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HÖLDER CONTINUITY AND BOUNDS FOR FUNDAMENTAL SOLUTIONS TO NONDIVERGENCE FORM PARABOLIC EQUATIONS

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We consider nondegenerate second-order parabolic partial differential equations in nondivergence form with bounded measurable coefficients (not necessary continuous). Under certain assumptions weaker than the Hölder continuity of the coefficients, we obtain Gaussian bounds and Hölder continuity of the fundamental solution with respect to the initial point. Our proofs employ pinned diffusion processes for the probabilistic representation of fundamental solutions and the coupling method.

1. Introduction and main result

Let $a(t,x)=(a_{ij}(t,x))$ be a symmetric $d\times d$-matrix-valued bounded measurable function on $[0,\infty)\times\mathbb{R}^d$ which is uniformly positive-definite, i.e.,

$$\Lambda^{-1}I\le a(t,x)\le\Lambda I, \tag{1-1}$$

where Λ is a positive constant and I is the unit matrix. Let $b(t,x)=(b_i(t,x))$ be an $\mathbb{R}^d$-valued bounded measurable function on $[0,\infty)\times\mathbb{R}^d$ and $c(t,x)$ a bounded measurable function on $[0,\infty)\times\mathbb{R}^d$. Consider the parabolic partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x)=\dfrac{1}{2}\displaystyle\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}u(t,x)+\sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}u(t,x)+c(t,x)u(t,x), \\ u(0,x)=f(x). \end{cases} \tag{1-2}$$

Generally, (1-2) does not have a unique solution. We will assume the continuity of a in spatial components uniformly in t, and this implies the uniqueness of the weak solution; see [Stroock and Varadhan 1979]. In the present paper, we always consider cases where the uniqueness of the weak solution holds. Set

$$L_tf(x):=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}f(x)+\sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}f(x)+c(t,x)f(x),\qquad f\in C_b^2(\mathbb{R}^d)$$

and denote the fundamental solution to (1-2) by $p(s,x;t,y)$, i.e., $p(s,x;t,y)$ is a measurable function defined for $s,t\in[0,\infty)$ such that $s<t$ and $x,y\in\mathbb{R}^d$ which satisfies

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d}f(y)p(s,\,\cdot\,;t,y)\,dy=L_t\biggl(\int_{\mathbb{R}^d}f(y)p(s,\,\cdot\,;t,y)\,dy\biggr)\quad\text{and}\quad\lim_{r\downarrow s}\int_{\mathbb{R}^d}f(y)p(s,\,\cdot\,;r,y)\,dy=f$$

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for $s, t \in [0, \infty)$ such that $s < t$ and a continuous function f with a compact support. In the present paper, we consider the existence and the regularity of $p(0, x; t, y)$.

The problem of regularity of the fundamental solutions to parabolic partial differential equations with bounded measurable coefficients has a long history. Parabolic equations in divergence form has been investigated more thoroughly than that those in nondivergence form, because the variational method is applicable to them. The Hölder continuity of the fundamental solution to $\partial_t u = \nabla \cdot a \nabla u$ for a matrix-valued bounded measurable function a with ellipticity condition $\Lambda^{-1} I \leq a \leq \Lambda I$ was originally obtained by De Giorgi [1957] and Nash [1958] independently. Precisely speaking, in their results the α-Hölder continuity of the fundamental solution, with some positive number $\alpha \in (0, 1]$, is obtained. The index α depends on many constants appearing in the Harnack inequality and so on. These results have been extended to the case of the more general equations $\partial_t u = \nabla \cdot a \nabla u + b \cdot \nabla u - cu$, where b, c are bounded measurable; see [Aronson 1967; Stroock 1988]. The case for unbounded coefficients is also studied; see, for example, [Metafune et al. 2009; Porper and Eidelman 1984; 1992]. An analogy to the case of a type of nonlocal generators (the associated stochastic processes are called stable-like processes) is given by Chen and Kumagai [2003]. In the results above, the index of the Hölder continuity of the fundamental solution depends on many constants appearing in the estimates, and it is difficult to calculate its exact value. Moreover, it is difficult to obtain even a lower bound for the index.

The fundamental solutions to parabolic equations in nondivergence form with low-regular coefficients have been studied mainly in the case of Hölder-continuous coefficients. One of the most powerful tools for the problem is the parametrix method, and it yields the existence, uniqueness and Hölder continuity of the fundamental solution; see [Friedman 1964; Ladyženskaja et al. 1967; Porper and Eidelman 1984, Chapter I]. Furthermore, an a priori estimate (the so-called Schauder estimate) is known for the solutions, and twice-continuous differentiability in x of the fundamental solution $p(s, x; t, y)$ to (1-2) is obtained; see, for example, [Ladyženskaja et al. 1967; Krylov 1996; Bogachev et al. 2009; Bogachev et al. 2005]. We remark that all the coefficients a, b, c need to be Hölder-continuous to apply the Schauder estimate. Even in the case that a is the unit matrix and b is not continuous, we cannot expect the continuous differentiability of the fundamental solution; see [Karatzas and Shreve 1991, Remark 5.2, Chapter 6].

In the present paper, we consider the Gaussian estimate and the lower bound of the index for the Hölder continuity in x of the fundamental solution $p(s, x; t, y)$ to (1-2) by a probabilistic approach.

Now we fix some assumptions. Let $B(x, R)$ be the open ball in $\mathbb{R}^d$ centered at x with radius R for $x \in \mathbb{R}^d$ and $R > 0$. We assume that

$$\sum_{i,j=1}^{d} \sup_{t \in [0,\infty)} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} a_{ij}(s, x) \right|^{\theta} e^{-m|x|} \, dx \leq M, \tag{1-3}$$

where the derivatives are in the weak sense, θ is a constant in $[d, \infty) \cap (2, \infty)$, and m and M are nonnegative constants. We also assume the continuity of a in spatial component uniformly in t, i.e., for any $R > 0$ there exists a continuous and nondecreasing function ρ_R on $[0, \infty)$ such that $\rho_R(0) = 0$ and

$$\sup_{t \in [0,\infty)} \sup_{i,j} |a_{ij}(t, x) - a_{ij}(t, y)| \leq \rho_R(|x - y|), \quad x, y \in B(0; R). \tag{1-4}$$

We remark that under the assumptions (1-1) and (1-4), the equation (1-2) under consideration is well-posed (see [Stroock and Varadhan 1979, Chapter 7]), and for fixed $s \in [0, \infty)$ the fundamental solution $p(s, \cdot\,; t, \cdot)$ exists for almost all $t \in (s, \infty)$ (see [Stroock and Varadhan 1979, Theorem 9.1.9]). However, the fundamental solution does not always exist for all $t \in (s, \infty)$ under assumptions (1-1) and (1-4); see [Fabes and Kenig 1981]. We remark that under assumptions (1-1), (1-3) and (1-4), neither existence of the fundamental solutions nor examples where the fundamental solution does not exist are known. In the case that a does not depend on the time t, the fundamental solution exists for all t; see [Stroock and Varadhan 1979, Theorem 9.2.6]. We also remark that (1-3) and (1-4) do not imply the local Hölder continuity of a in the spatial component.

Let $p^X(s, x; t, y)$ be the fundamental solution to the parabolic equation

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} u(t, x),$$

and let

$$L_t^X = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}. \tag{1-5}$$

Suppose that $\{a^{(n)}(t, x)\}$ is a sequence of symmetric $d \times d$-matrix-valued functions with components in $C_b^\infty([0, \infty) \times \mathbb{R}^d)$ such that $a^{(n)}(t, x)$ converges to $a(t, x)$ for each $(t, x) \in [0, \infty) \times \mathbb{R}^d$. We also assume that (1-1), (1-3) and (1-4) hold for $a^{(n)}$ instead of a, with the same constants m, M, θ, R and Λ, and the same function ρ_R. Denote the fundamental solution to the parabolic equation associated with the generator

$$\frac{1}{2} \sum_{i,j=1}^{d} a_{ij}^{(n)}(t, x) \frac{\partial^2}{\partial x_i \partial x_j}$$

by $p^{X,(n)}$. We assume the uniform Gaussian estimates for the fundamental solutions to $p^{X,(n)}$, i.e., there exist positive constants γ_{G}^-, γ_{G}^+, C_{G}^- and C_{G}^+ such that

$$\frac{C_{\mathrm{G}}^-}{(t-s)^{d/2}} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^- |x-y|^2}{t-s}\biggr) \le p^{X,(n)}(s, x; t, y) \le \frac{C_{\mathrm{G}}^+}{(t-s)^{d/2}} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ |x-y|^2}{t-s}\biggr) \tag{1-6}$$

for $s, t \in [0, \infty)$ such that $s < t$, $x, y \in \mathbb{R}^d$, and $n \in \mathbb{N}$. The Gaussian estimates for the fundamental solutions to parabolic equations in divergence form have been well investigated; see [Aronson 1967; Karrmann 2001; Porper and Eidelman 1984; 1992]. However, there are not many known results in the case of nondivergence form. A sufficient condition for the Gaussian estimate is obtained in [Porper and Eidelman 1992, Theorem 19] by means of Dini's continuity condition. The result includes the case of Hölder-continuous coefficients. We remark that two-sided estimates similar to the Gaussian estimates for the equations with general coefficients are obtained in [Escauriaza 2000].

Now we state the main theorem of this paper:

Theorem 1.1. *Assume (1-1), (1-3), (1-4) and (1-6). Then, there exist constants C_1, C_2, γ_1 and γ_2 depending on d, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$\frac{C_1 e^{-C_1(t-s)}}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_1|x-y|^2}{t-s}\right) \le p(s,x;t,y) \le \frac{C_2 e^{C_2(t-s)}}{(t-s)^{d/2}} \exp\left(-\frac{\gamma_2|x-y|^2}{t-s}\right)$$

for $s, t \in [0,\infty)$ such that $s < t$ and $x, y \in \mathbb{R}^d$. Moreover, for any $R > 0$ and sufficiently small $\varepsilon > 0$, there exists a constant C depending on d, ε, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that

$$|p(0,x;t,y) - p(0,z;t,y)| \le Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}$$

for $t \in (0,\infty)$, $x, z \in B(0;R/2)$ and $y \in \mathbb{R}^d$.

The first assertion of Theorem 1.1 is the Gaussian estimate for p. The advantage of the result is obtaining the Gaussian estimate of the fundamental solution to the parabolic equation in nondivergence form without the continuity of b and c. Such a result seems difficult to obtain via the parametrix method. The second assertion of Theorem 1.1 implies that $p(0,x;t,y)$ is $(1-\varepsilon)$-Hölder continuous in x, and this is a clear lower bound for the continuity. The approach in this paper is mainly probabilistic. The key method to prove Theorem 1.1 is the coupling method introduced by Lindvall and Rogers [1986]. This method enables us to discuss the Hölder continuity of $p(0,x;t,y)$ in x from the oscillation of the diffusion processes without regularity of the coefficients.

If a is uniformly continuous in the spatial component, our proof below follows without restriction on x, z, and the following corollary holds:

Corollary 1.2. *Assume (1-1), (1-3), (1-6) and that there exists a continuous and nondecreasing function ρ on $[0,\infty)$ such that $\rho(0) = 0$ and*

$$\sup_{t\in[0,\infty)} \sup_{i,j} |a_{ij}(t,x) - a_{ij}(t,y)| \le \rho(|x-y|), \quad x, y \in \mathbb{R}^d.$$

Then, for sufficiently small $\varepsilon > 0$, there exists a constant C such that

$$|p(0,x;t,y) - p(0,z;s,y)| \le Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}$$

for $t \in (0,\infty)$ and $x, y, z \in \mathbb{R}^d$.

The assumption (1-6) may seem strict. However, as mentioned above, Theorem 19 of [Porper and Eidelman 1992] gives a Gaussian estimate for the parabolic equations with coefficients which satisfy a version of Dini's continuity condition. From this sufficient condition and Theorem 1.1, we have the following corollary:

Corollary 1.3. *Assume (1-1), (1-3), and that there exists a continuous and nondecreasing function ρ on $[0,\infty)$ such that $\rho(0) = 0$,*

$$\int_0^1 \frac{1}{r_2}\left(\int_0^{r_2} \frac{\rho(r_1)}{r_1}\,dr_1\right) dr_2 < \infty \quad \text{and} \quad \sup_{t\in[0,\infty)} \sup_{i,j} |a_{ij}(t,x) - a_{ij}(t,y)| \le \rho(|x-y|), \quad x, y \in \mathbb{R}^d. \tag{1-7}$$

Then, for sufficiently small $\varepsilon > 0$, there exists a constant C such that

$$|p(0, x; t, y) - p(0, z; s, y)| \le Ct^{-d/2-1}e^{Ct}|x - z|^{1-\varepsilon}$$

for $t \in (0, \infty)$ and $x, y, z \in \mathbb{R}^d$.

We remark that, for $\alpha \in (0, 1]$ and a positive constant C, $\rho(r) = Cr^{\alpha}$ satisfies (1-7). Furthermore, $\rho(r) = C \min\{1, (-\log r)^{-\alpha}\}$ satisfies (1-7) for $\alpha \in (2, \infty)$. We also remark that continuity of b and c are not assumed in Corollary 1.3.

The organization of the paper is as follows:

In Section 2, we prepare the probabilistic representation of the fundamental solution to (1-2). It should be remarked that we consider the case where a is smooth in Sections 2–4, and the general case is considered only in Section 5. The representation enable us to consider the Hölder continuity of the fundamental solution by a probabilistic way, and actually in Section 4 we prove the constant appearing in the Hölder continuity of $p(0, x; t, y)$ in x depends only on the suitable constants. The representation is obtained by the Feynman–Kac formula and the Girsanov transformation, and in the end of this section $p(s, x; t, y)$ is represented by the functional of the pinned diffusion process.

In Section 3, we prepare some estimates. The goal of this section is Lemma 3.5, which concerns the integrability of a functional of the pinned diffusion process. Generally speaking, it is much harder to see the integrability with respect to conditional probability measures than with respect to the original probability measure. In our case, conditioning generates a singularity, and this fact makes the estimate difficult. To overcome the difficulty, we begin with Lemma 3.1, which is an estimate of the derivative of $p(s, x; t, y)$. The proof of this lemma is analytic, and (1-3) is assumed for the lemma. In this section, we also have the Gaussian estimate for $p(s, x; t, y)$.

In Section 4, we prove that the constant appearing in the $(1 - \varepsilon)$-Hölder continuity of $p(0, x; t, y)$ in x depends only on the suitable constants. This section is the main part of our argument. To show this, we apply the coupling method to diffusion processes. By virtue of the coupling method, the continuity problem of the fundamental solution is reduced to the problem of the local behavior of the pinned diffusion processes. To see the local behavior, (1-4) is needed. Finally, by showing an estimate of the coupling time, we obtain the $(1 - \varepsilon)$-Hölder continuity of $p(0, x; t, y)$ in x and the suitable dependence of the constant appearing in the Hölder continuity.

In Section 5, we consider the case of general a and prove Theorem 1.1. Our approach is just smoothly approximating a and using the result obtained in Section 4.

Throughout this paper, we denote the inner product in the Euclidean space $\mathbb{R}^d$ by $\langle \cdot , \cdot \rangle$, and all random variables are considered on a probability space $(\Omega, \mathcal{F}, P)$. We denote the expectation of random variables by $E[\,\cdot\,]$ and the expectation on the event $A \in \mathcal{F}$ (i.e., $\int_A \cdot \, dP$) by $E[\,\cdot\,; A]$. We denote the smooth functions with bounded derivatives on S by $C_b^{\infty}(S)$ and the smooth functions on S with compact support by $C_0^{\infty}(S)$.

2. Probabilistic representation of the fundamental solution

In this section, we assume that $a_{ij}(t, x) \in C_b^{0,\infty}([0, \infty) \times \mathbb{R}^d)$. Define a $d \times d$-matrix-valued function $\sigma(t, x)$ as the square root of $a(t, x)$. Then, (1-1) implies that $\sigma_{ij}(t, x) \in C_b^{0,\infty}([0, \infty) \times \mathbb{R}^d)$, $a(t, x) =$

$\sigma(t,x)\sigma(t,x)^T$ and

$$\sup_{t\in[0,\infty)}\sup_{i,j}|\sigma_{ij}(t,x)-\sigma_{ij}(t,y)|\le C\rho_R(|x-y|),\qquad x,y\in B(0;R), \tag{2-1}$$

where C is a constant depending on Λ. Note that (1-1) implies that

$$\Lambda^{-1/2}I\le\sigma(t,x)\le\Lambda^{1/2}I. \tag{2-2}$$

Consider the stochastic differential equation:

$$\begin{cases} dX_t^x=\sigma(t,X_t^x)\,dB_t,\\ \;X_0^x=x. \end{cases} \tag{2-3}$$

Lipschitz continuity of σ implies the existence of a solution and its pathwise uniqueness. Let $(\mathscr{F}_t)$ be the σ-field generated by $(B_s : s\in[0,t])$. Then, pathwise uniqueness implies that the solution X_t^x is $\mathscr{F}_t$-measurable. All stopping times appearing in this paper are associated with $(\mathscr{F}_t)$. We remark that the generator of (X_t^x) is given by (1-5), and therefore the transition probability density of (X_t^x) coincides with the fundamental solution p^X of the parabolic equation generated by (L_t^X). The smoothness of σ implies the smoothness of $p^X(s,x;t,y)$ on $(0,\infty)\times\mathbb{R}^d\times(0,\infty)\times\mathbb{R}^d$; see, for example, [Kusuoka and Stroock 1985] for the probabilistic proof and [Lax and Milgram 1954] for the analytic proof.

There is a relation between the fundamental solution and the generator, as follows. Since p^X is smooth, by the definition of p^X we have

$$\frac{\partial}{\partial t}p^X(s,x;t,y)=[L_t^Xp^X(s,\cdot\,;t,y)](x) \tag{2-4}$$

for $s,t\in[0,\infty)$ such that $s<t$ and $x,y\in\mathbb{R}^d$. Let $(L_t^X)^*$ be the dual operator of L_t^X on $L^2(\mathbb{R}^d)$. Define $T_{s,t}^X$ and $(T_{s,t}^X)^*$ as the semigroups generated by L_t^X and $(L_t^X)^*$, respectively. Since

$$\int_{\mathbb{R}^d}\phi(x)(T_{s,t}^X\psi)(x)\,dx=\int_{\mathbb{R}^d}\psi(x)[(T_{s,t}^X)^*\phi](x)\,dx,$$

we have

$$\int_{\mathbb{R}^d}\phi(x)\biggl(\int_{\mathbb{R}^d}\psi(y)p^X(s,x;t,y)\,dy\biggr)dx=\int_{\mathbb{R}^d}\psi(x)\biggl(\int_{\mathbb{R}^d}\phi(y)(p^X)^*(s,x;t,y)\,dy\biggr)dx,$$

where $(p^X)^*(s,x;t,y)$ is the fundamental solution associated with $(L_t^X)^*$. Hence, it holds that

$$p^X(s,x;t,y)=(p^X)^*(s,y;t,x)$$

for $s,t\in(0,\infty)$ such that $s<t$ and $x,y\in\mathbb{R}^d$. Differentiating both sides of this equation with respect to t, we obtain

$$[L_t^Xp^X(s,\cdot\,;t,y)](x)=[(L_t^X)^*(p^X)^*(s,\cdot\,;t,x)](y)=[(L_t^X)^*p^X(s,x;t,\cdot)](y) \tag{2-5}$$

for $s, t \in [0, \infty)$ such that $s < t$, and $x, y \in \mathbb{R}^d$. By the Chapman–Kolmogorov equation, we have, for $s, t, u \in [0, \infty)$ such that $u < s < t$ and $x, y \in \mathbb{R}^d$, that

$$p^X(u, x; t, y) = \int_{\mathbb{R}^d} p^X(u, x; s, \xi) p^X(s, \xi; t, y)\, d\xi.$$

Differentiating both sides of this equation with respect to s, we have

$$0 = \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial s} p^X(u, x; s, \xi) \right) p^X(s, \xi; t, y)\, d\xi + \int_{\mathbb{R}^d} p^X(u, x; s, \xi) \left(\frac{\partial}{\partial s} p^X(s, \xi; t, y) \right) d\xi$$

for $s, u \in [0, \infty)$ such that $u < s$ and $x, y \in \mathbb{R}^d$. Since (2-4) and (2-5) imply that

$$\frac{\partial}{\partial s} p^X(u, x; s, \xi) = [L_s^X p^X(u, \cdot\,; s, \xi)](x) = [(L_s^X)^* p^X(u, x; s, \cdot\,)](\xi),$$

we have, for $s, t, u \in [0, \infty)$ such that $u < s < t$ and $x, y \in \mathbb{R}^d$, that

$$\begin{aligned}\int_{\mathbb{R}^d} p^X(u, x; s, \xi) \left(\frac{\partial}{\partial s} p^X(s, \xi; t, y) \right) d\xi &= -\int_{\mathbb{R}^d} [(L_s^X)^* p^X(u, x; s, \cdot\,)](\xi) p^X(s, \xi; t, y)\, d\xi \\ &= -\int_{\mathbb{R}^d} p^X(u, x; s, \xi) [L_s^X p^X(s, \cdot\,; t, y)](\xi)\, d\xi.\end{aligned}$$

Noting that $p^X(u, x; s, \xi)$ converges to $\delta_x(\xi)$ as $u \uparrow s$ in the sense of Schwartz distributions, we obtain

$$\frac{\partial}{\partial s} p^X(s, x; t, y) = -[L_s^X p^X(s, \cdot\,; t, y)](x) \tag{2-6}$$

for $s, t \in (0, \infty)$ such that $s < t$ and $x, y \in \mathbb{R}^d$.

Next we study the probabilistic representation of $p(s, x; t, y)$ by $p^X(s, x; t, y)$. By the Feynman–Kac formula (see, for example, [Revuz and Yor 1999, Proposition 3.10, Chapter VIII]) and the Girsanov transformation (see, for example, [Ikeda and Watanabe 1989, Theorem 4.2, Chapter IV]), we have the following representation of $u(t, x)$ by X_t^x:

$$u(t, x) = E\left[f(X_t^x) \exp\left(\int_0^t \langle b_\sigma(s, X_s^x), dB_s \rangle - \frac{1}{2} \int_0^t |b_\sigma(s, X_s^x)|^2\, ds + \int_0^t c(s, X_s^x)\, ds \right) \right], \tag{2-7}$$

where $b_\sigma(t, x) := \sigma(t, x)^{-1} b(t, x)$. For $s \le t$ and $x \in \mathbb{R}^d$, let

$$\mathscr{E}(s, t; X^x) := \exp\left(\int_s^t \langle b_\sigma(u, X_u^x), dB_u \rangle - \frac{1}{2} \int_s^t |b_\sigma(u, X_u^x)|^2\, du + \int_s^t c(u, X_u^x)\, du \right).$$

Then, by the definition of the fundamental solution and (2-7), we obtain the probabilistic representation of the fundamental solution:

$$p(0, x; t, y) = p^X(0, x; t, y) E^{X_t^x = y}[\mathscr{E}(0, t; X^x)], \tag{2-8}$$

where $P^{X_t^x=y}$ is the conditional probability measure of P on $X_t^x = y$ and $E^{X_t^x=y}[\,\cdot\,]$ is the expectation with respect to $P^{X_t^x=y}$. Hence, to see the regularity of $p(0, x; t, y)$ in x, it is sufficient to see the regularity of the function $x \mapsto p^X(0, x; t, y)E^{X_t^x=y}[\mathscr{E}(0, t; X^x)]$. We prove Theorem 1.1 by studying the regularity of this function. The definition of $\mathscr{E}$ implies that

$$\mathscr{E}(0, t; X^x) - \mathscr{E}(\tau \wedge t, t; X^x) = \mathscr{E}(\tau \wedge t, t; X^x)(\mathscr{E}(0, \tau \wedge t; X^x) - 1) \tag{2-9}$$

for any stopping time τ and $t \in [0, \infty)$, and by Itō's formula we have

$$\mathscr{E}(s, t; X^x) - 1 = \int_s^t \mathscr{E}(s, u; X^x)\langle b_\sigma(u, X_u^x), dB_u\rangle + \int_s^t \mathscr{E}(s, u; X^x)c(u, X_u^x)\, du \tag{2-10}$$

for $s, t \in [0, \infty)$ such that $s \le t$. We use these equations in the proof.

Now we consider the diffusion process X^x pinned at y at time t. Let $s, t \in [0, \infty)$ such that $s < t$, $x, y \in \mathbb{R}^d$ and $\varepsilon > 0$. By the Markov property of X, we have for $A \in \mathscr{F}_s$ that

$$P(A \cap \{X_t^x \in B(y; \varepsilon)\}_{B(y;\varepsilon)}\left(\int_{\mathbb{R}^d} p^X(s, \xi; t, \xi')P(A \cap \{X_s^x \in d\xi\})\right) d\xi'.$$

Hence, we obtain

$$P^{X_t^x=y}(A) = \frac{1}{p^X(0, x; t, y)} \int_{\mathbb{R}^d} p^X(s, \xi; t, y)P(A \cap \{X_s^x \in d\xi\}) \tag{2-11}$$

for $s, t \in (0, \infty)$ such that $s < t$, $A \in \mathscr{F}_s$ and $x, y \in \mathbb{R}^d$. This formula enables us to see the generator of the pinned diffusion process. By Itō's formula, (2-6) and (2-11) we have, for $f \in C_b^2(\mathbb{R}^d)$, $s, t \in [0, \infty)$ such that $s < t$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&p^X(0, x; t, y)E^{X_t^x=y}[f(X_s^x)] - p^X(0, x; t, y)f(x) \\
&\quad = E[f(X_s^x)p^X(s, X_s^x; t, y)] - E[f(X_0^x)p^X(0, X_0^x; t, y)] \\
&\quad = E\left[\int_0^s (L_u^X f)(X_u^x)p^X(u, X_u^x; t, y)u\right] + E\left[\int_0^s f(X_u^x)\left(\frac{\partial}{\partial u}p^X(u, \xi; t, y)\right)\Big|_{\xi=X_u^x} du\right] \\
&\qquad\qquad + E\left[\int_0^s f(X_u^x)(L_u^X p^X(u, \cdot\,; t, y))(X_u^x)\, du\right] \\
&\qquad\qquad + \frac{1}{2}E\left[\int_0^s \langle \sigma(u, X_u^x)^T \nabla f(X_u^x), \sigma(u, X_u^x)^T(\nabla p^X(u, \cdot\,; t, y))(X_u^x)\rangle\, du\right] \\
&\quad = p^X(0, x; t, y)\int_0^s E^{X_t^x=y}[(L_u^X f)(X_u^x)]\, du \\
&\qquad\qquad + \frac{1}{2}p^X(0, x; t, y)\int_0^s E^{X_t^x=y}\left[\left\langle \nabla f(X_u^x), a(u, X_u^x)\frac{(\nabla p^X(u, \cdot\,; t, y))(X_u^x)}{p^X(u, X_u^x; t, y)}\right\rangle\right] du.
\end{aligned}$$

Hence, the generator of X pinned at y at time t is

$$\frac{1}{2}\sum_{i,j=1}^d a_{ij}(s, x)\frac{\partial^2}{\partial x_i \partial x_j} + \left\langle \frac{1}{2}a(s, x)\frac{\nabla p^X(s, \cdot\,; t, y)(x)}{p^X(s, x; t, y)}, \nabla\right\rangle$$

for $s \in [0, t)$ and $x \in \mathbb{R}^d$. Of course, pinned Brownian motion is an example of pinned diffusion processes; see [Ikeda and Watanabe 1989, Example 8.5, Chapter IV].

3. Estimates

In this section we prepare some estimates for the proof of the main theorem. Assume that a is smooth and fix notation as in Section 2.

Lemma 3.1. *Let $t \in (0, \infty)$ and ϕ be a nonnegative continuous function on $(0, t) \times \mathbb{R}^d$ such that $\phi(\,\cdot\,, x) \in W^{1,1}_{\mathrm{loc}}((0, t), ds)$ for $x \in \mathbb{R}^d$ and $\phi(s, \cdot\,) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, dx)$ for $s \in (0, t)$. Then, for $s_1, s_2 \in (0, t)$ such that $s_1 \le s_2$,*

$$\begin{aligned}
&\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2}\phi(u,\xi)\,d\xi\,du \\
&\le C(1+|\log(t-s_1)|)\int_{\mathbb{R}^d}\phi(s_1,\xi)\,d\xi + C(t-s_1)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\phi(s_1,\xi)\,d\xi \\
&\quad + C(1+|\log(t-s_2)|)\int_{\mathbb{R}^d}\phi(s_2,\xi)\,d\xi + C(t-s_2)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\phi(s_2,\xi)\,d\xi + C\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\phi(u,\xi)\,d\xi\,du \\
&\quad + C\int_{s_1}^{s_2}\int_{\operatorname{supp}\phi}\frac{|\nabla_\xi\phi(u,\xi)|^2}{\phi(u,\xi)}\,d\xi\,du + C\int_{s_1}^{s_2}(1+|\log(t-u)|)\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du \\
&\quad + C\int_{s_1}^{s_2}(t-u)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du + C\sum_{i,j=1}^d\int_0^t\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial\xi_j}a_{ij}(u,\xi)\right|^2\phi(u,\xi)d\xi\,du,
\end{aligned}$$

where C is a constant depending on d, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, and Λ, and $\operatorname{supp}\phi$ *is the support of ϕ.*

Remark 3.2. If ϕ is a continuous function on $\mathbb{R}^d$, the Lebesgue measure of $\operatorname{supp}\phi \setminus \{x \in \mathbb{R}^d : \phi(x) > 0\}$ is zero.

Proof of Lemma 3.1. It is sufficient to show the theorem for $\phi \in C_0^\infty([0, t] \times \mathbb{R}^d)$, because the general case is obtained by approximation. Let $u \in (0, t)$. Recall that the components of the coefficient σ of (2-3) are in $C_b^{0,\infty}([0, \infty) \times \mathbb{R}^d)$ and σ is uniformly positive-definite. Hence, the associated transition probability density $p^X(\,\cdot\,, \cdot\,; t, \cdot\,)$ is smooth on $(0, t) \times \mathbb{R}^d \times \mathbb{R}^d$, and $p^X(s, x; t, y) > 0$ for $s \in (0, t)$ and $x, y \in \mathbb{R}^d$; see, for example, [Aronson 1967]. By the Leibniz rule, we have

$$\begin{aligned}
&\frac{\partial}{\partial u}\int_{\mathbb{R}^d}(\log p^X(u,\xi;t,y))\phi(u,\xi)\,d\xi \\
&\qquad = \int_{\mathbb{R}^d}\frac{(\partial/\partial u)p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}\phi(u,\xi)\,d\xi + \int_{\mathbb{R}^d}(\log p^X(u,\xi;t,y))\frac{\partial}{\partial u}\phi(u,\xi)\,d\xi. \qquad (3\text{-}1)
\end{aligned}$$

The equality (2-6) and integration by parts imply

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{(\partial/\partial u) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi \\
&\qquad = -\frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{a_{ij}(u,\xi)(\partial^2/\partial\xi_i\partial\xi_j) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi \\
&\qquad = -\frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{(\partial/\partial\xi_i)(a_{ij}(u,\xi)(\partial/\partial\xi_j) p^X(u,\xi;t,y))}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi \\
&\qquad\qquad + \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{((\partial/\partial\xi_i)a_{ij}(u,\xi))(\partial/\partial\xi_j) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi \\
&\qquad = -\frac{1}{2}\int_{\mathbb{R}^d} \frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2} \phi(u,\xi)\,d\xi \\
&\qquad\qquad + \frac{1}{2}\int_{\mathbb{R}^d} \left\langle \frac{a(u,\xi)\nabla_\xi p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}, \nabla_\xi\phi(u,\xi)\right\rangle d\xi \\
&\qquad\qquad + \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{((\partial/\partial\xi_i)a_{ij}(u,\xi))(\partial/\partial\xi_j) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi.
\end{aligned}
$$

Hence, by (3-1) we have

$$
\begin{aligned}
\frac{\partial}{\partial u}\int_{\mathbb{R}^d} (\log p^X(u,\xi;t,y))\phi(u,\xi)\,d\xi &= -\frac{1}{2}\int_{\mathbb{R}^d} \frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2} \phi(u,\xi)\,d\xi \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d} \left\langle \frac{a(u,\xi)\nabla_\xi p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}, \nabla_\xi\phi(u,\xi)\right\rangle d\xi \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{((\partial/\partial\xi_i)a_{ij}(u,\xi))(\partial/\partial\xi_j) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi \\
&\quad + \int_{\mathbb{R}^d} (\log p^X(u,\xi;t,y))\frac{\partial}{\partial u}\phi(u,\xi)\,d\xi.
\end{aligned}
$$

Integrating both sides from s_1 to s_2 with respect to u, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2} \phi(u,\xi)\,d\xi\,du \\
&\qquad = \int_{\mathbb{R}^d} (\log p^X(s_1,\xi;t,y))\phi(s_1,\xi)\,d\xi - \int_{\mathbb{R}^d} (\log p^X(s_2,\xi;t,y))\phi(s_2,\xi)\,d\xi \\
&\qquad\quad + \frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \left\langle \frac{a(u,\xi)\nabla_\xi p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}, \nabla_\xi\phi(u,\xi)\right\rangle d\xi\,du \\
&\qquad\quad + \frac{1}{2}\sum_{i,j=1}^{d}\int_{s_1}^{s_2}\int_{\mathbb{R}^d} \frac{((\partial/\partial\xi_i)a_{ij}(u,\xi))(\partial/\partial\xi_j) p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)} \phi(u,\xi)\,d\xi\,du \\
&\qquad\quad + \int_{s_1}^{s_2}\int_{\mathbb{R}^d} (\log p^X(u,\xi;t,y))\frac{\partial}{\partial u}\phi(u,\xi)\,d\xi\,du. \qquad (3\text{-}2)
\end{aligned}
$$

Now we consider the estimates for the terms on the right-hand side of this equation. By (1-6), we have for $s \in (0, t)$ that

$$\left|\int_{\mathbb{R}^d} (\log p^X(s,\xi;t,y))\phi(s,\xi)\,d\xi\right| \le \int_{\mathbb{R}^d} \left(|\log C_G^+| + |\log C_G^-| + \frac{d}{2}|\log(t-s)| + \frac{\gamma_G^-|y-\xi|^2}{t-s}\right)\phi(s,\xi)\,d\xi.$$

Hence, there exists a constant C depending on d, γ_G^-, γ_G^+, C_G^-, C_G^+, and Λ such that, for $s \in (0, t)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (\log p^X(s,\xi;t,y))\phi(s,\xi)\,d\xi\right| \\ \le C(1+|\log(t-s)|)\int_{\mathbb{R}^d}\phi(s,\xi)d\xi + C(t-s)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\phi(s,\xi)\,d\xi.\end{aligned} \tag{3-3}$$

The third term of the right-hand side of (3-2) can be estimated as follows:

$$\begin{aligned}\frac{1}{2}\left|\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left\langle\frac{a(u,\xi)\nabla_z p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}, \nabla_\xi\phi(u,\xi)\right\rangle d\xi\,du\right| \\ \le \frac{1}{8}\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2}\phi(u,\xi)\,d\xi\,du \\ + 8\int_{s_1}^{s_2}\int_{\operatorname{supp}\phi}\frac{\langle a(u,\xi)\nabla_\xi\phi(u,\xi), \nabla_\xi\phi(u,\xi)\rangle}{\phi(u,\xi)}\,d\xi\,du.\end{aligned} \tag{3-4}$$

To estimate the fourth term of the right-hand side of (3-2), we observe that

$$\begin{aligned}\frac{1}{2}\left|\sum_{i,j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\frac{((\partial/\partial\xi_i)a_{ij}(u,\xi))(\partial/\partial\xi_j)p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}\phi(u,\xi)\,d\xi\,du\right| \\ \le \frac{1}{8\Lambda}\sum_{j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\frac{|(\partial/\partial\xi_j)p^X(u,\xi;t,y)|^2}{p^X(u,\xi;t,y)^2}\phi(u,\xi)\,d\xi\,du \\ + 8d\Lambda\sum_{i,j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial\xi_j}a_{ij}(u,\xi)\right|^2\phi(u,\xi)\,d\xi\,du.\end{aligned}$$

Hence, by (1-1) we have

$$\begin{aligned}\frac{1}{2}\left|\sum_{i,j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\frac{(\frac{\partial}{\partial\xi_i}a_{ij}(t,\xi))\frac{\partial}{\partial\xi_j}p^X(u,\xi;t,y)}{p^X(u,\xi;t,y)}\phi(u,\xi)\,d\xi\,du\right| \\ \le \frac{1}{8}\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\frac{\langle a(u,\xi)\nabla_\xi p^X(u,\xi;t,y), \nabla_\xi p^X(u,\xi;t,y)\rangle}{p^X(u,\xi;t,y)^2}\phi(u,\xi)\,d\xi\,du \\ + C\sum_{i,j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial\xi_j}a_{ij}(u,\xi)\right|^2\phi(u,\xi)\,d\xi\,du,\end{aligned} \tag{3-5}$$

where C depends on d, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+} and Λ. By using (1-6), we estimate the final term of the right-hand side of (3-2) as follows:

$$\left|\int_{s_1}^{s_2}\int_{\mathbb{R}^d}(\log p^X(u,\xi;t,y))\frac{\partial}{\partial u}\phi(u,\xi)\,d\xi\,du\right|$$
$$\leq \int_{s_1}^{s_2}\int_{\mathbb{R}^d}\left(|\log C_{\mathrm{G}}^{+}|+|\log C_{\mathrm{G}}^{-}|+\frac{d}{2}|\log(t-u)|+\frac{\gamma_{\mathrm{G}}^{-}|y-\xi|^2}{t-u}\right)\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|\,d\xi\,du.$$

Hence, there exists a constant C depending on d, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+} and Λ such that

$$\left|\int_{\mathbb{R}^d}(\log p^X(s,\xi;t,y))\phi(s_1,\xi)\,d\xi\right|$$
$$\leq C\int_{s_1}^{s_2}(1+|\log(t-u)|)\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du+C\int_{s_1}^{s_2}(t-u)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du. \quad (3\text{-}6)$$

Therefore, by (3-2), (3-3), (3-4), (3-5) and (3-6) we obtain the lemma. □

Next, we state the fact on the integrability of $\mathscr{E}$ as a lemma. The proof is obtained by the standard argument; see, for example, [Stroock and Varadhan 1979, Theorem 4.2.1]. So, we omit it.

Lemma 3.3. *Let τ_1, τ_2 be stopping times such that $0 \leq \tau_1 \leq \tau_2$ almost surely. It holds that, for any $q \in \mathbb{R}$,*

$$E[\mathscr{E}(t\wedge\tau_1, t\wedge\tau_2;\, X^x)^q] \leq e^{C(1+q^2)t}, \qquad t>0,\ x,y\in\mathbb{R}^d,$$

where C is a constant depending on d, Λ, $\|b\|_\infty$ and $\|c\|_\infty$.

Lemma 3.4. *Let τ_1, τ_2 be stopping times such that $0 \leq \tau_1 \leq \tau_2 \leq t$ almost surely. It holds that*

$$p^X(0,x;t,y)E^{X_t^x=y}\left[\int_0^t \mathscr{E}(u\wedge\tau_1, u\wedge\tau_2;\, X^x)^q\,du\right] \leq Ct^{-d/2+1-\varepsilon}e^{C(1+q^2)t}\exp\left(-\frac{\gamma|x-y|^2}{t}\right)$$

for $t\in(0,\infty)$, $x,y\in\mathbb{R}^d$, $q\in\mathbb{R}$ and sufficiently small $\varepsilon>0$, where C and γ are positive constants depending on d, ε, γ_{G}^{+}, C_{G}^{+}, Λ, $\|b\|_\infty$ and $\|c\|_\infty$.

Proof. In view of Fubini's theorem and (2-11), it is sufficient to show that there exist positive constants C and γ, depending on d, ε, γ_{G}^{+}, C_{G}^{+}, Λ, $\|b\|_\infty$ and $\|c\|_\infty$, such that

$$\int_0^t E[\mathscr{E}(u\wedge\tau_1, u\wedge\tau_2;\, X^x)^q p^X(u,X_u;t,y)]\,du \leq Ct^{-d/2+1-\varepsilon}e^{C(1+q^2)t}\exp\left(-\frac{\gamma|x-y|^2}{t}\right) \quad (3\text{-}7)$$

for $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$. By (1-6) and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t E[\mathscr{E}(u \wedge \tau_1, u \wedge \tau_2; X^x)^q p^X(u, X_u; t, y)]\,du \\
&\qquad \le C_{\mathrm{G}}^+ \int_0^t E\biggl[\mathscr{E}(u \wedge \tau_1, u \wedge \tau_2; X^x)^q (t-u)^{-\frac{d}{2}} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ |X_u - y|^2}{t-u}\biggr)\biggr]\,du \\
&\qquad \le C_{\mathrm{G}}^+ \biggl(\int_0^t E[\mathscr{E}(u \wedge \tau_1, u \wedge \tau_2; X^x)^{(d+\varepsilon)q/\varepsilon}]\,du\biggr)^{\frac{\varepsilon}{d+\varepsilon}} \\
&\qquad\qquad \times \biggl(\int_0^t E\biggl[(t-u)^{-(d+\varepsilon)/2} \exp\biggl(-\frac{(d+\varepsilon)\gamma_{\mathrm{G}}^+ |X_u^x - y|^2}{d(t-u)}\biggr)\biggr]\,du\biggr)^{\frac{d}{d+\varepsilon}}.
\end{aligned}$$

Hence, in view of Lemma 3.3, to show (3-7) it is sufficient to prove that

$$\int_0^t E\biggl[(t-u)^{-(d+\varepsilon)/2} \exp\biggl(-\frac{(d+\varepsilon)\gamma_{\mathrm{G}}^+ |X_u^x - y|^2}{d(t-u)}\biggr)\biggr]\,du \le C t^{-(d+\varepsilon)/2+1} \exp\biggl(-\gamma \frac{|x-y|^2}{t}\biggr) \tag{3-8}$$

for $t \in (0, \infty)$ and $x, y \in \mathbb{R}^d$, where C and γ are constants depending on d, ε, γ_{G}^+, C_{G}^+, Λ, $\|b\|_\infty$ and $\|c\|_\infty$.

Let $\tilde{\gamma} := (1 + \varepsilon/d)\gamma_{\mathrm{G}}^+$. By (1-6) again, we have for $u \in (0, t)$ that

$$\begin{aligned}
&E\biggl[(t-u)^{-d/2} \exp\biggl(-\frac{\tilde{\gamma}|X_u^x - y|^2}{t-u}\biggr)\biggr] \\
&\qquad \le C_{\mathrm{G}}^+ u^{-d/2} (t-u)^{-d/2} \int_{\mathbb{R}^d} \exp\biggl(-\frac{\tilde{\gamma}|\xi - y|^2}{t-u}\biggr) \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ |\xi - x|^2}{u}\biggr)\,d\xi \\
&\qquad = C_{\mathrm{G}}^+ u^{-d/2} (t-u)^{-d/2} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ \tilde{\gamma}}{u\tilde{\gamma} + (t-u)\gamma_{\mathrm{G}}^+} |x-y|^2\biggr) \\
&\qquad\qquad \times \int_{\mathbb{R}^d} \exp\biggl(-\frac{u\tilde{\gamma} + (t-u)\gamma_{\mathrm{G}}^+}{u(t-u)} \biggl|\xi - \frac{\gamma_{\mathrm{G}}^+ (t-u)x + \tilde{\gamma} u y}{u\tilde{\gamma} + (t-u)\gamma_{\mathrm{G}}^+}\biggr|^2\biggr)\,d\xi \\
&\qquad = (2\pi)^{d/2} C_{\mathrm{G}}^+ (u\tilde{\gamma} + (t-u)\gamma_{\mathrm{G}}^+)^{-d/2} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ \tilde{\gamma}}{u\tilde{\gamma} + (t-u)\gamma_{\mathrm{G}}^+} |x-y|^2\biggr) \\
&\qquad \le (2\pi)^{d/2} C_{\mathrm{G}}^+ {\gamma_{\mathrm{G}}^+}^{-d/2} t^{-d/2} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ |x-y|^2}{t}\biggr).
\end{aligned}$$

Hence, there exists a positive constant C depending on d, γ_{G}^+, C_{G}^+, Λ, $\|b\|_\infty$ and $\|c\|_\infty$, such that

$$E\biggl[(t-u)^{-d/2} \exp\biggl(-\frac{\tilde{\gamma}|X_u^x - y|^2}{t-u}\biggr)\biggr] \le C t^{-d/2} \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+ |x-y|^2}{t}\biggr), \quad u \in (0, t).$$

Thus, we obtain (3-8). $\square$

Lemma 3.5. *Let $t \in (0, \infty)$ and τ_1, τ_2 be stopping times such that $0 \le \tau_1 \le \tau_2 \le t$ almost surely. Then, it holds that*

$$p^X(0, x; t, y) E^{X_t^x = y}[\mathscr{E}(s \wedge \tau_1, s \wedge \tau_2; X^x)^q] \le C t^{-d/2} e^{C(1+q^2)t} \exp\biggl(-\frac{\gamma|x-y|^2}{t}\biggr)$$

for $t \in (0,\infty)$, $x, y \in \mathbb{R}^d$, $q \in \mathbb{R}$ *and* $s \in [0,t)$, *where* C *and* γ *are positive constants depending on* d, γ_G^-, γ_G^+, C_G^-, C_G^+, m, M, θ, Λ, $\|b\|_\infty$ *and* $\|c\|_\infty$.

Proof. Let $s_1, s_2 \in (0,t)$ such that $s_1 \le s_2$. In view of (2-10), by (2-11) and Itō's formula we have

$$\begin{aligned}
&p^X(0,x;t,y)E^{X_t^x=y}[\mathscr{E}((s\wedge\tau_1)\vee s_1,(s\wedge\tau_2)\wedge s_2;X^x)^q]\\
&\qquad = E[p^X(s_2,X_{s_2}^x;t,y)\mathscr{E}((s\wedge\tau_1)\vee s_1,(s\wedge\tau_2)\wedge s_2;X^x)^q]\\
&\qquad = p^X(0,x;t,y)+E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}\Big(\frac{\partial}{\partial u}p^X(u,\xi;t,y)\Big)\Big|_{\xi=X_u^x}\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q\,du\bigg]\\
&\qquad\quad +E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}(L_u^Xp^X(u,\cdot\,;t,y))(X_u^x)\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q\,du\bigg]\\
&\qquad\quad +\frac{q}{2}E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q\times\big\langle\sigma(u,X_u^x)^T\nabla_zp^X(u,z;t,y)|_{z=X_u^x},b_\sigma(u,X_u^x)\big\rangle\,du\bigg]\\
&\qquad\quad +\frac{q^2}{2}E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}p^X(u,X_u^x;t,y)\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q|b_\sigma(u,X_u^x)|^2\,du\bigg]\\
&\qquad\quad +qE\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}p^X(u,X_u^x;t,y)\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^qc(u,X_u^x)\,du\bigg].
\end{aligned}$$

Hence, by (2-6) we obtain

$$\begin{aligned}
&p^X(0,x;t,y)E^{X_t^x=y}[\mathscr{E}((s\wedge\tau_1)\vee s_1,(s\wedge\tau_2)\wedge s_2;X^x)^q]\\
&= p^X(0,x;t,y)\\
&\quad +\frac{q}{2}E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q\times\big\langle\sigma(u,X_u^x)^T\nabla_zp^X(u,z;t,y)|_{z=X_u^x},b_\sigma(u,X_u^x)\big\rangle\,du\bigg]\\
&\quad +\frac{q^2}{2}E\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}p^X(u,X_u^x;t,y)\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^q|b_\sigma(u,X_u^x)|^2\,du\bigg]\\
&\quad +qE\bigg[\int_{(s\wedge\tau_1)\vee s_1}^{(s\wedge\tau_2)\wedge s_2}p^X(u,X_u^x;t,y)\mathscr{E}((s\wedge\tau_1)\vee s_1,u;X^x)^qc(u,X_u^x)\,du\bigg].
\end{aligned}$$

In view of the boundedness of $\det\sigma$, b and c, the desired estimate is obtained, once we show the estimates

$$E\bigg[\int_{s_1}^{s_2}\mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1],u;X^x)^qp^X(u,X_u^x;t,y)\,du\bigg] \le Ct^{-d/2+1-\varepsilon}e^{C(1+q^2)t}\exp\Big(-\gamma\frac{|x-y|^2}{t}\Big), \tag{3-9}$$

$$E\bigg[\int_{s_1}^{s_2}\mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1],u;X^x)^q\big|\nabla_\xi p^X(u,\xi;t,y)|_{\xi=X_u^x}\big|\,du\bigg] \le Ct^{-d/2+(1-\varepsilon)/2}e^{C(1+q^2)t}\exp\Big(-\frac{\gamma|x-y|^2}{t}\Big) \tag{3-10}$$

for sufficiently small $\varepsilon > 0$, where C and γ are positive constants depending on d, ε, γ_G^-, γ_G^+, C_G^-, C_G^+, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. The first estimate (3-9) follows, because by (2-11) and Lemma 3.4 we have

$$\begin{aligned}
E\biggl[\int_{s_1}^{s_2} \mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^q p^X(u, X_u^x; t, y)\,du\biggr]
&= p^X(0, x; t, y) E^{X_t^x=y}\biggl[\int_{s_1}^{s_2} \mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^q\,du\biggr]\\
&\le Ct^{-d/2+1-\varepsilon} e^{C(1+q^2)t} \exp\Bigl(-\gamma\frac{|x-y|^2}{t}\Bigr),
\end{aligned}$$

where C and γ are positive constants depending on d, ε, γ_G^+, C_G^+, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Now we show (3-10). By (2-11) and Hölder's inequality, we have

$$\begin{aligned}
&E\biggl[\int_{s_1}^{s_2} \mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^q \bigl|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}\bigr|\,du\biggr]\\
&\quad= p^X(0, x; t, y) E^{X_t^x=y}\biggl[\int_{s_1}^{s_2} \mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^q \frac{|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}|}{p^X(u, X_u^x; t, y)}\,du\biggr]\\
&\quad\le p^X(0, x; t, y)\biggl(\int_{s_1}^{s_2} E^{X_t^x=y}\Bigl[\mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^{2q/(1-2\varepsilon)}\Bigr]\,du\biggr)^{1/2-\varepsilon}\\
&\quad\quad\times\biggl(\int_{s_1}^{s_2}[u(t-u)]^{-1/2}\,du\biggr)^{\varepsilon}\biggl(\int_{s_1}^{s_2}[u(t-u)]^{\varepsilon} E^{X_t^x=y}\biggl[\biggl(\frac{|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}|}{p^X(u, X_u^x; t, y)}\biggr)^2\biggr]\,du\biggr)^{1/2}.
\end{aligned}$$

Lemma 3.4 and (1-6) imply that

$$\begin{aligned}
E\biggl[\int_{s_1}^{s_2} \mathscr{E}(u\wedge[(s\wedge\tau_1)\vee s_1], u; X^x)^q \bigl|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}\bigr|\,du\biggr]
&\le Ct^{-d/4+1/2-\varepsilon} e^{C(1+q^2)t} \exp\Bigl(-\gamma\frac{|x-y|^2}{t}\Bigr)\\
&\quad\times\biggl(p^X(0, x; t, y)\int_{s_1}^{s_2}[u(t-u)]^{\varepsilon} E^{X_t^x=y}\biggl[\biggl(\frac{|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}|}{p^X(u, X_u^x; t, y)}\biggr)^2\biggr]\,du\biggr)^{1/2},
\end{aligned}$$

where C and γ are positive constants depending on d, ε, γ_G^+, C_G^+, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Hence, to show (3-10), it is sufficient to prove

$$\begin{aligned}
p^X(0, x; t, y)\int_0^t [u(t-u)]^{\varepsilon/2} E^{X_t^x=y}\biggl[\biggl(\frac{|\nabla_\xi p^X(u, \xi; t, y)|_{\xi=X_u^x}|}{p^X(u, X_u^x; t, y)}\biggr)^2\biggr]\,du&\\
\le C(t^{-d/2+\varepsilon} + t^{-d/2+\varepsilon}|\log t|),&\quad (3\text{-}11)
\end{aligned}$$

where C is a constant depending on d, ε, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ and Λ. Equation (2-11) implies that

$$\begin{aligned}
p^X(0,x;t,y)\int_0^t [u(t-u)]^{\varepsilon/2} E^{X_t^x=y}\Biggl[\biggl(\frac{|\nabla_\xi p^X(u,\xi;t,y)|_{\xi=X_u^x}|}{p^X(u,X_u^x;t,y)}\biggr)^2\Biggr]\,du \\
= \int_0^t [u(t-u)]^{\varepsilon/2} E\Biggl[\biggl(\frac{|\nabla_\xi p^X(u,\xi;t,y)|_{\xi=X_u^x}|}{p^X(u,X_u^x;t,y)}\biggr)^2 p^X(u,X_u^x;t,y)\Biggr]\,du \\
= \int_0^t [u(t-u)]^{\varepsilon/2}\int_{\mathbb{R}^d}\Biggl(\frac{\Bigl|\nabla_\xi p^X(u,\xi;t,y)\Bigr|}{p^X(u,\xi;t,y)}\Biggr)^2 p^X(u,\xi;t,y)\,p^X(0,x;u,\xi)\,d\xi\,du.
\end{aligned}$$

By (1-6), we have

$$\begin{aligned}
p^X(0,x;t,y)\int_0^t [u(t-u)]^{\varepsilon/2} E^{X_t^x=y}\Biggl[\biggl(\frac{|\nabla_\xi p^X(u,\xi;t,y)|_{\xi=X_u^x}|}{p^X(u,X_u^x;t,y)}\biggr)^2\Biggr]\,du \\
\le (C_{\mathrm{G}}^+)^2\int_0^t\int_{\mathbb{R}^d}\biggl(\frac{|\nabla_\xi p^X(u,\xi;t,y)|}{p^X(u,\xi;t,y)}\biggr)^2 [u(t-u)]^{-(d-\varepsilon)/2} \\
\times\exp\biggl[-\gamma_{\mathrm{G}}^+\biggl(\frac{|\xi-x|^2}{u}+\frac{|y-\xi|^2}{t-u}\biggr)\biggr]\,d\xi\,du. \qquad (3\text{-}12)
\end{aligned}$$

For fixed t, x and y, let

$$\phi(u,\xi) := [u(t-u)]^{-(d-\varepsilon)/2}\exp\biggl[-\gamma_{\mathrm{G}}^+\biggl(\frac{|\xi-x|^2}{u}+\frac{|y-\xi|^2}{t-u}\biggr)\biggr].$$

Denote the surface area of the unit sphere in $\mathbb{R}^d$ by ω_d for $d\ge 2$. In the case $d=1$, let $\omega_d = 2$. Explicit calculation implies that

$$\begin{aligned}
\int_0^t\biggl(\int_{\mathbb{R}^d} e^{2m|\xi|/(\theta-2)}\phi(u,\xi)^{\theta/(\theta-2)}\,d\xi\biggr)^{(\theta-2)/\theta}du \\
= \int_0^t [u(t-u)]^{-(d-\varepsilon)/2}\times\biggl(\int_{\mathbb{R}^d} e^{2m|\xi|/(\theta-2)}\exp\biggl[-\frac{\gamma_{\mathrm{G}}^+\theta}{\theta-2}\biggl(\frac{|\xi-x|^2}{u}+\frac{|y-\xi|^2}{t-u}\biggr)\biggr]\,d\xi\biggr)^{(\theta-2)/\theta}du \\
= \exp\biggl(-\frac{\gamma_{\mathrm{G}}^+|x-y|^2}{t}\biggr)\int_0^t [u(t-u)]^{-(d-\varepsilon)/2} \\
\times\biggl(\int_{\mathbb{R}^d} e^{2m|\xi|/(\theta-2)}\exp\biggl[-\frac{\gamma_{\mathrm{G}}^+\theta t}{(\theta-2)u(t-u)}\biggl|\xi-\frac{(t-u)x+uy}{t}\biggr|^2\biggr]\,d\xi\biggr)^{(\theta-2)/\theta}du.
\end{aligned}$$

Hence, noting that for $\mu_1 \in [0,\infty)$, $\mu_2 \in (0,\infty)$ and $\nu \in \mathbb{R}^d$

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\mu_1|\xi|}\exp(-\mu_2|\xi-\nu|^2)\,d\xi \\
&\quad= \int_{\mathbb{R}^d} e^{\mu_1|\xi+\nu|}\exp(-\mu_2|\xi|^2)\,d\xi \\
&\quad\le e^{\mu_1|\nu|}\int_{\mathbb{R}^d}\exp(\mu_1|\xi|-\mu_2|\xi|^2)\,d\xi \\
&\quad= \omega_d e^{\mu_1|\nu|}\int_{(0,\infty)} r^{d-1}\exp(\mu_1 r-\mu_2 r^2)\,dr \\
&\quad= \omega_d\mu_2^{-d/2}e^{\mu_1|\nu|}\int_{(0,\infty)} r^{d-1}\exp\Big(\frac{\mu_1}{\sqrt{\mu_2}}r-r^2\Big)\,dr \\
&\quad= \omega_d\mu_2^{-d/2}\int_0^{1+\mu_1/\sqrt{\mu_2}} r^{d-1}\exp\Big(\frac{\mu_1}{\sqrt{\mu_2}}r-r^2\Big)\,dr+\omega_d\mu_2^{-d/2}\int_{1+\mu_1/\sqrt{\mu_2}}^{\infty} r^{d-1}\exp\Big(\frac{\mu_1}{\sqrt{\mu_2}}r-r^2\Big)\,dr \\
&\quad\le \frac{\omega_d\mu_2^{-d/2}}{d}\Big(1+\frac{\mu_1}{\sqrt{\mu_2}}\Big)^d\exp\Big[\frac{\mu_1}{\sqrt{\mu_2}}\Big(1+\frac{\mu_1}{\sqrt{\mu_2}}\Big)\Big]+\omega_d\mu_2^{-d/2}\int_{1+\mu_1/\sqrt{\mu_2}}^{\infty} r^{d-1}e^{-r}\,dr \\
&\quad\le C\mu_2^{-d/2}\exp\Big[C\frac{\mu_1}{\sqrt{\mu_2}}\Big(1+\frac{\mu_1}{\sqrt{\mu_2}}\Big)\Big],
\end{aligned}$$

where C is a constant depending on d, we have

$$\begin{aligned}
&\int_0^t\Big(\int_{\mathbb{R}^d} e^{2m|\xi|/(\theta-2)}\phi(u,\xi)^{\theta/(\theta-2)}\,d\xi\Big)^{(\theta-2)/\theta}\,du \\
&\qquad\le C_1 t^{-d/2+d/\theta}\exp\Big(-\frac{\gamma_{\mathrm{G}}^{+}|x-y|^2}{t}\Big) \\
&\qquad\quad\times\int_0^t [u(t-u)]^{-d/\theta+\varepsilon/2}\exp\Big[C_1\sqrt{u\Big(1-\frac{u}{t}\Big)}\Big(1+\sqrt{u\Big(1-\frac{u}{t}\Big)}\Big)\Big]\,du \\
&\qquad\le C_2 t^{-d/2+d/\theta}e^{C_2 t}\int_0^t [u(t-u)]^{-d/\theta+\varepsilon/2}\,du \\
&\qquad\le C_3 t^{-d/2+1-d/\theta+\varepsilon/2}e^{C_2 t},
\end{aligned}$$

where C_1, C_2, C_3 are constants depending on d, ε, γ_{G}^{+}, m and θ. Hence, by Hölder's inequality and (1-3), we have

$$\begin{aligned}
&\sum_{i,j=1}^d\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\Big|\frac{\partial}{\partial\xi_j}a_{ij}(u,\xi)\Big|^2\phi(u,\xi)\,d\xi\,du \\
&\quad\le\int_0^t\Big(\int_{\mathbb{R}^d} e^{2m|\xi|/(\theta-2)}\phi(u,\xi)^{\theta/(\theta-2)}\,d\xi\Big)^{(\theta-2)/\theta}\,du\sum_{i,j=1}^d\Big(\sup_{u\in[0,t]}\int_{\mathbb{R}^d}\Big|\frac{\partial}{\partial\xi_j}a_{ij}(u,\xi)\Big|^\theta e^{-m|\xi|}\,d\xi\Big)^{2/\theta} \\
&\quad\le Ct^{-d/2+1-d/\theta+\varepsilon/2}e^{Ct},
\end{aligned}$$

where C is a constant depending on d, ε, γ_{G}^{+}, m, M and θ. On the other hand, by explicit calculation, we have

$$
\begin{aligned}
\lim_{s\downarrow 0}(1+|\log(t-s)|)\int_{\mathbb{R}^d}\phi(s,\xi)\,d\xi &= 0,\\
\lim_{s\downarrow 0}(t-s)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\phi(s,\xi)\,d\xi &= 0,\\
\lim_{s\uparrow t}(1+|\log(t-s)|)\int_{\mathbb{R}^d}\phi(s,\xi)\,d\xi &= 0,\\
\lim_{s\uparrow t}(t-s)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\phi(s,\xi)\,d\xi &= 0,\\
\int_0^t\int_{\mathbb{R}^d}\frac{|\nabla_\xi\phi(u,\xi)|^2}{\phi(u,\xi)}\,d\xi\,du &\le Ct^{-d/2+\varepsilon},\\
\int_0^t(1+|\log(t-u)|)\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du &\le Ct^{-d/2+\varepsilon}|\log t|,\\
\int_0^t(t-u)^{-1}\int_{\mathbb{R}^d}|y-\xi|^2\left|\frac{\partial}{\partial u}\phi(u,\xi)\right|d\xi\,du &\le Ct^{-d/2+\varepsilon},
\end{aligned}
$$

where C is a constant depending on d, ε and γ_{G}^{+}. In view of these results, applying Lemma 3.1 to (3-12), we obtain (3-11). □

From Lemma 3.5 we can easily show the Gaussian estimate for p with constants depending on the suitable constants.

Proposition 3.6. *For $s, t \in [0,\infty)$ such that $s<t$, and $x, y \in \mathbb{R}^d$,*

$$
\frac{C_1e^{-C_1(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_1|x-y|^2}{t-s}\right)\le p(s,x;t,y)\le\frac{C_2e^{C_2(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_2|x-y|^2}{t-s}\right),
$$

where γ_1, γ_2, C_1 and C_2 are positive constants depending on d, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+}, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$.

Proof. Since the argument follows even if a, b and c are replaced by $a(\cdot-s,\cdot)$, $b(\cdot-s,\cdot)$ and $c(\cdot-s,\cdot)$ respectively, it is sufficient to show that there exist positive constants γ_1, γ_2, C_1 and C_2 depending on d, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that

$$
C_1t^{-d/2}e^{-C_1t}\exp\left(-\frac{\gamma_1|x-y|^2}{t}\right)\le p(0,x;t,y)\le C_2t^{-d/2}e^{C_2t}\exp\left(-\frac{\gamma_2|x-y|^2}{t}\right) \tag{3-13}
$$

for $t\in(0,\infty)$ and $x, y\in\mathbb{R}^d$. The upper estimate in (3-13) follows immediately from (2-8) and Lemma 3.5.

Now we prove the lower estimate in (3-13). From Hölder's inequality, it follows that

$$
1\le E^{X_t^x=y}[\mathscr{E}(s\wedge\tau_1,s\wedge\tau_2;X^x)^{-1}]E^{X_t^x=y}[\mathscr{E}(s\wedge\tau_1,s\wedge\tau_2;X^x)].
$$

Hence, by Lemma 3.5, we have

$$\begin{aligned} p^X(0,x;t,y)E^{X_t^x=y}[\mathscr{E}(s\wedge\tau_1, s\wedge\tau_2; X^x)] &\geq \frac{p^X(0,x;t,y)^2}{p^X(0,x;t,y)E^{X_t^x=y}[\mathscr{E}(s\wedge\tau_1, s\wedge\tau_2; X^x)^{-1}]} \\ &\geq Ct^{d/2}e^{-C't}p^X(0,x;t,y)^2, \end{aligned}$$

where C and C' are positive constants depending on d, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. This inequality, (2-8) and (1-6) imply the lower bound in (3-13). ☐

4. The regularity of $p(0, x; t, y)$ in x

Assume that a is smooth and set notation as in Section 2. In this section, we prove the Hölder continuity of $p(0, x; t, y)$ in x, with constant depending only on suitable ones. The precise statement is as follows:

Proposition 4.1. *For any $R > 0$ and sufficiently small $\varepsilon > 0$, there exists a constant C depending on d, ε, γ_G^-, γ_G^+, C_G^-, C_G^+, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that*

$$|p(0,x;t,y) - p(0,z;t,y)| \leq Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}$$

for $t \in (0,\infty)$, $x, z \in B(0; R/2)$ and $y \in \mathbb{R}^d$.

We use the coupling method; see, for example, [Lindvall and Rogers 1986; Cranston 1991]. Let $x, z \in \mathbb{R}^d$. Given (X^x, B) defined by (2-3), we consider the stochastic process Z^z defined by

$$\begin{cases} Z_t^z = z + \displaystyle\int_0^{t\wedge\tau} \sigma(s, Z_s^z)\, d\widetilde{B}_s + \int_{t\wedge\tau}^t \sigma(s, Z_s^z)\, dB_s, \\ \widetilde{B}_t = \displaystyle\int_0^{t\wedge\tau}\left(I - \frac{2(\sigma(s,Z_s^z)^{-1}(X_s^x - Z_s^z))\otimes(\sigma(s,Z_s^z)^{-1}(X_s^x - Z_s^z))}{|\sigma(s,Z_s^z)^{-1}(X_s^x - Z_s^z)|^2}\right) dB_s, \end{cases} \tag{4-1}$$

where τ is the stopping time defined by $\tau := \inf\{t \geq 0 : X_t^x = Z_t^z\}$.

To see the existence and uniqueness of Z^z, for each $n \in \mathbb{N}$ consider the following stochastic differential equation for $\check{Z}^{n,z}$:

$$\begin{cases} d\check{Z}_t^{n,z} = \sigma(t, \check{Z}_t^{n,z}) \times \left(I - \dfrac{2(\sigma(t,\check{Z}_t^{n,z})^{-1}(X_t^x - \check{Z}_t^{n,z}))\otimes(\sigma(t,\check{Z}_t^{n,z})^{-1}(X_t^x - \check{Z}_t^{n,z}))}{|\sigma(t,\check{Z}_t^{n,z})^{-1}(X_t^x - \check{Z}_t^{n,z})|^2}\right)\mathbb{1}_{\{t<\check{\tau}_n\}}\, dB_t \\ \check{Z}_0^{n,z} = z, \end{cases}$$

where $\check{\tau}_n := \inf\{t \geq 0 : |X_t^x - \check{Z}_t^{n,z}| < 1/n\}$. Note that the equations have random coefficients, since we are considering equations where X^x is given. Now we see the existence and the uniqueness of $\check{Z}^{n,z}$. Let

$$G_n(t,\xi) := \begin{cases} \sigma(t,\xi)\left(I - \dfrac{2(\sigma(t,\xi)^{-1}(X_t^x - \xi))\otimes(\sigma(t,\xi)^{-1}(X_t^x - \xi))}{|\sigma(t,\xi)^{-1}(X_t^x - \xi)|^2}\right) & \text{if } (t,\xi) \in [0,\infty)\times\mathbb{R}^d \text{ and } |X_t^x - \xi| \geq 1/n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, there exists a constant C_n such that $|G_n(t,\xi)-G_n(t,\eta)|\le C_n$ for $t\in[0,\infty)$ and $\xi,\eta\in\mathbb{R}^d$. Note that C_n is nonrandom. Let Y, W be stochastic processes satisfying

$$\begin{cases} dY_t = G_n(t,Y_t)\mathbb{1}_{\{t<\tau_n^Y\}}\,dB_t, \\ Y_0 = z, \end{cases} \qquad \begin{cases} dW_t = G_n(t,W_t)\mathbb{1}_{\{t<\tau_n^W\}}\,dB_t, \\ W_0 = z, \end{cases} \tag{4-2}$$

where $\tau_n^Y := \inf\{t\ge 0 : |X_t^x - Y_t| < 1/n\}$ and $\tau_n^W := \inf\{t\ge 0 : |X_t^x - W_t| < 1/n\}$. Then, by Proposition 1.1(iv), Chapter II and (6.16) of Theorem 6.10, Chapter I of [Ikeda and Watanabe 1989], we have

$$\begin{aligned} E\Big[\sup_{s\in[0,t]} |Y_{s\wedge\tau_n^Y\wedge\tau_n^W} - W_{s\wedge\tau_n^Y\wedge\tau_n^W}|^2\Big] &= E\bigg[\sup_{s\in[0,t]}\bigg|\int_0^{s\wedge\tau_n^Y\wedge\tau_n^W} (G_n(v,Y_v)\mathbb{1}_{\{v<\tau_n^Y\}} - G_n(v,W_v)\mathbb{1}_{\{v<\tau_n^W\}})\,dB_v\bigg|^2\bigg] \\ &= E\bigg[\sup_{s\in[0,t]}\bigg|\int_0^{s} (G_n(v,Y_v) - G_n(v,W_v))\mathbb{1}_{\{v<\tau_n^Y\}}\mathbb{1}_{\{v<\tau_n^W\}}\,dB_v\bigg|^2\bigg] \\ &\le 4E\bigg[\int_0^t |G_n(v,Y_v) - G_n(v,W_v)|^2\mathbb{1}_{\{v<\tau_n^Y\}}\mathbb{1}_{\{v<\tau_n^W\}}\,dv\bigg] \\ &\le 4C_n^2\int_0^t E\Big[|Y_v-W_v|^2\mathbb{1}_{\{v<\tau_n^Y\}}\mathbb{1}_{\{v<\tau_n^W\}}\Big]\,dv \\ &\le 4C_n^2\int_0^t E\bigg[\sup_{s\in[0,v]}\Big|Y_{s\wedge\tau_n^Y\wedge\tau_n^W} - W_{s\wedge\tau_n^Y\wedge\tau_n^W}\Big|^2\bigg]\,dv. \end{aligned}$$

Hence, by Gronwall's inequality, we have

$$Y_{t\wedge\tau_n^Y\wedge\tau_n^W} = W_{t\wedge\tau_n^Y\wedge\tau_n^W}, \qquad t\in[0,\infty) \tag{4-3}$$

almost surely. If $\tau_n^Y\le\tau_n^W$ and $\tau_n^Y<\infty$ for some events, then by letting $t\to\infty$ in (4-3) we have $Y_{\tau_n^Y} = W_{\tau_n^Y}$. Hence, $\tau_n^Y = \tau_n^W$ for these events. Similarly, if $\tau_n^Y\ge\tau_n^W$ and $\tau_n^W<\infty$ for some events, then we have $\tau_n^Y=\tau_n^W$ for these events. Therefore, we obtain

$$\tau_n^Y = \tau_n^W \tag{4-4}$$

almost surely. On the other hand, (4-2) implies that $Y_{\tau_n^Y} = Y_{t\vee\tau_n^Y}$ and $W_{\tau_n^W} = W_{t\vee\tau_n^W}$ for $t\in[0,\infty)$. Hence, by (4-3) and (4-4) we obtain that $Y_t = W_t$ for $t\in[0,\infty)$ almost surely. Thus, we have uniqueness. To see existence, let

$$\widehat{G}_n(t,\xi) := \sigma(t,\xi)\bigg(I - \frac{2(\sigma(t,\xi)^{-1}(X_t^x-\xi))\otimes(\sigma(t,\xi)^{-1}(X_t^x-\xi))}{(|\sigma(t,\xi)^{-1}(X_t^x-\xi)|\vee(\Lambda^{-(1/2)}n^{-1}))^2}\bigg)$$

for $t\in[0,\infty)$ and $\xi\in\mathbb{R}^d$. Then, there exists a constant C_n such that $|\widehat{G}_n(t,\xi)-\widehat{G}_n(t,\eta)|\le C_n$ for $t\in[0,\infty)$ and $\xi,\eta\in\mathbb{R}^d$. Define a sequence of stochastic processes $\{Y^m : m\in\mathbb{N}\cup\{0\}\}$ by $Y_t^0 = z$ for $t\in[0,\infty)$ and

$$Y_t^m := z + \int_0^t \widehat{G}_n(s, Y_s^{m-1})\,dB_s \tag{4-5}$$

for $t \in [0,\infty)$ and $m \in \mathbb{N}$ by iteration. Then, by a similar calculation as above, we have for $m \in \mathbb{N}$ and $t \in [0,\infty)$ that

$$E\Big[\sup_{s\in[0,t]} |Y_s^{m+1} - Y_s^m|^2\Big] \le 4C_n^2 \int_0^t E\Big[\sup_{s\in[0,v]} |Y_s^m - Y_s^{m-1}|^2\Big]\, dv.$$

Applying this inequality iteratively, for $m \in \mathbb{N}$ and $t \in [0,\infty)$ we obtain

$$\begin{aligned} E\Big[\sup_{s\in[0,t]} |Y_s^{m+1} - Y_s^m|^2\Big] &\le (4C_n^2)^m \int_0^t \int_0^{v_m} \cdots \int_0^{v_2} E\Big[\sup_{s\in[0,v_1]} |Y_s^1 - Y_s^0|^2\Big]\, dv_1\, dv_2 \cdots dv_m \\ &= (4C_n^2)^m \int_0^t \int_0^{v_m} \cdots \int_0^{v_2} E\Big[\sup_{s\in[0,v_1]} \Big|\int_0^s \widehat{G}_n(w,z)\, dB_w\Big|^2\Big]\, dv_1\, dv_2 \cdots dv_m \\ &\le \frac{(4C_n^2)^m t^m}{(m+1)!} E\Big[\sup_{s\in[0,t]} \Big|\int_0^s \widehat{G}_n(w,z)\, dB_w\Big|^2\Big]. \end{aligned}$$

Since [Ikeda and Watanabe 1989, (6.16) of Theorem 6.10, Chapter I] implies

$$E\Big[\sup_{s\in[0,t]} \Big|\int_0^s \widehat{G}_n(w,z)\, dB_w\Big|^2\Big] \le 4E\Big[\int_0^t |\widehat{G}_n(w,z)|^2\, dw\Big] < \infty$$

for $t \in [0,\infty)$, we have

$$\sum_{m=1}^{\infty} E\Big[\sup_{s\in[0,t]} |Y_s^{m+1} - Y_s^m|^2\Big] < \infty$$

for $t \in [0,\infty)$. Hence, $\{Y^m\}$ is a Cauchy sequence in $L^2(\Omega;\mathscr{C})$, where $\mathscr{C}$ is the complete metric space $C([0,\infty);\mathbb{R}^d)$ with distance function given by

$$D(w,w') := \sum_{k=1}^{\infty} 2^{-k}\Big(\sup_{t\in[0,k]} |w(t) - w'(t)|\Big) \wedge 1, \qquad w, w' \in C([0,\infty);\mathbb{R}^d).$$

Therefore, there exists a stochastic process Y in $L^2(\Omega;\mathscr{C})$ which satisfies

$$\lim_{m\to\infty} E\Big[\sup_{s\in[0,t]} |Y_s - Y_s^m|^2\Big] = 0$$

for $t \in [0,\infty)$. By taking the limit in (4-5) as $m \to \infty$, we have

$$Y_t = z + \int_0^t \widehat{G}_n(s, Y_s)\, dB_s, \qquad t \in [0,\infty) \tag{4-6}$$

almost surely. Let $\tau_n^Y := \inf\{t \ge 0 : |X_t^x - Y_t| < 1/n\}$. Note that (2-2) implies that

$$|\sigma(t,Y_t)^{-1}(X_t^x - Y_t)| \vee (\Lambda^{-(1/2)} n^{-1}) = |\sigma(t,Y_t)^{-1}(X_t^x - Y_t)|, \qquad t \in [0, \tau_n^Y)$$

almost surely. Applying [Ikeda and Watanabe 1989, Proposition 1.1(iv), Chapter II] to (4-6), we see that the $Y_{\cdot \wedge \tau_n^Y}$ satisfy the stochastic differential equation for $\check{Z}^{n,z}$. Thus, we obtain existence.

We remark that $\{\check{Z}^{n,z} : n \in \mathbb{N}\}$ is consistent; i.e., $\check{Z}^{m,z}_{t\wedge\check{\tau}_n} = \check{Z}^{n,z}_{t\wedge\check{\tau}_n}$ for $m > n$ almost surely. This fact is immediately obtained by [Ikeda and Watanabe 1989, Proposition 1.1(iv), Chapter II] and uniqueness. Define the stochastic processes $(Z^z_t, \widetilde{B}_t; t \in [0, \tau))$ by

$$\begin{aligned} Z^z_t &= \check{Z}^{n,z}_t, \\ \widetilde{B}_t &= \int_0^t \left(I - \frac{2(\sigma(s, \check{Z}^{n,z}_s)^{-1}(X^x_s - \check{Z}^{n,z}_s)) \otimes (\sigma(s, \check{Z}^{n,z}_s)^{-1}(X^x_s - \check{Z}^{n,z}_s))}{|\sigma(s, \check{Z}^{n,z}_s)^{-1}(X^x_s - \check{Z}^{n,z}_s)|^2} \right) dB_s \end{aligned}$$

for $t \in [0, \tau^Z_n)$ and $n \in \mathbb{N}$, where $\tau^Z_n := \inf\{t \geq 0 : |X^x_t - Z^z_t| < 1/n\}$. Then, (4-1) holds for $t \in [0, \tau)$. On the other hand, by applying [Ikeda and Watanabe 1989, Proposition 1.1(iv), Chapter II], we have that $Z^z_{\cdot\wedge\tau^Z_n}$ solves the stochastic differential equation of $\check{Z}^{n,z}$ for $n \in \mathbb{N}$. Hence, $(Z^z_t, \widetilde{B}_t; t \in [0, \tau))$ are determined almost surely and uniquely. Let

$$H_t := I - \frac{2[\sigma(t, Z^z_t)^{-1}(X^x_t - Z^z_t)] \otimes [\sigma(t, Z^z_t)^{-1}(X^x_t - Z^z_t)]}{|\sigma(t, Z^z_t)^{-1}(X^x_t - Z^z_t)|^2}$$

for $t \in [0, \tau)$. Then, H_t is an orthogonal matrix for all $t \in [0, \tau)$, and hence $\widetilde{B}_t$ is a d-dimensional Brownian motion for $t \in [0, \tau)$. Hence, $(Z^z_t, \widetilde{B}_t; t \in [0, \tau))$ are extended to $(Z^z_t, \widetilde{B}_t; t \in [0, \tau])$ almost surely and uniquely. By the Lipschitz continuity of σ, (4-1) is solved almost surely and uniquely for $t \in [\tau, \infty)$, and thus we obtain $(Z^z_t; t \in [0, \infty))$ almost surely and uniquely; see [Stroock and Varadhan 1979, Section 6.6]. From this fact we have that Z^z_t is $\mathcal{F}_t$-measurable for $t \in [0, \infty)$. Hence, if $x = z$, X^x and Z^z have the same law. Moreover, $X^x_t = Z^z_t$ for $t \in [\tau, \infty)$ almost surely.

Lemma 4.2. *For $R > 0$ and sufficiently small $\varepsilon > 0$, there exist positive constants C and c_0 depending on d, ε, R, ρ_R and Λ such that*

$$E[t \wedge \tau] \leq C(1 + t^2)|x - z|^{1-\varepsilon} \tag{4-7}$$

for $t \in [0, \infty)$ and $x, z \in B(0; R/2)$ such that $|x - z| \leq c_0$.

Proof. Let $R > 0$ and $x, z \in B(0; R/2)$. Define

$$\xi_t := X^x_t - Z^z_t \quad \text{and} \quad \alpha_t := \sigma(t, X^x_t) - \sigma(t, Z^z_t)H_t.$$

Then, by Itō's formula we have, for $t \in [0, \tau)$,

$$d(|\xi_t|) = \left\langle \frac{\xi_t}{|\xi_t|}, \alpha_t \, dB_t \right\rangle + \frac{1}{2|\xi_t|} \left(\operatorname{tr}(\alpha_t \alpha_t^T) - \frac{|\alpha_t^T \xi_t|^2}{|\xi_t|^2} \right) dt, \tag{4-8}$$

where $\operatorname{tr}(A)$ is the trace of the matrix A. Now we follow the argument in [Lindvall and Rogers 1986, Section 3]. Since

$$\begin{aligned} \alpha_t &= \sigma(t, X^x_t) - \sigma(t, Z^z_t) + \frac{2\xi_t \otimes (\sigma(t, Z^z_t)^{-1}\xi_t)}{|\sigma(t, Z^z_t)^{-1}\xi_t|^2} \\ &= \sigma(t, X^x_t) - \sigma(t, Z^z_t) + \frac{2\xi_t \xi_t^T (\sigma(t, Z^z_t)^{-1})^T}{|\sigma(t, Z^z_t)^{-1}\xi_t|^2}, \end{aligned}$$

it holds that

$$\operatorname{tr}(\alpha_t\alpha_t^T)-\frac{|\alpha_t^T\xi_t|^2}{|\xi_t|^2}=\operatorname{tr}\bigl([\sigma(t,X_t^x)-\sigma(t,Z_t^z)][\sigma(t,X_t^x)-\sigma(t,Z_t^z)]^T\bigr)-\frac{|[\sigma(t,X_t^x)-\sigma(t,Z_t^z)]^T\xi_t|^2}{|\xi_t|^2}.$$

Hence, in view of (2-1), there exists a positive constant γ_1 depending on d and Λ such that

$$\left|\operatorname{tr}(\alpha_t\alpha_t^T)-\frac{|\alpha_t^T\xi_t|^2}{|\xi_t|^2}\right|\le\gamma_1\rho_R(|\xi_t|)\quad\text{for } t\in[0,\tau)\text{ such that } X_t^x,Z_t^z\in B(0;R). \tag{4-9}$$

On the other hand, following the argument in [Lindvall and Rogers 1986, Section 3], we have a positive constant γ_2 depending on d and Λ such that

$$\frac{|\alpha_t^T\xi_t|}{|\xi_t|}\ge\gamma_2^{-1}\quad\text{for } t\in[0,\tau)\text{ such that } |\sigma(t,X_t^x)-\sigma(t,Z_t^z)|\le 2\Lambda^{-1}. \tag{4-10}$$

Note that if $\rho_R(|\xi_t|)\le 2\Lambda^{-1}$ and $X_t^x,Z_t^z\in B(0;R)$, then $|\sigma(t,X_t^x)-\sigma(t,Z_t^z)|\le 2\Lambda^{-1}$. Let $\gamma:=\gamma_1\vee\gamma_2$. Define stopping times τ_n by $\tau_n:=\inf\{t>0:|X_t^x-Z_t^z|\le 1/n\}$ for $n\in\mathbb{N}$. For a given $\varepsilon>0$, let

$$\begin{aligned}\tilde\tau&:=\ \tau\wedge\inf\left\{t\in[0,\infty):\ \rho_R(|\xi_t|)>\frac{\varepsilon}{2\gamma^3}\wedge 2\Lambda^{-1},\ X_t^x\notin B(0;R)\text{ or } Z_t^z\notin B(0;R)\right\},\\ \tilde\tau_n&:=\tau_n\wedge\inf\left\{t\in[0,\infty):\ \rho_R(|\xi_t|)>\frac{\varepsilon}{2\gamma^3}\wedge 2\Lambda^{-1},\ X_t^x\notin B(0;R)\text{ or } Z_t^z\notin B(0;R)\right\}\end{aligned}$$

for $n\in\mathbb{N}$. Then, it holds that $\tilde\tau_n\uparrow\tilde\tau$ almost surely as $n\to\infty$. By Itō's formula, (4-8), (4-9) and (4-10), we have for $t\in[0,\infty)$ that

$$\begin{aligned}E[|\xi_{t\wedge\tilde\tau_n}|^{1-\varepsilon}]&=|x-z|^{1-\varepsilon}+(1-\varepsilon)E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-\varepsilon}\frac{1}{2|\xi_s|}\left(\operatorname{tr}(\alpha_s\alpha_s^T)-\frac{|\alpha_s^T\xi_s|^2}{|\xi_s|^2}\right)ds\right]\\&\qquad-\frac{\varepsilon(1-\varepsilon)}{2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\frac{|\alpha_s^T\xi_s|^2}{|\xi_s|^2}\,ds\right]\\&\le|x-z|^{1-\varepsilon}+\frac{(1-\varepsilon)\gamma}{2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\rho_R(|\xi_s|)\,ds\right]-\frac{\varepsilon(1-\varepsilon)}{2\gamma^2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\,ds\right]\\&\le|x-z|^{1-\varepsilon}+\frac{\varepsilon(1-\varepsilon)}{4\gamma^2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\,ds\right]-\frac{\varepsilon(1-\varepsilon)}{2\gamma^2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\,ds\right]\\&\le|x-z|^{1-\varepsilon}-\frac{\varepsilon(1-\varepsilon)}{4\gamma^2}E\left[\int_0^{t\wedge\tilde\tau_n}|\xi_s|^{-1-\varepsilon}\,ds\right]\\&\le|x-z|^{1-\varepsilon}-\frac{\varepsilon(1-\varepsilon)}{2^{3+\varepsilon}\gamma^2R^{1+\varepsilon}}E[t\wedge\tilde\tau_n].\end{aligned}$$

Hence, it holds that

$$E[t\wedge\tilde\tau]\le C|x-z|^{1-\varepsilon}\quad\text{for } t\in[0,\infty), \tag{4-11}$$

where C is a constant depending on d, ε, R and Λ.

Now we consider the estimate of the expectation of τ by using that of $\tilde{\tau}$. To simplify the notation, let

$$\delta_0 := \frac{1}{3}\rho_R^{-1}\left(\frac{\varepsilon}{2\gamma^3} \wedge 2\Lambda^{-1}\right).$$

Since

$$|\xi_t| > 3\delta_0 \text{ implies } |X_t^x - x| > \delta_0,\ |Z_t^z - z| > \delta_0,\ \text{or } |x - z| > \delta_0,$$
$$X_t^x \notin B(0;\, R) \text{ or } Z_t^z \notin B(0;\, R) \text{ implies } |X_t^x - x| > \frac{R}{2} \text{ or } |Z_t^z - z| > \frac{R}{2},$$

we have, for $x, z \in B(0;\, R/2)$ such that $|x - z| \le \delta_0$,

$$P(\tau \ge t) \le P(\tilde{\tau} \ge t) + P\left(\sup_{s\in[0,t]} |X_s^x - x| > \delta_0 \wedge \frac{R}{2}\right) + P\left(\sup_{s\in[0,t]} |Z_s^z - z| > \delta_0 \wedge \frac{R}{2}\right). \tag{4-12}$$

Let $\eta = x$ or z. By Chebyshev's inequality and Burkholder's inequality we have

$$\begin{aligned}
P\left(\sup_{s\in[0,t]} |X_s^\eta - \eta| > \delta_0 \wedge \frac{R}{2}\right) &\le \left(\delta_0 \wedge \frac{R}{2}\right)^{-2/\varepsilon} E\left[\sup_{s\in[0,t]} |X_s^\eta - \eta|^{2/\varepsilon}\right] \\
&\le \left(\delta_0 \wedge \frac{R}{2}\right)^{-2/\varepsilon} E\left[\sup_{s\in[0,t]} \left|\int_0^s \sigma(u, X_u^\eta) dB_u\right|^{2/\varepsilon}\right] \\
&\le \left(\delta_0 \wedge \frac{R}{2}\right)^{-2/\varepsilon} C E\left[\left(\sum_{i,j=1}^d \int_0^t \sigma_{ij}(u, X_u^\eta)\sigma_{ji}(u, X_u^\eta)\, du\right)^{1/\varepsilon}\right] \\
&\le d^{1/\varepsilon}\left(\delta_0 \wedge \frac{R}{2}\right)^{-2/\varepsilon} C\Lambda^{1/\varepsilon} t^{1/\varepsilon},
\end{aligned}$$

where C is a constant depending on ε. Hence, there exists a constant C depending on d, ε, R, ρ_R and Λ such that

$$P\left(\sup_{s\in[0,t]} |X_s^\eta - \eta| > \delta_0 \wedge \frac{R}{2}\right) \le C|x - z| \tag{4-13}$$

for $\eta = x, z$ and $t \in [0, |x - z|^\varepsilon]$. By (4-11), (4-12) and (4-13) we have, for $x, z \in B(0;\, R/2)$ such that $|x - z| \le \delta_0$, and $t \in [0, |x - z|^\varepsilon]$,

$$\begin{aligned}
E[t \wedge \tau] &\le \int_0^t P(\tau \ge s)\, ds \\
&\le \int_0^t P(\tilde{\tau} \ge s)\, ds + t\left[P\left(\sup_{s\in[0,t]} |X_s^x - x| > \delta_0 \wedge \frac{R}{2}\right) + P\left(\sup_{s\in[0,t]} |X_s^z - z| > \delta_0 \wedge \frac{R}{2}\right)\right] \\
&\le C(1 + t)|x - z|^{1-\varepsilon},
\end{aligned}$$

where C is a constant depending on d, ε, R, ρ_R and Λ. Therefore, we obtain

$$E[t \wedge \tau] \le C(1 + t)|x - z|^{1-\varepsilon} \tag{4-14}$$

for $x, z \in B(0; R/2)$ such that $|x-z| \le \delta_0$ and $t \in [0, |x-z|^\varepsilon]$. By using Chebyshev's inequality, we calculate $E[t \wedge \tau]$ as

$$\begin{aligned} E[t \wedge \tau] = \int_0^{|x-z|^\varepsilon} P(\tau \ge s)\, ds + \int_{|x-z|^\varepsilon}^t P(\tau \ge s)\, ds &\le E[|x-z|^\varepsilon \wedge \tau] + t P(\tau \ge |x-z|^\varepsilon) \\ &\le E[|x-z|^\varepsilon \wedge \tau] + \frac{t}{|x-z|^\varepsilon} E[\tau \wedge |x-z|^\varepsilon] \\ &\le (1 + t|x-z|^{-\varepsilon}) E[|x-z|^\varepsilon \wedge \tau]. \end{aligned}$$

Thus, applying (4-14) with $t = |x-z|^\varepsilon$ and choosing another small ε, we obtain (4-7) for all $t \in [0, \infty)$. □

Lemma 4.3. *For $R > 0$ and sufficiently small $\varepsilon > 0$, there exist positive constants C and c_0 depending on d, ε, C_{G}^+, R, ρ_R and Λ such that*

$$p^X(0, x; t, y) E^{X_t^x = y}[t \wedge \tau] \le C t^{-d/2}(1+t^2)|x-z|^{1-\varepsilon},$$
$$p^X(0, z; t, y) E^{Z_t^z = y}[t \wedge \tau] \le C t^{-d/2}(1+t^2)|x-z|^{1-\varepsilon}$$

for $t \in (0, \infty)$, $x, z \in B(0; R/2)$ such that $|x-z| \le c_0$, and $y \in \mathbb{R}^d$.

Proof. It holds that

$$E^{X_t^x = y}[t \wedge \tau] = E^{X_t^x = y}[(t \wedge \tau)\mathbb{1}_{[0,t/2]}(\tau)] + E^{X_t^x = y}[(t \wedge \tau)\mathbb{1}_{[t/2,\infty)}(\tau)]. \tag{4-15}$$

By (2-11) and (1-6), we have

$$p^X(0, x; t, y) E^{X_t^x = y}[(t \wedge \tau)\mathbb{1}_{[0,t/2]}(\tau)] = E\left[(t \wedge \tau)\mathbb{1}_{[0,t/2]}(\tau)\; p^X\left(\frac{t}{2}, X_{t/2}^x; t, y\right)\right] \le 2^{d/2} C_{\mathrm{G}}^+ t^{-d/2} E[t \wedge \tau].$$

Hence, in view of Lemma 4.2, there exists positive constants C and c_0 depending on d, ε, C_{G}^+, R, ρ_R and Λ such that

$$p^X(0, x; t, y) E^{X_t^x = y}[(t \wedge \tau)\mathbb{1}_{[0,t/2]}(\tau)] \le C t^{-d/2}(1+t^2)|x-z|^{1-\varepsilon} \tag{4-16}$$

for $x, z \in B(0; R/2)$ such that $|x-z| \le c_0$ and $y \in \mathbb{R}^d$.

On the other hand, by (2-11) and (1-6), we have

$$\begin{aligned} p^X(0, x; t, y) E^{X_t^x = y}[(t \wedge \tau)\mathbb{1}_{[t/2,\infty)}(\tau)] &\le t p^X(0, x; t, y) P^{X_t^x = y}\left(\tau > \frac{t}{2}\right) \\ &= t \int_{\mathbb{R}^d} p^X\left(\frac{t}{2}, z; t, y\right) P\left(\tau > \frac{t}{2},\; X_{t/2}^x \in dz\right) \\ &\le 2^{d/2} C_{\mathrm{G}}^+ t^{-d/2+1} P\left(\tau > \frac{t}{2}\right). \end{aligned}$$

Hence, by applying Chebyshev's inequality we have

$$p^X(0, x; t, y) E^{X_t^x = y}[(t \wedge \tau) \mathbb{1}_{[t/2,\infty)}(\tau)] \le C t^{-d/2} E[t \wedge \tau],$$

where C is a constant depending on d and C_{G}^{+}. Thus, Lemma 4.2 implies that

$$p^X(0, x; t, y) E^{X_t^x = y}[(t \wedge \tau) \mathbb{1}_{[t/2,\infty)}(\tau)] \le C t^{-d/2}(1 + t^2)|x - z|^{1-\varepsilon} \tag{4-17}$$

for $x, z \in B(0; R/2)$ such that $|x - z| \le c_0$, where C and c_0 are positive constants depending on d, ε, C_{G}^{+}, R, ρ_R and Λ. Therefore, we obtain the assertion for x by (4-15), (4-16) and (4-17). Similar argument yields the assertion for z. □

Lemma 4.4. *For $q \ge 1$, $R > 0$ and sufficiently small $\varepsilon > 0$, there exist positive constants C and c_0 depending on q, d, ε, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$, such that*

$$E\Big[\sup_{s \in [0,t]} |\mathscr{E}(0, \tau \wedge s; X^x) - 1|^q\Big] \le C e^{Ct} |x - z|^{2/(q \vee 2) - \varepsilon},$$
$$E\Big[\sup_{s \in [0,t]} |\mathscr{E}(0, \tau \wedge s; Z^z) - 1|^q\Big] \le C e^{Ct} |x - z|^{2/(q \vee 2) - \varepsilon}$$

for $t \in [0, \infty)$, $x, z \in B(0; R/2)$ such that $|x - z| \le c_0$, and $y \in \mathbb{R}^d$.

Proof. By (2-10) we have

$$\begin{aligned}
&E\Big[\sup_{v \in [0,\tau\wedge t]} |\mathscr{E}(0, v; X^x) - 1|^q\Big] \\
&= E\Big[\sup_{v \in [0,\tau\wedge t]} \Big| \int_0^v \mathscr{E}(0, u; X^x) \langle b_\sigma(u, X_u^x), dB_u \rangle + \int_0^v \mathscr{E}(0, u; X^x) c(u, X_u^x)\, du \Big|^q\Big] \\
&\le C E\Big[\sup_{v \in [0,\tau\wedge t]} \Big| \int_0^v \mathscr{E}(0, u; X^x) \langle b_\sigma(u, X_u^x), dB_u \rangle \Big|^q\Big] + C E\Big[\sup_{v \in [0,\tau\wedge t]} \Big| \int_0^v \mathscr{E}(0, u; X^x) c(u, X_u^x)\, du \Big|^q\Big],
\end{aligned}$$

where C is a constant depending on q. The terms of the right-hand side of this inequality are dominated as follows: By Burkholder's inequality and Hölder's inequality we have

$$\begin{aligned}
&E\Big[\sup_{v \in [0,\tau\wedge t]} \Big| \int_0^v \mathscr{E}(0, u; X^x) \langle b_\sigma(u, X_u^x), dB_u \rangle \Big|^q\Big] \\
&\qquad \le C E\Big[\Big(\int_0^{\tau\wedge t} \mathscr{E}(0, u; X^x)^2 |b_\sigma(u, X_u^x)|^2\, du\Big)^{q/2}\Big] \\
&\qquad \le C \Lambda^q \|b\|_\infty^q t^{1 - 1/[(q/2)\vee 1]} E\Big[\int_0^{\tau\wedge t} \mathscr{E}(0, u; X^x)^{2[(q/2)\vee 1]}\, du\Big]^{1/[(q/2)\vee 1]} \\
&\qquad \le C \Lambda^q \|b\|_\infty^q t^{1 - 2/[q\vee 2]} E[\tau \wedge t]^{2(1-\varepsilon)/(q\vee 2)} E\Big[\Big(\int_0^t \mathscr{E}(0, u; X^x)^{(q\vee 2)/\varepsilon}\, du\Big)\Big]^{2\varepsilon/(q\vee 2)},
\end{aligned}$$

where C is a constant depending on q, and by Hölder's inequality we have

$$E\bigg[\sup_{v\in[0,\tau\wedge t]}\bigg|\int_0^v \mathscr{E}(0,u;X^x)c(u,X_u^x)\,du\bigg|^q\bigg] \le \|c\|_\infty^q t^{1-1/q} E\bigg[\int_0^{\tau\wedge t}\mathscr{E}(0,u;X^x)^q\,du\bigg]$$
$$\le \|c\|_\infty^q t^{1-1/q} E[\tau\wedge t]^{1-\varepsilon} E\bigg[\int_0^t \mathscr{E}(0,u;X^x)^{q/\varepsilon}\,du\bigg]^{\varepsilon}.$$

Thus, applying by Lemmas 3.3 and 4.2 to these inequalities and choosing another small ε, we obtain

$$E\bigg[\sup_{v\in[0,\tau\wedge t]}|\mathscr{E}(0,v;X^x)-1|^q\bigg] \le Ce^{Ct}|x-z|^{2/(q\vee 2)-\varepsilon},$$

where C is a constant depending on q, d, ε, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Similar argument yields the same estimate for Z^z. □

Now we start the proof of Proposition 4.1. Let $t\in(0,\infty)$, $x,z\in B(0;R/2)$ such that $x\ne z$, $y\in\mathbb{R}^d$ and $s\in(t/2,t)$. Recall that X^z and Z^z have the same law. By (2-11) and (2-9) we have

$$\bigg|p^X(0,x;t,y)E^{X_t^x=y}\bigg[\mathscr{E}(0,s;X^x);\ \tau\le\frac{t}{2}\bigg]-p^X(0,z;t,y)E^{Z_t^z=y}\bigg[\mathscr{E}(0,s;Z^z);\ \tau\le\frac{t}{2}\bigg]\bigg|$$
$$=\bigg|E\bigg[\mathscr{E}(0,s;X^x)p^X(s,X_s^x;t,y);\ \tau\le\frac{t}{2}\bigg]-E\bigg[\mathscr{E}(0,s;Z^z)p^X(s,Z_s^z;t,y);\ \tau\le\frac{t}{2}\bigg]\bigg|$$
$$\le E\bigg[\mathscr{E}(0,s;Z^z)|p^X(s,X_s^x;t,y)-p^X(s,Z_s^z;t,y)|;\ \tau\le\frac{t}{2}\bigg]$$
$$+E\bigg[|\mathscr{E}(0,\tau\wedge s;X^x)-\mathscr{E}(0,\tau\wedge s;Z^z)|\mathscr{E}(\tau\wedge s,s;Z^z)p^X(s,X_s^x;t,y);\ \tau\le\frac{t}{2}\bigg]$$
$$+E\bigg[\mathscr{E}(0,\tau\wedge s;X^x)|\mathscr{E}(\tau\wedge s,s;X^x)-\mathscr{E}(\tau\wedge s,s;Z^z)|p^X(s,X_s^x;t,y);\ \tau\le\frac{t}{2}\bigg]$$

Noting that

$$X_s^x=Z_s^z \quad \text{for } s\ge\tau,$$

we obtain

$$\bigg|p^X(0,x;t,y)E^{X_t^x=y}\bigg[\mathscr{E}(0,s;X^x);\ \tau\le\frac{t}{2}\bigg]-p^X(0,z;t,y)E^{Z_t^z=y}\bigg[\mathscr{E}(0,s;Z^z);\ \tau\le\frac{t}{2}\bigg]\bigg|$$
$$\le E\bigg[|\mathscr{E}(0,\tau\wedge s;X^x)-\mathscr{E}(0,\tau\wedge s;Z^z)|\mathscr{E}(\tau\wedge s,s;Z^z)p^X(s,X_s^x;t,y);\ \tau\le\frac{t}{2}\bigg]. \quad (4\text{-}18)$$

By the triangle inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
&E\left[|\mathscr{E}(0, \tau \wedge s; X^x) - \mathscr{E}(0, \tau \wedge s; Z^z)|\mathscr{E}(\tau \wedge s, s; Z^z) p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right] \\
&\qquad \leq E\left[|\mathscr{E}(0, \tau \wedge s; X^x) - 1|\mathscr{E}(\tau \wedge s, s; Z^z) p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right] \\
&\qquad\quad + E\left[|\mathscr{E}(0, \tau \wedge s; Z^z) - 1|\mathscr{E}(\tau \wedge s, s; Z^z) p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right] \\
&\qquad \leq \left(E\left[|\mathscr{E}(0, \tau \wedge s; X^x) - 1|^{2/(2-\varepsilon)} p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right. \\
&\qquad\qquad \left. + E\left[|\mathscr{E}(0, \tau \wedge s; Z^z) - 1|^{2/(2-\varepsilon)} p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right) \\
&\qquad\quad \times E\left[\mathscr{E}(\tau \wedge s, s; Z^z)^{2/\varepsilon} p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right]^{\varepsilon/2}.
\end{aligned}
$$

Hence, by (2-11) and (1-6), we have

$$
\begin{aligned}
&E\left[\mathscr{E}(0, \tau \wedge s; X^x) - \mathscr{E}(0, \tau \wedge s; Z^z)|\mathscr{E}(\tau \wedge s, s; Z^z) p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right] \\
&\qquad \leq \left(E^{X_t^x=y}\left[|\mathscr{E}(0, \tau \wedge s; X^x) - 1|^{2/(2-\varepsilon)};\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right. \\
&\qquad\qquad \left. + E^{X_t^x=y}\left[|\mathscr{E}(0, \tau \wedge s; Z^z) - 1|^{2/(2-\varepsilon)};\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right) \\
&\qquad\quad \times p^X(0, x; t, y) E^{X_t^x=y}\left[\mathscr{E}(\tau \wedge s, s; Z^z)^{2/\varepsilon};\ \tau \leq \frac{t}{2}\right]^{\varepsilon/2} \\
&\qquad \leq \left(E\left[|\mathscr{E}(0, \tau \wedge s; X^x) - 1|^{2/(2-\varepsilon)} p^X(t/2, X_{t/2}^x; t, y);\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right. \\
&\qquad\qquad \left. + E\left[|\mathscr{E}(0, \tau \wedge s; Z^z) - 1|^{2/(2-\varepsilon)} p^X(t/2, X_{t/2}^x; t, y);\ \tau \leq \frac{t}{2}\right]^{1-\varepsilon/2} \right) \\
&\qquad\quad \times \left(p^X(0, x; t, y) E^{X_t^x=y}\left[\mathscr{E}(\tau \wedge s, s; Z^z)^{2/\varepsilon};\ \tau \leq \frac{t}{2}\right]\right)^{\varepsilon/2} \\
&\qquad \leq (C_{\mathrm{G}}^+)^{1-\varepsilon/2} t^{-d/2+d\varepsilon/4} (p^X(0, x; t, y) E^{X_t^x=y}[\mathscr{E}(\tau \wedge s, s; Z^z)^{2/\varepsilon}])^{\varepsilon/2} \\
&\qquad\quad \times (E[|\mathscr{E}(0, \tau \wedge s; X^x) - 1|^{2/(2-\varepsilon)}]^{1-\varepsilon/2} + E[|\mathscr{E}(0, \tau \wedge s; Z^z) - 1|^{2/(2-\varepsilon)}]^{1-\varepsilon/2}).
\end{aligned}
$$

Applying Lemmas 3.3 and 4.4 to this inequality, we obtain

$$
E\left[|\mathscr{E}(0, \tau \wedge s; X^x) - \mathscr{E}(0, \tau \wedge s; Z^z)|\mathscr{E}(\tau \wedge s, s; Z^z) p^X(s, X_s^x; t, y);\ \tau \leq \frac{t}{2}\right] \leq C t^{-d/2} e^{Ct} |x - z|^{1-\varepsilon} \tag{4-19}
$$

for $x, z \in B(0; R/2)$ such that $|x - z| \leq c_0$, where C and c_0 are constants depending on d, ε, C_{G}^+, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$.

Hölder's inequality and Chebyshev's inequality imply

$$E^{X_t^x=y}\left[\mathscr{E}(0,s;X^x);\ \tau\geq\frac{t}{2}\right]\leq P^{X_t^x=y}\left(\tau\geq\frac{t}{2}\right)^{1-\varepsilon/2}E^{X_t^x=y}[\mathscr{E}(0,s;X^x)^{2/\varepsilon}]^{\varepsilon/2}$$
$$\leq\frac{2^{1-\varepsilon/2}}{t^{1-\varepsilon/2}}E^{X_t^x=y}[\tau\wedge t]^{1-\varepsilon/2}E^{X_t^x=y}[\mathscr{E}(0,s;X^x)^{2/\varepsilon}]^{\varepsilon/2}.$$

Hence, by Lemmas 3.5 and 4.3, we obtain

$$p^X(0,x;t,y)E^{X_t^x=y}\left[\mathscr{E}(0,s;X^x);\ \tau\geq\frac{t}{2}\right]\leq Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}\tag{4-20}$$

for $x,z\in B(0;R/2)$ such that $|x-z|\leq c_0$, where C and c_0 are constants depending on d, ε, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Similarly we have

$$p^X(0,z;t,y)E^{Z_t^z=y}\left[\mathscr{E}(0,s;Z^z);\ \tau\geq\frac{t}{2}\right]\leq Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}\tag{4-21}$$

for $x,z\in B(0;R/2)$ such that $|x-z|\leq c_0$, where C and c_0 are constants depending on d, ε,γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Thus, (2-8), (4-18), (4-19), (4-20) and (4-21) imply

$$|p(0,x;t,y)-p(0,z;t,y)|\leq Ct^{-d/2-1+\varepsilon/2}e^{Ct}|x-z|^{1-\varepsilon}$$

for $t\in(0,\infty)$, $x,z\in B(0;R/2)$ such that $|x-z|\leq c_0$, and $y\in\mathbb{R}^d$, with constants C and c_0 depending on d, ε, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. By (1-6) we can remove the restriction on $|x-z|$, and, therefore, we obtain Proposition 4.1.

5. The case of general *a* (proof of the main theorem)

Let $a^{(n)}(t,x)=(a_{ij}^{(n)}(t,x))$ be symmetric $d\times d$-matrix-valued bounded measurable functions on $[0,\infty)\times\mathbb{R}^d$ which converge to $a(t,x)$ for each $(t,x)\in[0,\infty)\times\mathbb{R}^d$ and satisfy (1-1), (1-3) and (1-4). Consider the parabolic partial differential equation

$$\begin{cases}\dfrac{\partial}{\partial t}u(t,x)=\dfrac{1}{2}\displaystyle\sum_{i,j=1}^d a_{ij}^{(n)}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}u(t,x)+\sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}u(t,x)+c(t,x)u(t,x),\\ u(0,x)=f(x).\end{cases}\tag{5-1}$$

Denote the fundamental solution to (5-1) by $p^{(n)}(s,x;t,y)$. From (1-6) and Proposition 3.6 we have positive constants γ_1, γ_2, C_1 and C_2 depending on d, γ_{G}^-, γ_{G}^+, C_{G}^-, C_{G}^+, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that

$$\frac{C_1e^{-C_1(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_1|x-y|^2}{t-s}\right)\leq p^{(n)}(s,x;t,y)\leq\frac{C_2e^{C_2(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_2|x-y|^2}{t-s}\right)\tag{5-2}$$

for $s,t\in[0,\infty)$ such that $s<t$, $x,y\in\mathbb{R}^d$ and $n\in\mathbb{N}$.

It is known that local Hölder continuity of the fundamental solution follows, with index and constant depending only on the constants appearing in the Gaussian estimate; see [Stroock 1988]. This fact and (5-2) imply that the Arzelà–Ascoli theorem is applicable to $p^{(n)}$. Moreover, in view of Proposition 4.1, there exists a constant C depending on d, ε, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+}, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that

$$|p^{(n)}(0,x;t,y)-p^{(n)}(0,z;t,y)|\le Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon}$$

for $t\in(0,\infty)$, $y\in\mathbb{R}^d$ and $x,z\in B(0;R/2)$. Hence, there exists a continuous function $p^{(\infty)}(0,\cdot\,;\,\cdot,\cdot)$ on $\mathbb{R}^d\times(0,\infty)\times\mathbb{R}^d$ such that

$$\lim_{n\to\infty}\sup_{|x|\le R/2}|p^{(n)}(0,x;t,y)-p^{(\infty)}(0,x;t,y)|=0, \tag{5-3}$$

$$|p^{(\infty)}(0,x;t,y)-p^{(\infty)}(0,z;t,y)|\le Ct^{-d/2-1}e^{Ct}|x-z|^{1-\varepsilon},\qquad x,z\in B(0;R/2), \tag{5-4}$$

for $t\in(0,\infty)$ and $y\in\mathbb{R}^d$, where C is a constant depending on d, ε, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+}, m, M, θ, R, ρ_R, Λ, $\|b\|_\infty$ and $\|c\|_\infty$. Moreover, we have positive constants C_1, C_2, γ_1 and γ_2 depending on d, γ_{G}^{-}, γ_{G}^{+}, C_{G}^{-}, C_{G}^{+}, m, M, θ, Λ, $\|b\|_\infty$ and $\|c\|_\infty$ such that

$$\frac{C_1e^{-C_1(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_1|x-y|^2}{t-s}\right)\le p^{(\infty)}(s,x;t,y)\le\frac{C_2e^{C_2(t-s)}}{(t-s)^{\frac{d}{2}}}\exp\left(-\frac{\gamma_2|x-y|^2}{t-s}\right)$$

for $s,t\in[0,\infty)$ such that $s<t$, and $x,y\in\mathbb{R}^d$. To prove Theorem 1.1, we show that $p^{(\infty)}(0,\cdot\,;\,\cdot,\cdot)$ coincides with the fundamental solution $p(0,\cdot\,;\,\cdot,\cdot)$ of the original parabolic partial differential equation (1-2). Let $\phi,\psi\in C_0^\infty(\mathbb{R}^d)$, and set

$$P_t^{(n)}g(x):=\int_{\mathbb{R}^d}g(y)p^{(n)}(0,x;t,y)\,dy\quad\text{for } g\in C_b(\mathbb{R}^d),$$

$$L_t^{(n)}:=\frac12\sum_{i,j=1}^d a_{ij}^{(n)}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i}+c(t,x).$$

Noting that $p^{(n)}(s,x;t,y)$ is smooth in (s,x,t,y), we have $P_t^{(n)}L_t^{(n)}\phi(x)=(\partial/\partial t)P_t^{(n)}\phi(x)$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\phi(y)p^{(n)}(0,x;t,y)\,dy\right)\psi(x)\,dx-\int_{\mathbb{R}^d}\phi(x)\psi(x)\,dx\\
&=\int_{\mathbb{R}^d}[P_t^{(n)}\phi(x)]\psi(x)\,dx-\int_{\mathbb{R}^d}\phi(x)\psi(x)\,dx\\
&=\int_0^t\int_{\mathbb{R}^d}\left[\frac{\partial}{\partial s}P_s^{(n)}\phi(x)\right]\psi(x)\,dx\,ds\\
&=\int_0^t\int_{\mathbb{R}^d}[P_s^{(n)}L_s^{(n)}\phi(x)]\psi(x)\,dx\,ds\\
&=\int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left[\frac12\sum_{i,j=1}^d a_{ij}^{(n)}(s,y)\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^d b_i(s,y)\frac{\partial}{\partial y_i}+c(s,y)\right]\phi(y)p^{(n)}(0,x;s,y)\,dy\right)\psi(x)\,dx\,ds.
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\int_{\mathbb{R}^d}\biggl(\int_{\mathbb{R}^d}\phi(y)p^{(\infty)}(0,x;t,y)\,dy\biggr)\psi(x)\,dx-\int_{\mathbb{R}^d}\phi(x)\psi(x)\,dx$$
$$=\int_0^t\int_{\mathbb{R}^d}\biggl(\int_{\mathbb{R}^d}\biggl[\frac{1}{2}\sum_{i,j=1}^d a_{ij}(s,y)\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^d b_i(s,y)\frac{\partial}{\partial y_i}+c(s,y)\biggr]\phi(y)p^{(\infty)}(0,x;s,y)\,dy\biggr)\psi(x)\,dx\,ds.$$

This equality implies that $p^{(\infty)}(0, x; t, y)$ is also the fundamental solution to the parabolic partial differential equation (1-2). Since the weak solution to (1-2) is unique, $p^{(\infty)}(0, x; t, y)$ coincides with $p(0, x; t, y)$. Therefore, we obtain Theorem 1.1.

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