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SCATTERING FOR THE RADIAL 3D CUBIC WAVE EQUATION

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Consider the Cauchy problem for the radial cubic wave equation in $1+3$ dimensions with either the focusing or defocusing sign. This problem is critical in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$ and subcritical with respect to the conserved energy. Here we prove that if the critical norm of a solution remains bounded on the maximal time interval of existence, then the solution must in fact be global in time and must scatter to free waves as $t \to \pm\infty$.

1. Introduction

Consider the Cauchy problem for the cubic semilinear wave equation in $\mathbb{R}^{1+3}$, namely,

$$\begin{aligned} u_{tt} - \Delta u + \mu u^3 &= 0, \\ \vec{u}(0) &= (u_0, u_1), \end{aligned} \tag{1-1}$$

restricted to the radial setting and with $\mu \in \{\pm 1\}$. The case $\mu = 1$ yields what is referred to as the defocusing problem, since here the conserved energy,

$$E(\vec{u})(t) := \int_{\mathbb{R}^3} \left(\frac{1}{2}\left(|u_t(t)|^2 + |\nabla u(t)|^2\right) + \frac{1}{4}|u(t)|^4 \right) dx = \text{constant}, \tag{1-2}$$

is positive for sufficiently regular nonzero solutions, and the $\dot{H}^1 \times L^2(\mathbb{R}^3)$ norm of a solution,

$$\vec{u}(t) := (u(t), u_t(t)),$$

is bounded by its energy.

The case $\mu = -1$ gives the focusing problem, and the conserved energy for sufficiently regular solutions to (1-1) is given by

$$E(\vec{u})(t) := \int_{\mathbb{R}^3} \left(\frac{1}{2}\left(|u_t(t)|^2 + |\nabla u(t)|^2\right) - \frac{1}{4}|u(t)|^4 \right) dx = \text{constant}. \tag{1-3}$$

As we will only be considering radial solutions to (1-1), we will often slightly abuse notation by writing $u(t, x) = u(t, r)$, where (r, ω), with $r = |x|$, $x = r\omega$, $\omega \in \mathbb{S}^2$, are polar coordinates on $\mathbb{R}^3$. In this setting

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we can rewrite the Cauchy problem (1-1) as

$$\begin{aligned} u_{tt} - u_{rr} - \frac{2}{r}u_r \pm u^3 = 0,& \\ \vec{u}(0) = (u_0, u_1),& \end{aligned} \tag{1-4}$$

and the conserved energy (up to a constant multiple) as

$$E(\vec{u})(t) = \int_0^\infty \left(\frac{1}{2}\big(u_t^2(t) + u_r^2(t)\big) \pm \frac{1}{4}u^4(t)\right) r^2\, dr. \tag{1-5}$$

The Cauchy problem (1-4) is invariant under the scaling

$$\vec{u}(t,r) \mapsto \vec{u}_\lambda(t,r) := \left(\lambda^{-1} u\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right), \lambda^{-2} u_t\left(\frac{t}{\lambda}, \frac{r}{\lambda}\right)\right). \tag{1-6}$$

One can also check that this scaling leaves unchanged the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$-norm of the solution. It is for this reason that (1-4) is called *energy-subcritical*. It is natural to consider the Cauchy problem with initial data $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$. We remark that (1-4) is also invariant under conformal inversion:

$$u(t,r) \mapsto \frac{1}{t^2 - r^2} u\left(\frac{t}{t^2 - r^2}, \frac{r}{t^2 - r^2}\right). \tag{1-7}$$

A standard argument based on Strichartz estimates shows that both the defocusing and focusing problems are locally well-posed in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$. This means that for all initial data $\vec{u}(0) = (u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, there is a unique solution $\vec{u}(t)$ defined on a maximal interval of existence $I_{\max}$ with $\vec{u}(t) \in C(I_{\max}; \dot{H}^{1/2} \times \dot{H}^{-1/2})$. Moreover, for every compact time interval $J \subset I_{\max}$, we have $u \in S(J) := L_t^4(J; L_x^4)$. The Strichartz norm $S(J)$ determines criteria for both scattering and finite-time blow-up, and we make these statements precise in Proposition 2.4. Here we note that in particular, one can show that if the initial data $\vec{u}(0)$ has sufficiently small $\dot{H}^{1/2} \times \dot{H}^{-1/2}$-norm, then the corresponding solution $\vec{u}(t)$ has finite $S(\mathbb{R})$-norm and hence scatters to free waves as $t \to \pm\infty$.

The theory for solutions to (1-4) with initial data that is small in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ is thus very well understood: all solutions are global in time and scatter to free waves as $t \to \pm\infty$. However, much less is known regarding the *asymptotic dynamics* of solutions to either the defocusing or focusing problems once one leaves the perturbative regime.

It is well known that there are solutions to the focusing problem that blow up in finite time. To give an example,

$$\phi_T(t,r) = \frac{\sqrt{2}}{T - t} \tag{1-8}$$

solves the ODE $\phi_{tt} = \phi^3$. Using finite speed of propagation, one can construct from ϕ_T a compactly supported (in space) self-similar blow-up solution to (1-4). Indeed, define $\vec{u}_T(t)$ to be the solution to (1-4) with initial data $\vec{u}_T(0,x) = \chi_{2T}(x)\sqrt{2}$, where $\chi_{2T} \in C_0^\infty(\mathbb{R}^3)$ satisfies $\chi_{2T}(x) = 1$ if $|x| \le 2T$. Then $\vec{u}_T(t)$ equals $\phi_T(t)$ for all $r \le T$ and $0 \le t < T$, and blows up at time $t = T$. However, such a self-similar solution must have its critical $\dot{H}^{1/2} \times \dot{H}^{-1/2}$-norm tending to ∞ as $t \to T$:

$$\lim_{t \to T} \|\vec{u}_T(t)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} = \infty.$$

Indeed, one can show by a direct computation that the $L^3(\mathbb{R}^3)$-norm of $\vec{u}_T(t,x)$ tends to ∞ as $t \to T_+$. Since $\dot{H}^{1/2} \subset L^3$, this means that the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$-norm must blow up as well. Such behavior is typically referred to as type I, or ODE, blow-up.

One the other hand, type II solutions, $\vec{u}(t)$, are those whose critical norm remains bounded on their maximal interval of existence, $I_{\max}$:

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} < \infty. \tag{1-9}$$

In this paper we restrict our attention to type II solutions, i.e., those which satisfy (1-9). We prove that if a solution $\vec{u}(t)$ to (1-4) satisfies (1-9), then $\vec{u}(t)$ must in fact exist globally in time and scatter to free waves in both time directions. To be precise, we establish the following result.

Theorem 1.1. *Let $\vec{u}(t) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$ be a radial solution to (1-4) defined on its maximal interval of existence $I_{\max} = (T_-, T_+)$. Suppose in addition that*

$$\sup_{t \in I_{\max}} \|\vec{u}(t)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)} < \infty. \tag{1-10}$$

Then $I_{\max} = \mathbb{R}$, i.e., $\vec{u}(t)$ is defined globally in time. Moreover,

$$\|u\|_{L^4_{t,x}(\mathbb{R}^{1+3})} < \infty, \tag{1-11}$$

which means that $\vec{u}(t)$ scatters to a free wave in each time direction, i.e., there exist radial solutions $\vec{u}_L^{\pm}(t) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$ to the free wave equation, $\Box u_L^{\pm} = 0$, such that

$$\|\vec{u}(t) - \vec{u}_L^{\pm}(t)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)} \to 0 \quad \text{as} \quad t \to +\infty. \tag{1-12}$$

Remark 1. Theorem 1.1 is a conditional result. Other than the requirement that the initial data be small in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, there is no known general criterion that ensures that (1-10) is satisfied by the evolution for either the defocusing or the focusing equation. While the methods in this paper apply equally well to both the focusing and defocusing equations, one should expect drastically different behavior from generic initial data in these two cases.

Remark 2. The proof of Theorem 1.1 readily generalizes to all subcritical powers $p \leq 3$ for which there is a satisfactory small data/local well-posedness theory. In particular, the methods presented here allow one to deduce the exact analog of Theorem 1.1 for radial equations (1-13) for all powers p with $1 + \sqrt{2} < p \leq 3$; here $1 + \sqrt{2}$ is the F. John exponent [1979; Schaeffer 1985]. We have chosen to present the details for only the cubic equation to keep the exposition as simple as possible. We also remark that only the material in Section 4 relies on the assumption of radiality.

1A. ***History of the problem.*** The cubic wave equation on $\mathbb{R}^{1+3}$ has been extensively studied and we certainly cannot give a complete account of the vast body of literature devoted to this problem.

For the defocusing equation, the positivity of the conserved energy can be used to extend a local existence result to a global one if one begins with initial data that is sufficiently regular. Jörgens [1961] showed global existence for the defocusing equation for smooth compactly supported data. There has been a good deal of recent work extending the local existence result of Lindblad and Sogge [1995] in $H^s \times H^{s-1}$ for $s > \frac{1}{2}$ to an unconditional global well-posedness result, and we refer the reader to [Kenig et al. 2000; Gallagher and Planchon 2003; Bahouri and Chemin 2006; Roy 2009] and the references therein for details. However, since these works are not carried out in the scaling critical space, the issue of global dynamics, and in particular scattering, is not addressed.

For the focusing equation, type II finite-time blow-up has recently been ruled out for initial data that lies in $\dot{H}^1 \times L^2$ in the work of Killip, Stovall, and Vişan [Killip et al. 2012]. There are several works that open up interesting lines of inquiry related to the question of asymptotic dynamics. In two remarkable works, Merle and Zaag [2003; 2005] determined that all blow-up solutions must blow up *at the self-similar rate*. In the radial case, an infinite family of smooth self-similar solutions is constructed by Bizoń et al. [2010]. Bizoń and Zenginoğlu [2009] give numerical evidence to support a conjecture that a two-parameter family of solutions, obtained via time translation and conformal inversion of a self-similar solution, serves as a global attractor for a large set of initial data. In fact, Donninger and Schörkhuber [2012] showed that the blow-up profile (1-8) is stable under small perturbations in the energy topology.

Equations of the form

$$\Box u = \pm |u|^{p-1} u \tag{1-13}$$

for different values of p and for different dimensions have also been extensively studied. For $d = 3$, the energy-critical power, $p = 5$, exhibits quite different phenomena than both the subcritical and supercritical equations. Global existence and scattering for all finite-energy data was proved by Struwe [1988] for the radial defocusing equation and by Grillakis [1990] in the nonradial, defocusing case.

For the focusing energy-critical equation, type II blow-up can occur, as explicitly demonstrated by Krieger, Schlag, and Tataru [Krieger et al. 2009] via an energy concentration scenario resulting in the bubbling off of the ground state solution, W, for the underlying elliptic equation; see also [Krieger and Schlag 2014a; Donninger et al. 2014; Donninger and Krieger 2013].

Kenig and Merle [2008] initiated a powerful program of attack for semilinear equations (1-13) with the concentration compactness/rigidity method, giving a characterization of possible dynamics for solutions with energy below the threshold energy of the ground state elliptic solution. The subsequent work of Duyckaerts, Kenig, and Merle [Duyckaerts et al. 2011; 2012a; 2012b; 2013] resulted in a classification of possible dynamics for large energies. In particular, all type II radial solutions asymptotically resolve into a sum of rescaled solitons plus a radiation term at their maximal time of existence. Dynamics at the threshold energy of W have been examined by Duyckaerts and Merle [2008] and above the threshold by Krieger, Nakanishi, and Schlag [Krieger et al. 2013a; 2013b; 2014].

Analogs of Theorem 1.1 have been established for radial equations with different powers in 3 dimensions.

Shen [2012] proved the exact analog of Theorem 1.1 for subcritical powers $3 < p < 5$; and Kenig and Merle [2011], and then Duyckaerts, Kenig, and Merle [Duyckaerts et al. 2014], established the analog of Theorem 1.1 for all supercritical powers $p > 5$. Here we address type II behavior in the remainder of the subcritical range for the radial equation, $1+\sqrt{2} < p \leq 3$. While we focus on the cubic equation, our proof readily generalizes to other subcritical powers. The extra regularity for critical elements proved in Section 4 gives an extension and simplification of the argument in [Shen 2012] which allows us to treat the cubic and lower-power equations.

Leaving the setting of type II solutions, Krieger and Schlag [2014b] have recently constructed a family of solutions to the supercritical equation, $p > 5$, which are smooth, global in time, and stable under small perturbations, and have infinite critical norm.

1B. *Outline of the proof of Theorem 1.1.* The proof of Theorem 1.1 follows the concentration compactness/rigidity method developed in [Kenig and Merle 2006; 2008]. The proof follows a contradiction argument: if Theorem 1.1 were not true, the linear and nonlinear profile decompositions of Bahouri and Gérard would allow one to construct a minimal solution to (1-4), called the critical element, which does not scatter (here the minimality refers to the size of the norm in (1-10)). This construction, which is by now standard in the field and is outlined in Section 3, yields a critical element whose trajectory in the space $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ is precompact up to modulation. The goal is then to prove that this compactness property is too rigid a property for a nonzero solution and thus the critical element cannot exist.

A significant hurdle in the way of ruling out a critical element $\vec{u}_c(t)$ for the cubic equation (or any subcritical equation) is that $\vec{u}_c(t)$ is constructed in the space $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, and thus useful global monotone quantities that require more regularity, such as the conserved energy and virial type identities, are not, a priori, well defined. In general, solutions to the cubic wave equation are only as regular as their initial data, as evidenced by the presence of the free propagator $S(t)$ in the Duhamel representation for the solution

$$\vec{u}_c(t_0) = S(t_0 - t)\vec{u}_c(t) + \int_t^{t_0} S(t_0 - s)(0, \pm u^3)\, ds. \tag{1-14}$$

The critical element is rescued by the fact that the precompactness of its trajectory is at odds with the dispersive properties of the free part, $S(t_0 - t)\vec{u}(t)$, and thus the first term on the right of (1-14) is forced to vanish weakly as $t \to \sup I_{\max}$ and as $t \to \inf I_{\max}$. The second term on the right thus encodes the regularity of the critical element, and a gain can be expected due to the presence of the cubic term. The additional regularity is extracted by way of the "double Duhamel trick", which refers to the consideration of the pairing

$$\left\langle \int_{T_1}^{t_0} S(t_0 - s)(0, \pm u^3)\, ds, \int_{t_0}^{T_2} S(t_0 - \tau)(0, \pm u^3)\, d\tau \right\rangle_{\dot{H}^1 \times L^2},$$

where $T_1 < t_0$ and $T_2 > t_0$. This technique was developed by Tao [2007] and utilized in the Kenig–Merle framework for nonlinear Schrödinger problems by Killip and Vişan [2010a; 2010b; 2013], and for semilinear wave equations in [Killip and Vişan 2011; Bulut 2012a; 2012b]. This method is also closely related to the in/out decomposition used by Killip, Tao, and Vişan in [Killip et al. 2009, Section 6]. For more details on how to exploit the different time directions above, we refer the reader to Section 4, and in particular to the proof of Theorem 4.1.

Indeed, we bound the critical element in $\dot{H}^1 \times L^2$. We then use the conserved energy to rule out a critical element which fails to be compact by a low frequency concentration, as such a solution would have vanishing energy; see Section 5A. One is then left with a critical element that is global in time and evolves at a fixed scale. In Section 6, we prove that such a solution cannot exist by way of a virial identity. We note that this virial-based rigidity argument works for precompact solutions to (1-13) with powers $p \le 3$, but fails to produce useful estimates for powers $3 < p < 5$. However, in this range one can use the "channels of energy" method pioneered in [Duyckaerts et al. 2013; 2014]; see [Shen 2012]. For more on this, see Remark 12.

2. Preliminaries

2A. ***Harmonic analysis.*** In what follows we will denote by P_k the usual Littlewood–Paley projections onto frequencies of size $|\xi| \simeq 2^k$ and by $P_{\le k}$ the projection onto frequencies $|\xi| \lesssim 2^k$. These projections satisfy Bernstein's inequalities.

Lemma 2.1 (Bernstein's inequalities [Tao 2006, Appendix A]). *Let* $1 \le p \le q \le \infty$ *and* $s \ge 0$. *Let* $f : \mathbb{R}^d \to \mathbb{R}$. *Then*

$$\begin{aligned} \|P_{\ge N} f\|_{L^p} &\lesssim N^{-s} \||\nabla|^s P_{\ge N} f\|_{L^p}, \\ \|P_{\le N} |\nabla|^s f\|_{L^p} &\lesssim N^s \|P_{\le N} f\|_{L^p}, & \|P_N |\nabla|^{\pm s} f\|_{L^p} &\simeq N^{\pm s} \|P_N f\|_{L^p}, \\ \|P_{\le N} f\|_{L^q} &\lesssim N^{d/p-d/q} \|P_{\le N} f\|_{L^p}, & \|P_N f\|_{L^q} &\lesssim N^{d/p-d/q} \|P_N f\|_{L^p}. \end{aligned} \tag{2-1}$$

Next, we define the notion of a frequency envelope.

Definition 3 [Tao 2001, Definition 1]. We define a *frequency envelope* to be a sequence $\beta = \{\beta_k\}$ of positive real numbers with $\beta \in \ell^2$. Moreover, we require the local constancy condition

$$2^{-\sigma|j-k|}\beta_k \lesssim \beta_j \lesssim 2^{\sigma|j-k|}\beta_k,$$

where $\sigma > 0$ is a small fixed constant; in what follows we will use $\sigma = \frac{1}{8}$. If β is a frequency envelope and $(f, g) \in \dot{H}^s \times \dot{H}^{s-1}$, then we say that (f, g) *lies underneath* β if

$$\|(P_k f, P_k g)\|_{\dot{H}^s \times \dot{H}^{s-1}} \le \beta_k \quad \text{for all } k \in \mathbb{Z},$$

and we note that if (f, g) lies underneath β, then we have

$$\|(f, g)\|_{\dot{H}^s \times \dot{H}^{s-1}} \lesssim \|\beta\|_{\ell^2}.$$

We will require the following refinement of the Sobolev embedding for radial functions, which is a consequence of the Hardy–Littlewood–Sobolev inequality.

Lemma 2.2 (radial Sobolev embedding [Tao et al. 2007, Corollary A.3]). *Let $0 < s < 3$ and suppose $f \in \dot{H}^s(\mathbb{R}^3)$ is a radial function. Suppose that*

$$\beta > -\frac{3}{q}, \quad \frac{1}{2} - s \leq \frac{1}{q} \leq \frac{1}{2}, \quad \frac{1}{q} = \frac{1}{2} - \frac{\beta + s}{3},$$

and at most one of the equalities $q = 1$, $q = \infty$, $\frac{1}{q} + s = \frac{1}{2}$ holds. Then

$$\|r^\beta f\|_{L^q} \leq C\|f\|_{\dot{H}^s}. \tag{2-2}$$

2B. *Strichartz estimates.* An essential ingredient for the small data theory are Strichartz estimates for the linear wave equation in $\mathbb{R}^{1+3}$,

$$\begin{aligned} v_{tt} - \Delta v &= F, \\ \vec{v}(0) &= (v_0, v_1). \end{aligned} \tag{2-3}$$

A free wave will mean a solution to (2-3) with $F = 0$ and will be denoted by $\vec{u}(t) = S(t)\vec{u}(0)$. In what follows we will say that a pair (p, q) is admissible if

$$p, q \geq 2, \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}. \tag{2-4}$$

The Strichartz estimates we state below are standard and we refer the reader to [Keel and Tao 1998; Lindblad and Sogge 1995] or the book [Sogge 2008] and the references therein for more details.

Remark 4. We note that since we will only consider the waves with radial initial data and with F radial, we can allow the endpoint $(p, q) = (2, \infty)$ as an admissible pair. The admissibility of $(2, \infty)$ in the radial setting was established in [Klainerman and Machedon 1993]. This endpoint is of course forbidden for nonradial data in dimension $d = 3$.

Proposition 2.3 [Keel and Tao 1998; Klainerman and Machedon 1993; Lindblad and Sogge 1995; Sogge 2008]. *Let $\vec{v}(t)$ be a solution to (2-3) with initial data $\vec{v}(0) \in \dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^3)$ for $s > 0$. Let (p, q) and (a, b) be admissible pairs satisfying the gap condition*

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{a'} + \frac{3}{b'} - 2 = \frac{3}{2} - s, \tag{2-5}$$

where (a', b') are the conjugate exponents of (a, b). Then, for any time interval $I \ni 0$, we have the estimates

$$\|v\|_{L_t^p(I; L_x^q)} \lesssim \|(v_0, v_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|F\|_{L_t^{a'}(I; L_x^{b'})}. \tag{2-6}$$

2C. *Small data theory — global existence, scattering, perturbative theory.* A standard argument based on Proposition 2.3 with $s = \frac{1}{2}$, $(p, q) = (4, 4)$, and $(a', b') = \left(\frac{4}{3}, \frac{4}{3}\right)$ yields the following small data result.

Proposition 2.4 (small data theory). *Let $\vec{u}(0) = (u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$. Then there is a unique solution $\vec{u}(t) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$ defined on a maximal interval of existence $I_{\max}(\vec{u}) = (T_-(\vec{u}), T_+(\vec{u}))$. Moreover, for any compact interval $J \subset I_{\max}$, we have*

$$\|u\|_{L_t^4(J; L_x^4)} < \infty.$$

A globally defined solution $\vec{u}(t)$ for $t \in [0, \infty)$ scatters as $t \to \infty$ to a free wave, i.e., a solution $\vec{u}_L(t)$ of $\Box u_L = 0$, if and only if $\|u\|_{L_t^4([0,\infty), L_x^4)} < \infty$. In particular, there exists a constant $\delta > 0$ such that

$$\|\vec{u}(0)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} < \delta \implies \|u\|_{L_t^4(\mathbb{R};L_x^4)} \lesssim \|\vec{u}(0)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}} \lesssim \delta, \tag{2-7}$$

and hence $\vec{u}(t)$ scatters to free waves as $t \to \pm\infty$. Finally, we have the standard finite-time blow-up criterion:

$$T_+(\vec{u}) < \infty \implies \|u\|_{L_t^4([0,T_+(\vec{u}));L_x^4)} = +\infty. \tag{2-8}$$

A similar statement holds if $-\infty < T_-(\vec{u})$.

For the concentration compactness procedure in Section 3 one requires the following perturbation theory for approximate solutions to (1-4).

Lemma 2.5 (perturbation lemma). *There are continuous functions $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded); let $\vec{u}, \vec{v} \in C(I; \mathcal{H})$ satisfy, for some $A > 0$,*

$$\|\vec{v}\|_{L^\infty(I;\dot{H}^{1/2}\times\dot{H}^{-1/2})} + \|v\|_{L_t^4(I;L_x^4)} \le A,$$
$$\|\mathrm{eq}(u)\|_{L_t^{4/3}(I;L_x^{4/3})} + \|\mathrm{eq}(v)\|_{L_t^{4/3}(I;L_x^{4/3})} + \|w_0\|_{L_t^4(I;L_x^4)} \le \varepsilon \le \varepsilon_0(A),$$

where $\mathrm{eq}(u) := \Box u \pm u^3$ in the sense of distributions, and where $\vec{w}_0(t) := S(t-t_0)(\vec{u}-\vec{v})(t_0)$, with $t_0 \in I$ arbitrary but fixed. Then

$$\|\vec{u}-\vec{v}-\vec{w}_0\|_{L^\infty(I;\dot{H}^{1/2}\times\dot{H}^{-1/2})} + \|u-v\|_{L_t^4(I;L_x^4)} \le C_0(A)\varepsilon.$$

In particular, $\|u\|_{L_t^4(I;L_x^4)} < \infty$.

2D. *Blow-up for nonpositive energies.* Finally, we recall that in the case of the focusing equation, any nontrivial solution with negative energy must blow up in both time directions. This result was proved in [Killip et al. 2012] for solutions to (1-4).

Proposition 2.6 [Killip et al. 2012, Theorem 3.1]. *Let $\vec{u}(t)$ be a solution to (1-4) with the focusing sign and with maximal interval of existence $I_{\max} = (T_-, T_+)$. If $E(\vec{u}) \le 0$, then $\vec{u}(t)$ is either identically zero or blows up in finite time in both time directions, i.e., $T_+ < +\infty$ and $T_- > -\infty$.*

3. Concentration compactness

In this section we begin the proof of Theorem 1.1. We will follow the concentration compactness/rigidity method introduced by Kenig and Merle [2006; 2008]. The concentration compactness part of the argument, which is based on the profile decompositions of Bahouri and Gérard [1999], is by now standard, and we will essentially follow the scheme from [Kenig and Merle 2010], which is a refinement of the methods from [Kenig and Merle 2006; 2008]. Indeed, the main conclusion of this section is that in the event that Theorem 1.1 fails, there exists a minimal, nontrivial, nonscattering solution to (1-4), which we will call the critical element.

We begin with some notation, following [Kenig and Merle 2010] for convenience. For initial data $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, we let $\vec{u}(t) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$ be the unique solution to (1-4) with initial data $\vec{u}(0) = (u_0, u_1)$ defined on its maximal interval of existence $I_{\max}(\vec{u}) := (T_-(\vec{u}), T_+(\vec{u}))$. For $A > 0$, define

$$\mathcal{B}(A) := \left\{(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2} : \|\vec{u}(t)\|_{L_t^\infty(I_{\max}(\vec{u}); \dot{H}^{1/2} \times \dot{H}^{-1/2})} \leq A\right\}. \tag{3-1}$$

Definition 5. We say that $\mathcal{SC}(A)$ holds if for all $\vec{u} = (u_0, u_1) \in \mathcal{B}(A)$, we have $I_{\max}(\vec{u}) = \mathbb{R}$ and $\|u\|_{L_t^4 L_x^4} < \infty$. We also say that $\mathcal{SC}(A; \vec{u})$ holds if $\vec{u} \in \mathcal{B}(A)$, $I_{\max}(\vec{u}) = \mathbb{R}$, and $\|u\|_{L_t^4 L_x^4} < \infty$.

Remark 6. Recall from Proposition 2.4 that $\|u\|_{L_t^4; L_x^4} < \infty$ if and only if $\vec{u}$ scatters to a free wave as $t \to \pm\infty$. Therefore Theorem 1.1 is equivalent to the statement that $\mathcal{SC}(A)$ holds for all $A > 0$.

Now suppose that Theorem 1.1 is *false*. By Proposition 2.4, there is an $A_0 > 0$ small enough that $\mathcal{SC}(A_0)$ holds. Given that we are assuming that Theorem 1.1 fails, we can find a threshold, or critical value A_C such that for $A < A_C$, $\mathcal{SC}(A)$ holds, and for $A > A_C$, $\mathcal{SC}(A)$ fails. Note that $0 < A_0 < A_C$. The standard conclusion of this assumed failure of Theorem 1.1 is that there is a minimal nonscattering solution $\vec{u}(t)$ to (1-4) such that $\mathcal{SC}(A_C, \vec{u})$ fails, which enjoys certain compactness properties.

We will state a refined version of this result below, and we refer the reader to [Kenig and Merle 2010; Shen 2012; Tao et al. 2007; 2008] for the details of the argument. As usual, the main ingredients are the linear and nonlinear Bahouri–Gérard type profile decompositions [1999] used in conjunction with the perturbation theory in Lemma 2.5.

Proposition 3.1. *Suppose that Theorem 1.1 is false. Then there exists a solution $\vec{u}(t)$ such that $\mathcal{SC}(A_C; \vec{u})$ fails. Moreover, we can assume that $\vec{u}(t)$ does not scatter in either time direction*:

$$\|u\|_{L^4((T_-(\vec{u}),0]; L_x^4)} = \|u\|_{L^4([0,T_+(\vec{u})); L_x^4)} = \infty. \tag{3-2}$$

In addition, there exists a continuous function $N : I_{\max}(\vec{u}) \to (0, \infty)$ such that the set

$$K := \left\{\left(\frac{1}{N(t)} u\left(t, \frac{\cdot}{N(t)}\right), \frac{1}{N^2(t)} u_t\left(t, \frac{\cdot}{N(t)}\right)\right) \,\middle|\, t \in I_{\max}\right\} \tag{3-3}$$

is precompact in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$.

Remark 7. After passing to subsequences, scaling considerations, and possibly time reversal, we can assume, without loss of generality, that $T_+(\vec{u}) = +\infty$, and $N(t) \leq 1$ on $[0, \infty)$. We can further reduce this to two separate cases: either we have

- $N(t) \equiv 1$ for all $t \geq 0$, or
- $\limsup_{t\to\infty} N(t) = 0$.

These reductions follow from general arguments and are now standard. See, for example, [Kenig and Merle 2010; Killip et al. 2009; Shen 2012] for more details.

In what follows it will be convenient to give a name to the compactness property (3-3) satisfied by the critical element.

Definition 8. Let $I \ni 0$ be a time interval and suppose $\vec{u}(t)$ is a solution to (1-4) on I. We will say $\vec{u}(t)$ has the *compactness property on* I if there exists a continuous function $N : I \to (0, \infty)$ such that the set

$$K := \left\{ \left(\frac{1}{N(t)} u\left(t, \frac{\cdot}{N(t)}\right), \frac{1}{N^2(t)} u_t\left(t, \frac{\cdot}{N(t)}\right) \right) \,\Big|\, t \in I_{\max} \right\}$$

is precompact in $\dot{H}^{1/2} \times \dot{H}^{-1/2}(\mathbb{R}^3)$.

Remark 9. A consequence of a critical element having the compactness property on an interval I is that, after modulation, we can control the $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ tails uniformly in $t \in I$. Indeed, by the Arzelà–Ascoli theorem, for any $\eta > 0$ there exists $C(\eta) < \infty$ such that

$$\begin{aligned} \int_{|x| \geq C(\eta)/N(t)} \big| |\nabla|^{1/2} u(t, x) \big|^2 \, dx + \int_{|\xi| \geq C(\eta) N(t)} |\xi| |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta, \\ \int_{|x| \geq C(\eta)/N(t)} \big| |\nabla|^{-1/2} u_t(t, x) \big|^2 \, dx + \int_{|\xi| \geq C(\eta) N(t)} |\xi|^{-1} |\hat{u}_t(t, \xi)|^2 \, d\xi \leq \eta, \end{aligned} \tag{3-4}$$

for all $t \in I$.

Another standard fact about solutions to (1-4) with the compactness property is that any Strichartz norm of the linear part of the evolution vanishes as $t \to T_-$ and as $t \to T_+$. A concentration compactness argument then implies that the linear part of the evolution must vanish weakly in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$, i.e., for any $t_0 \in I$,

$$S(t_0 - t) u(t) \rightharpoonup 0 \tag{3-5}$$

weakly in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ as $t \nearrow \sup I$, $t \searrow \inf I$; see [Tao et al. 2008, Section 6; Shen 2012, Proposition 3.6]. This implies the following lemma, which will be crucial in the proof of Theorem 1.1.

Lemma 3.2 [Tao et al. 2008, Section 6; Shen 2012, Proposition 3.6]. *Let $\vec{u}(t)$ be a solution to (1-4) with the compactness property on an interval $I = (T_-, T_+)$. Then for any $t_0 \in I$, we can write*

$$\begin{aligned} \int_{t_0}^{T} S(t_0 - s)(0, \pm u^3) \, ds \rightharpoonup \vec{u}(t_0) \quad \text{as } T \nearrow T_+ \quad \text{weakly in } \dot{H}^{1/2} \times \dot{H}^{-1/2}, \\ -\int_{T}^{t_0} S(t_0 - s)(0, \pm u^3) \, ds \rightharpoonup \vec{u}(t_0) \quad \text{as } T \searrow T_- \quad \text{weakly in } \dot{H}^{1/2} \times \dot{H}^{-1/2}. \end{aligned} \tag{3-6}$$

4. Additional regularity for critical elements

In this section we show that the critical element $\vec{u}(t)$ from Section 3 has additional regularity for $t \in I$. In particular, we prove the following result.

Theorem 4.1. *Let $\vec{u}(t)$ be a solution to (1-4) defined on a time interval $I = (T_-, \infty)$ with $T_- < 0$ and suppose that $\vec{u}(t)$ has the compactness property on I with $N(t) \leq 1$ for all $t \in [0, \infty)$. Then for each $t \in I$ we have $\vec{u}(t) \in \dot{H}^1 \times L^2$, and the estimate*

$$\|\vec{u}(t)\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} \lesssim N(t)^{1/2} \tag{4-1}$$

holds with a constant that is uniform for $t \in I$.

Remark 10. We note that all constants this section implicit in the symbol $\lesssim$ will be allowed to depend on the $L_t^\infty(I; \dot{H}^{1/2} \times \dot{H}^{-1/2})$-norm of $\vec{u}$, which is fixed.

We will prove Theorem 4.1 using a bootstrap procedure with two steps. In particular, we will first show that if $\vec{u}(t)$ has the compactness property on an interval I as in Theorem 4.1, then $\vec{u}(t)$ must lie in $\dot{H}^{2/3} \times \dot{H}^{-1/3}$. We then use this result to attain Theorem 4.1.

Proposition 4.2. *Let $\vec{u}(t)$ be as in Theorem 4.1. Then for any $t_0 \in I$, we have*

$$\|\vec{u}(t_0)\|_{\dot{H}^{2/3}\times\dot{H}^{-1/3}(\mathbb{R}^3)} \lesssim N(t_0)^{1/6}. \tag{4-2}$$

We momentarily postpone the proof of Proposition 4.2 and use it to deduce Theorem 4.1.

4A. ***Proof of Theorem 4.1 assuming Proposition 4.2.*** The first step is to establish refined Strichartz estimates.

Lemma 4.3. *Let $\vec{u}(t)$ satisfy the conclusions of Proposition 4.2. Then there exists $\delta > 0$ sufficiently small that for any $t_0 \in I$,*

$$\|u\|_{L_t^3 L_x^6([t_0-\delta/N(t_0), t_0+\delta/N(t_0)]\times\mathbb{R}^3)} \lesssim N(t_0)^{1/6}. \tag{4-3}$$

Proof. To simplify notation, let $J = [t_0 - \delta/N(t_0), t_0 + \delta/N(t_0)]$. We also let $Y = L_t^\infty L_x^{18/5} \cap L_t^3 L_x^6$ with the natural norm. Using Strichartz estimates, we have

$$\begin{aligned}
\|u\|_{Y(J\times\mathbb{R}^3)} &\lesssim \|\vec{u}(t_0)\|_{\dot{H}^{2/3}\times\dot{H}^{-1/3}(\mathbb{R}^3)} + \|u^3\|_{L_t^{6/5} L_x^{3/2}(J\times\mathbb{R}^3)} \\
&\lesssim N(t_0)^{1/6} + (\delta/N(t_0))^{1/3} \|u\|^{3/2}_{L_t^3 L_x^6(J\times\mathbb{R}^3)} \|u\|^{3/2}_{L_t^\infty L_x^{18/5}(J\times\mathbb{R}^3)} \\
&\lesssim N(t_0)^{1/6} + \delta^{1/3} \big(N(t_0)^{-1/6} \|u\|_{Y(J\times\mathbb{R}^3)}\big)^2 \|u\|_{Y(J\times\mathbb{R}^3)}.
\end{aligned} \tag{4-4}$$

Choosing $\delta = \delta\big(N(t_0)^{-1/6}\|u(t_0)\|_{\dot{H}^{2/3}\times\dot{H}^{-1/3}}\big) > 0$ small enough, the lemma follows by a standard bootstrapping argument. We remark that here it is important that the constant in (4-2) is uniform in $t_0 \in I$. $\square$

An immediate consequence of Lemma 4.3 is the following estimate.

Corollary 4.4. *There exists $\delta > 0$ such that for each $t_0 \in I$, we have*

$$\|u^3\|_{L_t^1 L_x^2([t_0-\delta/N(t_0), t_0+\delta/N(t_0)]\times\mathbb{R}^3)} \lesssim N(t_0)^{1/2}. \tag{4-5}$$

We are now ready to begin the proof of Theorem 4.1 assuming Proposition 4.2.

Proof that Proposition 4.2 implies Theorem 4.1. Fix $t_0 \in I$. By translating in time, we can, without loss of generality, assume that $t_0 = 0$. Let

$$v = u + \frac{i}{\sqrt{-\Delta}} u_t. \tag{4-6}$$

Then we have

$$\|v(t)\|_{\dot{H}^1} \simeq \|\vec{u}(t)\|_{\dot{H}^1\times L^2}. \tag{4-7}$$

And if $\vec{u}(t)$ solves (1-4), then $v(t)$ is a solution to

$$v_t = -i\sqrt{-\Delta}v \pm \frac{i}{\sqrt{-\Delta}}u^3, \tag{4-8}$$

where $+$ corresponds to the focusing equation and $-$ to the defocusing equation. By Duhamel's principle, for any T such that $T_- < T < 0$,

$$v(0) = e^{iT\sqrt{-\Delta}}v(T) \pm \frac{i}{\sqrt{-\Delta}} \int_T^0 e^{i\tau\sqrt{-\Delta}}u^3(\tau)\,d\tau. \tag{4-9}$$

Next, we define an approximate identity $\{\psi_M\}_{M>0}$. Indeed, let $\psi \in C_0^\infty(\mathbb{R}^3)$ be a radial function, normalized in $L^1(\mathbb{R}^3)$ so that $\|\psi\|_{L^1} = 1$. Set $\psi_M(x) := M^3\psi(Mx)$. We then define the operator Q_M given by convolution with ψ_M:

$$Q_M f(x) := \int_{\mathbb{R}^3} \psi_M(x-y) f(y)\,dy. \tag{4-10}$$

Of course Q_M is also a Fourier multiplier operator, given by multiplication on the Fourier side by $\widehat{\psi_M}$, where $\widehat{\psi_M}(\xi) = \hat{\psi}\big(\frac{\xi}{M}\big)$. Since $\psi \in C_0^\infty$, we have $\hat{\psi} \in \mathcal{S}(\mathbb{R}^3)$.

With this setup, it clearly suffices to prove that there exists an $M_0 > 0$ such that

$$\|Q_M v(0)\|_{\dot{H}^1} \lesssim N(0)^{1/2} \tag{4-11}$$

for all $M \geq M_0 > 0$ with a constant that is independent of M.

To begin, let $T_- < T_1 < 0 < T_2 < \infty$ and let M be a large number to be determined below. By the Duhamel formula, we have

$$\begin{aligned}\langle Q_M v(0), Q_M v(0)\rangle_{\dot{H}^1} = \bigg\langle Q_M\bigg(e^{iT_2\sqrt{-\Delta}}v(T_2) \mp \frac{i}{\sqrt{-\Delta}} \int_0^{T_2} e^{it\sqrt{-\Delta}}u^3\,dt\bigg),&\\ Q_M\bigg(e^{iT_1\sqrt{-\Delta}}v(T_1) \pm \frac{i}{\sqrt{-\Delta}} \int_{T_1}^0 e^{i\tau\sqrt{-\Delta}}u^3\,d\tau\bigg)\bigg\rangle_{\dot{H}^1}&,\end{aligned} \tag{4-12}$$

where the bracket $\langle\cdot,\cdot\rangle_{\dot{H}^1}$ is the $\dot{H}^1$ inner product, namely

$$\langle f, g\rangle_{\dot{H}^1} = \operatorname{Re}\int_{\mathbb{R}^3} \sqrt{-\Delta}f \cdot \overline{\sqrt{-\Delta}g}.$$

We start by estimating the term that contains both Duhamel terms:

$$\begin{aligned}&\bigg|\bigg\langle Q_M\bigg(\frac{i}{\sqrt{-\Delta}} \int_0^{T_2} e^{it\sqrt{-\Delta}}u^3(t)\,dt\bigg), Q_M\bigg(\frac{i}{\sqrt{-\Delta}} \int_0^{T_1} e^{i\tau\sqrt{-\Delta}}u^3(\tau)\,d\tau\bigg)\bigg\rangle_{\dot{H}^1}\bigg|\\ &\qquad = \bigg|\bigg\langle Q_M\bigg(\int_0^{T_2} e^{it\sqrt{-\Delta}}u^3(t)\,dt\bigg), Q_M\bigg(\int_{T_1}^0 e^{i\tau\sqrt{-\Delta}}(u^3(\tau))\,d\tau\bigg)\bigg\rangle_{L^2}\bigg|.\end{aligned} \tag{4-13}$$

With $\delta > 0$ as in Corollary 4.4, we use (4-5) to deduce that

$$\int_{-\delta/N(0)}^{\delta/N(0)} \|Q_M(u^3(t))\|_{L_x^2(\mathbb{R}^3)}\,dt \lesssim N(0)^{1/2}. \tag{4-14}$$

Next, define a decreasing, smooth, radial function $\chi \in C_0^\infty(\mathbb{R}^3)$, with $\chi(x) \equiv 1$ for all $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Also, let $c > 0$ be a small constant, say $c = \frac{1}{4}$. We have

$$\left\| Q_M\left(\int_{\delta/N(0)}^\infty e^{it\sqrt{-\Delta}}\left(1-\chi\left(\frac{x}{c|t|}\right)\right)u^3(t)\,dt\right)\right\|_{L_x^2(\mathbb{R}^3)} \lesssim \int_{\delta/N(0)}^\infty \left\|(1-\chi)\left(\frac{x}{c|t|}\right)u^3(t)\right\|_{L_x^2(\mathbb{R}^3)}dt. \tag{4-15}$$

By the radial Sobolev embedding (i.e., Lemma 2.2), we note that

$$\||x|^{3/2}u^3\|_{L_x^2} \lesssim \||x|^{1/2}u\|_{L_x^6}^3 \lesssim \|u\|_{\dot H^{1/2}}. \tag{4-16}$$

Therefore,

$$\left\|(1-\chi)\left(\frac{x}{c|t|}\right)u^3\right\|_{L_x^2(\mathbb{R}^3)} \lesssim \frac{1}{|t|^{3/2}}\|u\|_{\dot H^{1/2}}^3. \tag{4-17}$$

Thus,

$$\int_{\delta/N(0)}^\infty \left\|Q_M\left((1-\chi)\left(\frac{x}{c|t|}\right)u^3(t)\right)\right\|_{L_x^2(\mathbb{R}^3)}dt \lesssim \delta^{-1/2}N(0)^{1/2}. \tag{4-18}$$

The same is also true in the negative time direction. With these estimates in hand, we write (4-13) as a pairing

$$\langle A+B, A'+B'\rangle = \langle A+B, A'\rangle + \langle A, A'+B'\rangle + \langle B, B'\rangle - \langle A, A'\rangle, \tag{4-19}$$

where

$$\begin{aligned} A &:= Q_M\left(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}}u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3\,dt\right), \\ B &:= Q_M\left(\int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}}\chi\left(\frac{x}{c|t|}\right)u^3(t)\,dt\right), \end{aligned} \tag{4-20}$$

and A', B' are the corresponding integrals in the negative time direction. We start by estimating the term $\langle A, A'\rangle$. By (4-14) and (4-18),

$$\begin{aligned} &\Big\langle Q_M\left(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}}u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3\,dt\right), \\ &\qquad Q_M\left(\int_{-\delta/N(0)}^0 e^{i\tau\sqrt{-\Delta}}u^3\,d\tau + \int_{T_1}^{-\delta/N(0)} e^{i\tau\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|\tau|}\right)u^3\,d\tau\right)\Big\rangle_{L^2} \lesssim N(0). \end{aligned} \tag{4-21}$$

Next, we examine the term $\langle B, B'\rangle$, which is given by

$$\begin{aligned} &\int_{T_1}^{-\delta/N(0)}\int_{\delta/N(0)}^{T_2}\Big\langle Q_M\left(e^{it\sqrt{-\Delta}}\chi\left(\frac{x}{c|t|}\right)u^3(t)\right), Q_M\left(e^{i\tau\sqrt{-\Delta}}\chi\left(\frac{x}{c|\tau|}\right)u^3(\tau)\right)\Big\rangle_{L^2}dt\,d\tau \\ &\qquad = \int_{T_1}^{-\delta/N(0)}\int_{\delta/N(0)}^{T_2}\Big\langle Q_M\left(\chi\left(\frac{x}{c|t|}\right)u^3(t)\right), Q_M\left(e^{i(\tau-t)\sqrt{-\Delta}}\chi\left(\frac{x}{c|\tau|}\right)u^3(\tau)\right)\Big\rangle_{L^2}dt\,d\tau. \end{aligned}$$

To estimate the above, we begin by noting that the sharp Huygens principle implies that

$$\left(e^{i(t-\tau)\sqrt{-\Delta}}\chi\left(\frac{\cdot}{c|\tau|}\right)u^3(\tau)\right)(x)$$

is supported on the set $|x| > \frac{3}{4}|t-\tau|$ for $c = \frac{1}{4}$. Also, we note that since $t > 0$ and $\tau < 0$, we have $|t-\tau| > |t|, |\tau|$. Next, recall that the kernel of Q_M is given by the function $\psi_M(x) = M^3\psi(Mx)$, where $\psi \in C_0^\infty$. This implies that for $M \gg N(0)^{-1}$ large enough, we have

$$\operatorname{supp}\left(\int_{\mathbb{R}^3} \psi_M(x-z)\chi\left(\frac{z}{c|t|}\right)u^3(t,z)\,dz\right) \subset \left\{|x| < \frac{1}{2}|t|\right\},$$
$$\operatorname{supp}\left(\int_{\mathbb{R}^3} \psi_M(x-y)\left(e^{i(\tau-t)\sqrt{-\Delta}}\chi\left(\frac{\cdot}{c|\tau|}\right)u^3(\tau)\right)(y)\,dy\right) \subset \left\{|x| > \frac{1}{2}|t-\tau|\right\}.$$

Therefore, as long as M is chosen large enough, say for $M \ge M_0 \gg N(0)^{-1}$, and since $|t| < |t-\tau|$ for $t > \delta/N(0)$ and $\tau < -\delta/N(0)$, we have

$$\left\langle \int \psi_M(x-z)\chi\left(\frac{z}{c|t|}\right)u^3(t,z)\,dz, \int \psi_M(x-y)\left(e^{i(\tau-t)\sqrt{-\Delta}}\chi\left(\frac{\cdot}{c|\tau|}\right)u^3(\tau)\right)(y)\,dy\right\rangle_{L^2} = 0. \tag{4-22}$$

It remains to estimate the terms $\langle A, A'+B'\rangle$ and $\langle A+B, A'\rangle$, which are given by

$$\left\langle Q_M\left(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}}u^3(t)\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3(t)\,dt\right),\right.$$
$$\left. Q_M\left(\int_{T_1}^0 e^{i\tau\sqrt{-\Delta}}u^3(\tau)\,d\tau\right)\right\rangle_{L^2} \tag{4-23}$$

and

$$\left\langle Q_M\left(\int_0^{T_2} e^{it\sqrt{-\Delta}}u^3(t)\,dt\right),\right.$$
$$\left. Q_M\left(\int_{T_1}^{-\delta/N(0)} e^{i\tau\sqrt{-\Delta}}(1-\chi)\left(\frac{\cdot}{c|\tau|}\right)u^3(\tau)\,d\tau + \int_{-\delta/N(0)}^0 e^{i\tau\sqrt{-\Delta}}u^3(\tau)\,d\tau\right)\right\rangle_{L^2}. \tag{4-24}$$

We provide the details for how to handle (4-23), as the estimates for (4-24) are similar. First recall that by the Duhamel principle (4-9), we can write

$$Q_M\int_{T_1}^0 e^{i\tau\sqrt{-\Delta}}u^3(\tau)\,d\tau = \mp i\sqrt{-\Delta}Q_M v(0) \pm i\sqrt{-\Delta}e^{iT_1\sqrt{-\Delta}}Q_M v(T_1). \tag{4-25}$$

Using again (4-14) and (4-18), we have

$$\left|\left\langle \sqrt{-\Delta}Q_M v(0), Q_M\left(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}}u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3\,dt\right)\right\rangle_{L^2}\right|$$
$$\lesssim N(0)^{1/2}\|Q_M v(0)\|_{\dot H^1(\mathbb{R}^3)}. \tag{4-26}$$

We remark that all of the estimates established so far have been uniform in $T_- < T_1 < 0 < T_2 < T_+$. This is important as we will now take limits, $T_1 \searrow T_-$ and $T_2 \nearrow T_+$. Indeed, using the weak convergence result in Lemma 3.2, we claim that for any fixed $T_2 \in (0, T_+)$, we have

$$\lim_{T_1 \searrow T_-} \Big\langle i\sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} Q_M v(T_1), \\ Q_M\Big(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}} u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}} (1-\chi)\Big(\frac{x}{c|t|}\Big) u^3\,dt\Big)\Big\rangle_{L^2} = 0. \tag{4-27}$$

In fact, (4-14) and (4-18) imply that letting $T_2 \nearrow T_+$, for M fixed,

$$(-\Delta)^{1/4} Q_M\Big(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}} u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}} (1-\chi)\Big(\frac{x}{c|t|}\Big) u^3\,dt\Big)$$

converges in $L^2(\mathbb{R}^3)$ to

$$(-\Delta)^{1/4} Q_M\Big(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}} u^3\,dt + \int_{\delta/N(0)}^{T_+} e^{it\sqrt{-\Delta}} (1-\chi)\Big(\frac{x}{c|t|}\Big) u^3\,dt\Big) \in L^2(\mathbb{R}^3).$$

Therefore, since Lemma 3.2 says that $e^{iT_1\sqrt{-\Delta}} v(T_1) \rightharpoonup 0$ weakly in $\dot H^{1/2}(\mathbb{R}^3)$ as $T_1 \searrow T_-$, we have

$$\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \Big\langle \sqrt{-\Delta} e^{iT_1\sqrt{-\Delta}} Q_M v(T_1), \\ Q_M\Big(\int_0^{\delta/N(0)} e^{it\sqrt{-\Delta}} u^3\,dt + \int_{\delta/N(0)}^{T_2} e^{it\sqrt{-\Delta}} (1-\chi)\Big(\frac{x}{c|t|}\Big) u^3\,dt\Big)\Big\rangle_{L^2} = 0. \tag{4-28}$$

Thus we have proved that

$$\Big|\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \langle A, A' + B'\rangle\Big| \lesssim N(0)^{1/2} \|Q_M v(0)\|_{\dot H^1}. \tag{4-29}$$

Using an identical argument, we can similarly prove that

$$\Big|\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \langle A + B, A'\rangle\Big| \lesssim N(0)^{1/2} \|Q_M v(0)\|_{\dot H^1}, \tag{4-30}$$

where A, B, A', B' are defined as in (4-20). Therefore, combining (4-21), (4-26), (4-29), and (4-30), we have proved that

$$\Big|\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \int_0^{T_2}\int_{T_1}^0 \big\langle e^{it\sqrt{-\Delta}} Q_M(u^3), e^{i\tau\sqrt{-\Delta}} Q_M(u^3)\big\rangle_{L^2}\,dt\,d\tau\Big| \lesssim \|v(0)\|_{\dot H^1(\mathbb{R}^3)} N(0)^{1/2} + N(0). \tag{4-31}$$

We are left to examine the terms in (4-12) (once expanded) that contain at most one Duhamel integral. Here we will rely heavily on the $\dot H^{1/2}$-weak convergence in Lemma 3.2.

Indeed, for a fixed T_1 and fixed M, we see that $\sqrt{-\Delta} Q_M v(T_1) \in \dot H^{1/2}(\mathbb{R}^3)$. Therefore, by Lemma 3.2, we have

$$\lim_{T_2 \nearrow T_+} \big\langle e^{iT_1\sqrt{-\Delta}} Q_M v(T_1),\ e^{iT_2\sqrt{-\Delta}} Q_M v(T_2)\big\rangle_{\dot H^1} = 0. \tag{4-32}$$

Next, for fixed $T_1 > T_-$, Corollary 4.4 and the bound (4-5) imply that

$$\|u^3\|_{L^1_t L^2_x([T_1,0]\times\mathbb{R}^3)} < \infty,$$

which in turn implies that $\int_{T_1}^{0} Q_M e^{it\sqrt{-\Delta}} u^3\,dt \in \dot H^{1/2}(\mathbb{R}^3)$, where again we are using that the multiplier $\widehat{\psi_M}$ is in $\mathcal{S}(\mathbb{R}^3)$. Therefore, Lemma 3.2 implies

$$\lim_{T_2 \nearrow T_+} \left\langle Q_M \left(\int_{T_1}^{0} e^{it\sqrt{-\Delta}} u^3\,dt \right), e^{iT_2\sqrt{-\Delta}} Q_M v(T_2) \right\rangle_{\dot H^{1/2}} = 0. \tag{4-33}$$

Finally, we claim that

$$\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \left\langle Q_M \big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), Q_M \left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3\,d\tau \right) \right\rangle_{\dot H^{1/2}} = 0. \tag{4-34}$$

To see this, we use (4-25). Indeed, using Lemma 3.2 again, we have

$$\begin{aligned} \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} & \left\langle Q_M \big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), \sqrt{-\Delta}\, Q_M \big(v(0) - e^{iT_2\sqrt{-\Delta}} v(T_2)\big) \right\rangle_{\dot H^{1/2}} \\ &= \lim_{T_1 \searrow T_-} \left\langle Q_M \big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), Q_M \big(\sqrt{-\Delta}\, v(0)\big) \right\rangle_{\dot H^{1/2}} = 0. \end{aligned} \tag{4-35}$$

Therefore, (4-12) together with (4-31)–(4-35) imply that

$$\|Q_M v(0)\|_{\dot H^1(\mathbb{R}^3)}^2 \lesssim \|Q_M v(0)\|_{\dot H^1(\mathbb{R}^3)} N(0)^{1/2} + N(0), \tag{4-36}$$

for all $M \geq M_0$ and with a uniform-in-M constant. We can then conclude that

$$\|Q_M v(0)\|_{\dot H^1(\mathbb{R}^3)} \lesssim N(0)^{1/2} \tag{4-37}$$

uniformly in $M \geq M_0$. Therefore, $\|v(0)\|_{\dot H^1(\mathbb{R}^3)} \lesssim N(0)^{1/2}$. This proves Theorem 4.1, assuming the conclusions of Proposition 4.2. □

4B. *Proof of Proposition 4.2.*

To complete the proof of Theorem 4.1 we prove Proposition 4.2. We begin with another refined Strichartz-type estimate.

Lemma 4.5. *Let $\eta > 0$. There exists $\delta = \delta(\eta) > 0$ such that for all $t_0 \in I$ we have*

$$\|u\|_{L^4_{t,x}([t_0 - \delta/N(t_0), t_0 + \delta/N(t_0)] \times \mathbb{R}^3)} \lesssim \eta. \tag{4-38}$$

Proof. Again, without loss of generality, suppose that $t_0 = 0$. Then define the interval $J = \left[-\frac{\delta}{N(0)}, \frac{\delta}{N(0)} \right]$. Using the Duhamel formula, we have

$$\|u\|_{L^4_{t,x}(J \times \mathbb{R}^3)} \leq \|S(t)\vec u(0)\|_{L^4_{t,x}(J \times \mathbb{R}^3)} + \left\| \int_0^t S(t-s)(0, \pm u^3)\,ds \right\|_{L^4_{t,x}(J \times \mathbb{R}^3)}. \tag{4-39}$$

We estimate the first term on the right side of (4-39) as follows. First choose $C(\eta)$ as in Remark 9, (3-4), so that

$$\|P_{\geq C(\eta)N(0)} \vec u(0)\|_{\dot H^{1/2} \times \dot H^{-1/2}(\mathbb{R}^3)} \leq \eta. \tag{4-40}$$

Note that by compactness, $C(\eta)$ above can be chosen uniformly in $t \in I$, which is why it suffices to only consider $t_0 = 0$ in this argument. Next, we have

$$\|S(t)\vec u(0)\|_{L^4_{t,x}(J \times \mathbb{R}^3)} \lesssim \|S(t) P_{\geq C(\eta)N(0)} \vec u(0)\|_{L^4_{t,x}(J \times \mathbb{R}^3)} + \|S(t) P_{\leq C(\eta)N(0)} \vec u(0)\|_{L^4_{t,x}(J \times \mathbb{R}^3)}. \tag{4-41}$$

We use (4-40) together with Strichartz estimates to handle the first term on the right side above:

$$\|S(t)P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{L^4_{t,x}(J\times\mathbb{R}^3)} \lesssim \|P_{\geq C(\eta)N(0)}\vec{u}(0)\|_{\dot{H}^{1/2}\times\dot{H}^{-1/2}(\mathbb{R}^3)} \lesssim \eta. \tag{4-42}$$

To control the second term we use Bernstein's inequalities, (2-1), and Sobolev embedding,

$$\|P_{\leq C(\eta)N(0)}S(t)\vec{u}(0)\|_{L^4_x(\mathbb{R}^3)} \lesssim C(\eta)^{1/4}N(0)^{1/4}\|u(0)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}. \tag{4-43}$$

Taking the $L^4_t(J)$-norm of both sides above gives

$$\|S(t)P_{\leq C(\eta)N(0)}\vec{u}(0)\|_{L^4_{t,x}(J\times\mathbb{R}^3)} \lesssim C(\eta)^{1/4}\delta^{1/4}. \tag{4-44}$$

Next we use Strichartz estimates on the second term on the right side of (4-39).

$$\left\| \int_0^t S(t-s)(0,\pm u^3)\,ds \right\|_{L^4_{t,x}(J\times\mathbb{R}^3)} \lesssim \|u^3\|_{L^{4/3}_{t,x}(J\times\mathbb{R}^3)} \lesssim \|u\|^3_{L^4_{t,x}(J\times\mathbb{R}^3)}. \tag{4-45}$$

Combining all of the above, we obtain

$$\|u\|_{L^4_{t,x}(J\times\mathbb{R}^3)} \lesssim \eta + C(\eta)^{1/4}\delta^{1/4} + \|u\|^3_{L^4_{t,x}(J\times\mathbb{R}^3)}. \tag{4-46}$$

The proof is concluded using the usual continuity argument after taking δ small enough. $\square$

Proof of Proposition 4.2. We can again, without loss of generality, just consider the case $t_0 = 0$. We will prove Proposition 4.2 by finding a frequency envelope $\alpha_k(0)$ such that

$$\|(P_k u(0), P_k u_t(0))\|_{\dot{H}^{2/3}\times\dot{H}^{-1/3}} \lesssim 2^{k/6}\alpha_k(0), \quad \|\{2^{k/6}\alpha_k(0)\}_{k\in\mathbb{Z}}\|_{\ell^2} \lesssim N(0)^{1/6}. \tag{4-47}$$

Once we find $\alpha_k(0)$ satisfying (4-47), Proposition 4.2 follows from Definition 3. With this in mind we first establish the following claim:

Claim 4.6. *There exists a number $\eta_0 > 0$ such that the following holds. Let $0 < \eta < \eta_0$ and let $J := [-\delta/N(0), \delta/N(0)]$, where $\delta = \delta(\eta) > 0$ is chosen as in Lemma 4.5. Define*

$$\begin{aligned} a_k &:= 2^{k/2}\|P_k u\|_{L^\infty_t L^2_x(J)} + 2^{-k/2}\|P_k u_t\|_{L^\infty_t L^2_x(J)} + 2^{k/4}\|P_k u\|_{L^8_t L^{8/3}_x(J)}, \\ a_k(0) &:= 2^{k/2}\|P_k u(0)\|_{L^2_x(\mathbb{R}^3)} + 2^{-k/2}\|P_k u_t(0)\|_{L^2_x(\mathbb{R}^3)}. \end{aligned} \tag{4-48}$$

Next define frequency envelopes α_k and $\alpha_k(0)$ by

$$\alpha_k := \sum_j 2^{-|j-k|/8}a_j, \quad \alpha_k(0) := \sum_j 2^{-|j-k|/8}a_j(0). \tag{4-49}$$

Then, as long as η_0 is chosen small enough, we have

$$a_k \lesssim a_k(0) + \eta^2 \sum_{j\geq k-3} 2^{(k-j)/4}a_j \tag{4-50}$$

and

$$\alpha_k \lesssim \alpha_k(0). \tag{4-51}$$

Proof of Claim 4.6. To prove (4-50), we note that Strichartz estimates, together with Lemma 4.5, imply that

$$\begin{aligned} a_k &= 2^{k/2}\|P_k u\|_{L_t^\infty L_x^2(J)} + 2^{k/4}\|P_k u\|_{L_t^8 L_x^{8/3}(J)} \\ &\lesssim 2^{k/2}\|P_k u(0)\|_{L_x^2(\mathbb{R}^3)} + 2^{-k/2}\|P_k u_t(0)\|_{L_x^2(\mathbb{R}^3)} + 2^{k/4}\|P_k(u^3)\|_{L_t^{8/5}L_x^{8/7}(J)} \\ &\lesssim a_k(0) + \eta^2 \sum_{j\geq k-3} 2^{(k-j)/4} a_j. \end{aligned} \tag{4-52}$$

To prove the last line above we note that it will suffice, by Hölder's inequality in time and Lemma 4.5, to show that

$$\|P_k(u^3)\|_{L_x^{8/7}} \lesssim \|u\|_{L^4}^2 \sum_{j\geq k-3} \|P_j u\|_{L^{8/3}}. \tag{4-53}$$

First, since $P_k((P_{\leq k-4}u)^3) = 0$, we have

$$\begin{aligned} \|P_k u^3\|_{L_x^{8/7}} &\lesssim \big\|P_k[(P_{\leq k-4}u)^2 P_{\geq k-3}u]\big\|_{L^{8/7}} + \big\|P_k[P_{\leq k-4}u(P_{\geq k-3}u)^2]\big\|_{L^{8/7}} + \|P_k[P_{\geq k-3}u]^3\|_{L^{8/7}} \\ &\lesssim \|u\|_{L^4}^2 \|P_{\geq k-3}u\|_{L^{8/3}}, \end{aligned}$$

where the last inequality follows from the boundedness of P_k on L^p and by Holder's inequality. This proves (4-53), and thus we have established (4-50).

To prove (4-51), we use (4-50) to obtain

$$\sum_j 2^{-|j-k|/8} a_j \lesssim \sum_j a_j(0) 2^{-|j-k|/8} + \eta^2 \sum_j 2^{-|j-k|/8} \sum_{j_1\geq j-3} 2^{(j-j_1)/4} a_{j_1}. \tag{4-54}$$

Reversing the order of summation in the second term above gives

$$\begin{aligned} \sum_{j_1\leq k}\sum_{j\leq j_1+3} 2^{(j-j_1)/4} 2^{(j-k)/8} a_{j_1} &\lesssim \sum_{j_1\leq k} 2^{(j_1-k)/8} a_{j_1} \lesssim \alpha_k, \\ \sum_{j_1> k}\sum_{j\leq j_1+3} 2^{(j-j_1)/4} 2^{-|j-k|/8} a_{j_1} &\lesssim \sum_{j_1>k} (2^{-(k-j_1)/4} + 2^{-(k-j_1)/8}) a_{j_1} \lesssim \alpha_k. \end{aligned} \tag{4-55}$$

Therefore, (4-54) implies that

$$\alpha_k \lesssim \alpha_k(0) + \eta^2 \alpha_k, \tag{4-56}$$

which in turn yields (4-51) as long as $\eta > 0$ is small enough. □

We now return to the proof of Proposition 4.2. We note that the calculation in the proof of Claim 4.6 also allows us to deduce that

$$2^{k/2}\left\| P_k \int_0^{\delta/N(0)} \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3(t)\,dt \right\|_{L_x^2(\mathbb{R}^3)} \lesssim \eta^2 \sum_{j\geq k-3} 2^{(k-j)/4} a_j. \tag{4-57}$$

Next, we claim that for any $s_0 \in \left(0, \tfrac{1}{2}\right]$ we have the estimate

$$\int_{\delta/N(0)}^\infty \left\| \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}} (1-\chi)\left(\frac{x}{c|t|}\right) u^3 \right\|_{\dot{H}^{1/2+s_0}(\mathbb{R}^3)} dt \lesssim N(0)^{s_0}\delta^{-s_0}, \tag{4-58}$$

where $c > 0$ is a fixed small constant ($c = \frac{1}{4}$ will do) and $\chi \in C_0^\infty(\mathbb{R}^3)$ is radial, $\chi(x) = 1$ for all $|x| \le 1$, and $\chi(x) = 0$ for all $|x| \ge 2$. To prove (4-58), we note that by Sobolev embedding,

$$\left\| \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3 \right\|_{\dot{H}^{1/2+s_0}} = \left\| (1-\chi)\left(\frac{x}{c|t|}\right)u^3 \right\|_{\dot{H}^{-1/2+s_0}} \lesssim \left\| (1-\chi)\left(\frac{x}{c|t|}\right)u^3 \right\|_{L^p},$$

where $\frac{1}{p} = \frac{2}{3} - \frac{s_0}{3}$. Then using the radial Sobolev embedding, i.e., Lemma 2.2, we have

$$(c|t|)^{1+s_0} \left\| (1-\chi)\left(\frac{x}{c|t|}\right)u^3 \right\|_{L^p} \lesssim \left\| (1-\chi)^{1/3}\left(\frac{x}{c|t|}\right)|x|^{(1+s_0)/3}u \right\|_{L^{3p}}^3 \lesssim \|u\|_{\dot{H}^{1/2}}^3.$$

Hence,

$$\left\| \frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\left(\frac{x}{c|t|}\right)u^3 \right\|_{\dot{H}^{1/2+s_0}} \lesssim |t|^{-1-s_0}\|u\|_{L_t^\infty \dot{H}^{1/2}}.$$

Integrating the above in time from $t = \delta/N(0)$ to $t = +\infty$ then yields (4-58).

Once again by the weak convergence result in Lemma 3.2, we have

$$\langle P_k v(0), P_k v(0)\rangle_{\dot{H}^{1/2}} = \left\langle P_k v(0), P_k\left(\lim_{T_2 \nearrow T_+} \pm \frac{i}{\sqrt{-\Delta}} \int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3(\tau)\, d\tau \right) \right\rangle_{\dot{H}^{1/2}},$$

which for all $T_- < T_1 < 0$ is equal to

$$\begin{aligned} &\lim_{T_2 \nearrow T_+} \left\langle P_k\big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), \frac{\pm i}{\sqrt{-\Delta}} P_k\left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3\, d\tau \right) \right\rangle_{\dot{H}^{1/2}} \\ &\qquad + \lim_{T_2 \nearrow T_+} \left\langle \frac{1}{\sqrt{-\Delta}} P_k\left(\int_{T_1}^0 e^{it\sqrt{-\Delta}} u^3\, dt \right), \frac{1}{\sqrt{-\Delta}} P_k\left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3\, d\tau \right) \right\rangle_{\dot{H}^{1/2}}. \end{aligned} \tag{4-59}$$

As $T_1 \searrow T_-$, we note that (4-59) $\to 0$. Indeed, by (4-9),

$$\begin{aligned} &\lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \left\langle P_k\big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), \frac{\pm i}{\sqrt{-\Delta}} P_k\left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3\, d\tau \right) \right\rangle_{\dot{H}^{1/2}} \\ &\qquad = \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \left\langle P_k\big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), P_k\big(v(0) - e^{iT_2\sqrt{-\Delta}} v(T_2)\big) \right\rangle_{\dot{H}^{1/2}} \\ &\qquad = \lim_{T_1 \searrow T_-} \left\langle P_k\big(e^{iT_1\sqrt{-\Delta}} v(T_1)\big), P_k(v(0)) \right\rangle_{\dot{H}^{1/2}} = 0. \end{aligned} \tag{4-60}$$

Therefore,

$$\langle P_k v(0), P_k v(0)\rangle_{\dot{H}^{1/2}} = \lim_{T_1 \searrow T_-} \lim_{T_2 \nearrow T_+} \left\langle \frac{1}{\sqrt{-\Delta}} P_k\left(\int_{T_1}^0 e^{it\sqrt{-\Delta}} u^3\, dt \right), \frac{1}{\sqrt{-\Delta}} P_k\left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}} u^3\, d\tau \right) \right\rangle_{\dot{H}^{1/2}}.$$

To estimate the right side above, we split each term into two pieces and use the identity

$$\langle A + B, A' + B'\rangle = \langle A + B, A'\rangle + \langle A, A' + B'\rangle - \langle A, A'\rangle + \langle B, B'\rangle, \tag{4-61}$$

where

$$A := P_k\biggl(\int_{-\delta/N(0)}^{0}\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}u^3\,dt\biggr)+P_k\biggl(\int_{T_1}^{-\delta/N(0)}\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Bigl(\frac{x}{c|t|}\Bigr)u^3\,dt\biggr),$$

$$B := P_k\biggl(\int_{T_1}^{-\delta/N(0)}\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}\chi\Bigl(\frac{x}{c|t|}\Bigr)u^3\,dt\biggr),$$

and A', B' are the analogous quantities in the positive time direction.

We begin by estimating the first two terms on the right side of (4-61). In fact, an identical argument applies to both of these terms, so we only provide details for the term $\langle A+B, A'\rangle$. To begin, we note that by (4-58), we have

$$\biggl\| P_k\biggl(\int_{\delta/N(0)}^{T_+}\frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Bigl(\frac{x}{c|\tau|}\Bigr)u^3\,d\tau\biggr)\biggr\|_{\dot H^{1/2}}\lesssim 2^{-k/6}b_k, \tag{4-62}$$

where $b_k = 2^{-k(s_0(k)-1/6)}N(0)^{s_0(k)}$ and we are free to choose any $s_0 = s_0(k)\in\bigl(0,\frac12\bigr]$. Setting $s_0(k)=\frac{5}{24}$ if $2^k \ge N(0)$ and $s_0(k)=\frac18$ if $2^k < N(0)$, we have

$$b_k := \begin{cases} 2^{-k/24}N(0)^{5/24}\delta^{-5/24} & \text{if } 2^k \ge N(0),\\ 2^{k/24}N(0)^{1/8}\delta^{-1/8} & \text{if } 2^k < N(0).\end{cases} \tag{4-63}$$

Then

$$\|\{b_k\}\|_{\ell^2}\lesssim N(0)^{1/6}. \tag{4-64}$$

Therefore with b_k as in (4-63), we can combine (4-57) and (4-62) to deduce that

$$\biggl\| P_k\biggl(\int_0^{\delta/N(0)}\frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}u^3\,dt\biggr)\biggr\|_{\dot H^{1/2}}+\biggl\| P_k\biggl(\int_{\delta/N(0)}^{T_+}\frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Bigl(\frac{x}{c|\tau|}\Bigr)u^3\,d\tau\biggr)\biggr\|_{\dot H^{1/2}} \lesssim \eta^2\sum_{j\ge k-3}2^{(k-j)/4}a_j+2^{-k/6}b_k.$$

Then since

$$\frac{\pm i}{\sqrt{-\Delta}}\int_{T_1}^{0}e^{it\sqrt{-\Delta}}u^3\,dt \rightharpoonup v(0) \quad \text{in } \dot H^{1/2} \quad \text{as } T_1\searrow T_-, \tag{4-65}$$

we can deduce the estimate

$$\begin{aligned}\lim_{T_1\searrow T_-}\lim_{T_2\nearrow T_+}\langle A+B, A'\rangle_{\dot H^{1/2}} &= \lim_{T_1\searrow T_-}\lim_{T_2\nearrow T_+}\biggl\langle P_k\biggl(\int_{T_1}^{0}\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}u^3\,dt\biggr),\\ &\qquad P_k\biggl(\int_0^{\delta/N(0)}\frac{e^{-i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}u^3\,d\tau\biggr)+P_k\biggl(\int_{\delta/N(0)}^{T_2}\frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Bigl(\frac{x}{c|\tau|}\Bigr)u^3\,d\tau\biggr)\biggr\rangle_{\dot H^{1/2}}\\ &\lesssim a_k(0)\biggl(\eta^2\sum_{j\ge k-3}2^{(k-j)/4}a_j+2^{-k/6}b_k\biggr).\end{aligned} \tag{4-66}$$

An identical calculation in the other time direction gives the same estimate for $\langle A, A'+B'\rangle$. Next, we estimate $\langle A, A'\rangle$, again using (4-57) and (4-62). We have

$$\begin{aligned}\Big\langle P_k\Big(\int_0^{\delta/N(0)} \frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3\,d\tau\Big) + P_k\Big(\int_{\delta/N(0)}^{T_2} \frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Big(\frac{x}{c|\tau|}\Big)u^3\,d\tau\Big),\\ P_k\Big(\int_{-\delta/N(0)}^{0} \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u^3\,dt\Big) + P_k\Big(\int_{T_1}^{-\delta/N(0)} \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}(1-\chi)\Big(\frac{x}{c|t|}\Big)u^3\,dt\Big)\Big\rangle_{\dot H^{1/2}}\\ \lesssim \Big(\eta^2\sum_{j\ge k-3} 2^{(k-j)/4}a_j\Big)^2 + 2^{-k/3}b_k^2. \qquad (4\text{-}67)\end{aligned}$$

Finally, it remains to estimate $\langle B, B'\rangle$, which is given by

$$\begin{aligned}&\int_{\delta/N(0)}^{T_2}\int_{T_1}^{-\delta/N(0)} \Big\langle P_k\Big(\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}\chi\Big(\frac{x}{c|t|}\Big)u^3(t)\Big), P_k\Big(\frac{e^{i\tau\sqrt{-\Delta}}}{\sqrt{-\Delta}}\chi\Big(\frac{x}{c|\tau|}\Big)u^3(\tau)\Big)\Big\rangle_{\dot H^{1/2}} dt\,d\tau\\ &\quad = \int_{\delta/N(0)}^{T_2}\int_{T_1}^{-\delta/N(0)} \Big\langle \chi\Big(\frac{x}{c|t|}\Big)u^3(t), P_k^2\Big(\frac{e^{i(\tau-t)\sqrt{-\Delta}}}{\sqrt{-\Delta}}\chi\Big(\frac{\cdot}{c|\tau|}\Big)u^3(\tau)\Big)(x)\Big\rangle_{L^2} dt\,d\tau. \qquad (4\text{-}68)\end{aligned}$$

Here we perform an argument similar to our use of the sharp Huygens principle in the proof of (4-22). The kernel of $P_k^2 e^{i(t-\tau)\sqrt{-\Delta}}(\sqrt{-\Delta})^{-1}$ is given by

$$K_k(x) = K_k(|x|) = c\int_0^{2\pi}\int e^{i|x|\rho\cos\theta}e^{i(\tau-t)\rho}\rho^{-1}\phi\Big(\frac{\rho}{2^k}\Big)\rho^2\,d\rho\,\sin\theta\,d\theta, \qquad (4\text{-}69)$$

where the integrand is written in polar coordinates on $\mathbb{R}^3$, where $\rho = |\xi|$. The function $\phi(\,\cdot\,/2^k)$ above is the Fourier multiplier for the Littlewood–Paley projection, P_k, and its support is contained in $\rho \in [2^{k-1}, 2^{k+1}]$. Integration by parts $L \in \mathbb{N}$ times in ρ gives the estimates

$$|K_k(x-y)| \lesssim_L \frac{2^{2k}}{\big\langle 2^k\big|(\tau-t) - |x-y|\big|\big\rangle^L}. \qquad (4\text{-}70)$$

In (4-68) we have $|x| \le \frac14|t|$, $|y| \le \frac14|\tau|$, and therefore $|x-y| \le \frac14|\tau-t|$. Thus we have

$$(\tau-t) - |x-y| \ge \tfrac12|\tau-t|$$

and hence

$$|K_k(x-y)| \lesssim_L \frac{2^{2k}}{\langle 2^k|\tau-t|\rangle^L}. \qquad (4\text{-}71)$$

If $2^k \gg N(0)$, we use (4-71) with $L=5$ to obtain

$$\begin{aligned}\int_{\delta/N(0)}^{T_2}\int_{T_1}^{-\delta/N(0)} \Big\langle \chi\Big(\frac{x}{c|t|}\Big)u^3(t), P_k^2\Big(\frac{e^{i(\tau-t)\sqrt{-\Delta}}}{\sqrt{-\Delta}}\chi\Big(\frac{\cdot}{c|\tau|}\Big)u^3(\tau)\Big)(x)\Big\rangle_{L^2} dt\,d\tau\\ \lesssim \|u\|_{L_t^\infty L_x^3}^6 2^{-3k}N(0)^3 \lesssim 2^{-3k}N(0)^3 \lesssim 2^{-1/2k}N(0)^{1/2}. \qquad (4\text{-}72)\end{aligned}$$

If $2^k \lesssim N(0)$, we use the crude estimate $|K_k(x-y)| \lesssim 2^{2k}$ in the (t,τ) region $|t-\tau| \lesssim 2^{-k}$, and we use (4-71) with $L=3$ in the region where $|\tau - t| \geq 2^{-k}$. We can then conclude that if $2^k \lesssim N(0)$, we have

$$\int_{\delta/N(0)}^{T_2} \int_{T_1}^{-\delta/N(0)} \left\langle \chi\left(\frac{x}{c|t|}\right) u^3(t),\, P_k^2\left(\frac{e^{i(\tau-t)\sqrt{-\Delta}}}{\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c|\tau|}\right) u^3(\tau)\right)(x)\right\rangle_{L^2} dt\, d\tau \lesssim 1. \tag{4-73}$$

Therefore, (4-66), (4-67), (4-72), and (4-73) imply that

$$a_k^2(0) \lesssim a_k(0)\left(\eta^2 \sum_{j\geq k-3} 2^{(k-j)/4} a_j + 2^{-k/6} b_k\right) + \left(\eta^2 \sum_{j \geq k-3} 2^{(k-j)/4} a_j\right)^2 + 2^{-k/3} b_k^2 + \min(2^{-k/2} N(0)^{1/2}, 1). \tag{4-74}$$

Hence we have

$$a_k(0) \lesssim \eta^2 \sum_{j\geq k-3} 2^{(k-j)/4} a_j + 2^{-k/6} b_k + \min(2^{-k/4} N(0)^{1/4}, 1). \tag{4-75}$$

Using the definitions of $\alpha_k(0)$, α_k and (4-55), we get

$$\alpha_k(0) \lesssim \eta^2 \alpha_k + \sum_j 2^{-|j-k|/8} 2^{-j/6} b_j + \sum_j 2^{-|j-k|/8} 2^{-j/6} \min(2^{-j/12} N(0)^{1/4}, 2^{j/6}).$$

Using (4-51) and choosing η small enough, we then have

$$\alpha_k(0) \lesssim \sum_j 2^{-|j-k|/8} 2^{-j/6} b_j + \sum_j 2^{-|j-k|/8} 2^{-j/6} c_j, \tag{4-76}$$

where the $c_j := \min(2^{-j/12} N(0)^{1/4}, 2^{j/6})$ satisfy

$$\|\{c_j\}\|_{\ell^2} \lesssim N(0)^{1/6}. \tag{4-77}$$

By Schur's test, using (4-64) and (4-77), we can finally conclude that

$$\|2^{k/6} \alpha_k(0)\|_{\ell^2} \lesssim N(0)^{1/6}, \tag{4-78}$$

as desired. This finishes the proof, since $\alpha_k(0)$ satisfies (4-47). □

5. No energy cascade and even more regularity when $N(t) \equiv 1$

In this section we begin by showing that an energy cascade, i.e., the case $\limsup_{t\to\infty} N(t) = 0$, is impossible. This leaves us with the soliton-like critical element $N(t) = 1$ for all $t \geq 0$. We can then reduce this situation to the case of a soliton-like critical element that is global in both time directions with $N(t) \equiv 1$ for all $t \in \mathbb{R}$. Finally, we show that such a solution is in fact uniformly bounded in $\dot{H}^2 \times \dot{H}^1$, which in turn means that $\vec{u}(t)$ satisfies the compactness property in $\dot{H}^1 \times L^2$.

5A. ***No energy cascade.*** We can quickly rule out the case of a critical element $\vec{u}(t)$ with scale $N(t)$ satisfying $\limsup_{t\to\infty} N(t) = 0$. We prove the following consequence of Theorem 4.1 and Proposition 2.6.

Lemma 5.1. *Let $\vec{u}(t)$ be a solution to (1-4) defined on a time interval $I = (T_-, +\infty)$ with $T_- < 0$ and suppose that $\vec{u}(t)$ has the compactness property on I with $N(t) \le 1$ for all $t \in [0, \infty)$. Then $\limsup\limits_{t\to\infty} N(t) = 0$ is impossible unless $\vec{u}(t) \equiv 0$.*

Proof. Since $\vec{u}(t)$ satisfies the conditions of (4-1), we see that

$$\limsup_{t\to\infty} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} \lesssim \limsup_{t\to\infty} N(t)^{1/2} = 0. \tag{5-1}$$

By Sobolev embedding and interpolation, we also have

$$\limsup_{t\to\infty} \|u(t)\|_{L^4} \lesssim \limsup_{t\to\infty} \|u(t)\|_{\dot{H}^{3/4}} \lesssim \limsup_{t\to\infty} \|u(t)\|_{\dot{H}^{1/2}}^{1/2} \|u(t)\|_{\dot{H}^1}^{1/2} \lesssim \limsup_{t\to\infty} N(t)^{1/4} = 0. \tag{5-2}$$

Therefore the conserved energy $E(\vec{u}(t))$ is well-defined and (5-1) and (5-2) imply that we must have $E(\vec{u}(t)) = 0$. If $\vec{u}(t)$ solves the defocusing equation, then $E(\vec{u}(t))$ is given by (1-2) and we can directly conclude that we must have $\vec{u}(t) \equiv 0$. If $\vec{u}(t)$ is a solution to the focusing equation, then we use Proposition 2.6 to deduce that $\vec{u}(t) \equiv 0$. □

5B. ***Additional regularity for a soliton-like critical element.*** For the case of a soliton-like critical element, i.e., $N(t) \equiv 1$, the rigidity argument in Section 6 will require that the trajectory $\vec{u}(t)$ be precompact in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ rather that just uniformly bounded in this norm, in time. This is not hard to do given our work in the previous sections.

Let $\vec{u}(t)$ be as in Proposition 3.1 and assume that $N(t) = 1$ for all $t \in [0, \infty)$. Then, without loss of generality, we can assume that $I_{\max}(\vec{u}) = \mathbb{R}$ and we have $N(t) \equiv 1$ for all $t \in \mathbb{R}$. Indeed, let $t_n \to \infty$ be any sequence. Since $\vec{u}(t)$ has the compactness property on $(T_-(\vec{u}), \infty)$ we can find a subsequence, still denoted by t_n so that $\vec{u}(t_n) \to \vec{u}_\infty$ in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$. Then, using the perturbation theory, one can readily check that the solution $\vec{u}_\infty(t)$ with initial data $\vec{u}_\infty(0) = \vec{u}_\infty$ is global in time and has the compactness property on $\mathbb{R}$ with $N(t) = 1$ for all $t \in \mathbb{R}$.

We can now establish the following proposition.

Proposition 5.2. *Let $\vec{u}(t)$ be the critical element and assume further that $\vec{u}(t)$ is soliton-like, i.e., $\vec{u}(t)$ is defined globally in time and $N(t) \equiv 1$. Then the trajectory*

$$K := \{\vec{u}(t) \mid t \in \mathbb{R}\} \tag{5-3}$$

is precompact in $(\dot{H}^{1/2} \times \dot{H}^{-1/2}) \cap (\dot{H}^1 \times L^2)(\mathbb{R}^3)$.

Proof. We prove that in fact we have a uniform-in-time bound on the $\dot{H}^2 \times \dot{H}^1$-norm of $\vec{u}(t)$. We only provide a sketch of this fact, as the proof is nearly identical to the proof of Theorem 4.1. The precompactness of $\{\vec{u}(t) \mid t \in \mathbb{R}\}$ in $\dot{H}^1 \times L^2$ then follows from its precompactness in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ and interpolation, as we have

$$\|\vec{u}(t)\|_{\dot{H}^1\times L^2} \lesssim \|\vec{u}(t)\|_{\dot{H}^{1/2}\times \dot{H}^{-1/2}}^{2/3} \|\vec{u}(t)\|_{\dot{H}^2\times \dot{H}^1}^{1/3}.$$

First note that by Theorem 4.1, we have

$$\|\vec{u}(t)\|_{\dot{H}^1\times L^2} \lesssim 1. \tag{5-4}$$

Claim 5.3. *There exists a $\delta > 0$ such that for all $t_0 \in R$ and for $J := (t_0-\delta, t_0+\delta)$, we have*

$$\|u\|_{L_t^2 L_x^\infty(J\times\mathbb{R}^3)} \lesssim 1. \tag{5-5}$$

Remark 11. In (5-5) we make use of the endpoint $L_t^2 L_x^\infty$ Strichartz estimate, which is valid in the radial setting; see [Klainerman and Machedon 1993]. However, this use of the endpoint is for convenience only, as it will allow for an upgrade of the uniform bound in $\dot{H}^1 \times L^2$ directly to a uniform bound in $\dot{H}^2 \times \dot{H}^1$. This implies that the trajectory is precompact in $\dot{H}^1 \times L^2$ using the precompactness in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ and interpolating with the $\dot{H}^2 \times \dot{H}^1$ bound. As we are only interested in proving the compactness property in $\dot{H}^1 \times L^2$, it would also suffice to prove a uniform bound in $\dot{H}^{1+\varepsilon} \times \dot{H}^\varepsilon$, and for this estimate we would not need the endpoint Strichartz estimate.

Proof of Claim 5.3. First we note that it suffices to prove the claim for $t_0 = 0$. We apply the endpoint Strichartz estimates, which are valid in the radial setting. Indeed, denote by $Z(J)$ the space $Z(J) := L_t^\infty(J; \dot{H}^1 \times L^2) \cap L_t^2(J; L_x^\infty)$. Then we have

$$\begin{aligned}\|u\|_{Z(J)} &\lesssim \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \|u^3\|_{L_t^1(J;L_x^2)} \lesssim \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \|u\|^3_{L_t^3(J;L_x^6)} \\ &\lesssim \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \delta\|u\|^3_{L_t^\infty(J;\dot{H}_x^1)} \lesssim \|\vec{u}(0)\|_{\dot{H}^1\times L^2} + \delta\|u\|_{Z(J)},\end{aligned} \tag{5-6}$$

where we remark that we have used the Sobolev inequality and the length of J in the third inequality above, and nothing else. In the last inequality we have used (5-4). Choosing $\delta = \delta(\|\vec{u}(0)\|_{\dot{H}^1\times L^2}) > 0$ small enough completes the proof. Note that here it is important that the constant in (5-4) is uniform in $t_0 \in I$. □

The proof of Proposition 5.2 now proceeds exactly as in the proof of Theorem 4.1, except here we seek an $\dot{H}^2$ bound. We give a brief sketch. Let $\vec{v}(t)$ be defined as in (4-6), and Q_M as in (4-10). We prove that

$$\langle Q_M v(t_0), Q_M v(t_0)\rangle_{\dot{H}^2} \lesssim 1 \tag{5-7}$$

for all $M \geq M_0$ with a constant that is uniform in M and in $t_0 \in \mathbb{R}$. Extracting weak limits using Lemma 3.2 as in the proof of Theorem 4.1, we note that it will suffice to prove the following estimate for the "double Duhamel" term:

$$\left|\left\langle Q_M\left(\int_{T_1}^0 e^{it\sqrt{-\Delta}}\nabla(u^3)(t)\,dt\right), Q_M\left(\int_0^{T_2} e^{i\tau\sqrt{-\Delta}}\nabla(u^3)(\tau)\,d\tau\right)\right\rangle_{L^2}\right| \lesssim 1, \tag{5-8}$$

where $T_1 < 0$ and $T_2 > 0$ and the constant above is uniform in such T_1, T_2. Note also that above we have set $t_0 = 0$, as again this case will be sufficient.

By (5-5), we see that for $\delta > 0$ as in Claim 5.3, we have

$$\left\|Q_M\left(\int_0^\delta e^{i\tau\sqrt{-\Delta}}\nabla(u^3)(\tau)\,d\tau\right)\right\|_{L^2} \lesssim \int_0^\delta \|\nabla(u^3)\|_{L^2} \lesssim \|\nabla u\|_{L_t^\infty L^2}\|u\|^2_{L_t^2([0,\delta);L_x^\infty)} \lesssim 1. \tag{5-9}$$

Next, by the radial Sobolev embedding, $\| |x|^{3/4} u \|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|u\|_{\dot H^{3/4}(\mathbb{R}^3)}$, we have

$$\left\| (1-\chi)\left(\frac{x}{c|t|}\right) \nabla u^3(t) \right\|_{L^2} \lesssim \frac{1}{c^{3/2}|t|^{3/2}} \|\nabla u\|_{L^\infty_t L^2(\mathbb{R}^3)} \|u(t)\|^2_{\dot H^{3/4}} \lesssim |t|^{-3/2}, \tag{5-10}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$, radial, satisfies $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$, and $c = \frac{1}{4}$. Therefore we have

$$\left\| Q_M\left(\int_\delta^{T_2} e^{i\tau\sqrt{-\Delta}} \left(1 - \chi\left(\frac{\cdot}{c|\tau|}\right)\right) \nabla(u^3)(\tau)\, d\tau \right) \right\|_{L^2} \lesssim \int_\delta^\infty \left\| \left(1 - \chi\left(\frac{\cdot}{c|\tau|}\right)\right) \nabla(u^3)(\tau) \right\|_{L^2} \lesssim \delta^{-1/2}. \tag{5-11}$$

Next, using the sharp Huygens principle exactly as in the proof of (4-22), the term

$$\begin{aligned} &\left\langle Q_M\left(e^{it\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c|t|}\right) \nabla(u^3)(t)\, dt \right), Q_M\left(e^{i\tau\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c|\tau|}\right) \nabla(u^3)(\tau)\, d\tau \right) \right\rangle \\ &\qquad = \left\langle Q_M\left(\chi\left(\frac{\cdot}{c|t|}\right) \nabla(u^3)(t)\, dt \right), Q_M\left(e^{i(\tau - t)\sqrt{-\Delta}} \chi\left(\frac{\cdot}{c|\tau|}\right) \nabla(u^3)(\tau)\, d\tau \right) \right\rangle \end{aligned} \tag{5-12}$$

is identically 0 for $t < -\delta$ and $\tau > \delta$. With (5-9), (5-11), and (5-12) playing the roles of (4-14), (4-18), and (4-22), the proof now proceeds exactly as the proof of Theorem 4.1. We omit the details. □

6. Rigidity via a virial identity

In this section we complete the rigidity argument by proving that a soliton-like critical element (i.e., $N(t) \equiv 1$) cannot exist. Indeed, we prove the following proposition:

Proposition 6.1. *Let $\vec u(t) \in (\dot H^{1/2} \times \dot H^{-1/2}) \cap (\dot H^1 \times L^2)(\mathbb{R}^3)$ be a global-in-time solution to (1-4) such that the trajectory*

$$K := \{\vec u(t) \mid t \in \mathbb{R}\} \tag{6-1}$$

is precompact in $(\dot H^{1/2} \times \dot H^{-1/2}) \cap (\dot H^1 \times L^2)(\mathbb{R}^3)$. Then $u(t) \equiv 0$.

The proposition will follow from a simple argument based on the following virial identity. We will fix a smooth radial cutoff function $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi(r) \equiv 1$ for $0 \le r \le 1$, $\operatorname{supp} \chi \subset [0, 2]$, and $|\chi'(r)| \le C$ for all $r > 0$. For each fixed $R > 0$, we will denote by χ_R the rescaling

$$\chi_R(r) := \chi(r/R). \tag{6-2}$$

Lemma 6.2 (virial identity). *Let $\vec u(t) \in \dot H^1 \times L^2(\mathbb{R}^3)$ be a solution to (1-4). Then for every $R > 0$,*

$$\begin{aligned} \frac{d}{dt}\langle u_t \mid \chi_R(u + r u_r)\rangle = -E(\vec u)(t) + \int_0^\infty (1-\chi_R)\left(\frac12 u_t^2 + \frac12 u_r^2 \pm \frac14 u^4\right) r^2\, dr \\ - \int_0^\infty \left(\frac12 u_t^2 + \frac12 u_r^2 \pm \frac14 u^4\right) r\chi_R' r^2\, dr - \int_0^\infty u u_r r \chi_R' r\, dr, \end{aligned} \tag{6-3}$$

where the bracket $\langle f \mid g\rangle$ is the radial $L^2(\mathbb{R}^3)$ inner product

$$\langle f \mid g\rangle := \int_0^\infty f(r)g(r)r^2\,dr.$$

Proof. The proof follows from (1-4) and integration by parts. □

Remark 12. In general, for a semilinear equation of the form

$$u_{tt} - \Delta u = \pm |u|^{p-1}u,$$

one has the formal virial identity

$$\frac{d}{dt}\langle u_t \mid u + x\cdot\nabla u\rangle = -E(u) \pm \left(\frac{p-3}{p+1}\right)\|u\|_{L^{p+1}}^{p+1}.$$

Note that the right side can be bounded from above by a negative constant times the *conserved* energy in the case $1+\sqrt{2} < p \le 3$, yielding a monotone quantity. But in the case $3 < p < 5$, the right side cannot be controlled by the conserved energy in the case of the focusing equation. However, for the range $3 < p < 5$, a different rigidity argument is available, based on the "channels of energy" method developed by Duyckaerts, Kenig, and Merle [2013; 2014]. For an implementation of this strategy for the range $p \in (3, 5)$, see [Shen 2012].

We also note that the virial identities and the argument in this section also readily extend to the nonradial setting.

The proof of Proposition 6.1 will now follow by applying the above lemma to our precompact trajectory $\vec{u}(t)$ in order to show that the energy must be nonpositive. One concludes the proof by noting that a solution to the defocusing equation with nonpositive energy must be identically zero. In the case of the focusing equation, we recall Proposition 2.6, which says that a solution with nonpositive energy must either be identically zero or blow up in both time directions, and the latter is impossible under the hypothesis of Proposition 6.1.

Proof of Proposition 6.1. Fix $\eta > 0$. We will show that for $\vec{u}(t)$ as in Proposition 6.1, we have

$$\mathscr{E}(\vec{u}) \le C\eta \tag{6-4}$$

for a fixed constant C which is independent of η. First, note that since $\{\vec{u}(t) \mid t \in \mathbb{R}\}$ is precompact in $(\dot{H}^{1/2}\times\dot{H}^{-1/2})\cap(\dot{H}^1\times L^2)(\mathbb{R}^3)$, we can find $R_0 = R_0(\eta)$ such that for all $R \ge R_0$ and for all $t \in \mathbb{R}$, we have

$$\int_R^\infty (u_t^2(t) + u_r^2(t))r^2\,dr \le \eta. \tag{6-5}$$

Moreover, due to the embeddings $\dot{H}^{1/2}\cap\dot{H}^1 \hookrightarrow \dot{H}^{3/4} \hookrightarrow L^4$, we can choose $R_0(\eta)$ large enough that we also have

$$\int_R^\infty u^4(t)r^2\,dr \le \eta \tag{6-6}$$

for all $R \geq R_0$ and for all $t \in \mathbb{R}$. Finally, we note that for any $R > 0$ and for any smooth radial function in $\dot{H}^1(\mathbb{R}^3)$, we have

$$\int_R^\infty f^2(r)\,dr + Rf^2(R) = -\int_R^\infty f_r(r)f(r)r\,dr,$$

which can be obtained by integrating by parts. This implies that

$$\int_R^\infty f^2(r)\,dr \leq \int_R^\infty f_r^2(r)r^2\,dr.$$

Therefore, for our precompact trajectory $\vec{u}(t)$, we can use (6-5) to obtain

$$\int_R^\infty u^2(t,r)\,dr \leq \eta \tag{6-7}$$

for all $R \geq R_0(\eta)$ and for all $t \in \mathbb{R}$. Letting $R \geq R_0(\eta)$, we can apply these estimates to the last three terms on the right side of (6-3):

$$\begin{aligned}\left\langle \int_0^\infty (1-\chi_R)\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_r^2 \pm \frac{1}{4}u^4\right)r^2\,dr\right\rangle &\leq C\eta,\\ \left\langle \int_0^\infty \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_r^2 \pm \frac{1}{4}u^4\right)r\chi_R' r^2\,dr\right\rangle &\leq C\int_R^{2R}\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_r^2 + \frac{1}{4}u^4\right)r^2\,dr \leq C\eta,\\ \left\langle \int_0^\infty uu_r r\chi_R' r\,dr\right\rangle &\leq \left(\int_R^{2R} u_r^2 r^2\,dr\right)^{1/2}\left(\int_R^{2R} u^2\,dr\right)^{1/2} \leq C\eta.\end{aligned}$$

Inserting the above estimates into (6-3) and averaging in time from 0 to T, we obtain the estimate

$$E(\vec{u}) \leq C\eta + \frac{1}{T}\left|\langle u_t(T) \mid \chi_R(u(T) + ru_r(T))\rangle\right| + \frac{1}{T}\left|\langle u_t(0) \mid \chi_R(u(0) + ru_r(0))\rangle\right|. \tag{6-8}$$

Now, set $R = T$ above with T large enough that $T \gg R_0(\eta)$. We have

$$\begin{aligned}E(\vec{u}) \leq C\eta + C\frac{1}{T}\int_0^T |u_t(T)||u(T)|r^2\,dr + C\frac{1}{T}\int_0^T |u_t(0)||u(0)|r^2\,dr\\ + C\frac{1}{T}\int_0^T |u_t(T)||u_r(T)|r^3\,dr + C\frac{1}{T}\int_0^T |u_t(0)||u_r(0)|r^3\,dr.\end{aligned} \tag{6-9}$$

We estimate the second and third terms on the right side of (6-9) by

$$\begin{aligned}\frac{1}{T}\int_0^T |u_t||u|r^2\,dr &\leq \frac{1}{T}\left(\int_0^T u_t^2 r^2\,dr\right)^{1/2}\left(\int_0^T |u|^3 r^2\,dr\right)^{1/3}\left(\int_0^T r^2\,dr\right)^{1/6}\\ &\leq C\frac{1}{T^{1/2}}\|u_t\|_{L^2}\|u\|_{\dot{H}^{1/2}} \to 0 \quad \text{as } T \to \infty,\end{aligned}$$

where in the last line we have used the embedding $\dot{H}^{1/2} \hookrightarrow L^3$, the fact that the critical element $\vec{u}(t)$ satisfies $\sup_{t\in\mathbb{R}} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \lesssim 1$, and $\sup_{t\in\mathbb{R}} \|u_t\|_{L^2} \lesssim 1$. To estimate the fourth and fifth terms in (6-9), we note that for $T \gg R(\eta)$ we have

$$\begin{aligned}\frac{1}{T}\int_0^T |u_t|\,|u_r| r^3\,dr &\le \frac{1}{T}\int_0^{R(\eta)} |u_t|\,|u_r| r^3\,dr + \frac{1}{T}\int_{R(\eta)}^T |u_t|\,|u_r| r^3\,dr \\ &\le \frac{R(\eta)}{T}\|u_t\|_{L^2}\|u\|_{\dot H^1} + \left(\int_{R(\eta)}^T u_t^2 r^2\,dr\right)^{1/2}\left(\int_{R(\eta)}^T u_r^2 r^2\,dr\right)^{1/2} \\ &= \eta + O(T^{-1}) \quad \text{as } T \to \infty.\end{aligned}$$

Thus, letting $T \to \infty$ in (6-9), we obtain

$$E(\vec u) \le C\eta,$$

as desired. Since this holds for all $\eta > 0$, we can conclude that

$$E(\vec u) \le 0. \tag{6-10}$$

In the case that $\vec u(t)$ is a solution to the defocusing equation, we are done, as we can conclude from (6-10) that $\vec u(t) \equiv 0$. In the case that $\vec u(t)$ is a solution to the focusing equation, we note that (6-10) together with Proposition 2.6 implies that either $\vec u(t) \equiv 0$ or $\vec u(t)$ blows up in finite time in both time directions. However, the latter case is impossible, as we have assumed that $\vec u(t)$ is global in time. This completes the proof of Proposition 6.1. □

7. Proof of Theorem 1.1

We provide a brief summary of the proof of Theorem 1.1, which is now complete. We argue by contradiction. If Theorem 1.1 were false, we could, by Proposition 3.1, find a critical element, i.e., a *nonzero* solution $\vec u(t)$ to (1-4) with the compactness property in $\dot H^{1/2} \times \dot H^{-1/2}$ on an open interval $I \ni 0$ with scale $N(t)$. By the remarks following the statement of Proposition 3.1, we can reduce to the case $I = (T_-, \infty)$ and $N(t) \le 1$ for $t \in [0, \infty)$. By (4-1), we then have

$$\|\vec u(t)\|_{\dot H^1 \times L^2} \lesssim N(t)^{1/2} \quad \text{for } t \in [0, \infty).$$

Then, since we are assuming $\vec u(t)$ is nonzero, by Section 5A, we can conclude that $N(t) \equiv 1$ for all $t \in [0, \infty)$. We can then ensure that $\vec u(t)$ is global in time for all $t \in \mathbb{R}$, and by Proposition 5.2, we know that $\vec u(t)$ has a precompact trajectory in $\dot H^{1/2} \times \dot H^{-1/2} \cap \dot H^1 \times L^2$. But then Proposition 6.1 shows that $\vec u(t) \equiv 0$, which is a contradiction.

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