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CLASSIFICATION OF BLOWUP LIMITS FOR SU(3) SINGULAR TODA SYSTEMS

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For singular SU(3) Toda systems, we prove that the limit of energy concentration is a finite set. In addition, for fully bubbling solutions we use a Pohozaev identity to prove a uniform estimate. Our results extend previous results of Jost, Lin and Wang on regular SU(3) Toda systems.

1. Introduction

Systems of elliptic equations in two-dimensional space with exponential nonlinearity are very commonly observed in physics, geometry, chemistry and biology. In this article we consider the following general system of equations defined in $\mathbb{R}^2$:

$$\Delta u_i + \sum_{j\in I} a_{ij} h_j e^{u_j} = 4\pi\gamma_i\delta_0 \quad \text{in } B_1\subset\mathbb{R}^2 \quad \text{for } i\in I, \tag{1-1}$$

where $I=\{1,\dots,n\}$, B_1 is the unit ball in $\mathbb{R}^2$, $h_1,\dots,h_n$ are smooth functions, $A=(a_{ij})_{n\times n}$ is a constant matrix, $\gamma_i > -1$ and δ_0 is the Dirac mass at 0. If $n=1$ and $a_{11}=1$, the system (1-1) is reduced to a single Liouville equation, which has vast background in conformal geometry and physics. The general system (1-1) is used for many models in different disciplines of science. If the coefficient matrix A is nonnegative, symmetric and irreducible, (1-1) is called a Liouville system and is related to models in the theory of chemotaxis [Childress and Percus 1981; Keller and Segel 1971], in the physics of charged particle beams [Bennet 1934; Debye and Huckel 1923; Kiessling and Lebowitz 1994] and in the theory of semiconductors [Mock 1975]; see [Chanillo and Kiessling 1995; Chipot et al. 1997; Lin and Zhang 2010] and the references therein for more applications of Liouville systems. If A is the Cartan matrix

$$A_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0\\ -1 & 2 & -1 & 0 & \cdots & 0\\ 0 & -1 & 2 & -1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & -1 & 2 & -1\\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix},$$

the system (1-1) is called an SU($n+1$) Toda system (which has n equations) and is related to the nonabelian gauge in Chern–Simons theory; see [Dunne et al. 1991; Dunne 1995; Ganoulis et al. 1982; Leznov 1980; Leznov and Saveliev 1992; Malchiodi and Ndiaye 2007; Malchiodi and Ruiz 2013; Mansfield

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1982; Nolasco and Tarantello 1999; 2000; Yang 1997; 2001] and the references therein. There are also many works on the relationship between SU$(n+1)$ Toda systems and holomorphic curves in $\mathbb{CP}^n$, the flat SU$(n+1)$ connection, complete integrability and harmonic sequences; see [Bolton and Woodward 1997; Bolton et al. 1988; Calabi 1953; Chern and Wolfson 1987; Doliwa 1997; Guest 1997; Leznov and Saveliev 1992; Lin et al. 2012a] for references.

After decades of extensive study, many important questions related to the scalar Liouville equation are answered and the behavior of blowup solutions is well understood (see [Bartolucci and Tarantello 2002a; 2002b; Bartolucci and Malchiodi 2013; Chen and Lin 2002; 2003] for related discussions). However, the understanding of blowup solutions to the more general systems (1-1) is far from complete. In recent years, much progress has been made on more general systems and we only mention a few works related to the topic of the current article. First, Lin and Zhang [2010; 2011] completed a degree-counting project for Liouville systems defined on Riemann surfaces. Second, for regular SU(3) Toda systems (which have two equations), Jost, Lin and Wang [Jost et al. 2006] proved some uniform estimates for fully bubbling solutions (see Section 4 for the definition) using holonomy theory. Later, Lin, Wei and Zhao [Lin et al. 2012b] improved the estimate of [Jost et al. 2006] to the sharp form using the nondegeneracy of the global SU(3) solutions, which was established by Lin, Wei and Ye [Lin et al. 2012a] among other things.

In this article we mainly focus on the asymptotic behavior of blowup solutions of (1-1) and the weak limit of energy concentration for SU$(n+1)$ Toda systems. More specifically, let $u^k=(u_1^k,\dots,u_n^k)$ be a sequence of solutions

$$\Delta u_i^k+\sum_{j=1}^n a_{ij}h_j^k e^{u_j^k}=4\pi\gamma_i^k\delta_0 \quad \text{in } B_1, \quad i=1,\dots,n, \tag{1-2}$$

with 0 being its only possible blowup point in B_1:

$$\max_{K\Subset B_1\setminus\{0\}} u_i^k\le C(K). \tag{1-3}$$

Since the right-hand side of (1-2) is a Dirac mass, we define the regular part of u_i^k to be

$$\tilde u_i^k(x)=u_i^k(x)-2\gamma_i^k\log|x|, \quad x\in B_1, \quad i=1,\dots,n. \tag{1-4}$$

Then $u^k=(u_1^k,\dots,u_n^k)$ is called *a sequence of blowup solutions* if $\max_i\max_{x\in B_1}\tilde u_i^k\to\infty$.

We assume that $\gamma_i^k\to\gamma_i>-1$, that $h_1^k,\dots,h_n^k$ are positive smooth functions with a uniform bound on their C^3 norm:

$$\frac{1}{C}\le h_i^k\le C, \qquad \|h_i^k\|_{C^3(B_1)}\le C \quad \text{in } B_1, \qquad \gamma_i^k\to\gamma_i>-1 \quad \text{for all } i\in I; \tag{1-5}$$

and we suppose that there is a uniform bound on the oscillation of u_i^k on ∂B_1 and its energy, $\int_{B_1}h_i^k e^{u_i^k}$:

$$|u_i^k(x)-u_i^k(y)|\le C \quad \text{for all } x,y\in\partial B_1, \qquad \int_{B_1}h_i^k e^{u_i^k}\le C, \quad i\in I, \tag{1-6}$$

where C is independent of k.

Note that the oscillation finiteness assumption in (1-6) is natural and generally satisfied in most applications. The energy bound in (1-6) is also natural for a system or equation defined in two-dimensional space.

If $A = A_2$, (1-2) describes SU(3) with sources. Our first main theorem is concerned with the energy limits of solutions to singular SU(3) Toda systems.

Given any $\delta > 0$, u^k has no blowup point in $B_1 \setminus B_\delta$ (in this article we use $B(x, r)$ to denote a ball centered at x with radius r and use B_r to denote $B(0, r)$). Thus we are interested in the following limit:

$$\sigma_i = \lim_{\delta \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B_\delta} h_i^k e^{u_i^k}, \quad i = 1, 2. \tag{1-7}$$

Since, for each $\delta > 0$, $\int_{B_\delta} h_i^k e^{u_i^k}$ is uniformly bounded, the $\lim_{k\to\infty}$ in (1-7) is understood as the limit of a subsequence of u^k. For convenience we don't distinguish u^k and its subsequences in this article.

Let

$$\mu_i = 1 + \gamma_i, \quad i = 1, 2,$$

and let

$$\Gamma = \{(\sigma_1, \sigma_2) : \sigma_1, \sigma_2 \geq 0,\ \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2\}$$

be a quadratic curve in the first quadrant. It is easy to see that Γ is contained in the box

$$\left[0, \tfrac{4}{3}\mu_1 + \tfrac{2}{3}\mu_2 + \tfrac{4}{3}\sqrt{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}\right] \times \left[0, \tfrac{2}{3}\mu_1 + \tfrac{4}{3}\mu_2 + \tfrac{4}{3}\sqrt{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}\right].$$

In Definition 1.1 below we shall define a finite set on Γ. In order to describe the relative positions of points, we say (c, d) is *in the upper right part of* (a, b) if $c \geq a$ and $d \geq b$.

Definition 1.1. It is easy to verify that the following six points are on Γ:

$$(0, 0) \quad (2\mu_1, 0), \quad (0, 2\mu_2), \quad (2\mu_1, 2(\mu_1 + \mu_2)), \quad (2(\mu_1 + \mu_2), 2\mu_2), \quad (2(\mu_1 + \mu_2), 2(\mu_1 + \mu_2)).$$

First we let the six points above belong to Σ, then we determine other points in Σ as follows: For $(a, b) \in \Sigma$, intersect Γ with $\sigma_1 = a + 2N$ and $\sigma_2 = b + 2N$ ($N = 0, 1, 2, \ldots$) and add the point(s) of intersection to Σ that belong to the upper right part of (a, b). For each new member $(c, d) \in \Sigma$ added by this process, we apply the same procedure based on (c, d) to obtain possible new members.

Theorem 1.2. *Let $A = A_2$, h_i^k and γ_i^k satisfy* (1-5). *Then, for u^k satisfying* (1-2), (1-3) *and* (1-6), *we have $(\sigma_1, \sigma_2) \in \Sigma$, where σ_i is defined by* (1-7) *and Σ is defined as in Definition 1.1.*

Remark 1.3. If $\gamma_1 = \gamma_2 = 0$, the system is a nonsingular SU(3) Toda system. One sees easily that

$$\Sigma = \{(0, 0),\ (2, 0),\ (0, 2),\ (2, 4),\ (4, 2),\ (4, 4)\}.$$

Indeed, when the procedure described in Definition 1.1 is applied to any of the six points in Σ, no extra point of intersection can be found. For example if we start from $(0, 0)$ and intersect Γ by lines $\sigma_1 = 2N$ (N being a nonnegative integer), then we see immediately that the intersection of Γ with $\sigma_1 = 2$ gives $(2, 0)$ and $(2, 4)$, which are already in Σ. The intersection with $\sigma_1 = 4$ gives $(4, 2)$ and $(4, 4)$, which also belong to the six types in Σ. There is no intersection between Γ and $\sigma_1 = 6$. Theorem 1.2 in this special case was proved in [Jost et al. 2006]. Recent work of Pistoia, Musso and Wei [Musso et al. 2015] proved that all six cases for nonsingular SU(3) Toda systems can occur.

Remark 1.4. It is easy to observe that the maximum value of σ_1 on Γ is

$$\tfrac{4}{3}\mu_1+\tfrac{2}{3}\mu_2+\tfrac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

The maximum value of σ_2 is

$$\tfrac{2}{3}\mu_1+\tfrac{4}{3}\mu_2+\tfrac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

Thus Σ is a finite set. As two special cases, we see that:

(1) If

$$\tfrac{4}{3}\mu_1+\tfrac{2}{3}\mu_2+\tfrac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}<2 \quad\text{and}\quad \tfrac{2}{3}\mu_1+\tfrac{4}{3}\mu_2+\tfrac{4}{3}\sqrt{\mu_1^2+\mu_1\mu_2+\mu_2^2}<2$$

then there are only six points in Σ:

$$\Sigma=\big\{(0,0),\ (2\mu_1,0),\ (0,2\mu_2),\ (2(\mu_1+\mu_2),2\mu_2),\ (2\mu_1,2(\mu_1+\mu_2)),\ (2(\mu_1+\mu_2),2(\mu_1+\mu_2))\big\}.$$

(2) For $\gamma_1=\gamma_2=1$, in addition to $(0,0)$, $(4,0)$, $(0,4)$, $(4,8)$, $(8,4)$ and $(8,8)$, Σ has other 14 points.

An earlier version of the current article was posted on the arXiv in March 2013. After that, some work has been done based on Theorem 1.2 (see [Battaglia and Malchiodi 2014] for example). Theorem 1.2 reflects some essential differences between Toda systems and Liouville systems. Lin et al. [2012a] proved that all the global solutions of SU$(n+1)$ Toda systems can be described by n^2+2n parameters and the energy of global solutions is a discrete set. On the other hand, the global solutions of Liouville systems all belong to a family of three parameters but their energy forms an $(n-1)$-dimensional hypersurface (see [Chipot et al. 1997; Lin and Zhang 2010]). These differences lead to very different approaches in their respective research. For example, [Lin et al. 2012b] obtained sharp estimates for fully bubbling solutions (see Section 4 for the definition) of SU(3) Toda systems using the discreteness of energy as a key ingredient in their proof.

Here we briefly describe the strategy used to prove Theorem 1.2. First we introduce a selection process suitable for SU$(n+1)$ Toda systems. The selection process has been widely used for prescribing curvature-type equations (see [Li 1995; Chen and Lin 1998], etc) and we modify it to locate the bubbling area, which is a union of finite disks. In each of the disks, the blowup solutions have roughly the energy of a global SU$(m+1)$ Toda system on $\mathbb{R}^2$ (with $m\le n$), which is the limit of the blowup solutions after scaling. If $m=n$, which means no component is lost after scaling and taking the limit, we say the sequence of solutions in the disk is fully bubbling, otherwise we call it partially bubbling. Next we introduce the "group" concept to place bubbling disks according to their relative locations. There are only finitely many bubbling disks and their relative distances may tend to 0 with very different speed. The name "group" is used to describe a few disks that are roughly closest to one another and much further from other disks. Lemma 2.4 is a Harnack-type result that plays an important role in determining the energy concentration around a group. Suppose there is a circle that surrounds a group and both components of the blowup solutions have fast decay (see Section 3 for the definition) on the circle. Then a Pohozaev identity can be computed on this circle to determine how much energy this group carries. Because of Lemma 2.4, such a circle can always be found, so the energy within the circle can be determined. Then we consider the combination of groups by scaling. The relationship among groups is similar to that of members in a same group. For example, if the distance between two groups is scaled to be 1, the bubbling disks of one group look like a Dirac mass from afar. We can similarly find circles surrounding groups that are also suitable for computing Pohozaev identities (i.e., both components of the blowup solutions have fast decay on these circles). From these Pohozaev identities, we determine how much energy is contained in each group and all the combinations of groups. One important fact is that one component of the blowup solutions always has fast decay, even though the other component may not. It is possible for the first (fast decay) component to turn to a slow decay component as the distance to a group becomes bigger, but before that happens the second component, which used to be a slow decay component, will turn to a fast decay component first.

As another application of the Pohozaev identity we establish some uniform estimates for fully bubbling solutions. These estimates were first obtained by Li [1999] for the scalar Liouville equation without singularity (using the method of moving planes) and [Bartolucci et al. 2004] for the scalar Liouville equation with singularity (using the Pohozaev identity and potential analysis). For regular SU(3) Toda systems, [Jost et al. 2006] established similar estimates using holonomy theory. Our results (Theorem 4.1 and Theorem 4.3) apply to general SU$(n+1)$ Toda systems with singularity.

This article is set out as follows. In Section 2 we introduce the selection process mentioned before and in Section 3 we prove the Pohozaev identity, which is crucial for the proof of Theorem 1.2. In Section 4 we prove a uniform estimate for fully bubbling solutions (Theorem 4.3 and Theorem 4.1). Then in Section 5 and Section 6 we finish the proof of Theorem 1.2 according to the strategy mentioned before.

2. A selection process for SU$(n+1)$ Toda systems

Clearly in the proof of Theorem 1.2 we can assume 0 to be a blowup point:

$$\max_{x\in B_1,\, i\in I}\{u_i^k - 2\gamma_i^k \log|x|\} \to \infty, \tag{2-1}$$

because otherwise the blowup type is (0, 0). So, from now on throughout the paper, (2-1) is assumed.

Case one: $\boldsymbol{\gamma_1^k = \cdots = \gamma_n^k = 0}$.

Proposition 2.1. *Let $A = (a_{ij})_{n\times n}$ be the Cartan matrix A_n, h_i^k satisfy (1-5) and $u^k = (u_1^k, \ldots, u_n^k)$ be a sequence of solutions to (1-2) with $\gamma_1^k = \cdots = \gamma_n^k = 0$ such that (1-6) and (1-3) hold. Then there exist finite sequences of points $\Sigma_k := \{x_1^k, \ldots, x_m^k\}$ (all $x_j^k \to 0$, $j = 1, \ldots, m$) and positive numbers $l_1^k, \ldots, l_m^k \to 0$ such that the following four properties hold*:

(1) $\max_{i\in I}\{u_i^k(x_j^k)\} = \max_{B(x_j^k, l_j^k), i\in I}\{u_i^k\}$ *for all* $j = 1, \ldots, m$.

(2) $\exp\big(\frac{1}{2}\max_{i\in I}\{u_i^k(x_j^k)\}\big) l_j^k \to \infty$, $j = 1, \ldots, m$.

(3) *There exists $C_1 > 0$ independent of k such that*

$$u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k) \le C_1 \quad \textit{for all } x \in B_1, \quad i \in I,$$

where dist *stands for distance.*

(4) *In each* $B(x_j^k, l_j^k)$ *let*

$$v_i^k(y) = u_i^k(\epsilon_k y + x_j^k) + 2\log\epsilon_k, \quad \epsilon_k = e^{-M_k/2}, \quad M_k = \max_i \max_{B(x_j^k, l_j^k)} u_i^k. \tag{2-2}$$

Then one of the following two alternatives holds:

(*a*) *The sequence is fully bubbling*: *along a subsequence*, $(v_1^k, \dots, v_n^k)$ *converges in* $C^2_{\text{loc}}(\mathbb{R}^2)$ *to* $(v_1, \dots, v_n)$ *which satisfies*

$$\Delta v_i + \sum_{j\in I} a_{ij} h_j e^{v_j} = 0 \quad \text{in } \mathbb{R}^2, \quad i \in I,$$

$$\lim_{k\to\infty} \int_{B(x_j^k, l_j^k)} \sum_{t\in I} a_{it} h_t^k e^{u_t^k} > 4\pi, \qquad i \in I.$$

(*b*) $I = J_1 \cup J_2 \cup \cdots \cup J_m \cup N$, *where* $J_1, J_2, \dots, J_m$ *and* N *are disjoint sets*, $N \neq \varnothing$ *and each* J_t $(t = 1, \dots, m)$ *consists of consecutive indices. For each* $i \in N$, v_j^k *tends to* $-\infty$ *over any fixed compact subset of* $\mathbb{R}^2$. *The components of* $v^k = (v_1^k, \dots, v_n^k)$ *corresponding to each* J_l $(l = 1, \dots, m)$ *converge in* $C^2_{\text{loc}}(\mathbb{R}^2)$ *to an* $\mathrm{SU}(|J_l|+1)$ *Toda system, where* $|J_l|$ *is the number of indices in* J_l. *For each* $i \in J_l$, *we have*

$$\lim_{k\to\infty} \int_{B(x_j^k, l_j^k)} \sum_{t\in J_l} a_{it} h_t^k e^{v_t^k} > 4\pi.$$

Remark 2.2. In this article we don't use different notations for sequences and subsequences.

Remark 2.3. For each $x_j^k \in \Sigma_k$, suppose $2t_j^k$ is the distance from x_j^k to $\Sigma_k \setminus \{x_j^k\}$. Then $t_j^k / l_j^k \to \infty$ as $k \to \infty$ if l_j^k is suitably chosen.

Proof of Proposition 2.1. Without loss of generality we assume

$$u_1^k(x_1^k) = \max_{i\in I,\, x\in B_1} u_i^k(x).$$

Clearly $x_1^k \to 0$, because $\max_i \max_{x\in B_1} u_i^k \to \infty$ and u^k is uniformly bounded from above away from the origin. Let $(v_1^k, \dots, v_n^k)$ be defined by (2-2) with x_j^k replaced by x_1^k. Immediately we observe that $|\Delta v_i^k|$ is bounded because each $v_i^k \le 0$. Consequently, $|v_i^k(z) - v_i^k(0)|$ is uniformly bounded in any compact subset of $\mathbb{R}^2$. Thus, since $v_1^k(0) = 0$, (along a subsequence) v_1^k converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to a function v_1. For the other components of $v^k = (v_1^k, \dots, v_n^k)$, either some of them tend to $-\infty$ over any compact subset of $\mathbb{R}^2$, or all of them converge to a system of n equations. Let $J \subset I$ be the set of indices corresponding to those convergent components. That is, for all $i \in J$, v_i^k converges to v_i in $C^2_{\text{loc}}(\mathbb{R}^2)$ and, for all $j \in I \setminus J$, v_i^k tends to $-\infty$ over any fixed compact subset of $\mathbb{R}^2$. For each $i \in I \setminus J$, there is $J_1 \subset J$ such that $i \in J_1$, the indices in J_1 are consecutive and the limit of the v_i^k is one component of an $\mathrm{SU}(|J_1|+1)$ Toda system:

$$\begin{cases} \Delta v_m + \sum_{j\in J} a_{ml} h_l e^{v_l} = 0 & \text{in } \mathbb{R}^2 \text{ for all } m \in J_1 \\ \int_{\mathbb{R}^2} h_m e^{v_m} \le C, & m \in J_1, \end{cases} \tag{2-3}$$

where $h_m = \lim_{k\to\infty} h_m^k(x_1^k)$, $(a_{ij}) = A_{|J_1|}$, and C is the same constant as in (1-6). By the classification theorem of [Lin et al. 2012a] (if the limit is a system) or [Chen and Li 1991] (if the limit is one equation) we have

$$\sum_{j\in J_1}\int_{\mathbb{R}^2} a_{ij}h_j e^{v_j} = 8\pi \quad \text{for all } i\in J_1 \tag{2-4}$$

and

$$v_i(x) = -4\log|x| + O(1), \quad |x|>2, \quad \text{for all } i\in J_1. \tag{2-5}$$

Thus, for any index $i\in I$, we can find $R_k\to\infty$ such that

$$v_i^k(y) + 2\log|y| \le C, \quad |y|\le R_k, \quad \text{for } i\in I. \tag{2-6}$$

Equivalently, for u^k there exist $l_1^k\to 0$ such that

$$u_i^k(x) + 2\log|x-x_1^k| \le C, \quad |x-x_1^k|\le l_1^k, \quad \text{for } i\in I$$

and

$$e^{u_1^k(x_1^k)/2} l_1^k \to \infty \quad \text{as } k\to\infty, \quad i\in J.$$

Next, we let q_k be the maximum point of $\max_{|x|<1,i\in I} u_i^k(x) + 2\log|x-x_1^k|$. If

$$\max_{|x|\le 1,i\in I} u_i^k(x) + 2\log|x-x_1^k| \to \infty,$$

we let j be the index such that

$$u_j^k(q_k) + 2\log|q_k-x_1^k| = \max_{i\in I} u_i^k(x) + 2\log|x-x_1^k| \to \infty.$$

The following localization is to adapt the original argument of R. Schoen [1988] for the scalar curvature equation (also see [Chen and Lin 1998]). Set

$$d_k = \tfrac{1}{2}|q_k - x_1^k|$$

and

$$S_i^k(x) = u_i^k(x) + 2\log(d_k - |x-q_k|) \quad \text{in } B(q_k,d_k).$$

Then clearly, for fixed k, $S_i^k\to -\infty$ as x tends to $\partial B(q_k,d_k)$. On the other hand, at least for j, we have

$$S_j^k(q_k) = u_j^k(q_k) + 2\log d_k \to \infty.$$

Let p_k be where

$$\max_i \max_{x\in\bar{B}(q_k,d_k)} S_i^k$$

is attained and i_0 be the index corresponding to where the maximum is taken:

$$u_{i_0}^k(p_k) + 2\log(d_k - |p_k-q_k|) \ge S_j^k(q_k) \to \infty. \tag{2-7}$$

Let

$$l_k = \tfrac{1}{2}(d_k - |p_k-q_k|).$$

Then for $y \in B(p_k, l_k)$, by the choice of p_k and l_k, we have

$$u_i^k(y) + 2\log(d_k - |y - q_k|) \le u_{i_0}^k(p_k) + 2\log(2l_k) \quad \text{for all } i \in I.$$

On the other hand, by the definition of l_k, we have

$$d_k - |y - q_k| \ge d_k - |p_k - q_k| - |y - p_k| \ge l_k \quad \text{if } |y - p_k| < l_k,$$

and

$$u_i^k(y) \le u_{i_0}^k(p_k) + 2\log 2 \quad \text{for all } y \in B(p_k, l_k). \tag{2-8}$$

Next, we set

$$\mathcal{R}_k = e^{u_{i_0}^k(p_k)/2} l_k \tag{2-9}$$

and scale u_i^k by

$$\tilde{v}_i^k(y) = u_i^k(p_k + e^{-u_{i_0}^k(p_k)/2} y) - u_{i_0}^k(p_k) \quad \text{for } i \in I.$$

From (2-7) we clearly have $\mathcal{R}_k \to \infty$. By (2-8) and standard elliptic estimates for the Laplacian, $\tilde{v}_i^k$ is bounded in $C^2_{\text{loc}}(\mathbb{R}^2)$ and there exists $\varnothing \ne J \subset I$ such that, for all $i \in J$, $\tilde{v}_i^k$ converges to a limit system like (2-3). On the other hand, $\tilde{v}_i^k$ converges uniformly to $-\infty$ over all compact subsets of $\mathbb{R}^2$ for all $i \in I \setminus J$. Clearly (2-6) holds for $\tilde{v}_i^k$. Going back to u^k, we have

$$u_i^k(x) + 2\log|x - x_2^k| \le C \quad \text{for } |x - x_2^k| \le l_2^k,$$

where x_2^k is the point where $\max_i \max_{B(p_k, l_2^k)} u_i^k$ is attained and $l_2^k = l_k$. Here we note that x_2^k is neither q_k nor p_k and the distance between p_k and x_2^k is small: $e^{u_{i_0}^k(p_k)/2}|x_2^k - p_k| = O(1)$. If we rescale u^k around x_2^k, then v^k defined as in (2-2) satisfies (a) and (b) in Proposition 2.1. Clearly $B(x_1^k, l_1^k) \cap B(x_2^k, l_2^k) = \varnothing$.

To continue with the selection process, we let $\Sigma_{k,2} := \{x_1^k, x_2^k\}$ and consider

$$\max_{i \in I, x \in B_1} u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_{k,2}).$$

If, along a subsequence, the quantity above tends to infinity, we apply the same procedure to get x_3^k and l_3^k. After each selection we add a new disjoint disk, say $B(x_m^k, l_m^k)$, in which the profile of bubbling solutions is like that of a global system, so from (2-4) we see that

$$\int_{B(x_m^k, l_m^k)} \sum_i h_i^k e^{u_i^k} \ge C \quad \text{for some } C > 0 \text{ independent of } k.$$

Therefore by (1-6) the process stops after finitely many steps and we have

$$u_i^k(x) + 2\log d(x, \Sigma_k) \le C, \quad i \in I. \tag{2-10}$$

Proposition 2.1 is established. □

2.1. *Case two: the singular case $\exists\gamma_i \neq 0$.* First, the selection process is almost the same. The difference is instead of taking the maximum of u_i^k over B_1 we require

$$0 \in \Sigma_k.$$

Clearly, in $B_1 \setminus \{0\}$, u^k satisfies the same equation as the nonsingular case. Then we consider the maximum of $u_i^k(x) + 2\log \operatorname{dist}(x, \Sigma_k) = u_i^k(x) + 2\log|x|$ and the selection proceeds the same as before. Therefore, in the singular case, $\Sigma_k = \{0, x_1^k, \dots, x_m^k\}$.

Lemma 2.4. *Let Σ_k be the blowup set (thus, if $\gamma_i^k = 0$ for all i, $\Sigma_k = \{x_1^k, \dots, x_m^k\}$, and if the system is singular, $\Sigma_k = \{0, x_1^k, \dots, x_m^k\}$). In either case, for all $x_0 \in B_1 \setminus \Sigma_k$, there exists C_0 independent of x_0 and k such that*

$$|u_i^k(x_1) - u_i^k(x_2)| \le C_0 \quad \textit{for all } x_1, x_2 \in B\big(x_0, \tfrac{1}{2}d(x_0, \Sigma_k)\big) \quad \textit{for all } i \in I.$$

Proof. We can assume $|x| < \frac{1}{10}$ because it is easy to see from Green's representation formula that the oscillation of u_i^k on $B_1 \setminus B_{1/10}$ is finite. Recall the regular part of u_i^k is defined in (1-4) and $\tilde{u}_i^k$ satisfies

$$\Delta\tilde{u}_i^k(x) + \sum_{j\in I} a_{ij} h_j^k(x)|x|^{2\gamma_j^k} e^{\tilde{u}_j^k(x)} = 0 \quad \text{in } B_1, \quad i \in I.$$

Let σ_k be the distance between x_0 and Σ_k. Clearly, for $x_0 \in B_1 \setminus \Sigma_k$ and $x_1, x_2 \in B\big(x_0, \frac{1}{2}d(x_0, \Sigma_k)\big)$,

$$u_i^k(x_1) - u_i^k(x_2) = \tilde{u}_i^k(x_1) - \tilde{u}_i^k(x_2) + O(1) = \int_{B_1} (G(x_1, \eta) - G(x_2, \eta)) \sum_{j\in I} a_{ij} h_j^k(\eta)|\eta|^{2\gamma_j^k} e^{\tilde{u}_j^k(\eta)}\, d\eta + O(1).$$

Here G is the Green's function on B_1. The last term on the above is $O(1)$ because it is the difference of two points of a harmonic function that has bounded oscillation on ∂B_1. Since both $x_1, x_2 \in B_{1/10}$, it is easy to use the uniform bound on the energy (1-6) to obtain

$$\int_{B_1} (\gamma(x_1, \eta) - \gamma(x_2, \eta)) \sum_{j\in I} a_{ij} h_j^k(\eta)|\eta|^{2\gamma_j^k} e^{\tilde{u}_j^k(\eta)}\, d\eta = O(1),$$

where $\gamma(\cdot, \cdot)$ is the regular part of G. Therefore, we only need to show

$$\int_{B_1} \log\frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k |\eta|^{2\gamma_j} e^{\tilde{u}_j}\, d\eta = O(1).$$

If $\eta \in B_1 \setminus B\big(x_0, \frac{3}{4}\sigma_k\big)$, we have $\log(|x_1 - \eta|/|x_2 - \eta|) = O(1)$, then the integration over $B_1 \setminus B\big(x_0, \frac{3}{4}\sigma_k\big)$ is uniformly bounded. Therefore, we only need to show

$$\int_{B(x_0, 3\sigma_k/4)} \log\frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k |\eta|^{2\gamma_j} e^{\tilde{u}_j^k}\, d\eta = \int_{B(x_0, 3\sigma_k/4)} \log\frac{|x_1 - \eta|}{|x_2 - \eta|} \sum_j a_{ij} h_j^k e^{u_j^k}\, d\eta = O(1).$$

To this end, let

$$v_i^k(y) = u_i^k(x_0 + \sigma_k y) + 2\log\sigma_k, \quad y \in B_{3/4}, \quad i \in I. \tag{2-11}$$

Then we just need to show

$$\int_{B_{3/4}} \log \frac{|y_1-\eta|}{|y_2-\eta|} \sum_j a_{ij} h_j^k(x_0+\sigma_k\eta) e^{v_j^k(\eta)}\, d\eta = O(1). \tag{2-12}$$

We assume, without loss of generality, that e_1 is the image of the closest blowup point in Σ_k. Thus, by the selection process,

$$v_i^k(\eta) \le -2\log|\eta-e_1| + C.$$

Therefore,

$$e^{v_i^k(\eta)} \le C|\eta-e_1|^{-2}.$$

With this estimate, we observe that $|\eta-e_1| \ge C > 0$ for $\eta \in B_{3/4}$. Thus, for $j=1,2$ and any fixed $i \in I$,

$$\int_{B_{3/4}} \big|\log|y_j-\eta|\big| e^{v_i^k(\eta)}\, d\eta \le C \int_{B_{3/4}} \frac{\big|\log|y_j-\eta|\big|}{|\eta-e_1|^2}\, d\eta \le C.$$

Lemma 2.4 is established. □

Remark 2.5. For systems with nonnegative coefficient matrix A, the selection process can also be applied. See [Chen and Li 1993] or [Lin and Zhang 2010] for more details.

3. Pohozaev identity and related estimates on the energy

In this section we derive a Pohozaev identity for u^k satisfying (1-2), (1-3) and (1-6), h_i^k and γ_i^k satisfying (1-5), and $A = A_n$.

Proposition 3.1. *Let $A = A_n$, σ_i be defined by (1-7). Suppose $u^k = (u_1^k, \dots, u_n^k)$ satisfy (1-2), (1-6),(1-3) and (2-1), h^k and γ_i^k satisfy (1-5). Then we have*

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^n (1+\gamma_i)\sigma_i.$$

Proof. We start with a lemma:

Lemma 3.2. *Given any $\epsilon_k \to 0$ such that $\Sigma_k \subset B\big(0, \frac{1}{2}\epsilon_k\big)$, there exist $l_k \to 0$ satisfying $l_k \ge 2\epsilon_k$ and*

$$\bar u_i^k(l_k) + 2\log l_k \to -\infty \quad \text{for all } i \in I, \quad \text{where} \quad \bar u_i^k(r) := \frac{1}{2\pi r}\int_{\partial B_r} u_i^k. \tag{3-1}$$

Remark 3.3. By Lemma 3.2 and Lemma 2.4,

$$u_i^k(x) + 2\log|x| \to -\infty \quad \text{for all } i \in I \text{ and } x \in \partial B_{l_k}.$$

This is crucial for evaluating the $\mathcal{R}_1$ term (the first term on the right) of (3-7) below.

Proof of Lemma 3.2. Since $\Sigma_k \subset B\big(0, \frac{1}{2}\epsilon_k\big)$, we have, by Proposition 2.1(3),

$$u_i^k(x) + 2\log|x| \le C, \quad |x| \ge \epsilon_k. \tag{3-2}$$

The key point of the argument below is that we can always use the finite energy assumption and Lemma 2.4 to make u_1^k satisfy (3-1). Then we can adjust the radius to make the other components satisfy (3-1) as well.

First we observe that, for each fixed i, there exists $r_{k,i} \geq \epsilon_k$ such that

$$\bar{u}_i^k(r_{k,i}) + 2\log r_{k,i} \to -\infty, \tag{3-3}$$

because otherwise we would have

$$\bar{u}_i^k(r) + 2\log r \geq -C \quad \text{for all } r \geq \epsilon_k$$

for some $C > 0$. By Lemma 2.4, u_i^k has bounded oscillation on each ∂B_r. Thus

$$u_i^k(x) + 2\log|x| \geq -C \quad \text{for all } x \in \partial B_r, \quad \epsilon_k < r < 1$$

for some C. Then

$$e^{u_i^k(x)} \geq C|x|^{-2}, \quad \epsilon_k \leq |x| \leq 1.$$

Integrating $e^{u_i^k}$ on $B_1 \setminus B_{\epsilon_k}$, we get a contradiction on the uniform energy bound of $\int_{B_1} h_i^k e^{u_i^k}$. Thus (3-3) is established.

Now, for u_1^k, we find $r_{k,1} \geq \epsilon_k$ so that

$$\bar{u}_1^k(r_{k,1}) + 2\log r_{k,1} \to -\infty.$$

Here we claim that we can assume $r_{k,1} \to 0$ as well. In fact, if $r_{k,1}$ does not tend to 0, by Lemma 2.4

$$\bar{u}_1^k(r) + 2\log r \leq -N_k + C, \quad \tfrac{1}{2}r_{k,1} < r < r_{k,1},$$

where $N_k \to \infty$ and satisfies

$$\bar{u}_1^k(r_{k,1}) + 2\log r_{k,1} \leq -N_k.$$

Using Lemma 2.4 again we have

$$\bar{u}_1^k(r) + 2\log r \leq -N_k + C, \quad \tfrac{1}{4}r_{k,1} < r < \tfrac{1}{2}r_{k,1}.$$

Obviously this process can be done $\bar{N}_k$ times, where $\bar{N}_k$ is chosen to tend to infinity slowly enough so that $\bar{r}_k := r_{k,1}2^{-\bar{N}_k}$ satisfies

$$\bar{u}_1^k(\bar{r}_k) + 2\log \bar{r}_k \leq -N_k + C\bar{N}_k \to -\infty.$$

We can use $\bar{r}_k$ to replace $r_{k,1}$. Exactly the same argument shows the existence of $s_k \to 0$, $\tilde{N}_k \to \infty$ such that

$$\begin{cases} s_k/r_{k,1} \to \infty, \\ \bar{u}_1^k(r) + 2\log r \leq -\tilde{N}_k, \quad r_{k,1} \leq r \leq s_k. \end{cases}$$

Next we claim that, between $r_{k,1}$ and s_k, there must be a $r_{k,2}$ such that

$$\bar{u}_2^k(r_{k,2}) + 2\log r_{k,2} \leq -N_{k,2} \tag{3-4}$$

for some $N_{k,2} \to \infty$ as $k \to \infty$. The proof of (3-4) is very similar to what has been used before: If this is not the case, $e^{u_2^k} \ge Cr^{-2}$ for some $C > 0$ and $r \in (r_{k,1}, s_k)$. The fact that $s_k/r_{k,1} \to \infty$ leads to a contradiction to the uniform bound of the energy of u_2^k.

Thus, we have proved that, for $r = r_{k,2}$ both u_1^k and u_2^k decay faster than $-2\log r$:

$$\bar{u}_i^k(r) + 2\log r \le -N_k, \quad r = r_{k,2}, \quad i = 1, 2,$$

for some $N_k \to \infty$. Then it is easy to see that there exist $s_k \to 0$ and $s_k/r_{k,2} \to \infty$ such that

$$\bar{u}_i^k(r) + 2\log r \le -N_k', \quad r_{k,2} \le r \le s_k, \quad i = 1, 2,$$

for some $N_k' \to \infty$ as well. The same argument as above guarantees the existence of $l_k \in (r_{k,2}, s_k)$ and some $N_k'' \to \infty$ such that

$$\bar{u}_3^k(l_k) + 2\log l_k \le -N_k''.$$

Clearly this argument can be applied finitely many times to exhaust all the components of the whole system. Lemma 3.2 is established. □

Now we continue with the proof of Proposition 3.1.

Case one: $\gamma_i^k \equiv 0$. Using the definition of σ_i in (1-7), we choose $l_k \to 0$ such that $\Sigma_k \subset B\big(0, \tfrac{1}{2}l_k\big)$ and

$$\frac{1}{2\pi}\int_{B_{l_k}} h_i^k e^{u_i^k} = \sigma_i + o(1) \quad \text{for } i \in I. \tag{3-5}$$

Here we claim that (3-1) also holds, because otherwise we would have

$$\bar{u}_i(l_k) + 2\log l_k \ge -C.$$

By Lemma 2.4

$$\bar{u}_i(r) + 2\log r \ge -C_1, \quad l_k \le r \le 2l_k,$$

which means there is a lower bound on the energy in the annulus $B_{2l_k} \setminus B_{l_k}$. Consequently

$$\frac{1}{2\pi}\int_{B_{2l_k}} h_i^k e^{u_i^k} > \sigma_i + \epsilon$$

for some $\epsilon > 0$ independent of k, a contradiction to the definition of σ_i in (1-7).

Let

$$v_i^k(y) = u_i^k(l_k y) + 2\log l_k, \quad i \in I.$$

Then clearly we have

$$\begin{cases} \Delta v_i^k(y) + \sum_{j=1}^n a_{ij} H_j^k(y) e^{v_j^k(y)} = 0, & |y| \le l_k^{-1}, \quad i \in I, \\ \bar{v}_i^k(1) \to -\infty, \end{cases} \tag{3-6}$$

where

$$H_i^k(y) = h_i^k(l_k y), \quad |y| \le l_k^{-1}, \quad i \in I.$$

The Pohozaev identity we use is

$$\sum_i \int_{B_{\sqrt{R_k}}} (x\cdot\nabla H_i^k)e^{v_i^k} + 2\sum_i \int_{B_{\sqrt{R_k}}} H_i^k e^{v_i^k} \\ = \sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}} \sum_i H_i^k e^{v_i^k} + \sqrt{R_k}\int_{\partial B_{\sqrt{R_k}}} \sum_{i,j}\bigl(a^{ij}\partial_\nu v_i^k \partial_\nu v_j^k - \tfrac{1}{2}a^{ij}\nabla v_i^k \nabla v_j^k\bigr), \quad (3\text{-}7)$$

where $R_k \to \infty$ will be chosen later and (a^{ij}) is the inverse matrix of (a_{ij}). The key point of the following proof is to choose R_k properly in order to estimate ∇v_i^k on $\partial B_{\sqrt{R_k}}$. In the estimate of $\partial B_{\sqrt{R_k}}$, the procedure is to get rid of unimportant parts and prove that the radial part of ∇v_i^k is the leading term. To estimate all the terms of the Pohozaev identity we first write (3-7) as

$$\mathscr{L}_1 + \mathscr{L}_2 = \mathscr{R}_1 + \mathscr{R}_2 + \mathscr{R}_3,$$

where $\mathscr{L}_1$ stands for “the first term on the left” and the other terms are understood similarly. First, we choose $R_k \to \infty$ such that $R_k^{3/2} = o(l_k^{-1})$, then use $l_k \to 0$ to show that $\mathscr{L}_1 = o(1)$. To evaluate $\mathscr{L}_2$, we observe that, by Lemma 2.4, $v_i^k(y) \to -\infty$ over all compact subsets of $\mathbb{R}^2 \setminus B_{1/2}$. Thus we further require R_k to satisfy

$$\int_{B_{R_k}\setminus B_{3/4}} H_i^k e^{v_i^k} = o(1) \qquad (3\text{-}8)$$

and, for $i \in I$, by (3-6) and Lemma 2.4,

$$v_i^k(y) + 2\log|y| \to -\infty \quad \text{uniformly in } 1 < |y| \le R_k. \qquad (3\text{-}9)$$

By the choice of l_k we clearly have

$$\frac{1}{2\pi}\int_{B_1} H_i^k e^{v_i^k} = \frac{1}{2\pi}\int_{B_{l_k}} h_i^k e^{u_i^k} = \sigma_i + o(1), \quad i \in I.$$

By (3-8), we have

$$\mathscr{L}_2 = 4\pi\sum_{i=1}^n \sigma_i + o(1).$$

For $\mathscr{R}_1$, we use (3-9) to conclude $\mathscr{R}_1 = o(1)$.

Therefore we are left with the estimates of $\mathscr{R}_2$ and $\mathscr{R}_3$, for which we shall estimate ∇v_i^k on ∂B_{R_k}. Let

$$G_k(y,\eta) = -\frac{1}{2\pi}\log|y-\eta| + \gamma_k(y,\eta)$$

be the Green’s function on $B_{l_k^{-1}}$ with respect to the Dirichlet boundary condition. Clearly

$$\gamma_k(y,\eta) = \frac{1}{2\pi}\log\frac{|y|}{l_k^{-1}}\left|\frac{l_k^{-2}y}{|y|^2} - \eta\right|,$$

and we have

$$\nabla_y\gamma_k(y,\eta) = O(l_k), \quad y \in \partial B_{\sqrt{R_k}}, \quad \eta \in B_{l_k^{-1}}. \qquad (3\text{-}10)$$

We first estimate ∇v_i^k on $\partial B_{R_k^{1/2}}$. By Green's representation formula,

$$v_i^k(y) = \int_{B_{l_k^{-1}}} G(y,\eta) \sum_{j=1}^n a_{ij} H_i^k e^{v_j^k}\, d\eta + H_{ik},$$

where H_{ik} is the harmonic function satisfying $H_{ik} = v_i^k$ on $\partial B_{l_k^{-1}}$. Since $H_{ik} - c_k = O(1)$ for some c_k, $|\nabla H_{ik}(y)| = O(l_k)$, so

$$\begin{aligned}\nabla v_i^k(y) &= \int_{B_{l_k^{-1}}} \nabla_y G_k(y,\eta) \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k}\, d\eta + \nabla H_{ik}(y)\\ &= -\frac{1}{2\pi}\int_{B_{l_k^{-1}}} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k}\, d\eta + O(l_k).\end{aligned} \tag{3-11}$$

We estimate the integral in (3-11) over a few subregions. First, the integral over $B_{l_k^{-1}} \setminus B_{R_k^{2/3}}$ is $o(1) R_k^{-1/2}$ because, over this region, $1/|y-\eta| \sim 1/|\eta| \le o(R_k^{-1/2})$. For the integral over B_1, we use

$$\frac{y-\eta}{|y-\eta|^2} = \frac{y}{|y|^2} + O\left(\frac{1}{|y|^2}\right)$$

to obtain

$$-\frac{1}{2\pi}\int_{B_1} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = \left(-\frac{y}{|y|^2} + O\left(\frac{1}{|y|^2}\right)\right)\left(\sum_{j=1}^n a_{ij}\sigma_j + o(1)\right).$$

This is the leading term. For the integral over the region $B(0, \sqrt{R_k}/2) \setminus B_1$, we use $1/|y-\eta| \sim 1/|y|$ and (3-8) to get

$$\int_{B_{R_k^{1/2}/2} \setminus B_1} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

By a similar argument we also have

$$\int_{B_{R_k^{2/3}} \setminus (B_{R_k^{1/2}/2} \cup B(y,|y|/2))} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

Finally, over the region $B(y,|y|/2)$ we use $e^{v_i^k(\eta)} = o(1)|\eta|^{-2}$ to get

$$\int_{B(y,|y|/2)} \frac{y-\eta}{|y-\eta|^2} \sum_{j=1}^n a_{ij} H_j^k e^{v_j^k} = o(1)|y|^{-1}.$$

Combining the estimates on all the subregions mentioned above, we have

$$\nabla v_i^k(y) = -\frac{y}{|y|^2}\left(\sum_{j=1}^n a_{ij}\sigma_j + o(1)\right) + o(|y|^{-1}), \quad |y| = R_k^{1/2}.$$

Using the above in $\mathcal{R}_2$ and $\mathcal{R}_3$, we have

$$\sum_{i,j=1}^{n} a_{ij}\sigma_i\sigma_j = 4\sum_{i=1}^{n}\sigma_i + \circ(1).$$

Proposition 3.1 is established for the nonsingular case.

Case two: the singular case $\exists\gamma_i \neq 0$.

Lemma 3.4. *For $\sigma \in (0, 1)$, the following Pohozaev identity holds*:

$$\sigma\int_{\partial B_\sigma}\sum_{i,j\in I} a^{ij}\big(\partial_\nu u_i^k\partial_\nu u_j^k - \tfrac{1}{2}\nabla u_i^k\cdot\nabla u_j^k\big) + \sum_{i\in I}\sigma\int_{\partial B_\sigma} h_i^k e^{u_i^k} = 2\sum_{i\in I}\int_{B_\sigma} h_i^k e^{u_i^k} + \sum_{i\in I}\int_{B_\sigma}(x\cdot\nabla h_i^k)e^{u_i^k} + 4\pi\sum_{i,j\in I} a^{ij}\gamma_i^k\gamma_j^k.$$

Proof. First, we claim that, for each fixed k,

$$\nabla u_i^k(x) = 2\gamma_i^k x/|x|^2 + O(1) \quad \text{near the origin.} \tag{3-12}$$

Indeed, recall the equation for the regular part $\tilde{u}_i^k$ is

$$\Delta\tilde{u}_i^k(x) + \sum_j |x|^{2\gamma_j^k} h_j^k(x)e^{\tilde{u}_j^k(x)} = 0 \quad \text{in } B_1.$$

By the argument of Lemma 4.1 in [Lin and Zhang 2010], for fixed k, $\tilde{u}_i^k$ is bounded above near 0, then an elliptic estimate leads to (3-12).

Let $\Omega = B_\sigma \setminus B_\epsilon$. The standard Pohozaev identity on Ω is

$$\sum_{i\in I}\bigg(\int_\Omega (x\cdot\nabla h_i^k)e^{u_i^k} + 2h_i^k e^{u_i^k}\bigg) = \int_{\partial\Omega}\bigg(\sum_i (x\cdot\nu)h_i^k e^{u_i^k} + \sum_{i,j} a^{ij}\big(\partial_\nu u_j^k(x\cdot\nabla u_i^k) - \tfrac{1}{2}(x\cdot\nu)(\nabla u_i^k\cdot\nabla u_j^k)\big)\bigg).$$

Let $\epsilon \to 0$, then the integration over Ω extends to B_σ by the integrability of $h_i^k e^{u_i^k}$ and (1-5). For the terms on the right-hand side, clearly $\partial\Omega = \partial B_\sigma \cup \partial B_\epsilon$. Thanks to (3-12), the integral on ∂B_ϵ is $-4\pi\sum_{i,j} a^{ij}\gamma_i^k\gamma_j^k$. Lemma 3.4 is established. □

Let

$$\sigma_i^k(r) = \frac{1}{2\pi}\int_{B_r} h_i^k e^{u_i^k}, \quad i \in I.$$

Lemma 3.5. *Let $\epsilon_k \to 0$ such that $\Sigma_k \subset B\big(0, \tfrac{1}{2}\epsilon_k\big)$ and*

$$u_i^k(x) + 2\log|x| \to -\infty, \quad |x| = \epsilon_k, \quad i \in I. \tag{3-13}$$

Then we have

$$\sum_{i,j\in I} a_{ij}\sigma_i^k(\epsilon_k)\sigma_j^k(\epsilon_k) = 4\sum_{i\in I}(1+\gamma_i^k)\sigma_i^k(\epsilon_k) + o(1). \tag{3-14}$$

Proof of Lemma 3.5. First the existence of ϵ_k that satisfies (3-13) is guaranteed by Lemma 2.4. In B_{ϵ_k}, we let $\tilde{u}_i^k(x)$ be defined as in (1-4). Then

$$v_i^k(y) = \tilde{u}_i^k(\epsilon_k y) + 2(1+\gamma_i^k)\log \epsilon_k.$$

Using $v_i^k \to -\infty$ on ∂B_1, we obtain, by Green's representation formula and standard estimates,

$$\nabla v_i^k(y) = \Big(\sum_{j\in I} a_{ij}\sigma_j^k(\epsilon_k) + o(1)\Big) y, \quad y \in \partial B_1.$$

After translating the above to estimates of u_i^k, we have

$$\nabla u_i^k(x) = \Big(\sum_{j\in I} (a_{ij}\sigma_j^k(\epsilon_k) - 2\gamma_j^k)\Big)\frac{x}{|x|^2} + \frac{o(1)}{|x|}, \quad |x| = \epsilon_k. \tag{3-15}$$

As we observe the Pohozaev identity in Lemma 3.4 with $\sigma = \epsilon_k$, we see easily that the second term on the left-hand side and the second term on the right-hand side are both $o(1)$. The first term on the right-hand side is clearly $4\pi \sum_i \sigma_i^k(\epsilon_k)$. Therefore we only need to evaluate the first term on the left-hand side, for which we use (3-15). Lemma 3.5 is established by similar estimates as in the nonsingular case. □

Thus Proposition 3.1 is established for the singular case as well. □

Remark 3.6. The proof of Proposition 3.1 clearly indicates the following statements when it is applied to an SU(3) Toda system. Let $B(p_k, l_k)$ be a circle centered at p_k with radius l_k. Let Σ_k' be a subset of Σ_k. Suppose $\operatorname{dist}(\Sigma_k', \partial B(p_k, l_k)) = o(1)\operatorname{dist}(\Sigma_k \setminus \Sigma_k', \partial B(p_k, l_k))$, and we consider the following two situations: If $p_k = 0$, we have

$$\tilde{\sigma}_1^k(l_k)^2 - \tilde{\sigma}_1^k(l_k)\tilde{\sigma}_2^k(l_k)^2 + \tilde{\sigma}_2^k(l_k) = 2\mu_1\tilde{\sigma}_1^k(l_k) + 2\mu_2\tilde{\sigma}_2^k(l_k) + o(1).$$

If $0 \in \Sigma_k \setminus \Sigma_k'$, then

$$\tilde{\sigma}_1^k(l_k)^2 - \tilde{\sigma}_1^k(l_k)\tilde{\sigma}_2^k(l_k) + \tilde{\sigma}_2^k(l_k)^2 = 2\tilde{\sigma}_1^k(l_k) + 2\tilde{\sigma}_2^k(l_k) + o(1),$$

where $\tilde{\sigma}_i^k(l_k) = (1/2\pi)\int_{B(p_k,l_k)} h_i^k e^{u_i^k}$. This fact will be used in the final step of the proof of Theorem 1.2.

Remark 3.7. From the proof of Proposition 3.1, we see that the Pohozaev identity has to be evaluated on *fast decay* components in order to rule out the $\mathcal{R}_1$ term. A component is called *fast decay* if the difference between itself and the threshold harmonic function tends to $-\infty$; for example, see (3-13). A component is called a *slow decay* component if it is not a fast decay component. Later, in the remaining part of the proof of Theorem 1.2, we shall derive Pohozaev identities over different regions and all of them will have to be evaluated on fast decay components.

4. Fully bubbling systems

Next we consider a typical blowup situation for systems: fully bubbling solutions. First, let $\gamma_i^k \equiv 0$ for all $i \in I$. Let

$$\lambda^k = \max\{\max_{B_1} u_1^k, \dots, \max_{B_1} u_n^k\} \tag{4-1}$$

and $x^k \to 0$ be where λ^k is attained. Let

$$v_i^k(y) = u_i^k(x_k + e^{-\lambda^k/2}y) - \lambda^k, \quad y \in \Omega_k, \quad i \in I, \tag{4-2}$$

where $\Omega_k = \{y : e^{-\lambda^k/2}y + x_k \in B_1\}$. The sequence is called fully bubbling if, along a subsequence,

$$\{v_1^k, \dots, v_n^k\} \quad \text{converge in } C^2_{\mathrm{loc}}(\mathbb{R}^2) \text{ to } (v_1, \dots, v_n) \tag{4-3}$$

that satisfies

$$\Delta v_i + \sum_{j\in I} a_{ij} h_j e^{v_j} = 0 \quad \text{in } \mathbb{R}^2, \quad i \in I, \tag{4-4}$$

where $h_i = \lim_{k\to\infty} h_i^k(0)$. Our next theorem is concerned with the closeness between $u^k = (u_1^k, \dots, u_n^k)$ and $v = (v_1, \dots, v_n)$.

Theorem 4.1. *Let $A = A_n$, u^k be a sequence of solutions to (1-2) with $\gamma_i^k = 0$ for all $i \in I$. Suppose u^k satisfies (1-3) and (1-6), h^k satisfies (1-5), and λ^k, x^k and v^k are described by (4-1) and (4-2), respectively. Suppose u^k is fully bubbling; then there exists $C > 0$ independent of k such that*

$$|u_i^k(e^{-\frac{1}{2}\lambda^k}y + x^k) - \lambda^k - v_i(y)| \le C + o(1)\log(1+|y|) \quad \textit{for } x \in \Omega_k, \quad i \in I. \tag{4-5}$$

Remark 4.2. If A is nonnegative, i.e., the system is a Liouville system, Theorem 4.1 and Theorem 4.3 below are established in [Lin and Zhang 2010]. For $A = A_2$, [Jost et al. 2006] proved

$$|u_i^k(e^{-\lambda^k/2}y + x^k) - \lambda^k - v_i(y)| \le C \quad \text{for } x \in \Omega_k, \quad i = 1, 2.$$

Clearly this estimate is slightly stronger than (4-5) for $n = 2$. The Jost–Lin–Wang proof uses holonomy theory but the proof of Theorem 4.1 is a simple application of the Pohozaev identity proved in Section 3.

If there is a $\gamma_i \neq 0$, we let

$$\tilde\lambda^k = \max\left\{\frac{\max_{B_1}\tilde u_1^k}{1+\gamma_1^k}, \dots, \frac{\max_{B_1}\tilde u_n^k}{1+\gamma_n^k}\right\},$$

and

$$\tilde v_i^k(y) = \tilde u_i^k(e^{-\tilde\lambda^k/2}y) - (1+\gamma_i^k)\tilde\lambda^k$$

for $i \in I$ and $y \in \Omega_k := \{y : e^{-\tilde\lambda^k/2}y \in B_1\}$. We assume

$$(\tilde v_1^k, \dots, \tilde v_n^k) \quad \text{converge in } C^2_{\mathrm{loc}}(\mathbb{R}^2) \text{ to } (\tilde v_1, \dots, \tilde v_n) \tag{4-6}$$

that satisfies

$$\Delta \tilde v_i + \sum_{j=1}^n a_{ij}|x|^{2\gamma_j} h_j e^{\tilde v_j} = 0 \quad \text{in } \mathbb{R}^2, \quad i \in I, \tag{4-7}$$

where $h_i = \lim_{k\to\infty} h_i^k(0)$.

Theorem 4.3. *Let $A = A_n$, $\tilde u^k$, $\tilde v^k$, $(\tilde v_1, \dots, \tilde v_n)$, $\tilde\lambda^k$, ϵ_k and Ω_k be as described above, and h_i^k and γ_i^k satisfy (1-5); then, under assumption (4-6), there exists $C > 0$ independent of k such that*

$$|\tilde u_i^k(e^{\tilde\lambda^k/2}y) - (1+\gamma_i^k)\tilde\lambda^k - \tilde v_i(y)| \le C + o(1)\log(1+|y|) \quad \textit{for } x \in \Omega_k. \tag{4-8}$$

Proof of Theorem 4.1. Recall that σ_i is defined in (1-7). By Proposition 3.1, we have

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i\in I}\sigma_i. \tag{4-9}$$

On the other hand, let

$$\sigma_{iv} := \frac{1}{2\pi}\int_{\mathbb{R}^2} h_i e^{v_i} \quad \text{for } i = 1,\dots,n,$$

where $v = (v_1,\dots,v_n)$ is the limit of the fully bubbling sequence after scaling. Clearly $\sigma_v = (\sigma_{1v},\dots,\sigma_{nv})$ also satisfies (4-9). We claim that

$$\sigma_i = \sigma_{iv} \quad \text{for } i = 1,\dots,n. \tag{4-10}$$

Let $s_i = \sigma_i - \sigma_{vi}$; we obviously have $s_i \ge 0$. The difference between σ and σ_v on (4-9) gives

$$\sum_{i,j\in I} a_{ij}s_is_j + 2\sum_{i\in I}\Big(\sum_{j\in I} a_{ij}\sigma_{vj} - 2\Big)s_i = 0. \tag{4-11}$$

First, by Proposition 2.1, we have $\sum_{j\in I} a_{ij}\sigma_{vj} - 2 > 0$. Next, if either A is nonnegative ($a_{ij} \ge 0$ for all $i, j = 1,\dots,n$) or A is positive definite, we have $\sum_{i,j\in I} a_{ij}s_is_j \ge 0$. Then (4-11) and $s_i \ge 0$ imply (4-10).

From the convergence from v_i^k to v_i in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, we can find $R_k \to \infty$ such that

$$|v_i^k(y) - v_i(y)| = o(1), \quad |y| \le R_k.$$

For $|y| > R_k$, let

$$\bar{v}_i^k(r) = \frac{1}{2\pi r}\int_{\partial B_r} v_i^k(y)\,dS_y.$$

Then

$$\frac{d}{dr}\bar{v}_i^k(r) = \frac{1}{2\pi r}\int_{B_r}\Delta v_i^k = -\frac{1}{2\pi r}\int_{B_r}\sum_{j\in I} a_{ij}h_j^k e^{v_j^k} = -\frac{\sum_j a_{ij}\sigma_j + o(1)}{r}.$$

Hence

$$\bar{v}_i^k(r) = -\Big(\sum_{j\in I} a_{ij}\sigma_j + o(1)\Big)\log r + O(1) \quad \text{for all } r > 2.$$

Since $v_i^k(y) = \bar{v}_i^k(|y|) + O(1)$ and

$$v_i(y) = -\Big(\sum_j a_{ij}\sigma_j\Big)\log|y| + O(1) \quad \text{for } |y| > 1,$$

we see that (4-5) holds. Theorem 4.1 is established. □

Proof of Theorem 4.3. By (3-14) we have

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i\in I}(1+\gamma_i)\sigma_i. \tag{4-12}$$

Recall that $v = (v_1,\dots,v_n)$ satisfies (4-7). Let

$$\sigma_{iv} = \frac{1}{2\pi}\int_{\mathbb{R}^2} h_i|x|^{2\gamma_i}e^{v_i}.$$

On the one hand, $(\sigma_{1v}, \dots, \sigma_{iv})$ also satisfies (4-12); on the other hand, the classification theorem of [Lin et al. 2012a] gives

$$\sum_{j\in I} a_{ij}\sigma_{jv} > 2+2\gamma_i, \quad i \in I. \tag{4-13}$$

Let $s_i = \sigma_i - \sigma_{iv}$ $(i \in I)$; then (4-12), which is satisfied by both $(\sigma_1, \dots, \sigma_n)$ and $(\sigma_{1v}, \dots, \sigma_{nv})$, gives

$$\sum_{i,j\in I} a_{ij}s_is_j + 2\sum_{i\in I}\Big(\sum_{j\in J} a_{ij}\sigma_{jv} - 2 - 2\gamma_i\Big)s_i = 0.$$

By (4-13) and the assumption on A, we have $s_i = 0$ for all $i \in I$. The remaining part of the proof is exactly like the last part of the proof of Theorem 4.1. Theorem 4.3 is established. □

5. Asymptotic behavior of solutions in each simple blowup area

In this section, we derive some results on the energy classification around each blowup point. First we let $A = A_n$ (the Cartan matrix) and consider:

***The neighborhood around* 0.** Since 0 is postulated to belong to Σ_k first, it means there may not be a bubbling picture in a neighborhood of 0.

Let $\tau_k = \frac{1}{2}\operatorname{dist}(0, \Sigma_k \setminus \{0\})$; we consider the energy limits of $h_i^k e^{u_i^k}$ in B_{τ_k}. By the selection process and Lemma 2.4,

$$u_i^k(x) + 2\log|x| \le C, \quad u_i^k(x) = \bar{u}_i^k(|x|) + O(1), \quad |x| \le \tau_k, \quad i \in I, \tag{5-1}$$

where $\bar{u}_i^k(|x|)$ is the average of u_i^k on $\partial B_{|x|}$. Let $\tilde{u}_i^k$ be defined by (1-4). Then we have

$$\Delta\tilde{u}_i^k(x) + \sum_{j\in I} a_{ij}|x|^{2\gamma_j}h_j^k(x)e^{\tilde{u}_j^k(x)} = 0, \quad |x| \le \tau_k.$$

Let

$$-2\log\delta_k = \max_{i\in I}\max_{x\in B(0,\tau_k)} \frac{\tilde{u}_i^k}{1+\gamma_i^k}$$

and

$$v_i^k(y) = \tilde{u}_i^k(\delta_k y) + 2(1+\gamma_i^k)\log\delta_k, \quad |y| \le \tau_k\delta_k^{-1}. \tag{5-2}$$

It is easy to see the equation for v_i^k is

$$\Delta v_i^k(y) + \sum_{j\in I} a_{ij}|y|^{2\gamma_j^k}h_j^k(\delta_k y)e^{v_j^k(y)} = 0, \quad |y| \le \tau_k\delta_k^{-1}.$$

Then we consider two trivial cases, first, $\tau_k\delta_k^{-1} \le C$. This is the case that there is no entire bubble after scaling.

Let f_i^k solve

$$\begin{cases} \Delta f_i^k + \sum_{j\in I} a_{ij}|y|^{2\gamma_j^k}h_j^k(\delta_k y)e^{v_j^k} = 0, & |y| \le \tau_k\delta_k^{-1}, \\ f_i^k = 0 & \text{on } |y| = \tau_k\delta_k^{-1}. \end{cases}$$

Using $v_i \le 0$ we have $|f_i^k| \le C$ on $B(0, \tau_k\delta_k^{-1})$. Since $v_i^k - f_i^k$ is harmonic and v_i^k has bounded oscillation on $\partial B(0, \tau_k\delta_k^{-1})$, we have

$$v_i^k(x) = \bar{v}_i^k(\partial B(0, \tau_k\delta_k^{-1})) + O(1) \quad \text{for all } x \in B(0, \tau_k\delta_k^{-1}), \tag{5-3}$$

where $\bar{v}_i^k(\partial B(0, \tau_k\delta_k^{-1}))$ stands for the average of v_i^k on $\partial B(0, \tau_k\delta_k^{-1})$. Direct computation shows that

$$\int_{B(0,\tau_k)} e^{u_i^k(x)}\,dx = \int_{B(0,\tau_k\delta_k^{-1})} e^{v_i^k(y)}|y|^{2\gamma_i^k}\,dy.$$

Therefore,

$$\int_{B_{\tau_k}} h_i^k e^{u_i^k}\,dx = O(1)e^{\bar{v}_i^k(\partial B(0,\tau_k\delta_k^{-1}))}. \tag{5-4}$$

So, if $\bar{v}_i^k(\partial B(0, \tau_k\delta_k^{-1})) \to -\infty$, then $\int_{B_{\tau_k}} h_i^k e^{u_i^k}\,dx = o(1)$.

The second trivial case is when the blowup sequence is fully bubbling. We now have

$$\tau_k\delta_k^{-1} \to \infty \tag{5-5}$$

and we assume that $(v_1^k, \dots, v_n^k) \to (v_1, \dots, v_n)$ in $C^2_{\text{loc}}(\mathbb{R}^2)$. Clearly,

$$\Delta v_i + \sum_{j=1}^n a_{ij}|x|^{2\gamma_j} h_j e^{v_j} = 0 \quad \text{in } \mathbb{R}^2, \quad i \in I,$$

where $h_i = \lim_{k\to\infty} h_i^k(0)$. By the classification theorem of [Lin et al. 2012a], we have

$$\frac{1}{2\pi}\sum_{j\in I} a_{ij}\int_{\mathbb{R}^2} |y|^{2\gamma_j} e^{v_j} h_j\,dy = 2(2+\gamma_i+\gamma_{n+1-i})$$

and

$$v_i(y) = -(4+2\gamma_{n+1-i})\log|y| + O(1), \quad |y| > 1, \quad i \in I.$$

By the proof of Theorem 4.3, there is only one bubble.

The final case we consider is a partially blown-up picture. Note that (5-5) is assumed. For the following two propositions we assume $n = 2$, i.e., we consider SU(3) Toda systems.

Proposition 5.1. *Suppose (1-2), (1-3), (1-5) and (1-6) hold for u^k, h_i^k and γ_i. The matrix A equals A_2, and (5-5) also holds. Suppose $s_k \in (0, \tau_k)$ satisfies*

$$u_i^k(x) \le -2\log|x| - N_k, \quad i = 1, 2,$$

for all $|x| = s_k$ and some $N_k \to \infty$. Then $(\sigma_1^k(s_k), \sigma_2^k(s_k))$ is an $o(1)$ perturbation of one of the following five types:

$$(2\mu_1, 0), \quad (0, 2\mu_2), \quad (2(\mu_1+\mu_2), 2\mu_2), \quad (2\mu_1, 2(\mu_1+\mu_2)), \quad (2\mu_1+2\mu_2, 2\mu_1+2\mu_2).$$

On $\partial B(0, \tau_k)$, for each i either

$$u_i^k(x) + 2\log|x| \ge -C, \quad |x| = \tau_k,$$

for some $C > 0$ *or*

$$u_i^k(x) + 2\log|x| < -(2+\delta)\log|x| + \delta\log\delta_k, \quad |x| = \tau_k, \tag{5-6}$$

for some $\delta > 0$. *If* (5-6) *holds for some* i, *then*

$$\sigma_i^k(\tau_k) = o(1),\ 2\mu_i + o(1) \ \text{or}\ 2\mu_1 + 2\mu_2 + o(1).$$

Moreover, there exists at least one i_0 *such that* (5-6) *holds for* $i = i_0$.

Similarly, for bubbles away from the origin we have:

Proposition 5.2. *Suppose* (1-2), (1-3), (1-5) *and* (1-6) *hold for* u^k, h_i^k *and* γ_i. *The matrix* A *equals* A_2. *Let* $x_k \in \Sigma_k \setminus \{0\}$, $\bar\tau_k = \frac{1}{2}\operatorname{dist}(x_k, \Sigma_k \setminus \{0, x_k\})$ *and*

$$\bar\delta_k = \exp\Bigl(-\tfrac{1}{2} \max_{\substack{i=1,2\\ x\in B(x_k,\bar\tau_k)}} u_i^k(x)\Bigr).$$

Then, for all $s_k \in (0, \bar\tau_k)$, *if*

$$u_i^k(x) + 2\log|x - x_k| \le -N_k \quad \text{for all } |x - x_k| = s_k, \quad i = 1, 2,$$

for some $N_k \to \infty$, *then* $\bigl((1/2\pi)\int_{B(x_k,s_k)} h_1^k e^{u_1^k}, (1/2\pi)\int_{B(x_k,s_k)} h_2^k e^{u_2^k}\bigr)$ *is an* $o(1)$ *perturbation of one of the following five types*:

$$(2, 0), \quad (0, 2), \quad (2, 4), \quad (4, 2), \quad (4, 4).$$

On $\partial B(x_k, \bar\tau_k)$, *for each* i *either*

$$u_i^k(x) + 2\log\bar\tau_k \ge -C \quad \text{for all } x \in \partial B(x_k, \bar\tau_k)$$

or

$$u_i^k(x) \le -(2+\delta)\log\bar\tau_k + \delta\log\bar\delta_k \quad \text{for all } x \in \partial B(x_k, \bar\tau_k). \tag{5-7}$$

If (5-7) *holds for some* i, *then* $(1/2\pi)\int_{B(x_k,\bar\tau_k)} h_i^k e^{u_i^k}$ *is* $o(1)$, $2 + o(1)$ *or* $4 + o(1)$. *Moreover, there exists at least one* i_0 *such that* (5-7) *holds for* i_0.

We shall only prove Proposition 5.1, as the proof for Proposition 5.2 is similar.

Proof of Proposition 5.1. Let v_i^k be defined by (5-2). Since we only need to consider a partially blown-up situation, without loss of generality we assume v_1^k converges to v_1 in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and v_2^k tends to $-\infty$ over any compact subset of $\mathbb{R}^2$. The equation for v_1 is

$$\Delta v_1 + 2h_1|y|^{2\gamma_1}e^{v_1} = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} h_1|y|^{2\gamma_1}e^{v_1} < \infty,$$

where $h_1 = \lim_{k\to\infty} h_1^k(0)$. By the classification result of [Prajapat and Tarantello 2001] we have

$$2\int_{\mathbb{R}^2} h_1|y|^{2\gamma_1}e^{v_1} = 8\pi\mu_1$$

and

$$v_1(y) = -4\mu_1\log|y| + O(1), \quad |y| > 1.$$

Thus we can find $R_k \to \infty$ (without loss of generality, $R_k = o(1)\tau_k\delta_k^{-1}$) such that

$$\frac{1}{2\pi}\int_{B_{R_k}} h_1^k(\delta_k y)|y|^{2\gamma_1^k} e^{v_1^k} = 2\mu_1 + o(1),$$

i.e., $\sigma_1^k(\delta_k R_k) = 2\mu_1 + o(1)$, and

$$\int_{B_{R_k}} h_2^k(\delta_k y)|y|^{2\gamma_2^k} e^{v_2^k} = o(1).$$

For $r \geq R_k$, recall that

$$\sigma_i^k(\delta_k r) = \frac{1}{2\pi}\int_{B_r} h_i^k(\delta_k y)|y|^{2\gamma_i^k} e^{v_i^k}\, dy;$$

then we have

$$\begin{aligned}
\frac{d}{dr}\bar{v}_1^k(r) &= \frac{-2\sigma_1^k(\delta_k r) + \sigma_2^k(\delta_k r)}{r},\\
\frac{d}{dr}\bar{v}_2^k(r) &= \frac{\sigma_1^k(\delta_k r) - 2\sigma_2^k(\delta_k r)}{r}, \qquad R_k \leq r \leq \tau_k\delta_k^{-1}.
\end{aligned}$$

Clearly we have

$$R_k\frac{d}{dr}\bar{v}_1^k(R_k) = -4\mu_1 + o(1), \qquad R_k\frac{d}{dr}\bar{v}_2^k(R_k) = 2\mu_1 + o(1). \tag{5-8}$$

The following lemma says that as long as both components stay well below the harmonic function $-2\log|y|$ (i.e., both of them are fast decay components), there is no essential change on the energy for either component:

Lemma 5.3. *Suppose $L_k \in (R_k, \tau_k\delta_k^{-1})$ satisfies*

$$v_i^k(y) + 2\gamma_i^k \log|y| \leq -2\log|y| - N_k, \qquad R_k \leq |y| \leq L_k, \qquad i = 1, 2, \tag{5-9}$$

for some $N_k \to \infty$, then

$$\sigma_i^k(\delta_k R_k) = \sigma_i^k(\delta_k L_k) + o(1), \qquad i = 1, 2.$$

Proof of Lemma 5.3. We aim to prove that σ_i^k does not change much from $\delta_k R_k$ to $\delta_k L_k$. Suppose this is not the case; then there exists i such that $\sigma_i^k(\delta_k L_k) > \sigma_i^k(\delta_k R_k) + \delta$ for some $\delta > 0$. Let $\tilde{L}_k \in (R_k, L_k)$ be such that

$$\max_{i=1,2}(\sigma_i^k(\delta_k\tilde{L}_k) - \sigma_i^k(\delta_k R_k)) = \epsilon \quad \text{for } i = 1, 2, \tag{5-10}$$

where $\epsilon > 0$ is sufficiently small. Then, for v_1^k,

$$\frac{d}{dr}\bar{v}_1^k(r) \leq \frac{-4(1+\gamma_1) + \epsilon}{r} \leq -\frac{2(1+\gamma_1) + \epsilon}{r}. \tag{5-11}$$

It is easy to see from Lemma 2.4 that

$$\int_{B_{\tilde{L}_k}\setminus B_{R_k}} |y|^{2\gamma_1^k} e^{v_1^k} = o(1),$$

which is $\sigma_1^k(\delta_k\tilde{L}_k)=\sigma_1^k(\delta_k R_k)+o(1)$. Indeed, by Lemma 2.4,

$$\int_{B_{L_k}\setminus B_{R_k}}|y|^{2\gamma_1^k}e^{v_1^k}=O(1)\int_{B_{L_k}\setminus B_{R_k}}|y|^{2\gamma_1^k}e^{\bar{v}_1^k}=o(1).$$

The second equality above is because, by (5-11),

$$\bar{v}_1^k(r)+2\gamma_1^k\log r\le -N_k-2\log R_k+\left(-2-\tfrac{1}{2}\epsilon\right)\log r,\qquad R_k\le r\le L_k.$$

Thus $\sigma_2^k(\delta_k\tilde{L}_k)=\sigma_2^k(\delta_k R_k)+\epsilon$. However, since (5-9) holds, by Remark 3.6 we have

$$\lim_{k\to\infty}(\sigma_1^k(\delta_k\tilde{L}_k),\sigma_2^k(\delta_k\tilde{L}_k))\in\Gamma.$$

The two points on Γ that have the first component equal to $2\mu_1$ are $(2\mu_1,0)$ and $(2\mu_1,2(\mu_1+\mu_2))$. Thus (5-10) is impossible. Lemma 5.3 is established. □

From Lemma 5.3 and (5-8) we see that, for $r\ge R_k$, either

$$v_i^k(y)+2\gamma_i^k\log|y|\le -2\log|y|-N_k,\qquad R_k\le|y|\le\tau_k\delta_k^{-1},\qquad i=1,2,\tag{5-12}$$

or there exists $L_k\in(R_k,\tau_k\delta_k^{-1})$ such that

$$v_2^k(y)+2\gamma_2^k\log L_k\ge -2\log L_k-C,\qquad |y|=L_k,\tag{5-13}$$

for some $C>0$, while, for $R_k\le|y|\le L_k$,

$$v_1^k(y)+2\gamma_1^k\log|y|\le -(2+\delta)\log|y|,\qquad R_k\le|y|\le L_k,\tag{5-14}$$

for some $\delta>0$. Indeed, from (5-8) we see that if the energy has to change, σ_2^k has to change first. L_k can be chosen so that $\sigma_2^k(\delta_k L_k)-\sigma_2^k(\delta_k R_k)=\epsilon$ for some $\epsilon>0$ small.

Lemma 5.4. *Suppose there exist $L_k\ge R_k$ such that (5-13) and (5-14) hold. For L_k, we assume $L_k=o(1)\tau_k\delta_k^{-1}$. Then there exist $\tilde{L}_k$ such that $\tilde{L}_k/L_k\to\infty$ and $\tilde{L}_k=o(1)\tau_k\delta_k^{-1}$ still holds. For $|y|=\tilde{L}_k$, we have*

$$v_i^k(y)+2(1+\gamma_i^k)\log|y|\le -N_k,\qquad |y|=\tilde{L}_k,\qquad i=1,2,\tag{5-15}$$

for some $N_k\to\infty$. In particular,

$$v_1^k(y)+2\left(1+\gamma_1^k+\tfrac{1}{4}\delta\right)\log|y|\le 0,\qquad |y|=\tilde{L}_k,\tag{5-16}$$

$$\sigma_1^k(\delta_k\tilde{L}_k)=2\mu_1+o(1),\qquad \sigma_2^k(\delta_k\tilde{L}_k)=2\mu_1+2\mu_2+o(1).\tag{5-17}$$

Remark 5.5. The statement of Lemma 5.4 can be understood as follows: Suppose, starting from ∂B_{L_k}, σ_2^k starts to change because (5-13) holds. Then, from L_k to $\tilde{L}_k$, σ_1^k does not change much and v_1^k is still far below $-2(1+\gamma_1^k)\log|y|$, but v_2^k has changed from decaying slowly (which is (5-13)) to a fast decay (the $i=2$ part of (5-16)). In other words, as σ_2^k changes from L_k to $\tilde{L}_k$, v_2^k changes from slow decay to fast decay but v_1^k still has fast decay in the meanwhile. The change of σ_2^k has influenced the derivative of $\bar{v}_1^k$, but has not made σ_1^k change much because σ_2^k changes too fast from L_k to $\tilde{L}_k$.

Proof of Lemma 5.4. First we observe that, by Lemma 5.3, the energy does not change if both components satisfy (5-12). Thus we can assume that $\sigma_2^k(\delta_k L_k) \le \epsilon$ for some $\epsilon > 0$ small. Consequently,

$$\frac{d}{dr}\bar{v}_1^k(r) \le \frac{-4(1+\gamma_1)+2\epsilon}{r}, \quad r \ge R_k.$$

Now we claim that there exists $N > 1$ such that

$$\sigma_2^k(\delta_k(L_k N)) \ge 2+\gamma_1+\gamma_2+o(1). \tag{5-18}$$

If this is not true, we would have $\epsilon_0 > 0$ and $\tilde{R}_k \to \infty$ such that

$$\sigma_2^k(\delta_k \tilde{R}_k L_k) \le 2+\gamma_1+\gamma_2-\epsilon_0. \tag{5-19}$$

On the other hand, $\tilde{R}_k$ can be chosen to tend to infinity slowly, so that, by Lemma 2.4 and (5-14),

$$v_1^k(y)+2(1+\gamma_1^k)\log|y| \le -\tfrac{1}{2}\delta\log|y|, \quad L_k \le |y| \le \tilde{R}_k L_k. \tag{5-20}$$

Clearly (5-20) implies $\sigma_1^k(\delta_k L_k) = \sigma_1^k(\delta_k \tilde{R}_k L_k)+o(1)$. Thus, by (5-19),

$$\frac{d}{dr}\bar{v}_2^k(r) \ge \frac{-2-2\gamma_2+\epsilon_0/2}{r}. \tag{5-21}$$

Using (5-21) and

$$v_2^k(y) = (-2-2\gamma_2^k)\log|y| + O(1), \quad |y| = L_k,$$

we see easily that

$$\int_{B(0,\tilde{R}_k L_k)\setminus B(0,L_k)} |y|^{2\gamma_2^k} e^{v_2^k} \to \infty,$$

a contradiction to (1-6). Therefore (5-18) holds.

By Lemma 2.4,

$$v_i^k(y)+2\log(NL_k) = \bar{v}_i^k(NL_k)+2\log(NL_k)+O(1), \quad |y| = NL_k, \quad i=1,2.$$

Thus we have

$$\bar{v}_1^k(NL_k) \le \bigl(-2-2\gamma_1^k-\tfrac{1}{2}\delta\bigr)\log(NL_k),$$
$$\bar{v}_2^k(NL_k) \ge (-2-2\gamma_2^k)\log(NL_k)-C.$$

Consequently,

$$\bar{v}_2^k((N+1)L_k) \ge (-2-2\gamma_2^k)\log L_k - C$$

leads to

$$\frac{1}{2\pi}\int_{B(0,(N+1)L_k)} h_2^k(\delta_k y)|y|^{2\gamma_2^k} e^{v_2^k(y)}\,dy \ge 2+\gamma_1+\gamma_2+\epsilon_0$$

for some $\epsilon_0 > 0$. Going back to the equation for $\bar{v}_2^k$, we have

$$\frac{d}{dr}\bar{v}_2^k(r) \le -\frac{2+2\gamma_2+\epsilon_0}{r}, \quad r = (N+1)L_k.$$

Therefore we can find $\tilde{R}_k \to \infty$ such that $\tilde{R}_k L_k = o(1)\tau_k \delta_k^{-1}$ and

$$\begin{aligned} v_2^k(y) &\le (-2-2\gamma_2^k-\epsilon_0)\log|y| - N_k, && |y| = \tilde{R}_k L_k,\\ v_1^k(y) &\le \bigl(-2-2\gamma_1^k-\tfrac{1}{4}\delta\bigr)\log|y|, && L_k \le |y| \le \tilde{R}_k L_k. \end{aligned}$$

Obviously,

$$\sigma_1^k(\delta_k \tilde{R}_k L_k) = \sigma_1^k(\delta_k L_k) + o(1) = \sigma_1^k(\delta_k R_k) + o(1) = 2(1+\gamma_1) + o(1).$$

By computing the Pohozaev identity on $\tilde{R}_k L_k$, we have

$$\sigma_2^k(\delta_k \tilde{R}_k L_k) = 2\mu_1 + 2\mu_2 + o(1).$$

Letting $\tilde{L}_k = \tilde{R}_k L_k$, we have proved Lemma 5.4. □

To finish the proof of Proposition 5.1, we need to consider the region $\tilde{L}_k \le |y| \le \tau_k\delta_k^{-1}$ if $L_k = o(1)\tau_k\delta_k^{-1}$ (in which case $\tilde{L}_k$ can be made to be $o(1)\tau_k\delta_k^{-1}$), or $L_k = O(1)\tau_k\delta_k^{-1}$. First we consider the region $\tilde{L}_k \le |y| \le \tau_k\delta_k^{-1}$ when $\tilde{L}_k = o(1)\tau_k\delta_k^{-1}$. It is easy to verify that

$$\begin{aligned} \frac{d}{dr}\bar{v}_1^k(r) &= -\frac{2\gamma_1 - 2\gamma_2}{r} + \frac{o(1)}{r}, && r = \tilde{L}_k,\\ \frac{d}{dr}\bar{v}_2^k(r) &= -\frac{6+2\gamma_1+4\gamma_2+o(1)}{r}, && r = \tilde{L}_k. \end{aligned}$$

The second equation above implies

$$\frac{d}{dr}\bar{v}_2^k(r) \le -\frac{2\mu_2+\delta}{r}, \quad r = \tilde{L}_k,$$

for some $\delta > 0$. So $\sigma_2^k(r)$ does not change for $r \ge \tilde{L}_k$ unless σ_1^k changes. By the same argument as before, either v_1^k rises to $-2\log|y| + O(1)$ on $|y| = \tau_k\delta_k^{-1}$, or there is $\hat{L}_k = o(1)\tau_k\delta_k^{-1}$ such that

$$\sigma_i^k(\delta_k \hat{L}_k) = 2\mu_1 + 2\mu_2 + o(1), \quad i = 1, 2.$$

Since this is the energy of a fully bubbling system, we have in this case both

$$v_i^k(y) \le -(2\mu_i + \delta)\log|y|, \quad |y| = \tau_k\delta_k^{-1}, \quad i = 1, 2,$$

for some $\delta > 0$.

If $L_k = O(1)\tau_k\delta_k^{-1}$, it is easy to use Lemma 2.4 to see that one component is $-2(1+\gamma_i^k)\log|y| + O(1)$ and the other component has the fast decay. Proposition 5.1 is established. □

6. Combination of bubbling areas

The following definition plays an important role:

Definition 6.1. Let $Q_k = \{p_1^k, \dots, p_q^k\}$ be a subset of Σ_k such that Q_k has more than one point in it and $\Sigma_k \setminus Q_k = \varnothing$. Q_k is called a group if:

(1) $$\operatorname{dist}(p_i^k, p_j^k) \sim \operatorname{dist}(p_s^k, p_t^k),$$

where $p_i^k, p_j^k, p_s^k, p_t^k$ are any points in Q_k such that $p_i^k \ne p_j^k$ and $p_t^k \ne p_s^k$.

(2) For any $p_k \in \Sigma_k \setminus Q_k$, $\mathrm{dist}(p_i^k, p_j^k)/\mathrm{dist}(p_i^k, p_k) \to 0$ for all $p_i^k, p_j^k \in Q_k$ with $p_i^k \neq p_j^k$.

Proof of Theorem 1.2. Let $2\tau_k$ be the distance between 0 and $\Sigma_k \setminus \{0\}$. For each $z_k \in \Sigma_k \cap \partial B(0, 2\tau_k)$, if $\mathrm{dist}(z_k, \Sigma_k \setminus \{z_k\}) \sim \tau_k$, let G_0 be the group that contains the origin. On the other hand, if there exists $z_k' \in \partial B(0, 2\tau_k)$ such that $\tau_k/\,\mathrm{dist}(z_k', \Sigma_k \setminus z_k') \to \infty$, we let G_0 be 0 itself. By the definition of a group, all members of G_0 are in $B(0, N\tau_k)$ for some N independent of k. Let

$$v_i^k(y) = u_i^k(\tau_k y) + 2\log \tau_k, \quad |y| \le \tau_k^{-1}.$$

Then we have

$$\Delta v_i^k(y) + \sum_{j=1}^{2} a_{ij} h_j^k(\tau_k y) e^{v_j^k(y)} = 4\pi \gamma_i^k \delta_0, \quad |y| \le \tau_k^{-1}. \tag{6-1}$$

Let $0, Q_1, \dots, Q_m$ be the images of members of G_0 after the scaling from y to $\tau_k y$. Then all $Q_i \in B_N$. By Proposition 5.1 and Proposition 5.2, at least one component decays fast on ∂B_1. Without loss of generality, we assume

$$v_1^k \le -N_k \quad \text{on } \partial B_1$$

for some $N_k \to \infty$, and

$$\sigma_1^k(\tau_k) = o(1),\ 2\mu_1 + o(1) \ \text{or}\ 2\mu_1 + 2\mu_2 + o(1).$$

Specifically, if $\tau_k \delta_k^{-1} \le C$, then $\sigma_1^k(\tau_k) = o(1)$. Otherwise, $\sigma_1^k(\tau_k)$ is equal to one of the two other cases mentioned above. By Lemma 2.4, $v_1^k \le -N_k + C$ on all $\partial B(Q_t, 1)$ $(t = 1, \dots, m)$; therefore, by Proposition 5.2,

$$\frac{1}{2\pi} \int_{B(Q_t,1)} h_1^k(\tau_k \cdot) e^{v_1^k} = 2m_t + o(1), \quad t = 1, \dots, m,$$

where, for each t, $m_t = 0$, 1 or 2. Let $2\tau_k L_k$ be the distance from 0 to the nearest group other than G_0. Then $L_k \to \infty$. By Lemma 2.4 and the proof of Lemma 3.2, we can find $\tilde{L}_k \le L_k$, $\tilde{L}_k \to \infty$, such that most of the energy of v_1^k in $B(0, \tilde{L}_k)$ is contributed by bubbles and v_2^k decays faster than $-2\log \tilde{L}_k$ on $\partial B(0, \tilde{L}_k)$:

$$\frac{1}{2\pi} \int_{B(0,\tilde{L}_k)} h_1^k(0) e^{v_1^k} = 2m + o(1),\ 2\mu_1 + 2m + o(1) \ \text{or}\ 2(\mu_1 + \mu_2) + 2m + o(1) \tag{6-2}$$

for some nonnegative integer m, and

$$v_2^k(y) + 2\log \tilde{L}_k \to -\infty, \quad |y| = \tilde{L}_k. \tag{6-3}$$

Then we evaluate the Pohozaev identity on $B(0, \tilde{L}_k)$. Since (6-3) holds, by Remark 3.6 we have

$$\lim_{k\to\infty} (\sigma_1^k(\tau_k \tilde{L}_k), \sigma_2^k(\tau_k \tilde{L}_k)) \in \Gamma.$$

Moreover, by (6-2) we see that $\lim_{k\to\infty}(\sigma_1^k(\tau_k \tilde{L}_k), \sigma_2^k(\tau_k \tilde{L}_k)) \in \Sigma$ because the limit point is the intersection between the line $\sigma_1 = \lim_{k\to\infty} \sigma_1^k(\tau_k \tilde{L}_k)$ and Γ.

The Pohozaev identity for $(\sigma_1^k(\tau_k\tilde{L}_k), \sigma_2^k(\tau_k\tilde{L}_k))$ can be written as

$$\sigma_1^k(\tau_k\tilde{L}_k)(2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k)-4\mu_1)+\sigma_2^k(\tau_k\tilde{L}_k)(2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k)-4\mu_2)=o(1).$$

Thus either

$$2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k)\geq 4\mu_1+o(1) \tag{6-4}$$

or

$$2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k)\geq 4\mu_2+o(1).$$

Moreover, if

$$2\sigma_1^k(\tau_k\tilde{L}_k)-\sigma_2^k(\tau_k\tilde{L}_k)\geq 2\mu_1+o(1) \quad \text{and} \quad 2\sigma_2^k(\tau_k\tilde{L}_k)-\sigma_1^k(\tau_k\tilde{L}_k)\geq 2\mu_2+o(1),$$

then, by the proof of Theorem 4.3,

$$\int_{B_{l_k}\setminus\tau_k\tilde{l}_k} h_i^k e^{u_i^k}=o(1), \quad i=1,2,$$

for any $l_k\to 0$. In this case we have

$$\sigma_i=\lim_{k\to\infty}\sigma_i^k(\tau_k\tilde{L}_k), \quad i=1,2,$$

and Theorem 1.2 is proved in this case.

Thus, without loss of generality, we assume that (6-4) holds. From the equation for u_1^k, this means that, for some $\delta>0$,

$$\bar{u}_1^k(\tau_k\tilde{L}_k)\leq -2\log(\tau_k\tilde{L}_k)-N_k, \quad \frac{d}{dr}\bar{u}_1^k(r)<\frac{-2-\delta}{r}, \quad r=\tau_k\tilde{L}_k. \tag{6-5}$$

The property above implies, by the proof of Proposition 5.1, that, as r grows from $\tau_k\tilde{L}_k$ to $\tau_k L_k$, the following three situations may occur:

Case one. Both u_i^k satisfy, for some $N_k\to\infty$, that

$$u_i^k(x)+2\log|x|\leq -N_k, \quad \tau_k\tilde{L}_k\leq|x|\leq\tau_k L_k, \quad i=1,2.$$

In this case,

$$\sigma_i^k(\tau_k\tilde{L}_k)=\sigma_i^k(\tau_k L_k)+o(1), \quad i=1,2.$$

So, on $\partial B(0,\tau_k L_k)$, u_1^k is still a fast decaying component.

Case two. There exist $L_{1,k}$ and $L_{2,k}\in(\tilde{L}_k, L_k)$ such that

$$\begin{aligned} u_2^k(x)&\geq -2\log(\tau_k L_{1,k})-C, \quad |x|=\tau_k L_{1,k},\\ u_i^k(x)&\leq -2\log(\tau_k L_{2,k})-N_k, \quad |x|=\tau_K L_{2,K}, \quad i=1,2, \end{aligned} \tag{6-6}$$

and

$$\sigma_1^k(\tau_k\tilde{L}_k)=\sigma_1^k(\tau_k L_{2,k})+o(1). \tag{6-7}$$

Since (6-6) holds, by Remark 3.6 we have $(\lim_{k\to\infty}\sigma_1^k(\tau_k L_{2,k}), \lim_{k\to\infty}\sigma_2^k(\tau_k L_{2,k}))\in\Gamma$. Then we further observe that, since (6-7) holds, $\lim_{k\to\infty}(\sigma_1^k(\tau_k L_{2,k}), \sigma_2^k(\tau_k L_{2,k}))\in\Sigma$, because this point is obtained by intersecting Γ with $\sigma_1=\lim_{k\to\infty}\sigma_1^k(\tau_k\tilde{L}_k)$. In other words, the new point $\lim_{k\to\infty}(\sigma_1^k(\tau_k L_{2,k}), \sigma_2^k(\tau_k L_{2,k}))$ is on the upper right part of the old point $\lim_{k\to\infty}(\sigma_1^k(\tau_k\tilde{L}_k), \sigma_2^k(\tau_k\tilde{L}_k))$.

Case three.
$$u_2^k(x)\ge -2\log\tau_k L_k - C, \quad |x|=\tau_k L_k,$$
for some $C>0$ and $\sigma_1^k(\tau_k\tilde{L}_k)=\sigma_1^k(\tau_k L_k)+o(1)$. This means that $\partial B(0,\tau_k L_k)$, u_1^k is still the fast decaying component.

If the second case above happens, the relationship between σ_1^k and σ_2^k on $B(0,\tau_k L_k)\setminus B(0,\tau_k L_{2,k})$ is the same as discussed before. In any case, on $\partial B(0,\tau_k L_k)$ at least one of the two components has fast decay and has its energy equal to a corresponding component of a point in Σ. For any group not equal to G_0, it is easy to see that the fast decay component has its energy equal to 0, 2 or 4. The combination of bubbles for groups is very similar to the combination of bubbling disks as we have done before. For example, let $G_0, G_1, \dots, G_t$ be groups in $B(0,\epsilon_k)$ for some $\epsilon_k\to 0$. Suppose the distances between any two of $G_0,\dots,G_t$ are comparable and
$$\mathrm{dist}(G_i,G_j)=o(1)\epsilon_k \quad \text{for all } i,j=0,\dots,t, \quad i\ne j.$$
Also we require $\big(\Sigma_k\setminus\big(\bigcup_{i=0}^t G_i\big)\big)\cap B(0,2\epsilon_k)=\varnothing$. Let $\epsilon_{1,k}=\mathrm{dist}(G_0,G_1)$; then all $G_0,\dots,G_t$ are in $B(0,N\epsilon_{1,k})$ for some $N>0$. Without loss of generality let u_1^k be a fast decaying component on $\partial B(0,N\epsilon_{1,k})$. Then we have
$$\sigma_1^k(N\epsilon_{1,k})=\sigma_1^k(\tau_k L_k)+2m+o(1),$$
where m is a nonnegative integer because, by Lemma 2.4, u_1^k is also a fast decaying component for $G_1,\dots,G_t$. Moreover, by Proposition 5.2, the energy of u_1^k in G_s $(s=1,\dots,t)$ is $o(1)$, $2+o(1)$ or $4+o(1)$. If u_2^k also has fast decay on $\partial B(0,N\epsilon_{1,k})$, then $\lim_{k\to\infty}(\sigma_1^k(N\epsilon_{1,k}), \sigma_1^k(N\epsilon_{1,k}))\in\Sigma$ because this is a point of intersection between Γ and $\sigma_1=\lim_{k\to\infty}\sigma_1^k(\tau_k L_k)+2m$. If
$$u_2^k(x)\ge -2\log N\epsilon_{1,k}-C, \quad |x|=N\epsilon_{1,k},$$
then, as before, we can find $\epsilon_{3,k}$ in $(N\epsilon_{1,k},\epsilon_k)$ such that, for some $N_k\to\infty$,
$$u_i^k(x)+2\log\epsilon_{3,k}\le -N_k, \quad |x|=\epsilon_{3,k}, \quad i=1,2,$$
and
$$\sigma_1^k(N\epsilon_{1,k})=\sigma_1^k(\epsilon_{3,k}).$$
Thus we have
$$\lim_{k\to\infty}(\sigma_1^k(\epsilon_{3,k}), \sigma_2^k(\epsilon_{3,k}))\in\Sigma,$$
because this point is the intersection between Γ and $\sigma_1=\lim_{k\to\infty}\sigma_1^k(N\epsilon_{1,k})$.

The last possibility on $B(0,\epsilon_k)\setminus B(0,\epsilon_{1,k})$ is
$$\sigma_1^k(\epsilon_k)=\sigma_1^k(N\epsilon_{1,k})+o(1)$$

and

$$u_2^k(x)+2\log\epsilon_k \geq -C, \quad |x|=\epsilon_k.$$

In this case, u_1^k is the fast decaying component on $\partial B(0,\epsilon_k)$.

Such a procedure can be applied to include groups further away from G_0. Since we have only finitely many blowup disks this procedure only needs to be applied finitely many times. Finally, let $s_k \to 0$ be such that

$$\sigma_i = \lim_{k\to\infty}\lim_{s_k\to 0}\sigma_i^k(s_k), \quad i=1,2,$$

and, for some $N_k \to \infty$,

$$u_i^k(x)+2\log s_k \leq -N_k, \quad |x|=s_k, \quad i=1,2.$$

Then we see that $(\sigma_1,\sigma_2)\in\Sigma$. Theorem 1.2 is established. □

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