ANALYSIS & PDE

Volume 8 No. 4 2015

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msp

ANALYSIS AND PDE
Vol. 8, No. 4, 2015
dx.doi.org/10.2140/apde.2015.8.839

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RICCI FLOW ON SURFACES WITH CONIC SINGULARITIES

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We establish short-time existence of the Ricci flow on surfaces with a finite number of conic points, all with cone angle between 0 and 2π, with cone angles remaining fixed or changing in some smooth prescribed way. For the angle-preserving flow we prove long-time existence; if the angles satisfy the Troyanov condition, this flow converges exponentially to the unique constant-curvature metric with these cone angles; if this condition fails, the conformal factor blows up at precisely one point. These geometric results rely on a new refined regularity theorem for solutions of linear parabolic equations on manifolds with conic singularities. This is proved using methods from geometric microlocal analysis, which is the main novelty of this article.

1. Introduction

This article studies the local and global properties of Ricci flow on compact surfaces with conic singularities. This is a natural continuation of various efforts, including recent work of Mazzeo and Sesum, to develop a comprehensive understanding of Ricci flow in two dimensions in various natural geometries. This work is also partly motivated by extensive recent efforts in higher-dimensional complex geometry toward finding Kähler–Einstein edge metrics with prescribed cone angle along a divisor, as approached by Mazzeo and Rubinstein using a stationary (continuity) method with features suggested by the Ricci flow, together with geometric microlocal techniques. A final motivation is the Hamilton–Tian conjecture, stipulating that Kähler–Ricci flow on Fano manifolds should converge in a suitable sense to a Kähler–Ricci soliton with mild singularities; we make some progress toward the analogue of this conjecture in our setting.

We investigate here the dynamical problem of Ricci flow on a Riemann surface (M, J), with conic singularities at a specified k-tuple of points $\vec{p}$, where the cone angle at p_j is $2\pi\beta_j$. Our main theorems provide optimal regularity for flow in this setting for cone angle smaller than 2π. We state these results, deferring explanation of the notations and terminology until later in the introduction and the next section.

MSC2010: 53C44, 58J35.
Keywords: Ricci flow, conic singularities, heat kernels.

Theorem 1.1. *Consider a set of conic data $(M, J, \vec{p}, \vec{\beta})$ with all $\beta_j \in (0, 1)$, and let g_0 be a $\mathcal{C}_b^{2,\gamma}$ conic metric compatible with this data (this regularity class is defined in Section 3B) with curvature $K_{g_0} \in \mathcal{C}_b^{0,\gamma}$. If $\beta_j(t) : [0, t_0] \to (0, 1)$ is a k-tuple of $\mathcal{C}^\infty$ functions with $\beta_j(0) = \beta_j$, then there exists a solution $g(t)$ of (2-1) defined on some interval $0 \le t < T \le t_0$ with conic singularities at the points p_j with cone angle $2\pi\beta_j(t)$ at time t. For $t > 0$, $g(t)$ is smooth away from the p_j and polyhomogeneous at these conic points, and satisfies $\lim_{t\searrow 0} g(t) = g_0$.*

The special case of this theorem when $\beta_j(t) \equiv \beta_j(0)$ is called the angle-preserving flow and is the two-dimensional case of a recent short-time existence result for the Yamabe flow with edge singularities by [Bahuaud and Vertman 2014]; the methods developed here to obtain the necessary bounds for the linear parabolic problem are somewhat more flexible than theirs and yield stronger estimates.

The key step in the proof of this short-time existence result is a new regularity statement for the linearized parabolic equation. This regularity is one of the main new technical contributions of this article.

Theorem 1.2. *Let $\beta \in (0, 1)$. Suppose that $(\partial_t - \Delta_g - V)u = f$, $u(0, \cdot\,) = \phi$, where $g, V, \phi \in \mathcal{C}_b^{0,\gamma}(M)$ and f lies in the parabolic regularity space $\mathcal{C}_b^{\gamma,\gamma/2}(M \times [0, T))$. Then, near each conic point,*

$$u = a_0(t) + r^{1/\beta}(a_{11}(t)\cos y + a_{12}(t)\sin y) + \mathcal{O}(r^2), \tag{1-1}$$

where $a_0, a_{ij}(t) \in \mathcal{C}^{1+\gamma/2}([0, T))$. When g, V, f and ϕ are all polyhomogeneous, then the solution u is polyhomogeneous on $[0, T) \times M$. If $\beta > 1$, a similar expansion holds but there exist additional terms of order $r^\delta(\log r)^k$ with $\delta \in (1/\beta, 2)$.

This refined regularity for solutions of singular parabolic equations seems to be new and requires some delicate analysis that is mostly contained in Propositions 3.6 and 3.9. We expect this type of estimate should be a standard tool in problems where such equations arise; see [Gell-Redman 2011] for a recent application.

We go beyond this short-time existence result only for the angle-preserving flow. Theorem 1.2 allows us to directly adapt Hamilton's method to get long-time existence of the normalized flow. Convergence, however, is more subtle. As we explain below, there is a set of linear inequalities (2-14), discovered by Troyanov, which is known to be necessary and sufficient for the existence of constant-curvature metrics with this prescribed conic data (for cone angles less than 2π).

Theorem 1.3. *Let $g(t)$ be the angle-preserving solution of the normalized Ricci flow from Theorem 1.1. Then $g(t)$ exists for all $t > 0$. If $\chi(M) \le 0$, or if $\chi(M) > 0$ and (2-14) holds, then $g(t)$ converges exponentially to the unique constant-curvature metric compatible with this conic data.*

In the remaining cases we have two parallel results.

Theorem 1.4. *Let $g(t)$ be the angle-preserving solution of the normalized Ricci flow, as above. Suppose that $\chi(M) > 0$ and (2-14) fails.*

- *Define $\psi(t)$ to be the t-dependent diffeomorphism generated by the vector field $\nabla f(t)$, where $\Delta f(t) = R_{g(t)} - \rho$ (where ρ is the average of R). Then $\hat{g}(t) := \psi^* g(t)$ satisfies $\partial\hat{g}(t)/\partial t = 2\hat{\mu}(t)$,* *where $\hat{\mu}$ is the tensor defined by (5-1) with respect to the metric $\hat{g}(t)$, and we prove that*

$$\lim_{t\to\infty}\int_M |\hat{\mu}(t)|^2_{\hat{g}(t)}\, d\hat{A}(t) = \lim_{t\to\infty}\int_M |\mu(t)|^2_{g(t)}\, dA(t) = 0.$$

Furthermore, the vector field $X = \nabla R + R\nabla f$ satisfies

$$\lim_{t\to\infty}\int_M |X(t)|^2_{g(t)}\, dA(t) = 0.$$

- *Returning to the unmodified normalized Ricci flow, and writing $g(t) = u(t,\cdot)g_0$, the conformal factor u blows up at precisely one point $q \in M$.*

The significance of the tensor μ and the vector field X, is that they both vanish on a Ricci soliton. It would be very interesting to connect the two different conclusions of this theorem.

Remark 1.5. It should be possible to show that there is a t-dependent family of conformal dilations $F(t)$ fixing the point of blowup of $u(t)$, and such that $F(t)^*g(t)$ converges (on every compact set $K \subset S^2\backslash\vec{p}$) to an eternal solution of normalized Ricci flow. One would hope to prove that the family of metrics $F(t)^*g(t)$ converges to a soliton metric, but, unfortunately, this does not seem to be possible with the present methods. It would also be quite interesting to identify the unique point of blowup of $u(t)$; the natural conjecture is that this blowup occurs at the unique conic point $p_j \in \vec{p}$ where the Troyanov condition fails.

We learned only in November 2014 of [Phong et al. 2014], where this conjecture is verified. The proof uses the machinery developed in the recent proof of the Yau–Tian–Donaldson conjecture.

Our goals are, first, to provide a clear and direct analytic treatment of the short-time existence for this problem, thus circumventing the approximation methods of [Yin 2010], and, second, to establish convergence to a constant-curvature metric when the Troyanov condition holds. This generalizes [Yin 2013], where only the negative case is handled. We assume throughout that all cone angles lie between 0 and 2π. As explained below, this restriction has significant geometric and analytical ramifications. The regularity theorem accounts for a substantial amount of the analysis here, and is one of our new innovations. Our methods provide a new approach for obtaining estimates for heat operators on conic spaces on the naturally associated Hölder spaces.

This article is organized as follows. In Section 2 we review some basic facts regarding the two-dimensional Ricci flow. The heart of the article, Section 3, develops the linear parabolic edge theory on Riemann surfaces. In particular, Sections 3A–3E review the relevant elliptic theory, based on the methods of [Mazzeo 1991; Jeffres et al. 2014], but emphasizing the simplifications that occur in this dimension compared to [Jeffres et al. 2014]. Building on this, Section 3F develops the corresponding parabolic regularity theory. Short-time existence, Theorem 1.1, is proved in Section 3G, while Section 3H contains Theorem 1.2 on the asymptotic expansion for solutions and the further results on higher regularity. The long-time existence of the flow is a fairly easy consequence of all of this and appears in Section 3J. The convergence result in the Troyanov regime is the subject of Section 4, while in Section 5 we study the complementary regime.

2. Preliminaries on Ricci flow

The normalized Ricci flow equation on surfaces is

$$\partial_t g(t) = (\rho - R(t, \cdot))g(t), \tag{2-1}$$

where R is the scalar curvature function of the metric $g(t)$ and ρ is the (time-independent!) average of the scalar curvature. For this choice of ρ, the area $A(M, g(t))$ remains constant in time. This flow preserves the conformal class of $g(t)$, so (2-1) can be written as a scalar equation for the conformal factor: if g_0 is the metric at $t = 0$ and $g(t) = u(t, \cdot)g_0$, then (2-1) is equivalent to

$$\partial_t u = \Delta_{g_0} \log u - R_{g_0} + \rho u, \quad u(0) \equiv 1. \tag{2-2}$$

This is the fundamental equation studied in this article.

2A. ***Miscellaneous formulae.*** In two dimensions, $\mathrm{Ric}(g) = \frac{1}{2} R\, g$, where R is the scalar curvature, so Ricci flow coincides with the Yamabe flow, and both are given by (2-1). This flow preserves the conformal class of the metric, and so can be written as a scalar parabolic equation. Indeed, if $g = e^{\phi} g_0$, then the scalar curvatures of these two metrics satisfy

$$\Delta_0 \phi - R_0 + R e^{\phi} = 0, \tag{2-3}$$

so, with $u = e^{\phi}$, (2-1) is equivalent to (2-2). (The reader should note that the conformal factor is often written $e^{2\phi}$ elsewhere, but this is compensated for here by the fact that $R = 2K$.)

If g_0 is any metric with finite Hölder regularity and isolated conic points, then its conformal class $[g_0]$ admits a representative $\bar{g}_0$ which is smooth on all of M. We can even assume that $\bar{g}_0$ is exactly Euclidean in a ball around each p_j. Fix any such metric, then choose a conformal factor $\phi_0 \in \mathcal{C}^\infty(M \setminus \{p_1, \dots, p_k\})$ which equals $(\beta_j - 1) \log r$ in Euclidean coordinates near each p_j. The metric $\tilde{g}_0 = e^{2\phi_0} \bar{g}_0$ is then smooth away from each p_j and has the exact conic form $dr^2 + \beta_j^2 r^2\, dy^2$ near p_j. Finally, write the metric g_0, the initial condition for the Ricci flow, as $u_0 \tilde{g}_0$. This encodes the finite regularity entirely in the conformal factor. Using this regular background conic metric $\tilde{g}_0$ allows for some technical simplifications in the presentation below. Henceforth we relabel $\tilde{g}_0$ as g_0, and then consider the initial value problem (2-2) with $u(0) = u_0$, assuming that g_0 is $\mathcal{C}^\infty$ on $M \setminus \{p_1, \dots, p_k\}$ and exactly conic near each p_j.

We record some other useful formulae. First, using (2-3) in (2-2) with $\phi = \frac{1}{2} \log u$ gives

$$\partial_t u = (\rho - R)u \iff \partial_t \log u = \rho - R. \tag{2-4}$$

Another formulation of the equations for the angle-fixing flow includes a distributional contribution from the cone points:

$$\partial_t \log u = \rho - R + 2\pi \sum (1 - \beta_i)\delta_{p_i}.$$

This conforms with a standard presentation in higher dimensions, but we primarily work with the equations (2-2) without the extra delta summands. Denoting the area form for $g(t)$ by dA, then

$$\frac{d}{dt} dA = (\rho - R)\, dA. \tag{2-5}$$

Consequently, the area $\mathcal{A}(t) := \int_M dA$ satisfies

$$\frac{d}{dt}\mathcal{A}(t) = \int_M (\rho - R)\, dA = \rho\mathcal{A}(t) - 4\pi\chi(M, \vec{\beta}),$$

so, if we now fix

$$\rho = \frac{4\pi\chi(M, \vec{\beta})}{\mathcal{A}(0)}, \tag{2-6}$$

then $\mathcal{A}(t) \equiv \mathcal{A}(0)$ for all t.

Note that, by (2-4), uniform bounds on $R(t)$ imply bounds and (at least subsequential) convergence for $\log u(t)$ as $t \nearrow \infty$. This means we can focus on the curvature function rather than the conformal factor. Differentiating (2-3), assuming that $g(t)$ is a solution of (2-1) on some interval $0 \le t < T$, we obtain

$$\partial_t R = \Delta_{g(t)} R + R(R - \rho). \tag{2-7}$$

When M is compact and smooth, (2-7) implies that the minimum of R is nondecreasing in t. Indeed, $R_{\min}(t) := \inf_M R(q, t)$ satisfies

$$\frac{d}{dt} R_{\min} \ge R_{\min}(R_{\min} - \rho).$$

Since $R_{min}(t)$ is only Lipschitz, the term on the left is defined as the limit infimum of the forward difference quotient of $R_{\min}(t)$. Since ρ is the average of R, $R_{\min} \le \rho$; hence, if $\rho \le 0$, then the right-hand side is nonnegative, and the claim about $R_{\min}$ being nondecreasing holds. If $\rho > 0$, then, choosing $r(t)$ so that $dr(t)/dt = r(t)(r(t) - \rho)$, $r(0) = R_{\min}(0)$, a similar argument applied to the difference $R_{\min} - r(t)$ leads to the same conclusion.

Estimating $R_{\max}$ is more difficult, especially when $R > 0$, and we discuss this later.

2B. ***Conic singularities.*** In two dimensions, there are two equivalent ways to describe conic singularities. The first is conformal: using a local holomorphic coordinate, we can write

$$g = e^{2\phi}|z|^{2\beta-2}|dz|^2, \tag{2-8}$$

where $\beta > 0$ and ϕ is a bounded function (with regularity to be specified later); the second is the polar coordinate model

$$g = dr^2 + r^2 h(r, y)^2 dy^2, \quad y \in S^1_{2\pi}, \tag{2-9}$$

where h is a strictly positive function with $h(0, y) = \beta$, again with regularity to be specified later. The equivalence between these two representations, at least in the model case where $\phi \equiv 0$ and $h \equiv 1$, is exhibited by writing $|dz|^2 = d\rho^2 + \rho^2 dy^2$, $y \in S^1_{2\pi} = \mathbb{R}/2\pi\mathbb{Z}$, and setting $r = \rho^\beta/\beta$, since then

$$dr = \rho^{\beta-1} d\rho \implies g = e^{2\phi}(dr^2 + \beta^2 r^2 dy^2).$$

The fact that more general conic metrics can be written in either of these two forms is considered in [Troyanov 1991]. We refer also to [Jeffres et al. 2014, §2.1] for a thorough discussion of this correspondence. Consequently, if g has a conic singularity at p, then the underlying conformal class $[g]$ extends smoothly across p, or, in other words, the conformal class $[g]$ determined by a conic metric contains a representative which is smooth across the conic points. (This holds for isolated conic singularities only in two dimensions, or, more generally, for nonisolated "edge" singularities in complex codimension one.)

It is also convenient to use

$$\alpha = \beta - 1,$$

and we refer to either α or β as the cone angle parameter, hopefully without causing confusion. We focus in this article exclusively on surfaces with conic singularities for which the equivalent conditions

$$2\pi\beta \in (0, 2\pi), \quad \beta \in (0, 1), \quad \alpha \in (-1, 0) \tag{2-10}$$

hold. There are good reasons for this restriction: for such cone angles, the uniformization results are definitive, and, in addition, conic surfaces with cone angles in this range have certain favourable geometric and analytic properties which are very helpful, and perhaps crucial, in certain parts of the analysis below. Related issues appear in [Jeffres et al. 2014].

2C. *Uniformization of conical Riemann surfaces.* Fix a smooth compact surface M, along with a conformal, or, equivalently, a complex structure J. Denote by $\vec{p} = \{p_1, \dots, p_k\} \subset M$ a collection of k *distinct* points, and let $\vec{\beta} = \{\beta_1, \dots, \beta_k\} \in (0, 1)^k$ be a corresponding set of cone angle parameters. As above, write $\alpha_j = \beta_j - 1$. The *conic Euler characteristic* associated to this data is the number

$$\chi(M, \vec{\beta}) = \chi(M) + \sum_{j=1}^{k} \alpha_j = \chi(M) + \sum_{j=1}^{k} \beta_j - k. \tag{2-11}$$

In the higher-dimensional language of [Jeffres et al. 2014], this is the twisted anticanonical class of the pair $\big(M, \sum(1-\beta_i)p_i\big)$, i.e., $-K_M - \sum(1-\beta_i)p_i$, where $K_M = T^{1,0,\star}M$ denotes the class of the canonical divisor of M.

The uniformization problem asks for the existence of a conic metric g compatible with the complex structure J with cone parameters β_j at p_j and with constant curvature away from these conic points. This can also be phrased in terms of the distributional equation

$$R_g - 2\pi \sum (1-\beta_i)\delta_{p_i} = \text{const}. \tag{2-12}$$

Indeed, in conformal coordinates, $R_g = -\Delta_g \log\gamma$ up to a constant factor, where $g = \sqrt{-1}\gamma\, dz \otimes \overline{dz} = \sqrt{-1}\gamma|dz|^2$, and the Poincaré–Lelong formula asserts that $-\Delta_g \log|z|$ is a multiple of the delta function at $\{z = 0\}$ (this can be seen by excising a small neighbourhood near the cone point and using Stokes' formula). Then (2-12) follows, since, for a conic metric, $\gamma = |z|^{2\beta-2}F$ near a cone point, with F bounded.

A consequence of this formulation is the Gauss–Bonnet theorem in this setting: if g is any metric with this conic data, then

$$2\pi\chi(M, \vec{\beta}) = \int_M K_g\, dA_g. \tag{2-13}$$

Therefore, if a constant-curvature metric with this conic data exists, then the sign of its curvature K_g agrees with the sign of $\chi(M) + \sum \alpha_i$. Note that, because of (2-10), this sign can be positive only when $M = S^2$ (or $\mathbb{R}P^2$, but for simplicity we always work in the oriented case).

The next theorem combines results of McOwen [1988; 1993] and Troyanov [Troyanov 1991] (the existence), Luo and Tian [1992] (uniqueness and nonexistence), and Jeffres, Mazzeo and Rubinstein [Jeffres et al. 2014] (higher regularity).

Theorem 2.1. *Let $(M, J, \vec{p}, \vec{\beta})$ be as above. Then there exists a conic metric with constant curvature associated to the data $(J, \vec{p}, \vec{\beta})$ if and only if either $\chi(M, \vec{\beta}) \leq 0$, in which case $\{\beta_j\} \in (0,1)^k$ can be arbitrary, or else $\chi(M, \vec{\beta}) > 0$ and, for each $j = 1, \dots, k$,*

$$\alpha_j > \sum_{i \neq j} \alpha_i \quad \text{or, equivalently,} \quad 2\alpha_j > \sum_{i=1}^{k} \alpha_i. \tag{2-14}$$

This metric, when it exists, is unique, except when $\chi(M, \vec{\beta}) = 0$, in which case it is unique up to a constant positive multiple, or when $M = S^2$ and there are no more than two conic singularities, in which case it is unique up to Möbius transformations which fix the cone points. Finally, the metric is polyhomogeneous with a complete asymptotic expansion of the form

$$g \sim \left(\sum_{j,k \geq 0} \sum_{\ell=0}^{N_{j,k}} a_{jk\ell}(y) r^{j+k/\beta} (\log r)^{\ell} \right) |z|^{2\beta-2} |dz|^2$$

The existence and regularity statements here were recently generalized to any dimension in [Jeffres et al. 2014, Theorems 1 and 2]; in that setting, the Troyanov condition is replaced by the coercivity of the twisted Mabuchi K-energy functional. Following [Ross and Thomas 2011], these conditions can also be reinterpreted as saying that the twisted Futaki invariant of the pair $\left(M, \sum(1-\beta_i)p_i\right)$ is nonnegative, or, equivalently, that this pair is logarithmically K-stable. The generalization of the uniqueness part of this result to higher dimensions has been accomplished by Berndtsson [2015]. Nonexistence when coercivity fails can be easily deduced from [Jeffres et al. 2014] together with work of Berman [2013]. We also remark that Berman's work gave a new proof of Troyanov's original results. We refer to [Rubinstein 2014] for a survey of the results mentioned in this paragraph and further references.

The rather curious linear inequalities (2-14) were discovered by Troyanov [1991, Theorem 5], and we refer to them henceforth as the *Troyanov conditions*. As just noted, they guarantee coercivity in the variational approach to this problem, which is key to proving existence, and which plays a key role in our considerations about the flow below. This coercivity is automatic when $\chi(M) \leq 0$, where simpler barrier methods suffice [McOwen 1988].

We also remark that, if $k > 2$, then (2-14) can fail for no more than one value of j. Indeed, if these inequalities fail for two distinct index values j, j', which we may as well take as $j = 1$ and $j' = 2$, then

$$\alpha_1 \leq \alpha_2 + \sum_{j=3}^{k} \alpha_i, \quad \alpha_2 \leq \alpha_1 + \sum_{j=3}^{k} \alpha_i \quad \Longrightarrow \quad 0 \leq \sum_{j=3}^{k} \alpha_i,$$

which is impossible since all the α_i are negative.

We discuss the cases $k = 1, 2$ separately. Using that a constant-curvature metric is rotationally symmetric near each conic point, we see that there can be no constant-curvature metric with only one conic point, while, if there are precisely two conic points, then the surface is globally rotationally symmetric, the cone angles are equal and the metric is the standard suspension $dr^2 + \beta^2 \sin^2 r\, dy^2$, $0 \le r \le \pi$. When $k \le 2$ and no constant-curvature metrics exist, there are well-known soliton metrics: the teardrop ($k = 1$ and any $\beta \in (0, 1)$) and the (American) football ($k = 2$ and any pair $0 < \beta_1 < \beta_2 < 1$). These can be obtained by ODEs methods; see [Hamilton 1988; Yin 2010; Ramos 2013]; Ramos's paper gives a particularly complete and incisive analysis.

The variational approach has recently been extended considerably through the work of Malchiodi et al. to allow angles bigger than 2π, even when coercivity fails; see, e.g., [Bartolucci et al. 2011; Carlotto and Malchiodi 2012]. Our regularity result, Theorem 1.2, holds for such angles, but our proofs of long-time existence and convergence do not carry over to that angle regime.

2D. *Optimal regularity.* We have already identified the central role of the refined regularity in Theorem 1.2. This result considerably sharpens the linear estimates proved by Jeffres and Loya [2003]. At the technical level, that paper establishes control on two "b-derivatives", i.e., with respect to the vector fields $r\partial_r$ and ∂_y, which vanish at the cone points, which imply only that $\partial_r u = O(r^{-1})$, for example. Our Theorem 1.2 shows that both $\partial_r u$ and $r^{1-1/\beta}\partial_r u$ are bounded. It also parallels the recent result [Jeffres et al. 2014, Proposition 3.3], which concerns the corresponding elliptic Poisson equation $\Delta_g u = f$ for the Laplacian of a Kähler edge metric g (generalizing the conic metrics considered here). This result in the elliptic case for smooth (or polyhomogeneous) edge metrics and with data lying in Sobolev spaces appears in [Mazzeo 1991].

These refined regularity statements represent basic phenomena associated to elliptic and parabolic edge operators. The fact that "singular" terms with noninteger exponents appear in solutions goes back to the work of Kondratiev and his school in the 1960s. However, since the methods and the particular choice of function spaces used here are less well known to geometric analysts, we pause to make some additional remarks. One key fact is that, even for the model (exact conic) case, if $\Delta_g u = f$ is Hölder continuous with respect to the metric g (i.e., defining Hölder seminorms using the distance determined by g), then it is *not* the case — unlike in the smooth setting — that all second derivatives of u are even bounded, let alone Hölder continuous. A basic example of this is the harmonic function $u = \operatorname{Re} z = r^{1/\beta}\cos y$, since, if $\frac{1}{2} < \beta < 1$, then $\partial_r^2 u \sim r^{1/\beta-2}$ blows up as $r \to 0$. The optimal regularity is that $[\partial_r u]_{g;0,1/\beta-1} < \infty$, where

$$[v]_{g;0,\gamma} = \sup \frac{|v(z) - v(z')|}{d_g(z, z')^\gamma}.$$

The results described above show that the phenomena in these examples provide the only mechanism through which control of second derivatives is lost. They also show that, if $\beta \in \left(0, \frac{1}{2}\right]$ (the easier "orbifold regime"), one has full control on the Hessian, since $1/\beta \ge 2$. One can obtain a slightly weaker statement using classical methods; see [Donaldson 2012]. As shown here, and in line with [Jeffres et al. 2014], one can go further by taking advantage of a detailed description of the structure of the Green function and heat kernel. Thus, we use here the so-called b-Hölder spaces $\mathcal{C}_b^{k,\gamma}$, which are defined using the slightly different seminorms

$$[v]_{b;0,\gamma} = \sup \frac{|v(z) - v(z')|(r + r')^\gamma}{d_g(z, z')^\gamma},$$

where $r = r(z)$ and $r' = r(z')$ are the g-distances of these respective points to the nearest conic points.

As already noted, [Bahuaud and Vertman 2014] contains a result similar to Theorem 1.1 for the higher-dimensional Yamabe flow for metrics with edges, while, as announced in [Mazzeo and Rubinstein 2012], direct analogues of Theorems 1.1 and 1.2 for the higher-dimensional Kähler–Ricci edge flow will appear in [Mazzeo and Rubinstein $\geq$ 2015].

2E. *Historical remarks.* The survey [Isenberg et al. 2011] provides a fairly recent account of what is known about Ricci flow on various classes of smooth surfaces, both compact and noncompact. The survey [Rubinstein 2014] reviews results on geometry and analysis related to Kähler edge metrics, including the special case of conic metrics on Riemann surfaces. The Ricci flow on conic surfaces presents several new challenges, some geometric and some analytic. For example, the uniformization problem in this setting is obstructed, in the sense that it is not always possible to find metrics of constant curvature in a given conformal class with certain prescribed cone angles. In addition, the flow starting at an initial singular surface is not uniquely defined: there are solutions which immediately smooth out the cone points [Simon 2002; Ramos 2011], and others which immediately become complete and send the cone points to infinity [Giesen and Topping 2010; 2011]. The solutions studied here, by contrast, either preserve the cone angles or allow them to change in some prescribed, smoothly varying manner. Our methods are drawn from geometric microlocal analysis, and are continuations of the elliptic methods used in [Jeffres et al. 2014; Mazzeo and Rubinstein 2012; $\geq$ 2015] to study the existence problem for Kähler–Einstein edge metrics. These provide very detailed information about the asymptotic behaviour of solutions near the conic points. Indeed, we have already noted that Theorem 1.2, concerning a regularity and asymptotics theorem for solutions of linear heat equations on manifolds with conic singularities, is a key ingredient, and should be useful elsewhere too.

The angle-preserving flow for Riemann surfaces with conic singularities was previously studied by Yin [2010]; his approach provides few details about the geometric nature of the solution and does not yield precise analytic or geometric control of the solution for positive time. More recently, in [Yin 2013], he establishes long-time existence of the normalized Ricci flow for conic surfaces, and proves convergence to a constant-curvature metric when the conic Euler characteristic (see Section 2C for the definition) is negative. However, he only establishes smooth convergence away from the conic points, and does not describe the precise limiting behaviour near these conic points. There is other work on this problem by Ramos, contained in his thesis but not yet released (see, however, [Ramos 2011; 2013]). Another related paper is [Bahuaud and Vertman 2014], which proves a short-time existence result for the Yamabe flow on higher-dimensional manifolds with edge singularities. Their methods are not far from the ones here, but our approach to regularity theory developed is simpler in many regards. Recently, Chen and Wang [2013] use quite different ideas to study the Kähler–Ricci flow on Kähler manifolds with edges.

We also mention the work of Rochon [2014], where a "propagation of polyhomogeneity" result is proved in the spirit of Theorem 1.2 but in the *complete* asymptotically hyperbolic setting; see also Albin, Aldana and Rochon [Albin et al. 2013], and also [Rochon and Zhang 2012] concerning a similar result in higher dimensions.

Finally, we make some remarks about the history of these results and of this particular work. The initial draft of this paper was completed in the Fall of 2011, though the work on it had started a few years before, and this material has been presented at conferences since then and announced in the survey article [Isenberg et al. 2011]. The appearance of this final draft was held up by other commitments of the authors, as well as our efforts to obtain the most incisive results possible. We now comment on the relationship between this work and other recent papers. These recent works include Yin's original [2010] paper and his very recent follow-up [2013]; these certainly have substantial overlap with the present work, although our more detailed treatment of the linear and nonlinear regularity theory should be useful in further and more refined investigations of this problem. In addition, some time ago we were informed that D. Ramos had obtained results on this problem, relying on the short-time existence results in [Yin 2010]. His work was done independently of ours and has many points of overlap as well, though we have not seen details beyond what is contained in [Ramos 2011; 2013]. We acknowledge some very interesting and helpful conversations with him, clarifying his work, shortly before this paper was initially posted. Finally, we mention the very recent paper by Chen and Wang [2013], which has made substantial inroads into the higher-dimensional Kähler–Ricci flow in the presence of edge singularities using rather different methods that do not give higher regularity, and the announcement of Tian and Zhang [2013] concerning the Hamilton–Tian conjecture in the smooth setting in dimension three.

3. Linear estimates and existence of the flow

We now review some of the basic theory of the Laplacian and its associated heat operator on manifolds with conic singularities. For brevity, we focus entirely on the two-dimensional case. The main part of this section is an extension of standard parabolic regularity estimates to this conic setting; the main goal is a refined regularity result which is necessary for understanding our particular geometric problem. These estimates also lead directly to a proof of short-time existence.

3A. ***Elliptic operators on conic manifolds.*** Let g be a metric on a compact two-dimensional surface M with a finite number of conic singularities; in fact, to simplify the discussion below, assume that there is only one conic point, p. Write $g = e^{\phi} g_0$, where g_0 is smooth and exactly conic near p. We now study some analytic properties of the operator $\Delta_g + V$, where g and V have some specified Hölder regularity. Since

$$(\Delta_g + V)u = (e^{-\phi}(\Delta_{g_0} + e^{\phi} V)u = f \implies (\Delta_{g_0} + e^{\phi} V)u = e^{\phi} f,$$

we may as well replace g by g_0 and the potential V by $e^{\phi} V$, and hence it suffices to study operators of the form $\Delta_g + V$, where g is smooth and exactly conic, and V satisfies an appropriate Hölder condition.

We use tools from geometric microlocal analysis to study elliptic operators on surfaces with cone points. As references for these results, see the monograph by Melrose [1993] and the articles of Mazzeo [1991], Gil, Krainer and Mendoza [Gil et al. 2006], and [Jeffres et al. 2014, §3] for a more extended expository review. This approach takes advantage of the approximate homogeneity of the Laplacian of a conic metric of the cone point, as well as the resulting approximate homogeneity of the Schwartz kernels of the corresponding Green functions. The strategy is to use these to obtain refined mapping properties of the operator, as well as regularity properties of its solutions.

In much of the following, it is convenient to replace the conic manifold M with a manifold with boundary $\widetilde{M}$ which is obtained by blowing up the conic point. This blowup procedure (which is described in more generality below) corresponds to introducing polar coordinates (r, y) around the conic point p and then replacing p by the circle $\{(0, y)\} = \{0\} \times S^1$ at $r = 0$. The space $\widetilde{M}$ is then given the smallest smooth structure for which these polar coordinate functions give a smooth chart.

3B. ***Function spaces.*** We first introduce various function spaces used later. The key to all these definitions is that it is advantageous to base them on differentiations with respect to the elements of $\mathcal{V}_b(\widetilde{M})$, the space of all smooth vector fields on $\widetilde{M}$ which are unconstrained in the interior but tangent to the boundary. In local coordinates, any element of this space is a linear combination, with $\mathcal{C}^\infty(\widetilde{M})$ coefficients, of the vector fields $r\partial_r$ and ∂_y. Natural differential operators are built out of these; for example, the Laplacian of an exactly conic metric with cone angle $2\pi\beta$ takes the form

$$\Delta_\beta = r^{-2}((r\partial_r)^2 + \beta^{-2}\partial_y^2)$$

near p, where $y \in S^1_{2\pi}$. In other words, up to the factor r^{-2}, this is an elliptic combination (sum of squares) of the basis elements of $\mathcal{V}_b$.

Now define

$$\mathcal{C}^k_b(\widetilde{M}) = \{u : V_1 \cdots V_\ell u \in \mathcal{C}^0(\widetilde{M}) \text{ for all } \ell \le k \text{ and } V_j \in \mathcal{V}_b(M)\}.$$

Because these spaces are based on differentiating by elements of $\mathcal{V}_b$, observe that $\mathcal{C}^k_b$ contains functions like $r^\zeta \psi(y)$, where $\psi \in \mathcal{C}^k(S^1)$ and $\operatorname{Re}\zeta > 0$. We also use the corresponding family of b-Hölder spaces $\mathcal{C}^{k+\delta}_b(\widetilde{M})$. The space $\mathcal{C}^\delta_b(\widetilde{M})$ consists of functions ϕ such that $\|\phi\|_{b,\delta} := \sup|\phi| + [\phi]_{b;\delta} < \infty$, where this Hölder seminorm is the ordinary one away from $\partial\widetilde{M}$, while, in a neighbourhood $\mathcal{U} = \{r < 2\}$,

$$[\phi]_{b,\delta,\mathcal{U}} = \sup_{(r,y)\neq(r',y')} \frac{|\phi(r, y) - \phi(r', y')|(r + r')^\delta}{|r - r'|^\delta + (r + r')^\delta |y - y'|^\delta}.$$

Observe that, if we decompose $\mathcal{U}$ into a union of overlapping dyadic annuli, $\bigcup_{\ell\ge0} A_\ell$, where each $A_\ell = \{(r, y) : 2^{-\ell-1} \le r \le 2^{-\ell+1}\}$, then this seminorm (for functions supported in $\mathcal{U}$) is equivalent to the supremum over ℓ of the Hölder seminorm on each annulus,

$$[\phi]_{\delta,\mathcal{U}} \approx \sup_{\ell\ge0} [\phi]_{\delta,A_\ell}. \tag{3-1}$$

Said differently, the seminorm can be computed assuming $\frac{1}{2} \le r/r' \le 2$. To verify this, simply note that, if $(r, y) \in A_\ell$ and $(r', y') \in A_{\ell'}$ with $|\ell - \ell'| \ge 2$, then

$$\frac{|r - r'|}{|r + r'|} \approx 1,$$

so that

$$\frac{|\phi(r, y) - \phi(r', y')|(r + r')^\delta}{|r - r'|^\delta + (r + r')^\delta |y - y'|^\delta} \le C \sup|\phi|$$

with C independent of ℓ and ℓ'.

We also let $\mathcal{C}_b^{k+\delta}(\widetilde{M})$ consist of the space of ϕ such that $V_1 \cdots V_\ell \phi \in \mathcal{C}_b^{\delta}(\widetilde{M})$ for all $\ell \leq k$, and where $V_j \in \mathcal{V}_b(\widetilde{M})$ for every j; finally, define $r^\gamma \mathcal{C}_b^{k+\delta}(\widetilde{M}) = \{\phi = r^\gamma \psi : \psi \in \mathcal{C}_b^{k+\delta}(\widetilde{M})\}$.

The intersection of all these spaces, $\bigcap_k \mathcal{C}_b^k(\widetilde{M})$, is the space of conormal functions, denoted by $\mathcal{A}(\widetilde{M})$. It contains the very useful subspace of polyhomogeneous functions, $\mathcal{A}_{\mathrm{phg}}$. By definition, $\mathcal{A}_{\mathrm{phg}}$ consists of all conormal functions which admit asymptotic expansions of the form

$$\phi \sim \sum_{\operatorname{Re}\gamma_j \nearrow \infty} \sum_{\ell=0}^{N_j} \phi_{j,\ell}(y) r^{\gamma_j} (\log r)^\ell.$$

Note that the conormality condition requires that each coefficient $\phi_{j,\ell}$ lies in $\mathcal{C}^\infty(S^1)$. As an important special case, $\mathcal{C}^\infty(\widetilde{M}) \subset \mathcal{A}_{\mathrm{phg}}(\widetilde{M})$, since smoothness corresponds to demanding that the exponents in the expansion above are all nonnegative integers, i.e., $\gamma_j = j$ and $N_j = 0$ for all $j \geq 0$. Finally, define $\mathcal{A}^0(\widetilde{M}) = \mathcal{A}(\widetilde{M}) \cap L^\infty$ and $\mathcal{A}^0_{\mathrm{phg}}(\widetilde{M}) = \mathcal{A}_{\mathrm{phg}}(\widetilde{M}) \cap L^\infty(M)$.

A metric g is $\mathcal{C}_b^{k+\delta}$, conormal, polyhomogeneous or smooth if $g = ug_0$, where the background metric g_0 is smooth and exactly conic, and where the function u satisfies any one of these regularity conditions.

3C. ***Mapping properties.*** Suppose that $L = \Delta_g + V$, where both g and V are polyhomogeneous (and V is real-valued). There is a canonical self-adjoint realization of this operator, which we still denote by L, defined via the Friedrichs construction associated to the quadratic form $\int |\nabla u|^2 - V|u|^2 \, dA_g$ and core domain $\mathcal{C}_0^\infty(M \setminus \{p\})$. It is well known that the Friedrichs domain of L obtained from this construction is compactly contained in L^2, so this operator has discrete spectrum. We let G denote its generalized inverse. As an operator on $L^2(\widetilde{M})$, this satisfies

$$\Delta_g \circ G = G \circ \Delta_g = \mathrm{Id} - \Pi, \tag{3-2}$$

where Π is the orthogonal projector onto the nullspace of L. Thus Π has finite rank, and a basic regularity theorem in the subject (see the references cited earlier) states that, if g and V are polyhomogeneous, then the range of Π, which is the nullspace of L, lies in $\mathcal{A}_{\mathrm{phg}}$. When $V \equiv 0$, we have $\operatorname{rank}(\Pi) = 1$ and Π projects onto the constant functions. We regard each of these integral operators as corresponding to a Schwartz kernel, which is an element of $\mathcal{D}'(\widetilde{M} \times \widetilde{M})$. The "integration", or distributional pairing, is taken with respect to the density dA_g. In local coordinates this equals $r\,dr\,dy$; the reader should note that this is *not* the standard b-density $r^{-1}\,dr\,dy$ that is commonly used in setting up the b-calculus. The differences are minor and notational only.

In this subsection we apply the theory of b-pseudodifferential operators to describe the fine structure of the Schwartz kernel of G. There are many reasons for wanting to know this structure, beyond the simplest statement that G is bounded on L^2. One example is that, once we know the pointwise structure of this Schwartz kernel, we can show that G and Π are bounded operators acting between certain weighted b-Hölder spaces. Since the equality of operators (3-2) remains true on these spaces as well, we deduce that the operator L is Fredholm between these weighted Hölder spaces as well as just on L^2 or Sobolev spaces. This is very helpful when studying nonlinear problems.

We are primarily interested in the mapping

$$L : \mathcal{C}_b^{2+\delta}(\widetilde{M}) \longrightarrow \mathcal{C}_b^{\delta}(\widetilde{M}). \tag{3-3}$$

This is unbounded because, for a general $u \in \mathcal{C}_b^{2+\delta}$, it need only be true that $\Delta_g u \in r^{-2}\mathcal{C}_b^{\delta}$. Thus the domain of (3-3) is

$$\mathcal{D}_b^{\gamma}(L) := \{u \in \mathcal{C}_b^{2+\delta}(\widetilde{M}) : Lu = f \in \mathcal{C}_b^{\delta}(\widetilde{M})\}, \tag{3-4}$$

which we call the *Friedrichs–Hölder domain* of L. This space is independent of the potential V. Indeed, if $u \in \mathcal{D}_b^{\delta}(L)$, then $\Delta_g u = f - Vu \in \mathcal{C}_b^{\delta}$, so $u \in \mathcal{D}_b^{\delta}(\Delta_g)$. Note finally that $\mathcal{D}_b^{\delta}(\Delta_g)$ is complete with respect to the Banach norm

$$\|u\|_{\mathcal{D}_b^{\delta}} := \|u\|_{\mathcal{C}_b^{\delta}} + \|\Delta_g u\|_{\mathcal{C}_b^{\delta}}.$$

An essentially tautological characterization of this space is that

$$\mathcal{D}_b^{\delta}(L) = \{u = Gf + w : f \in \mathcal{C}_b^{\delta} \text{ and } w \in \ker L \cap \mathcal{C}_b^{2+\delta}\}. \tag{3-5}$$

However, there is an even more explicit characterization of this space:

Proposition 3.1. *Suppose that $L = \Delta_g + V$ with $g, V \in \mathcal{C}_b^{\delta}$, and $u \in \mathcal{D}_b^{\delta}(L)$ satisfies $Lu = f \in \mathcal{C}_b^{\delta}(\widetilde{M})$. Then*

$$u = a_0 + (a_{11}\cos y + a_{12}\sin y)r^{1/\beta} + \tilde{u},$$

where a_0, a_{11}, a_{12} are constants and $\tilde{u} \in r^2\mathcal{C}_b^{2+\delta}$. (Note that the middle term on the right can be absorbed into $\tilde{u}$ if $\beta \le \frac{1}{2}$.)

To explain the relevance of the terms in this expansion, recall that, using the exactly conic structure of g near the conic points, we have that, if $\gamma \in \mathbb{R}$ and $\phi \in \mathcal{C}^{\infty}(S^1)$, then

$$\Delta_g r^{\gamma}\phi(y) = (\beta^{-2}\phi''(y) + \gamma^2\phi(y))r^{\gamma-2} \quad \text{and} \quad V r^{\gamma}\phi(y) = \mathcal{O}(r^{\gamma}).$$

Thus, in terms of its formal action on Taylor series, Δ_g is the principal part. The operator Δ_g has special locally defined solutions $r^{j/\beta}(a_{j1}\cos(jy) + a_{j2}\sin(jy))$, and the terms in the statement of this result are simply those special solutions with exponent less than 2.

The L^2 version of this proposition is a special case of Theorem 7.14 in [Mazzeo 1991], and it is not hard to deduce the corresponding statement in these b-Hölder spaces from that. We sketch a direct proof below in Section 3E.

Remark 3.2. The higher-dimensional version of this decomposition for solutions of Schrödinger-type equations on manifolds with edges plays a crucial role in [Jeffres et al. 2014].

3D. ***Structure of the generalized inverse.*** We now describe the detailed structure of G. First recall the definition of conormal and polyhomogeneous distributions. We say that u is conormal of order γ on $\widetilde{M}$, written $u \in \mathcal{A}^{\gamma}(\widetilde{M})$, if $V_1 \cdots V_{\ell} u \in r^{\gamma}L^{\infty}$ for every $\ell \ge 0$ and all $V_j \in \mathcal{V}_b(M)$. Such a u is smooth away from the conic points. Next, let E be an *index set*, i.e., a discrete subset $\{(\gamma_j, p_j)\} \subset \mathbb{C} \times \mathbb{N}_0$ such that there are only finitely many pairs with γ_j lying in any half-plane $\operatorname{Re} z < C$. We also assume that

$(\gamma_j, p_j) \in E$ implies that $(\gamma_j + \ell, p_j) \in E$ for every $\ell \in \mathbb{N}$. We then say that u is polyhomogeneous with index set E, written $u \in \mathcal{A}^E_{\mathrm{phg}}(M)$, if $u \in \mathcal{A}^\gamma(\widetilde{M})$ and

$$u \sim \sum_{(\gamma_j, p_j)\in E} \sum_{\ell \le p_j} a_{j\ell}(y) r^{\gamma_j} (\log r)^\ell,$$

where each $a_{j\ell} \in \mathcal{C}^\infty(S^1)$. Similarly, if X is any manifold with corners, then we can define the space of polyhomogeneous functions on X; these have the same type of asymptotic expansion at all boundary faces and product-type expansions at the corners of X.

The reason for introducing polyhomogeneity is that the Schwartz kernel G is polyhomogeneous, not on $(\widetilde{M})^2$, but rather on a certain manifold with corners $(\widetilde{M})^2_b$ which is obtained by blowing up $(\widetilde{M})^2$ along the codimension two corner $(\partial \widetilde{M})^2$. This new space has three boundary hypersurfaces: two are lifts of the faces $\partial \widetilde{M} \times \widetilde{M}$ and $\widetilde{M} \times \partial \widetilde{M}$ and called the left and right faces, lf and rf, respectively, and the third is the *front face*, ff, which is the one produced by the blowup. There is a natural blowdown map $\mathfrak{b} : (\widetilde{M})^2_b \to (\widetilde{M})^2$, and the precise statement is that $G = (\mathfrak{b})_* K_G$, where K_G is polyhomogeneous on $(\widetilde{M})^2_b$, with an additional conormal singularity along the lifted diagonal in $(\widetilde{M})^2_b$.

There are several useful coordinate systems on $(\widetilde{M})^2_b$. Using coordinates (r, y) near the boundary on the first copy of $\widetilde{M}$ and an identical set (r', y') on the second copy, this blowup is tantamount to introducing the polar coordinates $r = R\cos\theta$, $r' = R\sin\theta$ and replacing the corner $\{r = r' = 0\}$ by the hypersurface $\{R = 0,\ \theta \in [0, \pi/2]\}$. Thus lf corresponds to $\theta = \pi/2$, rf corresponds to $\theta = 0$, and the front face ff corresponds to $R = 0$. The lifted diagonal is the submanifold $\{\theta = \pi/4,\ y = y'\}$. If $\mathcal{E} = (E_{\mathrm{lf}}, E_{\mathrm{rf}})$ is a pair of index sets, the first for lf and the second for rf, then we say that a pseudodifferential operator A lies in the space $\Psi_b^{-\infty,r,\mathcal{E}}(\widetilde{M})$ if the lift K_A of its Schwartz kernel to $(\widetilde{M})^2_b$ lies in $\mathcal{A}^{r,\mathcal{E}}_{\mathrm{phg}}((\widetilde{M})^2_b)$, where the initial superscript r indicates that $K_A = R^{r-2} K'_A$, where K'_A is $\mathcal{C}^\infty$ up to the front face and is polyhomogeneous at the side faces with index sets E_{lf} and E_{rf}, respectively. Finally, $A \in \Psi_b^{m,r,\mathcal{E}}(\widetilde{M})$ if $K_A = R^{r-2}(K'_A + K''_A)$, where the first term lies in $\Psi_b^{-\infty,r,\mathcal{E}}$ and K''_A is supported in a small neighbourhood of the lifted diagonal, and in particular vanishes near $\mathrm{lf} \cup \mathrm{rf}$, has a conormal singularity of pseudodifferential order m along the lifted diagonal (so its Fourier transform on the fibres of the normal bundle to the lifted diagonal is a symbol of order $-2 + m$), and is smoothly extendible across the front face. The reason for the slightly odd normalization of the singularity along ff is to make the identity operator an element of $\Psi_b^{0,0,\varnothing,\varnothing}(M)$. Indeed, relative to the measure $r' dr' dy'$, the Schwartz kernel of Id is $r^{-1}\delta(r - r')\delta(y - y')$, and this lifts to $(\widetilde{M})^2_b$ as $R^{-2}\delta(\theta - \pi/4)\delta(y - y')$.

If g is a smooth conic metric and $\beta \notin \mathbb{Q}$, then the index set for the expansion of K_G at lb and rb is

$$E = \left\{ \left(\frac{j}{\beta} + \ell, 0 \right) : j,\ \ell \in \mathbb{N}_0,\ (j, \ell) \neq (0, 1) \right\}.$$

This excluded element $(0, 1)$ corresponds to requiring that the expansion not include the term $\log r$. If β is rational, or if g is only polyhomogeneous, then we are able to state that the generalized inverse G lies in $\Psi_b^{-2,2,E',E'}(M)$ for some index set E', which may contain extra terms, including log terms, high up in the index set; however, the initial part of this index set (and hence the exponents in the initial part of the expansion of any solution) up to order 2 remains the same. The fact that the index r in the general definition equals 2 for the particular kernel K_G turns out to be very helpful. This correspond to precisely the order of approximate homogeneity needed to compensate for the fact that the identity operator behaves like R^{-2} at the front face, and Δ_g is approximately homogeneous of order 2. The index sets of G at the left and right faces are equal to one another because G is a symmetric operator. The fact that E does not contain the term $(0, 1)$ is because G is the generalized inverse for the Friedrichs extension. It can also be verified by direct calculation that, in fact, E does not contain the element $(1, 0)$, for, if it did, then we could produce a polyhomogeneous element $u = Gf$ in the Friedrichs domain which contains a term $u_1(y)r$; this holds because $\Delta_g r = \mathcal{O}(r^{-1})$. We refer to [Jeffres et al. 2014, §3] for a more careful description of all of these facts.

Let us say that $A \in \Psi_b^{m,r,\mathcal{E}}$ is of nonnegative type if $m \le 0$, $r \ge 0$, all the terms (γ, s) in the index sets E_{lf} and E_{rf} are nonnegative and, if $(0, s)$ lies in either index set, then $s = 0$. Proposition 3.27 in [Mazzeo 1991] implies that, if A is of nonnegative type, then $A : \mathcal{C}_b^{0,\delta} \to \mathcal{C}_b^{0,\delta}$ is a bounded mapping.

3E. ***Mapping properties, revisited.*** We are now ready for:

Proof of Proposition 3.1. Rewrite $Lu = f$ as $\Delta_g u = f - Vu := \tilde{f} \in \mathcal{C}_b^\delta$. Let G denote the generalized inverse of the Friedrichs extension of Δ_g, so that $u = G\tilde{f} - \Pi u$; since Πu is a constant, we can concentrate on the first term.

Decompose the Schwartz kernel of G into a sum $G' + G''$, where G' is supported in a small neighbourhood of the lifted diagonal of $\widetilde{M}_b^2$ (and hence vanishes near $\mathrm{lf} \cup \mathrm{rf}$), and $G'' \in \mathcal{A}_{\mathrm{phg}}(\widetilde{M}_b^2)$; see Section 3F3, where the parabolic version of this decomposition is described more carefully. Since $G' \in \Psi_b^{-2,2,\varnothing,\varnothing}$, we can write $G' = r^2\hat{G}'$, where $\hat{G}' \in \Psi_b^{-2,0,\varnothing,\varnothing}$ and hence is nonnegative. Since $\hat{G}'\tilde{f} \in \mathcal{C}_b^{2+\delta}$, we obtain that $u' \in r^2\mathcal{C}_b^{2+\delta}$.

Turning now to u'', first observe that $r\partial_r$ and ∂_y lift to the left factor of $(\widetilde{M})_b^2$ as smooth vector fields on $\widetilde{M}_b^2$ that are tangent to all boundaries. It follows that $(r\partial_r)^j\partial_y^\ell G'' \in \Psi_b^{-\infty,2,0,0}$ for all $j, \ell \ge 0$, from which it follows that $u'' \in \mathcal{A}^0(\widetilde{M})$. Moreover, the initial part of the expansion G — and hence of G'' — at rf takes the form $A_0 r^0 + (A_{11}\cos y + A_{12}\sin y)r^{1/\beta} + \mathcal{O}(r^2)$, which means that the kernel $(r\partial_r - \beta^{-1})(r\partial_r) \circ G$ is not only of nonnegative type (and of pseudodifferential order $-\infty$), but in fact vanishes to order 2 at rf. Since G'' already vanishes to this order at ff, we can remove a factor of r^2, i.e., write $(r\partial_r - \beta^{-1})r\partial_r \circ G'' = r^2\hat{G}''$, where $\hat{G}''$ is of nonnegative type and smoothing. This means that

$$(r\partial_r)(r\partial_r - \beta^{-1})u'' \in r^2\mathcal{A}^0(\widetilde{M}).$$

Integrating in r gives that $u'' = a_0(y) + a_1(y)r^{1/\beta} + r^2\mathcal{A}^0$. Finally, since $\Delta_g u''$ is bounded, we conclude that a_0 is constant and $a_1(y) = a_{11}\cos y + a_{12}\sin y$, as claimed. ☐

We conclude this discussion with the following application of Proposition 3.1 to our geometric problem.

Proposition 3.3. *Let g_0 be a conic metric and suppose that its scalar curvature R_{g_0} lies in $\mathcal{C}_b^\delta$ and, in particular, is bounded near the conic points. If $g = e^\phi g_0$ is another conformally related metric, with $\phi \in \mathcal{C}_b^{2+\delta}$, then $R_g \in \mathcal{C}_b^\delta$ if and only if*

$$\phi = c_0 + r^{1/\beta}(c_{11}\cos y + c_{12}\sin y) + \tilde{\phi}, \quad \tilde{\phi} \in r^2\mathcal{C}_b^{2+\delta},$$

or, more succinctly, $\phi \in \mathcal{D}_b^\delta(\widetilde{M})$.

Proof. Apply the generalized inverse G for the Friedrichs extension of Δ_{g_0} to the curvature transformation equation

$$\Delta_{g_0}\phi = R_{g_0} - \tfrac{1}{2}R_g e^{\phi}$$

to get

$$\phi = \Pi\phi + G\big(R_{g_0} - \tfrac{1}{2}R_g e^{2\phi}\big).$$

Suppose now that $R_g \in \mathcal{C}_b^{\delta}$. The first term $\Pi\phi$ is just a constant, while, by Proposition 3.1, $G\big(R_{g_0} - \tfrac{1}{2}R_g e^{2\phi}\big)$ has an expansion up to order r^2.

On the other hand, if ϕ has an expansion as in the statement of this proposition, then $R_g \in \mathcal{C}_b^{\delta}$. □

Remark 3.4. The results in 3A–3E are special cases of the ones in [Jeffres et al. 2014, §3], which are proved for Kähler manifolds of arbitrary dimension. We have presented this material in some detail since the statements and proofs in the Riemann surface case are simpler than in higher dimensions, and also because the discussion above sets the stage for the derivation of the parabolic estimates, which occupies the remainder of this section.

3F. ***Parabolic Schauder estimates.*** We now turn to the parabolic problem, and in particular to the analogue of Proposition 3.1.

Let (M, g) be a smooth exactly conic metric with cone angle $2\pi\beta < 2\pi$, and set $L = \Delta_g + V$, where V is polyhomogeneous; later we relax this to assume that $V \in \mathcal{C}_b^{\delta}$. We are interested in the homogeneous and inhomogeneous problems

$$\begin{cases}(\partial_t - L)v = 0,\\ v(0, z) = \phi(z),\end{cases} \qquad \text{and} \qquad \begin{cases}(\partial_t - L)u = f,\\ u(0, z) = 0,\end{cases} \tag{3-6}$$

for which the solutions can be represented as

$$v(t, z) = \int_M H(t, z, z')\phi(z')\, dA_g(z'), \tag{3-7}$$

$$u(t, z) = \int_0^t \int_M H(t - t', z, z') f(t', z')\, dt'\, dA_g; \tag{3-8}$$

here H is the heat kernel associated to L. In order to study the regularity properties of the solution u, we describe a fine structure theorem for H, similar to the one for the Green function G above. This leads to a definition of parabolic weighted Hölder spaces, and finally a derivation of the estimates for solutions in these spaces. As in the previous section, we work exclusively with the Friedrichs extension of the Laplacian.

3F1. *Structure of the heat kernel.* Denote by g_β the complete flat conic metric $dr^2 + \beta^2 r^2\, dy^2$ and by Δ_β its Laplacian. The first observation is that the model heat operator $\partial_t - \Delta_\beta$ is homogeneous with respect to the dilation $(t, r, y) \mapsto (\lambda^2 t, \lambda r, y)$, $\lambda > 0$, and hence, if H_β is the heat kernel associated to (the Friedrichs realization of) Δ_β, then

$$H_\beta(\lambda^2 t, \lambda r, y, \lambda r', y') = \lambda^{-2} H_\beta(t, r, y, r', y'). \tag{3-9}$$

In fact, there are explicit expressions:

$$
\begin{aligned}
H_\beta(t,r,y,r',y') &= \frac{1}{\pi}\sum_{\ell=0}^{\infty}\biggl(\int_0^\infty e^{-\lambda^2 t} J_{\ell/\alpha}(\lambda r) J_{\ell/\alpha}(\lambda r')\lambda\, d\lambda\biggr)\cos\ell(y-y') \\
&= \sum_{\ell=0}^{\infty}\frac{1}{t}\exp\biggl(\frac{-(r^2+(r')^2)}{2t}\biggr) I_{\ell/\alpha}\biggl(\frac{rr'}{2t}\biggr)\cos\ell(y-y').
\end{aligned}
$$

These expressions are better suited for studying the action of H_β on L^2 Sobolev spaces than weighted Hölder spaces, so, just as for the operator G earlier, we describe this model heat kernel, and then the true heat kernel, using the language of blowups and polyhomogeneous distributions. This structure theory for the Laplacian on a conic space appears in [Mooers 1999], with basic mapping properties later determined by Jeffres and Loya [2003].

The function $H(t,z,z')$ is a distribution on $\mathbb{R}^+\times(\widetilde{M})^2$, but the key point is that its lift to the "conic heat space" $(\widetilde{M})^2_h$ is polyhomogeneous. This will be obvious for the model heat kernel H_β once we define $(\widetilde{M})^2_h$ and, conversely, starting from the ansatz that this lift is polyhomogeneous, we can construct (the lift of) H as a polyhomogeneous object by standard heat operator parametrix methods.

The conic heat space is defined, starting from $\mathbb{R}^+\times(\widetilde{M})^2$, through a sequence of blowups. The first step is to blow up the corner $r=r'=t=0$, with a parabolic homogeneity in the variable t, and, following that, to blow up the diagonal in $(\widetilde{M})^2$ at $t=0$. The first blowup is tantamount to introducing the parabolic spherical coordinates $\rho\geq 0$ and $\omega=(\omega_0,\omega_1,\omega_2)\in S^2_+=S^2\cap(\mathbb{R}^+)^3$, where

$$
\rho=\sqrt{t+r^2+(r')^2}\quad\text{and}\quad \omega=\biggl(\frac{t}{\rho^2},\frac{r}{\rho},\frac{r'}{\rho}\biggr).
$$

Thus ρ, ω, y, y' are nondegenerate local coordinates near the new face created by this first step. For the second blowup we use the coordinates

$$
R=\sqrt{t+|z-z'|^2},\quad \theta=\frac{z-z'}{R},\quad z',
$$

where z is any interior coordinate system and z' an identical chart on the second copy of $\widetilde{M}$. This sequence of blowups is summarized by the notation

$$
M^2_h := [\mathbb{R}^+\times\widetilde{M}^2;\{0\}\times(\partial\widetilde{M})^2,\{dt\};\{0\}\times\mathrm{diag}_{\widetilde{M}},\{dt\}].
$$

This manifold with corners has five boundary faces (see Figure 1): the left and right faces $\mathrm{lf}=\{\omega_2=0\}$ and $\mathrm{rf}=\{\omega_1=0\}$, which are the lifts of the faces $r'=0$ and $r=0$, respectively; the front face $\mathrm{ff}=\{\rho=0\}$; the temporal diagonal $\mathrm{td}=\{R=0\}$, which covers the diagonal at $t=0$, and bf, the original bottom face at $t=0$ away from the diagonal.

The construction in [Mooers 1999] shows that H is polyhomogeneous on $(\widetilde{M})^2_h$ with index set $E=\{(j/\beta,0): j\in\mathbb{N}_0\}$ at the left and right faces; note that these are exactly the same as the index sets for the Green function G at the corresponding faces. The kernel H vanishes to infinite order at bf, while at td it has an expansion in powers of R, starting with R^{-2} (in general, this is $R^{-\dim M}$). Finally, at ff it

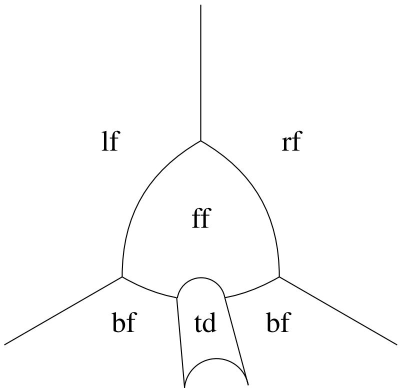

Figure 1. The faces of the manifold with corners.

has an expansion in integer powers of ρ, beginning with ρ^{-1}. The leading coefficient of the expansion at this face is precisely the model heat kernel H_β.

3F2. *Function spaces.* We now describe a family of function spaces commonly used in parabolic problems. We refer to [Lunardi 1995, Chapter 5] for a careful description of these (in the setting of interior and standard boundary problems). In the definitions and discussion below, we first introduce a scale of fully dilation-invariant spaces (jointly in the variables (t, r)), where the parabolic estimates are obtained by using scaling arguments to reduce to standard interior parabolic estimates. After that, we refine the estimates to obtain the maximal expected regularity in t.

First, for $0 < \delta < 2$, define $\mathcal{C}^{0,\delta/2}_{b0}([0, T] \times \widetilde{M})$ to consist of all $u \in \mathcal{C}^0([0, T] \times \widetilde{M})$ for which $u(\,\cdot\,, z) \in \mathcal{C}^{\delta/2}([0, T])$ for all $z \in \widetilde{M} \setminus \partial \widetilde{M}$ and

$$[u]_{b0;0,\delta/2} := \sup_z r^\delta [u(\,\cdot\,, z)]_{\delta/2,\,[0,T]} < \infty; \tag{3-10}$$

by contrast, the standard Hölder space in t, $\mathcal{C}^{0,\delta/2}([0, T] \times \widetilde{M})$ is defined using the usual seminorm

$$[u]_{0,\delta/2} := \sup_z [u(\,\cdot\,, z)]_{\delta/2,\,[0,T]}$$

(without the extra weight factor r^δ). Next, spatial regularity is measured using the spaces

$$\mathcal{C}^{\delta,0}_b([0, T] \times \widetilde{M}) = \{u \in \mathcal{C}^0([0, T] \times \widetilde{M}) : u(t, \cdot\,) \in \mathcal{C}^\delta_b(\widetilde{M}) \;\text{ for all }\; t \in [0, T]\},$$

where the norm is $\|u\|_{b;\delta,0} = \sup_t \|u(t, \cdot\,)\|_{b;\delta}$. We still let $0 < \delta < 2$, with the understanding that if $\delta = 1$ then this is the Zygmund space (so that interpolation arguments can be used). For simplicity below we omit discussion of this special case. Taking intersections yields the two natural parabolic Hölder spaces

$$\mathcal{C}^{\delta,\delta/2}_{b0}([0, T] \times \widetilde{M}) = \mathcal{C}^{0,\delta/2}_{b0}([0, T] \times \widetilde{M}) \cap \mathcal{C}^{\delta,0}_b([0, T] \times \widetilde{M}), \tag{3-11a}$$

$$\mathcal{C}^{\delta,\delta/2}_{b}([0, T] \times \widetilde{M}) = \mathcal{C}^{0,\delta/2}([0, T] \times \widetilde{M}) \cap \mathcal{C}^{\delta,0}_b([0, T] \times \widetilde{M}). \tag{3-11b}$$

Thus, functions in $\mathcal{C}_{b0}^{\delta,\delta/2}$ have no regularity in t at $r=0$, while functions in $\mathcal{C}_{b}^{\delta,\delta/2}$ satisfy the ordinary Hölder regularity in t even at $r=0$. The seminorms on these two spaces agree away from p, while, in a neighbourhood $\mathcal{U}$ of this conic point, these seminorms are described as follows. Decomposing $\mathcal{U}$ into a countable union of dyadic annuli, $\bigcup_{\ell\geq 0} A_\ell$, we have

$$[u]_{b0;\delta,\delta/2,\mathcal{U}} = \sup_{\ell\in\mathbb{N}_0} \sup_{|t-t'|<2^{-2\ell}} \sup_{z,z'\in A_\ell} \frac{|u(t,r,y)-u(t',r',y')|(r+r')^\delta}{|r-r'|^\delta + |t-t'|^{\delta/2} + (r+r')^\delta |y-y'|^\delta}$$

and

$$[u]_{b;\delta,\delta/2,\mathcal{U}} = \sup_{t,t'} \sup_{\ell\in\mathbb{N}_0} \sup_{z,z'\in A_\ell} \frac{|u(t,r,y)-u(t',r',y')|(r+r')^\delta}{|r-r'|^\delta + (r+r')^\delta(|t-t'|^{\delta/2} + |y-y'|^\delta)}.$$

These seminorms are equivalent to

$$\sup_{(t,z)\neq(t',z')} \frac{|u(t,z)-u(t',z')|\max\{r(z)^\delta, r'(z')^\delta\}}{|t-t'|^{\delta/2} + \mathrm{dist}_g(z,z')^\delta}$$

and

$$\sup_{(t,z)\neq(t',z')} \frac{|u(t,z)-u(t',z')|\max\{r(z)^\delta, r'(z')^\delta\}}{|t-t'|^{\delta/2}\max\{r(z)^\delta, r'(z')^\delta\} + \mathrm{dist}_g(z,z')^\delta},$$

respectively, where the radial function r has been extended from $\mathcal{U}$ to the rest of $\widetilde{M}$ to be smooth and strictly positive.

We also define higher regularity versions of these spaces,

$$\mathcal{C}_{b0}^{k+\delta,(k+\delta)/2}([0,T]\times\widetilde{M}) \quad\text{and}\quad \mathcal{C}_{b}^{k+\delta,(k+\delta)/2}([0,T]\times\widetilde{M}),$$

where k is an even positive integer and $0<\delta<2$. The former space consists of functions u such that $V_1\cdots V_i(r^2\partial_t)^j u \in \mathcal{C}_{b0}^{\delta,\delta/2}$ for $i+2j\leq k$, where every $V_\ell\in\mathcal{V}_b(\widetilde{M})$, while the latter consists of all u such that $V_1\cdots V_i\partial_t^j u\in\mathcal{C}_b^{\delta,\delta/2}$ for $i+2j\leq k$ and every $V_\ell\in\mathcal{V}_b(\widetilde{M})$. As before, these are Zygmund spaces when $\delta=1$. We also introduce weighted versions of these spaces, $r^\gamma\mathcal{C}_*^{k+\delta,(k+\delta)/2}$, $*=b0$ or b. For later reference, for the same ranges of k and δ, $\mathcal{C}^{0,(k+\delta)/2}([0,T]\times\widetilde{M})$ is the space of functions u with $\partial_t^j u\in\mathcal{C}^{0,\delta/2}([0,T]\times\widetilde{M})$ for $2j\leq k$.

Finally, we define the analogues of the Friedrichs–Hölder domain:

$$\mathcal{D}_*^{\delta,\delta/2}([0,T]\times\widetilde{M}) = \{u\in\mathcal{C}_*^{\delta,\delta/2} : \Delta u\in\mathcal{C}_*^{\delta,\delta/2}([0,T]\times\widetilde{M})\}, \quad *=b0 \text{ or } *=b,$$

again with the higher regularity analogues.

If $h(t,r,y)\in\mathcal{C}_{b0}^{k+\delta,(k+\delta)/2}$ is supported in $\mathbb{R}^+\times\mathcal{U}$, then the rescaled function $h_\lambda(t,r,y)=h(\lambda^2 t,\lambda r,y)$ satisfies

$$\|h_\lambda\|_{b0;k+\delta,(k+\delta)/2,\gamma} = \lambda^\gamma \|h\|_{b0;k+\delta,(k+\delta)/2,\gamma}$$

(the final subscript in the norms indicates the weight factor). In other words, these spaces are compatible with the approximate dilation invariance of the heat operator $\partial_t - L$, which means that we will be able to prove the basic a priori estimates on them by exploiting this scaling. On the other hand, it is clearly important to obtain better regularity of solutions in t near $r=0$. We obtain estimates on the b-spaces starting from the estimates on the $b0$-spaces and using induction and interpolation. Note that the analogue of (3-11b) is not true when $k > 0$; namely, there is a proper inclusion

$$\mathcal{C}_b^{k+\delta,(k+\delta)/2} \subsetneq \mathcal{C}_{b0}^{k+\delta,(k+\delta)/2} \cap \mathcal{C}^{0,(k+\delta)/2}, \quad k > 0.$$

3F3. *Estimates.* The basic Hölder estimates for the homogeneous problem were already determined by Jeffres and Loya [2003].

Proposition 3.5. *Suppose that $\phi \in \mathcal{C}_b^{k+\delta}(\widetilde{M})$ and*

$$(\partial_t - L)v = 0, \quad v|_{t=0} = \phi.$$

Then $v \in \mathcal{C}_b^{k+\delta,(k+\delta)/2}([0,T]\times\widetilde{M})$ and, furthermore, $v(t,\cdot) \in \mathcal{A}_{\mathrm{phg}}(\widetilde{M}) \cap \mathcal{D}_b^{0,\delta}(\widetilde{M})$ for all $t > 0$.

The proof in [Jeffres and Loya 2003] of the first assertion here proceeds by direct and rather intricate estimates in various local coordinate systems, but they do not consider the issue of membership in $\mathcal{D}_b^{0,\delta}$. The polyhomogeneity of v when $t > 0$ is immediate from the polyhomogeneous structure of H on M_h^2; also, $v \in \mathcal{D}_b^{0,\delta}$ implies that $v(t,\cdot) \sim c_0(t) + (c_{11}(t)\cos y + c_{12}(t)\sin y)r^{1/\beta}$ as $r \to 0$; using polyhomogeneity again, these coefficients are smooth when $t > 0$.

There are a couple of variants of the inhomogeneous problem, depending on the regularity assumptions placed on f. We start with the version in dilation-invariant spaces.

Proposition 3.6. *Let $f \in \mathcal{C}_{b0}^{k+\delta,(k+\delta)/2}([0,T]\times\widetilde{M})$ and suppose that u is the unique solution in the Friedrichs domain to $(\partial_t - L)u = f$, $u|_{t=0} = 0$. Then $u \in \mathcal{C}_{b0}^{k+2+\delta,(k+2+\delta)/2}([0,T]\times\widetilde{M})$ and*

$$\|u\|_{b0;k+2+\delta,(k+2+\delta)/2} \le C\|f\|_{b0;k+\delta,(k+\delta)/2}, \tag{3-12}$$

where C is a constant independent of u and f. In addition,

$$u(t,z) = \hat{u}(t,z) + \tilde{u}(t,z), \quad \text{where } \tilde{u} \in r^2\mathcal{C}_{b0}^{k+2+\delta,(k+2+\delta)/2}(\widetilde{M}) \text{ and } \hat{u}(t,z) \in \bigcap_{\ell\ge 0}\mathcal{C}_{b0}^{2\ell,\ell}.$$

The proof of this, which relies on the approximate homogeneity structure of H, adapts readily to the homogeneous case too, and gives a new proof of Proposition 3.5 which is conceptually simpler than the one in [Jeffres and Loya 2003].

Proof. Write u as in (3-8). We analyze this integral by decomposing H into a sum of two terms, as follows. Choose a smooth nonnegative cutoff function $\chi = \chi^{(1)}(\rho)\chi^{(2)}(\omega)$ on M_h^2, where $\chi^{(1)}(\rho)$ equals 1 for $\rho \le 1$ and vanishes for $\rho \ge 2$, and $\chi^{(2)}(\omega)$ has support in $\{\frac12 \le \omega_1/\omega_2 \le 2,\ \omega_0 \le \frac12\}$ and equals 1 near $(0, 1/\sqrt{2}, 1/\sqrt{2})$ (which is where the diagonal $\{t=0,\ r=r'\}$ intersects ff). Note that χ is (locally) invariant under the parabolic dilations $(t,r,y,r',y') \mapsto (\lambda^2 t, \lambda r, y, \lambda r', y')$. Then set

$$H = H_0 + H_1, \quad H_0 = (1-\chi(\rho,\omega))H, \quad H_1 = \chi(\rho,\omega)H,$$

and

$$u = u_0 + u_1, \quad u_j = H_j \star f, \quad j = 0,\ 1.$$

We study u_1 first. Introduce a partition of unity $\{\psi_\ell\}$ relative to the covering $\mathcal{U} = \bigcup A_\ell$; for example, take $\psi_\ell(r) = \psi(2^\ell r)$, where $\psi(r) \in \mathcal{C}_0^\infty\big(\big(\frac{1}{4}, 4\big)\big) \geq 0$ equals 1 for $\frac{1}{2} \leq r \leq 1$ and is chosen so that $\sum_{\ell \geq 0} \psi(2^\ell r) = 1$ for $0 < r \leq 1$. Now write

$$f = \sum f_\ell(t, r, y), \quad f_\ell = \psi_\ell f, \quad \text{and} \quad u_{1\ell} = H_1 \star f_\ell.$$

Thus f_ℓ has support in $\mathbb{R}^+ \times A_\ell$, while the support of $u_{1\ell}$ lies in $\mathbb{R}^+ \times (A_{\ell-1} \cup A_\ell \cup A_{\ell+1})$. We can also assume that f_ℓ is supported in some time interval $[\tau, \tau + 2^{2-2\ell}]$, since if $|t - t'| > (r + r')^2$ then the b-Hölder seminorm can be estimated by $C \sup |f_\ell|$. By the support properties of H_1, $u_{1\ell}$ is supported in a time interval of at most twice this length. We replace t by $t - \tau$ without further comment.

Fix $\ell \in \mathbb{N}_0$ and let $\lambda = 2^{\ell-1}$; for any function h, define $(D_\lambda h)(\bar{t}, \bar{r}, y) = h(\lambda^{-2}\bar{t}, \lambda^{-1}\bar{r}, y)$. Thus, if h is supported in A_ℓ, then $D_\lambda h$ is supported in $A_1 := \big\{(\bar{t}, \bar{r}, y) : \frac{1}{4} \leq \bar{r} \leq 1\big\}$. In particular, $D_\lambda f_\ell$ and $D_\lambda u_{1\ell}$ are supported in $[0, 1] \times A_1$ and $[0, 1] \times (A_0 \cup A_1 \cup A_2)$, respectively. We shall use that $\|D_\lambda u_{1\ell}\|_{b0;k+2+\delta,(k+2+\delta)/2} = \|u_{1\ell}\|_{k+2+\delta,(k+2+\delta)/2}$, and similarly for $D_\lambda f_\ell$.

For convenience in the next few paragraphs, drop the indices ℓ and 1, and simply write $D_\lambda u = u_\lambda$, $D_\lambda f = f_\lambda$. Since it also just complicates the notation, we also assume that $k = 0$. Using these conventions, change variables in $u = H_1 \star f$ by setting

$$\bar{t} = \lambda^2 t, \quad \hat{t} = \lambda^2 t', \quad \bar{r} = \lambda r, \quad \hat{r} = \lambda r'.$$

This yields

$$u_\lambda(\bar{t}, \bar{r}, y) = \int_0^{\bar{t}} \int \lambda^{-4} H_1(\lambda^{-2}(\bar{t} - \hat{t}), \lambda^{-1}\bar{r}, y, \lambda^{-1}\hat{r}, y') f_\lambda(\hat{t}, \hat{r}, y') \hat{r}\, d\hat{r}\, dy'\, d\hat{t}.$$

For simplicity we have replaced the measure $dA_g\, dt'$ in the initial integral by $r'\, dr'\, dy'\, dt'$.

The key point is that the polyhomogeneous structure of H_1 on M_h^2 implies that the family of dilated kernels

$$(H_1)_\lambda(\bar{t} - \hat{t}, \bar{r}, y, \hat{r}, y') := \lambda^{-2} H_1(\lambda^{-2}(\bar{t} - \hat{t}), \lambda^{-1}\bar{r}, y, \lambda^{-1}\hat{r}, y')$$

converges in $\mathcal{A}_{\mathrm{phg}}$ on the portion of the heat space with $\bar{r}, \hat{r} \in \big[\frac{1}{4}, 4\big]$ as $\lambda \to \infty$. In fact, its limit is simply the heat kernel for the model operator Δ_β on the complete warped product cone restricted to this range of radial variables. Since this region remains away from the vertex, we invoke the classical parabolic Schauder estimates to deduce that, as an operator between ordinary parabolic Hölder spaces, the norm of $(H_1)_\lambda$ restricted to functions supported in $[0, 1] \times (A_0 \cup A_1 \cup A_2)$ is uniformly bounded in λ. Hence, comparing the last two displayed formulae, we see that

$$\|u_\lambda\|_{b0;2+\delta,1+\delta/2} \leq C\lambda^{-2} \|f_\lambda\|_{b0;\delta,\delta/2} \implies \|r^{-2} u_\lambda\|_{b0;2+\delta,1+\delta/2} \leq C \|f_\lambda\|_{b0;\delta,\delta/2}$$

with C independent of λ. Restoring the indices, and using the fact that, analogous to (3-1),

$$\|h\|_{b0;k\delta,(k+\delta)/2} \approx \sup_\ell \|h\|_{b0;k+\delta,(k+\delta)/2}$$

for any function h and any $k \in \mathbb{N}_0$, we conclude finally that

$$\|r^{-2} u_1\|_{b0;2+\delta,1+\delta/2} \leq C \|f\|_{b0;\delta,\delta/2}, \tag{3-13}$$

so $u_1 \in r^2 \mathcal{C}_{b0}^{2+\delta,(1+\delta)/2}$.

We now turn to the estimate for $u_0 = H_0 \star f$, which is the same as the function $\hat{u}$ in the statement of the theorem. The polyhomogeneous structure of H_0 is slightly simpler than that for H; indeed, H_0 vanishes to infinite order not only along bf but along td as well. This means that H_0 is polyhomogeneous on the space obtained from M_h^2 by blowing down td. We first claim that $\|H_0 \star f\|_{\mathcal{C}^0} \leq C\|f\|_{\mathcal{C}^0}$. The proof reduces immediately to verifying that $\int_0^t \int_M H_0(t-s, r, y, r', y') r'\, dr'\, dy'\, ds \leq C$ independently of t, and this can be done by changing to polar coordinates in M_h^2 near ff to see that the integrand is actually bounded. Details are left to the reader. Since the vector fields $r^2\partial_t$, $r\partial_r$ and ∂_y lift to M_h^2 to be tangent to the side and front faces, and because of the infinite order vanishing along $t = 0$, the differentiated kernel $(r^2\partial_t)^i (r\partial_r)^j \partial_y^s H_0$ has the same polyhomogeneous structure as H_0 for any $i, j, s \in \mathbb{N}_0$. This means that $(r^2\partial_t)^i (r\partial_r)^j \partial_y^s u_0$ satisfies precisely the same estimates as u_0 does, whence $u_0 = \hat{u} \in \mathcal{C}_{b0}^{2\ell,\ell}$ for all $\ell \geq 0$, as claimed.

This discussion has focussed entirely on the behaviour of H near ff. This is because if we localize H by multiplying by a cutoff function which vanishes near ff and the side faces, then the estimates reduce to those for a standard local interior problem with no conic degeneracy. ☐

Remark 3.7. There is one other dilation-invariant vector field, namely $t\partial_t$, and it is natural to ask about the regularity of $t\partial_t u$ when $f \in \mathcal{C}_{b0}^{k+\delta,(k+\delta)/2}$. Write $t\partial_t = (t/r^2) r^2 \partial_t$, and note that, in the support of H_1, t/r^2 is a smooth bounded function; in addition, $t\partial_t$ is tangent to the front face of the heat space, and hence preserves the expansion of H_0. Taking these two facts together, we see that

$$(r\partial_r)^i (\partial_y)^j (r^2\partial_t)^\ell (t\partial_t)^m u \in \mathcal{C}_{b0}^{\delta,\delta/2}$$

provided $i + j + 2\ell + 2m \leq k + 2$. In particular, we see that u obtains more regularity in t than was initially apparent near $r = 0$ when $t > 0$.

The next estimate is for the Friedrichs–Hölder domain norm.

Proposition 3.8. *Suppose that $f \in \mathcal{C}_{b0}^{k+\delta,(k+\delta)/2}([0,T] \times \widetilde{M})$ and u is the unique solution to $(\partial_t - L)u = f$, $u|_{t=0} = 0$. Then u lies in the Friedrichs–Hölder domain $\mathcal{D}_{b0}^{k+\delta,(k+\delta)/2}$ and satisfies*

$$\|u\|_{\mathcal{D}_{b0}^{k+\delta,(k+\delta)/2}} := \|u\|_{b0;k+\delta,(k+\delta)/2} + \|\Delta_g u\|_{b0;k+\delta,(k+\delta)/2} \leq C\|f\|_{b0;k+\delta,(k+\delta)/2}. \tag{3-14}$$

Proof. We must estimate

$$\Delta_g u = \int_0^t \int_M \Delta_g H(t-t', z, z') f(s, z')\, dA_g\, dt'$$

in $\mathcal{C}_{b0}^{\delta,\delta/2}$. The key observation is that the Schwartz kernel K of $\Delta_g \circ H$ is an operator of heat type which we say is of "nonnegative type" (by analogy with the stationary case), and which therefore gives a bounded map of the spaces $\mathcal{C}_{b0}^{\delta,\delta/2}$. To be more specific, K is polyhomogeneous at all the faces of M_h^2, and the terms in its expansions at the left and right faces are nonnegative, while the leading terms at ff and td are $\rho^{-4} \cong t^{-2}$ and R^{-4}, respectively. To see this, note that Δ_g differentiates tangentially to the left face (where $r' \to 0$) so K has the same leading order as H there; at the right face ($r \to 0$), Δ_g annihilates the initial terms r^0 and $r^{1/\beta}\cos y$ and $r^{1/\beta}\sin y$ in the expansion of H, so the leading order of K is nonnegative here too; the leading orders exhibit the maximal drop in order to ρ^{-4} and R^{-2} at the other two faces because Δ_g is not tangent to these faces and acts as a second-order conic operator in (r, y), and the leading coefficients in the expansion of H there are not annihilated by this operator.

We now proceed as in the preceding proof, decomposing K into $K_0 + K_1$ and estimating the integrals corresponding to each. The details are almost exactly the same, except for two facts. First, the extra factor of $\lambda^{-2} = 2^{-2\ell}$ no longer appears when rescaling the terms $K_1 \star f_\ell$ because of the drop in leading order homogeneity (from ρ^{-2} to ρ^{-4}) at the front face. In addition, we appeal to the standard interior estimate $\|\Delta u\|_{\delta,\delta/2} \le C\|f\|_{\delta,\delta/2}$, where u and f are defined on the product of $[0, 1]$ with a ball of radius 1, Δ is a nondegenerate Laplacian on that ball, and, as usual, the norm on the left is only computed over a ball of radius $\frac{1}{2}$. A generalization of this interior estimate is that, if J is a kernel on the double heat space of $\mathbb{R}^2$ with compact support in all variables, and which vanishes to infinite order at $t = 0$ but blows up like t^{-2} at the new face, td, of the blowup, then $\|Jf\|_{\delta,\delta/2} \le C\|f\|_{\delta,\delta/2}$. The simpler integral estimate for $K_0 \star f$ is again essentially the same, since $\int K_0(t, z, z')\,dt\,dz'$ is still bounded as a function of z. This proves that $\|\Delta_g u\|_{b0;k+\delta,(k+\delta)/2} \le C\|f\|_{b0;k+\delta,(k+\delta)/2}$. □

We can now turn to the estimates in the b-spaces.

Proposition 3.9. *Suppose that $f \in \mathcal{C}_b^{k+\delta,(k+\delta)/2}([0, T] \times \widetilde{M})$ and u is the unique Friedrichs solution to $(\partial_t - L)u = f$, $u|_{t=0} = 0$. Then u lies in the Friedrichs–Hölder domain $\mathcal{D}_b^{k+\delta,(k+\delta)/2}$ and satisfies*

$$\|u\|_{b;k+2+\delta,(k+2+\delta)/2} \le C\|f\|_{b;k+\delta,(k+\delta)/2} \tag{3-15}$$

and

$$\|u\|_{\mathcal{D}_b^{k+\delta,(k+\delta)/2}} := \|u\|_{b;k+\delta,(k+\delta)/2} + \|\Delta_g u\|_{b;k+\delta,(k+\delta)/2} \le C\|f\|_{b;k+\delta,(k+\delta)/2}. \tag{3-16}$$

Moreover, $u = \hat{u} + \tilde{u}$, where $\tilde{u} \in r^2\mathcal{C}_b^{k+2+\delta,(k+2+\delta)/2}$ and

$$\hat{u}(t, z) = a_0(t) + (a_{11}(t)\cos y + a_{12}(t)\sin y)r^{1/\beta} \tag{3-17}$$

with $a_0, a_{11}, a_{12} \in \mathcal{C}^{1+\delta/2}([0, T])$.

Proof. First suppose that $k = 0$. We prove (3-16) using (3-11b). By Proposition 3.8, we already know that $u \in \mathcal{C}_{b0}^{2+\delta,1+\delta/2} \cap \mathcal{D}_{b0}^{\delta,\delta/2}$. Thus it suffices to show that u and $\Delta_g u$ lie in $\mathcal{C}^{0,\delta/2}$ as well. Defining $K = \Delta_g \circ H$, we first prove that

$$K\star : \mathcal{C}_{b0}^{\delta,\delta/2} \cap \mathcal{C}^{0,\ell} \longrightarrow \mathcal{C}_{b0}^{\delta,\delta/2} \cap \mathcal{C}^{0,\ell}$$

is bounded for $\ell = 0, 1$. For $\ell = 0$, observe first that if $f = C$ is constant then $K \star f \equiv 0$, since $H \star 1 = t$. This means that we may reduce to considering functions which vanish at $t = r = 0$. Next, if f vanishes near $t = r = 0$, then direct inspection of the integral defining $K \star f$ shows that this function also vanishes near $t = r = 0$; taking the closure in the $\mathcal{C}^0$ norm (or rather, the $\mathcal{C}^0 \cap \mathcal{C}_{b0}^{\delta,\delta/2}$) norm preserves the property of vanishing at $t = r = 0$. The case $\ell = 1$ follows by noting that ∂_t commutes with H and hence with K. By interpolation, we conclude the boundedness of

$$K\star : \mathcal{C}_{b0}^{\delta,\delta/2} \cap \mathcal{C}^{0,\delta/2} \longrightarrow \mathcal{C}_{b0}^{\delta,\delta/2} \cap \mathcal{C}^{0,\delta/2}.$$

This finishes the proof of (3-16).

To obtain (3-15) when $k=0$, we must show that $u\in\mathcal{C}_b^{2+\delta,1+\delta/2}$, or equivalently (in a neighbourhood of the conic point), that $(r\partial_r)^i\partial_y^j\partial_t^\ell u\in\mathcal{C}_b^{\delta,\delta/2}$ if $i+j+2\ell\le 2$. If $\ell=1$ (so $i=j=0$), we use that $\partial_t u=\Delta_g u+f\in\mathcal{C}_b^{\delta,\delta/2}$, as per the last paragraph. If $\ell=0$, we observe, as before, that $(r\partial_r)^i\partial_y^j\circ H$ is bounded on $\mathcal{C}_{b0}^{\delta,\delta/2}\cap\mathcal{C}^{0,\ell}$ for $\ell=0,1$, and hence, by interpolation, is bounded on $\mathcal{C}_b^{\delta,\delta/2}$.

Now suppose that k is a strictly positive even integer. We use induction, supposing that (3-16) and (3-15) have been proved for $0,2,\dots,k-2$. To prove that $K=\Delta_g\circ H\star$ is bounded on $\mathcal{C}_b^{k+\delta,(k+\delta)/2}$, we must show that $K_{i,j,\ell}:=(r\partial_r)^i\partial_y^j\partial_t^\ell\circ K\star:\mathcal{C}_b^{k+\delta,(k+\delta)/2}\to\mathcal{C}_b^{\delta,\delta/2}$ is bounded whenever $i+j+2\ell\le k$. There are three cases. First, if $1\le\ell\le k/2-1$, then $K_{i,j,\ell}:\mathcal{C}_b^{k+\delta,(k+\delta)/2}\to\mathcal{C}_b^{\delta,\delta/2}$ is bounded provided $K_{i,j,0}:\mathcal{C}_b^{k+\delta-2\ell,(k+\delta-2\ell)/2}\to\mathcal{C}_b^{\delta,\delta/2}$ is, and, since $i+j\le k-2\ell\le k-2$, this is known by induction. Next, if $\ell=k/2$ then, since $\partial_t^{k/2}\circ K=K\circ\partial_t^{k/2}$, we reduce directly to the boundedness of K on $\mathcal{C}_b^{\delta,\delta/2}$. Finally, when $\ell=0$, a bit more work is needed. If V is any b-vector field, we consider either the commutator $[V,H\star]$ or, more or less equivalently, the commutator $[V,\partial_t-\Delta]$. The latter is slightly more elementary, so we follow that route. Writing $g=e^\phi(dr^2+(1+\beta)^2r^2\,dy^2)$ near the conic point, it is easy to check that

$$[V,\Delta]=p\Delta+q+W,$$

where W is a second-order operator with coefficients supported away from $r=0$. Since the estimates we seek are standard in the support of W, we shall systematically neglect this term in the calculations below. For this part of the estimate we induct in integer steps, so, to unify the notation, assume that $k\in\mathbb{N}$ and $0<\delta<1$. Now, suppose that $f\in\mathcal{C}_b^{k+\delta,(k+\delta)/2}$ and that we have proved by induction that $u\in\mathcal{C}_b^{k+1+\delta,(k+1+\delta)/2}$ and $\Delta u\in\mathcal{C}_b^{k-1+\delta,(k-1+\delta)/2}$. We then compute that

$$(\partial_t-\Delta)Vu=Vf+(p\Delta+q)u\in\mathcal{C}_b^{k-1+\delta,(k-1+\delta)/2},$$

which implies that $Vu\in\mathcal{C}_b^{k+1+\delta,(k+1+\delta)/2}$ and $\Delta Vu\in\mathcal{C}_b^{k-1+\delta,(k-1+\delta)/2}$. Finally, $V\Delta u=\Delta Vu+(p\Delta+q)u$ is in $\mathcal{C}_b^{k-1+\delta,(k-1+\delta)/2}$. Since this is true for every b-vector field V, we conclude that $u\in\mathcal{C}_b^{k+2+\delta,(k+2+\delta)/2}$ and $\Delta u\in\mathcal{C}_b^{k+\delta,(k+\delta)/2}$, as required. This proves (3-16) and (3-15) in general.

It remains to study the expansion as $r\to 0$. We explain the case $k=0$ and leave the extension to spaces with higher regularity to the reader. Recalling the decomposition $H=H_0+H_1$ from the proof of Proposition 3.6, the same interpolation argument as earlier implies that

$$H_1\star:\mathcal{C}_b^{\delta,\delta/2}\longrightarrow r^2\mathcal{C}_b^{2+\delta,1+\delta/2}.$$

Next, similarly to what we did in the stationary (elliptic) case, note that $r\partial_r(r\partial_r-\beta^{-1})\circ H_0=r^2H_0'$, where H_0' has nonnegative index sets at $\mathrm{ff}\cup\mathrm{lf}\cup\mathrm{rf}$ (and vanishes to infinite order at td), which means that $r\partial_r(r\partial_r-\beta^{-1})u_0\in r^2\mathcal{C}_{b0}^{k,k/2}$ for all $k\ge 0$. Applying interpolation once more, this time for the mappings

$$(r\partial_r)^i\partial_y^j\partial_t^\ell r\partial_r(r\partial_r-\beta^{-1})H_0\star:\mathcal{C}_{b0}^{\delta,\delta/2}\cap\mathcal{C}^{0,m}\longrightarrow r^2\mathcal{C}_{b0}^{\delta,\delta/2}\cap\mathcal{C}^{0,m},$$

gives that $r\partial_r(r\partial_r-\beta^{-1})u_0\in r^2\mathcal{C}_b^{k+\delta,(k+\delta)/2}$ for every $k\ge 0$. Both this and the previous interpolation involving H_1 are complicated slightly by the fact that $[\partial_t,H_j\star]$ is no longer zero, but the extra terms can still be handled.

Finally, integrating in r gives that $u_0 = a_0(t, y) + a_1(t, y) r^{1/\beta} + \tilde{u}'$, where $\tilde{u}' \in r^2 \mathcal{C}_b^{2+\delta,1+\delta/2}$. Applying $(\partial_t - \Delta_g)$ to $u = u_0 + u_1$ shows first that $a_0 = a_0(t)$ and $a_1 = a_{11}(t)\cos y + a_{12}(t)\sin y$, and then that $a_0, a_{11}, a_{12} \in \mathcal{C}^{1+\delta/2}([0, T])$. □

Corollary 3.10. *Let u and f be as in Proposition 3.9. Then*

$$\|u\|_{b,k+\delta,(k+\delta)/2} \le CT \|f\|_{b,k+\delta,(k+\delta)/2}. \tag{3-18}$$

Proof. The inequality (3-18) is actually a formal consequence of (3-12) and (3-16). Indeed, since $u(0, z) = 0$,

$$u(t, z) = \int_0^t \partial_\tau u(\tau, z)\, d\tau \implies \|u\|_{b;\delta,0} \le \int_0^T \|\partial_\tau u(\tau, \cdot)\|_{b;\delta,0}\, d\tau \le T\|u\|_{b;2+\delta,1+\delta/2} \le CT\|f\|_{b;\delta,\delta/2}.$$

Similarly, since $\partial_\tau u(0, z) = \Delta_g u(0, z) = 0$,

$$\begin{aligned} |u(t, z) - u(t', z)| &\le \int_{t'}^t |\partial_\tau u(\tau, z)|\, d\tau = \int_{t'}^t |\partial_\tau u(\tau, z) - \partial_\tau u(0, z)|\, d\tau \\ &\le \|u\|_{b;2+\delta,1+\delta/2} \int_{t'}^t \tau^{\delta/2}\, d\tau \le C|t - t'| \cdot (|t + t'|^{\delta/2} + 1)\|u\|_{b;2+\delta,1+\delta/2} \end{aligned}$$

for some constant $C = C(\delta) > 0$, whence

$$[u]_{b;0,\delta/2} \le CT\|f\|_{b;\delta,\delta/2}.$$

Combining these two inequalities yields (3-18). □

We make a special note of the fact that the estimate (3-16) is the main one here, since both (3-15) and (3-18) follow from it.

Corollary 3.11. *Let g_0 be any smooth conic metric, and suppose that $g_1 = e^\phi g_0$ with $\phi \in \mathcal{C}_b^{k+\delta}(\widetilde{M})$, where $\phi = 0$ at $\partial\widetilde{M}$. For any $R_1 \in \mathcal{C}_b^{k+\delta}(\widetilde{M})$, i.e., not necessarily the scalar curvature of g_1, set $L_1 = \Delta_{g_1} + R_1$. Then the solution operator H_1 to $(\partial_t - L_1)u = f$, $u|_{t=0} = 0$, satisfies the same set of bounds (3-12), (3-14), (3-15), (3-16) and (3-18) (for that particular value of k, with constants depending only on g_0 and the norms $\|\phi\|_{b;k+\delta}$, $\|R_1\|_{b;k+\delta}$.*

Proof. We may as well absorb the term $R_1 u$ into f. Choose a function $\tilde{a} \in \mathcal{C}_b^{k+\delta}$ which agrees with e^ϕ in a small neighbourhood of $\partial\widetilde{M}$ and which is chosen uniformly close to 1 on the rest of $\widetilde{M}$, so that $\|(\tilde{a} - 1)\Delta_0 H_0\star\|_{b;k+\delta} < \epsilon$, where H_0 is the heat kernel for $\partial_t - \Delta_0$. Writing $\tilde{\Delta}_1 = \tilde{a}\Delta_0$,

$$(\partial_t - \tilde{\Delta}_1)H_0\star = \mathrm{Id} - (\tilde{a} - 1)\Delta_0 H_0\star;$$

by our choice of $\tilde{a}$, the right-hand side is invertible by Neumann series, so we may represent the heat kernel $\tilde{H}_1$ for $\tilde{\Delta}_1$ as

$$\tilde{H}_1 = H_0 \star (\mathrm{Id} - (\tilde{a} - 1)\Delta_0 H_0\star)^{-1}.$$

This shows that the solution $\tilde{u}$ to $(\partial_t - \tilde{\Delta}_1)\tilde{u} = f$ satisfies all the same estimates as the solution u to $(\partial_t - \Delta_0)u = f$, with constants depending only on the norm of ϕ.

Taking as given that the solution u exists, but may not satisfy the correct estimates near $\widetilde{M}$, observe that

$$(\partial_t - \tilde{\Delta}_1)(\tilde{u} - u) = b\Delta_0 u$$

for some function $b \in \mathcal{C}_b^{k+\delta}$ which vanishes in a fixed neighbourhood of the conic points. Noting that, by standard local parabolic regularity theory, u certainly satisfies the correct estimates on the support of b, we observe finally that

$$u = \tilde{u} - \tilde{H}_1 \star (b\Delta_0 u) = \tilde{H}_1 \star (f - b\Delta_0 u),$$

from which we again obtain all necessary estimates. It is clear that the constants depend on ϕ only through its norm $\|\phi\|_{b;k+\delta}$. □

3G. *Short-time existence.* We can now apply the mapping properties of the last section to establish the short-time existence for the angle-preserving solution of the flow (2-2). For this short-time result, we may as well assume that $\rho = 0$, and we consider the flow starting at any $\mathcal{D}_b^{k,\delta}$ metric g_0. Recall that this means that $g_0 = e^{w_0}\bar{g}_0$, where $\bar{g}_0$ is a smooth and exact conic and $w_0 \in \mathcal{D}_b^{k,\delta}$. Now let $g(t) = e^{\phi(t)}g_0$, so that (2-2) becomes

$$\begin{aligned}\partial_t\phi = e^{-\phi}\Delta_{g_0}\phi - R_0 e^{-\phi} &= (\Delta_{g_0} + R_0)\phi - R_0 + (e^{-\phi} - 1)\Delta_0\phi - R_0(e^{-\phi} - 1 + \phi) \\ &:= L\phi - R_0 + Q(\phi, \Delta_0\phi)\end{aligned} \tag{3-19}$$

with $\phi(0, \cdot) = 0$. By Corollary 3.11, the heat kernel H for $\partial_t - L$, $L = \Delta_{g_0} + R_{g_0}$, satisfies the same estimates as before.

Proposition 3.12. *Let g_0 be a $\mathcal{D}_b^{k,\delta}$ metric. Then there exists a unique solution $\phi \in \mathcal{D}_b^{k+\delta,(k+\delta)/2}([0, T] \times \widetilde{M})$ to (3-19) with $\phi|_{t=0} = 0$ for some $T > 0$ depending on the $\mathcal{D}_b^{k,\delta}$ norm of g_0.*

Proof. We suppose that $k = 0$, leaving the case of general k to the reader. The equation (3-19) is equivalent to the integral equation

$$\phi(t, z) = \int_0^t \int_M H(t - s, z, z')(Q(\phi, \Delta_0\phi)(s, z') - R_0(s, z'))\, ds\, dA_{z'}. \tag{3-20}$$

Denote the operator on the right by $\mathcal{T}(\phi)$. We claim that there are constants η and T so that the convex, closed set

$$\mathcal{J} = \left\{\phi \in \mathcal{D}_b^{\delta,\delta/2}([0, T] \times \widetilde{M}) : \|\phi\|_{b;\delta,\delta/2} + \|\Delta_0\phi\|_{b;\delta,\delta/2} \le \eta\right\}$$

is mapped to itself by $\mathcal{T}$ and, moreover, $\mathcal{T} : \mathcal{J} \to \mathcal{J}$ is a contraction.

For notational simplicity below, write

$$\|\phi\|_{b;\delta,\delta/2} + \|\Delta_0\phi\|_{b;\delta,\delta/2} := \|\phi\|_{\mathcal{D}}.$$

Denote by B the norm of $H\star : \mathcal{C}_b^{\delta,\delta/2} \to \mathcal{D}_b^{\delta,\delta/2}$; cf. Proposition 3.9. Writing $\Phi = H \star (-R_0)$, we then take $\eta = 2\|\Phi\|_{\mathcal{D}}$.

To proceed, recall first that, if $\chi \in \mathcal{C}_b^{\delta,\delta/2}$ vanishes at $t = 0$, then, for $0 \le t \le T$,

$$|\chi(t, z)| = |\chi(t, z) - \chi(0, z)| \le T^{\delta/2}\|\chi\|_{b;\delta,\delta/2},$$

and hence

$$[\chi_1\chi_2]_{b;\delta,\delta/2} \le \|\chi_1\|_\infty[\chi_2]_{b;\delta,\delta/2} + [\chi_1]_{b;\delta,\delta/2}\|\chi_2\|_\infty \le T^{\delta/2}\|\chi_1\|_{b;\delta,\delta/2}\|\chi_2\|_{b;\delta,\delta/2}.$$

Therefore,

$$\|(e^{-\phi}-1)\Delta_0\phi\|_{b;\delta,\delta/2} \le CT^{\delta/2}\|\phi\|_{b;\delta,\delta/2}\|\Delta_0\phi\|_{b;\delta,\delta/2},$$

where the constant C depends on η; hence,

$$\|Q(\phi,\Delta_0\phi)\|_{b;\delta,\delta/2} \le C_1T^{\delta/2}\eta^2.$$

Thus, if $\phi \in \mathcal{J}$, then

$$\|\mathcal{T}(\phi)\|_{\mathcal{D}} \le BC_1T^{\delta/2}\eta^2 + \|\Phi\|_{\mathcal{D}}.$$

By taking T sufficiently small, we can make this less than η again, so $\mathcal{T}$ maps $\mathcal{J}$ to itself.

By the same reasoning, adding and subtracting $(e^{-\phi_2}-1)\Delta_0\phi_1$ shows that

$$\|(e^{-\phi_1}-1)\Delta_0\phi_1 - (e^{-\phi_2}-1)\Delta_0\phi_2\|_{b;\delta,\delta/2} \le CT^{\delta/2}(\|\phi_1\|_{\mathcal{D}}+\|\phi_2\|_{\mathcal{D}})\|\phi_1-\phi_2\|_{\mathcal{D}}.$$

The identical estimate for the other term in $Q(\phi,\Delta_0\phi)$, which does not involve derivatives of the ϕ_j, is easier. We deduce that

$$\|\mathcal{T}(\phi_1)-\mathcal{T}(\phi_2)\|_{\mathcal{D}} \le BCT^{\delta/2}(2\eta)\|\phi_1-\phi_2\|_{\mathcal{D}},$$

so, by taking T still smaller, we can make the coefficient less than $\frac{1}{2}$. This proves that $\mathcal{T}$ is a contraction on $\mathcal{J}$, and hence that there exists a unique solution $\phi \in \mathcal{D}_b^{\delta,\delta/2}$ to (3-20) in $\mathcal{J}$. □

We now prove the short-time existence result for the angle-changing flow. Since this is a side note of the paper, we make some simplifying assumptions about the initial metric to remove some irrelevant details from the proof. We assume that the prescribed angle functions $\beta_i(t)$ are smooth functions of t, although the optimal result should allow these to have only finite Hölder regularity. Assume too that there is only one conic point, and that the initial metric g_0 is the exact conic metric $dr^2+\beta^2r^2\,dy^2$ near $r=0$. Reverting back to the conformal form of the metric, define

$$\hat{g}_0(t) = |z|^{2\beta(t)-2}|dz|^2.$$

We have $\hat{g}_0'(t) = 2\beta'(t)\log|z|\hat{g}_0(t)$, or, in terms of the (r,y) coordinates,

$$\hat{g}_0'(t) = \kappa\beta'(t)\log r\,\hat{g}_0(t), \qquad \kappa = \frac{2}{\beta}.$$

Setting $g(t) = u(t,\cdot)\hat{g}_0(t)$, the Ricci flow equation (with $\rho=0$) thus becomes

$$(\partial_t u + uC\kappa\beta'\log r) = \Delta_{\hat{g}_0(t)}\log u - R_{\hat{g}_0(t)},$$

or, finally, in terms of $\phi=\log u$,

$$\partial_t\phi = e^{-\phi}\Delta_{\hat{g}_0(t)}\phi - R_{\hat{g}_0(t)}e^{-\phi} - \kappa\beta'\log r. \tag{3-21}$$

We seek a local-in-t solution to this equation with initial value $\phi(0,\cdot)\equiv 0$.

Unlike the case considered before, the reference metric $\hat{g}_0(t)$ now depends on t, and there is an extra inhomogeneous term $-\kappa\beta'(t)\log r$. For the first issue we say nothing, because short-time existence for the heat operators associated to time-dependent metrics is standard; see [Chow et al. 2006]. Regarding the second issue, since this additional term is polyhomogeneous, we may choose a polyhomogeneous function $\hat{\phi}(t,\cdot)$ with leading term $C\kappa r^2\log r$ that satisfies

$$(\partial_t - e^{-\hat{\phi}}\Delta_{\hat{g}_0(t)})\hat{\phi} + R_{\hat{g}_0(t)}e^{-\hat{\phi}} = -\kappa\beta'(t)\log r + \chi,$$

where χ is smooth and vanishes to infinite order at $r=0$. Now set $\phi = \hat{\phi}+\psi$ and rewrite (3-21) as an equation for the unknown function ψ. It is straightforward to check that this equation is different from the one for the angle-fixing flow in only a few minor ways. There are additional terms in the coefficients of the nonlinear terms; these, however, are polyhomogeneous in (r, y, t) and vanish at least like $r^2\log r$. Next, there is an additional inhomogeneous term coming from the "error term" χ. The general structure of the equation is very similar to the one considered earlier in this section, and it is a straightforward exercise to check that this equation has a solution $\psi(t,\cdot)$ for $0\le t<T$ for T sufficiently small.

It is important to note that, unlike in the angle-changing flow, the fact that the conformal factor now includes a term $r^2\log r$ means that the curvature $R_{g(t)}$ is unbounded for $t>0$ near $r=0$. This is in accord with the results in the thesis of Ramos.

3H. ***Higher regularity.*** It will be very helpful for us later to be able to appeal to some higher regularity properties of the solution, so we prove these now.

Proposition 3.13. *Suppose that $g(t)$ is the solution to the Ricci flow equation with $g(t)=u(t)g_0$, where g_0 is smooth and exactly conic, $u(0)\in\mathcal{C}_b^{0,\delta}$, and $u\in\mathcal{D}_b^{\delta,\delta/2}$ is given by Proposition 3.12. Then u is polyhomogeneous on $(0,T)\times\widetilde{M}$.*

Proof. Write $u=e^{\phi}$ with ϕ satisfying (3-19) and $\phi(0)=\phi_0\in\mathcal{C}_b^{0,\delta}$. Since the initial condition is no longer zero, we have

$$\phi(t,z)=\int_M H_0(t,z,z')\phi_0(z')\,dA_{z'} + H_0\star(Q(\phi,\Delta_0\phi)-R_0).$$

The first term is polyhomogeneous when $t>0$ because of the polyhomogeneous structure of H_0. The second term lies in $\mathcal{C}_b^{2+\delta,1+\delta/2}$, so its restriction to any $t=\epsilon>0$ lies in $\mathcal{C}_b^{2,\delta}$. Consider the equation starting at $t=\epsilon$, i.e., replace t by $t+\epsilon$. Then Proposition 3.12 and the uniqueness of solutions shows that $u\in\mathcal{D}_b^{2+\delta,1+\delta/2}$ for $t\ge\epsilon$ and, since ϵ is arbitrary, this holds for all $t>0$. Bootstrapping in the obvious way gives that $u\in\mathcal{D}_b^{k+\delta,(k+\delta)/2}$ for every k, all in the same interval of existence $(0,T)$. In other words, $(r\partial_r)^j\partial_\theta^\ell\partial_t^r\Delta_0 u\in\mathcal{C}_b^{\delta,\delta/2}$ for all $j,\ell,s\ge 0$, which means that u is conormal when $t>0$.

From Proposition 3.6, $\phi=a_0(t)+r^{1/\beta}(a_{11}(t)\cos y+a_{12}(t)\sin y)+\tilde{\phi}$; by what we have just shown, $\tilde{\phi}\in r^2\mathcal{A}((0,T)\times\widetilde{M})$ and $a_0, a_{11}, a_{12}\in\mathcal{C}^\infty((0,T))$. In order to extend this expansion to all higher orders, assume g_0 is exactly conic (so $R_0\equiv 0$) in some neighbourhood of $r=0$ and write (3-19) there as

$$r^2\partial_t e^{\phi}=((r\partial_r)^2+\beta^{-2}\partial_y^2)\phi.$$

Since ϕ is conormal, we may study this formally. Taking advantage of information we have already obtained, inserting the expansion of ϕ to order 2 shows that the expression on the left has a finite expansion $r^2 a_0'(t) + r^{2+1/\beta}(a_{11}'(t)\cos y + a_{12}'(t)\sin y)$ and a conormal error term of order r^4. Using the operator on the right shows that ϕ must have an expansion up to order 4, with new terms of orders r^2 and $r^{2+1/\beta}$ as well as $r^{2/\beta}$ if $\beta > \frac{1}{2}$, with a conormal error term of order 4. Continuing in this way, we see that ϕ has an expansion to all orders, as claimed. □

Corollary 3.14. *Let $R(t)$ denote the curvature function of the solution metric $g(t)$. Then $R(t)$ is also polyhomogeneous on $(0, T) \times \widetilde{M}$, and the initial terms in its expansion have the form*

$$R(t) \sim b_0(t) + r^{1/\beta}(b_{11}(t)\cos y + b_{12}(t)\sin y) + \mathcal{O}(r^2).$$

In particular, $\Delta_0 R$ is bounded and polyhomogeneous for all $t > 0$.

Proof. This follows directly from the polyhomogeneity of ϕ and equation (2-4). □

3I. ***Maximum principles.*** Before embarking on the remainder of the proof of long-time existence and convergence, we present some results which show how the maximum principle may be extended to this conic setting. We adapt the trick of [Jeffres 2005].

The possible difficulty in applying the maximum principle directly is if the maximum of the solution were to occur at a conic point, so the idea is to perturb the solution slightly to ensure that the maximum cannot occur at the singular locus.

Lemma 3.15. *Suppose that $(M, g(t))$ is a family of metrics which is in $\mathcal{D}_b^{\delta,\delta/2}([0, T) \times \widetilde{M}$, polyhomogeneous on $(0, T) \times \widetilde{M})$, and that w satisfies*

$$\partial_t w \geq \Delta w + X \cdot \nabla w + a(w^2 - A^2),$$

where X and a are a given vector field and function, respectively, with the same regularity as $g(t)$ and with $a > 0$; here $A \geq 0$ is a constant. Suppose too that $w(0, \cdot) \geq -A$ and that $\sup\big(|w(t, \cdot)| + r^\sigma |\nabla w(t, \cdot)|\big) < \infty$ for every $t > 0$, where $0 < \sigma < 1$. Then $w \geq -A$ for all $t < T$.

Proof. Define $w_{\min}(t) = \inf_{q \in \widetilde{M}} w(t, q)$. By hypothesis, $w_{\min}(0) \geq -A$. Suppose that, at some time $t > 0$, this minimum is achieved at some point q. If q is not one of the conic points, then $\Delta w(t, q) \geq 0$ and $\nabla w(t, q) = 0$; hence

$$\frac{d}{dt} w_{\min}(t) \geq a(w_{\min}(t)^2 - A^2). \tag{3-22}$$

Suppose for the moment that we have established this differential inequality regardless of the location of the minimum. But then, if $w_{\min}(t)$ were ever to achieve a value less than $-A$ at some $t_0 > 0$, (3-22) would give that $w_{\min}'(t_0) > 0$, which is impossible (if $t_0 > 0$ is the smallest time at which $w_{\min}(t_0) < -A$).

Thus it suffices to show that (3-22) is always true. Fix γ with $0 < \gamma < 1 - \sigma$. Then, for any $k \geq 1$, define $w_k(q, t) = w(q, t) - r^\gamma / k$ (where r is a fixed radial function near each conic point such that r is smooth and strictly positive in the interior and $r = 0$ at a conic point). Suppose that $w_{\min}(t)$ is achieved at some conic point p. We first observe that, for q sufficiently near p, using the hypothesis on $|\nabla w|$,

$$w(t, q) \leq w_{\min}(t) + Cr^{1-\sigma} = w(t, p) + Cr^{1-\sigma},$$

where $r = r(q)$, and hence

$$w_k(t,q) \le w(t,p) + Cr^{1-\sigma} - \frac{1}{k}r^\gamma < w(t,p) = w_k(t,p)$$

for r sufficiently small. In other words, $(w_k)_{\min}(t)$ cannot occur at p. Now, the differential inequality satisfied by w_k is

$$\partial_t w_k \ge \Delta\Big(w_k + \frac{1}{k}r^\gamma\Big) + X\cdot\nabla\Big(w_k + \frac{1}{k}r^\gamma\Big) + a\Big(\Big(w_k + \frac{1}{k}r^\gamma\Big)^2 - A^2\Big).$$

At a spatial minimum (away from the conic point), $\Delta w_k \ge 0$ and $\nabla w_k = 0$. On the other hand, $\Delta r^\gamma \ge C r^{\gamma-2}$ and $|X\cdot\nabla r^\gamma| \le C r^{\gamma-1}$ near $r=0$, and, since the first of these terms is positive, these two terms together satisfy

$$\frac{1}{k}(\Delta r^\gamma + X\cdot\nabla r^\gamma) \ge \frac{C}{k}.$$

Thus altogether, applying the same reasoning as before (and using that $(w_k)_{\min}$ does not occur at a conic point), we deduce that

$$\frac{d}{dt}(w_k)_{\min} \ge \frac{C}{k} + a\Big(\Big((w_k)_{\min} + \frac{1}{k}r(q_k(t))^\gamma\Big)^2 - A^2\Big),$$

where the minimum of w_k is achieved at $q_k(t)$. The same arguments as above give $(w_k)_{\min} \ge -A - C'/k$, and hence $w_{\min} \ge -A - C''/k$. Letting $k \nearrow \infty$ proves the result. □

Essentially the same proof gives the following version of the maximum principle:

Lemma 3.16. *Suppose that the setup is exactly the same as in the previous lemma, and that*

$$\partial_t w = \Delta w + aw^2 + bw.$$

Then

$$\frac{d}{dt}w_{\max} \le a w_{\max}^2 + b w_{\max} \quad \textit{and} \quad \frac{d}{dt}w_{\min} \ge a w_{\min}^2 + b w_{\min}. \tag{3-23}$$

3J. ***Long-time existence.*** We are finally able to complete the proof of long-time existence of the solution of the Ricci flow with prescribed conic singularities. In fact, the proof is a straightforward adaptation of the original proof of this same fact for the Ricci flow on smooth compact surfaces in [Hamilton 1988]. We refer to that article as well as [Isenberg et al. 2011] for all the details of the proof. We supply here only the key results which then allow the proofs in those articles to be applied verbatim.

The strategy is to consider the "potential function" f for the metric $g(t)$. (In the language of [Jeffres et al. 2014], f is the Ricci potential for g.) By definition, this is a solution to the equation

$$\Delta_{g(t)} f = R_{g(t)} - \rho, \tag{3-24}$$

where ρ is the average scalar curvature. The crucial property that it must satisfy is that $|\nabla f| \le C$. Observe that f is only defined up to an arbitrary additive constant, which may depend on t, but that the proof in [Hamilton 1988] shows how to choose this constant using the evolution equation satisfied by f.

In any case, we now show that a potential function with bounded gradient exists. Interestingly, this is one place where the assumption that the cone angles are less than 2π plays a crucial role.

Proposition 3.17. *Suppose that g is a conic metric with all cone angles less than 2π; suppose too that $g = ug_0$, where g_0 is smooth (or polyhomogeneous) on $\widetilde{M}$, $u \in \mathcal{C}_b^{2,\delta}$ and, furthermore, $R_g \in \mathcal{C}_b^{0,\delta}$. Then the solution f to $\Delta_g f = R_g - \rho$ which lies in the Friedrichs domain and satisfies $\int f \, dA_g = 0$ has $|\nabla f| \leq C$.*

Proof. By Proposition 3.1 (as well as the fact that the integral of $R - \rho$ is zero), there exists a unique solution f which has integral zero, and this function has a partial expansion

$$f \sim a_0 + (a_{11} \cos y + a_{12} \sin y) r^{1/\beta} + \tilde{u}, \quad \tilde{u} \in r^2 \mathcal{C}_b^{2,\delta}.$$

Since $\beta < 1$, it follows immediately that $|\nabla f| \leq C$. □

We recall very briefly that the rest of the proof of long-time existence involves getting an priori uniform bound on $R_{g(t)}$ where $g(t)$ is the family of solution metrics, and then using (2-4) to find bounds for $\log u$. The bounds on $R_{\min}$ follow easily from the maximum principle, while the bound for $R_{\max}$ is derived by considering the evolution equation satisfied by $h := \Delta f + |\nabla f|^2$. For both of these steps, one needs the maximum principle from the previous subsection, which is permissible since R and h both satisfy the conditions of Lemma 3.15.

4. Convergence of the flow in the Troyanov case

We are now in a position to be able to prove that the solution $g(\cdot, t)$ converges exponentially as $t \to \infty$ to a constant-curvature metric with the same cone angles, provided the Troyanov condition (2-14) holds.

Let $W^{1,2}$ denote the usual Sobolev space of L^2 functions whose gradient is in L^2 (with respect to g_0). Following [Troyanov 1991; Struwe 2002], consider the energy functional $\mathcal{F} : W^{1,2} \to \mathbb{R}$,

$$\mathcal{F}(\phi) := \int_M (|\nabla_0 \phi|^2 + 2R_0 \phi) \, dA_0,$$

where the conformal factor has been rewritten as $u = e^\phi$. (The function spaces $W^{1,2}$ and $W^{2,2}$ used below are taken with respect to any fixed conic metric that is smooth in the (r, y) coordinates.) The next lemma says that the Ricci flow is the gradient flow of $\mathcal{F}$ with respect to the Calabi L^2 metric (see, e.g., [Clarke and Rubinstein 2013, §2]).

Lemma 4.1. *If u is a solution of (2-2), then*

$$\frac{d}{dt} \mathcal{F}(\phi) = -2 \int_M (R - \rho)^2 \, dA_g. \tag{4-1}$$

Proof. On smooth, closed surfaces the formula is well known [Struwe 2002, Equation (49)]. Indeed, recall that, using (2-1) and (2-2),

$$\partial_t \phi = e^{-\phi} (\Delta_0 u - R_0) + \rho = \rho - R;$$

from this we get

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(\phi) &= 2 \int_M (\nabla \phi \cdot \nabla \phi_t + R_0 \phi_t) \, dA_0 = 2 \int_M \phi_t (R_0 - \Delta_0 \phi) \, dA_0 \\ &= 2 \int_M R e^\phi \phi_t \, dA_0 = -2 \int_M (R - \rho) R \, dA_g, \end{aligned}$$

and the result follows since $\int (R-\rho)\, dA_g = 0$. Concealed here is the fact that these integrations by parts remain valid in this conic setting. This sort of computation will be used repeatedly in the remainder of this paper. The key point is that the functions involved enjoy sufficient regularity near the conic points that one may integrate by parts on the complement of an ϵ-neighbourhood of these points and show that the boundary term tends to 0 with ϵ. □

Troyanov [1991] proves that the conditions (2-14) ensure that there exists a constant C such that

$$\mathcal{F}(\phi(t)) \geq -C \quad \text{for all } t \geq 0.$$

(In fact, Troyanov considers the stationary problem from a variational point of view and proves that $\mathcal{F}$ is bounded below on $W^{1,2}$ if (2-14) holds.)

We now prove that $\phi(\,\cdot\,,t)$ is uniformly bounded in $W^{2,2}$. This too follows arguments in [Troyanov 1991; Struwe 2002].

Proposition 4.2. *With all notation as above, if the conditions (2-14) hold and ϕ is a solution to the flow, then*

$$\|\phi(\,\cdot\,,t)\|_{W^{2,2}} \leq C.$$

Proof. We sketch the argument and refer to [Troyanov 1991; Struwe 2002] for more details. The starting point is the uniform lower bound $\mathcal{F}(\phi(\,\cdot\,,t)) \geq -C$. We first claim that

$$\|\phi(\,\cdot\,,t)\|_{W^{1,2}} \leq C, \quad t \geq 0. \tag{4-2}$$

There are three cases to consider. We only give details for the case when $\chi(M,\vec{\beta}) > 0$, since the cases where $\chi(M,\vec{\beta}) \leq 0$ are similar but simpler. The Troyanov condition (2-14) is equivalent to $0 < 2\pi\gamma := 2\pi\chi(M,\vec{\beta}) < 4\pi \min_i\{\beta_i\}$. Choose b such that $\pi\gamma = \pi\chi(M,\vec{\beta}) < b < 2\pi \min_i\{\beta_i\}$ and set

$$I(\phi) := \frac{1}{2b}\int_M |\nabla\phi|^2\, dA_0 + \frac{1}{\pi\gamma}\int_M R_0\phi\, dA_0.$$

As in the proof of Theorem 5 in [Troyanov 1991], we have $I(\phi) \geq -C$ for all $\phi \in W^{1,2}$. But

$$\frac{1}{2\pi\gamma}\mathcal{F}(\phi) = I(\phi) + \frac{1}{2}\left(\frac{1}{\pi\gamma} - \frac{1}{b}\right)\int_M |\nabla\phi|^2\, dA_0 > I(\phi) \geq -C.$$

Since $\mathcal{F}(\phi) \leq m$,

$$\int_M |\nabla\phi|^2\, dA_0 \leq C, \quad t \geq 0,$$

and Troyanov's argument then shows that also the L^2 is uniformly bounded [Troyanov 1991, p. 817], whence $\|\phi(\,\cdot\,,t)\|_{W^{1,2}} \leq C$ for all $t \geq 0$.

It is proved in [Troyanov 1991] that, if $0 < b < 2\pi \min_i\{2+2\alpha_i\}$, then there exists a constant C such that

$$\int_M e^{bu^2}\, dA_0 \leq C$$

for all $u \in W^{1,2}$ such that $\int_M u\, dA_0 = 0$ and $\int_M |\nabla u|^2\, dA_0 \leq 1$. This is the Moser–Trudinger–Cherrier inequality for surfaces with conic singularities.

We now prove that

$$\int_M |\nabla^2\phi|^2 \, dA_0 \le C \quad \text{for all } t \in [0, \infty).$$

Carrying out a standard integration by parts argument over the complement of the ϵ-balls around the conic points, we obtain

$$\int_{M\setminus B(\vec{p},\epsilon)} |\nabla^2\phi|^2 \, dA_0$$
$$= \int_{M\setminus B(\vec{p},\epsilon)} |\Delta_0\phi|^2 \, dA_0 - \frac{1}{2}\int_{M\setminus B(\vec{p},\epsilon)} R|\nabla\phi|^2 \, dA_0 + \int_{\partial B(\vec{p},\epsilon)} \partial\nu\nabla\phi\cdot\nabla\phi \, d\sigma_0 - \int_{\partial B(\vec{p},\epsilon)} \Delta_0\phi \, \partial_\nu\phi \, d\sigma_0.$$

Using Proposition 3.3 and letting $\epsilon \to 0$ gives

$$\int_M |\nabla^2\phi|^2 \, dA_0 = \int_M |\Delta_0\phi|^2 \, dA_0 - \frac{1}{2}\int_M R|\nabla\phi|^2 \, dA_0. \tag{4-3}$$

By (2-3),

$$\int_M |\Delta_0\phi|^2 \, dA_0 \le 2\biggl(\int_M R_0^2 \, dA_0 + \int_M R^2 e^{2\phi} \, dA_0\biggr) \le C\biggl(1 + \int_M e^{2|\phi|} \, dA_0\biggr) \le C\biggl(1 + \int_M e^{b^2|\phi|^2} \, dA_0\biggr),$$

since, by Corollary 5.7 (proved later), the scalar curvature is uniformly bounded in time, where b is any real number such that $0 < b^2 < 2\pi \min_i\{2 + 2\alpha_i\}$ and C may depend on the choice of b. Now, by [Troyanov 1991, Proposition 11], the map $\phi \mapsto e^\phi$ is a compact embedding of $W^{1,2}$ in L^2, which thus yields

$$\int_M |\Delta_0\phi|^2 \, dA_0 \le C,$$

and hence, finally,

$$\int_M |\nabla^2\phi|^2 \, dA_0 \le C \quad \text{for all } t \ge 0. \qquad \square$$

Proposition 4.3. *Let $g(t)$ be the angle-preserving solution of (2-1) provided by Theorem 1.1. If (2-14) holds, then $g(t)$ converges exponentially to the unique constant-curvature metric in the conformal class of g_0 with specified conic data.*

Proof. We have already shown that $\phi(\,\cdot\,, t)$ exists and $\|\phi(\,\cdot\,, t)\|_{W^{2,2}} \le C$ for all $t \ge 0$. We now invoke the arguments of [Struwe 2002] verbatim to deduce that $g(t)$ converges exponentially to a constant-curvature metric g_∞ in the conformal class of g_0.

It remains to show that g_∞ has the same conic data $\{\vec{p}, \vec{\beta}\}$ as g_0. The $W^{2,2}$ bound and the Sobolev embedding theorem give a uniform $\mathcal{C}^0$ bound $|\phi(\,\cdot\,, t)| \le C$. This implies that the conic points do not merge in the limit. Indeed, if $i \ne j$ and γ_{ij}^t is the geodesic for $g(t)$ joining these two conic points, then

$$\operatorname{dist}_{g(t)}(p_i, p_j) = \int_{\gamma_{ij}^t} e^{\phi/2} \ge \bar{c}\int_{\gamma_{ij}^t} \ge \bar{c}\operatorname{dist}_{g(0)}(p_i, p_j).$$

Next, suppose that g_∞ has cone angle parameter $\tilde{\beta}_i$ at p_i. Thus, in local conformal coordinates,

$$g_0 = e^{2\phi_0}|z|^{2\beta_i - 2}|dz|^2 \quad \text{and} \quad g_\infty = e^{2\phi_\infty}|z|^{2\tilde{\beta}_i - 2}|dz|^2,$$

so by the uniform $\mathcal{C}^0$ bound it is clear that $\tilde{\beta}_i \geq \beta_i$ for all i. Since

$$\chi(M) + \sum \beta_i = \chi(M) + \sum \tilde{\beta}_i,$$

we see that $\beta_i = \tilde{\beta}_i$ for all i. □

5. Convergence in the non-Troyanov case

In this final section we consider the case where the Troyanov condition (2-14) fails. As remarked earlier, the angle inequality fails at just one of the points p_j, say p_1, and necessarily $M = S^2$. Then, $(M, J, \vec{p}, \vec{\beta})$ does not admit a constant-curvature metric, and hence, even if $g(\cdot, t)$ converges, its limit must either not be of constant curvature or else some of the conic data is destroyed in the limit. More precisely, the limit might be a surface with fewer conic points and different cone angles, and hence might conceivably still admit a constant-curvature metric. The existence of nonconstant-curvature, soliton metrics with one or two conic points (the teardrop or American football) on S^2 can be ascertained using ODEs arguments, [Yin 2010], and these are the reasonable candidates for limiting metrics in the non-Troyanov case. To this end, we first show that every compact two-dimensional shrinking Ricci soliton which does not have constant curvature has at most two conic points. Furthermore, if (2-14) holds, then any shrinking Ricci soliton must have constant curvature. The next lemma also appears in [Ramos 2013].

Lemma 5.1. *If g is a shrinking Ricci soliton metric on M with conic data $(\vec{p}, \vec{\beta})$ and there are at least three conic points, then g has constant curvature.*

Proof. View g as a Kähler–Ricci soliton; then

$$(R-1)g_{i\bar{j}} = \nabla_i \nabla_{\bar{j}} f,$$

where the vector field $X^i := \nabla^i f$ is a holomorphic vector field on $S^2 \backslash \vec{p}$. The trace of the soliton equation gives $\Delta f = R - 1$, and hence, using the static case of Theorem 1.2 — see also [Jeffres et al. 2014, Propositions 3.3 and 3.8] — it follows that $\nabla f = \mathcal{O}(r^{1/\beta - 1})$, so must vanish at each of the points p_i. This may also be deduced as in [Luo and Tian 1992, Lemma 3]. Using this same regularity, we can integrate by parts to get

$$\int_{S^2 \backslash \vec{p}} |X|^2 \, dA = \int_{S^2 \backslash \vec{p}} |\nabla f|^2 \, dA = \int_{S^2 \backslash \vec{p}} (1-R) f \, dA < \infty.$$

However, there is no nontrivial holomorphic vector field on S^2 which vanishes at more than two points, so $X = 0$ and hence $\nabla_{\bar{j}} \nabla_i f = 0$. Finally, using the soliton equation again, $R \equiv 1$. □

Lemma 5.2. *If $(M, J, \vec{\beta}, \vec{p})$ satisfies (2-14) and g is a shrinking Ricci soliton metric, then g has constant curvature, i.e., $f = \text{const}$.*

Proof. The argument carries over from the smooth setting, by virtue of Theorem 1.2. We already know that there exists a constant-curvature metric $\bar{g}$ with this prescribed data. By rescaling, assume $R_{\bar{g}} = 1$. Write $g = e^{\phi} \bar{g}$. Since g is a shrinking soliton, it moves under Ricci flow by a 1-parameter family of diffeomorphisms $\psi(t)$, so $g(t) = \psi(t)^* g$. Hence, $\phi(\,\cdot\,, t) = \psi^*\phi$ solves

$$\partial_t \phi = \langle \nabla f, \nabla \phi \rangle_g = e^{-\phi}(\Delta_{\bar{g}}\phi - R_{\bar{g}}) + 1.$$

However, $R_g = e^{-\phi}(1 - \Delta_{\bar{g}}\phi)$ and $R_{\bar{g}} = 1$, so $\langle \nabla f, \nabla \phi \rangle_g = -R_g + 1$, which implies that

$$\langle \nabla f, \nabla \phi \rangle_g = R_g - 1 = -\Delta_g f,$$

or, equivalently, $\operatorname{div}(e^{\phi}\nabla f) = 0$. Multiplying by f and integrating by parts on $M \setminus B_\epsilon(\vec{p})$ gives

$$\int_{M\setminus B_\epsilon(\vec{p})} |\nabla f|^2 e^{\phi}\, dA_g = \int_{\partial B_\epsilon(\vec{p})} f \partial_\nu f e^{\phi}\, d\sigma,$$

and this converges to 0 as $\epsilon \to 0$. Hence, $\int_M |\nabla f|^2\, dA_g = 0$, so $f = \text{const}$. Thus $R_g \equiv 1$ and, by the uniqueness of constant-curvature metrics with given conic data [Luo and Tian 1992], $g = \bar{g}$. □

Our goal in the remainder of this section is to prove:

Proposition 5.3. *Let $g(t)$ be the angle-preserving flow on $(M, J, \vec{p}, \vec{\beta})$ and assume that (2-14) fails. Define $\psi(t)$ to be the t-dependent diffeomorphism generated by the vector field $\nabla f(t)$, where $\Delta f(t) = R_{g(t)} - \rho$. Then $\hat{g}(t) := \psi^* g(t)$ satisfies $\partial \hat{g}(t)/\partial t = 2\hat{\mu}(t)$, where $\hat{\mu}$ is the tensor defined by (5-1) with respect to the metric $\hat{g}(t)$. We prove that*

$$\lim_{t\to\infty} \int_M |\hat{\mu}(t)|^2_{\hat{g}(t)}\, d\hat{A} = \lim_{t\to\infty} \int_M |\mu(t)|^2_{g(t)}\, dA = 0$$

and, moreover,

$$\lim_{t\to\infty} \int_M |X(t)|^2_{g(t)}\, dA = 0,$$

where $X = \nabla R + R\nabla f$.

In the next subsections we assemble various facts which lead to the proof of this proposition. These were all initially developed in the smooth case, and the main work here consists mainly in verifying that they remain true in this conic setting.

The outline of this proof is as follows: In Section 5A we adapt Perelman's arguments for volume noncollapsing for the Kähler–Ricci flow; see [Sesum and Tian 2008]. We then follow the arguments in [Hamilton 1988], making use of the entropy functional $N(g) = \int_M R \log R\, dV_g$, and showing that $N(g(t)) \le C$ here too. In Section 5C we explain how to apply the maximum principle in the proof of the Harnack inequality, and hence obtain that $R_{\sup} \le C R_{\inf}$. Area noncollapsing, entropy monotonicity and the Harnack estimate then show that $R \le C$ for all $t \in [0, \infty)$. We also show $R \ge c > 0$ for $t \ge t_0$.

5A. ***Area noncollapsing via Perelman's monotonicity formula.*** Our first goal is to prove an estimate on the area of small geodesic balls.

Lemma 5.4. *Let $(M, g(t))$ be a compact conic surface evolving by the angle-preserving area-normalized Ricci flow. Define $R_{\max}(t) = \sup_{q\in M} R_{g(t)}$. Then there exists $C > 0$ so that, for all $p \in M$ and $t > 0$, we have*

$$\operatorname{Area}_{g(t)} B(p, R_{\max}(t)^{-1/2}) \ge \frac{C}{R_{\max}(t)}.$$

Proof. The proof relies on monotonicity properties with respect to the unnormalized Ricci flow of Perelman's $\mathcal{W}$ functional,

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-1} \int_M [\tau(|\nabla f|^2 + R) + f - 2]e^{-f}\, dA_g,$$

where g is a metric, f is a function and $\tau \in \mathbb{R}^+$. For us, g is a polyhomogeneous conic metric and $f(z,t) = a_0(t) + (a_{11}(t)\cos y + a_{12}(t)\sin y)r^{1/\beta} + r^2 \tilde{f}(z,t)$, where both f and $\tilde{f}$ lie in $\mathcal{C}^1([0,\infty); \mathcal{C}_b^{2,\delta})$. This integral is convergent, since $|\nabla f|$ is bounded.

We review the proof of monotonicity of this functional to check that the singularities of g and f do not cause difficulties. We restrict to the space of triples (g, f, τ) such that the measure $(4\pi\tau)^{-1/2} e^{-f}\, dA_g$ is fixed. If (v, h, σ) is a tangent vector to this space then, by [Kleiner and Lott 2008, Propositions 5.3 and 12.1],

$$\delta\mathcal{W}|_{(g,f,\tau)}(v,h,\sigma) = (4\pi\tau)^{-1} \int_M [\sigma(R_g + |\nabla f|^2) - \tau\langle v, \mathrm{Ric}_g + \nabla^2 f\rangle + h]e^{-f}\, dA_g$$

This requires justifying the three integrations by parts

$$\int_M e^{-f}(-\Delta \operatorname{tr}_g v)\, dA = -\int_M \Delta(e^{-f}) \operatorname{tr}_g v\, dA,$$
$$\int_M e^{-f}\delta^*\delta^* v\, dA = \int_M \langle \nabla^2 e^{-f}, v\rangle\, dA,$$
$$\int_M e^{-f}\langle \nabla f, \nabla h\rangle\, dA = \int_M \Delta e^{-f} h\, dA,$$

which we do in the usual way, using the expansion for f.

Still following [Kleiner and Lott 2008, §12], set $v = -2(\mathrm{Ric}_g + \nabla^2 f)$, so $\operatorname{tr}_g v = -2(R_g + \Delta f)$, and also $h = -\Delta f + |\nabla f|^2 - R_g + 1/(2\tau)$, $\sigma = -1$. Then

$$\delta\mathcal{W}|_{(g,f,\tau)}(v,h,\sigma) = \int_M \tau\left|\mathrm{Ric}_g + \nabla^2 f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-1} e^{-f}\, dA_g \ge 0.$$

To recover the actual Ricci flow, we add to v and h the Lie derivative terms $\mathcal{L}_V g$ and $\mathcal{L}_V f = Vf$, respectively, where $V = \nabla f$. This new infinitesimal variation corresponds to the flow

$$\partial_t g = -2\,\mathrm{Ric}_g, \quad \partial_t f = -\Delta f + |\nabla f|^2 - R_g + \frac{1}{\tau} \quad \text{and} \quad \partial_t \tau = -1,$$

along which we have

$$\delta\mathcal{W}|_{(g,f,\tau)}(v,h,\sigma) = \int_M 2\tau\left|\nabla^2 f + \frac{1}{2}\Big(R_g - \frac{1}{\tau}\Big)g_{ij}\right|^2 (4\pi\tau)^{-1/2} e^{-f}\, dA_g.$$

Finally, define

$$\mu(g,\tau) := \inf_f \Big\{\mathcal{W}(g,f,\tau) : (4\pi\tau)^{-1/2} \textstyle\int_M e^{-f}\, dA_g = 1\Big\}.$$

We have proved that $\mu(g(t), \tau(t))$ increases along the Ricci flow. Using this monotonicity, we follow precisely the same arguments as in Perelman's proof of volume noncollapsing for the Kähler–Ricci flow (see [Sesum and Tian 2008] for details). □

5B. ***Entropy estimate.*** The potential function f satisfies $\Delta f = R - \rho$. Define the symmetric, trace-free 2-tensor

$$\mu = \nabla^2 f - \tfrac{1}{2}\Delta f\, g \tag{5-1}$$

and the vector field $X = \nabla R + R\nabla f$. As in the earlier part of this section, g is a gradient Ricci soliton if $\mu \equiv 0$; in fact, one also has $X \equiv 0$ on any soliton. The entropy function introduced by Hamilton [1988] when R_{g_0} is a strictly positive function on M is the quantity

$$N(t) = \int_M R \log R\, dA.$$

When R changes sign, Chow [1991b] considered the modified entropy

$$N(t) = \int_M (R-s)\log(R-s)\, dA, \tag{5-2}$$

where $s'(t) = s(s-r)$ with $s(0) < \min_{x\in M} R(x,0)$. In either case, if M is smooth, these authors showed that $N(t) \le C$ for $t \ge 0$; in the first case, this is based on the monotonicity of N, which follows from the formula

$$\frac{dN}{dt} = -\int \left(2|\mu|^2 + \frac{|X|^2}{R}\right) dA. \tag{5-3}$$

We now prove that this entropy function, or its modified form, is still bounded above even in the conic setting.

Lemma 5.5. *If $g(t)$ is an angle-preserving solution of the normalized Ricci flow, and if the entropy N is defined by* (5-3) *if $R > 0$ everywhere and by* (5-2) *if R changes signs, then $N(t) \le C$ for all $t < \infty$.*

Proof. The argument proceeds exactly as in the smooth case once we show that the various integrations by parts are justified. We assume that R does change signs, since the two cases are very similar, and follow Chow's [1991a] proof on orbifolds.

Define $L = \log(R-s)$. The proof relies on the following identities:

$$\int \Delta L(R-s) = -\int \frac{|\nabla R|^2}{R-s}, \qquad \int L\Delta R = -\int \frac{|\nabla R|^2}{R-s},$$

$$\int (\Delta f)^2 = -\int \langle \nabla f, \nabla\Delta f\rangle, \qquad \int \langle \nabla f, \Delta\nabla f\rangle = -\int |D^2 f|^2,$$

$$\int f\Delta f = -\int |\nabla f|^2, \qquad \int \langle \nabla R, \nabla f\rangle = -\int R\Delta f = -\int R(R-r),$$

$$\int \langle \nabla L, \nabla f\rangle = -\int L\Delta f = -\int L(R-r);$$

these are all proven using Green's identity on $M \setminus B(\vec{p}, \epsilon)$ and taking advantage of the expansions of f and R to show that the boundary terms vanish in the limit $\epsilon \to 0$. $\square$

5C. *Harnack estimate and curvature bound.* The proof of the Harnack estimate for R, when $R > 0$ everywhere, or for $R - s$ if R changes sign, again proceeds exactly as in the smooth [Chow 1991b] and orbifold [Chow and Wu 1991] cases, although now using the maximum principles from Lemmas 3.15 and 3.16. We outline the main step. Consider $P = Q + sL$, where

$$Q = \partial_t L - |\nabla L|^2 - s = \Delta L + R - \rho \quad \text{and} \quad L = \log(R - s).$$

One computes that

$$\partial_t P \geq \Delta P + 2\nabla L \cdot \nabla P + \tfrac{1}{2}(P^2 - C^2), \tag{5-4}$$

where C is a constant chosen so that $L \geq -C - Ct$. By Corollary 3.14, R is polyhomogeneous (for $t > 0$) and the only terms in its expansion less than r^2 are r^0 and $r^{1/\beta}$. Using (2-7), the initial terms in the expansion of ΔR have the same exponents. Thus, $|\nabla P|$ satisfies the conditions in these maximum principle lemmas, and we conclude that $Q \geq -C$, independently of t. The usual integration in spatial and time variables leads to the Harnack inequality — see [Chow 1991b] for details — and thus gives:

Lemma 5.6. *If $y \in B_{g(t)}\big(x, \frac{1}{8}\sqrt{R(x,t)}\big)$, then $R(y, t+1) \geq C R(x,t)$ for some universal constant $C > 0$.*

Using the entropy bound and area comparison, the boundedness of R follows as in [Hamilton 1988; Chow 1991b].

Corollary 5.7. *There exist constants c, $C > 0$ such that $|R(\cdot\,, t)| < C$ for all $t > 0$ and $R(\cdot\,, t) \geq c$ for $t \gg 1$.*

Proof of Proposition 5.3. Consider the following modification of the Ricci flow equation:

$$\frac{\partial}{\partial t}\hat{g}_{ij} = 2\hat{\mu}_{ij} = (\rho - \hat{R})\hat{g}_{ij} - 2\nabla_i \nabla_j f, \tag{5-5}$$

where $\hat{R}$ is the scalar curvature of $\hat{g}$ (the covariant derivatives in the last term are also with respect to $\hat{g}(t)$, but we omit this from the notation for simplicity) and f is the same potential function as before. This differs from the standard flow by the action of the one-parameter family of diffeomorphisms ψ_t generated by ∇f, i.e., $\hat{g}(t) = \psi_t^* g(t)$, where $g(t)$ is a solution of the original normalized (but unmodified) Ricci flow. According to Lemma 5.5, Corollary 5.7, and (5-3), $N(t)$ is monotone (for t sufficiently large) and converges to a finite limit, hence

$$\lim_{t \to \infty} \frac{dN}{dt} = 0;$$

recalling (5-3), the conclusion follows from this. $\square$

Remark 5.8. Hamilton's original argument showing that the pointwise norm of μ converges exponentially to zero breaks down in our setting for the following reason. As for many of the other quantities we consider here, the function f admits an expansion

$$f = a_0(t) + r^{1/\beta}(a_{11}(t)\cos y + a_{12}(t)\sin y) + a_2(t)r^2 + \mathcal{O}(r^{2+\epsilon}),$$

where the a_i and a_{ij} are smooth in t. This follows from the equation satisfied by R, the equation

$$\Delta_{g(t)} f(t) = R(t) - \rho = -\rho + \frac{R_0 - \Delta_{g(0)} \log u(t)}{u(t)},$$

the asymptotic expansion (1-1) of $u(t)$, and [Jeffres et al. 2014, Corollary 3.5]. However,

$$\mu = \nabla^2 f - \tfrac{1}{2}\Delta f\, g \tag{5-6}$$

is a second-order operator applied to f, and not all of these annihilate the troublesome term $r^{1/\beta}$ in the expansion of f. This means that, although μ, and hence $|\mu|$, has an asymptotic expansion, this expansion contains a singular term of the form $r^{1/\beta-2}$. This means that the maximum principle is not applicable, and we cannot conclude the exponential decay of $|\mu|$. Note that there is no difficulty with what we prove above, since this most singular term $r^{1/\beta-2}$ is square-integrable with respect to $r\,dr\,dy$.

5D. ***One concentration point.*** We now prove the second part of Theorem 1.4, concerning the divergence profile of the unmodified flow. Namely, we show that the conformal factor ϕ blows up at precisely one point q as $t \nearrow \infty$, but tends uniformly to zero on every compact set $K \subset S^2 \setminus \{q\}$. This argument is drawn from methods developed specifically for higher-dimensional complex analysis, so it is convenient to now change to the Kähler formalism.

Fix the initial conic metric g_0; since the flow immediately smooths out any initial metric, we may as well assume that g_0 is polyhomogeneous. Denote its associated Kähler form by ω. Define $\mathcal{H}_\omega$ to consist of all functions ϕ such that $\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, and then denote by PSH_ω the L^1 closure of $\mathcal{H}_\omega$. Observe that, since ω and ω_ϕ (or, rather, g_0 and g_ϕ) lie in the same Kähler class, they are conformally related; indeed,

$$\omega_\phi = (1 + \Delta_0\phi)\omega, \quad \text{and similarly} \quad \omega = (1 - \Delta_\phi\phi)\omega_\phi.$$

Here Δ_0 and Δ_ϕ are the Laplacians for ω and ω_ϕ, respectively. Note that this implies that

$$\Delta_0\phi > -1 \quad \text{and} \quad \Delta_\phi\phi < 1. \tag{5-7}$$

For any $\phi \in \mathrm{PSH}_\omega$, we define the multiplier ideal sheaf $\mathcal{I}(\phi)$ associated to the presheaf which assigns to any open set U the space of holomorphic functions

$$\mathcal{I}(\phi)(U) = \{h \in \mathcal{O}_{S^2}(U) : |h|^2 e^{-\phi} \in L^1_{\mathrm{loc}}(S^2, \omega)\}.$$

It is proved in [Nadel 1990] that $\mathcal{I}(\phi)$ is always coherent; moreover, it is called proper if it is neither the trivial (zero) sheaf nor the structure sheaf $\mathcal{O}_{S^2}$.

Definition 5.9 [Nadel 1990, Definition 2.4]. The multiplier ideal sheaf $\mathcal{I}(\phi)$ is called a Nadel sheaf if there exists an $\epsilon > 0$ such that $(1+\epsilon)\phi \in \mathrm{PSH}_\omega$.

A fundamental result of Nadel's [1990] is that any Nadel sheaf has connected support. The proof is not hard in this low dimension, so we give it below. This uses an extension (for the one-dimensional case only) of the result [Rubinstein 2009, Theorem 1.3], which in turn extends Nadel's work from the continuity method to the Ricci flow. Note too that [Rubinstein 2009] provided a new proof of the uniformization theorem in the smooth case using the Ricci flow (see also [Chen et al. 2006] for an earlier and different flow-based proof), and hence its use here is natural.

We write the flow equation in this setting in terms of the Kähler potential as

$$\omega+\sqrt{-1}\partial\bar{\partial}\phi := \omega_\phi = e^{f_\omega-\phi+\dot{\phi}}\omega, \quad \phi(0)=\phi_0, \quad \text{and} \quad \dot{\phi}=\partial_t\phi, \tag{5-8}$$

where f_ω is the initial value $f(0)$ of the Ricci potential, as defined in (3-24). There is a choice of constant in this initial condition, and it is explained in [Phong et al. 2007] how to choose this additional constant so that $\dot{\phi}$ remains bounded along the flow. We assume henceforth that this initial condition has been set properly. We also write A for the (constant value of the) area of $(S^2, g(t))$.

Theorem 5.10. *Suppose that $(S^2, J, \vec{p}, \vec{\beta})$ does not satisfy (2-14). Fix any $\gamma \in \left(\frac{1}{2}, 1\right)$. Then the solution $\phi(t)$, normalized as above, admits a subsequence $\phi_j := \phi_{t_j}$ for which $\hat{\phi}_j := \phi_{t_j} - A^{-1}\int \phi_{t_j}$ converges in L^1 to $\phi_\infty \in \mathrm{PSH}_\omega$. Finally, $\mathcal{I}(\gamma\phi_\infty)$ is a proper Nadel multiplier ideal sheaf with support equal to a single point.*

Proof. We proceed in a series of steps.

Step 1: $\mathrm{diam}(M, g(t)) \le C$. This is a special case of [Jeffres et al. 2014, Claim 6.4]. Indeed, since $\beta < 1$, if $p, q \in M$ are not conic points then the minimizing geodesic which connects them does not pass through a conic point. Thus we can apply the standard argument for Myers' theorem, using that $R > c > 0$ for large t. This can also be deduced by specializing Perelman's diameter estimate [Sesum and Tian 2008] to our setting, which is possible using Theorem 1.2.

Step 2: $-\inf\phi \le \sup\phi + C$. The proof of [Rubinstein 2009, Lemma 2.2] carries over without change by using the twisted Berger–Moser–Ding functional

$$D(\phi) = \frac{\sqrt{-1}}{2A}\int \partial\phi\wedge\bar{\partial}\phi - \log\left(\frac{1}{A}\int e^{f_\omega-\phi}\omega\right).$$

This is monotone along the flow, which gives, after some calculations, that [Rubinstein 2009, (15)]

$$\frac{1}{A}\int -\phi\omega_\phi \le \frac{1}{A}\int \phi\omega + C. \tag{5-9}$$

We next show that the average $A^{-1}\int\phi\omega$ is comparable to $\sup\phi$. Indeed, the inequality $A^{-1}\int\phi\omega \le \sup\phi$ is trivial. For the converse, recall that the Green function of Δ_0, normalized so that $\int G_0(q,q')\omega(q') = 0$ for every q and $G \searrow -\infty$ near the diagonal, is bounded from above by a constant E_0. We then write

$$\begin{aligned}\phi(q) - \frac{1}{A}\int\phi\omega &= \int G(q,q')\Delta_\omega\phi(q')\omega(q') \\ &= \int -(G(q,q')-E_0)(-\Delta_\omega\phi(q'))\omega(q') \le -\int (G(q,q')-E_0)\omega \le AE_0,\end{aligned}$$

using the first inequality in (5-7). Taking the supremum over the left side gives $\sup\phi \le (1/A)\int\phi\omega + C$, as claimed.

To estimate the infimum of ϕ we use a similar trick, but using the upper bound $G_\phi(q, q') \le E_\phi$ and the second inequality in (5-7). This gives

$$\phi(q) - \frac{1}{A}\int \phi\omega_\phi = \int G_\phi(q, q')\Delta_\phi\phi(q')\omega_\phi(q')$$
$$= \int (G(q, q') - E_0)\Delta_\omega\phi(q')\omega_\phi(q') \ge \int (G(q, q') - E_0)\omega_\phi(q') \ge -AE_\phi,$$

so, taking the infimum, $-\inf\phi \le -(1/A)\int \phi\omega_\phi + AE_\phi$.

It remains only to observe that, special to this dimension, $G_\phi(q, q') = G_0(q, q')$; this is because, if we write $\omega_\phi = F\omega$, and if $\int f\omega_\phi = 0$, then

$$\Delta_\phi \int G_\phi(q, q')f(q')\omega_\phi(q') = F^{-1}\Delta_0 \int G_\phi(q, q')f(q')F(q')\omega(q'),$$

and this equals $f(q)$ when $G_\phi = G_0$. This means that $E_\phi = E_0$ and the constant in this inequality does not vary along the flow.

Putting these inequalities together completes this step.

Step 3: $\sup_t \|\phi_t\|_0 = \infty$. Indeed, if this supremum were finite, then, by Step 2, ϕ would be bounded in $\mathcal{C}^0$, and standard regularity estimates would then show that some subsequence of the ϕ_t converges. The limiting metric (or rather, the limit of any one of these subsequences) would then need to have constant curvature. Furthermore, the uniform boundedness of the conformal factor shows that the cone angles do not change in the limit. This is a contradiction, since we are assuming that the Troyanov conditions (2-14) fail.

The construction of the Nadel sheaf now proceeds as in [Rubinstein 2009, p. 5846].

Step 4: If $\gamma \in \left(\frac{1}{2}, 1\right)$ and V_γ denotes the support of $\mathcal{I}_\gamma := \mathcal{I}(\gamma\phi_\infty)$, then V_γ is a single point. Recall that a coherent sheaf is locally free away from a complex codimension-two set, so, since we are in complex dimension one, $\mathcal{I}_\gamma$ is a sheaf of sections of a holomorphic line bundle $\mathcal{O}_{S^2}(-k)$, $k \ge 0$. By the properness assumption, $k \ge 1$. We claim that $k = 1$, which then implies that $\mathcal{I}_\gamma$ is spanned by a single holomorphic section, which vanishes to order one at precisely one point.

To do this, let U be a small open set and let $h \in \mathcal{I}_\gamma(U)$, and assume that h vanishes exactly to order one at a point $p \in U$. Then $\int_U |h|^2 e^{-\gamma\phi_\infty}\omega < \infty$. Now fix a local holomorphic coordinate z which vanishes at p and assume either that U contains no conic points or, if it does contain one, then p is that point. In the first of these cases, ω is locally equivalent to $|dz|^2$, while $\phi_\infty \ge 4\log|z|$, and $0 < \gamma < 1$. If $p = p_i$ is a conic point, then, assuming that U contains no other conic points, $\int_U |h|^2 e^{-\gamma\phi_\infty} z^{2\beta_i - 2}\,|dz|^2 < \infty$. This follows just as before but using that ϕ_∞ has a singularity of, at worst, $4\log|z| - 2(1-\beta_i)\log|z|$ (recall $[\omega] = \mathcal{O}_{S^2}(2) - \sum(1-\beta_i)[p_i]$) and $0 < \gamma < 1$. Thus k cannot be greater than one. Since $k > 0$, it follows that $k = 1$, as desired.

There is an alternative proof that does not rely on facts about coherent sheaves, using weighted L^2 estimates for the $\bar\partial$-equation. This proceeds as follows. Let η be a $(0, 1)$-form such that $\int |\eta|^2 e^{-\gamma\phi_\infty}\,|dz|^2 < \infty$. It is always possible [Berndtsson 2010, §1] to find a solution ρ to $\bar\partial\rho = \eta$ that satisfies $\int |\rho|^2 e^{-\gamma\phi_\infty}\,|dz|^2 \le$

$C_\gamma \int |\eta|^2 e^{-\gamma\phi_\infty} |dz|^2 < \infty$, where $C_\gamma = O((1-\gamma)^{-1})$. The same arguments can be used to verify that this estimate also holds with respect to the measure $|z|^{2\beta_i-2}|dz|^2$. This proves that $H^1(S^2, \mathcal{I}_\gamma) = 0$. From the long exact sequence in cohomology corresponding to the short exact sequence of sheaves $0 \to \mathcal{I}_\gamma \to \mathcal{O}_{S^2} \to \mathcal{O}_{V_\gamma} \to 0$, one concludes that $H^0(V_\gamma, \mathcal{O}_{V_\gamma}) \cong H^0(S^2, \mathcal{O}_{S^2}) \cong \mathbb{C}$, which means once again that the support of $\mathcal{I}_\gamma$ is connected, i.e., a single point.

These two methods of proof are closely related, of course, by virtue of the identification $H^1(S^2, \mathcal{I}_\gamma) \cong H^0(S^2, \mathcal{O}_{S^2}(K_{S^2} - \mathcal{I}_\gamma))$. □

Following [Clarke and Rubinstein 2013, Lemma 6.5], we can use Theorem 5.10 to deduce estimates on the conformal factor:

Corollary 5.11. *The conformal factor u blows up at exactly one point. On any compact set K disjoint from that point, $u \to 0$ uniformly, so, in particular, the area of K with respect to $g(t)$ tends to* 0.

Acknowledgements

The authors gratefully acknowledge support by the NSF under grants DMS-1105050 (Mazzeo), DMS-0802923,1206284 (Rubinstein), and DMS-0905749,1056387 (Sesum). Rubinstein is also grateful to the Stanford MRC and grant DMS-0603355 for support during the academic year 2009–2010. The authors also wish to acknowledge the careful reading by the anonymous referees as well as by J. Song, whose comments and suggestions helped us clarify and correct a number of points. In particular, this updated version of our paper contains a weaker version of what was originally stated as Proposition 5.3.

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Received 26 May 2014. Revised 27 Jan 2015. Accepted 6 Mar 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

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ANALYSIS & PDE

Volume 8 No. 4 2015