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ON THE UNCONDITIONAL UNIQUENESS OF SOLUTIONS TO THE INFINITE RADIAL CHERN–SIMONS–SCHRÖDINGER HIERARCHY

XUWEN CHEN AND PAUL SMITH

In this article, we establish the unconditional uniqueness of solutions to an infinite radial Chern–Simons–Schrödinger (IRCSS) hierarchy in two spatial dimensions. The IRCSS hierarchy is a system of infinitely many coupled PDEs that describes the limiting Chern–Simons–Schrödinger dynamics of infinitely many interacting anyons. The anyons are two-dimensional objects that interact through a self-generated field. Due to the interactions with the self-generated field, the IRCSS hierarchy is a system of *nonlinear* PDEs, which distinguishes it from the *linear* infinite hierarchies studied previously. Factorized solutions of the IRCSS hierarchy are determined by solutions of the Chern–Simons–Schrödinger system. Our result therefore implies the unconditional uniqueness of solutions to the radial Chern–Simons–Schrödinger system as well.

1. Introduction

1A. ***The Chern–Simons–Schrödinger system.*** The Chern–Simons–Schrödinger system is given by

$$\begin{cases} D_t\phi = i\sum_{\ell=1}^{2} D_\ell D_\ell \phi + ig|\phi|^2\phi, \\ \partial_t A_1 - \partial_1 A_0 = -\operatorname{Im}(\bar{\phi} D_2 \phi), \\ \partial_t A_2 - \partial_2 A_0 = \operatorname{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\tfrac{1}{2}|\phi|^2, \end{cases} \tag{1}$$

where the associated covariant differentiation operators are defined in terms of the potential A by

$$D_\alpha := \partial_\alpha + iA_\alpha, \quad \alpha \in \{0, 1, 2\}, \tag{2}$$

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and where we adopt the convention that $\partial_0 := \partial_t$ and $D_t := D_0$. The wavefunction ϕ is complex-valued, the potential A a real-valued 1-form, and the pair (A, ϕ) is defined on $I \times \mathbb{R}^2$ for some time interval I. The Lagrangian action for this system is

$$L(A,\phi) = \frac{1}{2}\int_{\mathbb{R}^{2+1}} \Big[\operatorname{Im}(\bar{\phi} D_t \phi) + |D_x \phi|^2 - \frac{g}{2}|\phi|^4\Big]\,dx\,dt + \frac{1}{2}\int_{\mathbb{R}^{2+1}} A \wedge dA, \tag{3}$$

where here $|D_x\phi|^2 := |D_1\phi|^2 + |D_2\phi|^2$. Although the potential A appears explicitly in the Lagrangian, it is easy to see that locally $L(A,\phi)$ only depends upon the field $F = dA$. Precisely, the Lagrangian is invariant with respect to the gauge transformations

$$\phi \mapsto e^{-i\theta}\phi, \qquad A \mapsto A + d\theta \tag{4}$$

for compactly supported real-valued functions $\theta(t,x)$. The Chern–Simons–Schrödinger system (1), obtained as the Euler–Lagrange equations of (3), inherits this gauge freedom.

The system (1) is a basic model of Chern–Simons dynamics [Jackiw and Pi 1992; Ezawa, Hotta, and Iwazaki 1991a; 1991b; Jackiw and Pi 1991]. It plays a role in describing certain physical phenomena, such as the fractional quantum Hall effect, high-temperature superconductivity, and Aharonov–Bohm scattering, and also provides an example of a Galilean-invariant planar gauge field theory [Jackiw and Templeton 1981; Deser, Jackiw, and Templeton 1982; Jackiw, Pi, and Weinberg 1991; Martina, Pashaev, and Soliani 1993; Wilczek 1990].

One interpretation of (1) is as a mean-field equation. Informally, one may consider (1) as describing the behavior of a large number of anyons, interacting with each other directly and through a self-generated field, in the case where the N-body wave function factorizes. There are a number of challenges one encounters in trying to formalize and prove this statement, and this paper addresses some of them. We will postpone further discussion of many-body dynamics to the next subsection and instead point out that, because the main evolution equation in (1) includes a cubic nonlinearity, one might hope to prove for (1) what one can prove for the cubic nonlinear Schrödinger equation (NLS). It is important to note, however, that (1) has many nonlinear terms, some nonlocal and some involving the derivative of the wave function. These terms appear because of the geometric structure that arises from modeling the interactions with the self-generated field. Due to the complexity of the nonlinearity in (1) and the gauge freedom (4), the system (1) is significantly more challenging to analyze than the cubic NLS. This difference is seen even at the level of the wellposedness theory, to which we now turn.

The system (1) is Galilean-invariant and has conserved *charge*

$$\operatorname{chg}(\phi) := \int_{\mathbb{R}^2} |\phi|^2\,dx \tag{5}$$

and *energy*

$$E(\phi) := \frac{1}{2}\int_{\mathbb{R}^2}\Big[|D_x\phi|^2 - \frac{g}{2}|\phi|^4\Big]\,dx. \tag{6}$$

Moreover, for each $\lambda > 0$, there is the scaling symmetry

$$\begin{aligned} \phi(t,x) &\mapsto \lambda\phi(\lambda^2 t, \lambda x), & A_j(t,x) &\mapsto \lambda A_j(\lambda^2 t, \lambda x), \quad j \in \{1,2\}, \\ \phi_0(x) &\mapsto \lambda\phi_0(\lambda x), & A_0(t,x) &\mapsto \lambda^2 A_0(\lambda^2 t, \lambda x), \end{aligned}$$

which preserves both the system and the charge of the initial data ϕ_0. Therefore, from the point of view of wellposedness theory, the system (1) is L^2-critical. We remark that system (1) is defocusing when $g < 1$ and focusing when $g \geq 1$. The defocusing/focusing dichotomy is most readily seen by rewriting the energy (6) using the so-called Bogomol'nyi identity. After using this identity, one may also see the dichotomy manifested in the virial and Morawetz identities. For more details, see [Liu and Smith 2014, §4, §5]. Note also that the sign convention for g that we adopt, which is the one used in the Chern–Simons literature, is opposite to the usual one adopted for the cubic NLS. A more significant difference between Chern–Simons systems and the cubic NLS is that, unlike the case for the cubic NLS, the coupling parameter g cannot be rescaled to belong to a discrete set of canonical values.

Nevertheless, (1) is ill-posed so long as it retains the gauge freedom (4). This freedom is eliminated by imposing an additional constraint equation. The most common gauge choice for studying (1) is the Coulomb gauge, which is the constraint

$$\partial_1 A_1 + \partial_2 A_2 = 0. \tag{7}$$

Coupling (7) with the field equations quickly leads to explicit expressions for A_α, $\alpha = 0, 1, 2$, in terms of ϕ. These expressions also happen to be nonlinear and nonlocal:

$$A_0 = \Delta^{-1}[\partial_1 \operatorname{Im}(\bar{\phi} D_2 \phi) - \partial_2 \operatorname{Im}(\bar{\phi} D_1 \phi)], \qquad A_1 = \tfrac{1}{2}\Delta^{-1}\partial_2|\phi|^2, \qquad A_2 = -\tfrac{1}{2}\Delta^{-1}\partial_1|\phi|^2. \tag{8}$$

Local wellposedness of (1) with respect to the Coulomb gauge at the Sobolev regularity of H^2 is established in [Bergé, De Bouard, and Saut 1995]. This is improved to H^1 in [Huh 2013]. Local wellposedness for data small in H^s, $s > 0$, is established in [Liu, Smith, and Tataru 2012] using the heat gauge, whose defining condition is $\partial_1 A_1 + \partial_2 A_2 = A_0$. This result relies upon various Strichartz-type spaces as well as more sophisticated U^p and V^p spaces. We refer the reader to [Liu, Smith, and Tataru 2012, §2] for a comparison of the Coulomb and heat gauges.

In symmetry-reduced settings, one may say more, and in particular, [Liu and Smith 2014] establishes large-data global wellposedness results at the critical regularity for the equivariant Chern–Simons–Schrödinger system. To introduce the equivariance (or *vortex*) ansatz, it is convenient to use polar coordinates. Define

$$A_r = \frac{x_1}{|x|}A_1 + \frac{x_2}{|x|}A_2, \qquad A_\theta = -x_2 A_1 + x_1 A_2. \tag{9}$$

We can invert the transform by writing

$$A_1 = A_r \cos\theta - \tfrac{1}{r}A_\theta \sin\theta, \qquad A_2 = A_r \sin\theta + \tfrac{1}{r}A_\theta \cos\theta. \tag{10}$$

Note that these relations are analogous to

$$\partial_r = \frac{x_1}{|x|}\partial_1 + \frac{x_2}{|x|}\partial_2, \qquad \partial_\theta = -x_2\partial_1 + x_1\partial_2$$

and

$$\partial_1 = (\cos\theta)\partial_r - \tfrac{1}{r}(\sin\theta)\partial_\theta, \qquad \partial_2 = (\sin\theta)\partial_r + \tfrac{1}{r}(\cos\theta)\partial_\theta.$$

The equivariant ansatz, then, is

$$\phi(t,x)=e^{im\theta}u(t,r),\quad A_1(t,x)=-\frac{x_2}{r}v(t,r),\quad A_2(t,x)=\frac{x_1}{r}v(t,r),\quad A_0(t,x)=w(t,r),\tag{11}$$

where we assume that m is a nonnegative integer, u is real-valued at time zero, and v and w are real-valued for all time. This ansatz implies that $A_r=0$ and that A_θ is a radial function. It also places us in the Coulomb gauge, i.e., $\partial_1 A_1+\partial_2 A_2=0$ or equivalently $\partial_r A_r+\frac{1}{r}A_r+\frac{1}{r^2}\partial_\theta A_\theta=0$. For some motivation for studying vortex solutions in Chern–Simons theories, see [Paul and Khare 1986; de Vega and Schaposnik 1986a; 1986b; Jackiw and Weinberg 1990; R. M. Chen and Spirn 2009; Byeon, Huh, and Seok 2012].

Converting (1) into polar coordinates and utilizing (11), we obtain the equivariant Chern–Simons–Schrödinger system (see [Liu and Smith 2014, §1] for full details):

$$\begin{cases}(i\partial_t+\Delta)\phi=\frac{2m}{r^2}A_\theta\phi+A_0\phi+\frac{1}{r^2}A_\theta^2\phi-g|\phi|^2\phi,\\ \partial_r A_0=\frac{1}{r}(m+A_\theta)|\phi|^2,\\ \partial_t A_\theta=r\,\mathrm{Im}(\bar\phi\partial_r\phi),\\ \partial_r A_\theta=-\frac{1}{2}|\phi|^2 r,\\ A_r=0.\end{cases}\tag{12}$$

Global wellposedness holds for equivariant L^2 data of arbitrary (nonnegative) charge in the defocusing case $g<1$ and for L^2 data with charge less than that of the ground state in the focusing case $g\geq 1$; this is the main result of [Liu and Smith 2014].

In this paper, we are interested in the radial case ($m=0$) of system (12), which is

$$\begin{cases}(i\partial_t+\Delta)\phi=A_0\phi+\frac{1}{r^2}A_\theta^2\phi-g|\phi|^2\phi,\\ \partial_r A_0=\frac{1}{r}A_\theta|\phi|^2,\\ \partial_t A_\theta=r\,\mathrm{Im}(\bar\phi\partial_r\phi),\\ \partial_r A_\theta=-\frac{1}{2}|\phi|^2 r,\\ A_r=0.\end{cases}\tag{13}$$

1B. ***The infinite Chern–Simons–Schrödinger hierarchy.*** The infinite Chern–Simons–Schrödinger hierarchy is a sequence of trace class nonnegative operator kernels that are symmetric in the sense that

$$\gamma^{(k)}(t,\boldsymbol{x}_k,\boldsymbol{x}_k')=\overline{\gamma^{(k)}(t,\boldsymbol{x}_k',\boldsymbol{x}_k)},$$

and

$$\gamma^{(k)}(t,x_{\sigma(1)},\dots,x_{\sigma(k)},x'_{\sigma(1)},\dots,x'_{\sigma(k)})=\gamma^{(k)}(t,x_1,\dots,x_k,x'_1,\dots,x'_k),\tag{14}$$

for any permutation σ, and which satisfy the two-dimensional infinite Chern–Simons–Schrödinger hierarchy of equations

$$\partial_t\gamma^{(k)}+\sum_{j=1}^{k}[iA_0(t,x_j),\gamma^{(k)}]=\sum_{j=1}^{k}\sum_{\ell=1}^{2}i[D_{x_j^{(\ell)}}D_{x_j^{(\ell)}},\gamma^{(k)}]+ig\sum_{j=1}^{k}B_{j,k+1}\gamma^{(k+1)},\tag{15}$$

where $\mathbb{R}^2 \ni x_j = (x_j^{(1)}, x_j^{(2)})$ for each j, as well as the corresponding field-current identities from [Jackiw and Pi 1990, (1.7a)–(1.7c)], i.e.,

$$\begin{cases} F_{01} = -P_2(t,x) - A_2(t,x)\rho(t,x), \\ F_{02} = P_1(t,x) + A_1(t,x)\rho(t,x), \\ F_{12} = -\frac{1}{2}\rho(t,x), \end{cases} \tag{16}$$

where, as before, $F := dA$. Here g is the coupling constant,

$$B_{j,k+1}\gamma^{(k+1)} := \mathrm{Tr}_{k+1}[\delta(x_j - x_{k+1}), \gamma^{(k+1)}], \tag{17}$$

the momentum $P(t,x)$ is given by

$$P(t,x) := \int e^{i(\xi-\xi')x} \frac{\xi+\xi'}{2} \hat{\gamma}^{(1)}(t,\xi,\xi')\, d\xi\, d\xi',$$

and $\rho(t,x)$ is a shorthand for

$$\rho(t,x) := \gamma^{(1)}(t,x,x). \tag{18}$$

Each $x_j \in \mathbb{R}^2$, and $\boldsymbol{x}_k := (x_1, \dots, x_k) \in \mathbb{R}^{2k}$. Given a compactly supported $\theta(t,x)$, the kernels $\gamma^{(k)}$ and potential A transform under a change of gauge according to

$$\gamma^{(k)} \mapsto \gamma^{(k)} \prod_{j=1}^{k} e^{-i\theta(t,x_j)} e^{i\theta(t,x_j')}, \qquad A \mapsto A + d\theta.$$

The invariance of (15) and (16) under such transformations can be checked straightforwardly.

For the purposes of our analysis, it is more convenient to write (15) as

$$\begin{aligned} i\partial_t \gamma^{(k)} + \sum_{j=1}^{k} [\Delta_{x_j}, \gamma^{(k)}] = \sum_{j=1}^{k}\sum_{\ell=1}^{2} [-2iA_{x_j^{(\ell)}}\partial_{x_j^{(\ell)}} - i\partial_{x_j^{(\ell)}} A_{x_j^{(\ell)}} + A^2_{x_j^{(\ell)}}, \gamma^{(k)}] \\ + \sum_{j=1}^{k} [A_0(t,x_j), \gamma^{(k)}] - g\sum_{j=1}^{k} B_{j,k+1}\gamma^{(k+1)}. \end{aligned} \tag{19}$$

The Coulomb gauge condition (7), upon being coupled to (16), leads to

$$A_0 = \Delta^{-1}[\partial_1(P_2 + A_2\rho) - \partial_2(P_1 + A_1\rho)], \qquad A_1 = \tfrac{1}{2}\Delta^{-1}\partial_2\rho, \qquad A_2 = -\tfrac{1}{2}\Delta^{-1}\partial_1\rho.$$

This is analogous to how (8) for the Chern–Simons–Schrödinger system (1) is obtained by coupling to the field equations in (1) the gauge condition (7). Because each A_α involves ρ, defined in (18), it is clear that each term involving $\gamma^{(k)}$ in the right-hand side of (19) is best thought of as a nonlinear term. This nonlinear dependence persists under changes of gauge, though some gauges lead to tamer nonlinearities than others.

We remark that, while the specific form the nonlinearity of (19) takes indeed depends upon the gauge selection made, the observables associated with the system do not depend upon the gauge choice.

We note that the system (1) generates a special solution to the infinite hierarchy (15)–(16). In particular, if (A, ϕ) solves (1), then $(A, \{\gamma^{(k)}\})$ solves (15)–(16), where each $\gamma^{(k)}$ is given by

$$\gamma^{(k)}(t, \boldsymbol{x}_k, \boldsymbol{x}_k') = \prod_{j=1}^{k} \phi(t, x_j)\bar{\phi}(t, x_j').$$

We start our analysis of many-body dynamics with the above infinite hierarchy. Ideally, one would prefer instead to begin with a many-body system with only finitely many quantum particles. Because the basic particles in question are neither bosons nor fermions, there are difficulties to overcome with such an approach. Concerning the difficulties in dealing with microscopic statistics, one can refer to [Benedetto, Castella, Esposito, and Pulvirenti 2005], for instance. Fortunately, as remarked in [Benedetto, Castella, Esposito, and Pulvirenti 2005], microscopic statistics disappear as the particle number tends to infinity. Thus, the infinite hierarchy satisfies the symmetry condition (14). We finally remark that the field equations (16) depend merely on the 1-particle density $\gamma^{(1)}$, as has been observed formally in the physics literature [Deser, Jackiw, and Templeton 1982; Jackiw, Pi, and Weinberg 1991; Jackiw and Pi 1991; Jackiw and Templeton 1981; Jackiw and Weinberg 1990].

One motivation for pursuing an analysis of the infinite hierarchy even without first specifying the finite hierarchy is that the known approaches to rigorously deriving mean-field equations, e.g., the Boltzmann equation and the cubic NLS, all require a uniqueness theorem for the corresponding infinite hierarchy. Establishing uniqueness of the infinite hierarchy is, moreover, a critical step. We therefore anticipate that our result in this article will be the linchpin of any future rigorous derivation of the Chern–Simons–Schrödinger system.

As remarked before, the analysis of the Chern–Simons–Schrödinger system with general data is, at the moment, very delicate. The same remark applies all the more to the associated infinite hierarchy, to which (1) is a special solution. Thus, we consider the radial version of the infinite Chern–Simons–Schrödinger hierarchy in this paper. The nonradial equivariant case ($m > 0$), though still much simpler than the general system, is slightly more challenging than the radial case. Unfortunately, the techniques we employ for studying the radial case do not immediately extend to the nonradial equivariant case due to certain logarithmic divergences.

The infinite radial Chern–Simons–Schrödinger hierarchy. The Chern–Simons–Schrödinger system (1) simplifies to (13) under the assumption of radiality. Similarly, by assuming radiality, we reduce Equations (15) through (18) to the infinite radial Chern–Simons–Schrödinger hierarchy

$$i\partial_t \gamma^{(k)} + \sum_{j=1}^{k} [\Delta_{x_j}, \gamma^{(k)}] = \sum_{j=1}^{k} \Big[A_0(t, |x_j|) + \frac{1}{|x_j|^2} A_\theta^2(t, |x_j|), \gamma^{(k)} \Big] - g \sum_{j=1}^{k} B_{j,k+1} \gamma^{(k+1)} \tag{20}$$

and the field equations

$$F_{r\theta}(t, |x|) = -\tfrac{1}{2}|x|\rho(t, |x|)$$

and

$$\begin{aligned} F_{0\theta}(t, |x|) &= |x| P_r(t, |x|), \\ F_{0r}(t, |x|) &= -\tfrac{1}{|x|} A_\theta(t, |x|)\rho(t, |x|), \end{aligned}$$

for $\gamma^{(k)} = \gamma^{(k)}(t, \boldsymbol{r}_k, \boldsymbol{r}'_k)$. In particular, here we assume that

$$\begin{aligned}\gamma^{(k)} &= u(t, \boldsymbol{r}_k, \boldsymbol{r}'_k),\\ A_r &= 0,\\ A_\theta &= v(t, r),\end{aligned}$$

where u is real-valued at time zero and v is real-valued for all time. This assumption enforces the Coulomb gauge. Recall that $B_{j,k+1}$ is defined in (17) and ρ is given by (18). As before, $F := dA$, though now we are adopting polar coordinates for A. Though we could rewrite everything exclusively in terms of polar coordinates, we choose instead to use both Cartesian and polar coordinates.

Putting everything together, we see that we are studying solutions $\gamma^{(k)} = \gamma^{(k)}(t, \boldsymbol{r}_k, \boldsymbol{r}'_k)$ of

$$\begin{cases} i\partial_t\gamma^{(k)} + \sum_{j=1}^k [\Delta_{x_j}, \gamma^{(k)}] = \sum_{j=1}^k \big[A_0(t, |x_j|) + \frac{1}{|x_j|^2} A_\theta^2(t, |x_j|), \gamma^{(k)}\big] - g\sum_{j=1}^k B_{j,k+1}\gamma^{(k+1)},\\ \partial_r A_0(t, |x|) = \frac{1}{|x|} A_\theta \rho(t, |x|),\\ \partial_t A_\theta(t, |x|) = |x| P_r(t, |x|),\\ \partial_r A_\theta(t, |x|) = -\frac{1}{2}|x|\rho(t, |x|),\\ A_r = 0.\end{cases} \tag{21}$$

We interpret $\gamma^{(k)}$ as a complex-valued function on $\mathbb{R}_t \times \mathbb{R}_+^k \times \mathbb{R}_+^k$ subject to the symmetries

$$\gamma^{(k)}(t, \boldsymbol{r}_k, \boldsymbol{r}'_k) = \overline{\gamma^{(k)}(t, \boldsymbol{r}'_k, \boldsymbol{r}_k)}$$

and

$$\gamma^{(k)}(t, r_{\sigma(1)}, \ldots, r_{\sigma(k)}, r'_{\sigma(1)}, \ldots, r'_{\sigma(k)}) = \gamma^{(k)}(t, r_1, \ldots, r_k, r'_1, \ldots, r'_k). \tag{22}$$

Though each $r_j \in \mathbb{R}_+$, we associate to this space the measure $r\,dr$, as indeed we think of $r_j = |x_j|$ for $x_j \in \mathbb{R}^2$.

Note that we can eliminate A_θ and A_0 in (21). In particular, we have

$$A_\theta(t, r) = -\frac{1}{2}\int_0^r \rho(t, s)s\,ds \tag{23}$$

and

$$A_0(t, r) = \frac{1}{2}\int_r^\infty \rho(t, s)\int_0^s \rho(t, u)u\,du\,\frac{ds}{s}, \tag{24}$$

which reflect the natural boundary conditions for A_θ and A_0 that we adopt for (1). Therefore, we may rewrite (21) as

$$\begin{aligned} i\partial_t\gamma^{(k)} + \sum_{j=1}^k [\Delta_{x_j}, \gamma^{(k)}] = \sum_{j=1}^k \Big[\frac{1}{2}\int_{r_j}^\infty \rho(t, s)\int_0^s \rho(t, u)u\,du\,\frac{ds}{s} + \frac{1}{r_j^2}\Big(-\frac{1}{2}\int_0^{r_j}\rho(t, s)s\,ds\Big)^2, \gamma^{(k)}\Big]\\ - g\sum_{j=1}^k B_{j,k+1}\gamma^{(k+1)},\\ \gamma^{(k)}(0) = \gamma_0^{(k)}, \quad k \in \mathbb{N}. \end{aligned} \tag{25}$$

1C. ***Main results.*** Our main theorem says that any admissible mild solution of the radial infinite CSS hierarchy is unconditionally unique in $L^\infty_{t\in[0,T)}\mathfrak{H}^{2/3}_{\mathrm{rad}}$. To explain what this means, for $s\in\mathbb{R}$, we define the space $\mathfrak{H}^s_{\mathrm{rad}}$ to be the collection of sequences $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of density matrices in $L^2_{\mathrm{sym}}(\mathbb{R}^{2k})$ such that $\gamma^{(k)}=\gamma^{(k)}(t,\boldsymbol{r}_k,\boldsymbol{r}'_k)$ and

$$\operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}|)<M^{2k}\quad\text{for all } k\in\mathbb{N} \text{ and for some constant } M>0,$$

where

$$S^{(k,s)}:=\prod_{j=1}^{k}(1-\Delta_{x_j})^{s/2}(1-\Delta_{x'_j})^{s/2}.$$

Here L^2_{sym} denotes the space of L^2 functions satisfying (14). Let $U^{(k)}(t)$ denote the propagator

$$U^{(k)}(t):=e^{it\Delta_{x_k}}e^{-it\Delta_{x'_k}}. \tag{26}$$

A *mild solution* of (25) in the space $L^\infty_{[0,T]}\mathfrak{H}^s_{\mathrm{rad}}$ is a sequence of marginal density matrices $\Gamma=(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ solving

$$\begin{aligned}\gamma^{(k)}(t)=U^{(k)}(t)\gamma^{(k)}(0)-i\int_0^t U^{(k)}(t-s)\Biggl(\sum_{j=1}^{k}\biggl[\frac{1}{2}\int_{r_j}^{\infty}\rho(t,v)\int_0^{v}\rho(t,u)u\,du\,\frac{dv}{v}\\+\frac{1}{r_j^2}\Bigl(-\frac{1}{2}\int_0^{r_j}\rho(t,v)v\,dv\Bigr)^2,\gamma^{(k)}\biggr]-g\sum_{j=1}^{k}B_{j,k+1}\gamma^{(k+1)}\Biggr)\,ds\end{aligned}$$

and satisfying

$$\sup_{t\in[0,T)}\operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|)<M^{2k}$$

for a finite constant M independent of k. Note that, if we are given factorized initial data

$$\gamma_0^{(k)}(\boldsymbol{r}_k,\boldsymbol{r}'_k)=\prod_{j=1}^{k}\phi_0(r_j)\overline{\phi_0(r'_j)},$$

then the condition that $(\gamma^{(k)}(0))\in\mathfrak{H}^s_{\mathrm{rad}}$ is equivalent to

$$\operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}(0)|)=\|\phi_0\|^{2k}_{H^s}<M^{2k},\quad k\in\mathbb{N},$$

which is to say that $\|\phi_0\|_{H^s}<M$ for some $M<\infty$. Then a solution to the IRCSS hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^s_{\mathrm{rad}}$ is given by the sequence of factorized density matrices

$$\gamma^{(k)}(t,\boldsymbol{r}_k,\boldsymbol{r}'_k)=\prod_{j=1}^{k}\phi_t(r_j)\overline{\phi_t(r'_j)}$$

provided the corresponding 1-particle wave function ϕ_t satisfies the radial Chern–Simons–Schrödinger system (13).

Admissibility we take to mean that $\operatorname{Tr}\gamma^{(k)} = 1$ for all $k \in \mathbb{N}$ and

$$\gamma^{(k)} = \operatorname{Tr}_{k+1}(\gamma^{(k+1)}), \quad k \in \mathbb{N}. \tag{27}$$

This is required in our application of the quantum de Finetti theorem. As there are weak analogues of the quantum de Finetti theorem applicable to limiting hierarchies, we expect our techniques to apply to the problem of rigorously deriving the radial CSS from large, finite systems.

Theorem 1.1 (unconditional uniqueness for the infinite hierarchy). *There is at most one $L^\infty_{t\in[0,T)}\mathfrak{H}^{2/3}_{\mathrm{rad}}$ admissible solution to the infinite radial Chern–Simons–Schrödinger hierarchy (21).*

Theorem 1.2 (unconditional uniqueness for the Chern–Simons–Schrödinger system). *There is at most one $L^\infty_{t\in[0,T)}H^{2/3}(\mathbb{R}^2)$ solution to the radial Chern–Simons–Schrödinger system (13).*

Before explaining our main theorem, we first remark that deriving mean-field equations from many-body systems by studying infinite hierarchies is a very rich subject. For works related to the Boltzmann equation, see [Lanford 1975; King 1975; Arkeryd, Caprino, and Ianiro 1991; Cercignani, Illner, and Pulvirenti 1994; Gallagher, Saint-Raymond, and Texier 2013]. For works related to the Hartree equation, see [Spohn 1980; Fröhlich, Knowles, and Schwarz 2009; Erdős and Yau 2001; Rodnianski and Schlein 2009; Knowles and Pickl 2010; Grillakis, Machedon, and Margetis 2010; 2011; X. Chen 2012b; L. Chen, Lee, and Schlein 2011; Michelangeli and Schlein 2012; Ammari and Nier 2008; 2011; Lewin, Nam, and Rougerie 2014]. For works related to the cubic NLS, see [Adami, Golse, and Teta 2007; Elgart, Erdős, Schlein, and Yau 2006; Erdős, Schlein, and Yau 2006; 2007; 2010; 2009; Klainerman and Machedon 2008; Kirkpatrick, Schlein, and Staffilani 2011; T. Chen and Pavlović 2011; 2010; T. Chen, Pavlović, and Tzirakis 2012; T. Chen and Pavlović 2014; Pickl 2011; X. Chen 2012a; 2013; Benedikter, Oliveira, and Schlein 2012; Grillakis and Machedon 2013; X. Chen and Holmer 2013c; 2013b; T. Chen, Hainzl, Pavlović, and Seiringer 2014; X. Chen and Holmer 2013a; Hong, Taliaferro, and Xie 2014; Gressman, Sohinger, and Staffilani 2014; Sohinger and Staffilani 2014; Sohinger 2014a; 2014b]. For works related to the quantum Boltzmann equation, see [Benedetto, Castella, Esposito, and Pulvirenti 2006; 2005; 2008; 2004]. The infinite hierarchies considered previously to the present one are all linear. In contrast to this, the infinite radial Chern–Simons–Schrödinger hierarchy is nonlinear.

For our problem, we have taken the phrase "unconditional uniqueness" from the study of the NLS. It is shown by Cercignani's counterexample [Cercignani, Illner, and Pulvirenti 1994] that solutions to infinite hierarchies like the Boltzmann hierarchy and the Gross–Pitaevskii hierarchy are generally not unconditionally unique in the sense that a solution is not uniquely determined by the initial datum unless one assumes appropriate space-time bounds on the solution. In the NLS literature, "unconditional uniqueness" usually means establishing uniqueness without assuming that some Strichartz norm is finite. Since we are using tools from the study of the NLS, we therefore call our main theorems unconditional uniqueness theorems.[1]

[1] In other words, the uniqueness theorems regarding the Gross–Pitaevskii hierarchies [Klainerman and Machedon 2008; Kirkpatrick, Schlein, and Staffilani 2011; X. Chen 2012a; X. Chen and Holmer 2013a; Gressman, Sohinger, and Staffilani 2014] are conditional, whereas [Adami, Golse, and Teta 2007; Erdős, Schlein, and Yau 2007; T. Chen, Hainzl, Pavlović, and Seiringer 2014;

Finally, we remark that, for the proof of the main theorems, we apply the quantum de Finetti theorem in a manner similar to [T. Chen, Hainzl, Pavlović, and Seiringer 2014; Hong, Taliaferro, and Xie 2014] but with adjustments tailored to deal with the nonlinearity in the infinite hierarchy that we consider. The quantum de Finetti theorem is a version of the classical Hewitt–Savage theorem. T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer are the first to apply the quantum de Finetti theorem to the study of infinite hierarchies in the quantum setting. For results regarding the uniqueness of the Boltzmann hierarchy using the Hewitt–Savage theorem, see [Arkeryd, Caprino, and Ianiro 1991].

2. Proof of the main theorem

We will prove that, if we are given two $L^\infty_{[0,T]}\mathfrak{H}^{2/3}_{\mathrm{rad}}$ solutions $\{\gamma_1^{(k)}\}$ and $\{\gamma_2^{(k)}\}$ to system (21) subject to the same initial datum, then the trace norm of the difference $\{\gamma^{(k)}=\gamma_1^{(k)}-\gamma_2^{(k)}\}$ is zero. In contrast to the usual infinite hierarchies (e.g., Boltzmann, Gross–Pitaevskii, ...), system (21) is nonlinear. Thus, $\gamma^{(k)}$ does not solve system (21). In order to show that $\gamma^{(k)}$ has zero trace norm, we first express $\gamma^{(k)}$ as a suitable Duhamel–Born series, which contains a nonlinear part and an interaction part (see Section 2A). These two parts we estimate separately with bounds contained respectively in Theorems 2.3 and 2.4, which together constitute our main estimates. In Section 2B, we prove the main theorem, Theorem 1.1, assuming the main estimates. The proof of Theorem 2.3 is postponed to Section 4 (and Theorem 2.4 we handle in this section).

2A. *Setup.* Set for short

$$a(r_j):=A_0(t,r_j)+\frac{1}{r_j^2}A_\theta^2(t,r_j) \tag{28}$$

and

$$a(\boldsymbol{r}_k):=\sum_{j=1}^k a(r_j). \tag{29}$$

Let $\mathscr{A}^{(k)}$ denote the operator that acts according to

$$\mathscr{A}^{(k)}f:=[a(\boldsymbol{r}_k),f]. \tag{30}$$

Also, set for short

$$B_{k+1}:=\sum_{j=1}^k B_{j,k+1}=\sum_{j=1}^k \mathrm{Tr}_{k+1}[\delta(x_j-x_{k+1}),\cdot\,]. \tag{31}$$

With these abbreviations, the first equation of (21) assumes the form

$$i\partial_t\gamma^{(k)}+[\Delta_{\boldsymbol{x}_k},\gamma^{(k)}]=\mathscr{A}^{(k)}\gamma^{(k)}-gB_{k+1}\gamma^{(k+1)}. \tag{32}$$

Remark 2.1. The operator $\mathscr{A}^{(k)}$ is linear but itself depends upon $\gamma^{(1)}$. In fact, it only depends upon the diagonal $\rho(t,r)=\gamma^{(1)}(t,r,r)$. The term $\mathscr{A}^{(k)}\gamma^{(k)}$ is therefore better thought of as a nonlinear term rather than a linear one.

Hong, Taliaferro, and Xie 2014; Sohinger 2014b] are unconditional in the NLS sense. Yet they are all considered conditional in the Boltzmann literature.

Let $\{\gamma_1^{(k)}\}$ and $\{\gamma_2^{(k)}\}$ be solutions subject to the same initial data with, respectively, $\rho_1(t,r) := \gamma_1^{(1)}(t,r,r)$ and $\rho_2(t,r) := \gamma_2^{(1)}(t,r,r)$. Let $\gamma^{(k)} := \gamma_1^{(k)} - \gamma_2^{(k)}$. Then

$$i\partial_t \gamma^{(k)} + [\Delta_{\boldsymbol{x}_k}, \gamma^{(k)}] = \mathscr{A}_1^{(k)}\gamma_1^{(k)} - \mathscr{A}_2^{(k)}\gamma_2^{(k)} - gB_{k+1}\gamma^{(k+1)}. \tag{33}$$

We can rewrite (33) using the relation

$$\mathscr{A}_1^{(k)}\gamma_1^{(k)} - \mathscr{A}_2^{(k)}\gamma_2^{(k)} = \mathscr{A}_1^{(k)}\gamma^{(k)} + \mathscr{A}^{(k)}\gamma_2^{(k)},$$

where now

$$\mathscr{A}^{(k)} := \mathscr{A}_1^{(k)} - \mathscr{A}_2^{(k)},$$

so that it becomes

$$i\partial_t \gamma^{(k)} + [\Delta_{\boldsymbol{x}_k}, \gamma^{(k)}] = \mathscr{A}_1^{(k)}\gamma^{(k)} + \mathscr{A}^{(k)}\gamma_2^{(k)} - gB_{k+1}\gamma^{(k+1)} \tag{34}$$

or, equivalently,

$$(i\partial_t + \Delta_{\boldsymbol{x}_k} - \Delta_{\boldsymbol{x}_k'})\gamma^{(k)} = \mathscr{A}_1^{(k)}\gamma^{(k)} + \mathscr{A}^{(k)}\gamma_2^{(k)} - gB_{k+1}\gamma^{(k+1)}.$$

Recalling the corresponding linear propagator $U^{(k)}(t)$ defined in (26), we write (34) in integral form, i.e.,

$$\gamma^{(k)}(t_k) = -ig\int_0^{t_k} dt_{k+1} U^{(k)}(t_k - t_{k+1})\big[\mathscr{A}_1^{(k)}\gamma^{(k)}(t_{k+1}) + \mathscr{A}^{(k)}\gamma_2^{(k)}(t_{k+1}) + B_{k+1}\gamma^{(k+1)}(t_{k+1})\big]. \tag{35}$$

In invoking this formula in future calculations, we set $g = -1$ for simplicity and we ignore the i in front so that we do not need to keep track of its exact power, as the precise power is not relevant to the estimates.

Remark 2.2. The choice of $g = -1$ corresponds to a defocusing case in (12). It is important to note, however, that the choice $g = -1$ at this step is purely for the sake of convenience; all subsequent arguments can accommodate any $g \neq -1$ at the cost of certain powers of $|g|$. In particular, our arguments apply to the self-dual case $g = 1$, which is the most interesting from the physical point of view.

For the purpose of proving unconditional uniqueness, it suffices to show $\gamma^{(1)} = 0$. Iterating (35) l_c times,[2] we obtain

$$\begin{aligned}
\gamma^{(1)}(t_1) &= \int_0^{t_1} dt_2 U^{(1)}(t_1 - t_2)\big(\mathscr{A}_1^{(1)}\gamma^{(1)}(t_2) + \mathscr{A}^{(1)}\gamma_2^{(1)}(t_2)\big) + \int_0^{t_1} dt_2 U^{(1)}(t_1 - t_2) B_2 \gamma^{(2)}(t_2) \\
&= \int_0^{t_1} dt_2 U^{(1)}(t_1 - t_2)\big(\mathscr{A}_1^{(1)}\gamma^{(1)}(t_2) + \mathscr{A}^{(1)}\gamma_2^{(1)}(t_2)\big) \\
&\qquad + \int_0^{t_1} dt_2 U^{(1)}(t_1 - t_2) B_2 \int_0^{t_2} dt_3 U^{(2)}(t_2 - t_3)\big(\mathscr{A}_1^{(2)}\gamma^{(2)}(t_3) + \mathscr{A}^{(2)}\gamma_2^{(2)}(t_3)\big) \\
&\qquad + \int_0^{t_1} dt_2 U^{(1)}(t_1 - t_2) B_2 \int_0^{t_2} dt_3 U^{(2)}(t_2 - t_3) B_3 \gamma^{(3)}(t_3) \\
&= \cdots \\
&= \mathrm{NP}^{(l_c)} + \mathrm{IP}^{(l_c)},
\end{aligned} \tag{36}$$

[2] Here, l_c stands for the level of coupling. When $l_c = 0$, one recovers (34).

where $\mathrm{NP}^{(l_c)}$ and $\mathrm{IP}^{(l_c)}$, the nonlinear part and the interaction part, respectively, are given by

$$\mathrm{NP}^{(l_c)} = G^{(1)} + \sum_{r=1}^{l_c} \int_0^{t_1} \cdots \int_0^{t_r} dt_2 \cdots dt_{r+1} U^{(1)}(t_1 - t_2) B_2 \cdots U^{(r)}(t_r - t_{r+1}) B_{r+1} G^{(r+1)}(t_{r+1}) \tag{37}$$

and

$$\mathrm{IP}^{(l_c)} = \int_0^{t_1} \cdots \int_0^{t_{l_c+1}} dt_2 \cdots dt_{l_c+1} U^{(1)}(t_1 - t_2) B_2 \cdots U^{(l_c+1)}(t_{l_c} - t_{l_c+1}) B_{l_c+2} \gamma^{(l_c+2)}(t_{l_c+2}), \tag{38}$$

where

$$G^{(k)}(t_k) := \int_0^{t_k} dt_{k+1} U^{(k)}(t_k - t_{k+1}) \big(\mathscr{A}_1^{(k)} \gamma^{(k)}(t_{k+1}) + \mathscr{A}^{(k)} \gamma_2^{(k)}(t_{k+1}) \big).$$

2B. ***Proof assuming the main estimates.***

Theorem 2.3. *There exists a constant $C > 0$ such that*

$$\mathrm{Tr}|\mathrm{NP}^{(l_c)}(t_1)| \le C t_1 \sup_{t \in [0, t_1]} \mathrm{Tr}|\gamma^{(1)}(t)|$$

for all coupling levels l_c and all sufficiently small t_1.

Proof. We postpone the proof to Section 3. □

Theorem 2.4. *There exists a constant $C > 0$ such that*

$$\mathrm{Tr}|\mathrm{IP}^{(l_c)}(t_1)| \le (C t_1^{1/3})^{l_c}$$

for all coupling levels l_c.

Proof. This estimate follows from the same method used for the corresponding term in [T. Chen, Hainzl, Pavlović, and Seiringer 2014], which relies on the quantum de Finetti theorem and on a combinatorial analysis of the graphs that one can associate to the Duhamel expansions. One merely needs to replace the three-dimensional trilinear estimates [T. Chen, Hainzl, Pavlović, and Seiringer 2014, (6.19), (6.20)] with (55) and (56), respectively, taking $s = \frac{2}{3}$, and replace the three-dimensional Sobolev estimate

$$\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|f\|_{H^1(\mathbb{R}^3)}$$

with the two-dimensional Sobolev estimate

$$\|f\|_{L^6(\mathbb{R}^2)} \lesssim \|f\|_{H^{2/3}(\mathbb{R}^2)}.$$

We remark that it is because of this Sobolev estimate that we take $s = \frac{2}{3}$ in H^s rather than a smaller s. □

With Theorems 2.3 and 2.4, we then infer from (36) that

$$\begin{aligned} \mathrm{Tr}|\gamma^{(1)}(t_1)| &\le \mathrm{Tr}|\mathrm{NP}^{(l_c)}(t_1)| + \mathrm{Tr}|\mathrm{IP}^{(l_c)}(t_1)| \\ &\le C t_1 \sup_{t \in [0, t_1]} \mathrm{Tr}|\gamma^{(1)}(t)| + (C t_1^{1/3})^{l_c} \\ &\le C T \sup_{t \in [0, T]} \mathrm{Tr}|\gamma^{(1)}(t)| + (C T^{1/3})^{l_c} \end{aligned}$$

for all $t_1 \in [0, T]$. Take the supremum in time on both sides to get

$$\sup_{t\in[0,T]} \operatorname{Tr}|\gamma^{(1)}(t)| \le CT \sup_{t\in[0,T]} \operatorname{Tr}|\gamma^{(1)}(t)| + (CT^{1/3})^{l_c}.$$

Therefore, for all T small enough, we obtain

$$\tfrac{1}{2} \sup_{t\in[0,T]} \operatorname{Tr}|\gamma^{(1)}(t)| \le (CT^{1/3})^{l_c} \to 0 \quad \text{as } l_c \to \infty,$$

i.e.,

$$\sup_{t\in[0,T]} \operatorname{Tr}|\gamma^{(1)}(t)| = 0.$$

Hence, we have finished the proof of the main theorem assuming Theorem 2.3. The bulk of the rest of the paper is devoted to proving Theorem 2.3.

3. Estimate for the nonlinear part

Recall

$$\begin{aligned}\mathrm{NP}^{(l_c)} &= G^{(1)} + \sum_{r=1}^{l_c} \int_0^{t_1} \cdots \int_0^{t_r} dt_2 \cdots dt_{r+1} U^{(1)}(t_1 - t_2) B_2 \cdots U^{(r)}(t_r - t_{r+1}) B_{r+1} G^{(r+1)}(t_{r+1}) \\ &=: \mathrm{I} + \mathrm{II},\end{aligned}$$

where

$$G^{(k)}(t_k) = \int_0^{t_k} dt_{k+1} U^{(k)}(t_k - t_{k+1}) \big(\mathcal{A}_1^{(k)} \gamma^{(k)}(t_{k+1}) + \mathcal{A}^{(k)} \gamma_2^{(k)}(t_{k+1})\big). \tag{39}$$

We will first treat $\operatorname{Tr}|G^{(1)}(t_1)|$ coming from part I and then, with some additional tools, the corresponding term coming from part II. Both of the estimates rely upon the quantum de Finetti theorem stated below.

Theorem 3.1 (quantum de Finetti theorem [Hudson and Moody 1976; Størmer 1969; Ammari and Nier 2008; 2011; Lewin, Nam, and Rougerie 2014]). *Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{H}^k = \bigotimes_{\mathrm{sym}}^k \mathcal{H}$ denote the corresponding bosonic k-particle space. Let Γ denote a collection of bosonic density matrices on $\mathcal{H}$, i.e.,*

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

with $\gamma^{(k)}$ a non-negative trace class operator on $\mathcal{H}^k$. If Γ is admissible, i.e., for all $k \in \mathbb{N}$ we have $\operatorname{Tr} \gamma^{(k)} = 1$ and $\gamma^{(k)} = \operatorname{Tr}_{k+1} \gamma^{(k+1)}$, where Tr_{k+1} denotes the partial trace over the $(k+1)$-th factor, then there exists a unique Borel probability measure μ, supported on the unit sphere in $\mathcal{H}$, and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that

$$\gamma^{(k)} = \int d\mu(\phi) (|\phi\rangle\langle\phi|)^{\otimes k} \quad \text{for all } k \in \mathbb{N}.$$

Remark 3.2. The μ determined by Theorem 3.1 is finite and so, in particular, σ-finite. Therefore, the Fubini–Tonelli theorem, which is crucial in the proof, applies. See [Dunford and Schwartz 1988, p. 190].

Using Theorem 3.1, we write

$$\gamma_j^{(k)}(t) = \int d\mu_t^{(j)}(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad j = 1, 2,$$

and

$$\gamma^{(k)}(t) = \int d\mu_t(\phi)(|\phi\rangle\langle\phi|)^{\otimes k},$$

where $\mu_t := \mu_t^{(1)} - \mu_t^{(2)}$ is a signed measure supported on the unit sphere of $L^2(\mathbb{R}^2)$. We remark that

$$\operatorname{Tr}|\gamma^{(1)}(t)| = \int d|\mu_t|(\phi)\|\phi\|_{L^2}^2 = \int d|\mu_t|(\phi)$$

while

$$\operatorname{Tr}|\gamma_j^{(1)}(t)| = \int d\mu_t^{(j)}\|\phi\|_{L^2}^2 = \int d\mu_t^{(j)} = 1.$$

Here $|\mu_t|$ is defined, in the usual way, as the sum of the positive part and the negative part of μ_t, which itself is another finite measure since $|\mu_t| \le \mu_t^{(1)} + \mu_t^{(2)}$. Write $\mu_t^{(0)} = \mu_t$ for convenience. The main properties of $\mu_t^{(i)}$ that we need are

$$\sup_{t\in[0,T]} \int d|\mu_t^{(i)}|(\phi)\|\phi\|_{H_x^{2/3}}^{2k} \le M^{2k} \quad \text{for } i = 0, 1, 2 \tag{40}$$

and

$$|\mu_t^{(i)}|\big(\big\{\phi \in L^2(\mathbb{R}^2) \bigm| \|\phi\|_{H^{2/3}} > M\big\}\big) = 0 \quad \text{for } i = 0, 1, 2, \tag{41}$$

where $|\mu_t^{(i)}|$ is of course $\mu_t^{(i)}$ if $i = 1$ or 2. For $i = 1, 2$, estimate (40) is equivalent to the energy condition

$$\sup_{t\in[0,T]} \operatorname{Tr}\left(\prod_{j=1}^{k}\langle\nabla_{x_j}\rangle^{2/3}\right)\gamma_i^{(k)}(t)\left(\prod_{j=1}^{k}\langle\nabla_{x_j'}\rangle^{2/3}\right) \le M^{2k} \quad \text{for } i = 1, 2, \tag{42}$$

and (41) then follows from (40) using Chebyshev's inequality.[3] The $i = 0$ case then follows from the definition.

Putting these structures into $\mathscr{A}$, for $\ell = 1, 2$, we have

$$\mathscr{A}_\ell^{(k)} f(t) = \iint d\mu_t^{(\ell)}(\psi)\, d\mu_t^{(\ell)}(\omega) \sum_{j=1}^{k}\big[a_{|\psi|^2,|\omega|^2}(r_j) - a_{|\psi|^2,|\omega|^2}(r_j')\big] f \tag{43}$$

and

$$\begin{aligned}\mathscr{A}^{(k)} f(t) &= (\mathscr{A}_1^{(k)} - \mathscr{A}_2^{(k)}) f \\ &= \iint d\mu_t^{(1)}(\psi)\, d\mu_t(\omega) \sum_{j=1}^{k}\big[a_{|\psi|^2,|\omega|^2}(r_j) - a_{|\psi|^2,|\omega|^2}(r_j')\big] f \\ &\qquad + \iint d\mu_t(\psi)\, d\mu_t^{(2)}(\omega) \sum_{j=1}^{k}\big[a_{|\psi|^2,|\omega|^2}(r_j) - a_{|\psi|^2,|\omega|^2}(r_j')\big] f,\end{aligned} \tag{44}$$

[3]See [T. Chen, Hainzl, Pavlović, and Seiringer 2014, Lemma 4.4] or [Hong, Taliaferro, and Xie 2014, (2.17)].

where $a_{|\psi|^2,|\omega|^2}$ is defined by

$$a_{|\psi|^2,|\omega|^2}(t,r) := A_0^{(|\psi|^2,|\omega|^2)}(t,r) + \frac{1}{r^2} A_\theta^{(|\psi|^2)}(t,r) A_\theta^{(|\omega|^2)}(t,r)$$

with

$$A_0^{(|\psi|^2,|\omega|^2)}(t,r) = -\int_r^\infty A_\theta^{(|\psi|^2)}(t,s)|\omega|^2(t,s)\,\frac{ds}{s}, \qquad A_\theta^{(\rho)}(t,r) = -\frac{1}{2}\int_0^r \rho(t,s)s\,ds.$$

Informally speaking, $a_{|\psi|^2,|\omega|^2}(r)$ is similar to $a(r)$ defined in (28) but is linear with respect to $|\psi|^2$ and $|\omega|^2$ independently rather than quadratic with respect to a single $|\phi|^2$.

This notation enables us to represent the core term of $G^{(k)}$ by

$$\begin{aligned}
&\mathscr{A}_1^{(k)}\gamma^{(k)}(t) + \mathscr{A}^{(k)}\gamma_2^{(k)}(t) \\
&\quad = \sum_{j=1}^{k} \sum_{(l,m,n)\in\mathscr{P}} \iiint d\mu_t^{(l)}(\psi)\, d\mu_t^{(m)}(\omega)\, d\mu_t^{(n)}(\phi)\big[a_{|\psi|^2,|\omega|^2}(r_j) - a_{|\psi|^2,|\omega|^2}(r_j')\big](|\phi\rangle\langle\phi|)^{\otimes k} \qquad (45)
\end{aligned}$$

if we take $\mathscr{P} = \{(1,1,0), (2,2,0), (1,0,2), (0,2,2)\}$. The set $\mathscr{P}$ is for bookkeeping, incorporating the terms from (43) and (44), and we remind the readers that $d\mu_t^{(0)} := d\mu_t$. We remark that, to reach (45), we used the quantum de Finetti theorem (i.e., Theorem 3.1) four times: twice for the $\gamma^{(k)}$ term (once for γ_1 and once for γ_2) and twice for the terms in the self-generated potential $\mathscr{A}$ (they are quadratic in ρ).

3A. *Estimate of* $\mathrm{Tr}|G^{(1)}(t_1)|$. Putting $k=2$ in (45) and replacing ψ, ω, and ϕ with ϕ_1, ϕ_2, and ϕ_3, respectively, we have

$$\begin{aligned}
\mathrm{Tr}|G^{(1)}(t_1)| &= \mathrm{Tr}\bigg|\int_0^{t_1} dt_2 U^{(1)}(t_1-t_2)\big(\mathscr{A}_1^{(1)}\gamma^{(1)}(t_2) + \mathscr{A}^{(1)}\gamma_2^{(1)}(t_2)\big)\bigg| \\
&\le \sum_{(l,m,n)\in\mathscr{P}} \int_0^{t_1} dt_2 \iiint d|\mu_{t_2}^{(l)}|(\phi_1)\, d|\mu_{t_2}^{(m)}|(\phi_2)\, d|\mu_{t_2}^{(n)}|(\phi_3) \\
&\qquad\qquad \times \mathrm{Tr}\big|U^{(1)}(t_1-t_2)\big[a_{|\phi_1|^2,|\phi_2|^2}(r_1) - a_{|\phi_1|^2,|\phi_2|^2}(r_1')\big]\phi_3(r_1)\bar{\phi}_3(r_1')\big|.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\mathrm{Tr}|U^{(1)}(t)f(r_1)g(r_1')| &= \int \big|e^{it\Delta}f(r_1)e^{-it\Delta}g(r_1)\big|\,dx_1 \\
&\le \|e^{it\Delta}f\|_{L_x^2}\|e^{it\Delta}g\|_{L_x^2} \\
&= \|f\|_{L_x^2}\|g\|_{L_x^2},
\end{aligned}$$

we have

$$\mathrm{Tr}|G^{(1)}(t_1)| \le \sum_{(l,m,n)\in\mathscr{P}} \int_0^{t_1} dt_2 \iiint d|\mu_{t_2}^{(l)}|(\phi_1)\, d|\mu_{t_2}^{(m)}|(\phi_2)\, d|\mu_{t_2}^{(n)}|(\phi_3) \|a_{|\phi_1|^2,|\phi_2|^2}\phi_3\|_{L_x^2}\|\phi_3\|_{L_x^2}.$$

Corollary 4.9, i.e., the main nonlinear estimate, turns the above into

$$\begin{aligned}
\mathrm{Tr}|G^{(1)}(t_1)| &\le \sum_{(l,m,n)\in\mathscr{P}} \int_0^{t_1} dt_2 \iiint d|\mu_{t_2}^{(l)}|(\phi_1)\, d|\mu_{t_2}^{(m)}|(\phi_2)\, d|\mu_{t_2}^{(n)}|(\phi_3) \times \|\phi_3\|_{L_x^2} \\
&\qquad \times \|\phi_1\|_{\dot{H}_x^{1/2}}\|\phi_2\|_{\dot{H}_x^{1/2}} \min_{\tau\in S_3}\|\phi_{\tau(1)}\|_{\dot{H}_x^{1/2}}\|\phi_{\tau(2)}\|_{\dot{H}_x^{1/2}}\|\phi_{\tau(3)}\|_{L_x^2}.
\end{aligned}$$

One of l, m, or n is zero, and we may put the corresponding term in L^2, i.e.,

$$\begin{aligned}\operatorname{Tr}|G^{(1)}(t_1)| \le \sum_{l=1}^{2} \int_0^{t_1} dt_2 \iiint d\mu_{t_2}^{(l)}(\phi_1)\, d\mu_{t_2}^{(l)}(\phi_2)\, d|\mu_{t_2}^{(0)}|(\phi_3) \|\phi_1\|_{\dot{H}_x^{1/2}}^2 \|\phi_2\|_{\dot{H}_x^{1/2}}^2 \|\phi_3\|_{L_x^2}^2 \\ + \int_0^{t_1} dt_2 \iiint d\mu_{t_2}^{(1)}(\phi_1)\, d|\mu_{t_2}|(\phi_2)\, d\mu_{t_2}^{(2)}(\phi_3) \|\phi_1\|_{\dot{H}_x^{1/2}}^2 \|\phi_2\|_{L_x^2} \|\phi_2\|_{\dot{H}_x^{1/2}} \|\phi_3\|_{\dot{H}_x^{1/2}} \|\phi_3\|_{L_x^2} \\ + \int_0^{t_1} dt_2 \iiint d|\mu_{t_2}|(\phi_1)\, d\mu_{t_2}^{(2)}(\phi_2)\, d\mu_{t_2}^{(2)}(\phi_3) \|\phi_1\|_{L_x^2} \|\phi_1\|_{\dot{H}_x^{1/2}} \|\phi_2\|_{\dot{H}_x^{1/2}}^2 \|\phi_3\|_{\dot{H}_x^{1/2}} \|\phi_3\|_{L_x^2}.\end{aligned}$$

Using the fact that each $\mu_t^{(j)}$ is supported on the unit sphere in L^2 and thanks to (40) and (41), we obtain

$$\operatorname{Tr}|G^{(1)}(t_1)| \le 4M^3 t_1 \sup_{t\in[0,t_1]} \int d|\mu_t|(\phi) \le CM^3 t_1 \Big(\sup_{t\in[0,t_1]} \operatorname{Tr}|\gamma^{(1)}(t)| \Big).$$

Thus, we have proved that

$$\operatorname{Tr}|G^{(1)}(t_1)| \le Ct_1 \Big(\sup_{t\in[0,t_1]} \operatorname{Tr}|\gamma^{(1)}(t)| \Big). \tag{46}$$

3B. ***Estimate for part II.*** Recall that

$$\mathrm{II} = \sum_{r=1}^{l_c} \int_0^{t_1} \cdots \int_0^{t_r} dt_2 \cdots dt_{r+1} U^{(1)}(t_1-t_2) B_2 \cdots U^{(r)}(t_r - t_{r+1}) B_{r+1} G^{(r+1)}(t_{r+1}).$$

Because each B_j is a sum of $2(j-1)$ terms (see (31)), integrands of summands of $\mathrm{NP}^{(l_c)}$ have up to $O(k!)$ summands themselves. We use the Klainerman–Machedon board game argument to combine them and hence reduce the number of terms that need to be treated. Define

$$J(\underline{t}_{j+1})(f^{(j+1)}) = U^{(1)}(t_1-t_2) B_2 \cdots U^{(j)}(t_j - t_{j+1}) B_{j+1} f^{(j+1)},$$

where $\underline{t}_{j+1}$ means $(t_2, \dots, t_{j+1})$. Then the Klainerman–Machedon board game argument implies the lemma.

Lemma 3.3 (Klainerman–Machedon board game [2008]). *One can express*

$$\int_0^{t_1} \cdots \int_0^{t_j} J(\underline{t}_{j+1})(f^{(j+1)})\, d\underline{t}_{j+1}$$

as a sum of at most 4^j *terms of the form*

$$\int_D J(\underline{t}_{j+1}, \sigma)(f^{(j+1)})\, d\underline{t}_{j+1},$$

or in other words,

$$\int_0^{t_1} \cdots \int_0^{t_j} J(\underline{t}_{j+1})(f^{(j+1)})\, d\underline{t}_{j+1} = \sum_\sigma \int_D J(\underline{t}_{j+1}, \sigma)(f^{(j+1)})\, d\underline{t}_{j+1}.$$

Here $D \subset [0, t_2]^j$, *the* σ *range over the set of maps from* $\{2, \dots, j+1\}$ *to* $\{1, \dots, j\}$ *satisfying* $\sigma(2) = 1$ *and* $\sigma(l) < l$ *for all* l, *and*

$$J(\underline{t}_{j+1}, \sigma)(f^{(j+1)}) = U^{(1)}(t_1-t_2) B_{1,2} U^{(2)}(t_2-t_3) B_{\sigma(3),3} \cdots U^{(j)}(t_j - t_{j+1}) B_{\sigma(j+1),j+1}(f^{(j+1)}).$$

With Lemma 3.3, we can write a typical summand of part II as

$$\int_0^{t_1}\cdots\int_0^{t_r} dt_2\cdots dt_{r+1}U^{(1)}(t_1-t_2)B_2\cdots U^{(r)}(t_r-t_{r+1})B_{r+1}G^{(r+1)}(t_{r+1}) = \sum_\sigma \int_D d\underline{t}_{r+1}J(\underline{t}_{r+1},\sigma)(G^{(r+1)}),$$

where the sum has at most 4^r terms inside. Let

$$\mathrm{II}^{(r,\sigma)} = \int_D d\underline{t}_{r+1}J(\underline{t}_{r+1},\sigma)(G^{(r+1)}). \tag{47}$$

To estimate part II, it suffices to prove the following lemma:

Lemma 3.4. *There is a C_0 depending on M in (42) such that, for all r, we have*

$$\mathrm{Tr}|\mathrm{II}^{(r,\sigma)}|(t_1) \le \big[(r+1)\big(C_0t_1^{1/3}\big)^r\big]t_1\Big(\sup_{t\in[0,t_1]}\mathrm{Tr}|\gamma^{(1)}(t)|\Big).$$

With the above lemma, we have

$$\begin{aligned}\mathrm{Tr}|\mathrm{II}|(t_1) &\le \sum_{r=1}^{l_c}\sum_\sigma\big[(r+1)\big(C_0t_1^{1/3}\big)^r\big]t_1\Big(\sup_{t\in[0,t_1]}\mathrm{Tr}|\gamma^{(1)}(t)|\Big)\\ &\le t_1\Big(\sup_{t\in[0,t_1]}\mathrm{Tr}|\gamma^{(1)}(t)|\Big)\sum_{r=1}^{\infty}4^r\big[(r+1)\big(C_0t_1^{1/3}\big)^r\big]\\ &\le Ct_1\Big(\sup_{t\in[0,t_1]}\mathrm{Tr}|\gamma^{(1)}(t)|\Big)\end{aligned} \tag{48}$$

for t_1 small enough so that the series converges.

Together the estimates (46) and (48) establish Theorem 2.3.

Before proving Lemma 3.4, we illustrate how to obtain the estimate for a specific example.

Example 3.5. To avoid heavy notation and demonstrate the main idea of the proof of Lemma 3.4, we first prove it for a concrete example. The general case uses the same underlying idea, which turns out to be quite simple as compared to what must be done for the interaction part IP. We adapt the example and use the notation in [T. Chen, Hainzl, Pavlović, and Seiringer 2014, §6.1] for our $\mathrm{II}^{(r,\sigma)}$. Denoting $U^{(j)}(t_k-t_l)$ by $U^{(j)}_{k,l}$, we consider

$$\begin{aligned}\mathrm{Tr}|\mathrm{II}^{(3,\sigma)}|(t_1) &= \int_D d\underline{t}_4U^{(1)}_{1,2}B_{1,2}U^{(2)}_{2,3}B_{2,3}U^{(3)}_{3,4}B_{3,4}[G^{(4)}(t_4)]\\ &\le \sum_{j=1}^{4}\sum_{(l,m,n)\in\mathcal{P}}\int_{[0,t_1]^3}d\underline{t}_4\int_0^{t_1}dt_5\iiint d|\mu^{(l)}_{t_5}|(\psi)\,d|\mu^{(m)}_{t_5}|(\omega)\,d|\mu^{(n)}_{t_5}|(\phi)\\ &\qquad\times\mathrm{Tr}\Big|U^{(1)}_{1,2}B_{1,2}U^{(2)}_{2,3}B_{2,3}U^{(3)}_{3,4}B_{3,4}U^{(4)}_{4,5}\big([a_{|\psi|^2,|\omega|^2}(r_j)-a_{|\psi|^2,|\omega|^2}(r_j')](|\phi\rangle\langle\phi|)^{\otimes 4}\big)\Big|.\end{aligned} \tag{49}$$

Remark 3.6. In the above, there is a $U^{(4)}_{4,5}$ after $B_{3,4}$. This is the main difference between the nonlinear part NP and the interaction part IP. As noted in [T. Chen, Hainzl, Pavlović, and Seiringer 2014], since the last B in IP is not followed by a Schrödinger propagator, it creates a factor of $|\phi|^2\phi$, which has to be handled by Sobolev embedding rather than Strichartz estimates.

It suffices to treat

$$\sum_{(l,m,n)\in\mathcal{P}} \int_{[0,t_1]^3} d\underline{t}_4 \int_0^{t_1} dt_5 \iiint d|\mu_{t_5}^{(l)}|(\psi)\, d|\mu_{t_5}^{(m)}|(\omega)\, d|\mu_{t_5}^{(n)}|(\phi) \\ \times \operatorname{Tr}\big|U_{1,2}^{(1)}B_{1,2}^{+}U_{2,3}^{(2)}B_{2,3}^{+}U_{3,4}^{(3)}B_{3,4}^{+}U_{4,5}^{(4)}\big([a_{|\psi|^2,|\omega|^2}(r_4)](|\phi\rangle\langle\phi|)^{\otimes 4}\big)\big|, \tag{50}$$

where $B_{1,2}^{+}$ is half of $B_{1,2}$, namely

$$B_{1,2}^{+}(\gamma^{(2)}) = \gamma^{(2)}(x_1, x_1, x_1', x_1).$$

When we plug the estimate of (50) into (49), we will pick up a 2^3 since there are three B's in (49). However, compensating for this is the factor $\big(t_1^{2/3}\big)^3$ that emerges by the end. Hence, our simplification is a valid one.

Step I (structure). We enumerate the four factors of $(|\phi\rangle\langle\phi|)^{\otimes 4}$ for the purpose of bookkeeping even though these factors are physically indistinguishable. So we write $\bigotimes_{i=1}^{4} u_i$, ordered with increasing index i. We first have

$$B_{3,4}^{+}U_{4,5}^{(4)}a_{|\psi|^2,|\omega|^2}(r_4)(|\phi\rangle\langle\phi|)^{\otimes 4} = \left(U_{4,5}^{(2)}\left(\bigotimes_{i=1}^{2} u_i\right)\right)\otimes\Theta_3,$$

where

$$\begin{aligned}\Theta_3 &= B_{1,2}^{+}(U_{4,5}^{(2)}(u_3\otimes a_{|\psi|^2,|\omega|^2}(r_4)u_4))\\ &= B_{1,2}^{+}(U_{4,5}\phi(x_3))(U_{5,4}\bar{\phi}(x_3'))(U_{4,5}[a_{|\psi|^2,|\omega|^2}(r_4)\phi(x_4)])(U_{5,4}\bar{\phi}(x_4'))\\ &= (U_{4,5}\phi(x_3))(U_{4,5}[a_{|\psi|^2,|\omega|^2}(r_3)\phi(x_3)])(U_{5,4}\bar{\phi}(x_3))(U_{5,4}\bar{\phi}(x_3'))\\ &\equiv T_3(x_3)(U_{5,4}\bar{\phi}(x_3'))\end{aligned} \tag{51}$$

with $U_{4,5} = e^{i(t_4-t_5)\Delta}$. Here T_3 stands for the trilinear form

$$(U_{4,5}[a_{|\psi|^2,|\omega|^2}(r_3)\phi(x_3)])(U_{5,4}\bar{\phi}(x_3))(U_{5,4}\bar{\phi}(x_3')).$$

We make similar substitutions below and, to bound these terms, shall invoke the trilinear estimate (56), which states that

$$\|T(f_1,f_2,f_3)\|_{L^1_{t\in[0,t_0)}L^2_x} \lesssim_s t_0^{s/2}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{\dot{H}^s}$$

for $0 < s \le 2$.

Applying $B_{2,3}^{+}U_{3,4}^{(3)}$, we reach

$$\begin{aligned}B_{2,3}^{+}U_{3,4}^{(3)}B_{3,4}^{+}U_{4,5}^{(4)}(a_{\rho_l,\rho_m}(r_j)(|\phi\rangle\langle\phi|)^{\otimes 4}) &= B_{2,3}^{+}U_{3,4}^{(3)}(U_{4,5}^{(1)}u_1\otimes U_{4,5}^{(1)}u_2\otimes\Theta_3)\\ &= U_{3,4}^{(1)}U_{4,5}^{(1)}u_1\otimes\Theta_2\\ &= U_{3,5}^{(1)}u_1\otimes\Theta_2,\end{aligned}$$

where

$$\begin{aligned}\Theta_2 &= B^+_{1,2}U^{(2)}_{3,4}(U^{(1)}_{4,5}u_2 \otimes \Theta_3)\\ &= B^+_{1,2}(U^{(1)}_{3,5}u_2 \otimes U^{(1)}_{3,4}\Theta_3)\\ &= B^+_{1,2}\big((U_{3,5}\phi(x_2))(U_{5,3}\bar{\phi}(x_2'))(U_{3,4}T_3(x_3))(U_{4,3}U_{5,4}\bar{\phi}(x_3'))\big)\\ &= (U_{3,5}\phi(x_2))(U_{3,4}T_3(x_2))(U_{5,3}\bar{\phi}(x_2))(U_{5,3}\bar{\phi}(x_2'))\\ &\equiv T_2(x_2)(U_{5,3}\bar{\phi}(x_2')).\end{aligned} \tag{52}$$

Finally, with $U^{(1)}_{1,2}B^+_{1,2}U^{(2)}_{2,3}$, we get

$$\begin{aligned}&U^{(1)}_{1,2}B^+_{1,2}U^{(2)}_{2,3}B^+_{2,3}U^{(3)}_{3,4}B^+_{3,4}U^{(4)}_{4,5}(a_{|\psi|^2,|\omega|^2}(r_j)(|\phi\rangle\langle\phi|)^{\otimes 4})\\ &\qquad= U^{(1)}_{1,2}B^+_{1,2}U^{(2)}_{2,3}(U^{(1)}_{3,5}u_1 \otimes \Theta_2)\\ &\qquad= U^{(1)}_{1,2}B^+_{1,2}(U^{(1)}_{2,5}u_1 \otimes U^{(1)}_{2,3}\Theta_2)\\ &\qquad= U^{(1)}_{1,2}B^+_{1,2}\big[(U_{2,5}\phi(x_1))(U_{5,2}\bar{\phi}(x_1'))(U_{2,3}T_2(x_2))(U_{3,2}U_{5,3}\bar{\phi}(x_2'))\big]\\ &\qquad= U^{(1)}_{1,2}\big[(U_{2,5}\phi(x_1))(U_{2,3}T_2(x_1))(U_{5,2}\bar{\phi}(x_1))(U_{5,2}\bar{\phi}(x_1'))\big]\\ &\qquad= U^{(1)}_{1,2}[T_1(x_1)U_{5,2}\bar{\phi}(x_1')].\end{aligned} \tag{53}$$

Step II (iterative estimate). Plugging the calculation in Step I into (50), we have

$$\begin{aligned}(50) &\le \sum_{(l,m,n)\in\mathcal{P}} \int_{[0,t_1]^3} d\underline{t}_4 \int_0^{t_1} dt_5 \iiint d|\mu^{(l)}_{t_5}|(\psi)\, d|\mu^{(m)}_{t_5}|(\omega)\, d|\mu^{(n)}_{t_5}|(\phi) \|T_1(x_1)\|_{L^2}\|\bar{\phi}\|_{L^2}\\ &\le \sum_{(l,m,n)\in\mathcal{P}} \int_{[0,t_1]^2} dt_3t_4 \int_0^{t_1} dt_5 \iiint d|\mu^{(l)}_{t_5}|(\psi)\, d|\mu^{(m)}_{t_5}|(\omega)\, d|\mu^{(n)}_{t_5}|(\phi) \times \|T_1\|_{L^1_{t_2}L^2},\end{aligned}$$

where

$$\|T_1\|_{L^1_{t_2}L^2} \le Ct_1^{1/3}\|\phi\|_{H^{2/3}}\|T_2\|_{L^2}\|\phi\|_{L^2}$$

by (56). Thus,

$$(50) \le Ct_1^{1/3} \sum_{(l,m,n)\in\mathcal{P}} \int_{[0,t_1]} t_4 \int_0^{t_1} dt_5 \iiint d|\mu^{(l)}_{t_5}|(\psi)\, d|\mu^{(m)}_{t_5}|(\omega)\, d|\mu^{(n)}_{t_5}|(\phi) \|\phi\|_{H^{2/3}}\|T_2\|_{L^1_{t_3}L^2}.$$

By (56) again,

$$\|T_2(x_2)\|_{L^1_{t_3}L^2_{x_2}} \le Ct_1^{1/3}\|\phi\|_{H^{2/3}}\|T_3\|_{L^2}\|\phi\|_{L^2},$$

and hence,

$$\begin{aligned}(50) &\le \big(Ct_1^{1/3}\big)^2 \sum_{(l,m,n)\in\mathcal{P}} \int_0^{t_1} dt_5 \iiint d|\mu^{(l)}_{t_5}|(\psi)\, d|\mu^{(m)}_{t_5}|(\omega)\, d|\mu^{(n)}_{t_5}|(\phi) \|\phi\|^2_{H^{2/3}}\|T_3\|_{L^1_{t_4}L^2_{x_2}}\\ &\le \big(Ct_1^{1/3}\big)^3 \sum_{(l,m,n)\in\mathcal{P}} \int_0^{t_1} dt_5 \iiint d|\mu^{(l)}_{t_5}|(\psi)\, d|\mu^{(m)}_{t_5}|(\omega)\, d|\mu^{(n)}_{t_5}|(\phi) \|\phi\|^3_{H^{2/3}}\big\|a_{|\psi|^2,|\omega|^2}(r_3)\phi(x_3)\big\|_{L^2}.\end{aligned}$$

By the fact that $|\mu_t^{(i)}|$ is supported in the set

$$\{\phi \in L^2(\mathbb{R}^2) \mid \|\phi\|_{H^{2/3}} \le M\},$$

we have

$$(50) \le \big(CMt_1^{1/3}\big)^3 \sum_{(l,m,n)\in\mathcal{P}} \int_0^{t_1} dt_5 \iiint d|\mu_{t_5}^{(l)}|(\psi)\, d|\mu_{t_5}^{(m)}|(\omega)\, d|\mu_{t_5}^{(n)}|(\phi) \big\| a_{|\psi|^2,|\omega|^2}(r_3)\phi(x_3)\big\|_{L^2}.$$

One then proceeds as in the estimate of $\mathrm{Tr}|G^{(1)}(t_1)|$ to reach

$$(50) \le \big(CMt_1^{1/3}\big)^3 M^3 t_1 \Big(\sup_{t\in[0,t_1]} \mathrm{Tr}|\gamma^{(1)}(t)|\Big).$$

Selecting a C_0 bigger than M^2 and 1, we obtain

$$(50) \le \big(C_0 t_1^{1/3}\big)^3 t_1 \Big(\sup_{t\in[0,t_1]} \mathrm{Tr}|\gamma^{(1)}(t)|\Big).$$

Plugging the above estimate back into (49), we get

$$\mathrm{Tr}|\mathrm{II}^{(3,\sigma)}|(t_1) \le \big[4\cdot 2^3 \cdot \big(C_0 t_1^{1/3}\big)^3\big] t_1 \Big(\sup_{t\in[0,t_1]} \mathrm{Tr}|\gamma^{(1)}(t)|\Big)$$

as desired. This finishes the proof of the example.

One observation to make concerning our approach in Example 3.5 is that the structure found in Step I is crucial. Such a structure generated by the collision operator B and propagator U is found in general, and we state its relevant properties in the following lemma:

Lemma 3.7. *Let $M \in \mathbb{N}$, $M > 1$, and for each j, $1 \le j \le M$, suppose that the two functions $f_j(x_j)$ and $f_j'(x_j')$ belong to*[4] *$L_t^1 H_x^s(\mathbb{R}^2)$, $\frac{1}{2} \le s \le \frac{2}{3}$. Then there exist $L_t^1 H_x^s(\mathbb{R}^2)$ functions h and h' such that*

$$B^{\pm}_{\sigma(M),M} U^{(M)}_{M,M+1}\left[\prod_{j=1}^{M} f_j(x_j) f_j'(x_j')\right] = h_{\sigma(M)}(x_{\sigma(M)}) h'_{\sigma(M)}(x'_{\sigma(M)}) U^{(M-2)}_{M,M+1}\left[\prod_{\substack{j=1\\ j\neq\sigma(M)}}^{M-1} f_j(x_j) f_j'(x_j')\right].$$

In the case where B is $B^+_{\sigma(M),M}$, h is a trilinear form of the type (54) and h' is a linear evolution. In the case where B is $B^-_{\sigma(M),M}$, the roles of h and h' are reversed.

Proof. The collision operator leaves untouched each term for which $j \notin \{M, \sigma(M)\}$. Only the propagator affects these terms. So we have

$$\begin{aligned} &B^{+}_{\sigma(M),M} U^{(M)}_{M,M+1}\left[\prod_{j=1}^{M} f_j(x_j) f_j'(x_j')\right] \\ &\qquad = U^{(M-2)}_{M,M+1}\left[\prod_{j\in\{1,\dots,M\}\setminus\{M,\sigma(M)\}} f_j(x_j) f_j'(x_j')\right] \cdot T_{\sigma(M),M}(x_{\sigma(M)}) e^{-i(t_M - t_{M+1})\Delta_{x'_{\sigma(M)}}} f'_{\sigma(M)}(x'_{\sigma(M)}), \end{aligned}$$

[4] We suppress the time dependence in the notation and allow restriction to time intervals, which may be achieved, for instance, by introducing sharp time cutoffs.

where

$$T_{\sigma(M),M}(x_{\sigma(M)}) := e^{i(t_M-t_{M+1})\Delta_{x_{\sigma(M)}}} f_{\sigma(M)}(x_{\sigma(M)}) \cdot e^{i(t_M-t_{M+1})\Delta_{x_{\sigma(M)}}} f_M(x_{\sigma(M)}) \cdot e^{-i(t_M-t_{M+1})\Delta_{x_{\sigma(M)}}} f'_M(x_{\sigma(M)}).$$

Similarly,

$$B^{-}_{\sigma(M),M} U^{(M)}_{M,M+1}\left[\prod_{j=1}^{M} f_j(x_j) f'_j(x'_j)\right] = U^{(M-2)}_{M,M+1}\left[\prod_{j\in\{1,\dots,M\}\setminus\{M,\sigma(M)\}} f_j(x_j) f'_j(x'_j)\right] \cdot T'_{\sigma(M),M}(x'_{\sigma(M)}) e^{i(t_M-t_{M+1})\Delta_{x_{\sigma(M)}}} f_{\sigma(M)}(x_{\sigma(M)}),$$

where

$$T'_{\sigma(M),M}(x'_{\sigma(M)}) := e^{i(t_M-t_{M+1})\Delta_{x'_{\sigma(M)}}} f_M(x'_{\sigma(M)}) \cdot e^{-i(t_M-t_{M+1})\Delta_{x'_{\sigma(M)}}} f'_{\sigma(M)}(x'_{\sigma(M)}) \cdot e^{-i(t_M-t_{M+1})\Delta_{x'_{\sigma(M)}}} f'_M(x'_{\sigma(M)}).$$

The $L^1_t H^s_x$ bounds follow from (55) and Strichartz. □

Proof of Lemma 3.4. Using (47), (39), and (45), we write

$$\mathrm{II}^{(r,\sigma)} = \sum_{j=1}^{r+1} \sum_{(l,m,n)\in\mathcal{P}} \int_D d\underline{t}_{r+1} J(\underline{t}_{r+1},\sigma)\Bigg\{\int_0^{t_{r+1}} dt_{r+2} U^{(r+1)}(t_{r+1}-t_{r+2}) \times \iiint d\mu^{(l)}_{t_{r+2}}(\psi)\, d\mu^{(m)}_{t_{r+2}}(\omega)\, d\mu^{(n)}_{t_{r+2}}(\phi)\big[a_{|\psi|^2,|\omega|^2}(|x_j|) - a_{|\psi|^2,|\omega|^2}(|x'_j|)\big](|\phi\rangle\langle\phi|)^{\otimes(r+1)}\Bigg\}.$$

We abbreviate

$$J(\underline{t}_{r+1},\sigma) = U^{(1)}_{1,2} B_{1,2} U^{(2)}_{2,3} B_{\sigma(3),3} \cdots U^{(r)}_{r,r+1} B_{\sigma(r+1),r+1}$$

and write

$$\mathrm{Tr}|\mathrm{II}^{(r,\sigma)}|(t_1) \le \sum_{j=1}^{r+1} \sum_{(l,m,n)\in\mathcal{P}} \int_{[0,t_1]^r} d\underline{t}_{r+1} \int_0^{t_1} dt_{r+2} \iiint d|\mu^{(l)}_{t_{r+2}}|(\psi)\, d|\mu^{(m)}_{t_{r+2}}|(\omega)\, d|\mu^{(n)}_{t_{r+2}}|(\phi) \times \mathrm{Tr}\Big|U^{(1)}_{1,2} B_{1,2} \cdots U^{(r)}_{r,r+1} B_{\sigma(r+1),r+1} U^{(r+1)}_{r+1,r+2}\big[a_{|\psi|^2,|\omega|^2}(|x_j|) - a_{|\psi|^2,|\omega|^2}(|x'_j|)\big](|\phi\rangle\langle\phi|)^{\otimes(r+1)}\Big|.$$

To simplify calculations, we drop, without loss of generality, the $-a_{|\psi|^2,|\omega|^2}(|x'_j|)$ term. Also, we split each $B_{j,k}$ into two pieces $B^{\pm}_{j,k}$ so that $B_{j,k} = B^{+}_{j,k} - B^{-}_{j,k}$.

Consider first the innermost terms

$$B^{\pm}_{\sigma(r+1),r+1} U^{(r+1)}_{r+1,r+2} a_{|\psi|^2,|\omega|^2}(|x_j|)(|\phi\rangle\langle\phi|)^{\otimes(r+1)}.$$

The index $j \in \{1, \dots, r+1\}$ and the permutation σ together determine at what point $a_{|\psi|^2,|\omega|^2}(|x_j|)$ is directly affected by a collision operator. In any case, we claim that, with respect to the variables $x_{\sigma(r+1)}$

and $x'_{\sigma(r+1)}$, the term

$$B^{+}_{\sigma(r+1),r+1}U^{(r+1)}_{r+1,r+2}a_{|\psi|^2,|\omega|^2}(|x_j|)(|\phi\rangle\langle\phi|)^{\otimes(r+1)}$$

is a trilinear form of the form T in (54) (see (51), (52), and (53) for examples of these trilinear forms) in the $x_{\sigma(r+1)}$ variable and a linear flow in the $x'_{\sigma(r+1)}$ variable (the term with B^- instead of B^+ is similar but with the roles of the primed and unprimed variables reversed). Note that precisely one of the terms in the trilinear form T involves $a_{|\psi|^2,|\omega|^2}(|x_j|)$. This follows from Lemma 3.7. Additionally, Lemma 3.7 is formulated so that we can apply it iteratively until termination, at which point we have one term that is trilinear of the form (54) in precisely one of x_1 or x'_1 and another term that is a linear evolution of a function of the remaining spatial variable. Step I of Example 3.5 illustrates such a process.

The final step is to iteratively bound the terms. We follow Step II of Example 3.5. The underlying idea behind the iterative bounds is relatively straightforward. We start by controlling the trace norm using Cauchy–Schwarz in space. One factor is simply a ϕ term associated to the measure and so will have L^2 norm equal to one. This leaves us with the other term in $L^1_tL^2$. The next step is to apply (56). This places one factor in $\dot{H}^s$ and the remaining ones in L^2. So that we can eventually apply (70), it is important to always place in L^2 the term appearing in the right-hand side that involves $a_{|\psi|^2,|\omega|^2}(|x_j|)$. To control the term placed in $\dot{H}^s$, we apply (55). For the terms in L^2, we use (56) or (70) as appropriate. □

Remark 3.8. We first remind the reader that, because at each step we are estimating a linear term of the type $e^{it\Delta}f$ or a trilinear term of the form (54), we do not need to apply Sobolev embedding as is necessary for estimating the interaction part. Secondly, the "a" term cannot be generated by B, and thus, we do not need to keep track of multiple "copies" of $|\phi|^2\phi$ generated by B in contrast to what must be done in controlling the interaction part. In particular, there is no need to introduce binary tree graphs or keep track of complicated factorization structures of kernels in controlling the nonlinear part.

4. Multilinear estimates

In this section, we will need the following fractional Leibniz rule from [Christ and Weinstein 1991, Proposition 3.3]:

Lemma 4.1. *Let $0 < s \le 1$ and $1 < r, p_1, p_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i = 1, 2$. Then*

$$\||\nabla|^s(fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}}\||\nabla|^s g\|_{L^{q_1}} + \||\nabla|^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}.$$

Define the trilinear form T by

$$T(f,g,h) = e^{i(t-t_1)\Delta}f \cdot e^{i(t-t_2)\Delta}g \cdot e^{i(t-t_3)\Delta}h. \tag{54}$$

Lemma 4.2. *Let $0 < s \le \frac{2}{3}$. The trilinear form T given by (54) satisfies*[5]

$$\|T(f,g,h)\|_{L^1_{t\in[0,t_0)}\dot{H}^s_x} \lesssim t_0^s\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}\|h\|_{\dot{H}^s}. \tag{55}$$

[5] Such trilinear estimates are the precursors to the Klainerman–Machedon collapsing estimates widely used in the literature. For those estimates, see [Klainerman and Machedon 2008; Kirkpatrick, Schlein, and Staffilani 2011; Grillakis and Margetis 2008; T. Chen and Pavlović 2011; X. Chen 2011; 2012a; Beckner 2014; Gressman, Sohinger, and Staffilani 2014].

Proof. By the fractional Leibniz rule, we have

$$\begin{aligned}\|T(f,g,h)\|_{L^1_t\dot H^s_x} \lesssim \|e^{i(t-t_1)\Delta}f\|_{L^3_t\dot W^{s,6}_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_tL^6_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_tL^6_x}\\ +\|e^{i(t-t_1)\Delta}f\|_{L^3_tL^6_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_t\dot W^{s,6}_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_tL^6_x}\\ +\|e^{i(t-t_1)\Delta}f\|_{L^3_tL^6_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_tL^6_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_t\dot W^{s,6}_x}.\end{aligned}$$

By Sobolev embedding, we bound the first term by

$$\|e^{i(t-t_1)\Delta}f\|_{L^3_t\dot W^{s,6}_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_t\dot W^{s,p}_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_t\dot W^{s,p}_x},$$

where $\frac{1}{p}=\frac{1}{6}+\frac{s}{2}$. Note that $2\le p<6$. Let q be given by $\frac{1}{q}+\frac{1}{p}=\frac{1}{2}$ so that (q,p) forms a Schrödinger-admissible Strichartz pair (see, for instance, [Tao 2006, §2]). So we use Hölder in time to bound the expression by

$$\|e^{i(t-t_1)\Delta}f\|_{L^3_t\dot W^{s,6}_x}t_0^{1/3-1/q}\|e^{i(t-t_2)\Delta}g\|_{L^q_t\dot W^{s,p}_x}t_0^{1/3-1/q}\|e^{i(t-t_3)\Delta}h\|_{L^q_t\dot W^{s,p}_x}.$$

Finally, we conclude by applying Strichartz estimates and noting that $\frac{1}{3}-\frac{1}{q}=\frac{s}{2}$. The second and third terms are similar. □

Lemma 4.3. *Let $0<s\le 2$. The trilinear form T given by (54) satisfies*

$$\|T(f,g,h)\|_{L^1_{t\in[0,t_0)}L^2_x}\lesssim t_0^{s/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{\dot H^s}. \tag{56}$$

Proof. By Hölder's inequality,

$$\|T(f,g,h)\|_{L^1_{t\in[0,t_0)}L^2_x}\le t_0^{s/2}\|e^{i(t-t_1)\Delta}f\|_{L^q_tL^r_x}\|e^{i(t-t_2)\Delta}g\|_{L^q_tL^r_x}\|e^{i(t-t_3)\Delta}h\|_{L^\infty_tL^p_x},$$

where $\frac{1}{q}=\frac{1}{2}-\frac{s}{4}$, $r=\frac{4}{s}$, and $p=2/(1-s)$. Using Strichartz estimates and Sobolev embedding, we control the right-hand side by

$$t_0^{s/2}\|f\|_{L^2_x}\|g\|_{L^2_x}\|e^{i(t-t_3)\Delta}h\|_{L^\infty_t\dot H^s_x}.$$

Finally, we conclude the bound stated in the lemma by noting that the Schrödinger propagator is an isometry on L^2-based spaces. □

Remark 4.4. From the proofs of both (55) and (56), it is evident that any of $e^{i(t-t_1)\Delta}f$, $e^{i(t-t_2)\Delta}g$, and $e^{i(t-t_3)\Delta}h$ can be replaced by its complex conjugate in the trilinear form (54).

For the next set of estimates, recall

$$\partial_rA_0=\tfrac{1}{r}A_\theta\rho,\qquad \partial_rA_\theta=-\tfrac{1}{2}r\rho$$

and

$$A_0(t,r):=-\int_r^\infty\frac{A_\theta(s)}{s}\rho(s)\,ds,\qquad A_\theta(t,r):=-\frac{1}{2}\int_0^r\rho(s)s\,ds. \tag{57}$$

When it is important to indicate the dependence upon the density function ρ, we write $A_\theta^{(\rho)}(t,r)$ for $A_\theta(t,r)$. Recall

$$A_0^{(\rho_1,\rho_2)}(t,r)=-\int_r^\infty A_\theta^{(\rho_1)}(s)\rho_2(s)\,\frac{ds}{s}, \tag{58}$$

where $A_\theta^{(\rho_1)}$ is defined using (57) but with ρ_1 in place of ρ in the right-hand side, i.e.,

$$A_\theta^{(\rho_1)}(t,r) = -\frac{1}{2}\int_0^r \rho_1(s)s\,ds.$$

Define the operators $[\partial_r]^{-1}$, $[r^{-n}\bar{\partial}_r]^{-1}$, and $[r\partial_r]^{-1}$ acting on radial functions by

$$[\partial_r]^{-1}f(r) = -\int_r^\infty f(s)\,ds, \qquad [r^{-n}\bar{\partial}_r]^{-1}f(r) = \int_0^r f(s)s^n\,ds, \qquad [r\partial_r]^{-1}f(r) = -\int_r^\infty \frac{1}{s}f(s)\,ds.$$

Then it follows by a direct argument that

$$\|[r\partial_r]^{-1}f\|_{L^p} \lesssim_p \|f\|_{L^p}, \quad 1 \le p < \infty, \tag{59}$$

$$\|r^{-n-1}[r^{-n}\bar{\partial}_r]^{-1}f\|_{L^p} \lesssim_p \|f\|_{L^p}, \quad 1 < p \le \infty, \tag{60}$$

$$\|[\partial_r]^{-1}f\|_{L^2} \lesssim \|f\|_{L^1}. \tag{61}$$

These estimates appear, for instance, in [Bejenaru, Ionescu, Kenig, and Tataru 2013, (1.5)] and also find application in [Liu and Smith 2014, §2].

Remark 4.5. In these estimates and those below, we use the Lebesgue measure on $\mathbb{R}^2$ for all L^p spaces. In particular, for radial functions of r, we essentially adopt the $r\,dr$ measure.

Lemma 4.6 (elementary bounds for A). *The connection coefficients A_θ and A_0, given by (57), satisfy*

$$\|A_\theta\|_{L_x^\infty} \lesssim \|\rho\|_{L_x^1}, \qquad \left\|\tfrac{1}{r}A_\theta\right\|_{L_x^\infty} \lesssim \|\rho\|_{L_x^2}, \qquad \left\|\tfrac{1}{r^2}A_\theta\right\|_{L_x^p} \lesssim \|\rho\|_{L_x^p}, \quad \text{where } 1 < p \le \infty, \tag{62}$$

and

$$\|A_0\|_{L_x^p} \lesssim \|\rho\|_{L_x^1}\|\rho\|_{L_x^p}, \quad \text{where } 1 \le p < \infty, \qquad \|A_0\|_{L_x^\infty} \lesssim \|\rho\|_{L_x^2}^2. \tag{63}$$

Moreover, A_θ^2 satisfies the bounds

$$\left\|\tfrac{1}{r^2}A_\theta^2\right\|_{L_x^p} \lesssim \|\rho\|_{L_x^1}\|\rho\|_{L_x^p}, \quad \text{where } 1 < p \le \infty, \qquad \left\|\tfrac{1}{r^2}A_\theta^2\right\|_{L_x^\infty} \lesssim \|\rho\|_{L_x^2}^2. \tag{64}$$

Proof. These estimates are essentially contained in [Liu and Smith 2014, §2].

The first inequality of (62) is trivial. The second follows from Cauchy–Schwarz:

$$|A_\theta(t,r)| \lesssim r\left(\int_0^\infty |\rho(s)|^2 s\,ds\right)^{1/2}.$$

The third is an application of (60) with $n = 1$.

The first inequality of (63) follows from the first inequality of (62) and from (59). The second is a consequence of Cauchy–Schwarz and the third inequality of (62) with $p = 2$.

The first inequality of (64) follows from the first and third inequalities of (62). The second follows from two applications of the second inequality of (62). □

Lemma 4.7 (weighted estimates). *Let $\frac{1}{p}+\frac{1}{q}=1$ with $1<q<\infty$, and suppose that $\rho=|\psi|^2$ and $\rho_j=|\psi_j|^2$ for $j=1,2$. Then*

$$\|r^{-2/q}A_\theta^{(\rho)}\|_{L_x^\infty}\lesssim\|\psi\|_{\dot H_x^{1/q}}^2, \tag{65}$$

$$\|r^{-1/q}A_\theta^{(\rho)}\|_{L_x^\infty}\lesssim\|\psi\|_{\dot H_x^{1/q}}\|\psi\|_{L_x^2} \tag{66}$$

and

$$\|r^{1/p}A_0^{(\rho_1,\rho_2)}\|_{L_x^\infty}\lesssim\min_{\tau\in S_2}\|\psi_{\tau(1)}\|_{\dot H_x^{1/q}}^2\|\psi_{\tau(2)}\|_{\dot H_x^{1/p}}\|\psi_{\tau(2)}\|_{L_x^2}, \tag{67}$$

where S_2 denotes the set of permutations on two elements.

Proof. To establish (66), use Hölder's inequality to obtain

$$|A_\theta|\lesssim r^{2/q}\|\psi\|_{L^{2p}}^2$$

and then use Sobolev embedding. The estimate (65) follows from Hölder's inequality, which yields

$$|A_\theta|\lesssim r^{1/q}\|r^{-1/q}\psi\|_{L_x^2}\|\psi\|_{L_x^2},$$

and Hardy's inequality.

To prove (67), use Hölder to write

$$|A_0^{(\rho_1,\rho_2)}|\lesssim\|r^{-2/q}A_\theta^{(\rho_1)}\|_{L_x^\infty}\|r^{-1/p}\psi_2\|_{L_x^2}\|\psi_2\|_{L_x^2}r^{-1/p}.$$

Then, using (65) and Hardy's inequality, we obtain

$$\|r^{1/p}A_0^{(\rho_1,\rho_2)}\|_{L_x^\infty}\lesssim\|\psi_1\|_{\dot H_x^{1/q}}^2\|\psi_2\|_{\dot H_x^{1/p}}\|\psi_2\|_{L_x^2}.$$

Finally, we may repeat the argument with the roles of ψ_1 and ψ_2 reversed. □

Lemma 4.8 (bounds for the nonlinear terms). *Suppose that $\rho_j=|\psi_j|^2$ for $j=1,2$. Then*

$$\|A_0^{(\rho_1,\rho_2)}\Theta\|_{L_x^2}+\Big\|\frac{1}{r^2}A_\theta^{(\rho_1)}A_\theta^{(\rho_2)}\Theta\Big\|_{L_x^2}\lesssim\|\psi_1\|_{\dot H_x^{1/2}}\|\psi_2\|_{\dot H_x^{1/2}}\|\Theta\|_{\dot H_x^{1/2}}\min_{\tau\in S_2}\|\psi_{\tau(1)}\|_{\dot H_x^{1/2}}\|\psi_{\tau(2)}\|_{L_x^2}. \tag{68}$$

Proof. We start with

$$\|A_0^{(\rho_1,\rho_2)}\Theta\|_{L_x^2}\lesssim\|r^{1/2}A_0^{(\rho_1,\rho_2)}\|_{L_x^\infty}\|r^{-1/2}\Theta\|_{L_x^2}\lesssim\|r^{1/2}A_0^{(\rho_1,\rho_2)}\|_{L_x^\infty}\|\Theta\|_{\dot H_x^{1/2}}$$

and then appeal to (67) with $p=q=2$.

Similarly,

$$\begin{aligned}\Big\|\frac{1}{r^2}A_\theta^{(\rho_1)}A_\theta^{(\rho_2)}\Theta\Big\|_{L_x^2}&\lesssim\|r^{-1}A_\theta^{(\rho_1)}\|_{L_x^\infty}\|r^{-1/2}A_\theta^{(\rho_2)}\|_{L_x^\infty}\|r^{-1/2}\Theta\|_{L_x^2}\\&\lesssim\|\psi_1\|_{\dot H_x^{1/2}}^2\|\psi_2\|_{\dot H_x^{1/2}}\|\psi_2\|_{L_x^2}\|\Theta\|_{\dot H_x^{1/2}},\end{aligned}$$

where we have used (66) and (65) with $p=q=2$ and Hardy's inequality. Finally, we may repeat the estimate but with the roles of ψ_1 and ψ_2 reversed. □

Now we introduce (see (28) to compare)

$$a_{\rho_1,\rho_2}(t,r) := A_0^{(\rho_1,\rho_2)}(t,r) + \tfrac{1}{r^2} A_\theta^{(\rho_1)}(t,r) A_\theta^{(\rho_2)}(t,r). \tag{69}$$

For the definitions of the terms on the right-hand side, see the equations and comments from (57) to (58).

Corollary 4.9. *Suppose $\rho_j = |\psi_j|^2$ for $j = 1, 2$. Then*

$$\|a_{\rho_1,\rho_2}\psi_3\|_{L_x^2} \lesssim \|\psi_1\|_{\dot{H}_x^{1/2}} \|\psi_2\|_{\dot{H}_x^{1/2}} \min_{\tau\in S_3} \|\psi_{\tau(1)}\|_{\dot{H}_x^{1/2}} \|\psi_{\tau(2)}\|_{\dot{H}_x^{1/2}} \|\psi_{\tau(3)}\|_{L_x^2}, \tag{70}$$

where S_3 denotes the set of permutations on three elements.

Proof. For all but two permutations, the estimate follows from (68). To establish the estimate for the remaining two cases, we need L_x^∞ bounds on $A_0^{(\rho_1,\rho_2)}$ and $\frac{1}{r^2} A_\theta^{(\rho_1)} A_\theta^{(\rho_2)}$. Using the second estimate of (62) twice and Sobolev embedding, we obtain

$$\Big\| \tfrac{1}{r^2} A_\theta^{(\rho_1)} A_\theta^{(\rho_2)} \Big\|_{L_x^\infty} \le \Big\| \tfrac{1}{r} A_\theta^{(\rho_1)} \Big\|_{L_x^\infty} \Big\| \tfrac{1}{r} A_\theta^{(\rho_2)} \Big\|_{L_x^\infty} \lesssim \|\psi_1\|_{L_x^4}^2 \|\psi_2\|_{L_x^4}^2 \lesssim \|\psi_1\|_{\dot{H}_x^{1/2}}^2 \|\psi_2\|_{\dot{H}_x^{1/2}}^2.$$

To bound $A_0^{(\rho_1,\rho_2)}$, we proceed in a manner similar to that of the second estimate of (63) and (67). In particular, invoking (66) with $q = 2$ and Hardy, we obtain

$$\|A_0^{(\rho_1,\rho_2)}\|_{L_x^\infty} = \left\| \int_r^\infty s^{-1} A_\theta^{(\rho_1)} s^{-1} |\psi_2|^2 s\, ds \right\|_{L_x^\infty} \lesssim \|r^{-1} A_\theta^{(\rho_1)}\|_{L_x^\infty} \|r^{-1/2}\psi_2\|_{L_x^\infty}^2 \lesssim \|\psi_1\|_{\dot{H}_x^{1/2}}^2 \|\psi_2\|_{\dot{H}_x^{1/2}}^2. \quad \square$$

Remark 4.10. From the proofs of these estimates, we see that the limiting factor in lowering the regularity of the unconditional uniqueness result lies in the interaction part, which requires $s = \frac{2}{3}$ rather than the $s = \frac{1}{2}$ required for the nonlinear part. By using negative-regularity Sobolev spaces, [Hong, Taliaferro, and Xie 2014] lowers the regularity required for the interaction part. Such a procedure does not seem to work, at least directly, for the problem at hand. This is because one would need to obtain the same negative-order Sobolev index in the right-hand side of (70) for the purpose of moving the term arising from controlling the nonlinear part back over to the left-hand side (see the argument following the proof of Theorem 2.4).

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