of x in $\overline{\Omega}_i$. Thus, we conclude that the set $U_\varepsilon(x) = \operatorname{Int}(\bigcup F_i)$ contains x. Clearly, the connected components of the complement $U_\varepsilon(x) \setminus \mathcal{N}(u)$ are precisely the domains Ω_i^j, and it remains to show that $U_\varepsilon(x)$ has finitely many boundary components. Choosing $\varepsilon > 0$ such that the metric ball $B_\varepsilon(x)$ is homeomorphic to a ball in $\mathbb{R}^2$, it is straightforward to see that any boundary component of $U_\varepsilon(x)$ that lies in $B_\varepsilon(x)$ bounds a union of nodal domains. Since the number of nodal domains is finite, then choosing $\varepsilon > 0$ even smaller we conclude that $U_\varepsilon(x)$ is simply connected, and hence its boundary is connected. Thus, the neighbourhood $U_\varepsilon(x)$ is indeed a proper subdomain with respect to a solution u. □

The second lemma says that if the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points, then it is necessarily finite, and the nodal set has the structure of a finite graph with the vertex set $\mathcal{N}^2_\Omega(u)$.

Lemma 4.2. *Under the hypotheses of Theorem 4.1, suppose that the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points. Then the set $\mathcal{N}^2_\Omega(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components.*

The proof of the last lemma appears at the end of the section. Now we proceed with the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.2 for a proof of the theorem it is sufficient to show that the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points in Ω. The second statement of the theorem is a direct consequence of Lemma 2.6. First, we consider the case of proper subdomains $\Omega \subset M$ whose closures are contained in the interior of M, $\bar\Omega \subset M$. Given such a subdomain Ω, it is straightforward to see that the *maximal vanishing order* $\ell = \max\{\mathrm{ord}_x u\}$, where $x \in \Omega$, is finite. Indeed, otherwise there exists a point $p \in \bar\Omega$ that is the limit of points $x_i \in \Omega$ such that $\mathrm{ord}_{x_i}(u) \to +\infty$ as $i \to +\infty$. Then, by Lemma 2.8, the solution u vanishes to an infinite order at p, and the strong unique continuation, Proposition 2.1, implies that u vanishes identically.

Let $\Omega \subset M$ be a proper subdomain whose closure is contained in the interior of M. We prove that the set $\mathcal{N}^2_\Omega(u)$ is finite by induction on the maximal vanishing order ℓ. Clearly, the statement holds for all solutions u and proper subdomains Ω such that the maximal vanishing order equals 2. Indeed, in this case by Lemma 2.9 the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points and, by Lemma 4.2, is finite. Now we perform an induction step. Suppose that the set $\mathcal{N}^2_\Omega(u)$ is finite for all solutions u to the Schrödinger equation (2-1) on M and all proper subdomains Ω whose closure is contained in the interior of M that satisfy

$$\max\{\mathrm{ord}_x u : x \in \Omega\} \leqslant \ell - 1.$$

Now let u be a solution on M and Ω a proper subdomain such that the maximal vanishing order equals ℓ,

$$\max\{\mathrm{ord}_x u : x \in \Omega\} = \ell.$$

By Lemma 2.9 the set $\mathcal{N}^\ell_\Omega(u)$ consists of isolated points in Ω. Pick a point $p \in \mathcal{N}^2_\Omega(u)$. By Lemma 4.1, it has a neighbourhood U that is a proper subdomain such that $\bar U \subset \Omega$. Then the neighbourhood U may contain only finitely many points $p_1, p_2, \dots, p_m$ whose vanishing order equals ℓ. Since the domain $U_0 = U \setminus \{p_1, \dots, p_m\}$ is proper with respect to u, the induction hypothesis implies that the set $\mathcal{N}^2(u) \cap U_0$ is finite. Hence, so is the set $\mathcal{N}^2(u) \cap U$. Thus, we conclude that $\mathcal{N}^2_\Omega(u)$ consists of isolated points in Ω and, by Lemma 4.2, is finite.

The statement that the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points in Ω for an arbitrary proper subdomain $\Omega \subset M$ follows directly from the case considered above together with Lemma 4.1. □

Proof of Theorem 1.2. Now we show how Theorem 1.3 implies the multiplicity bounds. We give an argument following the strategy described in [Karpukhin et al. 2013, Section 6]. It relies on two lemmas that appear below. The first lemma gives a lower bound for the number of nodal domains via the vanishing order of points $x \in \mathcal{N}^2(u)$.

Lemma 4.3. *Under the hypotheses of Theorem 1.2, for any nontrivial eigenfunction u of an eigenvalue $\lambda_k(g,\nu)$ the number of its nodal domains is at least $\sum(\operatorname{ord}_x u - 1) + \chi + l$, where the sum is taken over all points in $\mathcal{N}^2(u)$ and χ and l stand for the Euler–Poincaré number and the number of boundary components of M respectively.*

Before giving a proof we introduce some notation that is useful in the sequel. First, by Theorem 4.1, the nodal set $\mathcal{N}(u)$ of any eigenfunction u on M can be viewed as a finite graph, called a *nodal graph*. Its vertices are points in $\mathcal{N}^2(u)$ and the edges are connected components of $\mathcal{N}(u) \setminus \mathcal{N}^2(u)$. Below we denote by $\overline{M}$ a closed surface, viewed as the image of M under collapsing its boundary components to points, and by $\bar{\chi}$ its Euler–Poincaré number. Let $\bar{\mathcal{N}}(u)$ be the corresponding image of a nodal graph $\mathcal{N}(u)$, called the *reduced nodal graph*. Its edges are the same nodal arcs, and there are two types of vertices: vertices that correspond to the boundary components that contain limit points of nodal lines, called *boundary component vertices*, and genuine vertices that correspond to the points in $\mathcal{N}^2(u)$, called *interior vertices*. By *faces* of the graph $\bar{\mathcal{N}}(u)$ we mean the connected components of the complement $\overline{M} \setminus \bar{\mathcal{N}}(u)$. Clearly, they can be identified with the nodal domains of an eigenfunction u.

Proof of Lemma 4.3. Let $\bar{\mathcal{N}}(u)$ be a reduced nodal graph in $\overline{M}$. By Theorem 4.1 it is a finite graph, and let v, e and f be the number of its vertices, edges and faces respectively. We also denote by r the number of boundary component vertices in $\bar{\mathcal{N}}(u)$. Recall that the number of edges satisfies the relation $2e = \sum \deg x$, where the sum is taken over all vertices. Since an eigenfunction u has different signs on adjacent nodal domains, the degree of each boundary component vertex is at least two, and we obtain

$$e \geqslant r + \tfrac{1}{2}\sum \deg x \geqslant r + \sum \operatorname{ord}_x u,$$

where the sum is taken over all interior vertices $x \in \mathcal{N}^2(u)$. The second inequality above follows from the relation $\deg x \geqslant 2\operatorname{ord}_x u$; see Theorem 4.1. Viewing the number of vertices v as the sum $r + \sum 1$, where the sum is again taken over $x \in \mathcal{N}^2(u)$, by the Euler inequality [Giblin 2010, p. 207] we have

$$f \geqslant e - v + \bar{\chi} \geqslant \sum(\operatorname{ord}_x u - 1) + \bar{\chi},$$

where $\bar{\chi} = \chi + l$ is the Euler–Poincaré number of $\overline{M}$. Since f is precisely the number of nodal domains, we are done. □

We proceed with the second lemma. In the case when the potential of a Schrödinger equation is smooth it is due to [Nadirashvili 1987]; see also [Karpukhin et al. 2013]. The proof relies essentially on Proposition 2.2.

Lemma 4.4. *Let (M,g) be a compact Riemannian surface, possibly with boundary, and let $u_1, \dots, u_{2n}$ be a collection of nontrivial linearly independent solutions to the Schrödinger equation (2-1) with a* *potential* $V \in K^{2,\delta}(M)$, *where* $0 < \delta < 1$. *Then for a given interior point* $x \in M$ *there exists a nontrivial linear combination* $u = \sum \alpha_i u_i$ *whose vanishing order* $\operatorname{ord}_x u$ *at the point* x *is at least* n.

Proof. Let V be a linear space spanned by the functions $u_1, \dots, u_{2n}$, and V_i its subspace formed by solutions $u \in V$ whose vanishing order at x is at least i, $\operatorname{ord}_x u \geqslant i$. Clearly, the subspaces V_i form a nested sequence, $V_{i+1} \subset V_i$. The lemma claims that V_n is nontrivial. Suppose the contrary, that the subspace V_n is trivial. Then, it is straightforward to see that the dimension of V satisfies the inequality

$$\dim V \leqslant 1 + \sum_{i=1}^{n-1} \dim(V_i / V_{i+1});$$

the equality occurs if the space V does not coincide with V_1. By Proposition 2.2, the factor space V_i / V_{i+1} can be identified with a subspace of homogeneous harmonic polynomials on $\mathbb{R}^2$ of degree i. When the degree i is at least 1, the space of such polynomials has dimension two, and we obtain

$$\dim V \leqslant 1 + 2(n-1) = 2n - 1.$$

Thus, we arrive at a contradiction with the hypotheses of the lemma. □

Now we finish the proof of Theorem 1.2. Suppose the contrary to its statement. Then there exist at least $2(2 - \chi - l) + 2k + 2$ linearly independent eigenfunctions corresponding to the eigenvalue $\lambda_k(\mu, g)$. Pick an interior point $x \in M$. By Lemma 4.4 there exists a new eigenfunction u whose vanishing order at the point x is at least $2 - \chi - l + k + 1$. Now Lemma 4.3 implies that the number of the nodal domains of u is at least $k + 2$. Thus, we arrive at a contradiction with Courant's nodal domains theorem. □

Proof of Lemma 4.2. Since the set $\mathcal{N}^2_\Omega(u)$ consists of isolated points, we can view the nodal set $\mathcal{N}_\Omega(u)$ as a graph: the vertices are points in $\mathcal{N}^2_\Omega(u)$, and the edges are connected components of $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2_\Omega(u)$. Recall that the *degree* $\deg x$ of a vertex $x \in \mathcal{N}^2_\Omega(u)$ is defined as the number of edges incident to x; if one edge starts and ends at x, then it counts twice. The following lemma says that the degree of each vertex has to be finite.

Lemma 4.5. *Under the hypotheses of Theorem 4.1, suppose that the set* $\mathcal{N}^2_\Omega(u)$ *consists of isolated points. Then the degree* $\deg x$ *of any point* $x \in \mathcal{N}^2_\Omega(u)$ *is finite.*

Proof. By Corollary 2.4 it is sufficient to show that the number of nodal loops that start and end at a given point $x \in \mathcal{N}^2_\Omega(u)$ is finite. Suppose the contrary, that the number of such nodal loops is infinite. Let $\overline{\Omega}$ be a compactification of Ω, obtained by adding one point for each boundary component. By Proposition 3.1 it is homeomorphic to a closed surface, and we denote by $\bar{\chi}$ its Euler–Poincaré number. Let Γ be a subgraph in the nodal graph formed by one vertex x and $m + 2 - \bar{\chi}$ nodal loops that start and end at x, where m is the number of nodal domains of u in Ω. Denote by $v = 1$, $e = m + 2 - \bar{\chi}$ and f the number of vertices, edges and faces of Γ respectively. Here by the faces of Γ we mean the connected components of $\overline{\Omega} \setminus \Gamma$. Clearly, they are unions of nodal domains, and $f \leqslant m$. On the other hand, viewing Γ as a graph in $\overline{\Omega}$, by Euler's inequality [Giblin 2010, p. 207], we obtain

$$f \geqslant e - v + \bar{\chi} = m + 1.$$

This contradiction demonstrates the lemma. □

Now we prove the statement of Lemma 4.2: the set $\mathcal{N}^2_\Omega(u)$ is finite, and the complement $\mathcal{N}_\Omega(u) \setminus \mathcal{N}^2(u)$ has finitely many connected components. The argument below is based on the results in Section 2, and is close in the spirit to the one in [Karpukhin et al. 2013, Section 3].

Let $\bar{\Omega}$ be a closed surface obtained by collapsing boundary components of Ω to points. By $\bar{\mathcal{N}}_\Omega$ we denote the *reduced nodal graph* in $\bar{\Omega}$, defined in the proof of Theorem 1.2. Recall that its edges are the same nodal edges, and there are two types of vertices: vertices that correspond to the boundary components of Ω that contain limit points of nodal lines, called *boundary component vertices*, and genuine vertices that correspond to the points in $\mathcal{N}^2_\Omega(u)$, called *interior vertices*. For a proof of the lemma it is sufficient to show that $\bar{\mathcal{N}}_\Omega(u)$ is a finite graph. Our strategy is to show that

(i) each boundary component vertex has a finite degree, and

(ii) the number of interior vertices is finite in Ω.

We are going to construct new graphs in $\bar{\Omega}$ by *resolving interior vertices* in the following fashion. Let $x \in \mathcal{N}^2_\Omega(u)$ be an interior vertex. By Lemma 4.5 its degree is finite, and by Lemma 2.6 it is an even integer $2n$. Let B be a small disk centred at x that does not contain other vertices. By Corollary 2.5 we may assume that nodal edges nonincident to x lie in the complement $\Omega \setminus B$. Moreover, since the degree is finite, we may also assume that each nodal loop incident to x intersects ∂B in at least two points. Consider the intersections of nodal edges with B, and let Γ_i, where $i = 0, \dots, 2n-1$, be their connected components incident to x. Pick points $y_i \in \bar{\Gamma}_i \cap \partial B$, one for each $i = 0, \dots, 2n-1$. By the resolution of a vertex x we mean a new graph obtained by removing sub-arcs between x and y_i in each nodal edge incident to x and rounding them off by nonintersecting arcs in B joining the points y_{2j} and y_{2j+1}. If there was an edge that starts and ends at x, then such a procedure may make it into a loop. We remove all such loops, if they occur. A new graph, obtained by the resolution of one vertex, has one vertex less and at most as many faces as the original graph.

Proof of (i). Suppose the contrary. Let us resolve all interior vertices in $\bar{\mathcal{N}}_\Omega(u)$ in the way described above. The result is a graph Γ whose only vertices are boundary component vertices in $\bar{\mathcal{N}}_\Omega(u)$; let v be their number. Moreover, it has at most as many faces as $\bar{\mathcal{N}}_\Omega(u)$ — that is, no more than the number of nodal domains. Since there is a boundary component vertex in $\bar{\mathcal{N}}_\Omega(u)$ whose degree is infinite, the same vertex has an infinite degree in Γ. Let us remove all edges in Γ except for at least $v + m + 1 - \bar{\chi}$ of them, where m is the number of nodal domains and $\bar{\chi}$ is the Euler–Poincaré number of $\bar{\Omega}$. The result is a finite graph; it has precisely v vertices, and we denote by e and f the number of its edges and faces respectively. By Euler's inequality, we obtain

$$f \geqslant e - v + \bar{\chi} = m + 1.$$

On the other hand, since removing an edge does not increase the number of faces, we have $f \leqslant m$. Thus, we arrive at a contradiction.

Proof of (ii). Suppose the contrary, and let v be the number of boundary component vertices in $\bar{\mathcal{N}}_\Omega(u)$. Let us resolve all interior vertices except for $v + m + 1 - \bar{\chi}$ of them. The result is a finite graph; we denote by v', e' and f' the number of its vertices, edges and faces respectively. Clearly, we have

$$v' \leqslant 2v + m + 1 - \bar{\chi} \quad \text{and} \quad e' \geqslant 2(v + m + 1 - \bar{\chi}),$$

where in the second inequality we used Lemma 2.6, saying that the degree of each vertex $x \in \mathcal{N}^2_\Omega(u)$ is at least 4. Combining these inequalities with the Euler inequality, we obtain

$$f' \geqslant e' - v' + \bar{\chi} \geqslant m + 1.$$

On the other hand, we have $f' \leqslant m$. Thus, we arrive at a contradiction. □

5. Eigenvalue problems on singular Riemannian surfaces

Eigenvalue problems on surfaces with measures. The purpose of this section is to discuss multiplicity bounds on singular Riemannian surfaces. We start with recalling a useful general setting of eigenvalue problems on surfaces with measures, following [Kokarev 2014].

Let (M, g) be a compact Riemannian surface, possibly with boundary, and let μ be a finite absolutely continuous (with respect to $d\mathrm{Vol}_g$) Radon measure on M that satisfies the *decay condition*

$$\mu(B(x,r)) \leqslant Cr^\delta, \quad \text{for any } r > 0 \text{ and } x \in M, \tag{5-1}$$

and some constants C and $\delta > 0$. Denote by $L^1_2(M, \mathrm{Vol}_g)$ the space formed by distributions whose derivatives are in $L_2(M, \mathrm{Vol}_g)$. Then by the results of Maz'ja [1985] (see also [Kokarev 2014]) the embedding

$$L_2(M, \mu) \cap L^1_2(M, \mathrm{Vol}_g) \subset L_2(M, \mu)$$

is compact, the Dirichlet form $\int |\nabla u|^2 \, d\mathrm{Vol}_g$ is closable in $L_2(M, \mu)$, and its spectrum is discrete. We denote by

$$\lambda_0(g, \mu) < \lambda_1(g, \mu) \leqslant \cdots \leqslant \lambda_k(g, \mu) \leqslant \cdots$$

the corresponding eigenvalues, and by $m_k(g, \mu)$ their multiplicities. As above, we always suppose that the Dirichlet boundary hypothesis is imposed if the boundary of M is nonempty. The eigenfunctions corresponding to an eigenvalue $\lambda_k(g, \mu)$ are distributional solutions to the Schrödinger equation

$$-\Delta_g u = \lambda_k(g, \mu)\mu u \quad \text{on } M. \tag{5-2}$$

The latter fact ensures that the analysis in Sections 2–4 carries over to yield the following result.

Theorem 5.1. *Let (M, g) be a smooth compact Riemannian surface, possibly with boundary, endowed with a finite absolutely continuous Radon measure μ that satisfies hypothesis (5-1). Then the multiplicity $m_k(g, \mu)$ of a Laplace eigenvalue $\lambda_k(g, \mu)$ satisfies the inequality*

$$m_k(g, \mu) \leqslant 2(2 - \chi - l) + 2k + 1 \quad \textit{for any } k = 1, 2, \ldots,$$

where χ stands for the Euler–Poincaré number of M and l is the number of boundary components.

Proof. First, we claim that the decay hypothesis (5-1) on the measure μ implies that its density belongs to the space $K^{2,\delta'}(M)$ for some $0<\delta'<\delta$. Indeed, by Fubini's theorem and the change of variable formula, we obtain

$$\begin{aligned}\int_{B(x,r)}|x-y|^{-\delta'}\,d\mu &= \int_{r^{-\delta'}}^{+\infty}\mu\{y:|x-y|^{-\delta'}\geqslant t\}\,dt=\int_{r^{-\delta'}}^{+\infty}\mu(B(x,t^{-1/\delta'}))\,dt\\ &=\delta'\int_0^r s^{-\delta'-1}\mu(B(x,s))\,ds\leqslant C\delta'\int_0^r s^{\delta-\delta'-1}\,ds.\end{aligned}$$

Second, using a variational characterisation of eigenvalues $\lambda_k(g,\mu)$, it is also straightforward to check that the standard proof of Courant's nodal domains theorem carries over for eigenfunctions u which satisfy (5-2). Hence, Theorem 4.1 applies, and then the argument in the proof of Theorem 1.2 carries over directly to yield the multiplicity bounds. □

Note that, since the Dirichlet energy is conformally invariant, if the measure μ is the volume measure of a metric h conformal to g then the quantities $\lambda_k(g,\mu)$ are precisely the Laplace eigenvalues of a metric h. More generally, the eigenvalue problems on surfaces with singular metrics can be also often viewed as particular instances of the setting of eigenvalues on measures. Below we discuss this point of view in more detail.

Let (M,g) be a Riemannian surface and h be a Riemannian metric of finite volume defined on the set $M\setminus S$, where S is a closed nowhere dense subset of zero measure. Here the set S plays the role of a singular set of h on M. Denote by μ the volume measure of the metric h. In the literature, e.g., [Cheeger 1983], the Dirichlet spectrum of a singular metric h is normally defined as the spectrum of the Dirichlet form

$$u\mapsto\int_{M\setminus S}|\nabla u|^2\,d\mathrm{Vol}_h \tag{5-3}$$

defined on the space $\mathscr{C}\subset L_2(M,\mu)$ of smooth compactly supported functions in $M\setminus S$. Suppose that the set S has zero Dirichlet capacity, the metric h is conformal on $M\setminus S$ to the metric g, and its volume measure μ satisfies the decay hypothesis (5-1). Then, it is straightforward to see that the spectrum of h is discrete and coincides with the set of eigenvalues $\lambda_k(g,\mu)$ defined above. Moreover, the construction makes sense even if a metric h is not smooth on $M\setminus S$ as long as the Dirichlet form (5-3) is well-defined. Theorem 5.1 gives multiplicity bounds for such eigenvalue problems. We end by discussing two examples: metrics with conical singularities and, more generally, Alexandrov surfaces of bounded integral curvature.

Example I: metrics with conical singularities. Let M be a closed smooth surface and h be a metric on M with a number of conical singularities. Recall that a point $p\in M$ is called a *conical singularity* of order $\alpha>-1$ (or angle $2\pi(\alpha+1)$) if in an appropriate local complex coordinate the metric h has the form $|z|^{2\alpha}\rho(z)|dz|^2$, where $\rho(z)>0$. In other words, near p the metric is conformal to the Euclidean cone of total angle $2\pi(\alpha+1)$. As is known, such a metric h is conformal to a genuine Riemannian metric g on M away from the singularities. If a surface M has a nonempty boundary, we do not exclude an infinite number of conical singularities accumulating to the boundary, and suppose that the volume measure Vol_h satisfies the decay hypothesis (5-1). For a surface with a finite number of conical singularities the hypothesis on the volume measure is always satisfied. The Dirichlet integral with respect to the metric h is defined as an improper integral; by the conformal invariance, it satisfies the relation

$$\int_M |\nabla u|_h^2 \, d\mathrm{Vol}_h = \int_M |\nabla u|_g^2 \, d\mathrm{Vol}_g$$

for any smooth function u. Thus, we conclude that the Laplace eigenvalues and their multiplicities of a metric h coincide with the quantities $\lambda_k(g, \mathrm{Vol}_h)$ and $m_k(g, \mathrm{Vol}_h)$, defined above, and Theorem 5.1 yields the multiplicity bounds. Note that if a metric h has only a finite number of conical singularities, then the multiplicity bounds can also be obtained from arguments in [Karpukhin et al. 2013].

Example II: Alexandrov surfaces of bounded integral curvature. The most significant class of surfaces, illustrating our approach, is formed by the so-called Alexandrov surfaces of bounded integral curvature. Below we recall this notion and give a brief outline of its relevance to our setting; more details and references on the subject can be found in the surveys [Reshetnyak 1993; Troyanov 2009]. Eigenvalue problems on Alexandrov surfaces of bounded integral curvature are treated in detail in [Kokarev ≥ 2014].

Definition. A metric space (M, d), where M is a compact smooth surface, is called an *Alexandrov surface of bounded integral curvature* if:

(i) the topology induced by d coincides with the original surface topology on M;

(ii) the metric space (M, d) is a *geodesic length space*; that is, any two points x and $y \in M$ can be joined by a path whose length is $d(x, y)$;

(iii) the metric d is a C^0-limit of distances of smooth Riemannian metrics g_n on M whose integral curvatures are bounded; that is,

$$\sup_n \int_M |K_{g_n}| \, d\mathrm{Vol}_{g_n} < +\infty,$$

where K_{g_n} stands for the Gauss curvature of a metric g_n.

This is a large class of singular surfaces that contains, for example, all polyhedral surfaces as well as surfaces with conical singularities and their limits under the integral curvature bound. The hypothesis (iii) implies that after a selection of a subsequence the signed measures $K_{g_n} d\mathrm{Vol}_{g_n}$ converge weakly to a measure ω on M. By the result of Alexandrov [Alexandrov and Zalgaller 1967], the measure ω is an intrinsic characteristic of (M, g); it does not depend on an approximating sequence of Riemannian metrics g_n, and is called the *curvature measure* of an Alexandrov surface. As an example, consider the surface of a unit cube in $\mathbb{R}^3$. The metric on it is defined as the infimum of Euclidean lengths of all paths that lie on the surface of the cube and join two given points. As is known [Reshetnyak 1993; Troyanov 2009], its curvature measure is $\sum(\pi/2)\delta_p$, where δ_p is the Dirac mass and the sum runs over all vertices p of the cube.

Recall that a point $x \in M$ is called a *cusp* if $\omega(x) = 2\pi$. By the results of [Reshetnyak 1960; Huber 1960], any Alexandrov surface of bounded integral curvature and without cusps can be regarded as being "conformally equivalent" to a smooth Riemannian metric on a background compact surface. This means that the distance function on such a surface has the form

$$d(x, y) = \inf_{\gamma} \left\{ \int_0^1 e^{u(\gamma(t))} |\dot{\gamma}(t)|_g \, dt \right\}$$

for some function u and a smooth background Riemannian metric g; the infimum above is taken over smooth paths γ joining x and y. The conformal factor e^u here can be very singular, and is an L^{2p}-function, where $p > 1$. More precisely, the function u is the difference of weakly subharmonic functions [Reshetnyak 1960; 1993], and the set

$$S = \{x \in M : e^u(x) = 0\}$$

has zero capacity in M [Hayman and Kennedy 1976, Theorem 5.9].

Thus, an Alexandrov surface without cusps can be viewed as a surface with a "Riemannian metric" $h = e^u g$ on $M \setminus S$, whose distance function is precisely the original metric d. This "Riemannian metric" yields the *Alexandrov volume measure* $d\mu_h = e^{2u} d\mathrm{Vol}_g$, which is another intrinsic characteristic of (M, d); it can be also defined via approximations by Riemannian metrics. More precisely, Alexandrov and Zalgaller [1967] show that if g_n is a sequence of Riemannian metrics that satisfy the hypothesis (iii) in the definition of an Alexandrov surface, then its volume measures Vol_{g_n} converge weakly to μ_h.

Since the set S has zero capacity, by conformal invariance it is straightforward to conclude that the relation

$$\int_{M \setminus S} |\nabla u|_h^2 \, d\mu_h = \int_M |\nabla u|_g^2 \, d\mathrm{Vol}_g$$

holds for any smooth function u. Thus, the eigenvalues $\lambda_k(g, \mu_h)$ of the Dirichlet form $\int |\nabla u|^2 \, d\mathrm{Vol}_g$ in $L_2(M, \mu_h)$ are indeed natural versions of Laplace eigenvalues on an Alexandrov surface without cusps. Since e^{2u} is an L^p-function, where $p > 1$, we conclude that the Alexandrov volume measure μ_h satisfies the decay hypothesis (5-1). In particular, the multiplicities $m_k(g, \mu_h)$ are finite and satisfy the inequalities in Theorem 5.1.

Appendix: Cheng's structure theorem

The purpose of this section is to give details on Cheng's structure theorem, discussed in Section 1. It is based on the following lemma.

Lemma A.1. *Let u be a $C^{1,1}$-smooth function defined in a neighbourhood of the origin in $\mathbb{R}^n$ that satisfies the relation*

$$u(x) = P_N(x) + O(|x|^{N+\delta}) \quad \text{as } x \to 0, \tag{A-1}$$

where P_N is a homogeneous polynomial of order N such that $|\nabla P_N(x)| \geqslant C|x|^{N-1}$. Then there exists a neighbourhood U of the origin and a Lipschitz homeomorphism Φ of it that preserves the origin and such that $u(x) = P_N(\Phi(x))$ for any $x \in U$. Moreover, if u is C^2-smooth, then Φ is a C^1-diffeomorphism.

Proof outline. The second term on the right-hand side can be viewed as the product $\alpha(x)|x|^{N+\delta'-1}$, where $0 < \delta' < \delta$ and $\alpha(x)$ is a function that is C^1-smooth away from the origin and behaves like $O(|x|^{1+\delta-\delta'})$

as $x \to 0$. It is then straightforward to see that α is C^1-smooth in a neighbourhood of the origin and, differentiating relation (A-1), we obtain

$$\nabla u(x) = \nabla P_N(x) + O(|x|^{N+\delta'-1}) \quad \text{as } x \to 0.$$

Given the last relation, if u is C^2-smooth, the existence of the C^1-diffeomorphism Φ follows from the argument in the proof of [Cheng 1976, Lemma 2.4]. This argument also works when u is $C^{1,1}$-smooth, and in this case it yields a local Lipschitz homeomorphism Φ such that $u(x) = P_N(\Phi(x))$. □

In dimension two any homogeneous harmonic polynomial of degree $N \geqslant 1$ satisfies the hypothesis $|\nabla P_N(x)| \geqslant C|x|^{N-1}$ and, combining the lemma above with Proposition 2.2, we obtain the following improved version of Cheng's result.

Cheng's structure theorem. *Let u be a $C^{1,1}$-smooth solution of the Schrödinger equation*

$$(-\Delta + V)u = 0 \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{A-2}$$

where $V \in K^{2,\delta}(\Omega)$. Then for any nodal point $p \in \mathcal{N}(u)$ there is a neighbourhood U and a Lipschitz homeomorphism Φ of U onto a neighbourhood of the origin such that $u(x) = P_N(\Phi(x))$ for any $x \in U$, where P_N is an approximating homogeneous harmonic polynomial at p. Moreover, if u is C^2-smooth, then Φ is a C^1-diffeomorphism.

Cheng [1976] also states similar results in arbitrary dimension. However, in dimension $n > 2$ there are homogeneous harmonic polynomials for which the hypothesis $|\nabla P_N(x)| \geqslant C|x|^{N-1}$ fails, and thus Lemma A.1 cannot be used. As is shown in [Bérard and Meyer 1982, Appendix E], the latter hypothesis is necessary for the conclusion of Lemma A.1 to hold.

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Some of our arguments at the end of Section 4 (the proof of Lemma 4.2) are similar in spirit to the ones in [Karpukhin et al. 2013], and I am grateful to Mikhail Karpukhin and Iosif Polterovich for a number of discussions on the related topics. I am also grateful to Yuri Burago for a number of comments on Alexandrov surfaces.

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PARABOLIC BOUNDARY HARNACK PRINCIPLES IN DOMAINS WITH THIN LIPSCHITZ COMPLEMENT

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We prove forward and backward parabolic boundary Harnack principles for nonnegative solutions of the heat equation in the complements of thin parabolic Lipschitz sets given as subgraphs

$$E = \{(x,t) : x_{n-1} \le f(x'',t),\, x_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}$$

for parabolically Lipschitz functions f on $\mathbb{R}^{n-2} \times \mathbb{R}$.

We are motivated by applications to parabolic free boundary problems with thin (i.e., codimension-two) free boundaries. In particular, at the end of the paper we show how to prove the spatial $C^{1,\alpha}$-regularity of the free boundary in the parabolic Signorini problem.

1. Introduction

The purpose of this paper is to study forward and backward boundary Harnack principles for nonnegative solutions of the heat equation in certain domains in $\mathbb{R}^n \times \mathbb{R}$ which are, roughly speaking, complements of thin parabolically Lipschitz sets E. By the latter, we understand closed sets lying in the vertical hyperplane $\{x_n = 0\}$ which are locally given as subgraphs of parabolically Lipschitz functions (see Figure 1).

Such sets appear naturally in free boundary problems governed by parabolic equations, where the free boundary lies in a given hypersurface and thus has codimension two. Such free boundaries are also known as thin free boundaries. In particular, our study was motivated by the parabolic Signorini problem, recently studied in [Danielli et al. 2013].

The boundary Harnack principles that we prove in this paper provide important technical tools in problems with thin free boundaries. For instance, they open up the possibility of proving that the thin Lipschitz free boundaries have Hölder-continuous spatial normals, following the original idea in [Athanasopoulos and Caffarelli 1985]. In particular, we show that this argument can indeed be successfully carried out in the parabolic Signorini problem.

We have to point out that the elliptic counterparts of the results in this paper are very well known; see e.g. [Athanasopoulos and Caffarelli 1985; Caffarelli et al. 2008; Aikawa et al. 2003]. However, there are significant differences between the elliptic and parabolic boundary Harnack principles, mostly because of the time-lag in the parabolic Harnack inequality. This results in two types of boundary Harnack principles for parabolic equations: the forward one (also known as the Carleson estimate) and the backward one.

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Moreover, those results are known only for a much smaller class of domains than in the elliptic case. Thus, to put our results in a better perspective, we start with a discussion of the known results both in the elliptic and parabolic cases.

Elliptic boundary Harnack principle. The by-now classical boundary Harnack principle for harmonic functions [Kemper 1972a; Dahlberg 1977; Wu 1978] says that if D is a bounded Lipschitz domain in $\mathbb{R}^n$, $x_0 \in \partial D$, and u and v are positive harmonic functions on D vanishing on $B_r(x_0) \cap \partial D$ for a small $r > 0$, then there exist positive constants M and C, depending only on the dimension n and the Lipschitz constant of D, such that

$$\frac{u(x)}{v(x)} \le C\frac{u(y)}{v(y)} \quad \text{for } x, y \in B_{r/M}(x_0) \cap D.$$

Note that this result is scale-invariant, and hence, by a standard iterative argument, one then immediately obtains that the ratio u/v extends to $\bar{D} \cap B_{r/M}(x_0)$ as a Hölder-continuous function. Roughly speaking, this theorem says that two positive harmonic functions vanishing continuously on a certain part of the boundary will decay at the same rate near that part of the boundary.

This boundary Harnack principle depends heavily on the geometric structure of the domains. The scale-invariant boundary Harnack principle (among other classical theorems of real analysis) was extended in [Jerison and Kenig 1982] from Lipschitz domains to the so-called NTA (nontangentially accessible) domains. Moreover, if the Euclidean metric is replaced by the internal metric, then similar results hold for so-called uniform John domains [Aikawa et al. 2003; Aikawa 2005].

In particular, the boundary Harnack principle is known for domains of the type

$$D = B_1 \setminus E_f, \quad E_f = \{x \in \mathbb{R}^n : x_{n-1} \le f(x''), x_n = 0\},$$

where f is a Lipschitz function on $\mathbb{R}^{n-2}$ with $f(0) = 0$; it is used, for instance, in the thin obstacle problem [Athanasopoulos and Caffarelli 1985; Athanasopoulos et al. 2008; Caffarelli et al. 2008]. In fact, there is a relatively simple proof of the boundary Harnack principle for domains as above already indicated in [Athanasopoulos and Caffarelli 1985]: there exists a bi-Lipschitz transformation from D to a half-ball B_1^+, which is a Lipschitz domain. The harmonic functions in D transform to solutions of a uniformly elliptic equation in divergence form with bounded measurable coefficients in B_1^+, for which the boundary Harnack principle is known [Caffarelli et al. 1981].

Parabolic boundary Harnack principle. The parabolic version of the boundary Harnack principle is much more challenging than the elliptic one, mainly because of the time-lag issue in the parabolic Harnack inequality. The latter is called sometimes the forward Harnack inequality, to emphasize the way it works: for nonnegative caloric functions (solutions of the heat equation), if the earlier value is positive at some spatial point, after a necessary waiting time, one can expect that the value will become positive everywhere in a compact set containing that point. Under the condition that the caloric function vanishes on the lateral boundary of the domain, one may overcome the time-lag issue and get a backward-type Harnack principle (so, combining the two together, one gets an elliptic-type Harnack inequality).

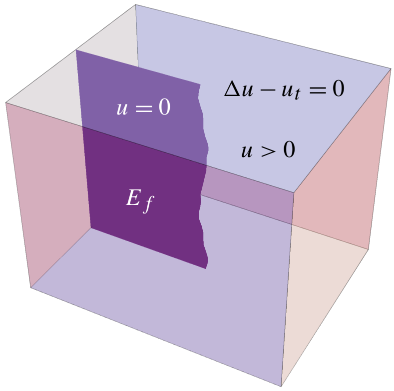

Figure 1. Domain with a thin Lipschitz complement.

The forward and backward boundary Harnack principle are known for parabolic Lipschitz domains, not necessarily cylindrical; see [Kemper 1972b; Fabes et al. 1984; Salsa 1981]. Moreover, they were shown more recently in [Hofmann et al. 2004] to hold for unbounded parabolically Reifenberg-flat domains. In this paper, we will generalize the parabolic boundary Harnack principle to the domains of the type (see Figure 1)

$$D = \Psi_1 \setminus E_f,$$

where

$$\Psi_1 = \{(x,t) : |x_i| < 1, i = 1, \dots, n-2, |x_{n-1}| < 4nL, |x_n| < 1, |t| < 1\},$$
$$E_f = \{(x,t) : x_{n-1} \le f(x'', t), x_n = 0\},$$

and $f(x'', t)$ is a parabolically Lipschitz function satisfying

$$|f(x'',t) - f(y'',s)| \le L(|x'' - y''|^2 + |t-s|)^{1/2}, \qquad f(0,0) = 0.$$

Note that D is not cylindrical (E_f is not time-invariant), and it does not fall into any category of domains on which the forward or backward Harnack principle is known. Inspired by the elliptic inner NTA domains (see e.g. [Athanasopoulos et al. 2008]), it seems natural to equip the domain D with the intrinsic geodesic distance $\rho_D((x,t),(y,s))$, where $\rho_D((x,t),(y,s))$ is defined as the infimum of the Euclidean length of rectifiable curves γ joining (x,t) and (y,s) in D, and consider the abstract completion D^* of D with respect to this inner metric ρ_D. We will not work directly with the inner metric in this paper since it seems easier to work with the Euclidean parabolic cylinders due to the time-lag issues and different scales in space and time variables. However, we do use the fact that the interior points of E_f (in the relative topology) correspond to two different boundary points in the completion D^*.

Even though we assume in this paper that E_f lies on the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n \times \mathbb{R}$, our proofs (except those on the doubling of the caloric measure and the backward boundary Harnack principle) are easily generalized to the case when E_f is a hypersurface which is Lipschitz in the space variable and independent of the time variable.

Structure of the paper. The paper is organized as follows.

In Section 2 we give basic definitions and introduce the notation used in this paper.

In Section 3 we consider the Perron–Wiener–Brelot (PWB) solution to the Dirichlet problem of the heat equation for D. We show that D is regular and has a Hölder-continuous barrier function at each parabolic boundary point.

In Section 4 we establish a forward boundary Harnack inequality for nonnegative caloric functions vanishing continuously on a part of the lateral boundary, following the lines of [Kemper 1972b].

In Section 5 we study the kernel functions for the heat operator. We show that each boundary point (y, s) in the interior of E_f (as a subset of the hyperplane $\{x_n = 0\}$) corresponds to two independent kernel functions. Hence, the parabolic Euclidean boundary for D is not homeomorphic to the parabolic Martin boundary.

In Section 6 we show the doubling property of the caloric measure with respect to D, which will imply a backward Harnack inequality for caloric functions vanishing on the whole lateral boundary.

Section 7 is dedicated to various forms of the boundary Harnack principle from Sections 4 and 6, including a version for solutions of the heat equation with a nonzero right-hand side. We conclude the section and the paper with an application to the parabolic Signorini problem.

2. Notation and preliminaries

2A. *Basic notation.*

$\mathbb{R}^n$	n-dimensional Euclidean space
$x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$	for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$
$x'' = (x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$	for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

Sometimes it will be convenient to identify x', x'' with $(x', 0)$ and $(x'', 0, 0)$, respectively.

$x \cdot y = \sum_{i=1}^{n} x_i y_i$	the inner product for $x, y \in \mathbb{R}^n$
$\lvert x\rvert = (x \cdot x)^{1/2}$	the Euclidean norm of $x \in \mathbb{R}^n$
$\lVert(x, t)\rVert = (\lvert x\rvert^2 + \lvert t\rvert)^{1/2}$	the parabolic norm of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$
$\bar{E}, E^\circ, \partial E$	the closure, the interior, the boundary of E
$\partial_p E$	the parabolic boundary of E in $\mathbb{R}^n \times \mathbb{R}$
$B_r(x) := \{y \in \mathbb{R}^n : \lvert x - y\rvert < r\}$	open ball in $\mathbb{R}^n$
$B'_r(x'), B''_r(x'')$	(thin) open balls in $\mathbb{R}^{n-1}, \mathbb{R}^{n-2}$
$Q_r(x, t) := B_r(x) \times (t - r^2, t)$	lower parabolic cylinders in $\mathbb{R}^n \times \mathbb{R}$
$\operatorname{dist}_p(E, F) = \inf_{\substack{(x,t)\in E \\ (y,s)\in F}} \lVert(x - y, t - s)\rVert$	the parabolic distance between sets E, F

We will also need the notion of a *parabolic Harnack chain* in a domain $D \subset \mathbb{R}^n \times \mathbb{R}$. For two points

(z_1, h_1) and (z_2, h_2) in D with $h_2 - h_1 \geq \mu^2 |z_2 - z_1|^2$, $0 < \mu < 1$, we say that a sequence of parabolic cylinders $Q_{r_i}(x_i, t_i) \subset D$, $i = 1, \dots, N$, is a Harnack chain from (z_1, h_1) to (z_2, h_2) with constant μ if:

$$\begin{gathered}(z_1, h_1) \in Q_{r_1}(x_1, t_1), \quad (z_2, h_2) \in Q_{r_N}(x_N, t_N),\\ \mu r_i \leq \operatorname{dist}_p(Q_{r_i}(x_i, t_i), \partial_p D) \leq \frac{1}{\mu} r_i, \quad i = 1, \dots, N,\\ Q_{r_{i+1}}(x_{i+1}, t_{i+1}) \cap Q_{r_i}(x_i, t_i) \neq \varnothing, \quad i = 1, \dots, N-1,\\ t_{i+1} - t_i \geq \mu^2 r_i^2, \quad i = 1, \dots, N-1.\end{gathered}$$

The number N is called the length of the Harnack chain. By the parabolic Harnack inequality, if u is a nonnegative caloric function in D and there is a Harnack chain of length N and constant μ from (z_1, h_1) to (z_2, h_2), then

$$u(z_1, h_1) \leq C(\mu, n, N)\, u(z_2, h_2).$$

Further, for given $L \geq 1$ and $r > 0$ we also introduce the (elongated) parabolic boxes, specifically adjusted to our purposes:

$$\begin{aligned}\Psi_r'' &= \{(x'', t) \in \mathbb{R}^{n-2} \times \mathbb{R} : |x_i| < r, i = 1, \dots, n-2, |t| < r^2\},\\ \Psi_r' &= \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : (x'', t) \in \Psi_r'', |x_{n-1}| < 4nLr\},\\ \Psi_r &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : (x', t) \in \Psi_r', |x_n| < r\},\\ \Psi_r(y, s) &= (y, s) + \Psi_r.\end{aligned}$$

We also define the neighborhoods

$$\mathcal{N}_r(E) := \bigcup_{(y,s) \in E} \Psi_r(y, s) \quad \text{for any set } E \subset \mathbb{R}^n \times \mathbb{R}.$$

2B. ***Domains with thin Lipschitz complement.*** Let $f : \mathbb{R}^{n-2} \times \mathbb{R} \to \mathbb{R}$ be a parabolically Lipschitz function with a Lipschitz constant $L \geq 1$ in the sense that

$$|f(x'', t) - f(y'', s)| \leq L(|x'' - y''|^2 + |t - s|)^{1/2}, \quad (x'', t), (y'', s) \in \mathbb{R}^{n-2} \times \mathbb{R}$$

Then consider the following two sets:

$$\begin{aligned}G_f &= \{(x, t) : x_{n-1} = f(x'', t), x_n = 0\},\\ E_f &= \{(x, t) : x_{n-1} \leq f(x'', t), x_n = 0\}.\end{aligned}$$

We will call them the *thin Lipschitz graph* and *subgraph* respectively (with "thin" indicating their lower dimension). We are interested in the behavior of caloric functions in domains of the type $\Omega \setminus E_f$, where Ω is open in $\mathbb{R}^n \times \mathbb{R}$. We will say that $\Omega \setminus E_f$ is a domain with a *thin Lipschitz complement*.

We are interested mostly in local behavior of caloric functions near the points on G_f and therefore we concentrate our study on the case

$$D = D_f := \Psi_1 \setminus E_f$$

with a normalization condition

$$f(0,0)=0 \Longleftrightarrow (0,0)\in G_f.$$

We will state most of our results for D defined as above; however, the results will still hold if we replace Ψ_1 in the construction above with a rectangular box

$$\widetilde{\Psi}=\left(\prod_{i=1}^{n}(a_i,b_i)\right)\times(\alpha,\beta)$$

such that, for some constants $c_0, C_0>0$ depending on L and n, we have

$$\widetilde{\Psi}\subset\Psi_{C_0},\quad \Psi_{c_0}(y,s)\subset\widetilde{\Psi}\quad\text{for all } (y,s)\in G_f,\ s\in[\alpha+c_0^2,\beta-c_0^2],$$

and consider the complement

$$\widetilde{D}=\widetilde{D}_f:=\widetilde{\Psi}\setminus E_f.$$

Even more generally, one may take $\widetilde{\Psi}$ to be a cylindrical domain of the type $\widetilde{\Psi}=\mathbb{O}\times(\alpha,\beta)$ where $\mathbb{O}\subset\mathbb{R}^n$ has the property that $\mathbb{O}_\pm=\mathbb{O}\cap\{\pm x_n>0\}$ are Lipschitz domains. For instance, we can take $\mathbb{O}=B_1$. Again, most of the results that we state will be valid also in this case, with a possible change in constants that appear in estimates.

2C. ***Corkscrew points.*** Since we will be working in $D=\Psi_1\setminus E_f$ as above, it will be convenient to redefine the sets E_f and G_f as follows:

$$G_f=\{(x,t)\in\overline{\Psi_1}:x_{n-1}=f(x'',t),\,x_n=0\},$$
$$E_f=\{(x,t)\in\overline{\Psi_1}:x_{n-1}\le f(x'',t),\,x_n=0\},$$

so that they are subsets of $\overline{\Psi_1}$. It is easy to see from the definition of D that it is connected and that its parabolic boundary is given by

$$\partial_p D=\partial_p\Psi_1\cup E_f.$$

As we will see, the domain D has a parabolic NTA-like structure, with the catch that at points on E_f (and close to it) we need to define two pairs of future and past corkscrew points, pointing into D_+ and D_-, respectively, where

$$D_+=D\cap\{x_n>0\}=(\Psi_1)_+,\qquad D_-=D\cap\{x_n<0\}=(\Psi_1)_-.$$

More specifically, fix $0<r<\frac{1}{4}$ and $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_p D$, and define

$$\overline{A}_r^{\pm}(y,s)=(y'',y_{n-1}+r/2,\pm r/2,s+2r^2)\quad\text{if } s\in[-1,1-4r^2),$$
$$\underline{A}_r^{\pm}(y,s)=(y'',y_{n-1}+r/2,\pm r/2,s-2r^2)\quad\text{if } s\in(-1+4r^2,1].$$

Note that, by definition, we always have $\bar{A}_r^+(y,s)$, $\underline{A}_r^+(y,s) \in D_+$ and $\bar{A}_r^-(y,s)$, $\underline{A}_r^-(y,s) \in D_-$. We also have that

$$\bar{A}_r^\pm(y,s),\ \underline{A}_r^\pm(y,s) \in \Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\bar{A}_r^\pm(y,s)) \cap \partial D = \Psi_{r/2}(\underline{A}_r^\pm(y,s)) \cap \partial D = \varnothing.$$

Moreover, the corkscrew points have the following property.

Lemma 2.1 (Harnack chain property I). *Let $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D \cap \mathcal{N}_r(E_f)$ and $(x,t) \in D$ be such that*

$$(x,t) \in \Psi_r(y,s) \quad \textit{and} \quad \Psi_{\gamma r}(x,t) \cap \partial_p D = \varnothing.$$

Then there exists a Harnack chain in D with constant μ and length N, depending only on γ, L, and n, from (x,t) to either $\bar{A}_r^+(y,s)$ or $\bar{A}_r^-(y,s)$, provided $s \le 1-4r^2$, and from either $\underline{A}_r^+(y,s)$ or $\underline{A}_r^-(y,s)$ to (x,t), provided $s \ge -1+4r^2$.

In particular, there exists a constant $C = C(\gamma, L, n) > 0$ such that, for any nonnegative caloric function u in D,

$$u(x,t) \le C \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} \quad \textit{if } s \le 1-4r^2,$$
$$u(x,t) \ge C^{-1} \min\{u(\underline{A}_r^+(y,s)), (\underline{A}_r^-(y,s))\} \quad \textit{if } s \ge -1+4r^2.$$

Proof. This is easily seen when $(y,s) \notin \mathcal{N}_r(G_f)$ (in this case the chain length N does not depend on L). When $(y,s) \in \mathcal{N}_r(G_f)$, one needs to use the parabolic Lipschitz continuity of f. □

Next, we want to define the corkscrew points when (y,s) is farther away from E_f. Namely, if $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$, we define a single pair of future and past corkscrew points by

$$\bar{A}_r(y,s) = (y(1-r), s+2r^2) \quad \text{if } s \in [-1, 1-4r^2),$$
$$\underline{A}_r(y,s) = (y(1-r), s-2r^2) \quad \text{if } s \in (-1+4r^2, 1].$$

Note that the points $\bar{A}_r(y,s)$ and $\underline{A}_r(y,s)$ will have properties similar to those of $\bar{A}_r^\pm(y,s)$ and $\underline{A}_r^\pm(y,s)$. That is,

$$\bar{A}_r(y,s),\ \underline{A}_r(y,s) \in \Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\bar{A}_r(y,s)) \cap \partial D = \Psi_{r/2}(\underline{A}_r(y,s)) \cap \partial D = \varnothing,$$

and we have the following version of Lemma 2.1 above.

Lemma 2.2 (Harnack chain property II). *Let $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$ and $(x,t) \in D$ be such that*

$$(x,t) \in \Psi_r(y,s) \quad \textit{and} \quad \Psi_{\gamma r}(x,t) \cap \partial_p D = \varnothing.$$

Then there exists a Harnack chain in D with constant μ and length N, depending only on γ, L, and n, from (x,t) to $\bar{A}_r(y,s)$, provided $s \le 1-4r^2$, and from $\underline{A}_r(y,s)$ to (x,t), provided $s \ge -1+4r^2$.

In particular, there exists a constant $C = C(\gamma, L, n) > 0$ such that, for any nonnegative caloric function u in D,

$$u(x,t) \le C\, u(\bar{A}_r(y,s)) \quad \text{if } s \le 1 - 4r^2,$$
$$u(x,t) \ge C^{-1} u(\underline{A}_r(y,s)) \quad \text{if } s \ge -1 + 4r^2. \qquad \square$$

To state our next lemma, we need to use a parabolic scaling operator on $\mathbb{R}^n \times \mathbb{R}$. For any $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, we define

$$T^r_{(y,s)} : (x,t) \mapsto \left(\frac{x-y}{r}, \frac{t-s}{r^2}\right).$$

Lemma 2.3 (localization property). *For $0 < r < \frac{1}{4}$ and $(y,s) \in \partial_p D$, there exists a point $(\tilde{y}, \tilde{s}) \in \partial_p D \cap \Psi_{2r}(y,s)$ and $\tilde{r} \in [r, 4r]$ such that*

$$\Psi_r(y,s) \cap D \subset \Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D \subset \Psi_{8r}(y,s) \cap D,$$

and the parabolic scaling $T^{\tilde{r}}_{(\tilde{y},\tilde{s})}(\Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D)$ is one of the following:

(1) *a rectangular box $\widetilde{\Psi}$ such that $\Psi_{c_0} \subset \widetilde{\Psi} \subset \Psi_{C_0}$ for some positive constants c_0 and C_0 depending on L and n, or*

(2) *a union of two rectangular boxes as in* (1) *with a common vertical side, or*

(3) *a domain $\widetilde{D}_{\tilde{f}} = \widetilde{\Psi} \setminus E_f$ with a thin Lipschitz complement, as defined at the end of Section 2B.*

Proof. Consider the following cases:

Case 1: $\Psi_r(y,s) \cap E_f = \varnothing$. In this case, we take $(\tilde{y},\tilde{s}) = (y,s)$ and $\rho = r$. Then $\Psi_r(y,s) \cap \Psi_1$ falls into category (1).

Case 2: $\Psi_r(y,s) \cap E_f \ne \varnothing$, but $\Psi_{2r}(y,s) \cap G_f = \varnothing$. In this case, we take $(\tilde{y},\tilde{s}) = (y,s)$ and $\rho = 2r$. Then $\Psi_{2r}(y,s) \cap D$ splits into the disjoint union of $\Psi_{2r}(y,s) \cap (\Psi_1)_\pm$, which falls into category (2).

Case 3: $\Psi_{2r}(y,s) \cap G_f \ne \varnothing$. In this case, choose $(\tilde{y},\tilde{s}) \in \Psi_{3r}(y,s) \cap G_f$ with the additional property that $-1 + r^2/4 \le \tilde{s} \le 1 - r^2/4$, and let $\rho = 4r$. Then $\Psi_\rho(\tilde{y},\tilde{s}) \cap D = (\Psi_\rho(\tilde{y},\tilde{s}) \setminus E_f) \cap \Psi_1$ falls into category (3). $\square$

3. Regularity of D for the heat equation

In this section we show that the domains D with thin Lipschitz complement E_f are regular for the heat equation by using the existence of an exterior thin cone at points on E_f and applying the Wiener-type criterion for the heat equation [Evans and Gariepy 1982]. Furthermore, we show the existence of Hölder-continuous local barriers at the points on E_f, which we will use in the next section to prove the Hölder continuity and regularity of the solutions up to the parabolic boundary.

3A. ***PWB solutions*** [Doob 1984; Lieberman 1996]. Given an open subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, let $\partial\Omega$ be its Euclidean boundary. Define the parabolic boundary $\partial_p\Omega$ of Ω to be the set of all points $(x,t) \in \partial\Omega$ such that for any $\varepsilon > 0$ the lower parabolic cylinder $Q_\varepsilon(x,t)$ contains points not in Ω.

We say that a function $u : \Omega \to (-\infty, +\infty]$ is supercaloric if u is lower semicontinuous, finite on dense subsets of Ω, and satisfies the comparison principle in each parabolic cylinder $Q \Subset \Omega$: if $v \in C(\bar{Q})$ solves $\Delta v - \partial_t v = 0$ in Q and $v = u$ on $\partial_p Q$, then $v \leq u$ in Q.

A subcaloric function is defined as the negative of a supercaloric function. A function is caloric if it is supercaloric and subcaloric.

Given any real-valued function g defined on $\partial_p \Omega$, we define the upper solution

$$\begin{aligned}\overline{H}_g = \inf\{u : u \text{ is supercaloric or identically } +\infty \text{ on each component of } \Omega,\\ \liminf_{(y,s)\to(x,t)} u(y,s) \geq g(x,t) \text{ for all } (x,t) \in \partial_p\Omega,\, u \text{ bounded below on } \Omega\},\end{aligned}$$

and the lower solution

$$\begin{aligned}\underline{H}_g = \sup\{u : u \text{ is subcaloric or identically } -\infty \text{ on each component of } \Omega,\\ \limsup_{(y,s)\to(x,t)} u(y,s) \leq g(x,t) \text{ for all } (x,t) \in \partial_p\Omega,\, u \text{ bounded above on } \Omega\}.\end{aligned}$$

If $\overline{H}_g = \underline{H}_g$, then $H_g = \overline{H}_g = \underline{H}_g$ is the Perron–Wiener–Brelot (PWB) solution to the Dirichlet problem for g. It is shown in §1.VIII.4 and §1.XVIII.1 in [Doob 1984] that if g is a bounded continuous function, then the PWB solution H_g exists and is unique for any bounded domain Ω in $\mathbb{R}^n \times \mathbb{R}$.

Continuity of the PWB solution at points of $\partial_p\Omega$ is not automatically guaranteed. A point $(x,t) \in \partial_p\Omega$ is a regular boundary point if $\lim_{(y,s)\to(x,t)} H_g(y,s) = g(x,t)$ for every bounded continuous function g on $\partial_p D$. A necessary and sufficient condition for a parabolic boundary point to be regular is the existence of a local barrier for earlier time at that point (Theorem 3.26 in [Lieberman 1996]). By a local barrier at $(x,t) \in \partial_p\Omega$, we mean here a nonnegative continuous function w in $\overline{Q_r(x,t) \cap \Omega}$ for some $r > 0$ that has the following properties: (i) w is supercaloric in $Q_r(x,t) \cap \Omega$, and (ii) w vanishes only at (x,t).

3B. ***Regularity of D and barrier functions.*** For the domain D defined in the introduction, we have $\partial_p D = \partial_p \Psi_1 \cup E_f$. The regularity of $(x,t) \in \partial_p\Psi_1$ follows immediately from the exterior cone condition for the Lipschitz domain. For $(x,t) \in E_f$, instead of the full exterior cone we only know the existence of a flat exterior cone centered at (x,t) by the Lipschitz nature of the thin graph. This will still be enough for the regularity, by the Wiener-type criterion for the heat equation. We give the details below.

For $(x,t) \in E_f$, with f parabolically Lipschitz, there exist $c_1, c_2 > 1$, depending on n and L, such that the exterior of D contains a flat parabolic cone $\mathscr{C}(x,t)$, defined by

$$\mathscr{C}(x,t) = (x,t) + \mathscr{C},$$
$$\mathscr{C} = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq 0,\, y_{n-1} \leq -c_1|y''| - c_2\sqrt{-s},\, y_n = 0\}.$$

Then by the Wiener-type criterion for the heat equation [Evans and Gariepy 1982], the regularity of $(x,t) \in E_f$ will follow once we show that

$$\sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}(\mathscr{A}(2^{-k}) \cap \mathscr{C}) = +\infty,$$

where

$$\mathscr{A}(c) = \{(y, s) : (4\pi c)^{-n/2} \le \Gamma(y, -s) \le (2\pi c)^{-n/2}\},$$

Γ is the heat kernel

$$\Gamma(y, s) = \begin{cases} (4\pi s)^{-n/2} e^{-|y|^2/4s} & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases}$$

and $\operatorname{cap}(K)$ is the thermal capacity of a compact set K, defined by

$$\operatorname{cap}(K) = \sup\{\mu(K) : \mu \text{ is a nonnegative Radon measure supported in } K, \text{ with } \mu * \Gamma \le 1 \text{ on } \mathbb{R}^n \times \mathbb{R}\}.$$

Since $\mathscr{C}$ is self-similar, it is enough to verify that

$$\operatorname{cap}(\mathscr{A}(1) \cap \mathscr{C}) > 0.$$

The latter is easy to see, since we can take as μ the restriction of H^n, the Hausdorff measure, to $\mathscr{A}(1) \cap \mathscr{C}$, and note that

$$(\mu * \Gamma)(x, t) = \int_{\mathscr{A}(1)\cap\mathscr{C}} \Gamma(x - y, t - s)\, dy'\, ds \le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(t - s)^+}}\, ds \le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(-s)}}\, ds < \infty$$

for any $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Since $H^n(\mathscr{A}(1) \cap \mathscr{C}) > 0$, we therefore conclude that $\operatorname{cap}(\mathscr{A}(1) \cap \mathscr{C}) > 0$. We have therefore established the following fact:

Proposition 3.1. *The domain $D = D_f$ is regular for the heat equation.* □

We next show that we can use the self-similarity of $\mathscr{C}$ to construct a Hölder-continuous barrier function at every $(x, t) \in E_f$.

Lemma 3.2. *There exists a nonnegative continuous function U on $\overline{\Psi_1}$ with the following properties*:

(i) $U > 0$ *in* $\overline{\Psi_1} \setminus \{(0, 0)\}$ *and* $U(0, 0) = 0$;

(ii) $\Delta U - \partial_t U = 0$ *in* $\Psi_1 \setminus \mathscr{C}$; *and*

(iii) $U(x, t) \le C(|x|^2 + |t|)^{\alpha/2}$ *for* $(x, t) \in \Psi_1$ *and some* $C > 0$ *and* $0 < \alpha < 1$ *depending only on n and L.*

Proof. Let U be a solution of the Dirichlet problem in $\Psi_1 \setminus \mathscr{C}$ with boundary values $U(x, t) = |x|^2 + |t|$ on $\partial_p(\Psi_1 \setminus \mathscr{C})$. Then U will be continuous on $\overline{\Psi_1}$ and will satisfy the following properties:

(i) $U > 0$ in $\overline{\Psi_1} \setminus \{(0, 0)\}$ and $U(0, 0) = 0$; and

(ii) $\Delta U - \partial_t U = 0$ in $\Psi_1 \setminus \mathscr{C}$.

In particular, there exists $c_0 > 0$ and $\lambda > 0$ such that

$$U \ge c_0 \text{ on } \partial_p \Psi_1 \quad \text{and} \quad U \le c_0/2 \text{ on } \Psi_\lambda.$$

We then can compare U with its own parabolic scaling. Indeed, let $M_U(r) = \sup_{\Psi_r} U$ for $0 < r < 1$. Then, by the comparison principle for the heat equation, we have

$$U(x, t) \le \frac{M_U(r)}{c_0} U(x/r, t/r^2) \quad \text{for } (x, t) \in \Psi_r.$$

(Carefully note that this inequality is satisfied on $\mathscr{C}$ by the homogeneity of the boundary data on $\mathscr{C}$.) Hence, we obtain that

$$M_U(\lambda r) \le \frac{M_U(r)}{2} \quad \text{for any } 0 < r < 1,$$

which implies the Hölder-continuity of U at the origin by the standard iteration. The proof is complete. □

4. Forward boundary Harnack inequalities

In this section, we show the boundary Hölder-regularity of the solutions to the Dirichlet problem and follow the lines of [Kemper 1972b] to show the forward boundary Harnack inequality (Carleson estimate).

We also need the notion of the caloric measure. Given a domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and $(x, t) \in \Omega$, the caloric measure on $\partial_p\Omega$ is denoted by $\omega_\Omega^{(x,t)}$. The following facts about caloric measures can be found in [Doob 1984]. For a Borel subset B of $\partial_p\Omega$, we have $\omega_\Omega^{(x,t)}(B) = H_{\chi_B}(x, t)$, which is the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \text{ in } \Omega; \quad u = \chi_B \text{ on } \partial_p\Omega,$$

where χ_B is the characteristic function of B. Given a bounded and continuous function g on $\partial_p\Omega$, the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \text{ in } \Omega; \quad u = g \text{ on } \partial_p\Omega$$

is given by $u(x, t) = \int_{\partial_p\Omega} g(y, s)\, d\omega_\Omega^{(x,t)}(y, s)$. For a regular domain Ω, one has the following useful property of caloric measures:

Proposition 4.1 [Doob 1984]. *If E is a fixed Borel subset of $\partial_p\Omega$, then the function $(x, t) \mapsto \omega_\Omega^{(x,t)}(E)$ extends to $(y, s) \in \partial_p\Omega$ continuously provided χ_E is continuous at (y, s).*

4A. ***Forward boundary Harnack principle.*** From now on, we will write the caloric measure with respect to $D = \Psi_1 \setminus E_f$ as $\omega^{(x,t)}$ for simplicity. Before we prove the forward boundary Harnack inequality, we first show the Hölder-continuity of the caloric functions up to the boundary, which follows from the estimates on the barrier function constructed in Section 3.

In what follows, for $0 < r < \frac{1}{4}$ and $(y, s) \in \partial_p D$, we will denote

$$\Delta_r(y, s) = \Psi_r(y, s) \cap \partial_p D,$$

and call it the *parabolic surface ball* at (y, s) of radius r.

Lemma 4.2. *Let $0 < r < \frac{1}{4}$ and $(y, s) \in \partial_p D$. Then there exist $C = C(n, L) > 0$ and $\alpha = \alpha(n, L) \in (0, 1)$ such that if u is positive and caloric in $\Psi_r(y, s) \cap D$ and u vanishes continuously on $\Delta_r(y, s)$, then*

$$u(x, t) \le C\left(\frac{|x - y|^2 + |t - s|}{r^2}\right)^{\alpha/2} M_u(r) \tag{4-1}$$

for all $(x, t) \in \Psi_r(y, s) \cap D$, where $M_u(r) = \sup_{\Psi_r(y,s)\cap D} u$.

Proof. Let U be the barrier function at $(0, 0)$ in Lemma 3.2 and $c_0 = \inf_{\partial_p \Psi_1} U > 0$. We then use the parabolic scaling $T^r_{(y,s)}$ to construct a barrier function at (y, s). If $(y, s) \in \mathcal{N}_r(E_f)$, then there is an exterior cone $\mathscr{C}(y, s)$ at (y, s) with a universal opening, depending only on n and L, and

$$U^r_{(y,s)} := U \circ T^r_{(y,s)}$$

will be a local barrier function at (y, s) and will satisfy

$$0 \le U^r_{(y,s)}(x, t) \le C\left(\frac{|x - y|^2 + |t - s|}{r^2}\right)^{\alpha/2} \quad \text{for } (x, t) \in \Psi_r(y, s). \tag{4-2}$$

This construction can be made also at $(y, s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$, as these points also have the exterior cone property, and we may still use the same formula for $U^r_{(y,s)}$, but after a possible rotation of the coordinate axes in $\mathbb{R}^n$.

Then, by the maximum principle in $\Psi_r(y, s) \cap D$, we easily obtain that

$$u(x, t) \le \frac{M_u(r)}{c_0} U^r_{(y,s)}(x, t) \quad \text{for } (x, t) \in \Psi_r(y, s) \cap D. \tag{4-3}$$

Combining (4-2) and (4-3), we obtain (4-1). □

The main result in this section is the following forward boundary Harnack principle, also known as the Carleson estimate.

Theorem 4.3 (forward boundary Harnack principle or Carleson estimate). *Let $0 < r < \frac{1}{4}$, $(y, s) \in \partial_p D$ with $s \le 1 - 4r^2$, and u be a nonnegative caloric function in D, continuously vanishing on $\Delta_{3r}(y, s)$. Then there exists $C = C(n, L) > 0$ such that, for $(x, t) \in \Psi_{r/2}(y, s) \cap D$,*

$$u(x, t) \le C \begin{cases} \max\{u(\bar{A}^+_r(y, s)), u(\bar{A}^-_r(y, s))\} & \text{if } (y, s) \in \partial_p D \cap \mathcal{N}_r(E_f), \\ u(\bar{A}_r(y, s)) & \text{if } (y, s) \in \partial_p D \setminus \mathcal{N}_r(E_f). \end{cases} \tag{4-4}$$

To prove the Carleson estimate above, we need the following two lemmas on the properties of the caloric measure in D, which correspond to Lemmas 1.1 and 1.2 in [Kemper 1972b], respectively.

Lemma 4.4. *For $0 < r < \frac{1}{4}$, $(y, s) \in \partial_p D$ with $s \le 1 - 4r^2$, and $\gamma \in (0, 1)$, there exists $C = C(\gamma, L) > 0$ such that*

$$\omega^{(x,t)}(\Delta_r(y, s)) \ge C \quad \text{for } (x, t) \in \Psi_{\gamma r}(y, s) \cap D.$$

Proof. Suppose first that $(y, s) \in \mathcal{N}_r(E_f)$. Consider the caloric function

$$v(x, t) := \omega^{(x,t)}_{\Psi_r(y,s) \setminus \mathscr{C}(y,s)}(\mathscr{C}(y, s)),$$

where $\mathscr{C}(y, s)$ is the flat exterior cone defined in Section 3. The domain $\Psi_r(y, s) \setminus \mathscr{C}(y, s)$ is regular; hence, by Proposition 4.1, $v(x, t)$ is continuous on $\overline{\Psi_{\gamma r}(y, s)}$. We next claim that there exists $C = C(\gamma, n, L) > 0$ such that

$$v(x, t) \ge C \quad \text{in } \Psi_{\gamma r}(y, s).$$

Indeed, consider the normalized version of v,

$$v_0(x,t) := \omega^{(x,t)}_{\Psi_1 \setminus \mathscr{C}}(\mathscr{C}),$$

which is related to v through the identity $v = v_0 \circ T^r_{(y,s)}$. Then, from the continuity of v_0 in $\overline{\Psi_\gamma}$, the equality $v_0 = 1$ on $\mathscr{C}$, and the strong maximum principle we obtain that $v_0 \geq C = C(\gamma, n, L) > 0$ on $\overline{\Psi_\gamma}$. Using the parabolic scaling, we obtain the claimed inequality for v. Moreover, applying the comparison principle to $v(x,t)$ and $\omega^{(x,t)}(\Delta_r(y,s))$ in $D \cap \Psi_r(y,s)$, we have

$$\omega^{(x,t)}(\Delta_r(y,s)) \geq v(x,t) \geq C \quad \text{for } (x,t) \in D \cap \Psi_{\gamma r}(y,s).$$

In the case when $(y,s) \in \partial_p D \setminus \mathscr{N}_r(E_f)$, we may modify the proof by changing the flat cone $\mathscr{C}(y,s)$ with the full cone contained in the complement of D, or directly applying Lemma 1.1 in [Kemper 1972b]. □

Lemma 4.5. *For $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D$ with $s \leq 1 - 4r^2$, there exists a constant $C = C(n, L) > 0$ such that, for any $r' \in (0,r)$ and $(x,t) \in D \setminus \Psi_r(y,s)$, we have*

$$\omega^{(x,t)}(\Delta_{r'}(y,s)) \leq C \begin{cases} \omega^{\bar{A}_r(y,s)}(\Delta_{r'}(y,s)) & \text{if } (y,s) \notin \mathscr{N}_r(E_f), \\ \max\{\omega^{\bar{A}^+_r(y,s)}(\Delta_{r'}(y,s)), \omega^{\bar{A}^-_r(y,s)}(\Delta_{r'}(y,s))\} & \text{if } (y,s) \in \mathscr{N}_r(E_f). \end{cases} \tag{4-5}$$

Proof. For notational simplicity, we define

$$\Delta' := \Delta_{r'}(y,s), \quad \Delta := \Delta_r(y,s), \quad \Psi^k := \Psi_{2^{k-1}r'}(y,s),$$
$$\bar{A}^\pm_k := \bar{A}^\pm_{2^{k-1}r'}(y,s) \quad \text{if } \Psi^k \cap E_f \neq \varnothing,$$
$$\bar{A}_k := \bar{A}_{2^{k-1}r'}(y,s) \quad \text{if } \Psi^k \cap E_f = \varnothing \text{ for } k = 0, 1, \dots, \ell \text{ with } 2^{\ell-1}r' < 3r/4 < 2^\ell r'.$$

We want to clarify here that for $(y,s) \notin E_f$ and small r' and k, it may happen that Ψ^k does not intersect E_f. To be more specific, let ℓ_0 be the smallest nonnegative integer such that $\Psi^{\ell_0} \cap E_f \neq \varnothing$. Then we define $\bar{A}_k$ for $0 \leq k \leq \min\{\ell_0 - 1, \ell\}$ and the pair $\bar{A}^\pm_k$ for $\ell_0 \leq k \leq \ell$.

To prove the lemma, we want to show that there exists a universal constant C, in particular independent of k, such that, for $(x,t) \in D \setminus \Psi^k$,

$$\omega^{(x,t)}(\Delta') \leq C \begin{cases} \omega^{\bar{A}_k}(\Delta') & \text{if } 1 \leq k \leq \min\{\ell_0 - 1, \ell\}, \\ \max\{\omega^{\bar{A}^+_k}(\Delta'), \omega^{\bar{A}^-_k}(\Delta')\} & \text{if } \ell_0 \leq k \leq \ell. \end{cases} \tag{S_k}$$

Once this is established, (4-5) will follow from (S_l) and the Harnack inequality.

The proof of (S_k) is going to be by induction in k. We start with the observation that, by the Harnack inequality, there is $C_1 > 0$, independent of k and r', such that

$$\begin{aligned} \omega^{\bar{A}_k}(\Delta') &\leq C_1 \omega^{\bar{A}_{k+1}}(\Delta') && \text{for } 0 \leq k \leq \min\{\ell_0 - 2, \ell - 1\}, \\ \omega^{\bar{A}_{\ell_0 - 1}}(\Delta') &\leq C_1 \max\{\omega^{\bar{A}^+_{\ell_0}}(\Delta'), \omega^{\bar{A}^-_{\ell_0}}(\Delta')\} && \text{if } \ell_0 \leq \ell, \\ \omega^{\bar{A}^\pm_k}(\Delta') &\leq C_1 \omega^{\bar{A}^\pm_{k+1}}(\Delta') && \text{for } \ell_0 \leq k \leq \ell - 1. \end{aligned} \tag{4-6}$$

Proof of (S_1): Without loss of generality, assume $(y,s) \in \partial_p D \cap \bar{D}_+$.

Case 1: Suppose first that $\Psi^1 \cap E_f = \varnothing$, i.e., $\ell_0 > 1$. In this case, $\bar{A}_0 = \bar{A}_{r'/2}(y, s) \in \Psi_{(3/4)r'}(y, s)$, and by Lemma 4.4 there exists a universal $C_0 > 0$ such that $\omega^{\bar{A}_0}(\Delta') \geq C_0$. By (4-6) we have $\omega^{\bar{A}_0}(\Delta') \leq C_1 \omega^{\bar{A}_1}(\Delta')$. Letting $C_2 = C_1/C_0$, we then have

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \omega^{\bar{A}_1}(\Delta'). \tag{4-7}$$

Case 2: Suppose now that $\Psi^1 \cap E_f \neq \varnothing$, but $\Psi^0 \cap E_f = \varnothing$, i.e., $\ell_0 = 1$. In this case, we start as in Case 1, and finish by applying the second inequality in (4-6), which yields

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \max\{\omega^{\bar{A}_1^+}(\Delta'), \omega^{\bar{A}_1^-}(\Delta')\}. \tag{4-8}$$

Case 3: Finally, assume that $\Psi^0 \cap E_f \neq \varnothing$, i.e., $\ell_0 = 0$. Without loss of generality, assume also that $(y, s) \in \partial_p D \cap \bar{D}_+$. In this case, $\bar{A}_0^+ \in \Psi_{(3/4)r'}(y, s)$, and therefore $\omega^{\bar{A}_0^+}(\Delta') \geq C_0$. Besides, by (4-6), we have that $\omega^{\bar{A}_0^+}(\Delta') \leq C_1 \omega^{\bar{A}_1^+}(\Delta')$, which yields

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \omega^{\bar{A}_1^+}(\Delta'). \tag{4-9}$$

This proves (S_1) with the constant $C = C_2$.

We now turn to the proof of the induction step.

Proof of $(S_k) \Longrightarrow (S_{k+1})$: More precisely, we will show that if (S_k) holds with some universal constant C (to be specified) then (S_{k+1}) also holds with the same constant.

By the maximum principle, we need to verify (S_{k+1}) for $(x, t) \in \partial_p(D \setminus \Psi^{k+1})$. Since $\omega^{(x,t)}(\Delta')$ vanishes on $(\partial_p D) \setminus \Psi^{k+1}$, we may assume that $(x, t) \in (\partial \Psi^{k+1}) \cap D$. We will need to consider three cases, as in the proof of (S_1):

1. $\Psi^{k+1} \cap E_f = \varnothing$, i.e., $\ell_0 > k + 1$;
2. $\Psi^{k+1} \cap E_f \neq \varnothing$, but $\Psi^k \cap E_f = \varnothing$, i.e., $\ell_0 = k + 1$;
3. $\Psi^k \cap E_f \neq \varnothing$, i.e., $\ell_0 \leq k$.

Since the proof is similar in all three cases, we will treat only Case 2 in detail.

Case 2: Suppose that $\Psi^{k+1} \cap E_f \neq \varnothing$ but $\Psi^k \cap E_f = \varnothing$. We consider two subcases, depending on whether $(x, t) \in \partial \Psi^{k+1}$ is close to $\partial_p D$ or not.

Case 2a: First, assume that $(x, t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D)$ for some small positive $\mu = \mu(L, n) < \frac{1}{2}$ (to be specified). Take $(z, h) \in \Psi_{\mu 2^k r'}(x, t) \cap \partial_p D$, and observe that $\omega^{(x,t)}(\Delta')$ is caloric in $\Psi_{2^{k-1} r'}(z, h) \cap D$ and vanishes continuously on $\Delta_{2^{k-1} r'}(z, h)$ (by Proposition 4.1). Besides, by the induction assumption that (S_k) holds, we have

$$\omega^{(x,t)}(\Delta') \leq C \omega^{\bar{A}_k}(\Delta') \quad \text{for } (x, t) \in \Psi_{2^{k-1} r'}(z, h) \cap D \subset D \setminus \Psi^k.$$

Hence, by Lemma 4.2, if $\mu = \mu(n, L) > 0$ is small enough, we obtain that

$$\omega^{(x,t)}(\Delta') \leq \frac{1}{C_1} C \omega^{\bar{A}_k}(\Delta') \quad \text{for } (x, t) \in \Psi_{\mu 2^k r'}(z, h).$$

Here C_1 is the constant in (4-6). This, combined with (4-6), gives

$$\omega^{(x,t)}(\Delta') \le \frac{C}{C_1}\omega^{\bar{A}_k}(\Delta') \le \frac{C}{C_1}\cdot C_1 \max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\} = C\max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\}.$$

This proves (S_{k+1}) for $(x,t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D) \cap \partial\Psi^{k+1}$.

Case 2b: Assume now that $\Psi_{\mu 2^k r'}(x,t) \cap \partial_p D = \varnothing$. In this case, it is easy to see that we can construct a parabolic Harnack chain in D of universal length from (x,t) to either $\bar{A}^+_{k+1}$ or $\bar{A}^-_{k+1}$, which implies that, for some universal constant $C_3 > 0$,

$$\omega^{(x,t)}(\Delta') \le C_3 \max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\}.$$

Thus, combining Cases 2a and 2b, we obtain that (S_{k+1}) holds provided $C = \max\{C_2, C_3\}$. This completes the proof of our induction step in Case 2. As we mentioned earlier, Cases 1 and 3 are obtained by a small modification from the respective cases in the proof of (S_1). This completes the proof of the lemma. □

Now we prove the Carleson estimate. With Lemma 4.4 and Lemma 4.5 at hand, we use ideas similar to those in [Salsa 1981].

Proof of Theorem 4.3. We start with the remark that if $(y,s) \notin \mathcal{N}_{r/4}(E_f)$ then we can restrict u to D_+ or D_- and obtain the second estimate in (4-4) from the known result for parabolic Lipschitz domains. We thus consider only the case $(y,s) \in \mathcal{N}_{r/4}(E_f)$. Besides, replacing (y,s) with $(y',s') \in \Psi_{r/4}(y,s) \cap E_f$, we may further assume that $(y,s) \in E_f$, but then we will need to change the assumption that u vanishes on $\Delta_{2r}(y,s)$ and prove the estimate (4-4) for $(x,t) \in \Psi_r(y,s) \cap D$.

With these assumptions in mind, let $0 < r < \frac{1}{4}$ and $R = 8r$. Let $\widetilde{D}_R(y,s) := \Psi_{\tilde{R}}(\tilde{y},\tilde{s}) \cap D$ be given by the localization property Lemma 2.3. Note that we will be either in Case (2) or (3) of that lemma; moreover, we can choose $(\tilde{y},\tilde{s}) = (y,s)$.

For notational brevity, let

$$\omega_R^{(x,t)} := \omega^{(x,t)}_{\widetilde{D}_R(y,s)}$$

be the caloric measure with respect to $\widetilde{D}_R(y,s)$. We will also omit the center (y,s) from the notations $\widetilde{D}_R(y,s)$, $\Psi_\rho(y,s)$ and $\Delta_\rho(y,s)$.

Since u is caloric in $\widetilde{D}_R$ and continuously vanishes up to Δ_{2r}, we have

$$u(x,t) = \int_{(\partial_p \widetilde{D}_R)\setminus\Delta_{2r}} u(z,h)\, d\omega_R^{(x,t)}(z,h), \quad (x,t) \in \widetilde{D}_R. \tag{4-10}$$

Note that for $(x,t) \in \Psi_r \cap D$, we have $(x,t) \notin \Psi_{r/2}(z,h)$ for any $(z,h) \in (\partial_p \widetilde{D}_R) \setminus \Delta_{2r}$. Hence, applying Lemma 4.5[1] to $\omega_R^{(x,t)}$ in $\widetilde{D}_R$, we will have that, for $(x,t) \in \Psi_r \cap D$ and sufficiently small r',

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \le C\max\left\{\omega_R^{\bar{A}^+_{r/2,R}(z,h)}(\Delta_{r'}(z,h)), \omega_R^{\bar{A}^-_{r/2,R}(z,h)}(\Delta_{r'}(z,h))\right\}$$

[1] We have to scale the domain $\widetilde{D}_R$ with $T^{\tilde{R}}_{(\tilde{y},\tilde{s})}$ first and apply Lemma 4.5 to $r/2\tilde{R} < \frac{1}{8}$ if we are in case (3) of the localization property Lemma 2.3; in the case (2) we apply the known results for parabolic Lipschitz domains.

for $(z,h) \in \mathcal{N}_{r/2}(E_f) \cap (\partial_p \widetilde{D}_R) \setminus \Delta_{2r}$, and

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \leq C\, \omega_R^{\bar{A}_{r/2,R}(z,h)}(\Delta_{r'}(z,h))$$

for $(z,h) \in \partial_p \widetilde{D}_R \setminus (\mathcal{N}_{r/2}(E_f) \cup \Delta_{2r})$, where $C = C(L,n)$ and by $\bar{A}^{\pm}_{r/2,R}$ and $\bar{A}_{r/2,R}$ we denote the corkscrew points with respect to the domain $\widetilde{D}_R$. To proceed, we note that, for $(z,h) \in \partial_p \widetilde{D}_R$ with $h > s + r^2$, by the maximum principle we have

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) = 0$$

for any $(x,t) \in \Psi_r \cap D$, provided r' is small enough. For $(z,h) \in (\partial_p \widetilde{D}_R) \setminus \Delta_{2r}$ with $h \leq s + r^2$, we note that with the help of Lemmas 2.1 and 2.2 we can construct a Harnack chain of controllable length in D from $\bar{A}^{\pm}_{r/2,R}(z,h)$ or $\bar{A}_{r/2,R}(z,h)$ to $\bar{A}^{+}_{r}(y,s)$ or $\bar{A}^{-}_{r}(y,s)$ (corkscrew points with respect to the original D). This implies that, for $(x,t) \in \Psi_r \cap D$ and $(z,h) \in \partial_p \widetilde{D}_R \setminus \Delta_{2r}$,

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \leq C \max\{\omega_R^{\bar{A}^{+}_{r}(y,s)}(\Delta_{r'}(z,h)), \omega_R^{\bar{A}^{-}_{r}(y,s)}(\Delta_{r'}(z,h))\}. \tag{4-11}$$

We now want to apply Besicovitch's theorem on the differentiation of Radon measures. However, since $\partial_p \widetilde{D}_R$ locally is not topologically equivalent to a Euclidean space, we make the following symmetrization argument. For $x \in \mathbb{R}^n$, let $\hat{x}$ be its mirror image with respect to the hyperplane $\{x_n = 0\}$. We then can write

$$\begin{aligned} u(x,t) + u(\hat{x},t) &= \int_{\partial_p \widetilde{D}_R \setminus \Delta_{2r}} [u(z,h) + u(\hat{z},h)]\, d\omega_R^{(x,t)}(z,h) \\ &= \frac{1}{2} \int_{\partial_p \widetilde{D}_R \setminus \Delta_{2r}} [u(z,h) + u(\hat{z},h)]\big(d\omega_R^{(x,t)}(z,h) + d\omega_R^{(\hat{x},t)}(z,h)\big) \\ &= \int_{\partial_p ((\widetilde{D}_R)_+) \setminus \Delta_{2r}} [u(z,h) + u(\hat{z},h)]\chi\big(d\omega_R^{(x,t)}(z,h) + d\omega_R^{(\hat{x},t)}(z,h)\big), \end{aligned}$$

where $\chi = \frac{1}{2}$ on $\partial_p((\widetilde{D}_R)_+) \cap \{x_n = 0\}$ and $\chi = 1$ on the remaining part of $\partial_p((\widetilde{D}_R)_+)$ and the measures $d\omega_R^{(x,t)}$ and $d\omega_R^{(\hat{x},t)}$ are extended as zero on the thin space outside E_f, i.e., on $\partial_p((\widetilde{D}_R)_+) \setminus \partial_p \widetilde{D}_R$. We then use the estimate (4-11) for (x,t) and $(\hat{x},t)$ in $\Psi_r \cap D$. Note that in this situation we can apply Besicovitch's theorem on differentiation, since we can locally project $\partial_p((\widetilde{D}_R)_+)$ to hyperplanes, similarly to [Hunt and Wheeden 1970]. This will yield

$$\frac{d\omega_R^{(x,t)}(z,h) + d\omega_R^{(\hat{x},t)}(z,h)}{d\omega_R^{\bar{A}^{+}_{r}(y,s)}(z,h) + d\omega_R^{\bar{A}^{-}_{r}(y,s)}(z,h)} \leq C \frac{d\omega_R^{(x,t)}(z,h)}{d\omega_R^{\bar{A}_{r}(y,s)}(z,h)} \leq C \tag{4-12}$$

for $(z,h) \in \partial_p((\widetilde{D}_R)_+) \setminus \Delta_{2r}$ and $(x,t) \in \Psi_r \cap D$. Hence, we obtain

$$\begin{aligned} u(x,t) + u(\hat{x},t) &\leq C \int_{\partial_p ((\widetilde{D}_R)_+) \setminus \Delta_{2r}} [u(z,h) + u(\hat{z},h)]\big(d\omega_R^{\bar{A}^{+}_{r}(y,s)}(z,h) + d\omega_R^{\bar{A}^{-}_{r}(y,s)}(z,h)\big) \\ &\leq C\big(u(\bar{A}^{+}_{r}(y,s)) + u(\bar{A}^{-}_{r}(y,s))\big) \\ &\leq C \max\{u(\bar{A}^{+}_{r}(y,s)), u(\bar{A}^{-}_{r}(y,s))\}, \quad (x,t) \in \Psi_r \cap D. \end{aligned}$$

This completes the proof of the theorem. □

The following theorem is a useful consequence of Theorem 4.3; with that in hand, its proof is similar to that of Theorem 1.1 in [Fabes et al. 1986]. Hence, we only state the theorem here without giving a proof.

Theorem 4.6. *For $0 < r < \frac{1}{4}$, $(y, s) \in \partial_p D$ with $s \le 1 - 4r^2$, let u be caloric in D and continuously vanishing on $\partial_p D \setminus \Delta_{r/2}(y, s)$. Then there exists $C = C(n, L)$ such that, for $(x, t) \in D \setminus \Psi_r(y, s)$, we have*

$$u(x,t) \le C \begin{cases} \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} & \text{if } (y,s) \in \mathcal{N}_r(E_f), \\ u(\bar{A}_r(y,s)) & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases} \tag{4-13}$$

Moreover, applying Lemma 4.4 and the maximum principle, for $(x, t) \in D \setminus \Psi_r(y, s)$, we have

$$u(x,t) \le C\omega^{(x,t)}(\Delta_{2r}(y,s)) \times \begin{cases} \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} & \text{if } (y,s) \in \mathcal{N}_r(E_f), \\ u(\bar{A}_r(y,s)) & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases}$$

5. Kernel functions

Before proceeding to the backward boundary Harnack principle, we need the notion of kernel functions associated to the heat operator and the domain D. In [Fabes et al. 1986], the backward Harnack principle is a consequence of the global comparison principle (Theorem 6.4) by a simple time-shifting argument. In our case, since D is not cylindrical, this simple argument does not work. So we will first prove some properties of the kernel functions which can be used to show the doubling property of the caloric measures, as in [Wu 1979]. Then, using arguments as in [Fabes et al. 1986], we obtain the backward Harnack principle.

5A. ***Existence of kernel functions.*** Let $(X, T) \in D$ be fixed. Given $(y, s) \in \partial_p D$ with $s < T$, a function $K(x, t; y, s)$ defined in D is called a kernel function at (y, s) for the heat equation with respect to (X, T) if:

(i) $K(\cdot, \cdot; y, s) \ge 0$ in D,

(ii) $(\Delta - \partial_t)K(\cdot, \cdot; y, s) = 0$ in D,

(iii) $\lim_{\substack{(x,t)\to(z,h)\\(x,t)\in D}} K(x, t; y, s) = 0$ for $(z, h) \in \partial_p D \setminus \{(y, s)\}$, and

(iv) $K(X, T; y, s) = 1$.

If $s \ge T$, $K(x, t; y, s)$ will be taken identically equal to zero. We note that, by the maximum principle, $K(x, t; y, s) = 0$ when $t < s$.

The existence of the kernel functions (for the heat operator on domain D) follows directly from Theorem 4.3. Let $(y, s) \in \partial_p D$ with $s < T - \delta^2$ for some $\delta > 0$, and consider

$$v_n(x,t) = \frac{\omega^{(x,t)}(\Delta_{1/n}(y,s))}{\omega^{(X,T)}(\Delta_{1/n}(y,s))}, \quad (x,t) \in D, \ \frac{1}{n} < \delta. \tag{5-1}$$

We clearly have $v_n(x, t) \ge 0$, $(\Delta - \partial_t)v_n(x, t) = 0$ in D and $v_n(X, T) = 1$. Given $\varepsilon \in \left(0, \frac{1}{4}\right)$ small, by Theorem 4.6 and the Harnack inequality, $\{v_n\}$ is uniformly bounded on $\overline{D \setminus \Psi_\varepsilon(y, s)}$ if $n \ge 2/\varepsilon$. Moreover, by the up-to-the-boundary regularity (see Proposition 4.1 and Lemma 4.2), the family $\{v_n\}$ is uniformly

Hölder in $\overline{D \setminus \Psi_\varepsilon(y,s)}$. Hence, up to a subsequence, $\{v_n\}$ converges uniformly on $\overline{D \setminus \Psi_\varepsilon(y,s)}$ to some nonnegative caloric function v satisfying $v(X,T)=1$. Since ε can be taken arbitrarily small, v vanishes on $\partial_p D \setminus \{(y,s)\}$. Therefore, $v(x,t)$ is a kernel function at (y,s).

Convention 5.1. From now on, to avoid cumbersome details we will make a time extension of the domain D for $1 \le t < 2$ by looking at

$$\widetilde{D} = \widetilde{\Psi} \setminus E_f, \quad \widetilde{\Psi} = (-1,1)^n \times (-1,2),$$

as in Section 2B. We then fix (X,T) with $T=\frac{3}{2}$ and $X \in \{x_n = 0\}$, $X_{n-1} > 3nL$, and normalize all kernels $K(\cdot,\cdot;\cdot,\cdot)$ at this point (X,T). In this way, we will be able to state the results in this section for our original domain D. Alternatively, we could fix $(X,T) \in D$, and then state the results in the part of the domain $D \cap \{(x,t) : -1 < t < T-\delta^2\}$ with some $\delta > 0$, with the additional dependence of constants on δ.

5B. ***Nonuniqueness of kernel functions at $E_f \setminus G_f$.*** The idea is this: if we consider the completion D^* of the domain D with respect to the inner metric ρ_D and let $\partial^* D = D^* \setminus D$, then it is clear that each Euclidean boundary point $(y,s) \in G_f$ and $(y,s) \in \partial_p \Psi_1$ will correspond to only one $(y,s)^* \in \partial^* D$, and each $(y,s) \in E_f \setminus G_f$ will correspond to exactly two points $(y,s)^*_+$, $(y,s)^*_- \in \partial^* D$. It is not hard to imagine that the kernel functions corresponding to $(y,s)^*_+$ and $(y,s)^*_-$ are linearly independent, and they are the two linearly independent kernel functions at (y,s). In this section we will make this idea precise by considering the two-sided caloric measures ϑ_+ and ϑ_-. We will study the properties of ϑ_+ and ϑ_- and their relationship with the caloric measure ω_D.

First we introduce some more notation. Given $(y,s) \in \partial_p D \setminus G_f$, let

$$r_0 = \sup\bigl\{r \in \bigl(0, \tfrac{1}{4}\bigr) : \Delta_{2r}(y,s) \cap G_f = \varnothing\bigr\}. \tag{5-2}$$

Note that r_0 is a constant depending on (y,s), and is such that, for any $0 < r < r_0$, $\Psi_{2r}(y,s) \cap D$ is either separated by E_f into two disjoint sets Ψ^+_{2r} and Ψ^-_{2r}, or $\Psi_{2r}(y,s) \cap D \subset D_+$ (or D_-). We define, for $0 < r < r_0$, the shifting operators F^+_r and F^-_r:

$$F^+_r(x,t) = (x'', x_{n-1} + 4nLr, x_n + r, t + 4r^2), \tag{5-3}$$

$$F^-_r(x,t) = (x'', x_{n-1} + 4nLr, x_n - r, t + 4r^2). \tag{5-4}$$

For any $0 < r < r_0$, define

$$D^+_r = D \setminus (E^+_{r,1} \cup E^+_{r,2} \cup E^+_{r,3} \cup E^+_{r,4}), \tag{5-5}$$

where

$$\begin{aligned} E^+_{r,1} &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x_{n-1} \le f(x'',t), -r \le x_n \le 0\},\\ E^+_{r,2} &:= \{(x,t) : 1-r \le x_n \le 1\},\\ E^+_{r,3} &:= \{(x,t) : 4nL(1-r) \le x_{n-1} \le 4nL\},\\ E^+_{r,4} &:= \{(x,t) : 1-4r^2 \le t \le 1\}. \end{aligned}$$

It is easy to see that $D_r^+ \subset D$ and $F_r^+(D_r^+) \subset D$. Similarly, we can define $D_r^- \subset D$ satisfying $F_r^-(D_r^-) \subset D$. Notice that $D_r^+ \nearrow D$, $D_r^- \nearrow D$ as $r \searrow 0$. Moreover, it is clear that, for each $r \in \big(0, \frac{1}{4}\big)$,

$$\mathcal{N}_{1/4}(E_f) \cap \partial_p D \subset (\partial_p D_r^+ \cup \partial_p D_r^-) \cap \partial_p D, \tag{5-6}$$

$$E_f \subset \partial_p D_r^+ \cap \partial_p D_r^-. \tag{5-7}$$

Let ω_r^+ and ω_r^- denote the caloric measures with respect to D_r^+ and D_r^-, respectively. Given $(x,t) \in D$ and $r > 0$ small enough such that $(x,t) \in D_r^+ \cap D_r^-$, $\omega_r^{\pm(x,t)}$ are Radon measures on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$ (recall $D_\pm = D \cap \{x_n \gtrless 0\}$. Moreover, if K is a relatively compact Borel subset of $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$, then, by the comparison principle, $\omega_r^{\pm(x,t)}(K) \le \omega_{r'}^{\pm(x,t)}(K)$ for $0 < r' < r$. Hence, there exist Radon measures $\vartheta_\pm^{(x,t)}$ on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$ such that

$$\omega_r^{\pm(x,t)}|_{\partial_p(D_r^\pm)\cap\partial_p(D_\pm)} \overset{*}{\rightharpoonup} \vartheta_\pm^{(x,t)}, \quad r \to 0.$$

For $(y,s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$ and $0 < r < r_0$, denote

$$\Delta_r^\pm(y,s) := \Delta_r(y,s) \cap \partial_p D_\pm \quad \text{if } \Delta_r(y,s) \cap \partial_p(D_\pm) \ne \varnothing.$$

Note that if $\Delta_r(y,s) \subset E_f$ then $\Delta_r^\pm(y,s) = \Delta_r(y,s)$. It is easy to see that $(x,t) \mapsto \vartheta_\pm^{(x,t)}(\Delta_r^\pm(y,s))$ are caloric in D.

To simplify the notation we will write Δ_r, $\Delta_r^\pm$ instead of $\Delta_r(y,s)$, $\Delta_r^\pm(y,s)$. If $\Delta_r(y,s) \cap \partial_p(D_+)$ (or $\Delta_r(y,s) \cap \partial_p(D_-)$) is empty, we set $\vartheta_+^{(x,t)}(\Delta_r^+(y,s)) = 0$ (or $\vartheta_-^{(x,t)}(\Delta_r^-(y,s)) = 0$).

We also note that, with Convention 5.1 in mind, the future corkscrew points $\bar{A}_r^\pm(y,s)$ or $\bar{A}_r(y,s)$, $0 < r < r_0$, are defined for all $s \in [-1,1]$.

Proposition 5.2. *Given $(y,s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$, for $0 < r < r_0$, we have*:

(i) $$\sup_{(x,t)\in\partial_p D_{r'}^+\cap D} \vartheta_+^{(x,t)}(\Delta_r^+) \to 0 \quad \textit{and} \quad \sup_{(x,t)\in\partial_p D_{r'}^-\cap D} \vartheta_-^{(x,t)}(\Delta_r^-) \to 0 \quad \textit{as } r' \to 0.$$

(ii) $\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-) = \omega^{(x,t)}(\Delta_r)$ *for* $(x,t) \in D$.

(iii) *There exists a constant $C = C(n,L)$ such that, for any $0 < r' < r$,*

$$\vartheta_+^{(x,t)}(\Delta_{r'}^+) \le C\vartheta_+^{\bar{A}_r^+(y,s)}(\Delta_{r'}^+)\vartheta_+^{(x,t)}(\Delta_{2r}^+) \quad \textit{for } (x,t) \in D \setminus \Psi_r^+(y,s),$$

$$\vartheta_-^{(x,t)}(\Delta_{r'}^-) \le C\vartheta_-^{\bar{A}_r^-(y,s)}(\Delta_{r'}^-)\vartheta_-^{(x,t)}(\Delta_{2r}^-) \quad \textit{for } (x,t) \in D \setminus \Psi_r^-(y,s).$$

(iv) *For (X,T) as defined above and $(y,s) \in E_f \setminus G_f$, there exists a positive constant $C = C(n,L,r_0)$ such that*

$$C^{-1}\vartheta_+^{(X,T)}(\Delta_r^+) \le \vartheta_-^{(X,T)}(\Delta_r^-) \le C\vartheta_+^{(X,T)}(\Delta_r^+).$$

Proof of (i). We assume that $\Delta_r^\pm \ne \varnothing$. If either of them is empty, the conclusion obviously holds.

For $0 < r < r_0$, we have

$$\partial_p D_r^+ \cap D = \{(x,t) \in D : x_{n-1} = 4nL(1-r) \text{ or } x_n = 1-r\}$$
$$\cup \{(x,t) \in D : x_{n-1} \le f(x'',t),\, x_n = -r \text{ or } x_{n-1} = f(x'',t),\, -r \le x_n < 0\}.$$

Given $(y,s)\in(\mathcal{N}_{1/4}(E_f)\cap\partial_p D)\setminus G_f$, let $0<r''<r'<r_0$; then $\omega^{+(x,t)}_{r''}(\Delta^+_r(y,s))$ is caloric in $D^+_{r''}$, and from the way r_0 is chosen, vanishes continuously on $\Delta_{r_0}(z,h)$ for each $(z,h)\in\partial_p D^+_{r''}\cap D$. Notice that

$$\partial_p D^+_{r'}\cap D\subset\bigcup_{(z,h)\in\partial_p D^+_{r''}\cap D}\Psi_{r_0}(z,h),$$

hence, applying Lemma 4.2 in each $\Psi_{r_0}(z,h)\cap D^+_{r''}$, we obtain constants $C=C(n,L)$ and $\gamma=\gamma(n,L)$, $\gamma\in(0,1)$, such that

$$\omega^{+(x,t)}_{r''}(\Delta^+_r)\le C\left(\frac{|x-z|+|t-h|^{1/2}}{r_0}\right)^{\gamma}\le C\left(\frac{r'}{r_0}\right)^{\gamma}\quad\text{for all }(x,t)\in\partial_p D^+_{r'}\cap D.\tag{5-8}$$

The constants C and γ above do not depend on $(z,h)\in\partial_p D^+_{r''}\cap D$, r or r'' because of the existence of the exterior flat parabolic cones centered at each (z,h) with an uniform opening depending only on n and L.

Let $r''\to 0$ in (5-8), we then get

$$\vartheta^{(x,t)}_{+}(\Delta^+_r)\le C\left(\frac{r'}{r_0}\right)^{\gamma}\quad\text{uniformly for }(x,t)\in\partial_p D^+_{r'}\cap D.$$

Therefore

$$\lim_{r'\to 0}\sup_{(x,t)\in\partial_p D^+_{r'}\cap D}\vartheta^{(x,t)}_{+}(\Delta^+_r)=0,$$

which finishes the proof.

Proof of (ii): Let χ_{Δ_r} be the characteristic function of Δ_r on $\partial_p D$. Let g_n be a sequence of nonnegative continuous functions on $\partial_p D$ such that $g_n\nearrow\chi_{\Delta_r}$. Let u_n be the solution to the heat equation in D with boundary values g_n. Then, by the maximum principle, $u_n(x,t)\nearrow\omega^{(x,t)}(\Delta_r)$ for $(x,t)\in D$.

Now we estimate $\vartheta^{(x,t)}_{+}(\Delta^+_r)+\vartheta^{(x,t)}_{-}(\Delta^-_r)$. Let $u^+_{n,r'}(x,t)$ be the solution to the heat equation in $D^+_{r'}$ with boundary value equal to g_n on $\partial_p D^+_{r'}\cap\partial_p D$ and equal to $\vartheta^{(x,t)}_{+}(\Delta^+_r)$ otherwise. Since $\vartheta^{(x,t)}_{+}(\Delta^+_r)=\lim_{r''\to 0}\omega^{+(x,t)}_{r''}(\Delta^+_r)$ takes the boundary value $\chi_{\Delta^+_r}$ on $\partial_p D^+_{r'}\cap\partial_p D$, then, by the maximum principle, we have $u^+_{n,r'}(x,t)\le\vartheta^{(x,t)}_{+}(\Delta^+_r)$ for $(x,t)\in D^+_{r'}$. Similarly, $u^-_{n,r'}(x,t)\le\vartheta^{(x,t)}_{-}(\Delta^-_r)$ for $(x,t)\in D^-_{r'}$. Therefore, for $(x,t)\in D^+_{r'}\cap D^-_{r'}$ and $0<r'<r$ sufficiently small, we have

$$u^+_{n,r'}(x,t)+u^-_{n,r'}(x,t)\le\vartheta^{(x,t)}_{+}(\Delta^+_r)+\vartheta^{(x,t)}_{-}(\Delta^-_r).\tag{5-9}$$

Let $r'\searrow 0$; then $D^+_{r'}\cap D^-_{r'}\nearrow D$. By the comparison principle, there is a nonnegative function $\tilde{u}_n$ in Ψ_1, caloric in D, such that

$$u^+_{n,r'}(x,t)+u^-_{n,r'}(x,t)\nearrow\tilde{u}_n(x,t)\quad\text{as }r'\searrow 0,\ (x,t)\in D.\tag{5-10}$$

By (i) just shown above and (5-9),

$$\sup_{\partial_p D^+_{r'}\cap D}u^+_{n,r'}(x,t)+\sup_{\partial_p D^-_{r'}\cap D}u^-_{n,r'}(x,t)\le\sup_{\partial_p D^+_{r'}\cap D}\vartheta^{(x,t)}_{+}(\Delta^+_r)+\sup_{\partial_p D^-_{r'}\cap D}\vartheta^{(x,t)}_{-}(\Delta^-_r)\to 0\quad\text{as }r'\to 0,$$

hence it is not hard to see that $\tilde{u}_n$ takes the boundary value g_n continuously on $\partial_p D$. Hence, by the maximum principle, $\tilde{u}_n = u_n$ in D. This, combined with (5-9) and (5-10), gives

$$u_n(x,t) \le \vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-). \tag{5-11}$$

Letting $n \to \infty$ in (5-11), we obtain

$$\omega^{(x,t)}(\Delta_r) \le \vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^+).$$

By taking the approximation $g_n \searrow \chi_{\Delta_r}$, $0 \le g_n \le 2$ and $\operatorname{supp} g_n \subset \mathcal{N}_{2r}(E_f) \cap \partial_p D$, we obtain the reverse inequality, and hence the equality.

Proof of (iii): We only show it for ϑ_+, and assume additionally that $\Delta_{r'}^{\pm} \neq \varnothing$.

First, for $0 < r'' < r' < r_0$, by Lemma 1.1 in [Kemper 1972b], there exists $C = C(n) \ge 0$ such that

$$\omega_{\Psi_{2r'}(y,s)\cap D_+}^{\bar{A}_{r'}^+(y,s)}(\Delta_{r'}^+) \ge C.$$

Applying the comparison principle in $\Psi_{2r'}(y,s) \cap D_+$, we have

$$\vartheta_+^{\bar{A}_{r'}^+(y,s)}(\Delta_{r'}^+) \ge C. \tag{5-12}$$

Next, for $0 < r'' < r' < r_0$, applying the same induction arguments as in Lemma 4.5, we have

$$\omega_{r''}^{+(x,t)}(\Delta_{r'}^+) \le C\omega_{r''}^{+\bar{A}_r^+(y,s)}(\Delta_{r'}^+) \quad \text{for } (x,t) \in D_{r''}^+ \setminus (\Psi_r(y,s))_+, \tag{5-13}$$

where $C = C(n, L)$ is independent of r' and r''. The reason that C is uniform in r'' is as follows. By the maximum principle, it is enough to show (5-13) for $(x,t) \in \partial(\Psi_r(y,s))_+ \cap D_{r''}^+$, which is contained in D_+. Hence, the same iteration procedure as in Lemma 4.5, but only on the D_+ side, gives (5-13), and the proof is uniform in r''. Therefore, letting $r'' \to 0$ in (5-13), we obtain

$$\vartheta_+^{(x,t)}(\Delta_{r'}^+) \le C\vartheta_+^{\bar{A}_r^+(y,s)}(\Delta_{r'}^+).$$

Applying Lemma 4.4 and the maximum principle, we deduce (iii).

Proof of (iv): Applying (iii), (ii), the Harnack inequality and Lemma 4.4, we have that, for given $(y,s) \in E_f \setminus G_f$ and $0 < r < r_0$,

$$\vartheta_-^{(X,T)}(\Delta_r^-) \le C\vartheta_-^{\bar{A}_{r_0}^-(y,s)}(\Delta_r^-) \le C\omega^{\bar{A}_{r_0}^-(y,s)}(\Delta_r) \le C\omega^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r) \le C\vartheta_+^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^+) \le C\vartheta_+^{(X,T)}(\Delta_r^+)$$

for $C = C(n, L, r_0)$. The second-last inequality holds because

$$\vartheta_+^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^+) \ge \vartheta_-^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^-), \tag{5-14}$$

which follows from the x_n-symmetry of D and the comparison principle. Equation (5-14), together with (ii) just shown above, yields the result. □

Now we use ϑ_+ and ϑ_- to construct two linearly independent kernel functions at $(y,s) \in E_f \setminus G_f$.

Theorem 5.3. *For $(y,s) \in E_f \setminus G_f$, there exist at least two linearly independent kernel functions at (y,s).*

Proof. Given $(y,s)\in E_f\setminus G_f$, let r_0 be as in (5-2). For $m>1/r_0$, we consider the sequence

$$v_m^+(x,t)=\frac{\vartheta_+^{(x,t)}(\Delta_{1/m}^+(y,s))}{\vartheta_+^{(X,T)}(\Delta_{1/m}^+(y,s))},\quad (x,t)\in D. \tag{5-15}$$

By Proposition 5.2(iii) and the same arguments as in Section 5A, we have, up to a subsequence, that $v_m(x,t)$ converges to a kernel function at (y,s) normalized at (X,T). We denote it by $K^+(x,t;y,s)$.

If we consider instead

$$v_m^-(x,t)=\frac{\vartheta_-^{(x,t)}(\Delta_{1/m}^-(y,s))}{\vartheta_-^{(X,T)}(\Delta_{1/m}^-(y,s))},\quad (x,t)\in D, \tag{5-16}$$

we will obtain another kernel function at (y,s), which we will denote $K^-(x,t;y,s)$.

We now show that, for fixed (y,s), $K^+(\,\cdot\,,\cdot\,;y,s)$ and $K^-(\,\cdot\,,\cdot\,;y,s)$ are linearly independent. In fact, by Proposition 5.2(i), (5-15) and (5-16), we have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_- and $K^-(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_+. If $K^+(\cdot,\cdot;y,s)=K^-(\cdot,\cdot;y,s)$, then we also have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_+, which will mean that $K^+(x,t;y,s)$ is a caloric function continuously vanishing on the whole of $\partial_p D$. By the maximum principle, K^+ will vanish in the entire domain D, which contradicts the normalization condition $K^+(X,T;y,s)=1$. Moreover, since $K^+(X,T;y,s)=K^-(X,T;y,s)=1$, it is impossible that $K^+(\,\cdot\,,\cdot\,;y,s)=\lambda K^-(\,\cdot\,,\cdot\,;y,s)$ for a constant $\lambda\neq 1$. Hence K^+ and K^- are linearly independent. □

Remark 5.4. The nonuniqueness of the kernel functions at (y,s) shows that the parabolic Martin boundary of D is not homeomorphic to the Euclidean parabolic boundary $\partial_p D$.

Next we show that K^+ and K^- in fact span the space of all the kernel functions at (y,s). We use an argument similar to the one in [Kemper 1972b].

Lemma 5.5. *Let $(y,s)\in E_f\setminus G_f$. There exists a positive constant $C=C(n,L,r_0)$ such that, if u is a kernel function at (y,s) in D, we have either*

$$u\geq CK^+ \tag{5-17}$$

or

$$u\geq CK^-. \tag{5-18}$$

Here K^+, K^- are the kernel functions at (y,s) constructed from (5-15) and (5-16).

Proof. For $0<r<r_0$, we consider $u_r^\pm : D_r^\pm\to\mathbb{R}$, where $u_r^\pm(x,t)=u(F_r^\pm(x,t))$. The functions $u_r^\pm$ are caloric in $D_r^\pm$ and continuous up to the boundary. Then, for $(x,t)\in D_r^\pm$,

$$\begin{aligned}
u_r^\pm(x,t)&=\int_{\partial_p D_r^\pm} u_r^\pm(z,h)\,d\omega_r^{\pm^{(x,t)}}(z,h)\\
&\geq\int_{\Delta_r^\pm(y,s)} u_r^\pm(z,h)\,d\omega_r^{\pm^{(x,t)}}(z,h)\\
&\geq\inf_{(z,h)\in\Delta_r^\pm(y,s)} u_r^\pm(z,h)\omega_r^{\pm^{(x,t)}}(\Delta_r^\pm(y,s)).
\end{aligned}$$

Note that the parabolic distance between $F_r^{\pm}(\Delta_r^{\pm}(y,s))$ and $\partial_p D$ is equivalent to r, and the time lag between it and $\bar{A}_r^{\pm}(y,s)$ is equivalent to r^2; hence, by the Harnack inequality, there exists $C = C(n, L)$ such that

$$\inf_{(z,h)\in\Delta_r^{\pm}(y,s)} u_r^{\pm}(z,h) \geq Cu(\bar{A}_r^{\pm}(y,s)).$$

Hence,

$$u_r^{\pm}(x,t) \geq Cu(\bar{A}_r^{\pm}(y,s))\omega_r^{\pm^{(x,t)}}(\Delta_r^{\pm}(y,s)) \quad \text{for } (x,t)\in D_r^{\pm}. \tag{5-19}$$

On the other hand, u is a kernel function at (y,s), and u vanishes on $\partial_p D \setminus \Delta_{r/4}(y,s)$ for any $0<r<1$. Applying Theorem 4.6, we obtain

$$u(x,t) \leq C \max\{u(\bar{A}_{r/2}^{+}(y,s)), u(\bar{A}_{r/2}^{-}(y,s))\}\omega^{(x,t)}(\Delta_r(y,s)) \quad \text{for } (x,t)\in D\setminus\Psi_{r/2}(y,s). \tag{5-20}$$

Case 1: $u(\bar{A}_{r/2}^{+}(y,s)) \geq u(\bar{A}_{r/2}^{-}(y,s))$ in (5-20).

By Proposition 5.2(ii) and the Harnack inequality,

$$u(x,t) \leq Cu(\bar{A}_r^{+}(y,s))(\vartheta_+^{(x,t)}(\Delta_r^{+}) + \vartheta_-^{(x,t)}(\Delta_r^{-})), \quad (x,t)\in D\setminus\Psi_{r/2}(y,s).$$

In particular,

$$1 = u(X,T) \leq Cu(\bar{A}_r^{+}(y,s))(\vartheta_+^{(X,T)}(\Delta_r^{+}) + \vartheta_-^{(X,T)}(\Delta_r^{-})). \tag{5-21}$$

Now (5-19) for u_r^{+}, (5-21) and Proposition 5.2(iv) yield the existence of $C_1 = C_1(n, L, r_0)$ such that, for any $0<r<r_0$,

$$u_r^{+}(x,t) \geq C\frac{\omega_r^{+^{(x,t)}}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+}) + \vartheta_-^{(X,T)}(\Delta_r^{-})} \geq C_1\frac{\omega_r^{+^{(x,t)}}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})}, \quad (x,t)\in D_r^{+}. \tag{5-22}$$

Since, by the maximum principle in D_r^{+},

$$\omega_r^{+^{(x,t)}}(\Delta_r^{+}) \geq \vartheta_+^{(x,t)}(\Delta_r^{+}) - \sup_{(z,h)\in\partial_p D_r^{+}\cap D} \vartheta_+^{(z,h)}(\Delta_r^{+}), \tag{5-23}$$

then (5-22) can be written as

$$u_r^{+}(x,t) \geq C_1\left(\frac{\vartheta_+^{(x,t)}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})} - \sup_{(z,h)\in\partial_p D_r^{+}\cap D}\frac{\vartheta_+^{(z,h)}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})}\right), \quad (x,t)\in D_r^{+}. \tag{5-24}$$

By Proposition 5.2(iii) and the Harnack inequality, there exists $C_2 = C_2(n, L, r_0)$ such that, for $(z,h)\in \partial_p D_r^{+}\cap D$,

$$\frac{\vartheta_+^{(z,h)}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})} \leq C\frac{\vartheta_+^{\bar{A}_{r_0}^{+}}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})}\cdot\vartheta_+^{(z,h)}(\Delta_{r_0}^{+}) \leq C_2\vartheta_+^{(z,h)}(\Delta_{r_0}^{+}). \tag{5-25}$$

Hence, (5-24) and (5-25) imply

$$u_r^{+}(x,t) \geq C_1\left(\frac{\vartheta_+^{(x,t)}(\Delta_r^{+})}{\vartheta_+^{(X,T)}(\Delta_r^{+})} - C_2\sup_{(z,h)\in\partial_p D_r^{+}\cap D}\vartheta_+^{(z,h)}(\Delta_{r_0}^{+})\right), \quad (x,t)\in D_r^{+}.$$

Case 2: $u(\bar{A}^+_{r/2}(y,s)) \le u(\bar{A}^-_{r/2}(y,s))$ in (5-20). Similarly,

$$u_r^-(x,t) \ge C_1\left(\frac{\vartheta_-^{(x,t)}(\Delta_r^-)}{\vartheta_-^{(X,T)}(\Delta_r^-)} - C_2 \sup_{(z,h)\in\partial_p D_r^-\cap D} \vartheta_-^{(z,h)}(\Delta_{r_0}^-)\right), \quad (x,t)\in D_r^-.$$

Note that as $r \searrow 0$, $D_r^\pm \nearrow D$ and $u_r^\pm \to u$. Let $r_j \to 0$ be such that either Case 1 applies for all r_j, or Case 2 applies. Hence, over a subsequence, it follows by Proposition 5.2(i) and (5-15) that either

$$u(x,t) \ge C_1 \lim_{r_j\to 0}\left(\frac{\vartheta_+^{(x,t)}(\Delta_{r_j}^+)}{\vartheta_+^{(X,T)}(\Delta_{r_j}^+)} - C_2 \sup_{(z,h)\in\partial_p D_{r_j}^+\cap D} \vartheta_+^{(z,h)}(\Delta_{r_0}^+)\right) = C_1K^+(x,t) \quad \text{for all } (x,t)\in D,$$

or

$$u(x,t) \ge C_1K^-(x,t) \quad \text{for all } (x,t)\in D. \qquad \square$$

The next theorem says that $K^+(\cdot,\cdot;y,s)$ and $K^-(\cdot,\cdot;y,s)$ span the space of kernel functions at (y,s).

Theorem 5.6. *If u is a kernel function at $(y,s)\in E_f\setminus G_f$ normalized at (X,T), then there exists a constant $\lambda\in[0,1]$, which may depend on (y,s), such that $u(\cdot,\cdot) = \lambda K^+(\cdot,\cdot;y,s) + (1-\lambda)K^-(\cdot,\cdot;y,s)$ in D, where K^+ and K^- are kernel functions obtained from (5-15) and (5-16).*

Proof. By Lemma 5.5, if u is a kernel function at (y,s), then either (i) $u \ge CK^+$ or (ii) $u \ge CK^-$ with $C = C(r_0,n,L)$.

If (i) holds, let

$$\lambda = \sup\{C : u(x,t) \ge CK^+(x,t) \text{ for all } (x,t)\in D\};$$

then we must have $\lambda \le 1$, because $u(X,T) = K^+(X,T) = 1$. If $\lambda = 1$, then $u(x,t) = K^+(x,t)$ for all $(x,t)\in D$, by the strong maximum principle, and we are done. If $\lambda < 1$, consider

$$u_1(x,t) := \frac{u(x,t) - \lambda K^+(x,t)}{1-\lambda},$$

which is another kernel function at (y,s) satisfying either (i) or (ii). If (i) holds for u_1 for some $C>0$, then $u(x,t) \ge (C(1-\lambda)+\lambda)K^+(x,t)$, with $C(1-\lambda)+\lambda > \lambda$, which contradicts the definition of λ as a supremum. Hence (ii) must be true for u_1. Let

$$\tilde\lambda = \sup\{C : u_1(x,t) \ge CK^-(x,t)\ \forall (x,t)\in D\}.$$

The same reason as above gives $\tilde\lambda \le 1$. We claim $\tilde\lambda = 1$.

Proof of the claim: If not, then $\tilde\lambda < 1$. We get that

$$u_2(x,t) := \frac{u_1(x,t) - \tilde\lambda K^-(x,t)}{1-\tilde\lambda}$$

is again a kernel function at (y,s). If u_2 satisfies (i) for some $C>0$, then

$$u_1(x,t) \ge u_1(x,t) - \tilde\lambda K^-(x,t) \ge C(1-\tilde\lambda)K^+(x,t),$$

which implies

$$u(x,t) \geq (\lambda + C(1-\tilde{\lambda}))K^+(x,t),$$

again a contradiction to the definition of λ. Hence, u_2 has to satisfy (ii) for some $C > 0$, and then we have

$$u_2(x,t) \geq (C(1-\tilde{\lambda}) + \tilde{\lambda})K^-(x,t),$$

but this contradicts the definition of $\tilde{\lambda}$. This completes the proof of the claim.

The fact that $\tilde{\lambda} = 1$ implies that $u_1(x,t) = K^-(x,t)$ in D, by the strong maximum principle. Hence, if (i) applies to u, we have $u(x,t) = \lambda K^+(x,t) + (1-\lambda)K^-(x,t)$ with $\lambda \in (0,1]$. If (ii) applies to u, we get the equality with $\lambda \in [0,1)$. □

5C. *Radon–Nikodym derivative as a kernel function.* We first show that the kernel function at $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$ is unique. The proof for the uniqueness is similar to Lemma 1.6 and Theorem 1.7 in [Kemper 1972b]. More precisely, we will need the direction-shift operator F_r^0:

$$\begin{aligned} F_r^0(x,t) &= (x'', x_{n-1} + 4nLr, x_n, t + 8r^2), \quad 0 < r < \tfrac{1}{4}, \\ D_r^0 &= \{(x,t) \in D : F_r^0(x,t) \in D\}. \end{aligned} \tag{5-26}$$

Let ω_r^0 denote the caloric measure for D_r^0. Note that D_r^0 is also a cylindrical domain with a thin Lipschitz complement.

Theorem 5.7. *For all $(y,s) \in \partial_p D$, the limit of (5-1) exists. If we denote the limit by $K_0(\cdot,\cdot;y,s)$, i.e.,*

$$K_0(x,t;y,s) = \lim_{n\to\infty} \frac{\omega^{(x,t)}(\Delta_{1/n}(y,s))}{\omega^{(X,T)}(\Delta_{1/n}(y,s))},$$

then:

(i) *For $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$, K_0 is the unique kernel function at (y,s).*

(ii) *If $(y,s) \in E_f \setminus G_f$, then K_0 is a kernel function at (y,s), and*

$$K_0(x,t;y,s) = \tfrac{1}{2}K^+(x,t;y,s) + \tfrac{1}{2}K^-(x,t;y,s), \tag{5-27}$$

where K^+ and K^- are kernel functions at (y,s) given by the limits of (5-15) and (5-16), respectively.

Proof. For $(y,s) \in G_f$ and r small enough, we denote $\bar{\bar{A}}_r(y,s) = (y'', y_{n-1} + 4nrL, 0, s + 4r^2)$, which is on $\{x_n = 0\}$ and has a time-lag $2r^2$ above $\bar{A}_r^{\pm}$. Then, by the Harnack inequality,

$$\omega^{\bar{A}_r^{\pm}(y,s)}(\Delta_{r'}(y,s)) \leq C(n,L)\,\omega^{\bar{\bar{A}}_r(y,s)}(\Delta_{r'}(y,s)) \qquad \text{for all } 0 < r' < r.$$

Then one can proceed as in Lemma 1.6 of [ibid.] by using F_r^0, D_r^0, ω^0 to show that any kernel function u (at (y,s)) satisfies $u \geq CK_0$ for some $C > 0$. Then the uniqueness follows from Theorem 1.7 and Remark 1.8 of [ibid.].

For $(y,s) \in \partial_p D \setminus E_f$, for r sufficiently small one has either $\Psi_r(y,s) \cap D \subset D_+$ or $\Psi_r(y,s) \cap D \subset D_-$. In either case, one can proceed as in Lemma 1.6, Theorem 1.7 and Remark 1.8 of [ibid.].

For $(y,s) \in E_f \setminus G_f$, by Theorem 5.6, $K_0(x,t;y,s) = \lambda K^+(x,t;y,s) + (1-\lambda)K^-(x,t;y,s)$ for some $\lambda \in [0,1]$. By Proposition 5.2(ii), the symmetry of the domain about x_{n-1} and the definitions of $K^\pm$, one has $\lambda = \frac{1}{2}$. □

Remark 5.8. From Theorem 5.7, we can conclude that the Radon–Nikodym derivative $d\omega^{(x,t)}/d\omega^{(X,T)}$ exists at every $(y,s) \in \partial_p D$ and it is the kernel function $K_0(x,t;y,s)$ with respect to (X,T).

The following corollary is an easy consequence of Theorems 5.6 and 5.7.

Corollary 5.9. *For fixed $(x,t) \in D$, the function $(y,s) \mapsto K_0(x,t;y,s)$ is continuous on $\partial_p D$, where K_0 is given by the limit of (5-1).*

Proof. Given $(y,s) \in \partial_p D$, let $(y_m, s_m) \in \partial_p D$ with $(y_m, s_m) \to (y,s)$ as $m \to \infty$.

If $(y,s) \in G_f$ or $\partial_p D \setminus E_f$, continuity follows from the uniqueness of the kernel function.

If $(y,s) \in E_f \setminus G_f$, by Theorem 5.7(ii), for each m we have

$$K_0(x,t;y_m,s_m) = \tfrac{1}{2}K^+(x,t;y_m,s_m) + \tfrac{1}{2}K^-(x,t;y_m,s_m). \tag{5-28}$$

Given $\varepsilon > 0$, $K^+(\cdot,\cdot;y_m,s_m)$ is uniformly bounded and equicontinuous on $D \setminus \Psi_\varepsilon(y,s)$ for m large enough. Hence, by a similar argument as in Section 5A, up to a subsequence, $K^+(\cdot,\cdot;y_m,s_m) \to v^+(\cdot,\cdot;y,s)$ uniformly on compact subsets, where $v^+(\cdot,\cdot;y,s)$ is some kernel function at (y,s). Moreover, by Theorem 5.6, we have

$$v^+(\cdot,\cdot;y,s) = \lambda K^+(\cdot,\cdot;y,s) + (1-\lambda)K^-(\cdot,\cdot;y,s) \quad \text{for some } \lambda \in [0,1]. \tag{5-29}$$

By Proposition 5.2(i),

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} K^+(x,t;y_m,s_m) \to 0 \quad \text{as } r \to 0,$$

which is uniform in m from the proof of the proposition. Hence, after $m \to \infty$, v^+ satisfies

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} v^+(x,t) \to 0 \quad \text{as } r \to 0,$$

which, combined with

$$K^-(x,t;y,s) \not\to 0 \quad \text{as } (x,t) \to (y,s), \text{ for } (x,t) \in D_-,$$

gives $\lambda = 1$ in (5-29).

Similarly, up to a subsequence, $K^-(x,t;y_m,s_m) \to K^-(x,t;y,s)$.

Thus, along a subsequence, $K(\cdot,\cdot;y_m,s_m) \to K_0(\cdot,\cdot;y,s)$ by (5-27). Since this holds for all the convergent subsequences, then $K_0(x,t;y,s)$ is continuous on $\partial_p D$ for fixed (x,t). □

By using Corollary 5.9, Remark 5.8 and Theorem 4.6, we can prove some uniform behavior of K_0 on $\partial_p D$, as in Lemmas 2.2 and 2.3 of [Kemper 1972b]. We state the results in the following two lemmas and omit the proof of the first.

Lemma 5.10. *Let* $(y,s) \in \partial_p D$. *Then, for* $0 < r < \frac{1}{4}$,

$$\sup_{(y',s') \in \partial_p D \setminus \Delta_r(y,s)} K_0(x,t;y',s') \to 0 \quad \text{as } (x,t) \to (y,s) \text{ in } D.$$

The following lemma says that if D' is a domain obtained by a perturbation of a portion of $\partial_p D$ where $\omega^{(x,t)}$ vanishes, then the caloric measure $\omega_{D'}$ is equivalent to ω_D on the common boundary of D' and D. We recall here that ω_r^0 is the caloric measure with respect to the domain D_r^0 defined in (5-26), and $\omega_r^\pm$ is the caloric measure with respect to $D_r^\pm$ defined in (5-5).

Lemma 5.11. (i) *Let* $0 < r < \frac{1}{4}$ *and* $(y,s) \in G_f \cup (\partial_p D \setminus E_f)$ *with* $s > -1 + 4r^2$. *Then there exist* $\rho_0 = \rho_0(n,L) > 0$ *and* $C = C(n,L) > 0$ *such that, for* $0 < \rho < \rho_0$, *we have*

$$\omega_\rho^{0^{(X',T')}}(\Delta_r(y,s)) \geq C\omega^{(X',T')}(\Delta_r(y,s)), \quad (X',T') \in \Psi_{1/4}(X,T), \tag{5-30}$$

provided also $r < |y_n|$ *for* $(y,s) \in \partial_p D \setminus E_f$.

(ii) *Let* $(y,s) \in (\mathcal{N}_r(E_f) \cap \partial_p D) \setminus G_f$. *Then there exists* $\delta_0 = \delta_0(n,L) > 0$, *such that, for* $0 < r' < \delta_0$, *we have*

$$\omega_{r'}^{+^{(X',T')}}(\Delta_r^+(y,s)) + \omega_{r'}^{-^{(X',T')}}(\Delta_r^-(y,s)) \geq \tfrac{1}{2}\omega^{(X',T')}(\Delta_r(y,s)) \tag{5-31}$$

for $(X',T') \in \Psi_{1/4}(X,T)$ *and* $0 < r < r_0$, *where* r_0 *is the constant defined in* (5-2).

Proof. To show (5-31) we first argue similarly as in [Kemper 1972b] to show there exists $\delta_0 = \delta_0(n,L) > 0$ such that, for any $0 < r' < \delta_0$,

$$\omega_{r'}^{\pm^{(X',T')}}(\Delta_r^\pm(y,s)) \geq \tfrac{1}{2}\vartheta_\pm^{(X',T')}(\Delta_r^\pm(y,s)) \tag{5-32}$$

for each $\Delta_r^\pm(y,s)$ with $0 < r < r_0$. Then using Proposition 5.2(ii) we get the conclusion. □

6. Backward boundary Harnack principle

In this section, we follow the lines of [Fabes et al. 1984] to build up a backward Harnack inequality for nonnegative caloric functions in D. To prove this kind of inequality, we have to ask that these functions vanish on the *lateral boundary*

$$S := \partial_p D \cap \{s > -1\},$$

or at least a portion of it. This will allow to control the time-lag issue in the parabolic Harnack inequality.

Some of the proofs in this section follow the lines of the corresponding proofs in [ibid.]. For that reason, we will omit the parts that don't require modifications or additional arguments.

For (x,t) and $(y,s) \in D$, denote by $G(x,t;y,s)$ the Green's function for the heat equation in the domain D. Since D is a regular domain, the Green's function can be written in the form

$$G(x,t;y,s) = \Gamma(x,t;y,s) - V(x,t;y,s),$$

where $\Gamma(\cdot\,,\cdot\,;y,s)$ is the fundamental solution of the heat equation with pole at (y,s), and $V(\cdot\,,\cdot\,;y,s)$ is a caloric function in D that equals $\Gamma(\cdot\,,\cdot\,;y,s)$ on $\partial_p D$. We note that, by the maximum principle, we have $G(x,t;y,s) = 0$ whenever $(x,t) \in D$ with $t \leq s$.

In this section, similarly to Section 5, we will work under Convention 5.1. In particular, in Green's function we will allow the pole (y,s) to be in $\widetilde{D}$ with $s\geq 1$. But in that case we simply have $G(x,t;y,s)=0$ for all $(x,t)\in D$.

Lemma 6.1. *Let $0<r<\frac{1}{4}$ and $(y,s)\in S$ with $s\geq -1+8r^2$. Then there exists a constant $C=C(n,L)>0$ such that, for $(x,t)\in D\cap\{t\geq s+4r^2\}$, we have*

$$C^{-1}r^n\max\{G(x,t;\bar{A}_r^{\pm}(y,s))\}\leq\omega^{(x,t)}(\Delta_r(y,s))\leq Cr^n\max\{G(x,t;\underline{A}_r^{\pm}(y,s))\}\quad \text{if } (y,s)\in\mathcal{N}_r(E_f),\tag{6-1}$$

and

$$C^{-1}r^nG(x,t;\bar{A}_r(y,s))\leq\omega^{(x,t)}(\Delta_r(y,s))\leq Cr^nG(x,t;\underline{A}_r(y,s))\quad \text{if } (y,s)\notin\mathcal{N}_r(E_f).\tag{6-2}$$

Proof. The proof uses Lemma 4.4 and Theorem 4.3, and is similar to that of Lemma 1 in [ibid.]. □

Theorem 6.2 (interior backward Harnack inequality). *Let u be a positive caloric function in D vanishing continuously on S. Then, for any compact $K\Subset D$, there exists a constant $C=C(n,L,\operatorname{dist}_p(K,\partial_pD))$ such that*

$$\max_K u\leq C\min_K u.$$

Proof. The proof is similar to that of Theorem 1 in [ibid.], and uses Theorem 4.3 and the Harnack inequality. □

Theorem 6.3 (local comparison theorem). *Let $0<r<\frac{1}{4}$, $(y,s)\in S$ with $s\geq -1+18r^2$, and u, v be two positive caloric functions in $\Psi_{3r}(y,s)\cap D$ vanishing continuously on $\Delta_{3r}(y,s)$. Then there exists $C=C(n,L)>0$ such that, for $(x,t)\in\Psi_{r/8}(y,s)\cap D$, we have*

$$\frac{u(x,t)}{v(x,t)}\leq C\frac{\max\{u(\bar{A}_r^+(y,s)),u(\bar{A}_r^-(y,s))\}}{\min\{v(\underline{A}_r^+(y,s)),v(\underline{A}_r^-(y,s))\}}\quad \text{if } (y,s)\in\mathcal{N}_r(E_f),\tag{6-3}$$

and

$$\frac{u(x,t)}{v(x,t)}\leq C\frac{u(\bar{A}_r(y,s))}{v(\underline{A}_r(y,s))}\quad \text{if } (y,s)\notin\mathcal{N}(E_f).\tag{6-4}$$

Proof. The proof is similar to that of Theorem 3 in [ibid.]. First, note that if $\Psi_{r/8}(y,s)\cap E_f=\varnothing$, we can consider the restrictions of u and v to D_+ or D_- (which are Lipschitz cylinders) and apply the arguments from [ibid.] directly there. Thus, we may assume that $\Psi_{r/8}(y,s)\cap E_f\neq\varnothing$. If we now argue as in the proof of the localization property (Lemma 2.3) by replacing (y,s) and r with $(\tilde{y},\tilde{s})\in\Psi_{(3/8)r}(y,s)\cap E_f$, we may further assume that $(y,s)\in E_f$, and that $\Psi_r(y,s)\cap D$ falls either into category (2) or (3) in the localization property. For definiteness, we will assume category (3). To account for the possible change in (y,s), we then change the hypothesis to assume that $u=0$ on $\Delta_{2r}(y,s)$, and prove (6-3) for $(x,t)\in\Psi_{r/2}(y,s)\cap D$.

With this simplification in mind, we proceed as in the proof of Theorem 3 in [ibid.]. By using Lemma 6.1 and Theorem 4.6, we first show

$$\omega_r^{(x,t)}(\alpha_r)\leq C\omega_r^{(x,t)}(\beta_r),\qquad (x,t)\in\Psi_{r/2}(y,s)\cap D,\tag{6-5}$$

where $\alpha_r = \partial_p(\Psi_r(y,s)\cap D)\setminus S$, $\beta_r = \partial_p(\Psi_r(y,s)\cap D)\setminus \mathcal{N}_{\mu r}(S)$ with a small fixed $\mu\in(0,1)$, and where ω_r denotes the caloric measure with respect to $\Psi_r(y,s)\cap D$. Then by Theorem 4.3, the Harnack inequality and the maximum principle, we obtain

$$u(x,t)\le C\max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\}\omega_r^{(x,t)}(\alpha_r),$$
$$v(x,t)\ge C\min\{v(\underline{A}_r^+(y,s)), v(\underline{A}_r^-(y,s))\}\omega_r^{(x,t)}(\beta_r),$$

which, combined with (6-5), completes the proof. □

Theorem 6.4 (global comparison theorem). *Let u, v be two positive caloric functions in D, vanishing continuously on S, and let (x_0,t_0) be a fixed point in D. If $\delta>0$, then there exists $C=C(n,L,\delta)>0$ such that*

$$\frac{u(x,t)}{v(x,t)}\le C\frac{u(x_0,t_0)}{v(x_0,t_0)}\quad \text{for all } (x,t)\in D\cap\{t>-1+\delta^2\}. \tag{6-6}$$

Proof. This is an easy consequence of Theorems 6.2 and 6.3. □

Now we show the doubling properties of the caloric measure at the lateral boundary points by using the properties of the kernel functions we showed in Section 5. The idea of the proof is similar to that of Lemma 2.2 in [Wu 1979], but with a more careful inspection of the different types of boundary points.

To proceed, we will need to define the time-invariant corkscrew points at (y,s) on the lateral boundary, in addition to future and past corkscrew points. Namely, for $(y,s)\in S$, we let

$$\begin{aligned} A_r(y,s) &= (y(1-r),s) && \text{if } \Psi_r(y,s)\cap E_f=\varnothing,\\ A_r^{\pm}(y,s) &= (y'', y_{n-1}+r/2, \pm r/2, s) && \text{if } \Psi_r(y,s)\cap E_f\ne\varnothing. \end{aligned}$$

Theorem 6.5 (doubling at the lateral boundary points). *For $0<r<\frac14$ and $(y,s)\in S$ with $s\ge -1+8r^2$, there exist $\varepsilon_0=\varepsilon_0(n,L)>0$ small and $C=C(n,L)>0$ such that, for any $r<\varepsilon_0$, we have:*

(i) *If $(y,s)\in E_f$ and $\Psi_{2r}(y,s)\cap G_f\ne\varnothing$, then*

$$C^{-1}r^nG(X,T;A_r^{\pm}(y,s))\le\omega^{(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r^{\pm}(y,s)). \tag{6-7}$$

(ii) *If $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_pD$ and $\Psi_{2r}(y,s)\cap G_f=\varnothing$, then*

$$C^{-1}r^nG(X,T;A_r^{+}(y,s))\le\vartheta_+^{(X,T)}(\Delta_r^+(y,s))\le Cr^nG(X,T;A_r^{+}(y,s)), \tag{6-8}$$
$$C^{-1}r^nG(X,T;A_r^{-}(y,s))\le\vartheta_-^{(X,T)}(\Delta_r^-(y,s))\le Cr^nG(X,T;A_r^{-}(y,s)). \tag{6-9}$$

(iii) *If $(y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f)$, then*

$$C^{-1}r^nG(X,T;A_r(y,s))\le\omega^{(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r(y,s)). \tag{6-10}$$

Moreover, there is a constant $C=C(n,L)>0$ such that:

- *For $(y,s)\in S\cap\{s\ge -1+8r^2\}$,*

$$\omega^{(X,T)}(\Delta_{2r}(y,s))\le C\omega^{(X,T)}(\Delta_r(y,s))u(x,t). \tag{6-11}$$

- *For* $(y,s)\in\mathcal{N}_r(E_f)\cap S\cap\{s\geq -1+8r^2\}$,

$$\vartheta_+^{(X,T)}(\Delta_{2r}^+(y,s))\leq C\vartheta_+^{(X,T)}(\Delta_r^+(y,s)),$$
$$\vartheta_-^{(X,T)}(\Delta_{2r}^-(y,s))\leq C\vartheta_-^{(X,T)}(\Delta_r^-(y,s)). \tag{6-12}$$

Proof. We start by showing the estimates from above in (6-7) and (6-8).

Case 1: $(y,s)\in E_f$ and $\Psi_{2r}(y,s)\cap G_f\neq\varnothing$. By Lemma 2.3, there is $(\tilde{y},\tilde{s})\in G_f$ such that

$$\Psi_r(y,s)\cap D\subset\Psi_{4r}(\tilde{y},\tilde{s})\cap D\subset\Psi_{8r}(y,s)\cap D.$$

It is not hard to check, by (5-26), that $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))\subset D$. Moreover, the parabolic distance between $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))$ and $\partial_p D$, and the t-coordinate distance from $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))$ down to $A_r^\pm$, are greater than cr for some universal c which only depends on n and L. Therefore, by the estimate of Green's function as in [Wu 1979], we have

$$G(x,t;A_r^\pm(y,s))\geq C(n,L)r^{-n},\quad (x,t)\in F_r^0(\Delta_{4r}(\tilde{y},\tilde{s})).$$

Applying the maximum principle to $F_r^0(D_r^0)$, we have

$$G(x,t;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(x,t)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

In particular,

$$G(X,T;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(X,T)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

Let $(X_r,T_r):=F_r^{0^{-1}}(X,T)$ and take $(X',T')\in D$ with $T'=T-\frac{1}{4}$, $X'=X$, so that $T'>\frac{1}{4}+T_r$. Then we obtain, by the Harnack inequality, that

$$G(X,T;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6-13}$$

By Lemma 5.11(i), for $0<r<\min\{\frac{1}{4},\rho_0\}$, there exists $C=C(n,L)$, independent of r, such that

$$\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq C\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6-14}$$

By Theorem 5.7, for each $(\tilde{y},\tilde{s})\in G_f$,

$$K_0(X',T';\tilde{y},\tilde{s})=\lim_{r\to 0}\frac{\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s}))}{\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s}))}>0,$$

and by Corollary 5.9, for (X',T') fixed, $K_0(X',T';\cdot,\cdot)$ is continuous on $\partial_p D$. Therefore, in the compact set G_f, there exists $c>0$, only depending on n, L, such that $K_0(X',T';\tilde{y},\tilde{s})\geq c>0$ for any $(\tilde{y},\tilde{s})\in G_f$. Hence, by the Radon–Nikodym theorem for $0<r<\min\{\frac{1}{4},\rho_0\}$, we have

$$\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq\frac{c}{2}\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq\frac{c}{2}\omega^{(X,T)}(\Delta_r(y,s)). \tag{6-15}$$

Combining (6-13), (6-14) and (6-15), we obtain the estimate from above in (6-7) for Case 1.

Case 2: $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_p D$ and $\Psi_{2r}(y,s)\cap G_f=\varnothing$.

In this case, $\Psi_{2r}(y,s)\cap D$ splits into the disjoint union of $\Psi_{2r}(y,s)\cap D_\pm$. We use F_r^+ and F_r^-, defined in (5-3) and (5-4), and apply the same arguments as in Case 1 in D_r^+ and D_r^-. Then

$$\omega_r^{\pm(X,T)}(\Delta_r^\pm(y,s))\le Cr^nG(X,T;A_r^\pm(y,s)).$$

Taking $0<r<\delta_0$, where $\delta_0=\delta_0(n,L)$ is the constant in Lemma 5.11(ii), we have

$$\vartheta_\pm^{(X,T)}(\Delta_r(y,s))\le 2\omega_r^{\pm(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r^\pm(y,s)).$$

Case 3: $(y,s)\in\partial_p D\setminus\mathcal{N}_r(E_f)$. We argue similarly to Cases 1 and 2.

Taking $\varepsilon_0=\min\{\rho_0,\delta_0,\frac14\}$, we complete the proof of the estimates from above in (6-7)–(6-10).

The proof of the estimate from below in (6-7)–(6-10) is the same as in [Wu 1979]. For (6-7) it is a consequence of Lemma 4.4 and the maximum principle. (6-8) and (6-9) follow from (5-12) and the maximum principle. The doubling properties of caloric measure $\omega^{(x,t)}$ and $\theta_\pm^{(x,t)}$ are easy consequences of (6-7)–(6-10) and Proposition 5.2(ii) for $0<r<\varepsilon_0/2$. For $r>\varepsilon_0/2$ we use Lemma 4.4 and (5-12). □

Theorem 6.5 implies the following backward Harnack principle.

Theorem 6.6 (backward boundary Harnack principle). *Let u be a positive caloric function in D vanishing continuously on S, and let $\delta>0$. Then there exists a positive constant $C=C(n,L,\delta)$ such that, for $(y,s)\in\partial_p D\cap\{s>-1+\delta^2\}$ and for $0<r<r(n,L,\delta)$ sufficiently small, we have*

$$\left.\begin{aligned} C^{-1}u(\underline{A}_r^+(y,s))\le u(\bar{A}_r^+(y,s))\le Cu(\underline{A}_r^+(y,s))\\ C^{-1}u(\underline{A}_r^-(y,s))\le u(\bar{A}_r^-(y,s))\le Cu(\underline{A}_r^-(y,s))\end{aligned}\right\}\quad \text{if } (y,s)\in\mathcal{N}_r(E_f)$$

and

$$C^{-1}u(\underline{A}_r(y,s))\le u(\bar{A}_r(y,s))\le Cu(\underline{A}_r(y,s))\quad \text{if } (y,s)\notin\mathcal{N}_r(E_f). \tag{6-16}$$

Proof. Once we have Theorem 6.5, which is an analogue of Lemma 2.2 in [Wu 1979], we can proceed as in Theorem 4 in [Fabes et al. 1984] to show the backward Harnack principle. □

Remark 6.7. From (6-7), and using the same proof as in Theorem 6.6, we can conclude that, for any positive caloric function u vanishing continuously on S and $(y,s)\in G_f$, there exists $C=C(n,L,\delta)>0$ such that

$$C^{-1}u(\bar{A}_r^-(y,s))\le u(\bar{A}_r^+(y,s))\le Cu(\bar{A}_r^-(y,s)),$$
$$C^{-1}u(\underline{A}_r^-(y,s))\le u(\underline{A}_r^+(y,s))\le Cu(\underline{A}_r^-(y,s)).$$

7. Various versions of boundary Harnack

In the applications, it is very useful to have a local version of the backward Harnack for solutions vanishing only on a portion of the lateral boundary S. For the parabolically Lipschitz domains this was proved in [Athanasopoulos et al. 1996] as a consequence of the (global) backward Harnack principle.

To state the results, we use the following corkscrew points associated with $(y,s)\in G_f$: for $0<r<\frac{1}{4}$, let

$$\begin{aligned}\bar{A}_r(y,s)&=(y'', y_{n-1}+4nLr, 0, s+2r^2),\\ \underline{A}_r(y,s)&=(y'', y_{n-1}+4nLr, 0, s-2r^2),\\ A_r(y,s)&=(y'', y_{n-1}+4nLr, 0, s).\end{aligned}$$

When $(y,s)=(0,0)$, we simply write $\bar{A}_r$, $\underline{A}_r$ and A_r, in addition to Ψ_r, Δ_r, $\bar{A}^{\pm}_r$, $\underline{A}^{\pm}_r$.

Theorem 7.1. *Let u be a nonnegative caloric function in D, vanishing continuously on E_f. Let $m=u(\underline{A}_{3/4})$, $M=\sup_D u$. Then there exists a constant $C=C(n,L,M/m)$ such that, for any $0<r<\frac{1}{4}$, we have*

$$u(\bar{A}_r)\le Cu(\underline{A}_r). \tag{7-1}$$

Proof. Using Theorems 6.6 and 6.5 and following the lines of Theorem 13.7 in [Caffarelli and Salsa 2005], we have

$$u(\bar{A}^{\pm}_{2r})\le Cu(\underline{A}^{\pm}_{2r}),\quad 0<r<\tfrac{1}{4},$$

for $C=C(n,L,M/m)$. Then (7-1) follows from Theorem 6.6 and the observation that there is a Harnack chain with constant $\mu=\mu(n,L)$ and length $N=N(n,L)$ joining $\bar{A}_r$ to $\bar{A}^{\pm}_{2r}$ and $\underline{A}^{\pm}_{2r}$ to $\underline{A}_r$. □

Theorem 7.1 implies the boundary Hölder-regularity of the quotient of two negative caloric functions vanishing on E_f. The proof of the following corollary is the same as for Corollary 13.8 in [Caffarelli and Salsa 2005], and is therefore omitted.

Theorem 7.2. *Let u_1, u_2 be nonnegative caloric functions in D continuously vanishing on E_f. Let $M_i=\sup_D u_i$ and $m_i=u_i(\underline{A}_{3/4})$ with $i=1,2$. Then we have*

$$C^{-1}\frac{u_1(A_{1/4})}{u_2(A_{1/4})}\le\frac{u_1(x,t)}{u_2(x,t)}\le C\frac{u_1(A_{1/4})}{u_2(A_{1/4})}\quad \textit{for } (x,t)\cap\Psi_{1/8}\cap D, \tag{7-2}$$

where $C=C(n,L,M_1/m_1,M_2/m_2)$. Moreover, if u_1 and u_2 are symmetric in x_n, then u_1/u_2 extends to a function in $C^\alpha(\Psi_{1/8})$ for some $0<\alpha<1$, where the exponent α and the C^α-norm depend only on n, L, M_1/m_1, M_2/m_2. □

Remark 7.3. The symmetry condition in the latter part of the theorem is important to guarantee the continuous extension of u_1/u_2 to the Euclidean closure $\overline{\Psi_{1/8}\setminus E_f}=\overline{\Psi_{1/8}}$, since the limits at $E_f\setminus G_f$ as we approach from different sides may be different. Without the symmetry condition, one may still prove that u_1/u_2 extends to a C^α function on the completion $(\Psi_{1/8}\setminus E_f)^*$ with respect to the inner metric.

For a more general application, we need to have a boundary Harnack inequality for u satisfying a nonhomogeneous equation with bounded right-hand side, but additionally with a nondegeneracy condition. The method we use here is similar to the one used in the elliptic case [Caffarelli et al. 2008].

Theorem 7.4. *Let u be a nonnegative function in D, continuously vanishing on E_f, and satisfying*

$$|\Delta u-\partial_t u|\le C_0\quad \textit{in } D, \tag{7-3}$$

$$u(x,t)\ge c_0\operatorname{dist}_p((x,t),E_f)^\gamma\quad \textit{in } D, \tag{7-4}$$

where $0 < \gamma < 2$, $c_0 > 0$, $C_0 \geq 0$. *Then there exists* $C = C(n, L, \gamma, C_0, c_0) > 0$ *such that, for* $0 < r < \frac{1}{4}$, *we have*

$$u(x,t) \leq C u(\bar{A}_r), \quad (x,t) \in \Psi_r. \tag{7-5}$$

Moreover, if $M = \sup_D u$, *then there exists a constant* $C = C(n, L, \gamma, C_0, c_0, M)$ *such that, for any* $0 < r < \frac{1}{4}$, *we have*

$$u(\bar{A}_r) \leq C u(\underline{A}_r). \tag{7-6}$$

Proof. Let u^* be a solution to the heat equation in $\Psi_{2r} \cap D$ that is equal to u on $\partial_p(\Psi_{2r} \cap D)$. Then, by the Carleson estimate, we have $u^*(x,t) \leq C(n, L) u^*(\bar{A}_r)$ for $(x,t) \in \Psi_r$.

On the other hand, we have

$$u^*(x,t) + C(|x|^2 - t - 8r^2) \leq u(x,t) \quad \text{on } \partial_p(\Psi_{2r} \cap D),$$
$$(\Delta - \partial_t)(u^*(x,t) + C(|x|^2 - t - 8r^2)) \geq C(2n-1) \geq (\Delta - \partial_t) u(x,t) \quad \text{in } \Psi_{2r} \cap D$$

for $C \geq C_0/(2n-1)$. Hence, by the comparison principle, we have $u^* - u \leq Cr^2$ in $\Psi_{2r} \cap D$ for $C = C(C_0, n)$. Similarly, $u - u^* \leq Cr^2$, and hence $|u - u^*| \leq Cr^2$ in $\Psi_{2r} \cap D$. Consequently,

$$u(x,t) \leq C(n, L)(u(\bar{A}_r) + C(C_0, n) r^2), \quad (x,t) \in \Psi_r. \tag{7-7}$$

Next, note that, by the nondegeneracy condition (7-4),

$$u(\bar{A}_r) \geq c_0 r^\gamma \geq c_0 r^2, \quad r \in (0, 1). \tag{7-8}$$

Thus, combining (7-7) and (7-8), we obtain (7-5).

The proof of (7-6) follows in a similar manner from Theorem 7.1 for u^*. □

Remark 7.5. In fact, the nondegeneracy condition (7-4) is necessary. An easy counterexample is $u(x,t) = x_{n-1}^2 x_n^2$ in Ψ_1 and $E_f = \{(x,t) : x_{n-1} \leq 0, x_n = 0\} \cap \Psi_1$. Then $u(\bar{A}_r) = 0$ for $r \in (0, 1)$, but obviously u does not vanish in $\Psi_r \cap D$.

We next state a generalization of the local comparison theorem.

Theorem 7.6. *Let* u_1, u_2 *be nonnegative functions in* D, *continuously vanishing on* E_f, *and satisfying*

$$|\Delta u_i - \partial_t u_i| \leq C_0 \quad \textit{in } D,$$
$$u_i(x,t) \geq c_0 \operatorname{dist}_p((x,t), E_f)^\gamma \quad \textit{in } D$$

for $i = 1, 2$, *where* $0 < \gamma < 2$, $c_0 > 0$, $C_0 \geq 0$. *Let* $M = \max\{\sup_D u_1, \sup_D u_2\}$. *Then there exists a constant* $C = C(n, L, \gamma, C_0, c_0, M) > 0$ *such that*

$$C^{-1} \frac{u_1(A_{1/4})}{u_2(A_{1/4})} \leq \frac{u_1(x,t)}{u_2(x,t)} \leq C \frac{u_1(A_{1/4})}{u_2(A_{1/4})}, \quad (x,t) \in \Psi_{1/8} \cap D. \tag{7-9}$$

Moreover, if u_1 *and* u_2 *are symmetric in* x_n, *then* u_1/u_2 *extends to a function in* $C^\alpha(\overline{\Psi_{1/8}})$ *for some* $0 < \alpha < 1$, *with* α *and the* C^α*-norm depending only on* $n, L, \gamma, C_0, c_0, M$.

To prove this theorem, we will also need the following two lemmas, which are essentially Lemmas 11.5 and 11.8 in [Danielli et al. 2013]. The proofs are therefore omitted.

Lemma 7.7. *Let Λ be a subset of $\mathbb{R}^{n-1} \times (-\infty, 0]$, and $h(x,t)$ a continuous function in Ψ_1. Then, for any $\delta_0 > 0$, there exists $\varepsilon_0 > 0$ depending only on δ_0 and n such that, if*:

(i) $h \geq 0$ *on* $\Psi_1 \cap \Lambda$,

(ii) $(\Delta - \partial_t)h \leq \varepsilon_0$ *in* $\Psi_1 \setminus \Lambda$,

(iii) $h \geq -\varepsilon_0$ *in* Ψ_1,

(iv) $h \geq \delta_0$ *in* $\Psi_1 \cap \{|x_n| \geq \beta_n\}$, $\beta_n = 1/(32\sqrt{n-1})$,

then $h \geq 0$ *in* $\Psi_{1/2}$. □

Lemma 7.8. *For any $\delta_0 > 0$, there exists $\varepsilon_0 > 0$ and $c_0 > 0$, depending only on δ_0 and n, such that, if h is a continuous function on $\Psi_1 \cap \{0 \leq x_n \leq \beta_n\}$, $\beta_n = 1/(32\sqrt{n-1})$, satisfying*:

(i) $(\Delta - \partial_t)h \leq \varepsilon_0$ *in* $\Psi_1 \cap \{0 < x_n < \beta_n\}$,

(ii) $h \geq 0$ *in* $\Psi_1 \cap \{0 < x_n < \beta_n\}$,

(iii) $h \geq \delta_0$ *on* $\Psi_1 \cap \{x_n = \beta_n\}$,

then

$$h(x,t) \geq c_0 x_n \quad \text{in } \Psi_{1/2} \cap \{0 < x_n < \beta_n\}. \qquad \square$$

Proof of Theorem 7.6. We first note that, arguing as in the proof of Theorem 7.4 and using Theorem 7.1, we will have that

$$u_i(x,t) \leq C u_i(A_{1/4}), \quad (x,t) \in \Psi_{1/8}, \tag{7-10}$$

for $C = C(n, L, \gamma, C_0, c_0, M)$. Next, dividing u_i by $u_i(A_{1/4})$, we can assume $u_i(A_{1/4}) = 1$. Then consider the rescalings

$$u_{i\rho}(x,t) = \frac{u_i(\rho x, \rho^2 t)}{\rho^\gamma}, \quad \rho \in (0,1),\ i = 1, 2.$$

It is immediate to verify that, for $(x,t) \in \Psi_{1/(8\rho)} \cap D$, the functions $u_{i\rho}$ satisfy

$$|(\Delta - \partial_t)u_{i\rho}(x,t)| \leq C_0 \rho^{2-\gamma}, \tag{7-11}$$

$$u_{i\rho}(x,t) \geq c_0 \operatorname{dist}_p((x,t), E_{f_\rho})^\gamma, \tag{7-12}$$

$$u_{i\rho}(x,t) \leq \frac{C}{\rho^\gamma}, \quad \text{where } C \text{ is the constant in (7-10)}, \tag{7-13}$$

where $f_\rho(x'', t) = (1/\rho) f(\rho x'', \rho^2 t)$ is the scaling of f. By (7-12), there exists $c_n > 0$ such that

$$u_{i\rho}(x,t) \geq c_0 c_n, \quad (x,t) \in \Psi_{1/(8\rho)} \cap \{|x_n| \geq \beta_n\}. \tag{7-14}$$

Consider now the difference

$$h = u_{2\rho} - s u_{1\rho}$$

for a small positive s, specified below. By (7-11), (7-14) and (7-13), one can choose a positive $\rho = \rho(n, L, \gamma, C_0, c_0, M) < \frac{1}{16}$ and $s = s(\rho, n, c_0, C) > 0$ such that

$$\begin{aligned} h(x,t) &\geq c_0 c_n - s \cdot \frac{C}{\rho^\gamma} \geq \frac{c_0 c_n}{2}, && (x,t) \in \Psi_{1/(8\rho)} \cap \{|x_n| \geq \beta_n\}, \\ h(x,t) &\geq -s \cdot \frac{C}{\rho^\gamma} \geq -\varepsilon_0, && (x,t) \in \Psi_{1/(8\rho)}, \\ |(\Delta - \partial_t)h(x,t)| &\leq C_0 \rho^{2-\gamma} \leq \varepsilon_0, && (x,t) \in \Psi_{1/(8\rho)} \cap D, \end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(c_0, c_n, n)$ is the constant in Lemma 11.5 of [Danielli et al. 2013]. Thus, by that result, $h > 0$ in $\Psi_{1/2} \cap D$, which implies

$$\frac{u_1(x,t)}{u_2(x,t)} \leq \frac{1}{s}, \qquad (x,t) \in \Psi_{\rho/2} \cap D. \tag{7-15}$$

By moving the origin to any $(z, h) \in \Psi_{1/8} \cap E_f$, we will therefore obtain the bound

$$\frac{u_1(x,t)}{u_2(x,t)} \leq C(n, L, \gamma, C_0, c_0, M) \tag{7-16}$$

for any $(x,t) \in \Psi_{1/8} \cap \mathcal{N}_{\rho/2}(E_f) \cap D$. On the other hand, for $(x,t) \in \Psi_{1/8} \setminus \mathcal{N}_{\rho/2}(E_f)$, the estimate (7-16) will follow from (7-4) and (7-10). Hence, (7-16) holds for any $(x,t) \in \Psi_{1/8} \cap D$, which gives the bound from above in (7-9). Changing the roles of u_1 and u_2, we get the bound from below.

The proof of C^α-regularity follows by iteration from (7-9), similarly to the proof of Corollary 13.8 in [Caffarelli and Salsa 2005]; however, we need to make sure that at every step the nondegeneracy condition is satisfied. We will only verify the Hölder-continuity of u_1/u_2 at the origin, the rest being standard.

For $k \in \mathbb{N}$ and $\lambda > 0$ to be specified below, let

$$l_k = \inf_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}, \qquad L_k = \sup_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}.$$

Then we know that $1/C \leq l_k \leq L_k \leq C$ for $\lambda \leq \frac{1}{8}$. Let also

$$\mu_k = \frac{u_1(\underline{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \in [l_k, L_k].$$

Then there are two possibilities:

$$\text{either} \quad L_k - \mu_k \geq \tfrac{1}{2}(L_k - l_k) \quad \text{or} \quad \mu_k - l_k \geq \tfrac{1}{2}(L_k - l_k).$$

For definiteness, assume that we are in the latter case, the former case being treated similarly. Then consider the two functions

$$v_1(x,t) = \frac{u_1(\lambda^k x, \lambda^{2k} t) - l_k u_2(\lambda^k x, \lambda^{2k} t)}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})}, \qquad v_2(x,t) = \frac{u_2(\lambda^k x, \lambda^{2k} t)}{u_2(\underline{A}_{\lambda^k/4})}.$$

In $\Psi_1 \setminus E_{f_{\lambda^k}}$, we will have

$$|(\Delta - \partial_t)v_1(x,t)| \le \frac{\lambda^{2k}(1+l_k)C_0}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})},$$
$$|(\Delta - \partial_t)v_2(x,t)| \le \frac{\lambda^{2k}C_0}{u_2(\underline{A}_{\lambda^k/4})}.$$

To proceed, fix a small $\eta_0 > 0$, to be specified below. From the nondegeneracy of u_2, we immediately have

$$|(\Delta - \partial_t)v_2(x,t)| \le C\lambda^{(2-\gamma)k} < \eta_0$$

if we take λ small enough. For v_1, we have a dichotomy:

$$\text{either} \quad |(\Delta - \partial_t)v_1(x,t)| \le \eta_0 \quad \text{or} \quad \mu_k - l_k \le C\lambda^{(2-\gamma)k}.$$

In the latter case, we obtain

$$L_k - l_k \le 2(\mu_k - l_k) \le C\lambda^{(2-\gamma)k}. \tag{7-17}$$

In the former case, we notice that both functions $v = v_1, v_2$ satisfy

$$v \ge 0, \quad v(\underline{A}_{1/4}) = 1 \quad \text{and} \quad |(\Delta - \partial_t)v(x,t)| \le \eta_0 \quad \text{in } \Psi_1 \setminus E_{f_{\lambda^k}},$$

and that v vanishes continuously on $\Psi_1 \cap E_{f_{\lambda^k}}$. We next establish a nondegeneracy property for such v. Indeed, first note that, by the parabolic Harnack inequality (see Theorems 6.17 and 6.18 in [Lieberman 1996]), for small enough η_0, we will have that

$$v \ge c_n \quad \text{on } \Psi_{1/8} \cap \{|x_n| \ge \beta_n/8\}.$$

Then, by invoking Lemma 7.8, we will obtain that

$$v(x,t) \ge c_n|x_n| \quad \text{in } \Psi_{1/16} \setminus E_{f_{\lambda^k}}. \tag{7-18}$$

We further claim that

$$v(x,t) \ge c\,\mathrm{dist}_p((x,t), E_{f_{\lambda^k}}) \quad \text{in } \Psi_{1/32} \setminus E_{f_{\lambda^k}}. \tag{7-19}$$

To this end, for $(x,t) \in \Psi_{1/32} \setminus E_{f_{\lambda^k}}$, let $d = \sup\{r : \Psi_r(x,t) \cap E_{f_{\lambda^k}} = \varnothing\}$, and consider the box $\Psi_d(x,t)$. Without loss of generality, assume $x_n \ge 0$. Then let $(x_*, t_*) = (x', x_n + d, t - d^2) \in \partial_p \Psi_d(x,t)$. From (7-18), we have that

$$v(x_*, t_*) \ge c_n(x_n + d) \ge c_n d,$$

and, applying the parabolic Harnack inequality, we obtain

$$v(x,t) \ge c_n v(x_*, t_*) - C_n \eta_0 d^2 \ge c_n d$$

provided η_0 is sufficiently small. Hence, (7-19) follows.

Having the nondegeneracy, we also have the bound from above for the functions v_1 and v_2. Indeed, by Theorem 7.4 for v_1 and v_2, we have

$$\sup_{\Psi_1} v_1 \le C v_1(\bar{A}_{1/4}) = C \frac{u_1(\bar{A}_{\lambda^k/4}) - l_k u_2(\bar{A}_{\lambda^k/4})}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})} \le C \frac{u_2(\bar{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \frac{L_k - l_k}{\mu_k - l_k} \le C \tag{7-20}$$

and

$$\sup_{\Psi_1} v_2 \le C v_2(\bar{A}_{1/4}) = C \frac{u_2(\bar{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \le C, \tag{7-21}$$

where we have also invoked the second part of Theorem 7.4 for u_2.

We have thus verified all conditions necessary for applying the estimate (7-9) to the functions v_1 and v_2. Particularly, the inequality from below, applied in $\Psi_{8\lambda} \setminus E_{f_{\lambda^k}}$, will give

$$\inf_{\Psi_\lambda \setminus E_{f_{\lambda^k}}} \frac{v_1}{v_2} \ge c \frac{v_1(A_{2\lambda})}{v_2(A_{2\lambda})} \ge c\lambda$$

for a small $c > 0$, or equivalently

$$l_{k+1} - l_k \ge c\lambda(\mu_k - l_k) \ge \frac{c\lambda}{2}(L_k - l_k).$$

Hence, we will have

$$L_{k+1} - l_{k+1} \le L_k - l_k - (l_{k+1} - l_k) \le \left(1 - \frac{c\lambda}{2}\right)(L_k - l_k). \tag{7-22}$$

Summarizing, (7-17) and (7-22) give a dichotomy: for any $k \in \mathbb{N}$,

$$\text{either} \quad L_k - l_k \le C\lambda^{(2-\gamma)k} \quad \text{or} \quad L_{k+1} - l_{k+1} \le (1 - c\lambda/2)(L_k - l_k).$$

This clearly implies that

$$L_k - l_k \le C\beta^k \quad \text{for some } \beta \in (0, 1),$$

for any $k \in \mathbb{N}$, which is nothing but the Hölder-continuity of u_1/u_2 at the origin. □

We next want to prove a variant of Theorem 7.6, but with the Ψ_r replaced with their lower halves

$$\Theta_r = \Psi_r \cap \{t \le 0\}.$$

Theorem 7.9. *Let u_1, u_2 be nonnegative functions in $\Theta_1 \setminus E_f$, continuously vanishing on $\Theta_1 \cap E_f$, and satisfying*

$$|\Delta u_i - \partial_t u_i| \le C_0 \quad \text{in } \Theta_1 \setminus E_f,$$
$$u_i(x, t) \ge c_0 \operatorname{dist}_p((x, t), E_f) \quad \text{in } \Theta_1 \setminus E_f$$

for $i = 1, 2$, for some $c_0 > 0$, $C_0 \ge 0$. Let also $M = \max\{\sup_D u_1, \sup_D u_2\}$. If u_1 and u_2 are symmetric in x_n, then u_1/u_2 extends to a function in $C^\alpha(\bar{\Theta}_{1/8})$ for some $0 < \alpha < 1$, with α and C^α-norm depending only on $n, L, \gamma, C_0, c_0, M$.

The idea is that the functions u_i can be extended to Ψ_δ, for some $\delta > 0$, while still keeping the same inequalities, including the nondegeneracy condition.

Lemma 7.10. *Let u be a nonnegative continuous function on Θ_1 such that*

$$\begin{aligned} u &= 0 && \text{in } \Theta_1 \cap E_f, \\ |(\Delta - \partial_t)u| &\le C_0 && \text{in } \Theta_1 \setminus E_f, \\ u(x,t) &\ge c_0 \operatorname{dist}_p((x,t), E_f) && \text{in } \Theta_1 \setminus E_f, \end{aligned}$$

for some $C_0 \ge 0$, $c_0 > 0$. Then there exist positive δ and $\tilde{c}_0$, depending only on n, L, c_0 and C_0, and a nonnegative extension $\tilde{u}$ of u to Ψ_δ, such that

$$\begin{aligned} \tilde{u} &= 0 && \text{in } \Psi_\delta \cap E_f, \\ |(\Delta - \partial_t)\tilde{u}| &\le C_0 && \text{in } \Psi_\delta \setminus E_f, \\ \tilde{u}(x,t) &\ge \tilde{c}_0 \operatorname{dist}_p((x,t), E_f) && \text{in } \Psi_\delta \setminus E_f. \end{aligned}$$

Moreover, we will also have that $\sup_{\Psi_\delta} \tilde{u} \le \sup_{\Theta_1} u$.

Proof. We first continuously extend the function u from the parabolic boundary $\partial_p \Theta_{1/2}$ to $\partial_p \Psi_{1/2}$ by keeping it nonnegative and bounded above by the same constant. Further, put $u = 0$ on $E_f \cap (\Psi_{1/2} \setminus \Theta_{1/2})$. Then extend u to $\Psi_{1/2}$ by solving the Dirichlet problem for the heat equation in $(\Psi_{1/2} \setminus \Theta_{1/2}) \setminus E_f$, with already defined boundary values. We still denote the extended function by u.

Then it is easy to see that u is nonnegative in $\Psi_{1/2}$, $\sup_{\Psi_{1/2}} u \le \sup_{\Theta_1} u$, u vanishes on $\Psi_{1/2} \cap E_f$ and $|(\Delta - \partial_t)u| \le C_0$ in $\Psi_{1/2} \setminus E_f$. Note that we still have the nondegeneracy property $u(x,t) \ge c_0 \operatorname{dist}_p((x,t), E_f)$ for in $\Theta_{1/2} \setminus E_f$, so it remains to prove the nondegeneracy for $t \ge 0$. We will be able to do it in a small box Ψ_δ as a consequence of Lemma 7.8.

For $0 < \delta < \frac{1}{2}$, consider the rescalings

$$u_\delta(x,t) = \frac{u(\delta x, \delta^2 t)}{\delta}, \qquad (x,t) \in \Psi_{1/(2\delta)}.$$

Then we have

$$\begin{aligned} |(\Delta - \partial_t)u_\delta| &\le C_0 \delta && \text{in } \Psi_1 \setminus E_{f_\delta}, \\ u_\delta(x,t) &\ge c_0 |x_n| && \text{in } \Theta_1, \end{aligned}$$

where $f_\delta(x'', t) = (1/\delta) f(\delta x'', \delta^2 t)$ is the rescaling of f. Then, by using the parabolic Harnack inequality (see Theorems 6.17 and 6.18 in [Lieberman 1996]) in $\Theta_1^{\pm}$, we obtain that

$$u_\delta(x,t) \ge c_n c_0 - C_n C_0 \delta > c_1 \quad \text{on } \{|x_n| = \beta_n/2\} \cap \Psi_{1/2}.$$

Further, choosing δ small and applying Lemma 7.8, we deduce that

$$u_\delta(x,t) \ge c_2 |x_n| \quad \text{in } \Psi_{1/4}.$$

Then, repeating the arguments based on the parabolic Harnack inequality, as for the inequality (7-19), we obtain

$$u(x,t) \geq C\,\mathrm{dist}_p((x,t), E_{f_\delta}) \quad \text{in } \Psi_{1/8}.$$

Scaling back, this gives

$$u(x,t) \geq C\,\mathrm{dist}_p((x,t), E_f) \quad \text{in } \Psi_{\delta/8}. \qquad \square$$

Proof of Theorem 7.9. Extend the functions u_i as in Lemma 7.10 and apply Theorem 7.6. If we repeat this at every $(y,s) \in \Theta_{1/8} \cap G_f$, we will obtain the Hölder-regularity of u_1/u_2 in $\mathcal{N}_{\delta/8}(\Theta_{1/8} \cap G_f) \cap \{t \leq 0\}$. For the remaining part of $\Theta_{1/8}$, we argue as in the proof of localization property Lemma 2.3, cases (1) and (2), and use the corresponding results for parabolically Lipschitz domains. $\square$

7A. *Parabolic Signorini problem.* In this subsection, we discuss an application of the boundary Harnack principle to the parabolic Signorini problem. The idea of such applications goes back to [Athanasopoulos and Caffarelli 1985]. The particular result that we will discuss here can be found also in [Danielli et al. 2013], with the same proof based on our Theorem 7.9.

In what follows, we will use $H^{\ell,\ell/2}$, $\ell > 0$, to denote the parabolic Hölder classes, as defined for instance in [Ladyženskaja et al. 1968].

For a given function $\varphi \in H^{\ell,\ell/2}(Q_1')$, $\ell \geq 2$, known as the *thin obstacle*, we say that a function v solves the *parabolic Signorini problem* if $v \in W_2^{2,1}(Q_1^+) \cap H^{1+\alpha,(1+\alpha)/2}(\overline{Q_1^+})$, $\alpha > 0$, and

$$(\Delta - \partial_t)v = 0 \quad \text{in } Q_1^+, \tag{7-23}$$

$$v \geq \varphi, \quad -\partial_{x_n} v \geq 0, \quad (v-\varphi)\partial_{x_n} v = 0 \quad \text{on } Q_1'. \tag{7-24}$$

This kind of problem appears in many applications, such as thermics (boundary heat control), biochemistry (semipermeable membranes and osmosis), and elastostatics (the original Signorini problem). We refer to the book [Duvaut and Lions 1976] for the derivation of such models as well as for some basic existence and uniqueness results.

The regularity that we impose on the solutions of (7-23)–(7-24) is also well known in the literature; see, e.g., [Athanasopoulous 1982; Ural′tseva 1985; Arkhipova and Uraltseva 1996]. It was proved recently in [Danielli et al. 2013] that one can actually take $\alpha = \frac{1}{2}$ in the regularity assumptions on v, which is the optimal regularity, as can be seen from the explicit example

$$v(x,t) = \mathrm{Re}(x_{n-1} + i x_n)^{3/2},$$

which solves the Signorini problem with $\varphi = 0$. One of the main objects of study in the Signorini problem is the *free boundary*

$$G(v) = \partial_{Q_1'}(\{v > \varphi\} \cap Q_1'),$$

where $\partial_{Q_1'}$ is the boundary in the relative topology of Q_1'.

As the initial step in the study, we make the following reduction. We observe that the difference

$$u(x,t) = v(x,t) - \varphi(x',t)$$

will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1^+, \tag{7-25}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\partial_{x_n} u = 0 \quad \text{on } Q_1', \tag{7-26}$$

where $g = -(\Delta_{x'} - \partial_t)\varphi \in H^{\ell-2,(\ell-2)/2}$. That is, one can make the thin obstacle equal to 0 at the expense of getting a nonzero right-hand side in the equation for u. For our purposes, this simple reduction will be sufficient, however, to take the full advantage of the regularity of φ. When $\ell > 2$, one may need to subtract an additional polynomial from u to guarantee the decay rate

$$|g(x,t)| \leq M(|x|^2 + |t|)^{(\ell-2)/2}$$

near the origin; see Proposition 4.4 in [Danielli et al. 2013]. With the reduction above, the free boundary $G(v)$ becomes

$$G(u) = \partial_{Q_1'}(\{u > 0\} \cap Q_1').$$

Further, it will be convenient to consider the even extension of u in the x_{n-1} variable to the entire Q_1, i.e., by putting $u(x', x_n, t) = u(x', -x_n, t)$. Then, such an extended function will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1 \setminus \Lambda(u),$$

where g has also been extended by even symmetry in x_n, and where

$$\Lambda(u) = \{u = 0\} \cap Q_1',$$

the so-called *coincidence set*.

As shown in [ibid.], a successful study of the properties of the free boundary near $(x_0, t_0) \in G(u) \cap Q_{1/2}'$ can be made by considering the rescalings

$$u_r(x,t) = u_r^{(x_0,t_0)}(x,t) = \frac{u(x_0 + rx, t_0 + r^2 t)}{H_u^{(x_0,t_0)}(r)^{1/2}}$$

for $r > 0$ and then studying the limits of u_r as $r = r_j \to 0+$ (so-called blowups). Here

$$H_u^{(x_0,t_0)}(r) := \frac{1}{r^2} \int_{t_0-r^2}^{t_0} \int_{\mathbb{R}^n} u(x,t)^2 \psi^2(x) \Gamma(x_0 - x, t_0 - t)\, dx\, dt,$$

where $\psi(x) = \psi(|x|)$ is a cutoff function that equals 1 on $B_{3/4}$. Then a point $(x_0, t_0) \in G(u) \cap B_{1/2}$ is called regular if u_r converges in the appropriate sense to

$$u_0(x,t) = c_n \operatorname{Re}(x_{n-1} + i x_n)^{3/2}$$

as $r = r_j \to 0+$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$. See [ibid.] for more details. Let $\mathcal{R}(u)$ be the set of regular points of u.

Proposition 7.11 [Danielli et al. 2013]. *Let u be a solution of the parabolic Signorini problem (7-25)–(7-26) in Q_1^+ with $g \in H^{1,1/2}(Q_1^+)$. Then the regular set $\mathcal{R}(u)$ is a relatively open subset of $G(u)$. Moreover, if $(0,0) \in \mathcal{R}(u)$, then there exists $\rho = \rho_u > 0$ and a parabolically Lipschitz function f such that*

$$G(u) \cap Q'_\rho = \mathcal{R}(u) \cap Q'_\rho = G_f \cap Q'_\rho$$
$$\Lambda(u) \cap Q'_\rho = E_f \cap Q'_\rho.$$

Furthermore, for any $0 < \eta < 1$, we can find $\rho > 0$ such that

$$\partial_e u \geq 0 \quad \text{in } Q_\rho$$

for any unit direction $e \in \mathbb{R}^{n-1}$ such that $e \cdot e_{n-1} > \eta$, and moreover

$$\partial_e u(x,t) \geq c \operatorname{dist}_p((x,t), E_f) \quad \text{in } Q_\rho$$

for some $c > 0$. □

We next show that an application of Theorem 7.9 implies the following result.

Theorem 7.12. *Let u be as in Proposition 7.11 and $(0,0) \in \mathcal{R}(u)$. Then there exists $\delta < \rho$ such that $\nabla'' f \in H^{\alpha,\alpha/2}(Q'_\delta)$ for some $\alpha > 0$, i.e., $\mathcal{R}(u)$ has Hölder-continuous spatial normals in Q'_δ.*

Proof. We will work in parabolic boxes $\Theta_\delta = \Psi_\delta \cap \{t \leq 0\}$ instead of cylinders Q_δ. For a small $\varepsilon > 0$, let $e = (\cos\varepsilon)e_{n-1} + (\sin\varepsilon)e_j$ for some $j = 1, \dots, n-2$, and consider the two functions

$$u_1 = \partial_e u \quad \text{and} \quad u_2 = \partial_{e_{n-1}} u.$$

Then, by Proposition 7.11, the conditions of Theorem 7.9 are satisfied (after a rescaling), provided $\cos\varepsilon > \eta$. Thus, if we fix such $\varepsilon > 0$, we will have that for some $\delta > 0$ and $0 < \alpha < 1$,

$$\frac{\partial_e u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta).$$

This gives that

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta), \qquad j = 1, \dots, n-2.$$

Hence the level surfaces $\{u = \sigma\} \cap \Theta'_\delta$ are given as graphs

$$x_{n-1} = f_\sigma(x'', t), \qquad x'' \in \Theta''_\delta,$$

with estimate on $\|\nabla'' f_\sigma\|_{H^{\alpha,\alpha/2}(\Theta''_\delta)}$ that is uniform in $\sigma > 0$. Consequently, this implies that

$$\nabla'' f \in H^{\alpha,\alpha/2}(\Theta''_\delta),$$

and completes the proof of the theorem. □

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