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We introduce a class of BMO spaces which interpolate with L_p and are sufficiently large to serve as endpoints for new singular integral operators. More precisely, let (Ω, Σ, μ) be a σ -finite measure space. Consider two filtrations of Σ by successive refinement of two atomic σ -algebras Σ_a and Σ_b having trivial intersection. Construct the corresponding truncated martingale BMO spaces. Then, the intersection seminorm only leaves out constants and we provide a quite flexible condition on (Σ_a, Σ_b) so that the resulting space interpolates with L_p in the expected way. In the presence of a metric d , we obtain endpoint estimates for Calderón–Zygmund operators on (Ω, μ, d) under additional conditions on (Σ_a, Σ_b) . These are weak forms of the “isoperimetric” and the “locally doubling” properties of Carbonaro, Mauceri and Meda which admit less concentration at the boundary. Examples of particular interest include densities of the form $e^{\pm|x|^\alpha}$ for any $\alpha > 0$ or $(1 + |x|^\beta)^{-1}$ for any $\beta \gtrsim n^{3/2}$. A (limited) comparison with Tolsa’s RBMO is also possible. On the other hand, a more intrinsic formulation yields a Calderón–Zygmund theory adapted to regular filtrations over (Σ_a, Σ_b) without using a metric. This generalizes well-known estimates for perfect dyadic and Haar shift operators. In contrast to previous approaches, ours extends to matrix-valued functions (via recent results from noncommutative martingale theory) for which only limited results are known and no satisfactory nondoubling theory exists so far.

Introduction

A BMO space is a set of functions that enjoy bounded mean oscillation in a certain sense. Both “mean” and “oscillation” can be measured in many different ways. Most frequently, we find BMO spaces refer to averages over balls in a metric measure space. In other notable scenarios, we may replace these averages by conditional expectations with respect to a martingale filtration, or even by the action of a nicely behaved semigroup of operators. These more abstract formulations are known to be very useful given the lack of appropriate metrics. The relation between metric and martingale BMO spaces is well understood for doubling spaces, that is, when the measure of a ball in the given metric is comparable with the measure of its concentric dilations up to constants depending on the dilation factor but not on the chosen ball. Indeed, in this case the metric BMO is equivalent to a finite intersection of martingale BMO spaces constructed out of dyadic two-sided filtrations of atomic σ -algebras whose atoms look like balls; see [Conde 2013; Garnett and Jones 1982; Hytönen and Kairema 2012; Mei 2003]. What is more relevant, however, is that any of these martingale BMO spaces satisfies the following fundamental properties:

- (i) Interpolation endpoint for the L_p scale.

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- (ii) John–Nirenberg inequalities and H_1 –BMO duality.
- (iii) CZ extrapolation: L_2 -boundedness $\Rightarrow L_\infty \rightarrow$ BMO boundedness.

Hence, these spaces yield at least as many endpoint estimates as the metric BMO.

The main goal of this paper is to construct BMO spaces satisfying the properties stated above for a larger class of measures, and to explore the implications of this construction to provide new endpoint estimates. The first attempts in this direction [Mateu et al. 2000; Nazarov et al. 2002] culminated in the work of Tolsa [2001] on so-called RBMO spaces. These spaces enjoy the above-mentioned properties for measures of polynomial growth. There are, however, a couple of open questions concerning Tolsa’s construction. In the first place, Calderón–Zygmund extrapolation holds under a Lipschitz kernel condition instead of the more flexible Hörmander condition. Second, only interpolation of operators, has been studied but it seems to be unknown whether these spaces interpolate with the L_p scale. These two problems were solved by Carbonaro, Mauceri and Meda [Carbonaro et al. 2009; 2010] for a different class of measures, based on similar results for the Gaussian measure on Euclidean spaces [Mauceri and Meda 2007]. The properties they imposed lead to locally doubling measures with certain concentration behavior at the boundary. In both cases — up to equivalence in the norm and additional conditions — only doubling balls are used to measure the mean oscillation of the function.

We present an alternative approach to these questions. Martingale BMO spaces always satisfy conditions (i) and (ii) above, with independence of the existence of a metric in the underlying measure space. The third property however requires additional structure on our BMO spaces. Indeed, assume for a moment that we work with a two-sided filtration $(\Sigma_k)_{k \in \mathbb{Z}}$ of atomic σ -subalgebras of Σ with corresponding conditional expectations E_{Σ_k} . If Π denotes the union of atoms in our filtration, the corresponding martingale BMO norm is given by

$$\|f\|_{\text{BMO}} = \sup_{k \in \mathbb{Z}} \|E_{\Sigma_k} |f|^2 - E_{\Sigma_{k-1}} |f|^2\|_\infty^{1/2},$$

which is larger than the function BMO norm

$$\sup_{A \in \Pi} \left(\frac{1}{\mu(A)} \int_A \left| f(w) - \frac{1}{\mu(A)} \int_A f d\mu \right|^2 d\mu(w) \right)^{\frac{1}{2}}.$$

Thus, if we admit from [Carbonaro et al. 2009; Tolsa 2001] that extrapolation for (nonlocal) Calderón–Zygmund operators imposes that our atoms be doubling — i.e., contained in a doubling ball of comparable measure or a union of at most C_0 sets of this kind; see below — we immediately find obstructions to constructing filtrations satisfying this assumption for nondoubling spaces. We propose to consider a sort of intersection of two large BMO spaces as follows. Consider a σ -finite measure space (Ω, Σ, μ) and two atomic σ -algebras Σ_a, Σ_b of measurable sets in Σ satisfying $\Sigma_a \cap \Sigma_b = \{\Omega, \emptyset\}$. Write BMO_j for any martingale BMO space over a filtration $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$; then the seminorm

$$\|f\|_{\text{BMO}_{\Sigma_{ab}}(\Omega)} = \max\{\|f - E_{\Sigma_a} f\|_{\text{BMO}_a}, \|f - E_{\Sigma_b} f\|_{\text{BMO}_b}\}$$

vanishes on constant functions precisely when $\Sigma_a \cap \Sigma_b$ is trivial. Let

$$\text{BMO}_{\Sigma_{ab}}(\Omega) = \{f \in L^1_{\text{loc}}(\Omega) \mid \|f\|_{\text{BMO}_{\Sigma_{ab}}(\Omega)} < \infty\} / \mathbb{C}.$$

This settles a model of “large BMO spaces” which easily satisfy property (ii) and leave some room for property (iii). The problem reduces then to identify conditions on the pair (Σ_a, Σ_b) so that $\text{BMO}_{\Sigma_{ab}}(\Omega)$ interpolates with the L_p scale. A standard argument shows that this is the case when

$$\|f\|_{L^{\circ}_p(\Omega)} := \inf_{k \in \mathbb{C}} \|f - k\|_p \sim \max\{\|f - E_{\Sigma_a} f\|_p, \|f - E_{\Sigma_b} f\|_p\} =: \|f\|_{L^p_{\Sigma_{ab}}(\Omega)}$$

for $2 \leq p < \infty$, where

$$\begin{aligned} L^{\circ}_p(\Omega) &= L_p(\Omega, \Sigma, \mu) / \mathbb{C}, \\ L^p_{\Sigma_{ab}}(\Omega) &= \{f \in L^1_{\text{loc}}(\Omega) \mid \|f\|_{L^p_{\Sigma_{ab}}(\Omega)} < \infty\} / \mathbb{C} \\ &= L_p(\Omega, \Sigma, \mu) / \Sigma_a \wedge L_p(\Omega, \Sigma, \mu) / \Sigma_b. \end{aligned}$$

Here, $L_p(\Omega, \Sigma, \mu) / \Sigma_i$ denotes the quotient space of $L_p(\Omega, \Sigma, \mu)$ by the subspace of Σ_i -measurable functions. More precisely, we have an isomorphism $L^{\circ}_p(\Omega) \simeq L^p_{\Sigma_{ab}}(\Omega)$. It should be mentioned that this isomorphism fails in general, even for the Lebesgue measure in \mathbb{R}^n and many “natural” choices of pairs (Σ_a, Σ_b) . Recall that $L^{\circ}_p(\Omega) = L_p(\Omega)$ for infinite measures. Note also that we use \wedge and not \cap since this space is not really an intersection; we shall also write $\text{BMO}_{\Sigma_{ab}}(\Omega) = \text{BMO}_a(\Omega) / \Sigma_a \wedge \text{BMO}_b(\Omega) / \Sigma_b$. To formulate a sufficient condition on (Σ_a, Σ_b) for $L^{\circ}_p(\Omega) \simeq L^p_{\Sigma_{ab}}(\Omega)$, let Π_j be the set of atoms in Σ_j . When $\mu(\Omega) < \infty$ we shall consider two distinguished atoms $(A_0, B_0) \in \Pi_a \times \Pi_b$, while for μ not finite we take $A_0 = B_0 = \emptyset$ for notation consistency. Given $(A, B) \in \Pi_a \times \Pi_b$, set

$$R_A = \{B' \in \Pi_b \mid \mu(A \cap B') > 0\} \quad \text{and} \quad R_B = \{A' \in \Pi_a \mid \mu(A' \cap B) > 0\}.$$

We will write $|R_A|$ and $|R_B|$ for the cardinality of these sets. The following is the main result of this paper, where we establish a condition on (Σ_a, Σ_b) which suffices to make intersections and quotients commute in L_p as described above. We will say that (Σ_a, Σ_b) is an *admissible covering* of (Ω, Σ, μ) when $\Sigma_a \cap \Sigma_b = \{\Omega, \emptyset\}$ and

$$\min \left\{ \sup_{A \in \Pi_a \setminus \{A_0\}} \sum_{B \in R_A} |R_B| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)}, \sup_{B \in \Pi_b \setminus \{B_0\}} \sum_{A \in R_B} |R_A| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)} \right\} < 1.$$

Theorem A. *Let (Ω, Σ, μ) be a σ -finite measure space equipped with an admissible covering (Σ_a, Σ_b) . Then, for each $2 \leq p < \infty$, there exists a constant c_p , depending only on p and the admissible covering, such that*

$$L^{\circ}_p(\Omega) \simeq_{c_p} L^p_{\Sigma_{ab}}(\Omega).$$

Moreover, we have the desired complex interpolation result,

$$[\text{BMO}_{\Sigma_{ab}}(\Omega), L^{\circ}_1(\Omega)]_{1/q} \simeq_{c_q} L^{\circ}_q(\Omega) \quad (1 < q < \infty),$$

with $\text{BMO}_{\Sigma_{ab}}(\Omega)$ defined as above for any two martingale BMO spaces over (Σ_a, Σ_b) .

The first assertion fails for $p = 1, \infty$. On the other hand, both the John–Nirenberg inequalities and H_1 –BMO duality are easily formulated for these spaces. Therefore, we shall focus in what follows on condition (iii). Calderón–Zygmund extrapolation means that under a certain mild smoothness condition on the kernel, L_2 -boundedness yields L_p -boundedness for $1 < p < \infty$. As usual, we handle it by providing an endpoint estimate for interpolation. Let d be a metric on Ω and denote by αB the α -dilation of a ball B . We impose the standard Hörmander kernel condition

$$\sup_{Bd\text{-ball}} \sup_{z_1, z_2 \in B} \int_{\Omega \setminus \alpha B} |k(z_1, x) - k(z_2, x)| + |k(x, z_1) - k(x, z_2)| d\mu(x) < \infty.$$

Define a CZO on (Ω, μ, d) as any linear map T satisfying the following properties:

- T is well-defined and bounded on $L_2(\Omega)$.
- The kernel representation for any $f \in C_c(\Omega)$,

$$Tf(x) = \int_{\Omega} k(x, y) f(y) d\mu(y) \quad \text{holds for } x \notin \text{supp } f$$

and some kernel $k : \Omega \times \Omega \setminus \Delta \rightarrow \mathbb{C}$ satisfying the Hörmander condition.

Given $C_0 > 0$, a Σ -measurable set A will be called (C_0, α, β) -doubling when it is the union of at most C_0 sets which are contained in (α, β) -doubling balls — balls B such that $\mu(\alpha B) \leq \beta \mu(B)$ — of comparable measure up to the constant C_0 . Recall that a filtration $(\Sigma_k)_{k \geq 1}$ is called *regular* if $E_k f \lesssim E_{k-1} f$ for all $k > 1$ and all $f \geq 0$.

Theorem B1. *Let (Σ_a, Σ_b) be an admissible covering of (Ω, Σ, μ) . Assume that (Ω, Σ, μ) admits regular filtrations $(\Sigma_{jk})_{k \geq 1}$ by successive refinement of $\Sigma_{j1} = \Sigma_j$ for $j = a, b$ and that each atom in Σ_{jk} is (C_0, α, β) -doubling for certain absolute constants $C_0, \alpha, \beta > 0$. Construct the spaces $\text{BMO}_{\Sigma_{ab}}(\Omega)$ which are defined over these filtrations. Then, every Calderón–Zygmund operator extends to a bounded map $L_\infty(\Omega) \rightarrow \text{BMO}_{\Sigma_{ab}}(\Omega)$, and $L_p(\Omega) \rightarrow L_p(\Omega)$ for $1 < p < \infty$.*

A few illustrations of [Theorem B1](#) are the following:

- *Doubling case:* [Theorem B1](#) recovers Calderón–Zygmund extrapolation on homogeneous spaces (Ω, μ, d) . We shall construct explicit pairs (Σ_a, Σ_b) and martingale filtrations satisfying our assumptions.
- *Polynomial growth:* Given any (Ω, μ, d) with polynomial growth, it is not difficult to construct atomic σ -algebras composed uniquely of doubling atoms, even giving admissible coverings. Under the existence of filtrations based on (Σ_a, Σ_b) and composed of doubling atoms — regular or not — we may prove that Tolsa’s RBMO sits inside our $\text{BMO}_{\Sigma_{ab}}(\Omega)$. This condition seems, unfortunately, a restrictive limit in [Theorem B1](#). However, it can be checked in some concrete scenarios, like for

$$d\mu(x) = \frac{dx}{1 + |x|^\beta} \quad \text{with } \beta \gtrsim n^{3/2}$$

in \mathbb{R}^n equipped with the Euclidean metric. Note that μ is doubling for $\beta < n$. The key advantage over Tolsa’s approach is that we only need to impose Hörmander kernel smoothness, instead of stronger Lipschitz conditions. This was also achieved by [[Carbonaro et al. 2009; 2010](#)] for another family of

measures (see below) but not for the measures considered above, since they are drastically less concentrated at the boundary for any β .

- *Concentration at the boundary*: Carbonaro et al. [2009; 2010] proved that when (Ω, μ, d) is locally doubling and the measure concentrates at the boundary of open sets in a certain sense — together with a purely metric condition that does not play any role here — a BMO space satisfying (i), (ii) and (iii) is possible. Their main examples in \mathbb{R}^n with a weighted Euclidean metric were $d\mu(x) = e^{\pm|x|^\alpha} dx$ and $\alpha > 1$. The exponentially decreasing ones behave in some sense like the Gaussian measure, which was studied a few years before by Mauceri and Meda. It is of polynomial growth, so that the kernel smoothness condition was the main advantage with respect to Tolsa’s approach. The exponentially increasing ones are not of polynomial growth. In this paper we shall remove their condition $\alpha > 1$.

In the literature, we find other families of operators — with no need of a metric in the underlying space — which are close to CZOs in spirit. Martingale transforms are the simplest ones, but are local and much easier to bound. Nonlocal models include the so-called perfect dyadic CZOs and, most notably, Haar shift operators, which include prominent examples like the discrete Hilbert transform and dyadic paraproducts. In these cases, the Hörmander kernel condition can be replaced by

$$\sup_Q \sup_{\text{dyadic cube } z_1, z_2 \in Q} \int_{\Omega \setminus \hat{Q}} |k(z_1, x) - k(z_2, x)| + |k(x, z_1) - k(x, z_2)| d\mu(x) < \infty,$$

where \hat{Q} denotes the dyadic father of Q . Our BMO spaces allow us to further replace dyadic cubes in dyadically doubling measure spaces — see [López-Sánchez et al. 2014] for recent progress on more general measures in this direction — by more general atoms. Namely, assume (Σ_a, Σ_b) gives an admissible covering of (Ω, Σ, μ) . Consider regular filtrations of atomic σ -algebras $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$. Let us write Π_{jk} for the family of atoms in the atomic σ -algebra Σ_{jk} and set $\mathbf{\Pi}_j = \bigcup_{k \geq 1} \Pi_{jk}$. Then, consider the following Hörmander-type kernel condition, where the former role of the metric d is replaced by the shape of our atoms in $\mathbf{\Pi} = \mathbf{\Pi}_a \cup \mathbf{\Pi}_b$:

$$\sup_{A \in \mathbf{\Pi}} \sup_{z_1, z_2 \in A} \int_{\Omega \setminus \hat{A}} |k(z_1, x) - k(z_2, x)| + |k(x, z_1) - k(x, z_2)| d\mu(x) < \infty.$$

Again, \hat{A} denotes the minimal atom in the filtration of A which contains A properly, unless there is no such atom, in which case we pick $\hat{A} = A$. If we replace the Hörmander condition by this one, we obtain another class of “atomic” CZOs, which will be denoted in what follows by ACZO.

Theorem B2. *Let (Σ_a, Σ_b) be an admissible covering of (Ω, Σ, μ) . Assume in addition that (Ω, Σ, μ) admits regular filtrations $(\Sigma_{jk})_{k \geq 1}$ by successive refinement of $\Sigma_{j1} = \Sigma_j$ for $j = a, b$. Construct the spaces $\text{BMO}_{\Sigma_{ab}}(\Omega)$ which are defined over these filtrations. Then, every ACZO extends to a bounded map $L_\infty(\Omega) \rightarrow \text{BMO}_{\Sigma_{ab}}(\Omega)$, and $L_p(\Omega) \rightarrow L_p(\Omega)$ for $1 < p < \infty$.*

An advantage of Theorem B2 is that our kernel conditions are flexible, since we may carefully choose (Σ_a, Σ_b) and the regular filtrations according to the concrete singular integral operator. It is worth mentioning that every σ -finite (atomless if μ is finite) measure space (Ω, Σ, μ) has nontrivial admissible

coverings. Of course, the regularity of the filtration is a light form of “doublingness” needed to emulate the classical argument in this setting. We will also provide weaker estimates for pseudolocal operators when the filtrations are not regular.

In contrast to [Carbonaro et al. 2009; 2010; Mauceri and Meda 2007; Tolsa 2001], our approach extends to matrix-valued functions, for which only limited results are known and no satisfactory nondoubling theory exists so far. In fact, this was our original motivation and the necessity of alternative arguments led to the results presented so far. We will postpone the discussion of the matrix-valued setting to the last section of this paper, which will allow those readers not familiar with noncommutative L_p theory to isolate these results.

Our results above give some insight on the relation between nondoubling and martingale BMO theories; see [Conde-Alonso and Parcet 2014; Junge et al. \geq 2015] for other results along this line. In [Conde-Alonso and Parcet 2014], we adapt Tolsa’s ideas to give an atomic block description of martingale H_1 . Semigroup BMO spaces are used in [Junge et al. \geq 2015] to construct a Calderón–Zygmund theory that incorporates noncommutative measure spaces (von Neumann algebras) to the picture.

1. Admissible coverings and BMO spaces

In this section we recall some basic background around martingale BMO spaces and introduce our new class of BMO spaces. We will study standard properties of this class, like the existence of admissible coverings, John–Nirenberg inequalities and H_1 –BMO duality. The proof of [Theorem A](#) is more technical and will be postponed to [Section 2](#).

Martingale BMO spaces. Let (Ω, Σ, μ) be a σ -finite measure space and consider a filtration $(\Sigma_k)_{k \geq 1}$ of Σ . In other words, we have $\Sigma_k \subset \Sigma_{k+1}$ and the union of the spaces $L_\infty(\Omega, \Sigma_k, \mu)$ is weak- $*$ dense in $L_\infty(\Omega, \Sigma, \mu)$. Let E_{Σ_k} denote the conditional expectation onto Σ_k -measurable functions. Then, define the martingale BMO space associated to this filtration as the space of locally integrable functions $f : \Omega \rightarrow \mathbb{C}$ whose BMO norm,

$$\|f\|_{\text{BMO}} = \sup_{k \geq 1} \left\| (E_{\Sigma_k} |f|^2 - E_{\Sigma_{k-1}} |f|^2)^{1/2} \right\|_\infty,$$

is finite, where we use the convention $E_{\Sigma_0} f = 0$; see [Garsia 1973]. Another expression for the norm is

$$\begin{aligned} \|f\|_{\text{BMO}} &= \sup_{k \geq 1} \left\| |df_k|^2 + \sum_{n > k} E_{\Sigma_k} |df_n|^2 \right\|_\infty^{1/2} \\ &\sim \left[\sup_{k \geq 1} \left\| (E_{\Sigma_k} |f|^2 - E_{\Sigma_{k-1}} |f|^2)^{1/2} \right\|_\infty + \|E_{\Sigma_1} f\|_\infty \right] + \sup_{k > 1} \|df_k\|_\infty, \end{aligned}$$

where $df_k = \Delta_k f = E_{\Sigma_k} f - E_{\Sigma_{k-1}} f$. According to [Janson and Jones 1982], $[\text{BMO}, L_1(\Omega)]_{1/p} \simeq L_p(\Omega)$ for any filtration we pick. The bracketed term in the right-hand side above is called the martingale bmo norm of f , and it is closer to the standard expressions to measure the mean oscillation of a function. Namely, if the σ -algebras Σ_k are atomic and if Π_k denotes the atoms in Σ_k and $\mathbf{\Pi} = \bigcup_{k \geq 1} \Pi_k$, we deduce

that

$$\begin{aligned} \|f\|_{\text{bmo}} &= \sup_{k \geq 1} \left\| (E_{\Sigma_k} |f - E_{\Sigma_k} f|^2)^{1/2} \right\|_{\infty} + \|E_{\Sigma_1} f\|_{\infty} \\ &= \sup_{A \in \Pi} \left(\frac{1}{\mu(A)} \int_A \left| f(w) - \frac{1}{\mu(A)} \int_A f d\mu \right|^2 d\mu(w) \right)^{\frac{1}{2}} + \sup_{A \in \Pi_1} \left| \frac{1}{\mu(A)} \int_A f d\mu \right|. \end{aligned}$$

Of course, using a selected family of atoms makes L_p -interpolation fail in general for bmo. The extra term in BMO corrects this. This should be compared with the extra condition in the definition of Tolsa’s RBMO. On the other hand, bmo spaces have good interpolation properties with little Hardy spaces h_p . Namely, according to [Bekjan et al. 2010] we have $[\text{bmo}, h_1]_{1/p} \simeq h_p$ for any filtration, where h_p is the closure of the space of finite martingales in L_p with respect to the norm

$$\|f\|_{h_p} = \left\| \left(\sum_{k \geq 1} E_{\Sigma_{k-1}} |df_k|^2 \right)^{\frac{1}{2}} \right\|_p;$$

this time the convention is $E_{\Sigma_0} |df_1|^2 = |E_{\Sigma_1} f|^2$. In contrast to other BMO spaces seminorms, paradoxically, we will need to quotient out certain spaces. Note that, for Σ_1 -measurable functions, the norms above coincide with the L_{∞} norm

$$\|E_{\Sigma_1} f\|_{\text{BMO}} = \|E_{\Sigma_1} f\|_{\text{bmo}} = \|E_{\Sigma_1} f\|_{L_{\infty}(\Omega)}.$$

If we define the seminorms

$$\begin{aligned} \|f\|_{\text{bmo}}^{\circ} &= \|f - E_{\Sigma_1} f\|_{\text{bmo}}, \\ \|f\|_{\text{BMO}}^{\circ} &= \|f - E_{\Sigma_1} f\|_{\text{BMO}}, \end{aligned}$$

we obtain complemented subspaces $\text{BMO}_{\Sigma_1} = J_{\Sigma_1}(\text{BMO})$ using the projection $J_{\Sigma_1} = \text{id} - E_{\Sigma_1}$. Indeed, it is a simple exercise using Jensen’s conditional inequality $|E_{\Sigma_1} f|^2 \leq E_{\Sigma_1} |f|^2$; details are left to the reader. Since J_{Σ_1} is also bounded on h_p and L_p , the previous interpolation results imply the following isomorphisms for $1 < p < \infty$:

$$\begin{aligned} [J_{\Sigma_1}(\text{bmo}), J_{\Sigma_1}(h_1(\Omega))]_{1/p} &\simeq J_{\Sigma_1}(h_p(\Omega)), \\ [J_{\Sigma_1}(\text{BMO}), J_{\Sigma_1}(L_1(\Omega))]_{1/p} &\simeq J_{\Sigma_1}(L_p(\Omega)). \end{aligned}$$

Note that $J_{\Sigma_1}(L_p(\Omega)) \simeq L_p(\Omega, \Sigma, \mu) / \Sigma_1$ in the terminology of the introduction.

Remark 1.1. It is worth mentioning that the Janson–Jones interpolation theorem [1982] holds for arbitrary filtrations. In particular, we could replace $(\Sigma_k)_{k \geq 1}$ by $(\Sigma_k)_{k \geq N}$ for some large N , and the latter BMO comes equipped with the norm

$$\sup_{k \geq N} \left\| (E_{\Sigma_k} |f - E_{\Sigma_k} f|^2)^{1/2} \right\|_{\infty} + \|E_{\Sigma_N} f\|_{\infty} + \sup_{k > N} \|df_k\|_{\infty}.$$

When N is large enough, the middle term dominates the others and we get spaces which are closer and closer to $L_{\infty}(\Omega)$. In contrast, when we quotient out the first σ -algebra by using the J -projections,

it follows from the interpolation identities above that the starting σ -algebra significantly affects the interpolated space. This justifies, in part, our need to intersect two such spaces in this paper.

BMO spaces for admissible coverings. Let (Ω, Σ, μ) be a σ -finite measure space and consider two atomic σ -algebras Σ_a, Σ_b of measurable sets in Σ . Let Π_j be the set of atoms in Σ_j for $j = a, b$. When $\mu(\Omega) < \infty$, we shall consider two distinguished atoms $(A_0, B_0) \in \Pi_a \times \Pi_b$. If μ is not finite, take $A_0 = B_0 = \emptyset$. Given $A \in \Pi_a$, set

$$R_A = \{B' \in \Pi_b \mid \mu(A \cap B') > 0\}.$$

Define R_B for $B \in \Pi_b$ similarly. The pair (Σ_a, Σ_b) is called an *admissible covering* of (Ω, Σ, μ) when $\Sigma_a \cap \Sigma_b = \{\Omega, \emptyset\}$ and

$$\min \left\{ \sup_{A \in \Pi_a \setminus \{A_0\}} \sum_{B \in R_A} |R_B| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)}, \sup_{B \in \Pi_b \setminus \{B_0\}} \sum_{A \in R_B} |R_A| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)} \right\} < 1.$$

One can view the condition above as a weak version of the concentration of measure near the boundary that appeared in [Carbonaro et al. 2009]. In particular, it is not a geometric notion, but only a measure-theoretic one (see Remark 3.3 for more details). Now, consider any pair of filtrations $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$, and construct the corresponding martingale BMO spaces BMO_a and BMO_b . As in the previous subsection, we quotient out the Σ_j -measurable functions and set, as we did in the introduction,

$$\begin{aligned} BMO_{\Sigma_j}(\Omega) &= J_{\Sigma_j}(BMO_j), \\ BMO_{\Sigma_{ab}}(\Omega) &= BMO_{\Sigma_a}(\Omega) \wedge BMO_{\Sigma_b}(\Omega) = \{f \in L^1_{loc}(\Omega) \mid \|f\|_{BMO_{\Sigma_{ab}}}(\Omega) < \infty\} / \mathbb{C} \end{aligned}$$

In the following, we construct admissible coverings for σ -finite measure spaces. The procedure we employ is quite general. In concrete scenarios, other admissible coverings can be constructed enjoying additional properties as required in Theorems B1 and B2; these examples will be given later in this paper.

Remark 1.2. The classical BMO on Euclidean spaces can be decomposed as an intersection of finitely many martingale BMO spaces, the number of which depends on the dimension [Conde 2013; Garnett and Jones 1982; Mei 2003]. In contrast, we just consider “intersections” of two martingale BMOs. Note this makes our spaces larger and still amenable for interpolation, which gives some extra room to obtain endpoint estimates for singular integral operators. The main reason why this is possible is that our approach just relies on measure-theoretic properties and does not rely on the geometry of the underlying space, as will become clear in the sequel.

Lemma 1.3. *Let (Ω, Σ, μ) be a σ -finite measure space. Then:*

- (i) *If $\mu(\Omega) = \infty$, it admits an admissible covering.*
- (ii) *If $\mu(\Omega) < \infty$ and μ is atomless, it admits an admissible covering.*

Proof. If $\mu(\Omega) = \infty$, pick $A_0 = \tilde{A}_0 = B_0 = \tilde{B}_0 = \emptyset$,

$$A_j = \tilde{A}_j \setminus \tilde{A}_{j-1} \quad \text{and} \quad B_j = \tilde{B}_j \setminus \tilde{B}_{j-1},$$

where $\emptyset \neq \tilde{A}_1 \subsetneq \tilde{B}_1 \subsetneq \tilde{A}_2 \subsetneq \tilde{B}_2 \subsetneq \tilde{A}_3 \subsetneq \dots$ are Σ -measurable sets chosen so that

$$\min \left\{ \frac{\mu(\tilde{B}_j \setminus \tilde{B}_{j-1})}{\mu(\tilde{A}_j)}, \frac{\mu(\tilde{A}_{j+1} \setminus \tilde{A}_j)}{\mu(\tilde{B}_j)} \right\} > \lambda > 4 \quad \text{for all } j \geq 1.$$

It is at this point that we have used that $\mu(\Omega) = \infty$. Let Σ_a be the atomic σ -algebra generated by the atoms $(A_j)_{j \geq 1}$. Similarly, define $\Sigma_b = \sigma(B_j : j \geq 1)$. It is clear by construction that

$$\Sigma_a \cap \Sigma_b = \{\Omega, \emptyset\}.$$

On the other hand, $|R_B| = 2$ for every atom B in Σ_b . Therefore, it remains to show that

$$\sup_{j \geq 1} \left[\frac{\mu(A_j \cap B_{j-1})^2}{\mu(A_j)\mu(B_{j-1})} + \frac{\mu(A_j \cap B_j)^2}{\mu(A_j)\mu(B_j)} \right] < \frac{1}{2}.$$

Note that the first summand above vanishes for $j = 1$. The rest of terms are smaller than $1/\lambda$, according to our conditions, so that $\lambda > 4$ suffices. When $\mu(\Omega) < \infty$ we may assume that $\mu(\Omega) = 1$, since renormalization does not affect our definition of admissible covering. We use again a ‘‘corona-type partition’’

$$\emptyset \neq \tilde{A}_0 \subsetneq \tilde{B}_0 \subsetneq \tilde{A}_1 \subsetneq \tilde{B}_1 \subsetneq \tilde{A}_2 \subsetneq \dots$$

satisfying $\mu(\tilde{A}_0) = 1 - \zeta$, $\mu(\tilde{B}_0 \setminus \tilde{A}_0) = \zeta(1 - \zeta)$ and the relations

$$\mu(\tilde{A}_{j+1} \setminus \tilde{B}_j) = \zeta \mu(\tilde{B}_j \setminus \tilde{A}_j) \quad \text{and} \quad \mu(\tilde{B}_{j+1} \setminus \tilde{A}_{j+1}) = \zeta \mu(\tilde{A}_{j+1} \setminus \tilde{B}_j) \quad \text{for } j \geq 0.$$

This is where we use the fact that μ has no atoms. Define $A_0 = \tilde{A}_0$, $B_0 = \tilde{B}_0$, $A_j = \tilde{A}_j \setminus \tilde{A}_{j-1}$ and $B_j = \tilde{B}_j \setminus \tilde{B}_{j-1}$ for $j \geq 1$. The σ -algebras Σ_a and Σ_b are the ones generated by $(A_j)_{j \geq 0}$ and $(B_j)_{j \geq 0}$, respectively. In order to show that $\Omega = \bigcup_{j \geq 0} A_j = \bigcup_{j \geq 0} B_j$, let us prove that we have

$$\sum_{j \geq 0} \mu(A_j) = \sum_{j \geq 0} \mu(B_j) = 1.$$

Indeed, if $j \geq 2$ we have

$$\begin{aligned} \mu(A_j) &= \mu(\tilde{A}_j \setminus \tilde{A}_{j-1}) \\ &= (1 + \zeta)\mu(\tilde{B}_{j-1} \setminus \tilde{A}_{j-1}) = \zeta(1 + \zeta)\mu(\tilde{A}_{j-1} \setminus \tilde{B}_{j-2}) \\ &= \zeta(1 + \zeta) \left[\mu(\tilde{A}_{j-1} \setminus \tilde{A}_{j-2}) - \frac{1}{\zeta} \mu(\tilde{A}_{j-1} \setminus \tilde{B}_{j-2}) \right] = \zeta^2 \mu(A_{j-1}). \end{aligned}$$

Therefore, since $\mu(A_0) = 1 - \zeta$ and $\mu(A_1) = \zeta(1 - \zeta^2)$, we deduce immediately that $\sum_{j \geq 0} \mu(A_j) = 1$. The sum $\sum_j \mu(B_j)$ also equals 1 since the two families are nested. The condition $\Sigma_a \cap \Sigma_b = \emptyset$ follows again by construction. Finally, since $|R_B| = 2$ for all atoms $B = B_j$, it suffices one more time to prove that

$$\sup_{j \geq 1} \left[\frac{\mu(A_j \cap B_{j-1})^2}{\mu(A_j)\mu(B_{j-1})} + \frac{\mu(A_j \cap B_j)^2}{\mu(A_j)\mu(B_j)} \right] < \frac{1}{2}.$$

According to our construction, the left-hand side can be majorized by

$$\frac{\mu(A_j \cap B_{j-1})^2}{\mu(A_j)\mu(B_{j-1})} + \frac{\mu(A_j \cap B_j)^2}{\mu(A_j)\mu(B_j)} \leq \frac{\mu(\tilde{B}_{j-1} \setminus \tilde{A}_{j-1})}{\mu(B_{j-1})} + \frac{\mu(\tilde{A}_j \setminus \tilde{B}_{j-1})}{\mu(A_j)}.$$

On the other hand, arguing as before, we may obtain the identities

$$\begin{aligned} \mu(A_j) &= \zeta^{2(j-1)}(\zeta - \zeta^3), & \mu(\tilde{A}_j \setminus \tilde{B}_{j-1}) &= \zeta^{2j}(1 - \zeta), \\ \mu(B_{j-1}) &= \zeta^{2(j-1)}(1 - \zeta^2), & \mu(\tilde{B}_{j-1} \setminus \tilde{A}_{j-1}) &= \zeta^{2(j-1)}(\zeta - \zeta^2). \end{aligned}$$

This gives a bound $2\zeta/(1 + \zeta)$. It suffices for $\zeta < \frac{1}{3}$. The proof is complete. □

Remark 1.4. All fully supported probability measures on \mathbb{R}^n are nondoubling. In fact, this also holds for probability measures supported on unbounded sets. In particular, we hope [Lemma 1.3](#) together with [Theorems B1](#) and [B2](#) might open a door to further insight into Calderón–Zygmund theory for these measures.

John–Nirenberg inequalities, atomic H_1 and duality. We now transfer some well-known properties of martingale BMO spaces to our new class of spaces. The analogue of John–Nirenberg inequalities [\[1961\]](#) for martingale BMO spaces can be stated as follows:

$$\sup_{k \geq 1} \sup_{A \in \Sigma_k} \frac{1}{\mu(A)} \mu(A \cap \{|f - E_{\Sigma_{k-1}} f| > \lambda\}) \lesssim \exp\left(-\frac{c\lambda}{\|f\|_{\text{BMO}}}\right) \quad \text{for all } \lambda > 0,$$

where the martingale BMO is constructed over the filtration $(\Sigma_k)_{k \geq 1}$ and we use the convention $E_{\Sigma_0} f = 0$. The proof can be found in [\[Garsia 1973\]](#). An important consequence of this inequality is the p -invariance of the BMO norm. To be more precise, the martingale BMO norm admits the equivalent expressions, for any $0 < p < \infty$,

$$\|f\|_{\text{BMO}} \sim \sup_{k \geq 1} \left\| (E_{\Sigma_k} |f - E_{\Sigma_{k-1}} f|^p)^{1/p} \right\|_{\infty}.$$

If we replace f by $J_{\Sigma_1} f = f - E_{\Sigma_1} f$ in both inequalities, we immediately obtain the corresponding analogues for the BMO spaces which quotient out Σ_1 -measurable functions, introduced above. Namely, the only difference is that we should read John–Nirenberg inequalities under the convention that $E_{\Sigma_0} f = E_{\Sigma_1} f$, and the BMO norm is given by $\|\cdot\|_{\text{BMO}}^\circ$ instead. If we intersect two of these BMO spaces, we get John–Nirenberg-type inequalities for our spaces $\text{BMO}_{\Sigma_{\text{ab}}}(\Omega)$ associated to an admissible covering $(\Sigma_{\text{a}}, \Sigma_{\text{b}})$ by taking again $E_{\Sigma_0} f = E_{\Sigma_1} f$:

$$\begin{aligned} \|f\|_{\text{BMO}_{\Sigma_{\text{ab}}}(\Omega)} &\sim \max_{j=\text{a,b}} \sup_{k \geq 1} \left\| (E_{\Sigma_{jk}} |f - E_{\Sigma_{j(k-1)}} f|^p)^{1/p} \right\|_{\infty}, \\ \sup_{j=\text{a,b}} \sup_{\substack{k \geq 1 \\ A \in \Sigma_{jk}}} \frac{1}{\mu(A)} \mu(A \cap \{|f - E_{\Sigma_{j(k-1)}} f| > \lambda\}) &\lesssim \exp\left(-\frac{c\lambda}{\|f\|_{\text{BMO}_{\Sigma_{\text{ab}}}(\Omega)}}\right). \end{aligned}$$

Let us now consider H_1 –BMO duality in our context. In the literature we find several equivalent descriptions of martingale H_1 spaces, via Doob’s maximal function, martingale square function or

conditional square function. Namely, H_1 can be defined as the closure of the space of finite L_1 martingales with respect to any of the norms

$$\left\| \sup_{k \geq 1} |E_{\Sigma_k} f| \right\|_1 \sim \left\| \left(\sum_{k \geq 1} |df_k|^2 \right)^{\frac{1}{2}} \right\|_1 \sim \sum_{k \geq 1} \|df_k\|_1 + \left\| \left(\sum_{k \geq 1} E_{\Sigma_{k-1}} |df_k|^2 \right)^{\frac{1}{2}} \right\|_1.$$

We refer to [Davis 1970] for the equivalences above and to [Garsia 1973] for the duality theorem, which claims that $H_1^* \simeq \text{BMO}$, a martingale analogue of the Fefferman–Stein duality theorem. Let us now consider atomic descriptions of these spaces. The term “atom” unfortunately appears here in several settings — σ -algebras, measures and Hardy spaces — with different meanings, but it will be clear which one is used from the context. Atomic descriptions are not possible for arbitrary H_1 — see [Conde-Alonso and Parcet 2014] for an “atomic block” description both in the commutative and noncommutative settings — but there are such results for h_1 (defined above). A Σ -measurable function $a \in L_2(\Omega)$ is called an atom when there exists $k \geq 1$ and $A \in \Sigma_k$ with

$$\text{supp}(a) \subset A, \quad E_{\Sigma_k}(a) = 0, \quad \|a\|_2 \leq \mu(A)^{-1/2}.$$

The atomic h_1 is defined as the space of functions of the form $f = \sum_j \lambda_j a_j$ with the a_j atoms. The norm is the infimum of $\sum_j |\lambda_j|$ over all such possible expressions for the function f . This space is isomorphic to h_1 ; see [Garsia 1973]. In particular, it is also isomorphic to H_1 when the filtration is regular. This will be enough for our purposes, since we will only use H_1 –BMO duality for regular filtrations. Now, given two filtrations $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$, let H_{1j} be the corresponding H_1 spaces. Define

$$H_{\Sigma_{ab}}^1(\Omega) = \left\{ f \in L_1(\Omega) \mid \|f\|_{H_1} = \inf_{\substack{f=f_1+f_2 \\ E_{\Sigma_a} f_1 = E_{\Sigma_b} f_2 = 0}} \|f_1\|_{H_{1a}} + \|f_2\|_{H_{1b}} < \infty \right\}.$$

Then, all the results above apply. In particular, we have

$$H_{\Sigma_{ab}}^1(\Omega)^* \simeq \text{BMO}_{\Sigma_{ab}}(\Omega).$$

2. Interpolation: proof of Theorem A

Proof of Theorem A. The argument is a bit lengthy, so we have divided it into several steps. We will assume that μ is a finite measure on Ω — normalized so that $\mu(\Omega) = 1$ — since this case is more technical. The slight modifications needed for the nonfinite case will be explained in the last step of the proof.

Step 1: Intersection of quotients. Let us first show that the interpolation result follows from the first assertion of Theorem A. Namely, given an admissible covering (Σ_a, Σ_b) of (Ω, Σ, μ) and filtrations $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$, let BMO_j be the corresponding martingale BMO spaces. It is clear that

$$\begin{aligned} \|f\|_{L_q^c(\Omega)} &= \|f\|_{[L_\infty^c(\Omega), L_1^c(\Omega)]_{1/q}} \gtrsim \|f\|_{[\text{BMO}_{\Sigma_{ab}}(\Omega), L_1^c(\Omega)]_{1/q}} \\ &\geq \max_{j=a,b} \|f - E_{\Sigma_j} f\|_{[J_{\Sigma_j}(\text{BMO}_j), J_{\Sigma_j}(L_1(\Omega))]_{1/q}} \\ &\simeq \max_{j=a,b} \|f - E_{\Sigma_j} f\|_{L_{\Sigma_j}^q(\Omega)} = \|f\|_{L_{\Sigma_{ab}}^q(\Omega)}. \end{aligned}$$

For $q \geq 2$, this implies

$$L_q^\circ(\Omega) \subset [\text{BMO}_{\Sigma_{ab}}(\Omega), L_1^\circ(\Omega)]_{1/q} \subset L_{\Sigma_{ab}}^q(\Omega).$$

Thus, the result follows from the isomorphism $L_q^\circ(\Omega) \simeq L_{\Sigma_{ab}}^q(\Omega)$. The interpolation result for $1 < q < 2$ follows from this and the well-known reiteration theorem [Bergh and Löfström 1976].

Step 2: Reduction to strict contractions. The rest of the proof will be devoted to justify the first assertion of Theorem A. We claim that such an isomorphism holds whenever we can find a constant $0 < c_p(\Sigma_{ab}) < 1$ such that, for every mean-zero function $f \in L_p(\Omega)$,

$$\min\{\|E_{\Sigma_a} E_{\Sigma_b} f\|_p, \|E_{\Sigma_b} E_{\Sigma_a} f\|_p\} \leq c_p(\Sigma_{ab}) \|f\|_p. \tag{2-1}$$

Indeed, if $E\phi = \int_\Omega \phi \, d\mu$, we first observe that

$$\begin{aligned} \|\phi\|_{L_p^\circ(\Omega)} &\sim \|\phi - E\phi\|_p \sim \inf_{k \in \mathbb{C}} \|\phi - k\|_p, \\ \|\phi\|_{J_{\Sigma_j}(L_p(\Omega))} &\sim \|\phi - E_{\Sigma_j} \phi\|_p \sim \inf_{\varphi \text{ } \Sigma_j\text{-measurable}} \|\phi - \varphi\|_p. \end{aligned}$$

Therefore, our goal in what follows is to show that

$$\|\phi - E\phi\|_p \sim \|\phi - E_{\Sigma_a} \phi\|_p + \|\phi - E_{\Sigma_b} \phi\|_p \quad \text{for every } \phi \in L_p(\Omega).$$

The lower estimate is trivial. For the upper estimate, we shall use (2-1). Assume that the minimum above is attained at the first term (say) and let $f = \phi - E\phi$ be a mean-zero function. We then find

$$\|E_{\Sigma_a} E_{\Sigma_b} f\|_p \leq c_p(\Sigma_{ab}) \|f\|_p \leq c_p(\Sigma_{ab}) [\|f - E_{\Sigma_a} f\|_p + \|E_{\Sigma_a}(f - E_{\Sigma_b} f)\|_p + \|E_{\Sigma_a} E_{\Sigma_b} f\|_p],$$

which implies

$$\|E_{\Sigma_a} E_{\Sigma_b} f\|_p \leq \frac{c_p(\Sigma_{ab})}{1 - c_p(\Sigma_{ab})} [\|\phi - E_{\Sigma_a} \phi\|_p + \|\phi - E_{\Sigma_b} \phi\|_p].$$

This inequality is all we need, since the upper estimate follows from it:

$$\begin{aligned} \|\phi - E\phi\|_p &\leq \|E_{\Sigma_a} \phi - E\phi\|_p + \|\phi - E_{\Sigma_a} \phi\|_p \leq \|E_{\Sigma_a} E_{\Sigma_b} f\|_p + \|E_{\Sigma_a}(\phi - E_{\Sigma_b} \phi)\|_p + \|\phi - E_{\Sigma_a} \phi\|_p \\ &\leq \frac{1}{1 - c_p(\Sigma_{ab})} [\|\phi - E_{\Sigma_a} \phi\|_p + \|\phi - E_{\Sigma_b} \phi\|_p]. \end{aligned}$$

Step 3: The case $p = 2$. Recall that we are assuming for the moment that $\mu(\Omega) = 1$, and in that case we may consider two distinguished atoms $(A_0, B_0) \in \Sigma_a \times \Sigma_b$. In accordance with the previous point, it suffices to show that

$$\min\{\|E_{\Sigma_a} E_{\Sigma_b} f\|_2, \|E_{\Sigma_b} E_{\Sigma_a} f\|_2\} \leq c_2(\Sigma_{ab}) \|f\|_2$$

for some $0 < c_2(\Sigma_{ab}) < 1$ and every mean-zero $f \in L_2(\Omega)$. We claim that this estimate follows if the same inequality holds for Σ_j -measurable functions which vanish on the corresponding distinguished atom. More precisely, it suffices to prove that one of the following conditions holds:

- $\|E_{\Sigma_a} \phi_b\|_2 \leq c_2(\Sigma_{ab}) \|\phi_b\|_2$ for ϕ_b Σ_b -measurable with $\phi_b(B_0) = 0$;
- $\|E_{\Sigma_b} \phi_a\|_2 \leq c_2(\Sigma_{ab}) \|\phi_a\|_2$ for ϕ_a Σ_a -measurable with $\phi_a(A_0) = 0$.

Indeed, assume the first condition holds and let $\phi_b \in L_2(\Omega, \Sigma_b, \mu)$ be mean-zero. Then

$$\begin{aligned} \|\phi_b\|_2^2 &= \|\phi_b - \phi_b(B_0)\|_2^2 - |\phi_b(B_0)|^2, \\ \|\mathbb{E}_{\Sigma_a} \phi_b\|_2^2 &= \|\mathbb{E}_{\Sigma_a}(\phi_b - \phi_b(B_0))\|_2^2 - |\phi_b(B_0)|^2. \end{aligned}$$

Subtracting and using the first condition, we get

$$\|\phi_b\|_2^2 - \|\mathbb{E}_{\Sigma_a} \phi_b\|_2^2 \geq (1 - c_2(\Sigma_{ab}))\|\phi_b - \phi_b(B_0)\|_2^2 \geq (1 - c_2(\Sigma_{ab}))\|\phi_b\|_2^2.$$

Here, $\phi_b(B_0)$ denotes the constant value of ϕ_b on B_0 . Rearranging, we get $\|\mathbb{E}_{\Sigma_a} \phi_b\|_2 \leq c_2(\Sigma_{ab})\|\phi_b\|_2$. Therefore, given any mean-zero $f \in L_2(\Omega)$, we may define $\phi_b = \mathbb{E}_{\Sigma_b} f$ and deduce that $\|\mathbb{E}_{\Sigma_a} \mathbb{E}_{\Sigma_b} f\|_2 \leq c_2(\Sigma_{ab})\|f\|_2$, as desired. Alternatively, if we use the second condition above, the roles of Σ_a and Σ_b are switched and we obtain the other sufficient inequality which is implicit in the minimum above. Thus we have reduced the proof to justify one of the two conditions above. It is at this point where our definition of admissible pair comes into play. Namely, we know that

$$\min \left\{ \sup_{A \in \Pi_a \setminus \{A_0\}} \sum_{B \in R_A} |R_B| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)}, \sup_{B \in \Pi_b \setminus \{B_0\}} \sum_{A \in R_B} |R_A| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)} \right\} = c(\Sigma_{ab})$$

for some $0 < c(\Sigma_{ab}) < 1$. Let us assume (say) that the minimum above is attained by the first term and let ϕ_a be a Σ_a -measurable function in $L_2(\Omega)$ that vanishes on A_0 . Then, if we write $\phi_a = \sum_{A \neq A_0} \alpha_A \chi_A$, we have the estimate

$$\begin{aligned} \|\mathbb{E}_{\Sigma_b} \phi_a\|_2^2 &= \sum_{A, A' \neq A_0} \bar{\alpha}_A \alpha_{A'} \sum_{B \in R_A \cap R_{A'}} \frac{\mu(A \cap B)}{\mu(B)^{1/2}} \frac{\mu(A' \cap B)}{\mu(B)^{1/2}} \\ &\leq \sum_{A, A' \neq A_0} \frac{1}{2} \sum_{B \in R_A \cap R_{A'}} \left(|\alpha_A|^2 \frac{\mu(A \cap B)^2}{\mu(B)} + |\alpha_{A'}|^2 \frac{\mu(A' \cap B)^2}{\mu(B)} \right) \\ &= \sum_{A \neq A_0} |\alpha_A|^2 \mu(A) \sum_{\substack{A' \neq A_0 \\ R_A \cap R_{A'} \neq \emptyset}} \sum_{B \in R_A \cap R_{A'}} \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)} \\ &= \sum_{A \neq A_0} |\alpha_A|^2 \mu(A) \sum_{B \in R_A} |R_B| \frac{\mu(A \cap B)^2}{\mu(A)\mu(B)} \leq c(\Sigma_{ab}) \sum_{A \neq A_0} |\alpha_A|^2 \mu(A). \end{aligned}$$

The right-hand side equals $c(\Sigma_{ab})\|\phi_a\|_2^2$, so we obtain the second condition. The first one follows when the minimum in our definition of admissible covering is attained by the second term. This proves that the first assertion of [Theorem A](#) holds for finite measures and $p = 2$. The case $p > 2$ requires some preliminaries.

Step 4: A mass absorption principle. Let us consider a particular ordering of the atoms in Σ_a and Σ_b . According to our assumption $\Sigma_a \cap \Sigma_b = \{\Omega, \emptyset\}$, we may order Π_a so that $\Pi_a = \{A_1, A_2, \dots\}$ and, for each $m \geq 0$, there exists $B \in \Pi_b$ such that $\mu(A_{m+1} \cap B)$ and $\mu(\bigcup_{s \leq m} A_s \cap B)$ are both strictly positive.

Similarly, we may order Π_b satisfying the symmetric condition. Define the atomic σ -algebras

$$\Sigma_a(m) = \sigma\left\langle \bigcup_{s=0}^m A_s, \{A_s\}_{s \geq m+1} \right\rangle,$$

$$\Sigma_b(m) = \sigma\left\langle \bigcup_{s=0}^m B_s, \{B_s\}_{s \geq m+1} \right\rangle.$$

In this step we will prove that

$$\|f\|_{L_{\Sigma_{ab}}^p(\Omega)} \simeq \|f - E_{\Sigma_a(m)}f\|_{L_p(\Omega)} + \|f - E_{\Sigma_b(m)}f\|_{L_p(\Omega)} \tag{2-2}$$

for any $m \geq 1$ and $2 < p < \infty$. The constants may depend on m, p and the covering (Σ_a, Σ_b) . Indeed, since the result is trivial for $m = 0$, we will proceed by induction and assume that the result holds for $m - 1$. Moreover, the upper estimate is straightforward and by symmetry it suffices to show that

$$\|f - E_{\Sigma_a(m)}f\|_p \lesssim \|f - E_{\Sigma_a(m-1)}f\|_p + \|f - E_{\Sigma_b}f\|_p.$$

Taking $A_0(m) = \bigcup_{s \leq m} A_s$, let $f = f\chi_{A_0(m)} + f\chi_{\Omega \setminus A_0(m)} = f_1 + f_2$. Since it is clear that $E_{\Sigma_a(m)}f_2 = E_{\Sigma_a(m-1)}f_2$, we may concentrate only on f_1 . The left-hand side for f_1 can be written as

$$\|f_1 - E_{\Sigma_a(m)}f_1\|_p = \|\chi_{A_0(m)}(f - E_{\Sigma_a(m)}f)\|_p = \|f\|_{L_p^{\circ}(A_0(m))} \sim \sup_{\substack{\|g\|_{L_{p'}(A_0(m))} \leq 1 \\ g \text{ mean-zero}}} \left| \int_{A_0(m)} fg \, d\mu \right|.$$

Approximating the right-hand side up to $\varepsilon > 0$ by some mean-zero g_0 in the unit ball of $L_{p'}(A_0(m))$, let B be an atom in Σ_b satisfying that $\mu(A_0(m-1) \cap B)$ and $\mu(A_m \cap B)$ are strictly positive. Recall that this can be done by the specific enumeration of atoms we picked. Then, define

$$g_1 = \chi_{A_m}g_0 - \frac{\chi_{A_m \cap B}}{\mu(A_m \cap B)} \int_{A_m} g_0 \, d\mu,$$

$$g_2 = \chi_{A_0(m-1)}g_0 - \frac{\chi_{A_0(m-1) \cap B}}{\mu(A_0(m-1) \cap B)} \int_{A_0(m-1)} g_0 \, d\mu,$$

$$g_3 = \frac{\chi_{A_0(m-1) \cap B}}{\mu(A_0(m-1) \cap B)} \int_{A_0(m-1)} g_0 \, d\mu + \frac{\chi_{A_m \cap B}}{\mu(A_m \cap B)} \int_{A_m} g_0 \, d\mu.$$

Obviously, $g_0 = g_1 + g_2 + g_3$ and each g_j is mean-zero. Moreover, we have

$$\begin{aligned} \|g_1\|_{L_{p'}(A_0(m))} &\leq \|\chi_{A_m}g_0\|_{L_{p'}(A_0(m))} + \left\| \frac{\chi_{A_m \cap B}}{\mu(A_m \cap B)} \int_{A_m} g_0 \, d\mu \right\|_{L_{p'}(A_0(m))} \\ &\leq \left(1 + \frac{\mu(A_m)^{1/p}}{\mu(A_m \cap B)^{1/p}} \right) \|g_0\|_{L_{p'}(A_0(m))} \lesssim \|g_0\|_{L_{p'}(A_0(m))} \leq 1. \end{aligned}$$

Similar computations apply to g_2 and g_3 . In summary, we obtain the estimate below, where we write f_Q to denote the average of f over a given measurable set Q :

$$\begin{aligned} \|f_1 - E_{\Sigma_a(m)} f_1\|_p &\sim \left| \int_{A_0(m)} f g_0 d\mu \right| \\ &\leq \left| \int_{A_m} (f - f_{A_m}) g_1 d\mu \right| + \left| \int_{A_0(m-1)} (f - f_{A_0(m-1)}) g_2 d\mu \right| + \left| \int_B (f - f_B) g_3 d\mu \right| \\ &\lesssim \|\chi_{A_m} (f - f_{A_m})\|_p + \|\chi_{A_0(m-1)} (f - f_{A_0(m-1)})\|_p + \|\chi_B (f - f_B)\|_p \\ &\lesssim \|f - E_{\Sigma_a(m-1)} f\|_p + \|f - E_{\Sigma_b} f\|_p. \end{aligned}$$

This completes the proof of the norm equivalence (2-2).

Step 5: The case $p > 2$. We now complete the proof of Theorem A for probability measures. According to (2-2), it suffices to show that there exists $0 < c_p(\Sigma_{ab}) < 1$ and $m = m(p) \geq 1$ such that, for any mean-zero function $f \in L_p(\Omega)$,

$$\min\{\|E_{\Sigma_a(m)} E_{\Sigma_b(m)} f\|_p, \|E_{\Sigma_b(m)} E_{\Sigma_a(m)} f\|_p\} \leq c_p(\Sigma_{ab}) \|f\|_p.$$

Pick $m = m(p)$ as the smallest possible value of m satisfying

$$\min\{\mu(A_0(m)), \mu(B_0(m))\} > \max\left\{\left(\frac{2 \cdot 4^{-p}}{1 - 2 \cdot 4^{-p}}\right)^{\frac{1}{p-1}}, (1 - 4^{-p})^{1/p}\right\}$$

and $\varepsilon = \varepsilon(p) > 0$ small enough so that

$$(1 - 2 \cdot 4^{-p})^{1/2} \leq (1 - 4^{-p})^{1/(2p)} (1 - \varepsilon^3)^{1/2} - \varepsilon^{3/2}.$$

Since $L_p(\Omega) \subset L_2(\Omega)$, we know from Step 3 that f always satisfies the above inequality for $p = 2$. Assume that the minimum for $p = 2$ is attained (say) at the first term, so that $\|E_{\Sigma_a(m)} E_{\Sigma_b(m)} f\|_2 \leq c_2(\Sigma_{ab}) \|f\|_2$. Recall that $E f = \int_{\Omega} f d\mu$. When

$$E(|E_{\Sigma_b(m)} f|^{p/2})^2 < (1 - \varepsilon^3) \|E_{\Sigma_b(m)} f\|_p^p,$$

we proceed as follows:

$$\begin{aligned} \|E_{\Sigma_a(m)} E_{\Sigma_b(m)} f\|_p^p &\leq \|E_{\Sigma_a(m)} |E_{\Sigma_b(m)} f|^{p/2}\|_2^2 - (E|E_{\Sigma_b(m)} f|^{p/2})^2 + (E|E_{\Sigma_b(m)} f|^{p/2})^2 \\ &= \|E_{\Sigma_a(m)} (|E_{\Sigma_b(m)} f|^{p/2} - E|E_{\Sigma_b(m)} f|^{p/2})\|_2^2 + (E|E_{\Sigma_b(m)} f|^{p/2})^2 \\ &\leq c_2^2(\Sigma_{ab}) \| |E_{\Sigma_b(m)} f|^{p/2} - E|E_{\Sigma_b(m)} f|^{p/2} \|_2^2 + (E|E_{\Sigma_b(m)} f|^{p/2})^2 \\ &\leq [c_2^2(\Sigma_{ab}) + (1 - c_2^2(\Sigma_{ab}))(1 - \varepsilon^3)] \|f\|_p^p = c_p^p(\Sigma_{ab}) \|f\|_p^p. \end{aligned}$$

If $E(|E_{\Sigma_b(m)} f|^{p/2})^2 \geq (1 - \varepsilon^3) \|E_{\Sigma_b(m)} f\|_p^p$, then one can easily show that

$$\| |E_{\Sigma_b(m)} f|^{p/2} - E|E_{\Sigma_b(m)} f|^{p/2} \|_2^2 \leq \varepsilon^3 \|E_{\Sigma_b(m)} f\|_p^p.$$

Now, decomposing $E_{\Sigma_b(m)} f = E_{\Sigma_b(m)} f(B_0(m))\chi_{B_0(m)} + E_{\Sigma_b(m)} f \chi_{\Omega \setminus B_0(m)}$, we get

$$\begin{aligned} \sqrt{\mu(B_0(m))|E_{\Sigma_b(m)} f(B_0(m))|^p} &= \| |E_{\Sigma_b(m)} f|^{p/2} \chi_{B_0(m)} \|_2 \\ &\geq \| E |E_{\Sigma_b(m)} f|^{p/2} \chi_{B_0(m)} \|_2 - \| |E_{\Sigma_b(m)} f|^{p/2} - E |E_{\Sigma_b(m)} f|^{p/2} \|_2 \\ &\geq \mu(B_0(m))^{1/2} E |E_{\Sigma_b(m)} f|^{p/2} - \varepsilon^{3/2} \| E_{\Sigma_b(m)} f \|_p^{p/2} \\ &\geq [(1 - 4^{-p})^{1/(2p)} (1 - \varepsilon^3)^{1/2} - \varepsilon^{3/2}] \| E_{\Sigma_b(m)} f \|_p^{p/2} \\ &\geq (1 - 2 \cdot 4^{-p})^{1/2} \| E_{\Sigma_b(m)} f \|_p^{p/2}. \end{aligned}$$

This also implies

$$\| E_{\Sigma_b(m)} f \chi_{\Omega \setminus B_0(m)} \|_p^p \leq 2 \cdot 4^{-p} \| E_{\Sigma_b(m)} f \|_p^p.$$

On the other hand, since f is mean-zero we have

$$E_{\Sigma_b(m)} f(B_0(m))\mu(B_0(m)) + E(E_{\Sigma_b(m)} f \chi_{\Omega \setminus B_0(m)}) = 0.$$

Rearranging and raising to the power p then gives

$$\mu(B_0(m))^p |E_{\Sigma_b(m)} f(B_0(m))|^p \leq \| E_{\Sigma_b(m)} f \chi_{\Omega \setminus B_0(m)} \|_p^p \leq 2 \cdot 4^{-p} \| E_{\Sigma_b(m)} f \|_p^p.$$

Finally, combining our two estimates so far for $\mu(B_0(m))$, we obtain

$$\mu(B_0(m)) \leq \left(\frac{2 \cdot 4^{-p}}{1 - 2 \cdot 4^{-p}} \right)^{\frac{1}{p-1}},$$

which contradicts our choice of $m = m(p)$. This shows that $E(|E_{\Sigma_b(m)} f|^{p/2})^2$ cannot be larger than $(1 - \varepsilon^3) \| E_{\Sigma_b(m)} f \|_p^p$ and completes the proof in the case the minimum for $p = 2$ is attained at the first term. When the minimum is attained at the second term, a symmetric argument applies.

Step 6: The nonfinite case. When $\mu(\Omega) = \infty$ the proof of [Theorem A](#) is a bit simpler. In the first place, note that $L_p^\circ(\Omega) = L_p(\Omega)$ in this case. In particular, the goal is to show that

$$\begin{aligned} L_q(\Omega) &\simeq [\text{BMO}_{\Sigma_{\text{ab}}}(\Omega), L_1(\Omega)]_{1/q}, \\ L_p(\Omega) &\simeq L_p(\Omega, \Sigma, \mu) / \Sigma_a \wedge L_p(\Omega, \Sigma, \mu) / \Sigma_b. \end{aligned}$$

Since $L_\infty(\Omega) \subset \text{BMO}_{\Sigma_{\text{ab}}}(\Omega)$, our argument in Step 1 can be easily adapted and interpolation follows from the second isomorphism above. To prove it, we follow essentially the same argument as for finite measures. Indeed, arguing as in Step 2, we see that it suffices to show that

$$\min\{ \| E_{\Sigma_a} E_{\Sigma_b} f \|_p, \| E_{\Sigma_b} E_{\Sigma_a} f \|_p \} \leq c_p(\Sigma_{\text{ab}}) \| f \|_p$$

for some constant $0 < c_p(\Sigma_{\text{ab}}) < 1$ and every function $f \in L_p(\Omega)$. The only difference is that here it must hold for every f , not just mean-zero elements as in the finite case. The case $p = 2$ is proved following Step 3. The fact that we do not assume f to be mean-zero — or ultimately to vanish at A_0 or B_0 — is compensated by our definition of admissible coverings, which does not consider distinguished atoms

for infinite measures. Finally, once we know the case $p = 2$ holds — for arbitrary functions, not only mean-zero ones — we conclude that

$$\|E_{\Sigma_a} E_{\Sigma_b} f\|_p^p \leq \|E_{\Sigma_a} E_{\Sigma_b} |f|^{p/2}\|_2^2 \leq c_2^2(\Sigma_{ab}) \| |f|^{p/2} \|_2^2 \leq c_p^p(\Sigma_{ab}) \|f\|_p^p$$

or a similar estimate for $E_{\Sigma_b} E_{\Sigma_a} f$. The proof of [Theorem A](#) is now complete. □

3. Calderón–Zygmund operators, I

Let (Ω, Σ, μ) be a measure space and consider a metric d on Ω . Assume that μ is σ -finite with respect to the metric topology. In this section we will be interested in Calderón–Zygmund operators on the metric measure space (Ω, μ, d) , as defined in the introduction. More precisely, we prove [Theorem B1](#) below and, after that, we shall illustrate this result with a few constructions of admissible coverings.

Proof of [Theorem B1](#). Our definition of CZO includes a symmetric Hörmander kernel condition. This implies that the class of Calderón–Zygmund operators is closed under taking adjoints. In particular, the L_p -boundedness for $1 < p < 2$ can be deduced by duality from the case $p > 2$. On the other hand, according to [Theorem A](#), the latter follows by interpolation if we can prove that any CZO extends to a bounded map $L_\infty(\Omega) \rightarrow \text{BMO}_{\Sigma_{ab}}(\Omega)$. Indeed, since T is L_2 -bounded, [Theorem A](#) yields that $T : L_p(\Omega) \rightarrow L_p^\circ(\Omega)$. This is enough when the measure μ is infinite, since in that case $L_p(\Omega) = L_p^\circ(\Omega)$. When μ is finite we use L_2 -boundedness once again together with Hölder’s inequality to deduce that

$$\|Tf\|_p \leq \|Tf - ETf\|_p + \mu(\Omega)^{1/p} |ETf| \lesssim \|f\|_p + \mu(\Omega)^{1/p-1/2} \|f\|_2 \lesssim \|f\|_p.$$

This completes the proof of our claim. Let us then prove the $L_\infty \rightarrow \text{BMO}$ estimate. Consider an auxiliary BMO space which arises by averaging over the family of doubling balls in (Ω, Σ, μ) ,

$$\|f\|_{\text{DBMO}} = \sup_{\substack{\text{Bd-ball} \\ \text{doubling}}} \left(\frac{1}{\mu(B)} \int_B |f(w) - \frac{1}{\mu(B)} \int_B f \, d\mu|^2 \, d\mu(w) \right)^{\frac{1}{2}}.$$

Following the standard argument, it is easily checked that

$$T : L_\infty(\Omega) \rightarrow \text{DBMO}.$$

Indeed, in the first place we may observe as usual that we have the equivalence

$$\|f\|_{\text{DBMO}} \sim \sup_{\substack{\text{Bd-ball} \\ \text{doubling}}} \inf_{k_B \in \mathbb{C}} \left(\frac{1}{\mu(B)} \int_B |f(w) - k_B|^2 \, d\mu(w) \right)^{\frac{1}{2}}.$$

Second, we decompose $f = f\chi_{\alpha B} + f\chi_{\Omega \setminus \alpha B} = \phi_{1B} + \phi_{2B}$ and pick the constant k_B to be the average of $T\phi_{2B}$ over B . Then, we may estimate the norm of Tf in DBMO by using the L_2 -boundedness of T for

$T\phi_{1B}$ and the Hörmander kernel condition for $T\phi_{2B}$. More precisely, we get

$$\|Tf\|_{\text{DBMO}} \leq \sup_{\substack{\text{Bd-ball} \\ \text{doubling}}} \left(\frac{1}{\mu(B)} \int_B \left| T(f\chi_{\alpha B})(w) \right|^2 d\mu(w) \right)^{\frac{1}{2}} + \left(\frac{1}{\mu(B)} \int_B \left| T(f\chi_{\Omega \setminus \alpha B})(w) - \frac{1}{\mu(B)} \int_B T(f\chi_{\Omega \setminus \alpha B}) d\mu \right|^2 d\mu(w) \right)^{\frac{1}{2}}.$$

Since we just use (α, β) -doubling balls, the first term is dominated by

$$\left(\frac{\mu(\alpha B)}{\mu(B)} \right)^{\frac{1}{2}} \|T\|_{2 \rightarrow 2} \|f\|_{\infty} \lesssim \|f\|_{\infty}.$$

On the other hand, using the kernel representation of T we may write

$$T(f\chi_{\Omega \setminus \alpha B})(w) - \frac{1}{\mu(B)} \int_B T(f\chi_{\Omega \setminus \alpha B}) d\mu = \frac{1}{\mu(B)} \int_B \int_{\Omega \setminus \alpha B} (k(w, \zeta) - k(\xi, \zeta)) f(\zeta) d\mu(\zeta) d\mu(\xi)$$

for $w \in B$. In particular, the last term above can be majorized by $\|f\|_{\infty}$ using the Hörmander condition for k . This proves the $L_{\infty}(\Omega) \rightarrow \text{DBMO}$ boundedness of our CZO. Therefore, it suffices to show that $\text{DBMO} \subset \text{BMO}_{\Sigma_{\text{ab}}}(\Omega)$. This follows from the chain of inclusions

$$\text{DBMO} \subset \text{bmo}_{\Sigma_a} \wedge \text{bmo}_{\Sigma_b} \subset \text{BMO}_{\Sigma_a} \wedge \text{BMO}_{\Sigma_b} = \text{BMO}_{\Sigma_{\text{ab}}}(\Omega).$$

Let us recall in passing the terminology we are using, namely

$$\text{bmo}_{\Sigma_j} = J_{\Sigma_j}(\text{bmo}_j) \quad \text{and} \quad \text{BMO}_{\Sigma_j} = J_{\Sigma_j}(\text{BMO}_j)$$

for $j = a, b$. Here, bmo_j and BMO_j are the martingale bmo and BMO spaces constructed over the filtrations $(\Sigma_{jk})_{k \geq 1}$ described in the statement of [Theorem B1](#). If Π_j denotes the atoms in such a filtration, the norm in bmo_{Σ_j} is given by

$$\|f\|_{\text{bmo}_{\Sigma_j}} = \sup_{k \geq 1} \|E_{\Sigma_{jk}}|f - E_{\Sigma_{jk}}f|^2\|_{\infty}^{1/2} = \sup_{A \in \Pi_j} \left(\frac{1}{\mu(A)} \int_A \left| f(w) - \frac{1}{\mu(A)} \int_A f d\mu \right|^2 d\mu(w) \right)^{\frac{1}{2}}.$$

Now, since we assume that all atoms in $\Pi = \Pi_a \cup \Pi_b$ are doubling, the seminorm above is majorized (up to absolute constants) by the seminorm in DBMO . As this holds for both $j = a, b$, we have proved the first inclusion. Now, for the second inclusion, we recall the seminorm in BMO_{Σ_j} ,

$$\|f\|_{\text{BMO}_{\Sigma_j}} = \sup_{k \geq 1} \|E_{\Sigma_{jk}}|f - E_{\Sigma_{jk-1}}f|^2\|_{\infty}^{1/2},$$

where $E_{\Sigma_{j0}}f = E_{\Sigma_{j1}}f$ since we quotient out Σ_{j1} -measurable functions. Note also that we are requiring the filtrations $(\Sigma_{jk})_{k \geq 1}$ to be regular. In other words, there exist absolute constants $c_j > 0$ such that $E_{\Sigma_{jk}}|f| \leq c_j E_{\Sigma_{jk-1}}|f|$ for $j = a, b$ and $k \geq 1$. This yields the inequality

$$\|f\|_{\text{BMO}_{\Sigma_j}} \leq c_j \|f\|_{\text{bmo}_{\Sigma_j}}.$$

Thus, $\text{BMO}_{\Sigma_{\text{ab}}}(\Omega) \simeq \text{bmo}_{\Sigma_a} \wedge \text{bmo}_{\Sigma_b}$ for regular filtrations, and we are done. □

Remark 3.1. Under the same assumptions, every CZO extends to a bounded map

$$H^1_{\Sigma_{ab}}(\Omega) \rightarrow L_1(\Omega).$$

Indeed, this follows at once by duality and [Theorem B1](#). Alternatively, since we need to work with regular filtrations, we may use the atomic description of $H^1_{\Sigma_{ab}}(\Omega)$ given in [Section 1](#), from which an easy argument arises; details are left to the reader.

In the following subsections we shall illustrate [Theorem B1](#) with a few examples.

Doubling case. Admissible coverings fulfilling the assumptions in [Theorem B1](#) can always be constructed on every doubling space, so that Calderón–Zygmund extrapolation for homogeneous spaces appears as a particular application of our approach. For clarity of the exposition, we shall just indicate how to construct such admissible coverings in \mathbb{R}^2 with the Lebesgue measure m and the Euclidean metric, although a similar construction works in the general case. Let us pick $Q_0 = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, the unit cube, and set $Q_s = 3^s Q_0$ for $s \geq 1$. Consider the σ -algebras

$$\Sigma_a = \sigma\langle A_s \mid s \geq 1 \rangle \quad \text{and} \quad \Sigma_b = \sigma\langle B_s \mid s \geq 1 \rangle,$$

where $(A_1, B_1) = (Q_0, Q_1)$ and $(A_s, B_s) = (Q_{2s-2} \setminus Q_{2s-4}, Q_{2s-1} \setminus Q_{2s-3})$ for $s \geq 2$. Then it follows from the proof of [Lemma 1.3](#) that (Σ_a, Σ_b) is an admissible covering of the Euclidean space (\mathbb{R}^2, m) .

Next, we define the filtrations $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$. Except for A_1 and B_1 — which are ordinary cubes — the atoms A_s and B_s ($s \geq 2$) are punctured cubes in which we remove a concentric cube with side-length $\frac{1}{9}$ times the side-length of the larger one. To define Σ_{j2} for $j = a, b$, we break each A_s, B_s into a disjoint union of 80 equal cubes of side-length $\frac{1}{9}$ times the side-length of the original punctured cube — i.e., all except for the one in the center — unless $s = 1$, in which case we also pick the center and get 81 subcubes (see [Figure 1](#)). The next generations are simpler. Indeed, since all our atoms in Σ_{j2} are already cubes, we perform dyadic partitions in each of them to provide the next generations of our filtration. This procedure completely defines two filtrations respectively based on Σ_a and Σ_b . It remains to check that these filtrations are regular and the atoms are doubling. The regularity constant is dominated uniformly by 81 when $(k - 1, k) = (1, 2)$ and by 4 otherwise. On the other hand, our atoms for $k = 1$ are punctured cubes which are comparable to the corresponding unpunctured ones, which in turn are doubling with constant 4. This proves that all conditions in [Theorem B1](#) are satisfied. In the general case, we just need to use Christ dyadic cubes [[1990](#)] and adapt our choice according to the finiteness or nonfiniteness of μ as we did in [Lemma 1.3](#).

Polynomial growth. Assume that we have (Ω, μ, d) of k -th degree polynomial growth with $\mu(\Omega) = \infty$. The associated RBMO norm can be defined as follows:

$$\|f\|_{\text{RBMO}} = \max \left\{ \|f\|_{\text{DBMO}}, \sup_{\substack{B \subset B' \\ B, B' \text{ } d\text{-balls} \\ \text{doubling}}} \left| \frac{1}{\mu(B)} \int_B f \, d\mu - \frac{1}{\mu(B')} \int_{B'} f \, d\mu \right| / K_{B, B'} \right\},$$

with $1 \leq K_{B, B'} = 1 + \sum_{2^j B \subset B'} \mu(2^j B) / r(2^j B)^k$. For such measures, we may easily construct an admissible covering of (Ω, μ) composed of doubling atoms. Indeed, the construction above can easily be modified

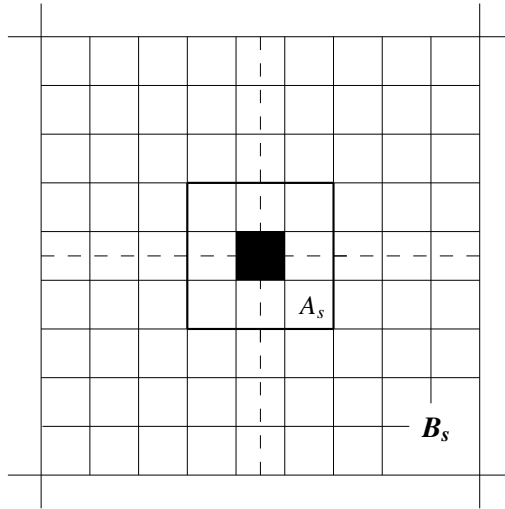


Figure 1. The admissible covering and the second generation of one of the filtrations.

using the existence of arbitrarily large doubling cubes centered at almost every point in the support of μ ; see [Tolsa 2001] for details. The main difficulty relies in the construction of filtrations $(\Sigma_{jk})_{k \geq 1}$ satisfying the assumptions in Theorem B1. Note that, whenever that holds, we find

$$\text{RBMO} \subset \text{DBMO} \subset \text{BMO}_{\Sigma_{\text{ab}}}(\Omega).$$

In particular, we deduce that $[\text{RBMO}, L_1(\Omega)]_{1/q} \simeq L_q(\Omega)$ when this happens. As far as we know, such interpolation identities are new since Tolsa [2001] studied interpolation of operators. Unfortunately, the construction of such filtrations seems to be a difficult task in the general case. For instance, the corona-type construction described above finds some obstructions when the measure μ is supported in Cantor-like sets. Nevertheless, we may construct these filtrations in some other cases. Let us consider the following family of measures on \mathbb{R}^n equipped with the Euclidean distance

$$d\mu_\beta(x) = \frac{dx}{1 + |x|^\beta}.$$

These measures are nondoubling only for $\beta > n$. We will construct an admissible covering for $\beta \gtrsim n^{3/2}$ satisfying the hypotheses of Theorem B1 when d is the Euclidean metric in \mathbb{R}^n . We will work with the equivalent measure

$$dv_\beta(x) = \min\{1, |x|^{-\beta}\} dx$$

for convenience. Note that this does not affect the conclusions in Theorem B1.

Pick $Q_0 = [-\lambda, \lambda]^n$ with $\lambda > 1$ to be fixed, and set $Q_s = 2^s Q_0$. Consider the σ -algebras $\Sigma_a = \sigma\langle A_s \mid s \geq 1 \rangle$ and $\Sigma_b = \sigma\langle B_s \mid s \geq 1 \rangle$, where $(A_0, B_0) = (Q_0, Q_1)$ and $(A_s, B_s) = (Q_{2s} \setminus Q_{2s-2}, Q_{2s+1} \setminus Q_{2s-1})$ for any $s \geq 1$. We clearly have $\Sigma_a \cap \Sigma_b = \{\mathbb{R}^n, \emptyset\}$ and $\max\{|R_A|, |R_B|\} \leq 2$ for $(A, B) \in \Pi_a \times \Pi_b$, by

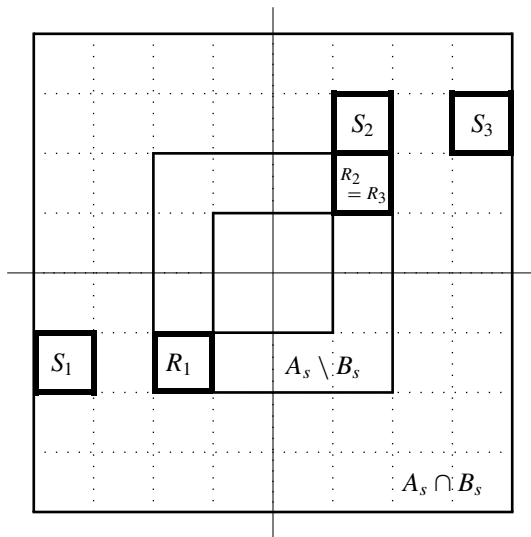


Figure 2. There is a cube R_j for each cube S_j .

construction of (Σ_a, Σ_b) . Thus, it suffices to show that

$$\sup_{\substack{(A, B) \in \Pi_a \times \Pi_b \\ A \neq A_0}} \frac{v_\beta(A \cap B)^2}{v_\beta(A)v_\beta(B)} < \frac{1}{2}.$$

We will prove, in fact, the apparently stronger inequalities

$$\sup_{s \geq 1} \frac{v_\beta(A_s \cap B_{s-1})}{v_\beta(B_{s-1})} < \frac{1}{2} \quad \text{and} \quad \sup_{s \geq 1} \frac{v_\beta(A_s \cap B_s)}{v_\beta(A_s)} < \frac{1}{2}.$$

By symmetry of the argument, we just prove the second inequality above. Denote by L the side length of the smallest cube Q_{2s} containing A_s . Then we have that $A \cap B$ can be decomposed into $C_n = 8^n - 4^n$ cubes S_j , each of which satisfies that $S_j = R_j + a_{S_j}$ for some cube $R_j = R_j(S_j) \subset A_s \setminus B_s$ of side length equal to $L/8$ and such that the angle between any point in R_j and a_{S_j} is smaller than $\pi/3$. We can also impose that $|a_{S_j}| \geq L/8$; see Figure 2. This implies that, for each x in R_j , we have

$$|x + a_{S_j}| \geq |x| + |a_{S_j}| \cos \angle(x, a_{S_j}) \geq |x| + \frac{1}{2}|a_{S_j}|.$$

Since $A_s \subset \mathbb{R}^n \setminus B_1(0)$ for $s \geq 1$, we have

$$\begin{aligned} v_\beta(A_s \cap B_s) &= \int_{A_s \cap B_s} |x|^{-\beta} dx = \sum_{j=1}^{C_n} \int_{S_j} |x|^{-\beta} dx \\ &= \sum_{j=1}^{C_n} \int_{R_j} |x + a_{S_j}|^{-\beta} dx \leq \sum_{j=1}^{C_n} \int_{R_j} (|x| + \frac{1}{2}|a_{S_j}|)^{-\beta} dx. \end{aligned}$$

Using that $|x| \leq \sqrt{n}L/2$ for $x \in A_s$ and $|a_{S_j}| \geq L/8$,

$$\frac{1}{v_\beta(R_j)} \int_{R_j} (|x| + \frac{1}{2}|a_{S_j}|)^{-\beta} dx \leq \sup_{x \in R_j} \frac{(|x| + \frac{1}{2}|a_{S_j}|)^{-\beta}}{|x|^{-\beta}} \leq \left(\frac{\sqrt{n}}{\sqrt{n} + \frac{1}{8}}\right)^\beta.$$

Therefore, we obtain

$$\frac{v_\beta(A_s \cap B_s)}{v_\beta(A_s)} \leq C_n \left(\frac{\sqrt{n}}{\sqrt{n} + \frac{1}{8}}\right)^\beta \leq 8^n \left(\frac{\sqrt{n}}{\sqrt{n} + \frac{1}{8}}\right)^\beta < \frac{1}{2} \quad \text{for } \beta \gtrsim n^{3/2}.$$

A similar argument shows that

$$\frac{v_\beta(A_s \cap B_{s-1})}{v_\beta(B_{s-1})} < \frac{1}{2}$$

for $\beta \gtrsim n^{3/2}$ and $s \geq 2$, whereas the same estimate holds for $s = 1$ as a consequence of the fact that B_0 contains $[-\lambda, \lambda]^n$ for $\lambda > 1$ large enough. This completes the construction of an admissible covering. It remains to construct filtrations $(\Sigma_{jk})_{k \geq 1}$ for $j = a, b$, that are regular and composed of doubling atoms. Recall that we set $\Sigma_{j1} = \Sigma_j$ and define Σ_{j2} by splitting each atom in Σ_j into a disjoint union of cubes. Namely, for $j = a$ we keep A_0 and divide A_s into the cubes R_j, S_j in [Figure 2](#). We proceed similarly for $j = b$. Once we have defined Σ_{j2} , we construct Σ_{jk} by dyadic splitting of the cubes in $\Sigma_{j(k-1)}$. Note that the atoms in $\Sigma_{j1} \setminus \{A_0, B_0\}$ split at most into 8^n cubes K centered at c_K which are away from the origin. Thus

$$v_\beta(2K) = \int_{2K} |x|^{-\beta} dx \lesssim |2K| |c_K|^{-\beta} \lesssim |K| |c_K|^{-\beta} \lesssim \int_K |x|^{-\beta} dx = v_\beta(K).$$

It easily follows from this that all the atoms in Σ_{jk} are doubling up to absolute constants independent of $k \geq 1$ and that both filtrations are regular. This shows that [Theorem B1](#) applies to $(\mathbb{R}^n, \mu_\beta)$ with the Euclidean metric.

Remark 3.2. A few comments are in order:

- (i) In the light of the example above, one could wonder what happens with the positive powers $d\mu_\gamma(x) = |x|^\gamma dx$ for $\gamma > 0$, but it is straightforward to show that these measures are doubling, so that we can handle them following the construction of the previous section (see p. [731](#)).
- (ii) Our proof of [Theorem B1](#) relies crucially on the embedding of the space DBMO in $BMO_{\Sigma_{ab}}(\Omega)$ under suitable conditions. When the metric measure space (Ω, μ, d) is of polynomial growth, we know from [\[Tolsa 2001\]](#) that CZOs are $L_\infty \rightarrow RBMO$ bounded. Since $RBMO \subset DBMO$, it is natural to wonder if we have

$$RBMO \subset BMO_{\Sigma_{ab}}(\Omega)$$

under weaker assumptions than in [Theorem B1](#). It turns out that this is the case when there exists filtrations composed of doubling atoms, no matter whether they are regular or not. Indeed, noticing that RBMO can be described as a subspace of DBMO with an additional condition, it is this crucial extra condition introduced by Tolsa that allows an embedding into $BMO_{\Sigma_{ab}}(\Omega)$ and not into $bmo_{\Sigma_{ab}}(\Omega)$ for nonregular filtrations.

Concentration at the boundary. Let

$$d\mu_{\pm\alpha}(x) = e^{\pm|x|^\alpha} dx$$

on \mathbb{R}^n equipped with the Euclidean metric. Carbonaro et al. [2009; 2010] proved that these measures satisfy their concentration condition when $\alpha > 1$. In this subsection we shall prove that our hypotheses in [Theorem B1](#) hold for any $\alpha > 0$, hence extending their results for measures with less concentration at the boundary. Let us start with the probability measure $\mu_{-\alpha}$. Pick $K = K(n, \alpha) > 0$, a large constant of the form 2^k for some $k \geq 1$ to be fixed below. Denote by $\mathbf{D}(\mathbb{R}^n)$ the standard filtration of dyadic cubes in \mathbb{R}^n . We consider the distinguished atom $A_0 = [-K, K]^n$. The other atoms $A_s \in \Pi_a$ for $s \geq 1$ are chosen to be the cubes in $\mathbf{D}(\mathbb{R}^n \setminus A_0)$ which are maximal under the following constraint on the side-length $\ell(A_s)$ in terms of the modulus of its center c_{A_s} :

$$\Pi_a = \{A_0\} \cup \{A_s \text{ maximal in } \mathbf{D}(\mathbb{R}^n \setminus A_0) \mid \ell(A_s) \leq K|c_{A_s}|^{1-\alpha}, s \geq 1\}.$$

Before defining Π_b , we also need another dyadic filtration $\mathbf{D}'(\mathbb{R}^n)$ satisfying some specific properties which we now detail. Given cubes $(A, B) \in \mathbf{D}(\mathbb{R}^n) \times \mathbf{D}'(\mathbb{R}^n)$ of comparable size — $2^{-k_0} \leq \ell(A)/\ell(B) \leq 2^{k_0}$ for some absolute constant k_0 — with nonempty intersection, there exists a parallelepiped $R \subset A \triangle B$ such that:

- (1) R is “substantially closer” than $A \cap B$ to the origin;
- (2) there exists $a = a(R) \in \mathbb{R}^n$ such that $A \cap B \subset \bigcup_{j=1}^N R + ja$;
- (3) $|a| \geq \frac{1}{N} \max\{\ell(A), \ell(B)\}$ and $|x + ja| \geq |x| + \frac{1}{2}|a|$ for every $x \in R$.

Let A_1 be the cube in $\Pi_a \setminus \{A_0\}$ whose center is the closest to the origin. Let $L = \ell(A_1)$ and pick $B_0 = A_0 + \frac{1}{3}Le_d$ with $e_d = (1, 1, \dots, 1)$. Then, the dyadic filtration $\mathbf{D}'(\mathbb{R}^n)$ is defined as one of the shifted dyadic filtrations in [\[Conde 2013\]](#) with the initial cube being B_0 . The fact that the properties above hold follows ultimately from the “good separation” between $\mathbf{D}(\mathbb{R}^n)$ and $\mathbf{D}'(\mathbb{R}^n)$. Here we pick K large enough so that the estimate $\mu_{-\alpha}(\frac{1}{2}A_0) > (1 - \varepsilon)\mu_{-\alpha}(\mathbb{R}^n)$ holds. In particular, we get

$$\frac{\mu_{-\alpha}(A_0 \cap B_0)}{\mu_{-\alpha}(\mathbb{R}^n)} > 1 - \varepsilon$$

for some small $\varepsilon > 0$ to be fixed. The family Π_b is defined similarly by

$$\Pi_b = \{B_0\} \cup \{B_s \text{ maximal in } \mathbf{D}'(\mathbb{R}^n \setminus B_0) \mid \ell(B_s) \leq K|c_{B_s}|^{1-\alpha}, s \geq 1\}.$$

Set $\Sigma_j = \sigma(\Pi_j)$ for $j = a, b$ and observe that $\Sigma_a \cap \Sigma_b = \{\mathbb{R}^n, \emptyset\}$ by construction. Therefore, to prove that (Σ_a, Σ_b) yields an admissible covering we only need to check that we have

$$\sup_{A \in \Pi_a \setminus \{A_0\}} \sum_{B \in R_A} |R_B| \frac{\mu_{-\alpha}(A \cap B)^2}{\mu_{-\alpha}(A)\mu_{-\alpha}(B)} < 1.$$

According to our definition of A_s , it is a simple exercise to check that we have $\ell(A_s) \geq \frac{1}{3}K|c_{A_s}|^{1-\alpha}$ for all $s \geq 1$ but for a finite number (independent of K) of cubes close to the origin. The same argument

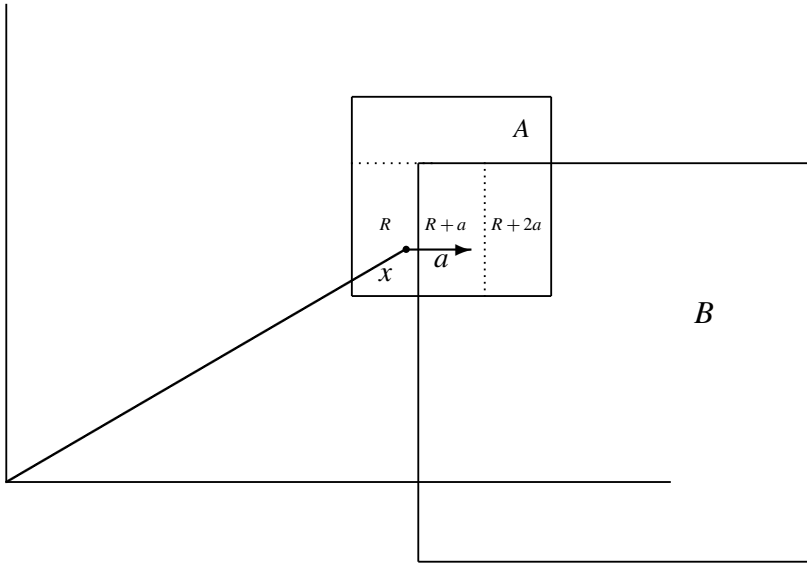


Figure 3. $A \cap B$ is covered by at most N a -translates of R ; R is “substantially closer” to the origin than $A \cap B$; a is parallel to a coordinate axis for cubes A, B in a sector around that axis. In particular, x and a are “close” to being parallel, so that $|x + ja| \geq |x| + \frac{1}{2}|a|$.

holds for atoms in Π_b . In particular, we have $|R_A|, |R_B| \leq C_n$ for all $(A, B) \in \Pi_a \times \Pi_b$. Therefore, when $B = B_0$ we obtain

$$|R_B| \frac{\mu_{-\alpha}(A \cap B)^2}{\mu_{-\alpha}(A)\mu_{-\alpha}(B)} \leq C_n \frac{\mu_{-\alpha}(A \cap B)}{\mu_{-\alpha}(B)} < C_n \frac{\varepsilon}{1 - \varepsilon} < \frac{1}{2}$$

for $\varepsilon < \frac{1}{3}C_n^{-1}$. Otherwise, when $B \neq B_0$, we obtain

$$\begin{aligned} \frac{\mu_{-\alpha}(A \cap B)}{\mu_{-\alpha}(R)} &= \frac{1}{\mu_{-\alpha}(R)} \int_{A \cap B} e^{-|x|^\alpha} dx \\ &\leq \sum_{j=1}^N \frac{1}{\mu_{-\alpha}(R)} \int_R e^{-|x+ja|^\alpha} dx \\ &\leq \frac{N}{\mu_{-\alpha}(R)} \int_R e^{-(|x|+|a|/2)^\alpha} dx \leq N \sup_{x \in R} e^{-(|x|+|a|/2)^\alpha + |x|^\alpha}. \end{aligned}$$

If $\alpha = 1$, we get an estimate $Ne^{-|a|/2} \geq Ne^{-C'_n K}$. For other values of $\alpha > 0$, a straightforward application of the mean value theorem gives

$$\left(|x| + \frac{1}{2}|a|\right)^\alpha - |x|^\alpha \geq \frac{\alpha}{18} K |c_A|^{1-\alpha} |x|^{\alpha-1} \geq C'_n K$$

since $|x| \sim |c_A|$. Hence we get, for $A \neq A_0$,

$$\sum_{B \in R_A} |R_B| \frac{\mu_{-\alpha}(A \cap B)^2}{\mu_{-\alpha}(A)\mu_{-\alpha}(B)} < \frac{1}{2} + C_n^2 \sup_{B \neq B_0} \frac{\mu_{-\alpha}(A \cap B)^2}{\mu_{-\alpha}(A)\mu_{-\alpha}(B)} \leq \frac{1}{2} + C_n^2 Ne^{-C'_n K} < 1,$$

picking $K = K(n, \alpha)$ large enough. This shows that we have an admissible covering. Note that our choice of cubes for $\alpha > 1$ is a family which becomes smaller and smaller when we get away from the origin. This is in the spirit of the Mauceri–Meda [2007] construction for the Gaussian measure. In contrast, when $\alpha < 1$ we pick larger and larger cubes as we get away from the origin. This construction seems not to be useful in [Carbonaro et al. 2009; 2010], since we may not use the locally doubling property for arbitrarily large cubes. Let us complete the proof by showing that the other hypotheses in Theorem B1 hold. Our choice of filtrations $(\Sigma_{jk})_{k \geq 1}$ for $j = a, b$ is by dyadic splitting of the cubes in Π_a and Π_b , respectively. The regularity of such filtrations will follow from the fact that every atom in $(\Sigma_{jk})_{k \geq 1}$ is $(3, \beta)$ -doubling for some absolute constant β , and this suffices to complete the proof. If Q is any subcube of $A_0 \cup B_0$, there are dimensional constants k_n and K_n such that

$$k_n |Q| \leq \int_Q e^{-|x|^\alpha} dx \leq K_n |Q|,$$

and hence Q is trivially $(3, \beta)$ -doubling. Otherwise, we compute

$$\begin{aligned} \frac{\mu_{-\alpha}(3Q)}{\mu_{-\alpha}(Q)} &\leq \frac{|3Q|}{|Q|} \sup_{x \in 3Q} e^{-|x|^\alpha} \sup_{x \in Q} e^{|x|^\alpha} \\ &\leq 3^n \exp\left(\left(|x_Q| + \frac{1}{2}\sqrt{n}\ell(Q)\right)^\alpha - \left(|x_Q| - \frac{3}{2}\sqrt{n}\ell(Q)\right)^\alpha\right) \leq \beta \end{aligned}$$

for some absolute constant $\beta > 0$, using the mean value theorem one more time.

Remark 3.3. A few comments are in order:

(i) Given $\alpha > 0$ and by minor modifications in the above arguments, we may also produce an admissible covering for $(\mathbb{R}^n, e^{|x|^\alpha} dx)$ which satisfies the hypotheses of Theorem B1 with respect to the Euclidean metric.

(ii) In this paper we \wedge -intersect two truncated martingale BMO spaces, but our results also hold for finite \wedge -intersections; details are simple and not very relevant. The Mauceri–Meda BMO space for the Gaussian measure [2007] can be described as such a finite intersection of BMO spaces using a construction similar to the one above for μ_{-2} but intersecting $n + 1$ BMO spaces instead of 2. Namely, one uses as many filtrations as needed to cover all cubes in \mathbb{R}^n with dyadic cubes of comparable size; see, for instance, [Conde 2013] for the optimal choice. This establishes an inclusion of their BMO space into our 2-intersection $\text{BMO}_{\Sigma_{ab}}$ associated to μ_{-2} , which still interpolates and is strictly larger. The latter assertion can be proved following the argument which shows that classical BMO is strictly contained in dyadic BMO.

(iii) A geometric interpretation of our definition of admissible covering could be that we still impose certain concentration at the boundary, but much less than [Carbonaro et al. 2009; 2010]. In support of this, let us consider an admissible covering (Σ_a, Σ_b) . Let \mathcal{A} be a finite family in $\Pi_a \setminus \{A_0\}$ and let $R_{\mathcal{A}}$ be the union $\bigcup_{A \in \mathcal{A}} R_A$. If we consider the set $R_{\mathcal{A}}$ as a measurable set and interpret $R_{\mathcal{A}} \setminus \mathcal{A}$ as the region “close to the boundary”, then we can prove that

$$\mu(R_{\mathcal{A}}) \leq \frac{1}{1 - c(\Sigma_{ab})} \mu(R_{\mathcal{A}} \setminus \mathcal{A})$$

or, equivalently, $\mu(\mathcal{A}) \leq c(\Sigma_{ab})\mu(R_{\mathcal{A}})$. Indeed, we have

$$\begin{aligned} \mu(\mathcal{A}) &= \sum_{A \in \mathcal{A}} \sum_{B \in R_{\mathcal{A}}} \mu(A \cap B) \\ &\leq c(\Sigma_{ab})^{1/2} \sum_{A \in \mathcal{A}} \mu(A)^{1/2} \left(\sum_{B \in R_{\mathcal{A}}} \frac{\mu(B)}{|R_B|} \right)^{\frac{1}{2}} \\ &\leq c(\Sigma_{ab})^{1/2} \mu(\mathcal{A})^{1/2} \left(\sum_{A \in \mathcal{A}} \sum_{B \in R_{\mathcal{A}}} \frac{\mu(B)}{|R_B|} \right)^{\frac{1}{2}} \leq (c(\Sigma_{ab})\mu(\mathcal{A})\mu(R_{\mathcal{A}}))^{1/2}. \end{aligned}$$

(iv) In the case of the Gaussian measure on \mathbb{R}^n , Mauceri, Meda and Sjögren [Mauceri et al. 2012] proved that R_i, S_i , the Riesz transforms associated with the Ornstein–Uhlenbeck semigroups, are bounded from L_∞ to Mauceri–Meda BMO spaces, but their adjoint operators R_i^*, S_i^* are not when $n \geq 2$. As explained in (ii), our BMO spaces are strictly larger than Mauceri and Meda’s if the σ -algebras Σ_a, Σ_b are picked as in the beginning of this subsection. Therefore, the Riesz transforms R_i, S_i studied in [Mauceri et al. 2012] are bounded from L^∞ to our BMO spaces as well. Pierre Portal [2014] introduced a different type of Hardy spaces using truncated maximal functions and square functions. He proved that the Riesz transforms R_i, S_i and their adjoint operators R_i^*, S_i^* are all bounded from his H_1 space to L_1 [Portal 2014, Theorem 6.1]. It is interesting to determine whether R_i^*, S_i^* are L^∞ -BMO bounded for our BMO spaces with carefully picked Σ_a, Σ_b .

4. Calderón–Zygmund operators, II

In this section we will study the class of atomic Calderón–Zygmund operators (ACZO) defined in the introduction over a given measure space (Ω, Σ, μ) . More precisely, we shall prove Theorem B2 and illustrate it with a few constructions of dyadic operators satisfying its hypotheses.

Proof of Theorem B2. Following the same argument as in the proof of Theorem B1, we can use duality and our interpolation result in Theorem A to reduce the L_p -boundedness in the assertion to the $L_\infty(\Omega) \rightarrow \text{BMO}_{\Sigma_{ab}}(\Omega)$ boundedness of our ACZO. This is however standard. Indeed, since the filtration is regular we know that $\text{BMO}_{\Sigma_{ab}}(\Omega) \simeq \text{bmo}_{\Sigma_{ab}}(\Omega)$. Up to absolute constants, the norm in the latter space is given by

$$\|Tf\|_{\text{bmo}_{\Sigma_{ab}}(\Omega)} = \sup_{Q \in \Pi} \inf_{k_Q \in \mathbb{C}} \left(\frac{1}{\mu(Q)} \int_Q |f(\omega) - k_Q|^2 d\mu(\omega) \right)^{\frac{1}{2}},$$

where $\Pi = \Pi_a \cup \Pi_b$ is the set of atoms in any of the two filtrations. Decompose

$$f = f \chi_{\hat{Q}} + f \chi_{\Omega \setminus \hat{Q}} = f_1 + f_2.$$

As usual, we pick $k_Q = (Tf_2)_Q$. Then, we control the term for Tf_1 using the L_2 -boundedness of T and the regularity of the filtrations. The term $Tf_2 - k_Q$ is dominated by means of the Hörmander kernel

condition given in the definition of ACZO. Namely,

$$\left(\frac{1}{\mu(Q)} \int_Q |Tf_1(\omega)|^2 d\mu(\omega) \right)^{\frac{1}{2}} \leq \|T\|_{2 \rightarrow 2} \sqrt{\frac{\mu(\hat{Q})}{\mu(Q)}} \|f\|_\infty \lesssim \|f\|_\infty$$

by regularity of the filtrations. On the other hand

$$\begin{aligned} & \left(\frac{1}{\mu(Q)} \int_Q |Tf_2(\omega) - (Tf_2)_Q|^2 d\mu(\omega) \right)^{\frac{1}{2}} \\ & \leq \left(\frac{1}{\mu(Q)^2} \int_Q \int_Q \left[\int_{\Omega \setminus \hat{Q}} |K(z_1, x) - K(z_2, x)| d\mu(x) \right]^2 d\mu(z_1) d\mu(z_2) \right)^{\frac{1}{2}} \|f\|_\infty, \end{aligned}$$

which is dominated by $\|f\|_\infty$ according to the Hörmander condition for ACZOs. □

Remark 4.1. As mentioned in the introduction, standard prototypes of atomic Calderón–Zygmund operators include martingale transforms, perfect dyadic CZOs and Haar shift operators. These are usually defined on the Euclidean space \mathbb{R}^n equipped with a dyadic filtration. Nevertheless, the exact same arguments apply on any dyadically doubling measure space or even for any measure space equipped with a two-sided regular filtration. López-Sánchez et al. [2014] have studied those nondoubling measure spaces for which Haar shift operators satisfy weak type-(1, 1) estimates. **Theorem B2** provides a tool to produce nondoubling measure spaces over which Haar shifts, or more general atomic CZOs, are $L_\infty \rightarrow$ BMO bounded. In the case of martingale transforms, Haar shift operators and perfect dyadic CZOs in (\mathbb{R}^n, μ) , all of them satisfy the Hörmander-like condition

$$\sup_{Q \in \mathcal{D}(\mathbb{R}^n)} \sup_{z_1, z_2 \in Q} \int_{\mathbb{R}^n \setminus \hat{Q}} |k(z_1, x) - k(z_2, x)| + |k(x, z_1) - k(x, z_2)| d\mu(x) < \infty.$$

This means that these operators are ACZOs satisfying **Theorem B2** as long as we can find an admissible covering (Σ_a, Σ_b) and regular filtrations over it such that all the atoms are cubes in $\mathcal{D}(\mathbb{R}^n)$ or suitable unions of those. If we review our examples in **Section 3**, this is not the case for our construction for $d\mu_{\pm\alpha}(x) = e^{\pm|x|^\alpha} dx$. It is however quite simple to adapt our construction for

$$d\mu_\beta(x) = \frac{dx}{1 + |x|^\beta}$$

so that it satisfies the hypotheses of **Theorem B2**. In particular, the Haar shift operators defined on $(\mathbb{R}^n, \mu_\beta)$ are $L_\infty(\mathbb{R}^n) \rightarrow$ $\text{BMO}_{\Sigma_{ab}}(\mathbb{R}^n)$ bounded. It remains open to decide whether an admissible covering exists on the exponential measure spaces $(\mathbb{R}^n, \mu_{\pm\alpha})$ using only atoms associated to one and not two dyadic systems.

5. Matrix-valued forms of our results

In this section, we extend our main results to the context of operator-valued functions. Noncommutative forms of Calderón–Zygmund theory have been recently studied in [Junge et al. \geq 2015; Mei and Parcet 2009; Parcet 2009; Mei 2007]. There are however no specific results in the context of nondoubling metric

measure spaces. Unfortunately, it seems difficult to extend the approach of [Tolsa 2001] or [Mauceri and Meda 2007] to the operator-valued or even the noncommutative setting, since their interpolation results rest on good- λ inequalities which do not have a noncommutative analogue so far. On the other hand, the semicommutative approach in [Parcet 2009] is valid for doubling spaces, but again presents serious obstacles to be extended to the nondoubling setting. The crucial aspect of our approach is that it ultimately rests on martingale inequalities that have been successfully transferred to the noncommutative setting. Namely, after Pixier and Xu’s [1997] seminal contribution on Burkholder–Gundy inequalities for noncommutative martingales, we find analogues of Doob’s maximal inequalities, Gundy, Davis and atomic decompositions, Burkholder conditional square functions, John–Nirenberg inequalities, L_p /BMO interpolation results; see [Hong and Mei 2012; Junge 2002; Junge and Mei 2010; Junge and Musat 2007; Junge and Perrin 2014; Junge and Xu 2003; Mei 2007; Musat 2003; Parcet and Randrianantoanina 2006; Perrin 2009] and the references therein.

Let us briefly introduce the framework for our results in this section; we refer to [Parcet 2009, Section 1] for a rather complete review of the necessary background adapted to our necessities. We also refer the reader to Pisier and Xu’s survey [2003] for more on noncommutative L_p theory. Let (Ω, Σ, μ) be a σ -finite measure space and consider any pair (\mathcal{M}, τ) given by a von Neumann algebra \mathcal{M} equipped with a normal, semifinite, faithful trace τ . This is sometimes called a noncommutative measure space. We will write (\mathcal{R}, φ) to denote the von Neumann algebra generated by essentially bounded functions $f : \Omega \rightarrow \mathcal{M}$ equipped with the trace

$$\varphi(f) = \int_{\Omega} \tau(f(\omega)) d\mu(\omega).$$

\mathcal{R} is the von Neumann algebra tensor product $\mathcal{R} = L_{\infty}(\Omega) \bar{\otimes} \mathcal{M}$ and we may consider the corresponding noncommutative spaces $L_p(\mathcal{R}, \varphi)$. This semicommutative model is the context where we intend to generalize our main results. Apart from its own interest as an operator-valued model, it constitutes a first step towards further results for more general von Neumann algebras. In particular, as [Junge et al. 2014] demonstrates, certain fully noncommutative questions can be reduced to the semicommutative setting. Readers not familiar with von Neumann algebra theory are encouraged to read this section restricting their attention to matrix-valued functions. In other words, replace \mathcal{M} by the algebra M_m of $m \times m$ matrices and τ by the standard trace tr . The difficulties are similar in this case to in the general setting, as long as we provide results with constants independent of m . We also refer to [Parcet 2009] for a comparison between this model and the vector-valued setting, which differs substantially in the endpoint estimates.

The BMO spaces. First we review the definitions and results in Section 1 for the semicommutative setting described above. Given a filtration $(\Sigma_k)_{k \geq 1}$ of (Ω, Σ, μ) , we consider the conditional expectations

$$f \mapsto E_{\Sigma_k} \otimes \text{id}_{\mathcal{M}}(f) \in \mathcal{R} \quad \text{for } f \in \mathcal{R},$$

still denoted by E_{Σ_k} . The martingale bmo and BMO norms are

$$\begin{aligned} \|f\|_{\text{bmo}} &= \max\{\|f\|_{\text{bmo}_c}, \|f^*\|_{\text{bmo}_c}\}, \\ \|f\|_{\text{BMO}} &= \max\{\|f\|_{\text{BMO}_c}, \|f^*\|_{\text{BMO}_c}\}, \end{aligned}$$

where the column norms are defined as in the commutative case, taking into account that we use $|x|^2 = x^*x$ for any operator x on a Hilbert space. The interpolation result $[\text{BMO}, L_1(\mathcal{R})]_{1/p} \simeq L_p(\mathcal{R})$ was proved by Musat [2003] for any semifinite von Neumann algebra \mathcal{R} . This is the noncommutative analogue of the Janson–Jones interpolation theorem. If we set $\|f\|_{h_p^r} = \|f^*\|_{h_p^c}$, where the norm in h_p^c is defined as in the commutative case, then the noncommutative Hardy spaces have the form

$$h_p = \begin{cases} h_p^r + h_p^c & \text{if } 1 \leq p \leq 2, \\ h_p^r \cap h_p^c & \text{if } 2 \leq p \leq \infty. \end{cases}$$

This combination of row and column square functions is known to be the right one for L_p inequalities, as was discovered for the first time with the noncommutative Khintchine inequalities [Lust-Piquard 1986; Lust-Piquard and Pisier 1991]. The interpolation result $[\text{bmo}, h_1]_{1/p} \simeq h_p$ was proved in [Bekjan et al. 2010] for noncommutative martingales. As in the commutative case, the projections $J_{\Sigma_1} = \text{id} - E_{\Sigma_1}$ are bounded on bmo , BMO , L_p and h_p , so that we will be working with these complemented subspaces which enjoy the same interpolation and duality properties as the original spaces. Note that the identity

$$\begin{aligned} \|f\|_{J_{\Sigma_1}(\text{bmo}_c)} &= \sup_{k \geq 1} \left\| (E_{\Sigma_k} |f - E_{\Sigma_k} f|^2)^{1/2} \right\|_{\mathcal{M}} \\ &= \sup_{A \in \bar{\Pi}} \left\| \left(\frac{1}{\mu(A)} \int_A |f(w) - \frac{1}{\mu(A)} \int_A f d\mu|^2 d\mu(w) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \end{aligned}$$

still holds and we have $J_{\Sigma_1}(\text{bmo}) \simeq J_{\Sigma_1}(\text{BMO})$ for regular filtrations. Consider an admissible covering (Σ_a, Σ_b) of (Ω, Σ, μ) and any pair of filtrations $(\Sigma_{jk})_{k \geq 1}$ with $\Sigma_{j1} = \Sigma_j$ for $j = a, b$. Denote by BMO_a and BMO_b the BMO spaces associated to these filtrations in the semicommutative algebra \mathcal{R} and set

$$\text{BMO}_{\Sigma_j}(\mathcal{R}) = J_{\Sigma_j}(\text{BMO}_j) \quad \text{and} \quad \text{BMO}_{\Sigma_{ab}}(\mathcal{R}) = \text{BMO}_{\Sigma_a}(\mathcal{R}) \wedge \text{BMO}_{\Sigma_b}(\mathcal{R}).$$

The John–Nirenberg inequalities, atomic descriptions of H_1 and duality results have also been transferred to the context of noncommutative martingales [Bekjan et al. 2010; Hong and Mei 2012; Junge and Musat 2007; Pisier and Xu 1997] and we will not review these results here, since they will not play a crucial role.

The interpolation theorem. Let us now state the analogue of Theorem A in the operator-valued setting. As usual, we will write $L_p^\circ(\mathcal{R})$ for the subspace of mean-zero elements with respect to μ . In the terminology we use for admissible coverings,

$$L_p^\circ(\mathcal{R}) \simeq J_{\Sigma_a \cap \Sigma_b}(L_p(\mathcal{R})).$$

Theorem 5.1. *Let (Σ_a, Σ_b) be an admissible covering in (Ω, Σ, μ) and consider the semicommutative space $\mathcal{R} = L_\infty(\Omega) \bar{\otimes} \mathcal{M}$. Then, for each $2 \leq p < \infty$, there exists a constant $c_p \geq 1$ such that*

$$L_p^\circ(\mathcal{R}) \simeq_{c_p} J_{\Sigma_a}(L_p(\mathcal{R})) \wedge J_{\Sigma_b}(L_p(\mathcal{R})).$$

In particular, we have by complex interpolation that

$$[\text{BMO}_{\Sigma_{ab}}(\mathcal{R}), L_1^\circ(\mathcal{R})]_{1/q} \simeq_{c_q} L_q^\circ(\mathcal{R}) \quad (1 < q < \infty),$$

with $\text{BMO}_{\Sigma_{ab}}(\mathcal{R})$ constructed with any two filtrations over (Σ_a, Σ_b) .

Sketch of the proof. Thanks to the close connection with martingales, the proof is entirely parallel to the one given in the classical case. Indeed, combining standard facts from noncommutative L_p theory with the martingale results reviewed in the previous subsection, it is a simple exercise to adapt our proof of [Theorem A](#) to the present case. The only subtle point is the inequality

$$\|E_{\Sigma_a} E_{\Sigma_b} f\|_p^p \leq \|E_{\Sigma_a} |E_{\Sigma_a} f|^{p/2}\|_2^2,$$

which is used in the last two steps of our argument. Namely, in the classical case this is due to the conditional Jensen’s inequality $\phi(E_{\Sigma_k} f) \leq E_{\Sigma_k} \phi(f)$ for convex functions ϕ . In contrast, its non-commutative form does not hold for all $p \geq 2$, since we need the operator-convexity of the function $\phi(x) = |x|^\beta$ for $\beta = p/2$ and x not necessarily positive. This is the case for $\beta \geq 2$, or equivalently $p \geq 4$, but it fails for $2 \leq p < 4$. Note however that the ultimate goal in Steps 5 and 6 is to show that $\|E_{\Sigma_a} E_{\Sigma_b} f\|_p \leq c_p(\Sigma_{ab}) \|f\|_p$ for some $0 < c_p(\Sigma_{ab}) < 1$. To prove it, we observe that

$$\begin{aligned} E_{\Sigma_a}(g_1^* g_2) &= \xi_k(g_1)^* \xi_k(g_2), \\ E_{\Sigma_a} E_{\Sigma_b}(f_1^* f_2) &= \omega_k(f_1)^* \omega_k(f_2) \end{aligned}$$

for certain right \mathcal{R}_k -module maps $\xi_k, \omega_k : L_q(\mathcal{R}) \rightarrow C_q(L_q(\mathcal{R}))$ with $\mathcal{R}_k = E_{\Sigma_k}(\mathcal{R})$. This follows from standard factorization properties of completely positive unital maps in terms of Hilbert modules; see, for instance, [\[Junge 2002\]](#). Let us consider the polar decompositions $f = u|f|$ and $g = v|g|$ of $g = E_{\Sigma_b} f$. Then we can factorize $E_{\Sigma_a} E_{\Sigma_b} f$ in two ways:

$$\begin{aligned} E_{\Sigma_a} g &= E_{\Sigma_a}(v|g|^{1/2}|g|^{1/2}) = \xi_k(|g|^{1/2}v^*)^* \xi_k(|g|^{1/2}), \\ E_{\Sigma_a} E_{\Sigma_b} f &= E_{\Sigma_a} E_{\Sigma_b}(u|f|^{1/2}|f|^{1/2}) = \omega_k(|f|^{1/2}u^*)^* \omega_k(|f|^{1/2}). \end{aligned}$$

This yields the estimates

$$\begin{aligned} \|E_{\Sigma_a} E_{\Sigma_b} f\|_p &\leq \|\xi_k(|g|^{1/2}v^*)\|_{2p} \|\xi_k(|g|^{1/2})\|_{2p} \\ &\leq \|\xi_k(|g|^{1/2}v^*)^* \xi_k(|g|^{1/2}v^*)\|_p^{1/2} \|\xi_k(|g|^{1/2})^* \xi_k(|g|^{1/2})\|_p^{1/2} \\ &= \|E_{\Sigma_a}(v|g|v^*)\|_p^{1/2} \|E_{\Sigma_a}(|g|)\|_p^{1/2} \leq \|f\|_p^{1/2} \|E_{\Sigma_a} |E_{\Sigma_b} f|^{p/2}\|_2^{1/p}, \end{aligned}$$

and

$$\begin{aligned} \|E_{\Sigma_a} E_{\Sigma_b} f\|_p &\leq \|\omega_k(|f|^{1/2}u^*)\|_{2p} \|\omega_k(|f|^{1/2})\|_{2p} \\ &\leq \|\omega_k(|f|^{1/2}u^*)^* \omega_k(|f|^{1/2}u^*)\|_p^{1/2} \|\omega_k(|f|^{1/2})^* \omega_k(|f|^{1/2})\|_p^{1/2} \\ &= \|E_{\Sigma_a} E_{\Sigma_b}(u|f|u^*)\|_p^{1/2} \|E_{\Sigma_a} E_{\Sigma_b}(|f|)\|_p^{1/2} \leq \|f\|_p^{1/2} \|E_{\Sigma_a} E_{\Sigma_b} |f|^{p/2}\|_2^{1/p}. \end{aligned}$$

The last inequality in both estimates follows from the Kadison–Schwarz inequality for operator-convex functions, since $\phi(x) = x^\beta$ is operator-convex on \mathbb{R}_+ for $\beta \geq 1$. The first estimate is the right one to use in Step 5 and the second one in Step 6. □

The Calderón–Zygmund operators. We now consider Calderón–Zygmund operators in semicommutative algebras associated to operator-valued kernels. Our construction is standard; we refer to [\[Duoandikoetxea 2001; Junge et al. 2014; Rubio de Francia et al. 1986\]](#) for further details. Let us write $L_0(\mathcal{M})$ for the $*$ -algebra of τ -measurable operators affiliated with \mathcal{M} and consider kernels $k : (\Omega \times \Omega) \setminus \Delta \rightarrow \mathcal{L}(L_0(\mathcal{M}))$

defined away from the diagonal Δ of $\Omega \times \Omega$ and which take values in linear maps on τ -measurable operators. If d is a metric in Ω , the standard Hörmander kernel condition takes the same form in this setting when we replace the absolute value by the norm in the algebra $\mathcal{B}(\mathcal{M})$ of bounded linear operators acting on \mathcal{M} :

$$\sup_{\substack{\text{Bd-ball} \\ z_1, z_2 \in \mathbb{B}}} \int_{\Omega \setminus \alpha \mathbb{B}} \|k(z_1, x) - k(z_2, x)\|_{\mathcal{B}(\mathcal{M})} + \|k(x, z_1) - k(x, z_2)\|_{\mathcal{B}(\mathcal{M})} d\mu(x) < \infty.$$

Define a CZO in $(\mathcal{R}, \varphi, d)$ as any linear map T satisfying the following properties:

- T is bounded on $L_\infty(\mathcal{M}; L_2^r(\Omega))$,

$$\left\| \left(\int_{\Omega} Tf(x)Tf(x)^* d\mu(x) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \lesssim \left\| \left(\int_{\Omega} f(x)f(x)^* d\mu(x) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

- T is bounded on $L_\infty(\mathcal{M}; L_2^c(\Omega))$,

$$\left\| \left(\int_{\Omega} Tf(x)^*Tf(x) d\mu(x) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \lesssim \left\| \left(\int_{\Omega} f(x)^*f(x) d\mu(x) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

- The kernel representation

$$Tf(x) = \int_{\Omega} k(x, y)(f(y)) d\mu(y) \quad \text{holds for } x \notin \text{supp } f$$

and some kernel $k : (\Omega \times \Omega) \setminus \Delta \rightarrow \mathbb{C}$ satisfying the Hörmander condition.

The first two conditions replace the usual L_2 -boundedness; see [Junge et al. 2014] for explanations.

Theorem 5.2. *Let (Σ_a, Σ_b) be an admissible covering of (Ω, Σ, μ) . Assume that Σ admits regular filtrations $(\Sigma_{jk})_{k \geq 1}$ by successive refinement of $\Sigma_{j1} = \Sigma_j$ for $j = a, b$ and that each atom in Σ_{jk} is a (C_0, α, β) -doubling set for certain absolute constants $C_0, \alpha, \beta > 0$. Let $\text{BMO}_{\Sigma_{ab}}(\mathcal{R})$ denote the \wedge -intersection of the BMO spaces defined over these filtrations. Then, every CZO extends to a bounded map:*

- (i) $H_{\Sigma_{ab}}^1(\mathcal{R}) \rightarrow L_1(\mathcal{R})$;
- (ii) $L_\infty(\mathcal{R}) \rightarrow \text{BMO}_{\Sigma_{ab}}(\mathcal{R})$;
- (iii) $L_p^\circ(\mathcal{R}) \rightarrow L_p^\circ(\mathcal{R})$ for $1 < p < \infty$.

Moreover, if T is $L_2(\mathcal{R})$ -bounded then $T : L_p(\mathcal{R}) \rightarrow L_p(\mathcal{R})$ for all $1 < p < \infty$.

Proof. According to Theorem 5.1 (interpolation) and the semicommutative form of Remark 3.1 (duality), it turns out that $L_\infty(\mathcal{R}) \rightarrow \text{BMO}_{\Sigma_{ab}}(\mathcal{R})$ boundedness automatically implies $H_{\Sigma_{ab}}^1(\mathcal{R}) \rightarrow L_1(\mathcal{R})$ boundedness, as well as $L_p^\circ(\mathcal{R}) \rightarrow L_p^\circ(\mathcal{R})$ boundedness. Moreover, if T is also L_2 -bounded we may reproduce the argument given in the proof of Theorem B1 to obtain L_p -boundedness for all $1 < p < \infty$. Let us then focus on the $L_\infty \rightarrow \text{BMO}$ boundedness. Define

$$\text{DBMO} = \text{DBMO}_r \cap \text{DBMO}_c$$

with $\|f\|_{\text{DBMO}_r} = \|f^*\|_{\text{DBMO}_c}$ and

$$\|f\|_{\text{DBMO}_c} = \sup_{\substack{\mathbf{B} \text{ ball} \\ d\text{-doubling}}} \left\| \left(\frac{1}{\mu(\mathbf{B})} \int_{\mathbf{B}} |f(w) - \frac{1}{\mu(\mathbf{B})} \int_{\mathbf{B}} f d\mu|^2 d\mu(w) \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

As usual, we write $|x|^2$ for x^*x . The assertion follows from

$$L_\infty(\mathcal{R}) \xrightarrow{T} \text{DBMO} \xrightarrow{\text{id}} \text{bmo}_{\Sigma_{\text{ab}}}(\mathcal{R}) \simeq \text{BMO}_{\Sigma_{\text{ab}}}(\mathcal{R}).$$

The boundedness of the chain above can be justified as in the proof of [Theorem B1](#). Indeed, the analogies in the argument lead us to apply the new conditions which appear in our definition of semicommutative CZO; see [\[Junge et al. 2014\]](#). \square

Remark 5.3. [Theorem B2](#) also admits a straightforward generalization to the semicommutative setting. Again, our use of martingale techniques makes the proof entirely analogous, so that we think it would be too repetitive to include it here.

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