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BOUNDARY BLOW-UP UNDER SOBOLEV MAPPINGS

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We prove that for mappings in $W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$, continuous up to the boundary and with modulus of continuity satisfying a certain divergence condition, the image of the boundary of the unit ball has zero n-Hausdorff measure. For Hölder continuous mappings we also prove an essentially sharp generalised Hausdorff dimension estimate.

1. Introduction

Throughout this paper $\mathcal{B}^n$ denotes the unit ball in $\mathbb{R}^n$ and $W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ is the Sobolev space of $L^n(\mathcal{B}^n, \mathbb{R}^m)$-functions $f : \mathcal{B}^n \to \mathbb{R}^m$ with weak first-order derivatives in $L^n(\mathcal{B}^n)$.

If $f : \mathcal{B}^2 \to \Omega \subset \mathbb{R}^2$ is a conformal mapping, then the boundary of Ω can have positive Lebesgue measure even if f extends continuously up to the boundary of the disk. If one requires more, for example uniform Hölder continuity, then $\partial\Omega$ is necessarily of Lebesgue measure zero. In fact, Jones and Makarov proved [1995, Theorem C.1] that $\partial\Omega$ has measure zero if f satisfies $|f(z) - f(w)| \le \psi(|z - w|)$ in $\mathcal{B}^2$ for $\psi : [0, \infty) \to [0, \infty)$ with

$$\int_0 \left| \frac{\log \psi(t)}{\log t} \right|^2 \frac{\mathrm{d}t}{t} = \infty. \tag{1}$$

This condition is very sharp: if the integral in (1) converges then [Jones and Makarov 1995, Section 6] provides us with a simply connected domain Ω and a conformal mapping $f : \mathcal{B}^2 \to \Omega$ such that the boundary of Ω has positive Lebesgue measure and f has the modulus of continuity ψ.

Our first result gives a surprisingly general extension of the conformal setting; notice that each uniformly continuous conformal mapping $f : \mathcal{B}^2 \to \Omega$ belongs to $W^{1,2}(\mathcal{B}^2, \mathbb{R}^2)$.

Theorem 1.1. *Let $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ be a continuous mapping that satisfies*

$$|f(z) - f(w)| \le \psi(|z - w|) \tag{2}$$

for all $z, w \in \overline{\mathcal{B}}^n$, where $\psi : (0, \infty) \to (0, \infty)$ is an allowable modulus of continuity with

$$\int_0 \left| \frac{\log \psi(t)}{\log t} \right|^n \frac{dt}{t} = \infty. \tag{3}$$

Then $\mathcal{H}^n(f(\partial\mathcal{B}^n)) = 0$.

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Recall that every uniformly continuous map defined on $\mathcal{B}^n$ has a continuous extension to all of $\overline{\mathcal{B}}^n$. In the above, f on $\partial\mathcal{B}^n$ refers to this extension, $\mathcal{H}^n(A)$ denotes the n-dimensional Hausdorff measure of a set A, and the definition of an *allowable modulus of continuity* is given in Definition 2.2 of Section 2. For example, both $\psi(t) = Ct^\gamma$, $0 < \gamma \leq 1$, and

$$\psi_{l,s}(t) = \exp\left(-C \frac{(\log(C_l/t))^{(n-1)/n}}{(\log^{(l)}(C_l/t))^{s/n}\left(\prod_{k=2}^{l-1}\log^{(k)}(C_l/t)\right)^{1/n}}\right)$$

are allowable, where $l \geq 2$ is an integer and $s > 0$. Notice that $\psi_{l,s}$ satisfies (3) if and only if $s \leq 1$. Here $C > 0$, $\log^{(k)} t$ is the k-times iterated logarithm and C_l can be any constant with $\log^{(l)}(C_l/2) \geq 1$.

Let us look at the special case $n = m = 2$ of Theorem 1.1 in the Hölder continuous setting: $\psi(t) = Ct^\gamma$, where $0 < \gamma \leq 1$. Consider a space-filling (Peano) curve, i.e., a continuous mapping g from the unit circle onto a square. In one of the standard constructions, g is Hölder continuous with exponent $\gamma = \frac{1}{2}$; see, for example, [Buckley 1996, Theorem 3]. If one takes, say, the Poisson extension f of g to the unit disk, then f is also Hölder continuous. It is easy to check by hand that the partial derivatives of f do not belong to $L^2(\mathcal{B}^2)$. By Theorem 1.1, no Hölder continuous (or even continuous with control function satisfying (3)) extension f of a space filling curve can satisfy $|Df| \in L^2(\mathcal{B}^2)$.

In the Hölder continuous case, Jones and Makarov actually proved that the Hausdorff dimension of $f(\partial\mathcal{B}^2)$ is strictly less than two for conformal f. Contrary to the area zero results, this dimension estimate is truly conformal in the following sense:

Example 1.2. Let $p > 1$. There exists a locally Hölder continuous homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f \in W^{1,2}_{\mathrm{loc}}(\mathcal{B}^2, \mathbb{R}^2)$, which maps $\partial\mathcal{B}^2$ onto a set of positive $\mathcal{H}^g$-measure for the gauge function $g(t) = t^2(\log(1/t))^p$.

This construction can be found in Section 4. Here $\mathcal{H}^g$ denotes the generalised Hausdorff measure with the function g as the dimension gauge. The precise definitions are given in Section 2.

Our second result gives a rather optimal positive result.

Theorem 1.3. *Fix $\gamma \in (0, 1]$, $C > 0$, and let $g(t) = t^n \log(1/t)$. Suppose that $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ satisfies*

$$|f(z) - f(w)| \leq C|z - w|^\gamma$$

for all $z, w \in \mathcal{B}^n$. Then $\mathcal{H}^g(f(\partial\mathcal{B}^n)) = 0$.

Jones and Makarov proved their result via harmonic measure and hence this technique does not work in the setting of Theorem 1.1. An alternate approach, relying on the conformal (quasi)invariance of the (quasi)hyperbolic metric, was given in [Koskela and Rohde 1997]; see also [Nieminen 2006]. Furthermore, Malý and Martio [1995] established Theorem 1.1 in the Hölder continuous case via a technique that we have not been able to push further.

Let us briefly describe the idea of the proof of Theorem 1.1. We consider a Whitney decomposition $\mathcal{W}$ of $\mathcal{B}^n$ and assign to each $Q \in \mathcal{W}$ a vector $f_Q \in \mathbb{R}^m$ and a radius r_Q. The vector f_Q will simply be the "average" of f over Q and r_Q the maximum of $|f_Q - f_{\widetilde{Q}}|$ over all neighbours $\widetilde{Q}$ of Q. Then the n-integrability of the weak derivatives of f guarantees, via the Poincaré inequality, that the sequence

$\{r_Q\}_{Q\in\mathscr{W}}$ belongs to l^n. We realise $f(\partial\mathscr{B}^n)$ as (a part of) the closure of $\{f_Q\}_{Q\in\mathscr{W}}$ in $\mathbb{R}^m$. Those $f(\omega)$, $\omega\in\partial\mathscr{B}^n$, for which one can find a sequence of $Q\in\mathscr{W}$ with $|f_Q-f(\omega)|\lesssim r_Q$ are easily handled. For the remaining $\omega\in\partial\mathscr{B}^n$ we modify our centres f_Q and radii r_Q, while still retaining the l^n-condition, so that suitably blown-up balls cover these points sufficiently many times. This is where the nonintegrability condition (3) kicks in. One cannot fully follow the above idea, and so our proof, given below in Section 3, is more complicated.

Our approach is flexible and applies to many related problems. In order to avoid extra technicalities we do not record such applications here. Let us simply mention that the dimension gap phenomenon from [Hencl et al. 2012] can be shown to extend from conformal mappings to general Sobolev mappings [Koskela and Zapadinskaya 2014].

2. Preliminaries

Let us first agree on some basic notation. Given a number $a>0$, we write $\lfloor a\rfloor$ for the largest integer less than or equal to a. Similarly, $\lceil a\rceil$ is the smallest integer greater than or equal to a. If A is a finite set, $\sharp A$ is the number of elements in A. If $A\subset\mathbb{R}^n$ has finite and strictly positive Lebesgue measure and $f:\mathbb{R}^n\to\mathbb{R}$ is a Lebesgue integrable function, we denote the average $(1/|A|)\int_A f$ of f over the set A by $⨍_A f$ or f_A, where $|A|$ is the n-dimensional Lebesgue measure of the set A. For $f:\mathbb{R}^n\to\mathbb{R}^m$, f_A is then defined via the component functions of f. Given a point $x\in\mathbb{R}^n$ and a nonnegative number r, $B(x,r)$ denotes the open ball with centre x and radius r and $Q(x,r)$ denotes the cube $\{y\in\mathbb{R}^n:\max\{|x_i-y_i|\}_{i=1,2,\dots,n}\le r\}$. If $B=B(x,r)$ is a ball and a is a positive number, the notation aB stands for the ball $B(x,ar)$. We denote the radius of a ball B by $r(B)$. When we write $L=L(\cdot)$, we mean that the positive constant L depends only on the parameters listed inside the parentheses. Finally, C denotes a positive constant, which may depend only on n and m, the dimensions of the domain space and the image space, and may differ from occurrence to occurrence.

We write $\mathscr{H}^h(A)$ for the generalised Hausdorff measure of a set $A\subset\mathbb{R}^n$, given by

$$\mathscr{H}^h(A)=\lim_{\delta\to0}\mathscr{H}^h_\delta(A),\quad \text{where } \mathscr{H}^h_\delta(A)=\inf\left\{\sum_{i=1}^\infty h(\operatorname{diam}U_i):A\subset\bigcup_{i=1}^\infty U_i,\ \operatorname{diam}U_i\le\delta\right\}$$

and h is a dimension gauge (a nondecreasing function with $\lim_{t\to0+}h(t)=h(0)=0$ and with $h(t)>0$ for all $t>0$). If $h(t)=t^a$ for some $a\ge0$ we simply write $\mathscr{H}^a$ for $\mathscr{H}^h$ and call it the a-dimensional Hausdorff measure.

A sequence of pairs $(c_i,U_i)_{i=1}^\infty$, where $c_i\ge0$ and $U_i\subset\mathbb{R}^n$, that satisfies $\chi_A(x)\le\sum_{i=1}^\infty c_i\chi_{U_i}(x)$ for all $x\in\mathbb{R}^n$ is called a weighted cover of the set A. We also need a generalised weighted Hausdorff content of a set $A\subset\mathbb{R}^n$, given by

$$\lambda^h_\infty(A)=\inf\left\{\sum_{i=1}^\infty c_i h(\operatorname{diam}U_i):(c_i,U_i)_{i=1}^\infty \text{ is a weighted cover of } A\right\}.$$

Here also h is a gauge function. Again we write $\lambda^h_\infty=\lambda^a_\infty$ if $h(t)=t^a$.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be bounded. Let h be a continuous gauge function with $h(2t) \le ch(t)$ for some $c > 0$. Then $\mathcal{H}^h_\infty(E) \le c\lambda^h_\infty(E)$.*

Proof. The lemma follows from Corollary 8.2 and the proof of Theorem 9.7 of [Howroyd 1994]; see also [Federer 1969, 2.10.24]. □

Recall that for each open subset U of $\mathbb{R}^n$ there exists a Whitney decomposition $\mathcal{W}$ given by $U = \bigcup_{i=1}^\infty Q_i$, where $Q_i \in \mathcal{W}$ are cubes with mutually parallel sides, pairwise disjoint interiors and each of edge-length 2^k for some integer k, such that the relation

$$\frac{1}{4} \le \frac{\operatorname{diam} Q_i}{\operatorname{dist}(Q_i, \partial\Omega)} \le 1 \tag{4}$$

holds for all $i = 1, 2, \dots$. We write $Q_1 \backsim Q_2$ if the Whitney cubes $Q_1 \neq Q_2$ share at least one point (the so-called neighbour cubes). We have

$$\frac{1}{4} \le \frac{\operatorname{diam} Q}{\operatorname{diam} \widetilde{Q}} \le 4$$

whenever $Q \backsim \widetilde{Q}$. Therefore, the total number $\sharp\{\widetilde{Q} : \widetilde{Q} \backsim Q\}$ of all neighbours of a fixed cube Q does not exceed C. See [Stein 1970] for details.

Let $\omega \in \partial\mathcal{B}^n$. By $(Q_j(\omega))_{j=1}^\infty$ we mean the sequence of all Whitney cubes in a fixed Whitney decomposition of $\mathcal{B}^n$ intersecting the radius $[0, \omega]$. This sequence starts with a central cube and tends to ω. For a point $x \in [0, \omega]$, we denote the number of Whitney cubes intersecting the segment $[0, x]$ by $\sharp q(0, x)$. It is easy to see that

$$c_1 \le \frac{\sharp q(0, x)}{\log(1/(1 - |x|))} \le c_2 \tag{5}$$

whenever $\sharp q(0, x) > c_3$, where $c_i > 0$, $i = 1, 2, 3$ are constants that may depend on n.

Finally, we define the allowable moduli of continuity:

Definition 2.2. A continuously differentiable increasing bijection $\psi : (0, \infty) \to (0, \infty)$ is an *allowable modulus of continuity* if there exists $t_0 < 1$ and $\beta > 0$ such that for every $t \le t_0$ the following conditions hold:

$$\log \frac{1}{\psi^{-1}(t)} \text{ is differentiable and } \frac{(\psi^{-1})'(t)}{\psi^{-1}(t)}\, t \text{ is a decreasing function;} \tag{6}$$

$$\log \frac{1}{\psi^{-1}(t)} \le \beta \log \frac{1}{\psi^{-1}(\sqrt{t})}; \tag{7}$$

$$\frac{(\log \psi(t))' t \log t}{\log \psi(t)} \text{ is a monotone function.} \tag{8}$$

Remark 2.3. (i) One could replace the monotonicity conditions in (6) and (8) with a *pseudomonotonicity* condition (e.g., there exists a constant $C > 0$ such that $u(t) \le Cu(s)$ if $t \le s$). This would only affect the constants in the proofs.

(ii) The conditions (6) and (7) mean that the function $\log(1/\psi^{-1}(t))$ is a function of logarithmic type in the sense of [Nieminen 2006, Definition 4.2].

3. Proofs

Proof of Theorem 1.1. We may assume that $m, n \geq 2$. Let $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ and ψ be as in the statement of Theorem 1.1. Denote $\psi^{-1}(t)$ by $u(t)$. It follows from our assumptions (3), (6), (7), (8) and [Nieminen 2006, Remark 5.3.] that

$$\int_0 \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n} = \infty. \tag{9}$$

We define $\alpha(t) = u(t)/u'(t)$ and $\lambda(k) = 2^{-k}/\alpha(2^{-k})$ for $k \in \mathbb{N}$. By (6), λ is increasing for large k. For simplicity we assume λ to be increasing.

Let $\mathcal{W}$ be a fixed Whitney decomposition of $\mathcal{B}^n$. For each cube $Q \in \mathcal{W}$ we define a corresponding centre f_Q and a corresponding radius $r_Q = \max\{|f_Q - f_{\widetilde{Q}}| : Q \backsim \widetilde{Q}\}$, which determine a family of balls on the image side indexed by $\mathcal{W}$:

$$\mathcal{B} = \{(Q, B(f_Q, r_Q)) : Q \in \mathcal{W},\ r_Q > 0\}.$$

To simplify our notation we abbreviate $(Q, B(f_Q, r_Q))$ to $B(f_Q, r_Q)$ in what follows.

We assign two new weighted collections of balls to each element in $\mathcal{B}$. Given $B = B(x, r) \in \mathcal{B}$, we define concentric subballs $S_i(B) = B(x, r/2^i)$ for all $i \in \mathbb{N}$ and assign the weight $w_{S_i(B)} = 2^i$ to each $S_i(B)$. We set $\mathcal{S}_B = \{S_i(B) : i \in \mathbb{N}\}$. Then

$$\sum_{B' \in \mathcal{S}_B} w_{B'} r(B')^n = \sum_{i=1}^{\infty} w_{S_i(B)} r(S_i(B))^n = \sum_{i=1}^{\infty} 2^i \frac{r(B)^n}{2^{ni}} \leq r(B)^n.$$

The second collection is defined in a similar way. If $B = B(x, r)$ is a ball in $\mathcal{B}$, we choose the smallest number $k_0(r) \in \mathbb{N}$ such that $2^{-k_0(r)} \leq r$. Next, for each $k = k_0(r), k_0(r)+1, \ldots,$ we choose $R_k(B) = B(x, \alpha(2^{-k}))$ and set $\mathcal{R}_B = \{R_k(B) : k = k_0(r), k_0(r)+1, \ldots\}$. The weights we assign this time are $w_{R_k(B)} = \lambda(k)$ for all $k = k_0(r), k_0(r)+1, \ldots$. Similarly to the above,

$$\begin{aligned}
\sum_{B' \in \mathcal{R}_B} w_{B'} r(B')^n &= \sum_{k=k_0(r)}^{\infty} w_{R_k(B)} r(R_k(B))^n = \sum_{k=k_0(r)}^{\infty} (\alpha(2^{-k}))^n \lambda(k) \\
&\leq \sum_{k=k_0(r)}^{\infty} (\alpha(2^{-k}))^n \frac{\lambda(k)^n}{\lambda(0)^{n-1}} = \frac{1}{\lambda(0)^{n-1}} \sum_{k=k_0(r)}^{\infty} 2^{-nk} \leq \frac{2 \cdot 2^{-nk_0(r)}}{\lambda(0)^{n-1}} \leq \frac{2}{\lambda(0)^{n-1}} r(B)^n.
\end{aligned}$$

Finally, we define our weighted collection of balls by setting $\mathcal{F} = \bigcup_{B \in \mathcal{B}} (\mathcal{S}_B \cup \mathcal{R}_B)$.

Let us now estimate the weighted sums of the n-th powers of the radii of the balls in $\mathcal{F}$. Let $N(Q) = Q \cup \bigcup_{\widetilde{Q} \backsim Q} \widetilde{Q}$ be the union of $Q \in \mathcal{W}$ and all neighbours $\widetilde{Q}$ of Q. For neighbouring cubes Q and $\widetilde{Q}$, we obtain, via the Hölder and Poincaré inequalities, that

$$\begin{aligned}
|f_Q - f_{\widetilde{Q}}| &\leq \fint_Q |f - f_{N(Q)}| + \fint_{Q'} |f - f_{N(Q)}| \leq C \fint_{N(Q)} |f - f_{N(Q)}| \leq C \left(\fint_{N(Q)} |f - f_{N(Q)}|^n \right)^{1/n} \\
&\leq C \left(\fint_{N(Q)} |Df|^n \right)^{1/n}.
\end{aligned}$$

Hence, we have the estimate

$$r_Q^n = \max\{|f_Q - f_{\widetilde{Q}}|^n : Q \backsim \widetilde{Q}\} \le C \int_{N(Q)} |Df|^n$$

for each $Q \in \mathcal{W}$ and some constant $C > 0$. Next, using the fact that the inequality $\sum_{Q\in\mathcal{W}} \chi_{N(Q)}(y) \le C$ holds for every $y \in \mathbb{R}^n$, we estimate

$$\begin{aligned}\sum_{B\in\mathcal{F}} w_B r(B)^n &\le C(\lambda(0)) \sum_{B\in\mathcal{B}} r(B)^n = C(\lambda(0)) \sum_{Q\in\mathcal{W}} r_Q^n \le C(\lambda(0)) \sum_{Q\in\mathcal{W}} \int_{N(Q)} |Df|^n \\ &\le C_1 \int_{\bigcup_{Q\in\mathcal{W}} N(Q)} |Df|^n \le C_1 \int_{\mathcal{B}^n} |Df|^n < \infty, \end{aligned} \tag{10}$$

where $C_1 > 0$ is some constant depending on n, m and $\lambda(0)$ only.

We may assume that there is at least one $Q \in \mathcal{W}$ with $r_Q > 0$; otherwise $f(\partial\mathcal{B}^n)$ is a singleton. Let $\omega \in \partial\mathcal{B}^n$. We consider the radius $[0, \omega]$ and the sequence $(Q_j(\omega))_{j=1}^\infty$. We fix a large integer $l_0 = l_0(\omega, f) \in \mathbb{N}$ so that there are elements of the sequence $(f_{Q_j(\omega)})_{j=1}^\infty$ outside $B(f(\omega), 2^{-l_0+1})$ if $(f_{Q_j(\omega)})_{j=1}^\infty$ contains at least one element different from $f(\omega)$. If such an integer does not exist there necessarily is some $Q = Q_w \in \mathcal{W}$ with $f_Q = f(\omega)$ and $r_Q > 0$. In this case, we choose $l_0 = l_0(\omega, f) \in \mathbb{N}$ so that $2^{-l_0} < r_{Q_\omega}$. In both cases we also require that $2^{-l_0+1} < t_0$. This allows us to use the properties (6) and (7).

For the purposes of our "porosity argument", we would like to make the number l_0 independent of the point ω. This is done by considering the decomposition

$$\partial\mathcal{B}^n = \bigcup_{l\in\mathbb{N}} E_l, \quad \text{where } E_l = \{\omega \in \partial\mathcal{B}^n : l_0(\omega, f) \le l\}.$$

Setting $F_l = f(E_l)$, we then have $f(\partial\mathcal{B}^n) = \bigcup_{i\in\mathbb{N}} F_l$.

Let us fix $l_0 \in \mathbb{N}$. Our aim is to prove that $\mathcal{H}_\infty^n(F_{l_0}) = 0$.

Fix $x \in F_{l_0}$. Take any $\omega \in E_{l_0}$ such that $x = f(\omega)$ and define the sequence of concentric annuli $A_l(x) = B(x, 2^{-l+1}) \setminus B(x, 2^{-l})$ with $l = l_0, l_0 + 1, \dots$. Next, we assign a suitable set $P_l(x)$ of cubes from $\mathcal{W}$ to each annulus $A_l(x)$, $l = l_0, l_0 + 1, \dots$. If $f_{Q_j(\omega)} = x$ for all $j \in \mathbb{N}$, we put $P_l(x) = \{Q_\omega\}$ for each $l \ge l_0$, where Q_ω is the cube defined earlier. Otherwise, all the sets $P_l(x)$ with $l \ge l_0$ consist of elements from $(Q_j(\omega))_{j=1}^\infty$. If an annulus $A_l(x)$ with some $l \ge l_0$ contains no centres from $(f_{Q_j(\omega)})_{j=1}^\infty$ we define $P_l(x) = \{Q_m(\omega)\}$, where an integer $m \in \mathbb{N}$ is chosen so that $f_{Q_{m-1}(\omega)} \notin B(x, 2^{-l+1})$ but $f_{Q_m(\omega)} \in B(x, 2^{-l})$; if, in contrast, there is at least one centre $f_{Q_j(\omega)}$ in $A_l(x)$ we take $P_l(x) = \{Q_k(\omega) : k = m_1, \dots, m_2\}$, where $m_1, m_2 \in \mathbb{N}$ are such that $f_{Q_{m_1-1}(\omega)} \notin B(x, 2^{-l+1})$, $f_{Q_{m_2+1}(\omega)} \in B(x, 2^{-l})$ and $f_{Q_k(\omega)} \in A_l(x)$ for all $k = m_1, \dots, m_2$. Moreover, it is possible to choose the sets $P_l(x)$ above so that the inequality $k_1 \le k_2$ is valid whenever $Q_{k_1}(\omega) \in P_{l_1}(x)$, $Q_{k_2}(\omega) \in P_{l_2}(x)$ and $l_1 < l_2$.

Denoting

$$\theta_l(x) = \begin{cases} 1 & \text{if } \sharp P_l(x) \le \tilde{c}_0 \lambda(l), \\ 0 & \text{otherwise} \end{cases}$$

for $l \geq l_0$ and a constant $\tilde{c}_0 > \lambda^{-1}(0)$, which we will specify later, we would like to prove that there exists an integer $l_1 \geq 2l_0$ such that

$$\sum_{k=l_0}^{l} \theta_k(x) \geq \frac{l}{2} \tag{11}$$

for each $l \geq l_1$. In other words, at least half of the annuli do not contain too many centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. There is nothing to prove if $f_{Q_j(\omega)} = x$ for all $j \in \mathbb{N}$; otherwise, the proof is by contradiction:

Let us assume that (11) does not hold for some $l \geq 2l_0$. Take the smallest number $J \in \mathbb{N}$ such that $f_{Q_j(\omega)} \in B(x, 2^{-l})$ for all $j > J$ and let $\omega' \in [0, \omega]$ be the point of $Q_J(\omega) \cap [0, \omega]$ which is closest to ω. Now, the assumption on the continuity of f and the properties of our Whitney decomposition imply

$$2^{-l} \leq |f_{Q_J(\omega)} - x| = |f_{Q_J(\omega)} - f(\omega)| \leq \fint_{Q_J} |f(y) - f(\omega)|\, dy \leq \psi(2(1 - |\omega'|)).$$

That is,

$$\frac{u(2^{-l})}{2} \leq 1 - |\omega'|.$$

Next, we connect this estimate to the number of Whitney cubes that precede Q_J in $(Q_j(\omega))_{i=1}^{\infty}$.

Using (5), we observe that

$$\log \frac{2}{u(2^{-l})} \geq \log \frac{1}{1 - |\omega'|} \geq \frac{1}{c_2} \sharp q(0, \omega').$$

In the calculation above we may have to adjust the choice of l_0 to ensure $\sharp q(0, \omega') > c_3$ (see (5)). Finally, we obtain a lower bound for $\sharp q(0, \omega')$ using the assumption that we have at least $\lfloor l/2 \rfloor - l_0 + 2$ annuli $A_k(x)$ with $\theta_k(x) = 0$. We notice that the sets $P_k(x)$ with $\theta_k(x) = 0$ contain different cubes for different k, and if $k \leq l$ then the cubes in $P_k(x)$ precede $Q_J(\omega)$ in $(Q_j(\omega))_{j=1}^{\infty}$. We have

$$\begin{aligned} c_2 \log \frac{2}{u(2^{-l})} &\geq \sharp q(0, \omega') \geq \sum_{\substack{k=l_0,\dots,l \\ \theta_k(x)=0}} \sharp P_k(x) \geq \sum_{k=l_0}^{\lfloor l/2 \rfloor + 1} \tilde{c}_0 \lambda(k) \geq \tilde{c}_0 \sum_{k=l_0}^{\lfloor l/2 \rfloor + 1} \frac{2^{-k} u'(2^{-k})}{u(2^{-k})} \\ &\geq \tilde{c}_0 \left(\log \frac{1}{u(2^{-l/2})} - \log \frac{1}{u(2^{-l_0})} \right) \geq \tilde{c}_0 \beta^{-1} \log \frac{1}{u(2^{-l})} - \tilde{c}_0 \log \frac{1}{u(2^{-l_0})}. \end{aligned}$$

Choosing $\tilde{c}_0 > c_2 \beta$, this cannot hold when l is large enough. Thus there is a number $l_1 = l_1(\tilde{c}_0, l_0, u)$ such that (11) holds for all $l \geq l_1$.

Our next step is to prove that, if $\theta_k(x) = 1$ for some k and $P_k(x) = \{Q_1, \dots, Q_m\}$, then it is possible to find a collection of balls $\{B_1, \dots, B_{m'}\}$ from the families $\mathcal{S}_{B(f_{Q_i}, r_{Q_i})}$ or $\mathcal{R}_{B(f_{Q_i}, r_{Q_i})}$ having radii at least a constant times $\alpha(2^{-k})$ and such that $\sum_{i=1}^{m'} w_{B_i}$ is at least a constant times $\lambda(k)$. Moreover, we choose different balls for different k.

Let us fix $k \geq l_0$ such that $\theta_k(x) = 1$. Suppose first that the annulus $A_k(x)$ contains no centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. Then the set $P_k(x)$ consists of a single cube $Q \in \mathcal{W}$ with $f_Q \in B(x, 2^{-k})$. The definitions of r_Q and l_0 imply that $r_Q > 2^{-k}$ and hence $k \geq k_0(r_Q)$. Thus, we may choose the ball $R_k(B(f_Q, r_Q))$, which, by definition, has radius $\alpha(2^{-k})$ and weight $\lambda(k)$. In addition, the centre of this ball lies in $B(x, 2^{-k})$.

Assume now that the annulus $A_k(x)$ contains at least one of the centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. Then, by the definitions of $P_k(x)$ and r_Q,

$$\sum_{Q\in P_k(x)} 2r_Q \geq 2^{-k}.$$

Since $\sharp P_k(x) \leq \tilde{c}_0\lambda(k)$, we observe that

$$\sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2r_Q \geq \frac{2^{-k}}{2}.$$

For each $Q \in P_k(x)$ with $2r_Q \geq \alpha(2^{-k})/2\tilde{c}_0$ we choose a number $n_Q \in \mathbb{N}$ so that

$$2^{n_Q-1}\frac{\alpha(2^{-k})}{2\tilde{c}_0} \leq 2r_Q < 2^{n_Q}\frac{\alpha(2^{-k})}{2\tilde{c}_0}$$

and pick a ball $\widetilde{B} = S_{n_Q}(B(f_Q, r_Q)) = B(f_Q, r_Q/2^{n_Q}) \in \mathcal{S}_{B(f_Q,r_Q)}$. By the definition of $S_i(B)$, we have $w_{\widetilde{B}} = 2^{n_Q}$ and

$$r(\widetilde{B}) = \frac{r_Q}{2^{n_Q}} \geq \frac{\alpha(2^{-k})}{8\tilde{c}_0}.$$

For the sum of the weights $\sum_Q 2^{n_Q}$ of all the balls obtained in such a manner, we observe that

$$\frac{\alpha(2^{-k})}{2\tilde{c}_0}\sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2^{n_Q} > \sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2r_Q \geq \frac{2^{-k}}{2}.$$

Hence, we have a collection of balls $\{B_1, \ldots, B_m\} \subset \mathcal{F}$ with weights sum $\sum_{i=1}^m w_{B_i} > \tilde{c}_0\lambda(k)$ and of radii at least $\alpha(2^{-k})/8\tilde{c}_0$. Moreover, all these balls have their centres in the annulus $A_k(x)$ and hence in the ball $B(x, 2^{-k+1})$.

We have proved that there exists a number $l_1 = l_1(l_0, \tilde{c}_0)$ such that, for each $\omega \in E_{l_0}$ and $l \geq l_1$, among the numbers $l_0, \ldots, l$ there are at least $\lceil l/2\rceil$ integers $k \in \{l_0, \ldots, l\}$ such that $\theta_k(x) = 1$. For these k we are able to find a finite collection of balls $\{B_i\}_{i\in I} \subset \mathcal{F}$ with weight-sum $\sum_{i\in I} w_{B_i}$ at least $\lambda(k)$ and of radii at least $\alpha(2^{-k})/8\tilde{c}_0$, so that the centres of the balls B_i, $i \in I$, lie in the ball $B(x, 2^{-k+1})$. Here $\tilde{c}_0$ is a positive constant depending only on β, n and $\lambda(0)$, and the balls are different for a fixed ω and different k.

Fix $l \geq l_1$. We modify our family $\mathcal{F}$ according to l. If $B \in \mathcal{F}$ and there is $k \in \{l_0+1, \ldots, l\}$ such that $\alpha(2^{-k})/8\tilde{c}_0 \leq r(B) < \alpha(2^{-k+1})/8\tilde{c}_0$, we replace B with the ball $\widetilde{B} = (\lambda(k)/\lambda(l))B$ and set $w_{\widetilde{B}} = (\lambda(l)/\lambda(k))^n w_B$. The radius of $\widetilde{B}$ satisfies $r(\widetilde{B}) \geq (\lambda(k)/\lambda(l))\alpha(2^{-k})/8\tilde{c}_0 = 2^{-k}/8\tilde{c}_0\lambda(l)$ and the equality $w_{\widetilde{B}} r(\widetilde{B})^n = w_B r(B)^n$ holds. Similarly, we replace a ball B with $r(B) \geq \alpha(2^{-l_0})/8\tilde{c}_0$ with the ball $\widetilde{B} = (\lambda(l_0)/\lambda(l))B$ and set $w_{\widetilde{B}} = (\lambda(l)/\lambda(l_0))^n w_B$. Again, we have $r(\widetilde{B}) \geq 2^{-l_0}/8\tilde{c}_0\lambda(l)$ and $w_{\widetilde{B}} r(\widetilde{B})^n = w_B r(B)^n$. Finally, $\mathcal{F}_l$ is the collection of balls obtained in this manner from the balls in $\mathcal{F}$. For this family of balls, we notice (see (10)) that

$$\sum_{B\in\mathcal{F}_l} w_B r(B)^n \leq \sum_{B\in\mathcal{F}} w_B r(B)^n < \infty. \tag{12}$$

If $\omega \in E_{l_0}$, $x = f(\omega)$ and $k \in \{l_0, \dots, l\}$ is such that $\theta_k(x) = 1$, then there is a collection $\{B_i\}_{i \in I} \subset \mathcal{F}$ with the properties mentioned above. If for some $i \in I$ the ball B_i is replaced by the ball $\widetilde{B}_i = (\lambda(k_i)/\lambda(l))B_i$ while creating $\mathcal{F}_l$, we necessarily have $k_i \le k$. Therefore, the inequalities

$$\sum_{i \in I} w_{\widetilde{B}_i} = \sum_{i \in I} \left(\frac{\lambda(l)}{\lambda(k_i)}\right)^n w_{B_i} \ge \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \sum_{i \in I} w_{B_i} \ge \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \lambda(k) = \lambda(l)^n \frac{1}{\lambda(k)^{n-1}}$$

and $r(\widetilde{B}_i) \ge 2^{-k_i}/(8\tilde{c}_0\lambda(l)) \ge 2^{-k}/(8\tilde{c}_0\lambda(l))$ hold (by (6), λ is increasing). Since, for each $i \in I$, the centre of a ball $\widetilde{B}_i$ is contained in $B(x, 2^{-k+1})$, we have $x \in 16\tilde{c}_0\lambda(l)\widetilde{B}_i$. Hence, we observe that

$$\sum_{B \in \mathcal{F}_l} w_B \chi_{16\tilde{c}_0\lambda(l)B}(y) \ge \sum_{\substack{k=l_0,\dots,l \\ \theta_k(y)=1}} \lambda(l)^n \frac{1}{\lambda(k)^{n-1}} \ge \frac{\lambda(l)^n}{4} \sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}} \ge \frac{\lambda(l)^n}{4} G_l$$

for each $y \in F_{l_0}$, where $G_l = \sum_{k=l_1}^{l} 1/\lambda(k)^{n-1}$. That is, $(4w_B/(\lambda(l)^n G_l), 16\tilde{c}_0\lambda(l)B)_{B \in \mathcal{F}_l}$ is a weighted cover of the set F_{l_0}. We observe also that the diameters of all balls in this cover are at least 2^{-l}. This information will be used in the proof of Theorem 1.3 below.

Finally, using the weighted cover obtained above and (12), we estimate the weighted Hausdorff n-content $\lambda^n_\infty(F_{l_0})$:

$$\begin{aligned}\lambda^n_\infty(F_{l_0}) &\le \frac{4}{\lambda(l)^n G_l} \sum_{B \in F_l} w_B \big(\operatorname{diam}(16\tilde{c}_0\lambda(l)B)\big)^n \le \frac{4^{2n+1}\tilde{c}_0^n}{G_l} \sum_{B \in F_l} w_B (\operatorname{diam} B)^n \\ &\le \frac{2^{5n+2}\tilde{c}_0^n}{G_l} \sum_{B \in F_l} w_B r(B)^n \le \frac{A}{G_l},\end{aligned}$$

where the constant A depends on β, n, m, $\|f\|_{W^{1,n}(\mathcal{B}^n,\mathbb{R}^m)}$ and $\lambda(0)$ but not on l_0 or l.

Now Lemma 2.1 implies $\mathcal{H}^n_\infty(F_{l_0}) \le CA/G_l$. Here C depends only on the dimension n. Hence, we are done as soon as we can show that $G_l \to \infty$ as $l \to \infty$. Towards this end, we have

$$G_l = \sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}} = \sum_{k=l_1}^{l} \frac{u(2^{-k})^{n-1}}{2^{-k(n-1)}u'(2^{-k})^{n-1}} \ge \int_{2^{-l}}^{2^{-l_1}} \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n},$$

and the right-hand side diverges as $l \to \infty$ by the assumptions on the modulus of continuity. □

The proof of Theorem 1.3 is similar to the proof of Theorem 1.1. We only point out the required changes.

Proof of Theorem 1.3. Let f be as in statement of the theorem. Our notation will be the same as in previous proof. That is, $\alpha(t) = \gamma t$ and $\lambda(k) = 1/\gamma$.

Fix a small $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\int_{\mathcal{B}^n \setminus B(0,1-\delta)} |Df|^n \le \varepsilon. \tag{13}$$

Let $\mathcal{W}^\delta$ be the set of the cubes in $\mathcal{W}$ which are contained in $\mathcal{B}^n \setminus B(0, 1-\delta)$ and whose neighbour cubes are also contained in $\mathcal{B}^n \setminus B(0, 1-\delta)$. We define our collection of balls to be $\mathcal{B}^\delta = \{B(f_Q, r_Q) : Q \in \mathcal{W}^\delta\}$.

Then, proceeding as in the previous proof, we define $\mathscr{F}^\delta$ analogously to $\mathscr{F}$ and obtain the estimate (see (10))

$$\sum_{B\in\mathscr{F}^\delta} w_B r(B)^n \le C_1\varepsilon. \tag{14}$$

Let $\omega\in\partial\mathscr{B}^n$. We define the number $l_0=l_0(\omega,f,\delta)$ as in the previous proof, but instead of all cubes in $(Q_j(\omega))_{j=1}^\infty$ we consider only those which are contained in $\mathscr{W}^\delta$. Again, we split $\partial\mathscr{B}^n$ into sets $E_l=\{\omega\in\partial\mathscr{B}^n : l_0(\omega)\le l\}$ and consider a fixed $f(E_{l'})$. With the same method as earlier we find for large l a collection of balls $\mathscr{F}_l^\delta$ with weights such that $(8w_B\gamma/(l-l_1),(16\tilde c_0/\gamma)B)_{B\in\mathscr{F}_l^\delta}$ is a weighted cover of the set $f(E_{l'})$, the radii of the balls $(16\tilde c_0/\gamma)B$ are at least 2^{-l} and

$$\sum_{B\in\mathscr{F}_l^\delta} w_B r(B)^n \le C_1\varepsilon.$$

We may assume that our $\varepsilon>0$ is so small that all balls in our weighted cover have radii smaller than $\frac12$. With this weighted cover, we obtain

$$\begin{aligned}\lambda_\infty^g(f(E_{l'})) &\le \frac{4\gamma}{l-l_1}\sum_{B\in\mathscr{F}_l^\delta} w_B\biggl(\operatorname{diam}\Bigl(\frac{16\tilde c_0}{\gamma}B\Bigr)\biggr)^n \log\frac{1}{\operatorname{diam}((16\tilde c_0/\gamma)B)}\\ &\le \frac{4\gamma}{l-l_1}\sum_{B\in\mathscr{F}_l^\delta} w_B\biggl(\operatorname{diam}\Bigl(\frac{16\tilde c_0}{\gamma}B\Bigr)\biggr)^n \log 2^l \le \frac{2^{2+5n}\tilde c_0^n}{\gamma^{n-1}}\,\frac{l}{l-l_1}\sum_{B\in\mathscr{F}_l^\delta} w_B r(B)^n \le \frac{2^{3+5n}\tilde c_0^n C_1}{\gamma^{n-1}}\,\varepsilon.\end{aligned}$$

Here we assumed l to be so large that $l/(l-l_1)\le 2$. Lemma 2.1 implies $\mathscr{H}_\infty^g(f(E_{l'}))\le A\varepsilon$. Here A depends on γ, n and m but not on l' or l; therefore, we have $\mathscr{H}_\infty^g(f(\partial\mathscr{B}^n))\le A\varepsilon$; see [Howroyd 1994, Corollary 8.2] or [Federer 1969, 2.10.22]. Letting ε tend to zero gives $\mathscr{H}_\infty^g(f(\partial\mathscr{B}^n))=0$, which implies $\mathscr{H}^g(f(\partial\mathscr{B}^n))=0$. □

4. Example

In this section, we work in $\mathbb{R}^2$ and use the notation $\|x\|=\max\{|x_1|,|x_2|\}$. Let $p>\frac12$. We will construct a locally Hölder continuous mapping $f:\mathbb{R}^2\to\mathbb{R}^2$ that belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ and maps $\partial\mathscr{B}^2$ onto a set of positive $\mathscr{H}^g$-measure, where $g(t)=t^2(\log(1/t))^{2p}$.

The mapping is a composition of two locally Hölder continuous mappings. The second mapping is defined in [Herron and Koskela 2003, Proposition 5.1]. It is a homeomorphism $h:\mathbb{R}^2\to\mathbb{R}^2$ that is the identity mapping outside $[0,1]^2$ and maps a small Cantor set $\mathscr{C}\subset[0,1]^2$ onto a large Cantor set $\mathscr{C}'\subset[0,1]^2$ with positive $\mathscr{H}^g$-measure. It was checked in [Koskela et al. 2009] that this mapping belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ if $p>\frac12$.

Next, we elaborate on the construction of h and prove that it is Hölder continuous in $[0,1]^2$. Let $\sigma<\frac12$. We use the notation $2r_k=\sigma^k$ and $2R_k=\frac12\sigma^{k-1}$ for $k\in\mathbb{N}$. The set $\mathscr{C}$ is defined as follows: In the first generation we have one square $Q_0=[0,1]^2$ with side length $2r_0$. We split this square into four subsquares P_{1i}, $i=1,2,3,4$, of side length $2R_1$. We define Q_{1i} to be the square of side length $2r_1$ centred at the centre of P_{1i}. Then P_{1i} and Q_{1i} generate the frame $A_{1i}=P_{1i}\setminus Q_{1i}$. Next, we divide all squares Q_{1i} into squares P_{2j}, $j=1,\dots,4^2$. Then we define Q_{2j} and A_{2j} as in the first step. We proceed inductively. Thus, we obtain for all $k\in\mathbb{N}$ sets Q_{ki}, P_{ki} and A_{ki}, where $i=1,\dots,2^{2k}$, and we set $\mathscr{C}=\bigcap_k\bigcup_i Q_{ki}$.

The set $\mathscr{C}'$ and sets Q'_{ki}, P'_{ki} and A'_{ki} with $k\in\mathbb{N}$ and $i=1,\dots,2^{2k}$ are defined in the same way, using $2r'_1=\frac{1}{2}(\log 4)^{-p}$, $2R'_2=r'_1$, and $2r'_k=(\log 4)^{-p}2^{-k}k^{-p}$ and $2R'_k=(\log 4)^{-p}2^{-k}(k-1)^{-p}$ for other $k\in\mathbb{N}$.

The mapping h is defined so that it maps the frame A_{ki} to the frame A'_{ki} via a "radial" stretching and is continuous in $[0,1]^2$. The radial stretching which maps $A=\{x: r_k\le\|x\|\le R_k\}$ to $A'=\{x: r'_k\le\|x\|\le R'_k\}$ is

$$\rho(x)=(a\|x\|+b)\frac{x}{\|x\|},\quad \text{where } a=\frac{R'_k-r'_k}{R_k-r_k} \text{ and } b=\frac{R_kr'_k-R'_kr_k}{R_k-r_k}.$$

If $x,y\in A$ then $\|x-y\|\le 2R_k=\frac{1}{2}\sigma^{k-1}$ and

$$a\le\frac{4\sigma}{1-2\sigma}(2\sigma)^{-k}\le C(\sigma)\sigma^{-(1-\beta)k}\le C(\sigma)\|x-y\|^{\beta-1},$$

where $\beta=\log 2/\log(1/\sigma)$. Similarly,

$$\frac{|b|}{|r_k|}\le\frac{4}{1-2\sigma}(2\sigma)^{-k}\le C(\sigma)\|x-y\|^{\beta-1}.$$

The mapping ρ is Hölder continuous with exponent β, as

$$\|\rho(x)-\rho(y)\|\le Ca\|x-y\|+2\frac{|b|}{|r_k|}\|x-y\|\le C(\sigma)\|x-y\|^{\beta}.$$

If $x\in A_{ki}$ and $y\in Q_{k+1,j}\subset P_{ki}$, then $\|x-y\|\ge R_{k+1}-r_{k+1}=C(\sigma)\sigma^k$ and $\|h(x)-h(y)\|\le 2R'_k\le 2^{-k}$. These imply

$$\frac{\|h(x)-h(y)\|}{\|x-y\|^{\beta}}\le C(\sigma).$$

The β-Hölder continuity of h easily follows from the continuity estimates obtained above.

The first mapping $G:\mathbb{R}^2\to\mathbb{R}^2$ is a (locally Hölder continuous) quasiconformal mapping for which $\mathscr{C}\subset G(\partial\mathscr{B}^2)$. Such a mapping was constructed in [Gehring and Väisälä 1973].

Finally, the composition $h\circ G:\mathbb{R}^2\to\mathbb{R}^2$ is a homeomorphism with $h\circ G(\partial\mathscr{B}^2)\supset\mathscr{C}'$. Moreover, it is locally Hölder continuous and $h\circ G\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ by quasiconformality of G and the change of variable formula; see, for example, [Astala et al. 2009, Section 3.8].

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