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We prove the existence and uniqueness of a traveling front and of its speed for the homogeneous heat equation in the half-plane with a Neumann boundary reaction term of unbalanced bistable type or of combustion type. We also establish the monotonicity of the front and, in the bistable case, its behavior at infinity. In contrast with the classical bistable interior reaction model, its behavior at the side of the invading state is of power type, while at the side of the invaded state its decay is exponential. These decay results rely on the construction of a family of explicit bistable traveling fronts. Our existence results are obtained via a variational method, while the uniqueness of the speed and of the front rely on a comparison principle and the sliding method.

1. Introduction

This paper concerns the problem

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}_+^2 \times (0, \infty), \\ \frac{\partial v}{\partial \nu} = f(v) & \text{on } \partial \mathbb{R}_+^2 \times (0, \infty) \end{cases} \quad (1-1)$$

for the homogeneous heat equation in a half-plane with a nonlinear Neumann boundary condition. To study the propagation of fronts given an initial condition, it is important to understand first the existence and properties of traveling fronts — or traveling waves — for (1-1). Taking $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, these are solutions of the form $v(x, y, t) := u(x, y - ct)$ for some speed $c \in \mathbb{R}$. Thus, the pair (c, u) must solve the elliptic problem

$$\begin{cases} \Delta u + cu_y = 0 & \text{in } \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial \mathbb{R}_+^2, \end{cases} \quad (1-2)$$

where $\partial u / \partial \nu = -u_x$ is the exterior normal derivative of u on $\partial \mathbb{R}_+^2 = \{x = 0\}$, u is real valued, and $c \in \mathbb{R}$.

We look for solutions u with $0 < u < 1$ and having the limits

$$\lim_{y \rightarrow -\infty} u(0, y) = 1 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u(0, y) = 0. \quad (1-3)$$

Our results apply to nonlinearities f of unbalanced bistable type or of combustion type, as defined next.

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Definition 1.1. Let f in $C^{1,\gamma}([0, 1])$ for some $\gamma \in (0, 1)$ satisfy

$$f(0) = f(1) = 0 \tag{1-4}$$

and, for some $\delta \in (0, \frac{1}{2})$,

$$f' \leq 0 \quad \text{in } (0, \delta) \cup (1 - \delta, 1). \tag{1-5}$$

(a) We say that f is of positively balanced bistable type if it satisfies (1-4) and (1-5), it has a unique zero — named α — in $(0, 1)$, and that it is “positively balanced” in the sense that

$$\int_0^1 f(s) ds > 0. \tag{1-6}$$

(b) We say that f is of combustion type if it satisfies (1-4) and (1-5), and that there exists $0 < \beta < 1$ (called the ignition temperature) such that $f \equiv 0$ in $(0, \beta)$ and

$$f > 0 \quad \text{in } (\beta, 1). \tag{1-7}$$

In (1-2) one must find not only the solution u but also the speed c , which is a priori unknown. We will establish that there is a unique speed $c \in \mathbb{R}$ for which (1-2) admits a solution u satisfying the limits (1-3). For this speed c , using variational techniques we show the existence of a solution u which is decreasing in y , with limits 1 and 0 at infinity on every vertical line. Moreover, we prove the uniqueness (up to translations in the y variable) of a solution u with limits 1 and 0 as $y \rightarrow \mp\infty$.

The speed c of the front will be shown to be positive. Hence, since $c > 0$, we have that

$$v(x, y, t) = u(x, y - ct) \rightarrow 1 \quad \text{as } t \rightarrow +\infty.$$

That is, the state $u \equiv 1$ invades the state $u \equiv 0$.

For unbalanced bistable nonlinearities which satisfy in addition $f'(0) < 0$ and $f'(1) < 0$, we find the behaviors of the front at $y = \pm\infty$. In contrast with the classical bistable interior reaction model, its behavior at the side of the invading state $u = 1$, i.e., as $y \rightarrow -\infty$, is of power type, while its decay is exponential as $y \rightarrow +\infty$.

Our results are collected in the following theorem. Since f is in $C^{1,\gamma}$, weak solutions to (1-2) can be shown to be classical, indeed $C^{2,\gamma}$ up to $\partial\mathbb{R}_+^2$. This is explained in the beginning of Section 4.

Theorem 1.2. *Let f be of positively balanced bistable type or of combustion type as in Definition 1.1. We have:*

(i) *There exists a solution pair (c, u) to (1-2), where $c > 0$, $0 < u < 1$, and u has the limits (1-3). The solution u lies in the weighted Sobolev space*

$$H_c^1(\mathbb{R}_+^2) := \left\{ w \in H_{loc}^1(\mathbb{R}_+^2) : \|w\|_c := \int_{\mathbb{R}_+^2} e^{cy} \{w^2 + |\nabla w|^2\} dx dy < \infty \right\}.$$

(ii) *Up to translations in the y variable, (c, u) is the unique solution pair to (1-2) among all constants $c \in \mathbb{R}$ and solutions u satisfying $0 \leq u \leq 1$ and the limits (1-3).*

- (iii) For all $x \geq 0$, u is decreasing in the y variable and has limits $u(x, -\infty) = 1$ and $u(x, +\infty) = 0$. Furthermore, $\lim_{x \rightarrow +\infty} u(x, y) = 0$ for all $y \in \mathbb{R}$. If f is of combustion type then we have, in addition, $u_x \leq 0$ in \mathbb{R}_+^2 .
- (iv) If f_1 is of positively balanced bistable type or of combustion type, if another nonlinearity f_2 is of the same type, with $f_1 \geq f_2$ and $f_1 \not\equiv f_2$, then their corresponding speeds satisfy $c_1 > c_2$.
- (v) Assume that f is of positively balanced bistable type and that

$$f'(0) < 0 \quad \text{and} \quad f'(1) < 0. \tag{1-8}$$

Then, there exists a constant $b > 1$ such that:

$$\frac{1}{b} \frac{e^{-cy}}{y^{3/2}} \leq -u_y(0, y) \leq b \frac{e^{-cy}}{y^{3/2}} \quad \text{for } y > 1, \tag{1-9}$$

$$\frac{1}{b} \frac{1}{(-y)^{3/2}} \leq -u_y(0, y) \leq b \frac{1}{(-y)^{3/2}} \quad \text{for } y < -1, \tag{1-10}$$

$$\frac{1}{b} \frac{e^{-cy}}{y^{3/2}} \leq u(0, y) \leq b \frac{e^{-cy}}{y^{3/2}} \quad \text{for } y > 1, \quad \text{and} \tag{1-11}$$

$$\frac{1}{b} \frac{1}{(-y)^{1/2}} \leq 1 - u(0, y) \leq b \frac{1}{(-y)^{1/2}} \quad \text{for } y < -1. \tag{1-12}$$

The lower bounds for $-u_y$, u , and $1 - u$ in (1-9)–(1-12) hold for any f of positively balanced bistable type or of combustion type as in Definition 1.1.

Our result on the existence of the traveling front will be proved using a variational method introduced by Steffen Heinze [2001]. It is explained later in this section. Heinze studied (1-2) in infinite cylinders of \mathbb{R}^n instead of half-spaces. For these domains and for both bistable and combustion nonlinearities, he showed the existence of a traveling front. Using a rearrangement technique after making the change of variables $z = e^{ay}/a$, he also proved the monotonicity of the front. In addition, [Heinze 2001] found an interesting formula, (1-24), for the front speed in terms of the minimum value of the variational problem. The formula has interesting consequences, such as part (iv) of our theorem, the relation between the speeds for two comparable nonlinearities.

For our existence result, we will proceed as in [Heinze 2001]. The weak lower semicontinuity of the problem will be more delicate in our case due to the unbounded character of the problem in the x variable — a feature not present in cylinders. The rearrangement technique will produce a monotone front. Its monotonicity will be crucial in order to establish that it has limits 1 and 0 as $y \rightarrow \mp\infty$. On the other hand, the front being in the weighted Sobolev space $H_c^1(\mathbb{R}_+^2)$ will lead easily to the fact that $u \rightarrow 0$ as $x \rightarrow +\infty$. While $u_x \leq 0$ in the case of combustion nonlinearities — as stated in part (iii) of the theorem — this property is not true for bistable nonlinearities since the normal derivative $-u_x = f(u)$ changes sign on $\{x = 0\}$.

The variational approach has another interesting feature. Obviously, the solutions that we produce in the half-plane are also traveling fronts for the same problem in a half-space \mathbb{R}_+^n for $n \geq 3$: they only depend on two Euclidean variables. However, if the minimization problem is carried out directly in \mathbb{R}_+^n

for $n \geq 3$, then it produces a different type of solution that decays to 0 in all variables but one; see [Remark 1.5](#) for more details.

In the case of combustion nonlinearities, (1-2) in a half-plane has been studied by Caffarelli, Mellet, and Sire [[Caffarelli et al. 2012](#)], a paper developed at the same time as most of our work. They establish the existence of a speed admitting a monotone front. As mentioned in [[Caffarelli et al. 2012](#)], our approaches towards the existence result are different. Their work does not use minimization methods, but instead approximation by truncated problems in bounded domains — as in [[Berestycki and Nirenberg 1992](#)]. They also rely in an interesting explicit formula for traveling fronts of a free boundary problem obtained as a singular limit of (1-2). In addition, [[Caffarelli et al. 2012](#)] establishes the following precise behavior of the combustion front at the side of the invaded state $u = 0$. For some constant $\mu_0 > 0$,

$$u(0, y) = \mu_0 \frac{e^{-cy}}{y^{1/2}} + O\left(\frac{e^{-cy}}{y^{3/2}}\right) \quad \text{as } y \rightarrow +\infty \tag{1-13}$$

(here we follow our notation; [[Caffarelli et al. 2012](#)] reverses the states $u = 0$ and $u = 1$). The decay for combustion fronts is different than ours: $y^{1/2}$ in (1-13) is replaced by $y^{3/2}$ in the bistable case. Note however that the main order in the decays is e^{-cy} and that the exponent c depends in a highly nontrivial way on each nonlinearity f .

Uniqueness issues for the speed or for the front in (1-2) are treated for first time in the present paper. Our result on uniqueness of the speed and of the front relies heavily on the powerful sliding method of Berestycki and Nirenberg [[1991](#)]. We also use a comparison principle analogue to one in the paper by Cabré and Solà-Morales [[2005](#)], which studied (1-2) with $c = 0$. Among other things, [[Cabré and Solà-Morales 2005](#)] established the existence, uniqueness, and monotonicity of a front for (1-2) when $c = 0$ and f is a balanced bistable nonlinearity. It was also shown there that, in the balanced bistable case, the front reaches its limits 1 and 0 at the power rate $1/|y|$. We point out that the variational method in the present paper requires f to be unbalanced. It cannot be carried out in the balanced case.

Suppose now that f satisfies the assumptions made above for bistable nonlinearities except for condition (1-6), and assume instead that

$$\int_0^1 f(s) ds \leq 0.$$

First, if the above integral is zero (i.e., f is balanced), [[Cabré and Solà-Morales 2005](#)] established the existence of a monotone front for f with speed $c = 0$. Suppose now that the above integral is negative. Then, the nonlinearity $\tilde{f}(s) := -f(1 - s)$ has positive integral and is of bistable type. Thus, it produces a solution pair (\tilde{c}, \tilde{u}) for (1-2) with positive speed \tilde{c} . Then, if $u(x, y) := 1 - \tilde{u}(x, -y)$, $(-\tilde{c}, u)$ is a solution pair to (1-2) for the original f . The traveling speed $-\tilde{c}$ is now negative.

To prove our decay estimates as $y \rightarrow \pm\infty$, we use ideas from [[Caffarelli et al. 2012](#); [Cabré and Sire 2015](#)]. The estimates rely on the construction of a family of explicit fronts for some bistable nonlinearities. Their formula and properties are stated in [Theorem 1.3](#) below. To see how we construct these fronts, note

that u is a solution of (1-2) if and only if its trace $v(y) := u(0, y)$ solves the fractional diffusion equation

$$(-\partial_{yy} - c\partial_y)^{1/2}v = f(v) \quad \text{in } \mathbb{R} \quad \text{for } v(y) := u(0, y). \tag{1-14}$$

This follows from two facts. First, if u solves the first equation in (1-2), then so does $-u_x$. Second, we have $(-\partial_x)^2 = \partial_{xx} = -\partial_{yy} - c\partial_y$. Our main result states that there is a unique $c \in \mathbb{R}$ for which the fractional equation (1-14) admits a solution connecting 1 and 0.

As in the paper by Cabré and Sire [2015], which studied problem $(-\partial_{yy})^s v = f(v)$ in \mathbb{R} for balanced bistable nonlinearities, the construction of explicit fronts will be based on the fundamental solution for the homogeneous heat equation associated to the fractional operator in (1-14), that is, equation

$$\partial_t v + (-\partial_{yy} - c\partial_y)^{1/2}v = 0.$$

The process to find such heat kernel uses an idea from [Caffarelli et al. 2012], and it is explained in Section 6 below.

Regarding the resulting decays (1-11) and (1-12) for bistable fronts, note that the exponential decay at the side of the invaded state $u = 0$ is much faster than the power decay (1-12) at the side of the invading state $u = 1$. This large difference of rates is clearly seen in the explicit fronts that we built. See Figure 1 for the plots of one such front, where the much steeper decay on the right is clearly seen.

These decays are also in contrast with the classical ones for the bistable equation $u_{yy} + cu_y + f(u) = 0$ in \mathbb{R} , which are both pure exponentials — with exponents that may be different at $+\infty$ and $-\infty$. Note however that the exponent c in the exponential term at $+\infty$ for our problem will be different, in general, than the corresponding exponent in the classical case — taking the same nonlinearity f for both problems.

The next theorem concerns the explicit bistable fronts that we construct. They will lead to the decay bounds of Theorem 1.2 for general bistable fronts. They involve the modified Bessel function of the second kind K_1 with index $\nu = 1$. We recall that $K_1(s)$ is a positive and decreasing function of $s > 0$ (see [Abramowitz and Stegun 1964]).

Theorem 1.3. *For every $c > 0$ and $t > 0$, let*

$$u^{t,c}(x, y) := u^t\left(\frac{c}{2}x, \frac{c}{2}y\right),$$

where

$$u^t(x, y) := \int_y^{+\infty} e^{-z} \frac{x+t}{\pi\sqrt{(x+t)^2+z^2}} K_1(\sqrt{(x+t)^2+z^2}) dz$$

and K_1 is the modified Bessel function of the second kind with index $\nu = 1$.

Then, there exists a nonlinearity $f^{t,c}$ of positively balanced bistable type for which $(c, u^{t,c})$ is the unique solution pair to (1-2) with $0 \leq u^{t,c} \leq 1$ satisfying the limits (1-3). In addition, we have

$$(f^{t,c})'(0) = (f^{t,c})'(1) = -\frac{c}{2t}.$$

and that $f^{t,c} = (c/2)f^t$ for a nonlinearity f^t independent of c .

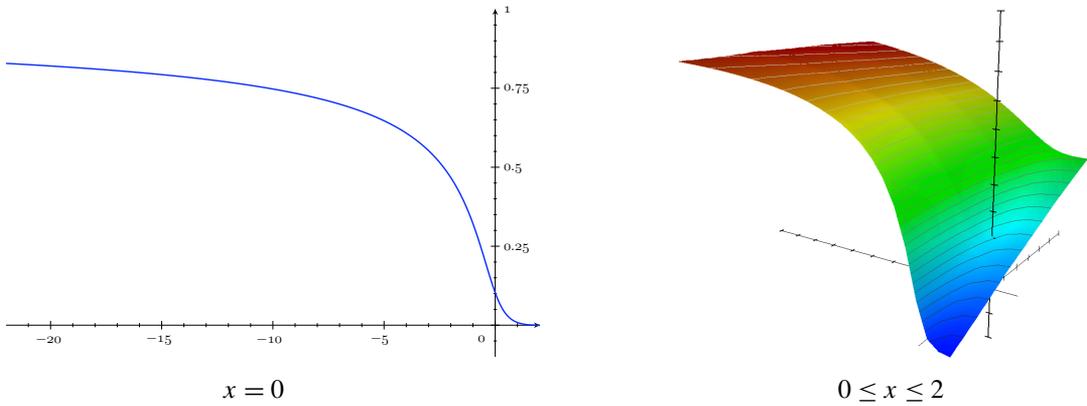


Figure 1. The explicit bistable front u^1 , for $y \in [-22, 2]$. Left: $x = 0$. Right: $x \in [0, 2]$.

Furthermore, on $\partial\mathbb{R}_+^2$ the derivative of $u^{t,c}$ satisfies

$$\begin{aligned}
 -u_y^{t,c}(0, y) &= \frac{t}{(\pi c)^{1/2}} \frac{e^{-cy}}{y^{3/2}} + o\left(\frac{e^{-cy}}{y^{3/2}}\right) \quad \text{as } y \rightarrow +\infty, \quad \text{and} \\
 -u_y^{t,c}(0, y) &= \frac{t}{(\pi c)^{1/2}} \frac{1}{(-y)^{3/2}} + o\left(\frac{1}{(-y)^{3/2}}\right) \quad \text{as } y \rightarrow -\infty.
 \end{aligned}$$

Figure 1 shows plots of the explicit bistable front $u^1 = u^{1,2}$, a front with speed $c = 2$. In both of them we have $-22 \leq y \leq 2$. The much faster decay for positive values of y than for negative values is clearly appreciated. In the three-dimensional plot, the steepest profile corresponds to $x = 0$ while the profile in the back of the picture is for $x = 2$.

Kyed [2008] also studies problem (1-2) in infinity cylinders of \mathbb{R}^3 . It deals with nonlinearities f that vanish only at 0 and that appear in models of boiling processes. Kyed also uses the variational principle of Heinze. In addition, [Kyed 2008] contains some exponential decay bounds. Landes [2009; 2012] studies problem (1-1) in finite cylinders, with special interest in bistable nonlinearities. These articles establish the presence of wavefront-type solutions for some initial conditions and give in addition bounds for their propagation speed. For this, appropriate sub- and supersolutions are constructed.

The variational method in [Heinze 2001] has also been used by Lucia, Muratov, and Novaga [Lucia et al. 2004] to study the classical interior reaction equation $u_{yy} + cu_y + f(u) = 0$ for monostable-type nonlinearities f . Their paper gives a very interesting characterization of the phenomenon of linear versus nonlinear selection for the front speed.

In relation to fractional diffusions — such as (1-14) — the existence of traveling fronts for

$$\partial_t v + (-\partial_{yy})^s v = f(v) \quad \text{in } \mathbb{R} \tag{1-15}$$

has been established in [Mellet et al. 2010] when $s \in (\frac{1}{2}, 1)$ and f is a combustion nonlinearity. This article also shows that v tends to 0 at $+\infty$ at the power rate $1/|y|^{2s-1}$. Note that the equation for traveling fronts of (1-15) is

$$\{(-\partial_{yy})^s - c\partial_y\}v = f(v) \quad \text{in } \mathbb{R},$$

which should be compared with (1-14). In the case of bistable nonlinearities, [Gui and Zhao 2014] establishes that (1-15) admits a unique traveling front and a unique speed for any $s \in (0, 1)$. In contrast with the decay in [Mellet et al. 2010] for combustion nonlinearities, in the bistable case [Gui and Zhao 2014] shows that the front reaches its two limiting values at the rate $1/|y|^{2s}$ — as in [Cabré and Solà-Morales 2005; Cabré and Sire 2015] for balanced bistable nonlinearities.

Next, let us describe the structure of the variational problem that will lead to our existence result. First, we enumerate the five concrete properties of the nonlinearity needed for all the results of the paper.

Remark 1.4. Even though we state our main result for bistable- and combustion-type nonlinearities (for the clarity of reading), all our proofs require only the following five conditions on f :

$$(1-4), (1-5), (1-6), (1-7), \text{ and } \int_0^s f(\sigma) d\sigma \leq 0 \text{ for all } s \in (0, \beta). \tag{1-16}$$

We claim that both positively balanced bistable nonlinearities and combustion nonlinearities as in Definition 1.1 satisfy the above five assumptions.

The claim is easily seen. For a combustion nonlinearity, (1-6) is obviously true, while the last condition in (1-16) holds (indeed with an equality) for the same β as in part (b) of the definition. On the other hand, for f of bistable type, since f has a unique zero α in $(0, 1)$ and (1-5) holds, it follows that $f < 0$ in $(0, \alpha)$ and $f > 0$ in $(\alpha, 1)$. Thus, by (1-6) there exists a unique $\beta \in (\alpha, 1)$ such that $\int_0^\beta f(s) ds = 0$. As a consequence, (1-7) and the last condition in (1-16) hold for such β .

To describe the potential energy of our problem, we first extend f linearly to $(-\infty, 0)$ and to $(1, +\infty)$, keeping its $C^{1,\gamma}$ character. Consider now the potential $G \in C^2(\mathbb{R})$ defined by

$$G(s) := - \int_0^s f(\sigma) d\sigma \text{ for } s \in \mathbb{R}.$$

Note that $G' = -f$ in $[0, 1]$. Since $f'(0) \leq 0$ and $f'(1) \leq 0$ due to hypothesis (1-5), we have that

$$G(s) \geq \begin{cases} 0 = G(0) & \text{for } s \leq 0, \\ G(1) & \text{for } s \geq 1. \end{cases} \tag{1-17}$$

Two other important properties of G are the following. First, by (1-6), we have

$$G(1) < G(0) = 0 \text{ and } G'(0) = -f(0) = 0. \tag{1-18}$$

Second, the last condition in (1-16) reads

$$G \geq 0 \text{ in } [0, \beta]. \tag{1-19}$$

On the other hand, since $G \in C^2(\mathbb{R})$ and $G(0) = G'(0) = 0$, we have that

$$-Cs^2 \leq G(s) \leq Cs^2 \text{ for all } s \in \mathbb{R}, \tag{1-20}$$

for some constant C .

Figure 2 shows the shape of the potential G for a typical positively balanced bistable nonlinearity.

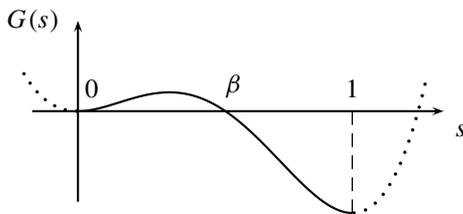


Figure 2. The potential G for a positively balanced bistable f .

For $a > 0$, consider the weighted Sobolev space $(H_a^1(\mathbb{R}_+^2), \|\cdot\|_a)$ defined by

$$H_a^1(\mathbb{R}_+^2) = \{w \in H_{\text{loc}}^1(\mathbb{R}_+^2) : \|w\|_a < \infty\},$$

where the norm $\|\cdot\|_a$ is defined by

$$\|w\|_a^2 = \int_{\mathbb{R}_+^2} e^{ay} \{w^2 + |\nabla w|^2\} dx dy.$$

Notice that, by first truncating and then smoothing, the set $C_c^\infty(\overline{\mathbb{R}_+^2})$ — smooth functions with compact support in $\overline{\mathbb{R}_+^2}$ — is dense in $H_a^1(\mathbb{R}_+^2)$ with the $\|\cdot\|_a$ norm.

In both the bistable and combustion cases, the traveling front u will be constructed from a minimizer \underline{u} to the constraint problem

$$E_a(\underline{u}) = \inf_{w \in B_a} E_a(w) =: I_a \tag{1-21}$$

after scaling its independent variables x and y . This is the method introduced by Heinze [2001] to study (1-2) in cylinders instead of half-spaces. The energy functional

$$E_a(w) = \frac{1}{2} \int_{\mathbb{R}_+^2} e^{ay} |\nabla w|^2 dx dy + \int_{\partial\mathbb{R}_+^2} e^{ay} G(w(0, y)) dy \tag{1-22}$$

will be minimized over the submanifold

$$B_a = \{w \in H_a^1(\mathbb{R}_+^2) : \Gamma_a(w) = 1\},$$

where

$$\Gamma_a(w) = \int_{\mathbb{R}_+^2} e^{ay} |\nabla w|^2 dx dy.$$

To carry out this program, we will need to take a constant $a > 0$ small enough, depending only on f .

Note an important feature of the functionals E_a and Γ_a . For $w \in H_a^1(\mathbb{R}_+^2)$ and $t \in \mathbb{R}$, define

$$w^t(x, y) := w(x, y + t)$$

(throughout the paper there is no risk of confusion with the same notation used for the explicit front u^t of Theorem 1.3). We then have

$$E_a(w^t) = e^{-at} E_a(w) \quad \text{and} \quad \Gamma_a(w^t) = e^{-at} \Gamma_a(w). \tag{1-23}$$

The shape of the potential G will lead to the existence of functions u in $H_a^1(\mathbb{R}_+^2)$ with negative energy $E_a(u) < 0$. This will be essential in order to prove that our variational problem attains its infimum. In addition, the constraint will introduce a Lagrange multiplier and, through it, the a priori unknown speed c of the traveling front.

As already noted by Heinze [2001], the above variational method produces an interesting formula for the speed c . One has

$$c = a(1 - 2I_a), \quad (1-24)$$

where I_a is the minimum value (1-21) of the constraint variational problem; see Remark 2.8 below. This formula leads to the comparison result between the front speeds for two different comparable nonlinearities — part (iv) of Theorem 1.2. Note also that the value $a(1 - 2I_a)$ in (1-24) does not depend on which constant a is chosen to carry out the minimization problem. The reason is that this value coincides with the speed c , and we prove uniqueness of the speed.

Remark 1.5. Obviously, the traveling front found in our paper on \mathbb{R}_+^2 is also a traveling front for problem (1-1) in the half-space \mathbb{R}_+^n , $n \geq 3$, traveling in any given unit direction e of \mathbb{R}^{n-1} . That is, it is a solution of the problem

$$\begin{cases} \Delta u + c\partial_e u = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\mathbb{R}_+^n, \end{cases}$$

where $\mathbb{R}_+^n := \{(x, y) \in (0, +\infty) \times \mathbb{R}^{n-1}\}$. An interesting application of the constraint minimization method is that, when carried out in \mathbb{R}_+^n and $n \geq 3$, it produces another type of traveling front. Their trace $u(0, \cdot)$ will not depend only on one Euclidean variable e of \mathbb{R}^{n-1} (as the solutions in the present paper do), but they will be monotone with limits 1 and 0 at infinity in the direction e and will be even with limits 0 at $\pm\infty$ on the y variables orthogonal to e . The reason is that here one minimizes the energy functional

$$E_a(w) = \frac{1}{2} \int_{\mathbb{R}_+^n} e^{a \cdot y} |\nabla w|^2 dx dy + \int_{\partial\mathbb{R}_+^n} e^{a \cdot y} G(w) dy$$

under the constraint on the Dirichlet energy, where $a \in \mathbb{R}^{n-1}$ is a nonunitary direction parallel to e . Now, note that the solutions built in this paper, which are constant in the y variables orthogonal to a , do not belong to the corresponding energy space (since they have infinite Dirichlet energy).

The article is organized as follows. In Section 2 we study the variational structure of problem (1-2) and prove the existence of a solution pair. Section 3 uses the variational characterization and a monotone decreasing rearrangement of the minimizer to show that the front may be taken to be monotone in the y direction. In Section 4 we establish the limits at infinity for the obtained solution. In Section 5 we prove a monotonicity and comparison result by means of a maximum principle and the sliding method. This result is the key ingredient to prove uniqueness of speed and of the front. Section 6 deals with the explicit fronts and supersolutions; here we give the proof of Theorem 1.3. Finally, in Section 7 we collect all results in the paper to establish Theorem 1.2.

2. The variational solution

Two inequalities. We prove trace and Poincaré-type inequalities for functions in $H_a^1(\mathbb{R}_+^2)$. The existence of a lower bound for E_a on B_a will be a consequence of the following lemma:

Lemma 2.1. *Let $a > 0$. Then, for every $u \in H_a^1(\mathbb{R}_+^2)$, we have*

$$\int_{\mathbb{R}} e^{ay} u^2(0, y) dy \leq \|u\|_a^2 = \int_{\mathbb{R}_+^2} e^{ay} \{u^2 + |\nabla u|^2\} dx dy$$

and

$$\int_{\mathbb{R}_+^2} e^{ay} u^2 dx dy \leq \frac{4}{a^2} \int_{\mathbb{R}_+^2} e^{ay} |\nabla u|^2 dx dy.$$

Proof. By density, it suffices to establish both inequalities for $u \in C_c^\infty(\overline{\mathbb{R}_+^2})$. Since u has compact support, we have

$$\int_{\mathbb{R}} e^{ay} u^2(0, y) dy = - \int_{\mathbb{R}_+^2} (e^{ay} u^2)_x dx dy = -2 \int_{\mathbb{R}_+^2} e^{ay} u u_x dx dy \leq \int_{\mathbb{R}_+^2} e^{ay} (u^2 + u_x^2) dx dy.$$

This proves the first inequality.

Next, for every $x \geq 0$ we have that

$$\int_{\mathbb{R}} e^{ay} u^2(x, y) dy = -\frac{2}{a} \int_{\mathbb{R}} e^{ay} u(x, y) u_y(x, y) dy.$$

Thus, by Cauchy–Schwarz,

$$\int_{\mathbb{R}} e^{ay} u^2(x, y) dy \leq \frac{4}{a^2} \int_{\mathbb{R}} e^{ay} u_y^2(x, y) dy.$$

Integrating in x from 0 to ∞ , we obtain the second inequality. □

Construction of functions with negative energy. It is fundamental to show that E_a takes a negative value somewhere on the constraint B_a . This will be accomplished by constructing test functions $u_0 \in H_a^1(\mathbb{R}_+^2)$ for which $E_a(u_0) < 0$ if a is positive and small enough. We undertake this task next.

Let u_0 be defined by

$$u_0(x, y) := e^{-dx} h(y), \tag{2-1}$$

where h is given by

$$h(y) := \begin{cases} 1 & \text{if } y \leq 0, \\ e^{-amy} & \text{if } y > 0, \end{cases}$$

and the values of $d > 0$ and $m \geq 1$ are to be determined.

The following proposition applies to a class of nonlinearities which includes those of positively balanced bistable type and of combustion type.

Proposition 2.2. *Let G satisfy (1-18). Let $a > 0$, E_a be defined by (1-22), and u_0 by (2-1). Then, for small positive values of a and d , and for large values of m (all depending only on f), we have that $u_0 \in H_a^1(\mathbb{R}_+^2)$ and $E_a(u_0) < 0$. In addition,*

$$-\infty < \inf_{w \in B_a} E_a(w) < 0.$$

Proof. A simple calculation for the Dirichlet energy shows that, for $m \geq 1$,

$$\Gamma_a(u_0) = \frac{d}{2a} \left(1 + \frac{1}{2m-1} \right) + \frac{am^2}{2d(2m-1)}.$$

The potential energy can be computed as follows:

$$\begin{aligned} \int_{\mathbb{R}} e^{ay} G(u_0(0, y)) dy &= \frac{G(1)}{a} + \int_0^{+\infty} e^{ay} G(e^{-amy}) dy \\ &= \frac{1}{a} \left\{ G(1) + \int_0^{+\infty} (e^{ay})' G(e^{-amy}) dy \right\} \\ &= \frac{1}{a} \int_0^{+\infty} e^{ay} f(e^{-amy})(-am)e^{-amy} dy = -\frac{1}{a} \int_0^1 s^{-1/m} f(s) ds, \end{aligned}$$

where we have used property (1-20) of G in order to integrate by parts. Note that (1-20) follows from assumption (1-18). Therefore,

$$aE_a(u_0) = \frac{d}{4} \left(1 + \frac{1}{2m-1} \right) + \frac{a^2m^2}{4d(2m-1)} - \int_0^1 s^{-1/m} f(s) ds.$$

Note that

$$\lim_{m \rightarrow +\infty} \int_0^1 s^{-1/m} f(s) ds = \int_0^1 f(s) ds > 0,$$

since $G(1) < G(0)$. It follows that $E_a(u_0) < 0$ if we first choose m large enough, then choose d small enough to make the first term above small, and finally a also small to handle the second term.

It follows from property (1-23) that there exists a unique value t such that $\Gamma_a(u_0^t) = e^{-at} \Gamma_a(u_0) = 1$. Since $u_0^t \in B_a$ and $E_a(u_0^t) = e^{-at} E_a(u_0) < 0$, we have shown that

$$\inf_{w \in B_a} E_a(w) < 0.$$

As a consequence of the two inequalities in Lemma 2.1 and of (1-20), we obtain that E_a is bounded below on B_a . Therefore

$$-\infty < \inf_{w \in B_a} E_a(w) < 0,$$

as claimed. □

A special minimizing sequence. To establish that our constraint variational problem (1-21) achieves its infimum it will be important to work with the following type of minimizing sequences.

Lemma 2.3. *Let G satisfy (1-17), (1-18), and (1-20). Then, there exists a minimizing sequence $\{u_k\} \subset B_a$ of problem (1-21) such that, for all k , $u_k \in C_c(\mathbb{R}_+^2)$ has compact support in \mathbb{R}_+^2 and $0 \leq u_k \leq 1$.*

Proof. By the previous proposition we know that

$$-\infty < I_a := \inf_{w \in B_a} E_a(w) < 0.$$

Let $\{w_k\} \subset B_a$ be any minimizing sequence. Approximating each w_k by a function v_k in $C_c^\infty(\overline{\mathbb{R}_+^2})$ (by first truncating and then smoothing), we may assume that $\lim_{k \rightarrow +\infty} E_a(v_k) = I_a$ and $\Gamma_a(v_k) = 1 + \tau_k$ with $\tau_k \rightarrow 0$. Consider now $t_k = \log\{(1 + \tau_k)^{1/a}\}$. As a consequence of (1-23), we obtain $\Gamma_a(v_k^{t_k}) = 1$ — thus $\{v_k^{t_k}\} \subset B_a \cap C_c^\infty(\overline{\mathbb{R}_+^2})$ — and $\lim_{k \rightarrow +\infty} E_a(v_k^{t_k}) = I_a$.

Next, to show that we can restrict ourselves to minimizing sequences taking values in $[0, 1]$, let us rename $\{v_k^{t_k}\}$ by $\{v_k\}$. We truncate $\{v_k\}$ and define \tilde{v}_k by

$$\tilde{v}_k = \begin{cases} 0 & \text{if } v_k < 0, \\ v_k & \text{if } v_k \in [0, 1], \\ 1 & \text{if } v_k > 1. \end{cases}$$

It is easy to see that $\tilde{v}_k \in C_c(\overline{\mathbb{R}_+^2}) \cap H_a^1(\mathbb{R}_+^2)$, $\Gamma_a(\tilde{v}_k) \leq \Gamma_a(v_k)$ and, using (1-17), $E_a(\tilde{v}_k) \leq E_a(v_k)$.

Next we claim that we may choose $s_k \leq 0$ so that $u_k(x, y) := \tilde{v}_k(x, y + s_k)$ satisfies $\{u_k\} \subset B_a$. This claim follows from (1-23) and the fact that $0 < \Gamma_a(\tilde{v}_k) \leq 1$ for k large. To show this last assertion, note that $\Gamma_a(\tilde{v}_k) \leq 1$ is a consequence of $\Gamma_a(\tilde{v}_k) \leq \Gamma_a(v_k) = 1$. On the other hand, if $\Gamma_a(\tilde{v}_k) = 0$ then $\tilde{v}_k \equiv 0$ and thus $v_k \leq 0$. From this and (1-17), we would get $E_a(v_k) \geq 0$. This is a contradiction if k is large, since $I_a < 0$ and $\{v_k\}$ is a minimizing sequence.

Finally, since $s_k \leq 0$ and $E_a(\tilde{v}_k) \leq E_a(v_k) < 0$ for k large, (1-23) gives $E_a(u_k) \leq E_a(\tilde{v}_k) \leq E_a(v_k)$. Therefore, $\{u_k\} \subset B_a$ is a minimizing sequence made of continuous functions with compact support and satisfying $0 \leq u_k \leq 1$. □

Weak lower semicontinuity. Due to the unbounded character of \mathbb{R}_+^2 , a delicate issue in this paper is to prove the weak lower semicontinuity (WLSC) of E_a in B_a . The key point is to establish the WLSC result for the potential energy — the difficulty being that the potential $G(s)$ is negative near $s = 1$.

Note that for any sequence $\{u_k\} \in H_a^1(\mathbb{R}_+^2)$ converging weakly in $H_a^1(\mathbb{R}_+^2)$ to a function \underline{u} , $u_k \rightharpoonup \underline{u}$, we have $\Gamma_a(\underline{u}) \leq \liminf_{k \rightarrow \infty} \Gamma_a(u_k)$. Also, if we split $G = G^+ - G^-$ into its positive and negative parts, Fatou’s lemma gives

$$\int_{\partial \mathbb{R}_+^2} e^{ay} G^+(\underline{u}) \, dy \leq \liminf_{k \rightarrow \infty} \int_{\partial \mathbb{R}_+^2} e^{ay} G^+(u_k) \, dy.$$

Thus, we need to study the convergence of

$$\int_{\partial \mathbb{R}_+^2} e^{ay} G^-(u_k) \, dy.$$

To do this, the key observation (that already appears in [Heinze 2001]) is that any minimizing sequence cannot spend “too much time” (time meaning “positive y variable”) in $(\beta, 1)$, where G may be negative, even if the state $u = 1$ invades $u = 0$. This key fact will be a consequence of the presence of the weight e^{ay} .

We will see that, for any given $R > 0$ and any minimizing sequence $\{u_k\}$, the Lebesgue measure of the sets $\{y > R : u_k(0, y) \geq \beta\}$ and $\{x > 0, y > R : u_k(x, y) \geq \beta\}$ both decrease to zero as $R \rightarrow +\infty$, uniformly in k .

To proceed from this, our analysis must be more delicate than in [Heinze 2001] — which deals with cylinders — due to the unbounded character of \mathbb{R}_+^2 in the x variable. To handle this difficulty, we need to recall some facts about Riesz potentials; see [Gilbarg and Trudinger 1983, Section 7.8]. Let Ω be a bounded domain in \mathbb{R}^2 . Consider the operator on $L^2(\Omega)$ defined by

$$Vw(z) := \int_{\Omega} \frac{w(\bar{z})}{|z - \bar{z}|} d\bar{z}.$$

It is well known (see [Gilbarg and Trudinger 1983, Section 7.8]) that

$$\|Vw\|_{L^2(\Omega)} \leq 2\sqrt{\pi}|\Omega|^{1/2}\|w\|_{L^2(\Omega)}. \tag{2-2}$$

Next, we use this inequality to prove a proposition that will be important to control the superlevel sets of u_k mentioned above. The proposition will be applied to the functions $v = e^{ay/2}(u_k - \beta)^+$.

Proposition 2.4. *Given any constant $a > 0$, let $v \in C_c(\overline{\mathbb{R}_+^2}) \cap H_a^1(\mathbb{R}_+^2)$ have compact support with $\text{supp}(v) \subset \bar{\Omega}$ for some bounded domain $\Omega \subset \mathbb{R}_+^2$. For any $R > 0$, define*

$$\Omega_R := \Omega \cap \{(x, y) \in \mathbb{R}_+^2 : y > R\} \quad \text{and} \quad \partial^0\Omega_R := \overline{\Omega_R} \cap \partial\mathbb{R}_+^2.$$

Then, we have

$$\|v\|_{L^2(\Omega_R)} \leq \frac{4}{\sqrt{\pi}}|\Omega_R|^{1/2}\|\nabla v\|_{L^2(\Omega_R)} \quad \text{and} \quad \left(\int_{\partial^0\Omega_R} v^2(0, y) dy\right)^{1/2} \leq \frac{\sqrt{8}}{\sqrt[4]{\pi}}|\Omega_R|^{1/4}\|\nabla v\|_{L^2(\Omega_R)}.$$

Proof. By density, it is enough to consider $v \in C_c^\infty(\overline{\mathbb{R}_+^2})$. Since v has compact support, for $z \in \Omega_R$ and $\omega = (\omega_1, \omega_2) \in S^1$ with $\omega_1 > 0$ and $\omega_2 > 0$, we have

$$v(z) = - \int_0^\infty D_r v(z + r\omega) dr.$$

Integrating with respect to ω on the quarter of circle

$$S_+^1 := \{\omega \in S^1 : \omega_1 > 0 \text{ and } \omega_2 > 0\},$$

we obtain

$$v(z) = -\frac{2}{\pi} \int_0^\infty dr \int_{S_+^1} D_r v(z + r\omega) d\omega.$$

This leads to

$$|v(z)| \leq \frac{2}{\pi} \int_{\Omega_R} \frac{|\nabla v(\bar{z})|}{|z - \bar{z}|} d\bar{z}.$$

From this, the first inequality in Proposition 2.4 is now a consequence of (2-2).

As for the second inequality, we have

$$\begin{aligned} \int_{\partial^0 \Omega_R} v^2(0, y) dy &= - \int_{\Omega_R} (v^2)_x dx dy = -2 \int_{\Omega_R} v v_x dx dy \\ &\leq 2 \|v\|_{L^2(\Omega_R)} \|\nabla v\|_{L^2(\Omega_R)} \leq \frac{8}{\sqrt{\pi}} |\Omega_R|^{1/2} \|\nabla v\|_{L^2(\Omega_R)}, \end{aligned}$$

as claimed. □

Next, note that, for any fixed $R > 0$, the embedding

$$H_a^1(\mathbb{R}_+^2 \cap \{y < R\}) \cap L^\infty(\mathbb{R}_+^2 \cap \{y < R\}) \hookrightarrow L_a^2(\partial \mathbb{R}_+^2 \cap \{y < R\})$$

is compact (L_a^2 is the L^2 space for the measure $e^{ay} dy$). Indeed, a bounded sequence in $H_a^1(\mathbb{R}_+^2 \cap \{y < R\})$ is also bounded in $H^1((0, 1) \times (-M, R))$ for all $M > 0$. This last space is compactly embedded in $L^2(\partial \mathbb{R}_+^2 \cap \{-M < y < R\})$, and thus also in $L_a^2(\partial \mathbb{R}_+^2 \cap \{-M < y < R\})$. In addition, since the sequence of functions is bounded in $L^\infty(\partial \mathbb{R}_+^2)$, their $L_a^2(\partial \mathbb{R}_+^2 \cap \{-\infty < y < -M\})$ norms are as small as wished as $M \rightarrow \infty$ — since $e^{ay} \leq e^{-aM}$ in this set.

Thanks to the previous compact embedding, to achieve the desired WLSC result for E_a , it is enough to prove that

$$\int_R^{+\infty} e^{ay} G^-(u_k(0, y)) dy$$

can be made — uniformly on k — as small as we want, provided that R is large enough. This is the content of the next proposition.

Proposition 2.5. *Let G satisfy (1-19), i.e., $G \geq 0$ in $[0, \beta]$. Let $\{u_k\} \subset B_a \subset H_a^1(\mathbb{R}_+^2)$ be a minimizing sequence for $\inf\{E_a(w) : w \in B_a\}$ such that $u_k \in C_c(\overline{\mathbb{R}_+^2})$ and $0 \leq u_k \leq 1$ for all k .*

Then, given $\varepsilon > 0$, there exists $R > 0$ such that we have

$$\int_R^{+\infty} e^{ay} G^-(u_k(0, y)) dy \leq \varepsilon$$

for all k .

Proof. For $R > 0$, we define

$$A_k := \{(x, y) \in \mathbb{R}_+^2 : y > R, u_k(x, y) > \beta\}.$$

We can estimate the measures of $\partial^0 A_k$ and A_k respectively as follows (recall that the notation ∂^0 was introduced in Proposition 2.4). First,

$$e^{aR} |\partial^0 A_k| \leq \int_{\partial^0 A_k} e^{ay} dy \leq \frac{1}{\beta^2} \int_{\partial^0 A_k} e^{ay} u_k^2(0, y) dy \leq \frac{C}{\beta^2}.$$

The last inequality is a consequence of $\Gamma_a(u_k) = 1$ and the trace inequality in Lemma 2.1. In what follows, C denotes different constants depending only on a (and thus not on k). Therefore we have

$$|\partial^0 A_k| \leq \frac{C}{\beta^2} e^{-aR}$$

and

$$\int_{\partial^0 A_k} e^{ay} dy \leq \frac{C}{\beta^2}. \tag{2-3}$$

In an analogous way — integrating now on all of A_k and not on its boundary, and using again [Lemma 2.1](#) — we obtain that

$$|A_k| \leq \frac{C}{\beta^2} e^{-aR}. \tag{2-4}$$

By (1-19), there exists a constant C such that $G^-(s) \leq C(s - \beta)^+$ for $s \in [0, 1]$. This and $0 \leq u_k \leq 1$ lead to

$$\begin{aligned} \int_R^{+\infty} e^{ay} G^-(u_k(0, y)) dy &= \int_{\partial^0 A_k} e^{ay} G^-(u_k) dy \leq C \int_{\partial^0 A_k} e^{ay} (u_k - \beta) dy \\ &\leq C \left(\int_{\partial^0 A_k} e^{ay} (u_k - \beta)^2 dy \right)^{1/2} \left(\int_{\partial^0 A_k} e^{ay} dy \right)^{1/2}. \end{aligned}$$

Because of (2-3) above, the last factor is bounded by C/β , a constant independent of k . Using the second inequality in [Proposition 2.4](#) applied to the function $e^{ay/2}(u_k - \beta)^+$, we get

$$\int_R^{+\infty} e^{ay} G^-(u_k(0, y)) dy \leq \frac{C}{\beta} |A_k|^{1/4} \left(\int_{A_k} |\nabla(e^{ay/2}(u_k - \beta))|^2 dx dy \right)^{1/2}. \tag{2-5}$$

Using Cauchy–Schwartz and the trace inequality of [Lemma 2.1](#), we see that the integral on the right-hand side of (2-5) is bounded by a constant independent of k . In addition, as a consequence of inequality (2-4) we have that

$$\lim_{R \rightarrow \infty} |A_k| = 0.$$

Thus, the result follows from (2-5). □

We can now show that the infimum is achieved.

Corollary 2.6. *Let f be of positively balanced bistable type or of combustion type as in [Definition 1.1](#). Then, for every $a > 0$ small enough (depending only on f), there exists $\underline{u} \in B_a$ such that*

$$E_a(\underline{u}) = \inf_{w \in B_a} E_a(w).$$

In addition, $0 \leq \underline{u} \leq 1$, $|\{\underline{u}(0, y) : y \in \mathbb{R}\} \setminus [0, \beta]| > 0$, and \underline{u} is not identically constant.

Proof. For $a > 0$ small enough, [Proposition 2.2](#) shows that $-\infty < I_a < 0$, where

$$I_a := \inf_{w \in B_a} E_a(w). \tag{2-6}$$

By [Lemma 2.3](#), there exists $\{u_k\} \subset B_a \subset H_a^1(\mathbb{R}_+^2)$ such that $E_a(u_k) \rightarrow I_a$, $u_k \in C_c(\overline{\mathbb{R}_+^2})$, and $0 \leq u_k \leq 1$. Since $\{u_k\} \subset B_a$, by [Lemma 2.1](#) $\{u_k\}$ is bounded in $H_a^1(\mathbb{R}_+^2)$. Therefore, there exists a weakly convergent subsequence (still denoted by $\{u_k\}$) such that $u_k \rightharpoonup \underline{u}$ and $\underline{u} \in H_a^1(\mathbb{R}_+^2)$.

By the WLSC comments made on the beginning of this subsection on the kinetic energy and the potential energy corresponding to G^+ , and by the compactness result of [Proposition 2.5](#), we have that $E_a(\underline{u}) \leq \liminf_k E_a(u_k)$. Thus, we will have that \underline{u} is a minimizer if we show that

$$\underline{u} \in B_a.$$

To show this claim, recall that $\Gamma_a(\underline{u}) \leq \liminf_k \Gamma_a(u_k) = 1$. If $\Gamma_a(\underline{u}) = 0$ then $\underline{u} \equiv 0$; thus, in that case we would have $0 = E_a(\underline{u}) \leq \liminf_k E_a(u_k) = I_a < 0$, a contradiction. Hence, $\Gamma_a(\underline{u}) \in (0, 1]$.

Let us see now that $\Gamma_a(\underline{u}) \in (0, 1)$ is not possible either. Indeed, assume that $\Gamma_a(\underline{u}) < 1$. Then, for some $t < 0$, the function \underline{u}^t defined by $\underline{u}^t(x, y) := \underline{u}(x, y + t)$ satisfies $\Gamma_a(\underline{u}^t) = e^{-at}\Gamma_a(\underline{u}) = 1$, and hence $\underline{u}^t \in B_a$. In addition, $E_a(\underline{u}^t) = e^{-at}E_a(\underline{u}) < E_a(\underline{u}) = I_a$, which is a contradiction. Therefore, we have shown our claim $\underline{u} \in B_a$.

To prove the last statements of the corollary, since $0 \leq u_k \leq 1$ the same holds for \underline{u} . Moreover, since $\underline{u} \in B_a$, \underline{u} is not identically constant. Finally, if we had $|\{\underline{u}(0, y) : y \in \mathbb{R}\} \setminus [0, \beta]| = 0$, then $E_a(\underline{u}) \geq 0$ by (1-19) and this is a contradiction. \square

Solving the PDE. In this part we show that there exists a solution pair (c, u) to (1-2) with $c > 0$ and u not identically constant. The solution is constructed from a minimizer \underline{u} of our variational problem, after scaling its independent variables (x, y) to take care of a Lagrange multiplier λ_a . The speed turns out to be $c = a(1 - 2I_a) = a(1 - 2\lambda_a)$; see (2-11).

Proposition 2.7. *Let f be of positively balanced bistable type or of combustion type as in [Definition 1.1](#). Let \underline{u} be a minimizer for problem (1-21) as given by [Corollary 2.6](#). Then, there exists $c > 0$ and $\mu > 0$ such that, defining*

$$u(x, y) = \underline{u}(\mu x, \mu y),$$

we have that (c, u) is a solution pair for problem (1-2), u is not identically constant, $0 \leq u \leq 1$, and $u \in H_c^1(\mathbb{R}_+^2)$.

Proof. Let $\underline{u} \in B_a$ be a minimizer as in [Corollary 2.6](#). We have that $D\Gamma_a(\underline{u}) \neq 0$ because

$$D\Gamma_a(\underline{u}) \cdot \underline{u} = \int_{\mathbb{R}_+^2} 2e^{ay} |\nabla \underline{u}|^2 dx dy = 2.$$

Therefore, there exists a Lagrange multiplier $\lambda_a \in \mathbb{R}$ such that $DE_a(\underline{u}) \cdot \phi = \lambda_a D\Gamma_a(\underline{u}) \cdot \phi$ for all $\phi \in H_a^1(\mathbb{R}_+^2)$, that is,

$$(1 - 2\lambda_a) \int_{\mathbb{R}_+^2} e^{ay} \nabla \underline{u} \nabla \phi dx dy - \int_{\partial \mathbb{R}_+^2} e^{ay} f(\underline{u}(0, y)) \phi(0, y) dy = 0. \tag{2-7}$$

Let us see that $\lambda_a \neq \frac{1}{2}$. Indeed, otherwise, from (2-7) we deduce $f(\underline{u}(0, \cdot)) \equiv 0$ in \mathbb{R} . Thus, by assumption (1-7) on f , we would have that either $\underline{u}(0, \cdot) \equiv 1$ or that $0 \leq \underline{u}(0, \cdot) \leq \beta$. The first of these is not possible since $\underline{u} \equiv 1 \notin H_a^1(\partial \mathbb{R}_+^2)$. On the other hand, $0 \leq \underline{u}(0, \cdot) \leq \beta$ is ruled out by the last statement of [Corollary 2.6](#).

Let us consider arbitrary functions $\varphi \in C_c^\infty(\mathbb{R}_+^2)$ vanishing on $\partial\mathbb{R}_+^2$, and also functions $\psi \in C_c^\infty(\overline{\mathbb{R}_+^2})$. From (2-7) and $\lambda_a \neq \frac{1}{2}$, we have

$$\int_{\mathbb{R}_+^2} e^{ay} \{ \Delta \underline{u} + a\underline{u}_y \} \varphi \, dx \, dy = 0 \quad \text{and} \quad \int_{\partial\mathbb{R}_+^2} e^{ay} \{ (1 - 2\lambda_a)\underline{u}_x + f(\underline{u}) \} \psi \, dy = 0.$$

As a consequence, and since $\lambda_a \neq \frac{1}{2}$, the pair (a, \underline{u}) is a solution of

$$\begin{cases} \Delta \underline{u} + a\underline{u}_y = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial \underline{u}}{\partial \nu} = \frac{1}{1 - 2\lambda_a} f(\underline{u}) & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Let us now show that $\lambda_a < \frac{1}{2}$. Consider the test function $(\underline{u} - \beta)^+ \in H_a^1(\mathbb{R}_+^2)$. Plugging it into (2-7), we get

$$(1 - 2\lambda_a) \int_{\{\underline{u} > \beta\}} e^{ay} |\nabla \underline{u}|^2 \, dx \, dy - \int_{\{\underline{u}(0, \cdot) > \beta\}} e^{ay} f(\underline{u}(0, y)) (\underline{u}(0, y) - \beta) \, dy = 0. \tag{2-8}$$

Recall that $|\{\underline{u}(0, y) : y \in \mathbb{R}\} \setminus [0, \beta]| > 0$, and thus

$$\int_{\{\underline{u} > \beta\}} e^{ay} |\nabla \underline{u}|^2 \, dx \, dy > 0. \tag{2-9}$$

Since $f(\underline{u}(0, y))(\underline{u}(0, y) - \beta) > 0$ in $\{\underline{u} > \beta\}$ by (1-7), (2-8) and (2-9) lead to $1 - 2\lambda_a > 0$.

Let $\mu := 1 - 2\lambda_a > 0$ and define

$$u(x, y) := \underline{u}(\mu x, \mu y) \quad \text{and} \quad c := a(1 - 2\lambda_a) > 0. \tag{2-10}$$

We then have a solution pair (c, u) for (1-2).

Note that $u \in H_c^1(\mathbb{R}_+^2)$ since

$$\int_{\mathbb{R}_+^2} e^{cy} \{ |\nabla u|^2 + u^2 \} \, dx \, dy = \int_{\mathbb{R}_+^2} e^{a\bar{y}} \{ |\nabla \underline{u}|^2 + \mu^{-2} \underline{u}^2 \} \, d\bar{x} \, d\bar{y} < \infty$$

and $\underline{u} \in H_a^1(\mathbb{R}_+^2)$.

Finally, since $f \in C^{1,\gamma}$, the weak solution that we have found can be shown to be classical, indeed $C^{2,\gamma}$ in all $\overline{\mathbb{R}_+^2}$. This is explained in the beginning of Section 4. □

Remark 2.8. It is interesting to note the following relation, already noted in [Heinze 2001], between the infimum value I_a of our problem (2-6) and the speed c of the traveling front. The formula, which is not strictly needed anywhere else in this paper, provides however with an alternative proof of part (iv) of Theorem 1.2 on the comparison of the front speeds for different nonlinearities.

We claim that

$$c = a(1 - 2I_a) = a(1 - 2\lambda_a), \tag{2-11}$$

where a and λ_a are the parameter and the multiplier in the proof of Proposition 2.7. To show this formula, we take a minimizing sequence $\{u_k\}$ made of C^∞ functions with compact support, and we test (2-7) with

$\phi = \partial_y u_k \in H_a^1(\mathbb{R}_+^2)$. Integrating by parts in order to pass to the limit as $k \rightarrow \infty$, we obtain

$$\begin{aligned} 0 &= (1 - 2\lambda_a) \int_{\mathbb{R}_+^2} e^{ay} \partial_y \frac{|\nabla \underline{u}|^2}{2} dx dy + \int_{\partial \mathbb{R}_+^2} e^{ay} \partial_y G(\underline{u}(0, y)) dy \\ &= -a \frac{1 - 2\lambda_a}{2} - a \int_{\partial \mathbb{R}_+^2} e^{ay} G(\underline{u}(0, y)) dy, \end{aligned}$$

where in the last equality we have also integrated by parts. We deduce that

$$I_a = E_a(\underline{u}) = \frac{1}{2} \Gamma_a(\underline{u}) - \frac{1}{2} (1 - 2\lambda_a) = \frac{1}{2} - \frac{1}{2} (1 - 2\lambda_a) = \lambda_a,$$

which together with (2-10) shows the claim.

3. Monotonicity

In this section we show that the front u built in the previous section can be taken to be nonincreasing in the y variable. This fact will be crucial to show in the next section that such a nonincreasing front u has limits 1 and 0 as $y \rightarrow \mp \infty$.

Note that it suffices to show the existence of a nonincreasing minimizer, since the scaling used in the proof of Proposition 2.7 does not change the monotonicity of the front. As in [Heinze 2001], the existence of a nonincreasing minimizer will be a consequence of an inequality for monotone decreasing rearrangements in a new variable z , defined by

$$z = \frac{e^{ay}}{a}.$$

Proposition 3.1. *The minimizer \underline{u} of Corollary 2.6 can be taken to be nonincreasing in the y variable.*

Proof. We follow ideas in [Heinze 2001] and perform the change of variables $(x, z) := (x, e^{ay}/a)$, which takes \mathbb{R}_+^2 into $(\mathbb{R}_+)^2 = \{(x, z) : x > 0, z > 0\}$, and the functionals Γ_a, E_a , into $\tilde{\Gamma}_a, \tilde{E}_a$, where

$$\tilde{\Gamma}_a(v) := \iint_{(\mathbb{R}_+)^2} \{|\partial_x v|^2 + a^2 z^2 |\partial_z v|^2\} dx dz \tag{3-1}$$

and

$$\tilde{E}_a(v) := \frac{1}{2} \tilde{\Gamma}_a(v) + \int_0^{+\infty} G(v(0, z)) dz.$$

Let $\{u_k\}$ be the minimizing sequence for problem (1-21) given by Lemma 2.3. The functions $\{u_k\}$ take values in $[0, 1]$, are continuous, and have compact support in \mathbb{R}_+^2 . Let v_k be defined by $v_k(x, z) := u_k(x, y)$. Since v_k is nonnegative, continuous, and with compact support in $[0, +\infty)^2$, we may consider its monotone decreasing rearrangement in the z variable, that we denote by v_k^* ; see [Kawohl 1985]. That is, for each $x \geq 0$, we make the usual one-dimensional monotone decreasing rearrangement of the function $v_k(x, \cdot)$ of $z > 0$. Recall also that if we consider the even extension of v_k across $\{z = 0\}$, then v_k^* coincides with the Steiner symmetrization of v_k with respect to $\{z = 0\}$.

As a consequence of equimeasurability, we have

$$\int_0^{+\infty} G(v_k^*(0, z)) dz = \int_0^{+\infty} G(v_k(0, z)) dz.$$

On the other hand, the inequality

$$\tilde{\Gamma}_a(v_k^*) \leq \tilde{\Gamma}_a(v_k)$$

— and thus $\tilde{E}_a(v_k^*) \leq \tilde{E}_a(v_k)$ — follows from a result of Landes [2007] for monotone decreasing rearrangements since the weight $w(x, z) = a^2z^2$ in (3-1) is nonnegative and nondecreasing in $z \in (0, +\infty)$. It also follows from a previous result of Brock [1999] on Steiner symmetrization, which requires w to be nonnegative and $w^{1/2}(x, z) = a|z|$ to be even and convex. These results require that the weight in front of $|\partial_x v|^2$ (which in our case is identically one) does not depend on z .

Finally, we pull back the sequence v_k^* to the (x, y) variables and name these functions u_k^* . We have that

$$\Gamma_a(u_k^*) = \tilde{\Gamma}_a(v_k^*) \leq \tilde{\Gamma}_a(v_k) = 1$$

and

$$E_a(u_k^*) = \tilde{E}_a(v_k^*) \leq \tilde{E}_a(v_k) = E_a(u_k).$$

Let \underline{u}^* be a weak limit in $H_a^1(\mathbb{R}_+^2)$ of a subsequence of $\{u_k^*\}$. By the WLSC results of the previous section, it is easy to prove that we necessarily have $\underline{u}^* \in B_a$. This is done exactly as in the proof of Corollary 2.6. Thus, \underline{u}^* is a minimizer which is nonincreasing in the y variable. Note also that it still takes values in $[0, 1]$. □

4. Limits at infinity

In this section we prove that the front u for (1-2) constructed in the previous sections satisfies

$$\lim_{y \rightarrow -\infty} u(x, y) = 1 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u(x, y) = 0 \quad \text{for all } x \geq 0, \tag{4-1}$$

and

$$\lim_{x \rightarrow +\infty} u(x, y) = 0 \quad \text{for all } y \in \mathbb{R}. \tag{4-2}$$

To establish (4-1), it will be crucial to use that u is nonincreasing in the y variable.

In what follows, we will be using the following regularity fact. Assume that u is a bounded C^2 function in \mathbb{R}_+^2 , C^1 up to the boundary $\partial\mathbb{R}_+^2$, that satisfies our nonlinear problem

$$\begin{cases} \Delta u + cu_y = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Since f is $C^{1,\gamma}$ for some $\gamma \in (0, 1)$, we have that, for every $R > 0$, $u \in C^{2,\gamma}(\overline{B_R^+})$ and

$$\|u\|_{C^{2,\gamma}(\overline{B_R^+})} \leq CR \tag{4-3}$$

for some constant C_R depending only on c, γ, R , and on upper bounds for $\|u\|_{L^\infty(B_{4R}^+)}$ and $\|f\|_{C^{1,\gamma}}$. Here $B_R^+ = \{(x, y) \in \mathbb{R}^2 : x > 0, |(x, y)| < R\}$. This estimate is established by easily adapting the proof of [Cabré and Solà-Morales 2005, Lemma 2.3(a)]. As a consequence of the estimate, we also deduce that

$$|\nabla u| \in L^\infty(\mathbb{R}_+^2). \tag{4-4}$$

To establish (4-1), we first need the following easy result on limits as $|y| \rightarrow \infty$. It applies to any solution, not only to the variational one constructed in previous sections.

Lemma 4.1. *Assume that $f(0) = f(1) = 0$ and that $0 \leq u \leq 1$ is a solution of (1-2) satisfying*

$$\lim_{y \rightarrow -\infty} u(0, y) = 1 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u(0, y) = 0.$$

Then, for all $R > 0$, we have

$$\lim_{y \rightarrow -\infty} u(x, y) = 1, \quad \lim_{y \rightarrow +\infty} u(x, y) = 0, \quad \text{and} \quad \lim_{|y| \rightarrow \infty} |\nabla u(x, y)| = 0 \tag{4-5}$$

uniformly for $x \in [0, R]$.

Proof. For $t \in \mathbb{R}$ let us define $u^t(x, y) := u(x, y + t)$, also a solution of (1-2). We claim that

$$\|u^t - 1\|_{L^\infty(B_R^+)} + \|\nabla u^t\|_{L^\infty(B_R^+)} \rightarrow 0 \quad \text{as } t \rightarrow -\infty.$$

Assume, to the contrary, that there exist $\varepsilon > 0$ and $\{t_k\} \subset \mathbb{R}$ with $t_k \rightarrow -\infty$ such that

$$\|u^{t_k} - 1\|_{L^\infty(B_R^+)} + \|\nabla u^{t_k}\|_{L^\infty(B_R^+)} \geq \varepsilon. \tag{4-6}$$

The estimates (4-3) lead to the existence of a subsequence $\{t_{k_j}\}$ for which $u^{t_{k_j}}$ converges in $C^2(\overline{B_R^+})$ to u^∞ . By the hypothesis of the lemma, we will have $0 \leq u^\infty \leq 1$ and

$$\begin{cases} \Delta u^\infty + cu_y^\infty = 0 & \text{in } \mathbb{R}_+^2, \\ u^\infty = 1 & \text{on } \partial\mathbb{R}_+^2, \\ \frac{\partial u^\infty}{\partial \nu} = f(u^\infty) = f(1) = 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

From this and Hopf’s boundary lemma, we deduce $u^\infty \equiv 1$ on \mathbb{R}_+^2 , which contradicts (4-6).

In an analogous way we can show the limits as $y \rightarrow +\infty$. □

We can now prove the existence of limits as $y \rightarrow \pm\infty$ for the variational solution constructed in the last sections.

Lemma 4.2. *Let f be of positively balanced bistable type or of combustion type as in Definition 1.1. Let u be any front constructed as in Proposition 2.7 from the nonincreasing minimizer of Proposition 3.1. Then,*

$$\lim_{y \rightarrow -\infty} u(x, y) = 1 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u(x, y) = 0 \quad \text{for all } x \geq 0.$$

Proof. By [Corollary 2.6](#) we know that $0 \leq u \leq 1$, $u \in H_c^1(\mathbb{R}_+^2)$, and that the set $\{u(0, y) : y \in \mathbb{R}\}$ is not contained in $[0, \beta]$. We also know that $u_y(x, y) \leq 0$. Therefore, for all $x \geq 0$, there exist $L^-(x) \in (\beta, 1]$ and $L^+(x) \in [0, 1]$ such that

$$\lim_{y \rightarrow -\infty} u(x, y) = L^-(x) \quad \text{and} \quad \lim_{y \rightarrow +\infty} u(x, y) = L^+(x).$$

Note that $L^+(x) \equiv 0$ is a consequence of the inequalities of [Lemma 2.1](#).

To prove that $L^-(x) \equiv 1$, by [Lemma 4.1](#) it is enough to show that $L^-(0) = 1$. To do this, we consider the sequence of solutions $\{u^k\}$ defined by $u^k(x, y) := u(x, y + k)$. Then, as in the proof of [Lemma 4.1](#), as $k \rightarrow -\infty$ there exists a convergent subsequence to a solution u^∞ of

$$\begin{cases} \Delta u^\infty + cu_y^\infty = 0 & \text{in } \mathbb{R}_+^2, \\ u^\infty = L^-(0) & \text{on } \partial\mathbb{R}_+^2, \\ \frac{\partial u^\infty}{\partial \nu} = f(L^-(0)) & \text{on } \partial\mathbb{R}_+^2. \end{cases} \tag{4-7}$$

Since $u^\infty(x, y) = L^-(x)$, we have that $\partial_y u^\infty(x, y) = \partial_{yy} u^\infty(x, y) \equiv 0$. Therefore, the first equation in [\(4-7\)](#) leads to $\partial_{xx} u^\infty(x, y) = 0$ for all $x > 0$ and $y \in \mathbb{R}$. Since u^∞ is bounded, then it must be constant and equal to $L^-(0)$, its value at $x = 0$.

This and the last equation in [\(4-7\)](#) lead to $0 = -\partial_x u^\infty(0, y) = f(L^-(0))$. Since $L^-(0) \in (\beta, 1]$ and, on this interval, f vanishes only at 1 by hypothesis [\(1-7\)](#), we conclude that $L^-(0) = 1$. □

It remains to prove [\(4-2\)](#) on the limits as $x \rightarrow +\infty$. This is a simple consequence of the Harnack inequality and the fact that the variational solution lies in $H_c^1(\mathbb{R}_+^2)$:

Lemma 4.3. *Let f be of positively balanced bistable type or of combustion type as in [Definition 1.1](#). Let u be any front constructed as in [Proposition 2.7](#). Then,*

$$\lim_{x \rightarrow +\infty} u(x, y) = 0 \quad \text{for all } y \in \mathbb{R}.$$

Proof. Take any $y_0 \in \mathbb{R}$. Since $u \in H_c^1(\mathbb{R}_+^2)$ we have that

$$\lim_{x_0 \rightarrow +\infty} \int_{y_0-1}^{y_0+1} dy \int_{x_0-1}^{+\infty} e^{cy} u^2 dx = 0. \tag{4-8}$$

Recall that $0 \leq u \leq 1$ satisfies $\Delta u + cu_y = 0$ in \mathbb{R}_+^2 . Thus, by the Harnack inequality, for any $x_0 > 2$ we have

$$\sup_{B_1(x_0, y_0)} u \leq C \inf_{B_1(x_0, y_0)} u \leq C \int_{B_1(x_0, y_0)} u dx dy \leq C \left(\int_{B_1(x_0, y_0)} u^2 dx dy \right)^{1/2}$$

for different constants C independent of x_0 . Using [\(4-8\)](#), it follows that

$$\lim_{x_0 \rightarrow +\infty} u(x_0, y_0) = 0,$$

as claimed. □

5. Uniqueness of speed and of solution with limits

In the first part of this section we establish a useful comparison principle, [Proposition 5.2](#), in the spirit of one in [\[Cabr e and Sol a-Morales 2005\]](#). It will lead first to the asymptotic bounds on fronts stated in our main theorem (after building appropriate comparison barriers in next section). Then, it will be used in the second part of this section to establish a key result, [Proposition 5.3](#) below.

[Proposition 5.3](#) will have several important applications: first, the monotonicity in y of every solution with limits; second, the uniqueness of a speed and of a front with limits; and third, the comparison result between speeds corresponding to different ordered nonlinearities. The proof of the proposition follows the powerful sliding method of Berestycki and Nirenberg [\[1991\]](#).

A maximum principle. We start with the following easy lemma:

Lemma 5.1. *Let w be a C^2 function in \mathbb{R}_+^2 , bounded below, continuous up to $\partial\mathbb{R}_+^2$, and satisfying*

$$\Delta w + cw_y \leq 0 \quad \text{in } \mathbb{R}_+^2$$

for some constant $c \in \mathbb{R}$. Assume also that $w \geq 0$ on $\partial\mathbb{R}_+^2$ and that, for every $R > 0$,

$$\liminf_{|y| \rightarrow +\infty} w(x, y) \geq 0 \quad \text{uniformly in } x \in [0, R]. \tag{5-1}$$

Then, $w \geq 0$ in \mathbb{R}_+^2 .

Proof. Consider the new function

$$\bar{w} = \frac{w}{x+1} \quad \text{for } x \geq 0, y \in \mathbb{R}.$$

It satisfies

$$\Delta \bar{w} + \frac{2}{x+1} \bar{w}_x + c\bar{w}_y \leq 0 \quad \text{in } \mathbb{R}_+^2.$$

Let $\varepsilon > 0$. Since w is bounded below, if R is sufficiently large we have that

$$\bar{w}(x, y) \geq -\varepsilon \tag{5-2}$$

for $x = R$. By assumption [\(5-1\)](#), we also have [\(5-2\)](#) for $x \in [0, R]$ and $|y| = S$, if S is large enough (depending on R). Since, [\(5-2\)](#) also holds for $x = 0$, the maximum principle applied in $(0, R) \times (-S, S)$ gives that $\bar{w} \geq -\varepsilon$ in $(0, R) \times (-S, S)$.

Letting $S \rightarrow \infty$ we deduce that $\bar{w} \geq -\varepsilon$ in $(0, R) \times \mathbb{R}$. Now, letting $R \rightarrow \infty$ we conclude that $\bar{w} \geq -\varepsilon$ in \mathbb{R}_+^2 for any $\varepsilon > 0$. Thus $\bar{w} \geq 0$ in \mathbb{R}_+^2 and this finishes the proof. \square

The following maximum principle (in the spirit of one in [\[Cabr e and Sol a-Morales 2005\]](#)) is a key ingredient in the remaining of this section. It will be applied to the difference of two solutions (and also of a supersolution and a solution) of our nonlinear problem.

Proposition 5.2. *Let $c \in \mathbb{R}$ and v be a C^2 bounded function in $\overline{\mathbb{R}_+^2}$ satisfying*

$$\Delta v + cv_y \leq 0 \quad \text{in } \mathbb{R}_+^2$$

and that, for all $R > 0$,

$$\lim_{|y| \rightarrow \infty} v(x, y) = 0 \quad \text{uniformly in } x \in [0, R]. \tag{5-3}$$

Finally, assume that there exists a nonempty set $H \subset \mathbb{R}$ such that $v(0, y) > 0$ for $y \in H$,

$$\frac{\partial v}{\partial \nu} + d(y)v \geq 0 \quad \text{if } y \notin H \tag{5-4}$$

and

$$d(y) \geq 0 \quad \text{if } y \notin H \tag{5-5}$$

for some continuous function d defined on $\mathbb{R} \setminus H$.

Then, $v > 0$ in $\overline{\mathbb{R}^2_+}$.

Proof. We need to prove that $v \geq 0$ in $\overline{\mathbb{R}^2_+}$. It then follows that $v > 0$ in $\overline{\mathbb{R}^2_+}$: Indeed, since H is nonempty, v cannot be identically zero. If we assume that $v = 0$ at some point $(x_1, y_1) \in \overline{\mathbb{R}^2_+}$, we obtain a contradiction using the strong maximum principle (if $x_1 > 0$) and using the Hopf’s boundary lemma and (5-4) (if $x_1 = 0$, since then $y_1 \notin H$ because $v(0, y_1) = 0$).

Let

$$A := \inf_{\partial \mathbb{R}^2_+} v.$$

By (5-3) used with $x = 0$, we have $A \leq 0$. Thus, we can apply Lemma 5.1 to $w := v - A$. We deduce that $v \geq A$ in all of \mathbb{R}^2_+ .

It only remains to prove that $A \geq 0$. By contradiction, assume that $A < 0$. Then, by its definition and since $v(0, y) \rightarrow 0$ as $|y| \rightarrow \infty$, we have that the infimum A of v on $\partial \mathbb{R}^2_+$ is achieved at some point $(0, y_0)$. Since we have proved that $v \geq A$ in all of \mathbb{R}^2_+ , then $(0, y_0)$ is also a minimum of v in all $\overline{\mathbb{R}^2_+}$. Since $v(0, y_0) = A < 0$, v is not identically constant, and thus the Hopf’s boundary lemma gives that $-v_x(0, y_0) < 0$. This is a contradiction with (5-4) and (5-5) — since $y_0 \notin H$ because $v(0, y_0) < 0$. \square

Uniqueness. The goal of this section is to establish uniqueness of the traveling speed, as well as uniqueness — up to vertical translations — of solutions to (1-2) which have limits 1 and 0 as $y \rightarrow \mp\infty$ on $\partial \mathbb{R}^2_+$.

We also prove in this section that every solution with the above limits is necessarily decreasing in y .

All these three results will follow from the following proposition — an analogue of Lemma 5.2 in [Cabré and Solà-Morales 2005].

Proposition 5.3. *Assume that f satisfies (1-4) and (1-5), and let $c \in \mathbb{R}$. Let u_1 and u_2 be, respectively, a supersolution and a solution of (1-2) such that*

$$0 \leq u_i \leq 1 \quad \text{and} \quad u_i(0, 0) = \frac{1}{2}$$

for $i = 1, 2$. Assume that, for $i = 1, 2$ and all $R > 0$,

$$\lim_{y \rightarrow -\infty} u_i(x, y) = 1 \quad \text{and} \quad \lim_{y \rightarrow +\infty} u_i(x, y) = 0 \quad \text{uniformly in } x \in [0, R]. \tag{5-6}$$

For $t > 0$, consider

$$u_2^t(x, y) := u_2(x, y + t).$$

Then,

$$u_2^t \leq u_1 \quad \text{in } \overline{\mathbb{R}_+^2} \quad \text{for every } t > 0. \tag{5-7}$$

As a consequence, $u_2 \equiv u_1$ in $\overline{\mathbb{R}_+^2}$.

In addition, for any solution u_2 satisfying $0 \leq u_2 \leq 1$ and (5-6), we have $\partial_y u_2 < 0$ in $\overline{\mathbb{R}_+^2}$.

If we apply the proposition to $u_1 = u_2 = u$, where u is a solution to (1-2) taking values in $[0, 1]$ and with limits 1 and 0, the conclusion (5-7) establishes that u is nonincreasing in y . From this, the strong maximum principle and Hopf’s boundary lemma applied to the linearized problem satisfied by u_y , we deduce that $u_y < 0$, as claimed in the last statement of the proposition.

Second, by letting $t \rightarrow 0^+$ in (5-7) we deduce that $u_2 \leq u_1$. But u_2 is a solution and u_1 a supersolution, with $u_2(0, 0) = u_1(0, 0)$. Again the strong maximum principle and the Hopf’s boundary lemma give that $u_2 \equiv u_1$, as stated in the proposition.

The proposition also gives the uniqueness (up to vertical translations) of a solution with limits for a given speed c . For this, apply the proposition to two solutions after translating them in the y direction.

In the proof of our main theorem in Section 7, we will give two other important applications of the proposition. First, the uniqueness of a speed admitting a solution with limits. This will follow from the fact that any front u_1 with limits 1 and 0 is necessarily decreasing, and hence the terms $c_1 \partial_y u_1$ and $c_2 \partial_y u_1$ will be comparable for two different speeds. Since one of the functions in the proposition may be taken to be only a supersolution, this will lead to the uniqueness of speed. A similar argument will show the comparison of speeds corresponding to two different ordered nonlinearities.

Proof of Proposition 5.3. As explained above, we only need to prove (5-7). The subsequent statements follow easily from this.

Note first that u_i are not identically constant, by the assumption in (5-6) about their limits as $y \rightarrow \pm\infty$. Therefore, since $0 \leq u_i \leq 1$ and $f(0) = f(1) = 0$, the strong maximum principle leads to $0 < u_i < 1$ for $i = 1, 2$.

Let $\delta > 0$ be the constant in assumption (1-5) for f . By hypothesis (5-6), there exists a compact interval $[a, b]$ in \mathbb{R} such that, for $i = 1, 2$,

$$\begin{aligned} u_i(0, y) &\in (1 - \delta, 1) && \text{if } y \leq a, && \text{and} \\ u_i(0, y) &\in (0, \delta) && \text{if } y \geq b. \end{aligned}$$

Note that u_2^t is also a solution of (1-2), and hence

$$\begin{cases} \Delta(u_1 - u_2^t) + c(u_1 - u_2^t)_y \leq 0 & \text{in } \mathbb{R}_+^2, \\ -(u_1 - u_2^t)_x \geq -d^t(y)(u_1 - u_2^t) & \text{on } \partial\mathbb{R}_+^2, \end{cases}$$

where

$$d^t(y) = -\frac{f(u_1) - f(u_2^t)}{u_1 - u_2^t}(0, y)$$

if $(u_1 - u_2^t)(0, y) \neq 0$, and $d^t(y) = -f'(u_1(0, y)) = -f'(u_2^t(0, y))$ otherwise. Note that d^t is a continuous function since f is C^1 .

We also have, for all $R > 0$,

$$\lim_{|y| \rightarrow \infty} (u_1 - u_2^t)(x, y) = 0 \quad \text{uniformly in } x \in [0, R].$$

We finish the proof in three steps.

Step 1. We claim that $u_2^t < u_1$ in $\overline{\mathbb{R}_+^2}$ for $t > 0$ large enough.

To show this, we take $t > 0$ sufficiently large so that $u_2^t(0, y) < u_1(0, y)$ for $y \in [a, b]$. This is possible since $u_2(0, y + t) \rightarrow 0$ as $t \rightarrow +\infty$ and $u_1 > 0$. We apply [Proposition 5.2](#) to $v = u_1 - u_2^t$ with

$$H = (a, b) \cup \{y \in \mathbb{R} : (u_1 - u_2^t)(0, y) > 0\}.$$

Clearly, $v(0, y) > 0$ in H .

To show that $d^t \geq 0$ in $\mathbb{R} \setminus H$, let $y \notin H$. There are two possibilities. First, if $y \geq b$ then $y + t \geq b$ also. Therefore, $u_1(0, y) \leq \delta$ and $u_2^t(0, y) = u_2(0, y + t) \leq \delta$. We conclude that $d^t(y) \geq 0$, since $f' \leq 0$ in $(0, \delta)$ by [\(1-5\)](#).

The other possibility is that $y \leq a$. In this case, we have $u_1(0, y) \geq 1 - \delta$, and since $y \notin H$ then $(u_1 - u_2^t)(0, y) \leq 0$. Therefore $u_2^t(0, y) \geq u_1(0, y) \geq 1 - \delta$, and we conclude $d^t(y) \geq 0$, again by [\(1-5\)](#).

[Proposition 5.2](#) gives that $u_1 - u_2^t > 0$ in $\overline{\mathbb{R}_+^2}$.

Claim. *If $u_2^t \leq u_1$ for some $t > 0$, then $u_2^{t+\mu} \leq u_1$ for every μ small enough (with μ either positive or negative).*

This statement will finish the proof of the proposition, since then $\{t > 0 : u_2^t \leq u_1\}$ is a nonempty, closed and open set in $(0, \infty)$, and hence equal to this interval. We conclude $u_2^t \leq u_1$ for all $t > 0$.

Step 2. To prove the claim, we will show that

$$\text{if } t > 0 \text{ and } u_2^t \leq u_1, \text{ then } u_2^t \not\equiv u_1. \tag{5-8}$$

Once [\(5-8\)](#) is known, we can finish the proof of the claim as follows: First, by the strong maximum principle and Hopf's boundary lemma, $u_2^t < u_1$ in $\overline{\mathbb{R}_+^2}$. Let K_t be a compact interval such that, on $\mathbb{R} \setminus K_t$, both u_1 and u_2^t take values in $(0, \delta/2) \cup (1 - \delta/2, 1)$. Recall that $(u_1 - u_2^t)(0, \cdot) > 0$ in the compact set K_t . By continuity and the existence of limits at infinity, we have that, if $|\mu|$ is small enough, then $(u_1 - u_2^{t+\mu})(0, y) > 0$ for $y \in K_t$ and $u_2^{t+\mu}(0, y)$ takes values in $(0, \delta) \cup (1 - \delta, 1)$ for $y \notin K_t$. Hence, we can apply [Proposition 5.2](#) to $v = u_1 - u_2^{t+\mu}$ with $H = K_t$, since $d^{t+\mu} \geq 0$ outside K_t . We conclude that $u_1 - u_2^{t+\mu} > 0$ in $\overline{\mathbb{R}_+^2}$.

Step 3. Here we establish [\(5-8\)](#), therefore completing the proof of the claim and of the proposition. We assume that $t > 0$ and $u_2^t \leq u_1$, and we need to show that $u_2^t \not\equiv u_1$.

To prove this, consider first the case when both functions in the proposition are the same, that is, $u_1 \equiv u_2$. Assume that $t > 0$ and $u_2^t \equiv u_1 \equiv u_2$. Then, the function $u_2(0, y)$ is t -periodic. But this is a contradiction with the hypothesis [\(5-6\)](#) on limits. Therefore, in the case $u_1 \equiv u_2$, the two steps above can be carried out. We conclude that, for every solution u_2 as in the lemma, we have $u_2^t \leq u_2$ for every $t > 0$. In particular, $\partial_y u_2 \leq 0$ and, by the strong maximum principle and Hopf's boundary lemma, $\partial_y u_2 < 0$.

Finally, consider the general case of a supersolution u_1 and a solution u_2 . Assume that $t > 0$ and $u_2^t \equiv u_1$. Then $\frac{1}{2} = u_1(0, 0) = u_2^t(0, 0) = u_2(0, t)$. Moreover, $u_2(0, 0) = \frac{1}{2}$ by hypothesis. Hence, $u_2(0, 0) = u_2(0, t)$. This is a contradiction, since in the previous paragraph we have established that u_2 is decreasing in y . \square

6. Explicit traveling fronts

In this section we construct an explicit supersolution of the linearized problem for (1-2) in the case of positively balanced bistable nonlinearities satisfying

$$f'(0) < 0 \quad \text{and} \quad f'(1) < 0. \tag{6-1}$$

It will lead to our result on the asymptotic behavior of traveling fronts. In addition, we construct a family of explicit traveling fronts corresponding to some positively balanced bistable nonlinearities satisfying (6-1).

To simplify the notation in this section, by rescaling the independent variables we may assume that the speed of the front is

$$c = 2.$$

Recall that our nonlinear problem, when written for the trace $v = v(y)$ of functions on $x = 0$, becomes (1-14) with $c = 2$, i.e.,

$$(-\partial_{yy} - 2\partial_y)^{1/2}v = f(v) \quad \text{in } \mathbb{R}.$$

As in [Cabré and Sire 2015], the construction of explicit fronts will be based on the fundamental solution for the homogeneous heat equation associated to the previous fractional operator in \mathbb{R} , that is, equation

$$\partial_t v + (-\partial_{yy} - 2\partial_y)^{1/2}v = 0 \tag{6-2}$$

for functions $v = v(y, t)$. Taking one more derivative ∂_t in (6-2), we see that the solution of this problem at time t (given an initial condition v_0) coincides with the value of $w(x = t, \cdot)$ for the solution of

$$\begin{cases} L_2 w := \Delta w + 2w_y = 0 & \text{in } \mathbb{R}_+^2, \\ w(0, \cdot) = w_0 & \text{on } \mathbb{R}, \end{cases} \tag{6-3}$$

where the operator L_2 acts on functions $w = w(x, y)$. Thus, the heat kernel for (6-2) coincides with the Poisson kernel for (6-3).

To compute such a Poisson kernel, as in [Caffarelli et al. 2012] we start with the observation that, if

$$w = e^{-y}\phi,$$

then

$$L_2 w = \Delta w + 2w_y = 0 \quad \text{if and only if} \quad -\Delta\phi + \phi = 0. \tag{6-4}$$

The fundamental solution of Helmholtz’s equation, the solution of

$$-\Delta\Phi + \Phi = \delta_0,$$

is given by

$$\Phi(r) = \frac{1}{2\pi} K_0(r),$$

where $r = \sqrt{x^2 + y^2}$ and K_0 is the modified Bessel function of the second kind with index $\nu = 0$ (see [Abramowitz and Stegun 1964]). The function $K_0 = K_0(s)$ is a positive and decreasing function of $s > 0$, whose asymptotic behavior at $s = 0$ is given by

$$K_0(s) = -\log s + o(|\log s|) \quad \text{as } s \rightarrow 0.$$

For $s \rightarrow +\infty$, all modified Bessel functions of the second kind K_ν have the same behavior

$$K_\nu(s) = \sqrt{\frac{\pi}{2}} s^{-1/2} e^{-s} + o(s^{-1/2} e^{-s}) \quad \text{as } s \rightarrow +\infty. \tag{6-5}$$

By considering the fundamental solution Φ but now with pole at a point $(x_0, y_0) \in \mathbb{R}_+^2$, subtracting from it Φ with pole at the reflected point $(-x_0, y_0)$, and applying the divergence theorem, one sees that the Poisson kernel for the Helmholtz’s equation $-\Delta\phi + \phi = 0$ in the half-plane \mathbb{R}_+^2 is given by

$$-2\Phi_x = -\frac{1}{\pi} \frac{x}{r} K'_0(r).$$

Writing this convolution formula for $w = e^{-y}\phi$, we deduce that the Poisson kernel for (6-3) is given by

$$-2e^{-y}\Phi_x = -e^{-y} \frac{x}{\pi r} K'_0(r).$$

To avoid its singularity at the origin, given any constant $t > 0$ we consider the Poisson kernel after “time” $x = t$ and define

$$\begin{aligned} P^t(x, y) &:= -2G_x^t = -e^{-y} \frac{x+t}{\pi\sqrt{(x+t)^2 + y^2}} K'_0(\sqrt{(x+t)^2 + y^2}) \\ &= e^{-y} \frac{x+t}{\pi\sqrt{(x+t)^2 + y^2}} K_1(\sqrt{(x+t)^2 + y^2}), \end{aligned} \tag{6-6}$$

where $K_1 = -K'_0$ is the modified Bessel function of the second kind with index $\nu = 1$, and where

$$G^t(x, y) := \frac{1}{2\pi} e^{-y} K_0(\sqrt{(x+t)^2 + y^2}).$$

By (6-4), G^t is a solution of the homogeneous equation $L_2w = 0$ in \mathbb{R}_+^2 . Thus, so is $P^t = -2G_x^t$.

Finally, the explicit traveling front will be given by

$$u^t(x, y) := \int_y^{+\infty} P^t(x, z) dz = \int_y^{+\infty} e^{-z} \frac{x+t}{\pi\sqrt{(x+t)^2 + z^2}} K_1(\sqrt{(x+t)^2 + z^2}) dz.$$

Next, let us check all the properties of P^t for $t > 0$ that will be needed in order to use it as a supersolution of the linearized problem for (1-2). We know that $L_2P^t = 0$ in \mathbb{R}_+^2 . Using (6-5) we see that P_t is bounded in all \mathbb{R}_+^2 . We also have that $P^t > 0$ in \mathbb{R}_+^2 since K_0 is radially decreasing. Next, we have that, for every $R > 0$, $P^t(x, y) \rightarrow 0$ as $|y| \rightarrow \infty$ uniformly in $x \in [0, R]$. This follows from the last equality in (6-6) and from (6-5). Also from the last equality in (6-6), we see that

$$-\frac{\partial_x P^t}{P^t}(0, y) = \frac{-1}{t} \left\{ 1 - \frac{t^2}{t^2 + y^2} + \frac{t^2}{\sqrt{t^2 + y^2}} \frac{K'_1(\sqrt{t^2 + y^2})}{K_1(\sqrt{t^2 + y^2})} \right\}. \tag{6-7}$$

Now, using that $K'_1 = -\frac{1}{2}(K_0 + K_2)$ and the asymptotic behavior (6-5), we deduce that

$$\lim_{|y| \rightarrow \infty} \frac{-\partial_x P^t}{P^t}(0, y) = \frac{-1}{t}. \tag{6-8}$$

Finally, since

$$P^t(0, y) = e^{-y} \frac{t}{\pi \sqrt{t^2 + y^2}} K_1(\sqrt{t^2 + y^2})$$

and K_1 has the asymptotic behavior (6-5), we deduce

$$P^t(0, y) = \frac{t}{\sqrt{2\pi}} y^{-3/2} e^{-2y} + o(y^{-3/2} e^{-2y}) \quad \text{as } y \rightarrow +\infty, \quad \text{and} \tag{6-9}$$

$$P^t(0, y) = \frac{t}{\sqrt{2\pi}} (-y)^{-3/2} + o((-y)^{-3/2}) \quad \text{as } y \rightarrow -\infty. \tag{6-10}$$

We can now establish our result on explicit traveling fronts. We need to verify that each u^t is a traveling front for some nonlinearity f^t of positively balanced bistable type satisfying $(f^t)'(0) < 0$ and $(f^t)'(1) < 0$.

Proof of Theorem 1.3. The statements for $u^{t,c}$ follow from the corresponding ones for $u^{t,2} = u^t$. To prove them for u^t , note first that the solution of (6-3) when $w_0 \equiv 1$ is $w \equiv 1$. We deduce that its Poisson kernel satisfies

$$\int_{-\infty}^{+\infty} P^t(x, z) dz = 1$$

for all $x > 0$. It follows that $0 < u^t < 1$ and that $\lim_{y \rightarrow -\infty} u^t(x, y) = 1$ for all $x \geq 0$. Clearly we also have $\lim_{y \rightarrow +\infty} u^t(x, y) = 0$. In addition, $\partial_y u^t = -P^t < 0$ in \mathbb{R}_+^2 .

Next, let us see that we have $L_2 u^t = 0$ in \mathbb{R}_+^2 . Indeed, $\partial_y L_2 u^t = L_2 u_y^t = -L_2 P^t = 0$. Thus, $L_2 u^t$ is a function of x alone. But

$$\begin{aligned} L_2 u^t &= \int_y^{+\infty} \partial_{xx} \left\{ e^{-z} \frac{x+t}{\pi \sqrt{(x+t)^2 + z^2}} K_1(\sqrt{(x+t)^2 + z^2}) \right\} dz \\ &\quad - \partial_y \left\{ e^{-y} \frac{x+t}{\pi \sqrt{(x+t)^2 + y^2}} K_1(\sqrt{(x+t)^2 + y^2}) \right\} - 2e^{-y} \frac{x+t}{\pi \sqrt{(x+t)^2 + y^2}} K_1(\sqrt{(x+t)^2 + y^2}). \end{aligned}$$

Since $L_2 u^t$ does not depend on y , we may let $y \rightarrow +\infty$ in this expression. From this, and since $K'_j = -\frac{1}{2}(K_{j-1} + K_{j+1})$ for all j and all functions $K_\nu(s)$ have the asymptotic behavior (6-5), we deduce that $L_2 u^t \equiv 0$ in \mathbb{R}_+^2 .

The asymptotic behaviors for $-u_y^t = P^t$ in the statement of the theorem follow from (6-9) and (6-10).

Next, we find the expression for the nonlinearity f^t . Since $u^t(0, \cdot)$ is decreasing from 1 to 0, f^t is implicitly well defined in $[0, 1]$ by

$$\begin{aligned} f^t(u^t(0, y)) &:= -u_x^t(0, y) = \int_y^{+\infty} 2G_{xx}^t(0, z) dz = \int_y^{+\infty} 2(-G_{yy}^t - 2G_y^t)(0, z) dz \\ &= 2(G_y^t + 2G^t)(0, y) \\ &= \frac{1}{\pi} e^{-y} \left\{ K_0(\sqrt{t^2 + y^2}) - \frac{y}{\sqrt{t^2 + y^2}} K_1(\sqrt{t^2 + y^2}) \right\}. \end{aligned} \tag{6-11}$$

From this, we clearly see that $f^t(0) = f^t(1) = 0$, again by (6-5).

The remaining properties of f^t will be deduced from the following implicit formula for its derivative. We have $(f^t)'(u^t(0, y))u_y^t(0, y) = -u_{xy}^t(0, y)$ and thus, by (6-7),

$$\begin{aligned} (f^t)'(u^t(0, y)) &= \frac{-u_{xy}^t}{u_y^t}(0, y) \\ &= \frac{-\partial_x P^t}{P^t}(0, y) = \frac{-1}{t} \left\{ 1 - \frac{t^2}{t^2 + y^2} + \frac{t^2}{\sqrt{t^2 + y^2}} \frac{K_1'(\sqrt{t^2 + y^2})}{K_1(\sqrt{t^2 + y^2})} \right\} \\ &= \frac{-1}{t} \left\{ 1 - \frac{t^2}{t^2 + y^2} - \frac{t^2}{\sqrt{t^2 + y^2}} \frac{K_0(\sqrt{t^2 + y^2}) + K_2(\sqrt{t^2 + y^2})}{2K_1(\sqrt{t^2 + y^2})} \right\} \\ &= \frac{t}{\sqrt{t^2 + y^2}} \left\{ -\frac{\sqrt{t^2 + y^2}}{t^2} + \frac{1}{\sqrt{t^2 + y^2}} + \frac{K_0(\sqrt{t^2 + y^2}) + K_2(\sqrt{t^2 + y^2})}{2K_1(\sqrt{t^2 + y^2})} \right\} \\ &=: \frac{t}{\sqrt{t^2 + y^2}} h^t(y). \end{aligned}$$

It turns out that the function $\{K_0(s) + K_2(s)\}/(2K_1(s))$ is a decreasing function of $s \in (0, +\infty)$ which behaves as $1/s$ as $s \rightarrow 0^+$ and as $1 + 1/(2s)$ as $s \rightarrow +\infty$. Therefore,

$$(f^t)'(0) = (f^t)'(1) = -\frac{1}{t} < 0.$$

It also follows that h^t is a decreasing function of $y \in [0, +\infty)$ with $h^t(0) = (K_0(t) + K_2(t))/(2K_1(t)) > 0$ and $\lim_{y \rightarrow +\infty} h^t(y) = -\infty$. Therefore, since h^t is an even function of y , there exists a $y^t > 0$ such that h^t is negative in $(-\infty, -y^t)$, positive in $(-y^t, y^t)$, and negative in $(y^t, +\infty)$. As a consequence, for some $0 < \gamma_1 < \gamma_2 < 1$, we have that $(f^t)'$ is negative in $(0, \gamma_1)$, positive in (γ_1, γ_2) , and negative in $(\gamma_2, 1)$. This gives that f^t has a unique zero in $(0, 1)$ and that f^t is of bistable type.

We finally check the positively balanced character of f^t (after the end of the proof we give an alternative, more synthetic argument for this). Using formula (6-11) for f^t in terms of G^t and that $G_{xx}^t = -\partial_y(G_y^t + 2G^t)$, we have

$$\begin{aligned} \int_0^1 f^t(s) ds &= \int_{\mathbb{R}} f(u^t(0, y))(-u_y^t)(0, y) dy = -4 \int_{\mathbb{R}} \{(G_y^t + 2G^t)G_x^t\}(0, y) dy \\ &= 4 \int_{\mathbb{R}_+^2} \{(G_y^t + 2G^t)G_{xx}^t + (G_{xy}^t + 2G_x^t)G_x^t\} dx dy \\ &= 4 \int_{\mathbb{R}_+^2} \left\{ -\frac{1}{2}\partial_y(G_y^t + 2G^t)^2 + \frac{1}{2}\partial_y(G_x^t)^2 + 2(G_x^t)^2 \right\} dx dy \\ &= 8 \int_{\mathbb{R}_+^2} (G_x^t)^2 dx dy > 0, \end{aligned}$$

which finishes the proof. □

The following is an alternative way to prove that the integral of f^t is positive. It is more synthetic but it relies on deeper results. Assume by contradiction that $\int_0^1 f^t(s) ds \leq 0$. Then, by the remarks made after

Theorem 1.2, the existence part of **Theorem 1.2**, and the results of [Cabré and Solà-Morales 2005], there exists a solution of (1-2) for some $c \leq 0$ which satisfies the limits (1-3). But u^t is also a solution of (1-2), now with $c = 2$, and satisfying the limits (1-3). By the uniqueness of the speed proved in **Theorem 1.2**, we arrive at a contradiction.

7. Proof of Theorem 1.2

In this section we use all the previous results to establish our main theorem.

Proof of Theorem 1.2. Let f be of positively balanced bistable type or of combustion type as in **Definition 1.1**.

Part (i). This first part has been established in **Proposition 2.7**, together with **Lemma 4.2**, where we proved the existence of limits as $y \rightarrow \pm\infty$. That $0 < u < 1$ follows from the strong maximum principle and the Hopf’s boundary lemma, since we know that $0 \leq u \leq 1$ and $f(0) = f(1) = 0$.

Part (ii). Let (c_1, u_1) and (c_2, u_2) be two solution pairs with u_i taking values in $[0, 1]$ and having limits 1 and 0 as $y \rightarrow \mp\infty$. By **Lemma 4.1**, both u_i satisfy the uniform limits assumption (5-6). Translate each one in the y variable so that both satisfy $u_i(0, 0) = \frac{1}{2}$. Assume that $c_1 \leq c_2$. **Proposition 5.3** applied with $c = c_1$ gives that the solution u_1 is decreasing in y . Thus

$$0 = \Delta u_1 + c_1 \partial_y u_1 \geq \Delta u_1 + c_2 \partial_y u_1,$$

and hence u_1 is a supersolution for the problem with $c = c_2$. **Proposition 5.3** applied with $c = c_2$ leads to $u_1 \equiv u_2$. As a consequence, since $\partial_y u_1 < 0$, we deduce from the equations that $c_1 = c_2$.

Part (iii). The monotonicity in y of a variational solution as in part (i) was established in **Proposition 3.1**. From $u_y \leq 0$ we deduce $u_y < 0$ using the strong maximum principle and the Hopf’s boundary lemma for the linearized problem satisfied by u_y . The existence of its vertical and horizontal limits has been proved in **Lemmas 4.2** and **4.3**. Let us now show that $u_x \leq 0$ in the case that f is of combustion type — this fact is not true for bistable nonlinearities since the normal derivative $-u_x = f(u)$ changes sign on $\{x = 0\}$. Note that u_x is a solution of $(\Delta + c \partial_y)u_x = 0$ in \mathbb{R}_+^2 , it is bounded by (4-4), and has limits 0 as $|y| \rightarrow \infty$ uniformly on compact sets of x by (4-5). In addition, $u_x = -f(u) \leq 0$ on $x = 0$. **Lemma 5.1** leads to $u_x \leq 0$ in \mathbb{R}_+^2 .

Part (iv). We can give two different proofs of this part. Let f_1 and f_2 be not identically equal with $f_1 \geq f_2$. Let (c_i, u_i) be the unique solution pair for the nonlinearity f_i with u_i taking values in $[0, 1]$ and having limits as $y \rightarrow \pm\infty$.

The first proof is variational and uses formula (2-11) for the speed. Take $a > 0$ small enough so that both problems (1-21), for f_1 and for f_2 , can be minimized in B_a . Since $G_1 \leq G_2$, the minimum values satisfy $I_{1,a} \leq I_{2,a}$, and the inequality is in fact strict since $G_1 \not\equiv G_2$ and the minimizers \underline{u}_1 and \underline{u}_2 take all values in $(0, 1)$. Thus, from (2-11) we deduce $c_1 > c_2$.

The second proof of (iv) is nonvariational. Recall that, by **Lemma 4.1**, both u_i satisfy the uniform limits assumption (5-6). Translate each front in the y variable so that both satisfy $u_i(0, 0) = \frac{1}{2}$. Assume,

arguing by contradiction, that $c_1 \leq c_2$. Since u_1 is decreasing in y , we have

$$0 = \Delta u_1 + c_1 \partial_y u_1 \geq \Delta u_1 + c_2 \partial_y u_1.$$

In addition,

$$\frac{\partial u_1}{\partial \nu} = f_1(u_1) \geq f_2(u_1) \quad \text{on } \partial \mathbb{R}_+^2.$$

Hence, u_1 is a supersolution for the problem with $c = c_2$ and $f = f_2$. Proposition 5.3 leads to $u_1 \equiv u_2$. As a consequence, we obtain $f_1 \equiv f_2$ — since the image of u_1 is the whole of $(0, 1)$. This is a contradiction.

Part (v). To establish this part, it suffices to show the bounds for $-u_y$. From them, the bounds for u and $1 - u$ follow by integration. Defining \tilde{u} by $u(x, y) = \tilde{u}(cx/2, cy/2)$, we see that \tilde{u} is a front with speed 2 for (1-2) with nonlinearity $(2/c)f$. Since the constants on the bounds of part (v) do not reflect the dependence on f , we may rename \tilde{u} by u and $(2/c)f$ by f , and assume that u is a front for the nonlinearity f with speed $c = 2$. Note however that the factor e^{-2y} in (1-9) will change to e^{-cy} after the scaling.

As stated in the theorem, the lower bounds for $-u_y$ hold for any f of positively balanced bistable type or of combustion type. To prove them, we take $t > 0$ small enough that

$$-\frac{1}{2t} \leq \min_{[0,1]} f'.$$

For such t , consider the Poisson kernel P^t defined by (6-6) and, for any positive constant $C > 0$, the function

$$v := C(-u_y) - P^t.$$

Note that $\Delta v + 2v_y = 0$ in \mathbb{R}_+^2 . By using (6-8) and that $-u_y$ and P^t are positive, it follows that $\{-\partial_x v + (2t)^{-1}v\}(0, y) \geq 0$ for $|y|$ large enough, say for y in the complement of a compact interval H . Next, take the constant $C > 0$ large enough so that $v > 0$ in the compact set H . By (4-4), the limits of u_y established in (4-5), and the properties of P^t checked in Section 6, we can apply Proposition 5.2 with $c = 2$ and $d(y) = (2t)^{-1}$ to deduce that $v > 0$ in \mathbb{R}_+^2 . By using the asymptotic behaviors (6-9) and (6-10) of P^t at $\pm\infty$, we conclude the two lower bounds for $-u_y$.

To prove the upper bounds for $-u_y$ we need to assume that $f'(0) < 0$ and $f'(1) < 0$. We proceed in the same way as for the lower bounds, but replacing the roles of $-u_y$ and P^t . We now take $t > 0$ large enough that

$$\max\{f'(0), f'(1)\} < -\frac{2}{t}.$$

Using (6-8), for any $C > 0$, $\tilde{v} := CP^t - (-u_y)$ satisfies $\{-\partial_x \tilde{v} + (2/t)\tilde{v}\}(0, y) \geq 0$ for $|y|$ large enough, say for y in the complement of a compact interval H . One proceeds exactly as before to obtain $\tilde{v} > 0$ in \mathbb{R}_+^2 for C large enough. This gives the desired upper bounds for $-u_y$. □

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