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LOCAL SPECTRAL ASYMPTOTICS FOR METRIC PERTURBATIONS OF THE LANDAU HAMILTONIAN

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We consider metric perturbations of the Landau Hamiltonian. We investigate the asymptotic behavior of the discrete spectrum of the perturbed operator near the Landau levels, for perturbations of compact support, and of exponential or power-like decay at infinity.

1. Introduction

Let

$$H_0 := (-i\nabla - A_0)^2$$

with $A_0 = (A_{0,1}, A_{0,2}) := \frac{1}{2}b(-x_2, x_1)$ be the Landau Hamiltonian, self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. In other words, H_0 is the two-dimensional Schrödinger operator with constant scalar magnetic field $b > 0$, that is, the Hamiltonian of a two-dimensional, spinless, nonrelativistic quantum particle subject to a constant magnetic field. As is well known, the spectrum $\sigma(H_0)$ consists of infinitely degenerate eigenvalues $\Lambda_q := b(2q+1)$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, called *Landau levels* (see, e.g., [Fock 1928; Landau 1930]).

In the present article we consider metric perturbations of H_0. Namely, let

$$m(x) = \{m_{jk}(x)\}_{j,k=1,2}, \quad x \in \mathbb{R}^2,$$

be a Hermitian 2×2 matrix such that $m(x) \geq 0$ for all $x \in \mathbb{R}^2$. Throughout the article we assume that $m_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$, i.e., $m_{jk} \in C^\infty(\mathbb{R}^2)$, and the entries m_{jk} together with all their derivatives are bounded on $\mathbb{R}^2$. Set

$$\Pi_j := -i\frac{\partial}{\partial x_j} - A_{0,j}, \quad j = 1, 2, \tag{1-1}$$

so that $H_0 = \Pi_1^2 + \Pi_2^2$. On Dom H_0, define the operators

$$H_\pm := \sum_{j,k=1,2} \Pi_j(\delta_{jk} \pm m_{jk})\Pi_k = H_0 \pm W,$$

where $W := \sum_{j,k=1,2} \Pi_j m_{jk} \Pi_k$; in the case of H_-, we suppose additionally that $\sup_{x\in\mathbb{R}^2} |m(x)| < 1$. Thus, the matrices $g_\pm(x) = \{g_{jk}^\pm(x)\}_{j,k=1,2}$ with $g_{jk}^\pm := \delta_{jk} \pm m_{jk}$ are positive definite for each $x \in \mathbb{R}^2$. Under these assumptions, the operators $H_\pm$ are self-adjoint in $L^2(\mathbb{R}^2)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see the Appendix).

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From a mathematical physics point of view, the operators $H_\pm$ are special cases of Schrödinger operators with *position-dependent mass*, which have a long history (see, e.g., [Bastard et al. 1975; von Roos 1983]), but have received increased attention during the last decade (see, e.g., [Midya et al. 2010; Gadella and Smolyanov 2008; Killingbeck 2011]). We would like to mention especially [de Souza Dutra and de Oliveira 2009], where the model considered is quite close to the operators $H_\pm$ discussed here.

The operators $H_\pm$ admit also a geometric interpretation, since they are related to the Bochner Laplacians corresponding to connections with constant nonvanishing curvature (see, e.g., [Rosenberg 1997; Colin de Verdière 1986]); we discuss this relation in more detail at the end of Section 2. Further, assume that

$$\lim_{|x|\to\infty} m_{jk}(x) = 0, \qquad j, k = 1, 2. \tag{1-2}$$

Thus m models a localized perturbation with respect to a reference medium. Under condition (1-2), the resolvent difference $H_\pm^{-1} - H_0^{-1}$ is a compact operator (see the Appendix), and therefore the essential spectra of $H_\pm$ and H_0 coincide:

$$\sigma_{\text{ess}}(H_\pm) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}.$$

The spectrum $\sigma(H_\pm)$ on $\mathbb{R} \setminus \bigcup_{q=0}^{\infty}\{\Lambda_q\}$ may consist of discrete eigenvalues whose only possible accumulation points are the Landau levels. Moreover, taking into account that $W \geq 0$, and applying [Birman and Solomjak 1987, Section 9.4, Theorem 7], we find that the eigenvalues of H_+ (resp. H_-) may accumulate to a given Landau level Λ_q only from above (resp. from below). Fix $q \in \mathbb{Z}_+$. Let $\{\lambda_{k,q}^-\}$ be the eigenvalues of H_- lying on the interval $(\Lambda_{q-1}, \Lambda_q)$ with $\Lambda_{-1} := -\infty$, counted with multiplicities and enumerated in increasing order. Similarly, let $\{\lambda_{k,q}^+\}$ be the eigenvalues of H_+ lying on the interval $(\Lambda_q, \Lambda_{q+1})$, counted with multiplicities and enumerated in decreasing order.

The aim of the article is to investigate the rate of convergence of $\lambda_{k,q}^\pm - \Lambda_q$ as $k \to \infty$, with $q \in \mathbb{Z}_+$ fixed, for perturbations m of compact support, of exponential decay, or of power-like decay at infinity.

The properties of the discrete spectrum generated by perturbative second-order differential operators with decaying coefficients have been considered also in [Alama et al. 1994; Boyarchenko and Levendorskiĭ 1997; Briet et al. 2009; Raikov 2015].

The article is organized as follows. In Section 2 we formulate our main results and briefly comment on them. In Section 3 we reduce our analysis to the study of operators of Berezin–Toeplitz type, and in Section 4 we establish several useful unitary equivalences for these operators. Section 5 contains the proofs of our results in the case of rapid decay, i.e., of compact support or exponential decay, while the proofs for slow, i.e., power-like decay, can be found in Section 6. Finally, in the Appendix we address some standard issues concerning the domain of the operators $H_\pm$ and the compactness of the resolvent difference $H_0^{-1} - H_\pm^{-1}$.

2. Main results

First, we formulate our results concerning perturbations m of compact support. Denote by $m_<(x)$ and $m_>(x)$, with $m_<(x) \leq m_>(x)$, the two eigenvalues of the matrix $m(x)$, $x \in \mathbb{R}^2$.

Theorem 2.1. *Assume that the support of the matrix m is compact, and its smaller eigenvalue $m_<$ does not vanish identically. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\ln\left(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)\right) = -k \ln k + O(k), \quad k \to \infty. \tag{2-1}$$

Remarks. (i) Under additional technical hypotheses on $m_{\gtrless}$, we could make the asymptotic relation (2-1) more precise. Namely, assume that there exists a nonincreasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $s_j > 0$, $j \in \mathbb{N}$, $\lim_{j\to\infty} s_j = 0$, and the level lines

$$\{x \in \mathbb{R}^2 \mid m_<(x) = s_j\}, \quad j \in \mathbb{N},$$

are bounded Lipschitz curves. In particular, the existence of such a sequence follows from the Sard lemma (see, e.g., [Sternberg 1964, Chapter 2, Theorem 3.1]) if we assume that $m_< \in C^2(\mathbb{R}^2)$. Further, denote by $\mathscr{C}_{\gtrless}$ the *logarithmic capacities* (see, e.g., [Landkof 1972, Chapter II, Section 4]) of $\operatorname{supp} m_{\gtrless}$. Then we have

$$\left(1 + \ln\left(\frac{b\mathscr{C}_<^2}{2}\right)\right)k + o(k) \le \ln\left(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)\right) + k\ln k \le \left(1 + \ln\left(\frac{b\mathscr{C}_>^2}{2}\right)\right)k + o(k) \tag{2-2}$$

as $k \to \infty$. We omit the details of the proof of (2-2), which is inspired by [Filonov and Pushnitski 2006].

(ii) For $q \in \mathbb{Z}_+$ and $\lambda > 0$, set

$$\mathcal{N}_q^{\pm}(\lambda) := \#\{k \in \mathbb{Z}_+ \mid \pm(\lambda_{k,q}^{\pm} - \Lambda_q) > \lambda\}. \tag{2-3}$$

Then, a less precise version of (2-1), namely

$$\ln\left(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)\right) = -k \ln k\,(1 + o(1)), \quad k \to \infty,$$

is equivalent to

$$\mathcal{N}_q^{\pm}(\lambda) = \frac{|\ln\lambda|}{\ln|\ln\lambda|}(1 + o(1)), \quad \lambda \downarrow 0. \tag{2-4}$$

Further, we state our results concerning perturbations of exponential decay. Assume that there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\ln m_{\gtrless}(x) = -\gamma|x|^{2\beta} + O(\ln|x|), \quad |x| \to \infty. \tag{2-5}$$

Remark. In (2-5), we suppose that the values of γ and β are the same for $m_<$ and $m_>$. Of course, the remainder $O(\ln|x|)$ could be different for $m_<$ and $m_>$.

Given $\beta > 0$ and $\gamma > 0$, set $\mu := \gamma(2/b)^{\beta}$, $b > 0$ being the constant magnetic field.

Theorem 2.2. *Let $m_{\gtrless}$ satisfy (2-5). Fix $q \in \mathbb{Z}_+$.*

(i) *If $\beta \in (0, 1)$, then there exist constants $f_j = f_j(\beta, \mu)$, $j \in \mathbb{N}$, with $f_1 = \mu$, such that*

$$\ln\left(\pm(\lambda_{k,q}^{\pm} - \Lambda_q)\right) = -\sum_{1\le j<1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{2-6}$$

(ii) *If* $\beta = 1$, *then*

$$\ln(\pm(\lambda^{\pm}_{k,q} - \Lambda_q)) = -(\ln(1+\mu))k + O(\ln k), \quad k \to \infty. \tag{2-7}$$

(iii) *If* $\beta \in (1, \infty)$, *then there exist constants* $g_j = g_j(\beta, \mu)$, $j \in \mathbb{N}$, *such that*

$$\ln(\pm(\lambda^{\pm}_{k,q} - \Lambda_q)) = -\frac{\beta-1}{\beta} k \ln k + \left(\frac{\beta - 1 - \ln(\mu\beta)}{\beta}\right)k - \sum_{1 \le j < \beta/(\beta-1)} g_j k^{(1/\beta-1)j+1} + O(\ln k), \quad k \to \infty. \tag{2-8}$$

Remarks. (i) Let us describe explicitly the coefficients f_j and g_j, $j \in \mathbb{N}$, appearing in (2-6) and (2-8) respectively. Assume first that $\beta \in (0, 1)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| \ll 1$, introduce the function

$$F(s; \epsilon) := s - \ln s + \epsilon\mu s^{\beta}. \tag{2-9}$$

Denote by $s_<(\epsilon)$ the unique positive solution of the equation $s = 1 - \epsilon\beta\mu s^{\beta}$, so that $\partial F(s_<(\epsilon); \epsilon)/\partial s = 0$. Set

$$f(\epsilon) := F(s_<(\epsilon); \epsilon). \tag{2-10}$$

Note that f is a real analytic function for small $|\epsilon|$. Then $f_j := (1/j!)\, d^j f(0)/d\epsilon^j$, $j \in \mathbb{N}$.

Let now $\beta \in (1, \infty)$. For $s > 0$ and $\epsilon \in \mathbb{R}$, $|\epsilon| \ll 1$, introduce the function

$$G(s; \epsilon) := \mu s^{\beta} - \ln s + \epsilon s. \tag{2-11}$$

Denote by $s_>(\epsilon)$ the unique positive solution of the equation $\beta\mu s^{\beta} = 1 - \epsilon s$, so that $\partial G(s_>(\epsilon); \epsilon)/\partial s = 0$. Define

$$g(\epsilon) := G(s_>(\epsilon); \epsilon), \tag{2-12}$$

which is a real analytic function for small $|\epsilon|$. Then $g_j := (1/j!)\, d^j g(0)/d\epsilon^j$, $j \in \mathbb{N}$.

(ii) If, instead of (2-5), we assume that

$$\ln m_{\gtrless}(x) = -\gamma |x|^{2\beta}(1 + o(1)), \quad |x| \to \infty, \tag{2-13}$$

then we can prove less precise versions of (2-6), (2-7), and (2-8), namely

$$\ln(\pm(\lambda^{\pm}_{k,q} - \Lambda_q)) = \begin{cases} -\mu k^{\beta}(1 + o(1)) & \text{if } \beta \in (0, 1), \\ -(\ln(1+\mu))k(1 + o(1)) & \text{if } \beta = 1, \\ -\dfrac{\beta-1}{\beta} k \ln k (1 + o(1)) & \text{if } \beta \in (1, \infty), \end{cases} \quad k \to \infty,$$

which are equivalent to

$$\mathcal{N}^{\pm}_q(\lambda) = \begin{cases} \mu^{-1/\beta} |\ln\lambda|^{1/\beta}(1 + o(1)) & \text{if } \beta \in (0, 1), \\ \dfrac{1}{\ln(1+\mu)} |\ln\lambda|(1 + o(1)) & \text{if } \beta = 1, \\ \dfrac{\beta}{\beta-1} \dfrac{|\ln\lambda|}{\ln|\ln\lambda|}(1 + o(1)) & \text{if } \beta \in (1, \infty), \end{cases} \quad \lambda \downarrow 0. \tag{2-14}$$

Note that in (2-13), similarly to (2-5), we assume that the values of γ and β are the same for $m_<$ and $m_>$. However, since the coefficient in (2-14) with $\beta > 1$ does not depend on γ, in this case we could assume different values of $\gamma > 0$ for $m_<$ and $m_>$.

Finally, we consider perturbations m which admit a power-like decay at infinity. For $\rho > 0$ recall the definition of the Hörmander class

$$\mathcal{S}^{-\rho}(\mathbb{R}^2) := \{\psi \in C^\infty(\mathbb{R}^2) \mid |D^\alpha \psi(x)| \le c_\alpha \langle x \rangle^{-\rho-|\alpha|},\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{Z}_+^2\},$$

where $\langle x \rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^2$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ satisfy $\lim_{|x|\to\infty} \psi(x) = 0$. Set

$$\Phi_\psi(\lambda) := |\{x \in \mathbb{R}^2 \mid \psi(x) > \lambda\}|, \quad \lambda > 0, \tag{2-15}$$

where $|\cdot|$ denotes the Lebesgue measure. Fix $q \in \mathbb{Z}_+$, and introduce the function

$$\mathcal{T}_q(x) := \tfrac{1}{2}(\Lambda_q \operatorname{Tr} m(x) - 2b \operatorname{Im} m_{12}(x)), \quad x \in \mathbb{R}^2. \tag{2-16}$$

Note that $\mathcal{T}_q(x) \ge 0$ for any $x \in \mathbb{R}^2$ and $q \in \mathbb{Z}_+$.

Theorem 2.3. *Let $m_{jk} \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $j,k = 1,2$, with $\rho > 0$. Fix $q \in \mathbb{Z}_+$. Suppose that there exists a function $0 < \tau_q \in C^\infty(\mathbb{S}^1)$ such that*

$$\lim_{|x|\to\infty} |x|^\rho \mathcal{T}_q(x) = \tau_q\left(\frac{x}{|x|}\right).$$

Then we have

$$\mathcal{N}_q^\pm(\lambda) = \frac{b}{2\pi}\Phi_{\mathcal{T}_q}(\lambda)(1+o(1)) \asymp \lambda^{-2/\rho}, \quad \lambda \downarrow 0, \tag{2-17}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho}\mathcal{N}_q^\pm(\lambda) = \mathcal{C}_q := \frac{b}{4\pi}\int_0^{2\pi} \tau_q(\cos\theta, \sin\theta)^{2/\rho}\, d\theta, \tag{2-18}$$

or to

$$\pm(\lambda_{k,q}^\pm - \Lambda_q) = \mathcal{C}_q^{\rho/2} k^{-\rho/2}(1+o(1)), \quad k \to \infty. \tag{2-19}$$

Remarks. (i) Relation (2-17) could be regarded as a semiclassical one, although here the semiclassical interpretation is somewhat implicit. In Propositions 4.1 and 4.3 below, we show that the effective Hamiltonian, which governs the asymptotics of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$, is a pseudodifferential operator with anti-Wick symbol $w_{q,b} := w_q \circ \mathcal{R}_b$ defined by (4-8) and (4-31). Under the assumptions of Theorem 2.3, $\mathcal{T}_{q,b} := \mathcal{T}_q \circ \mathcal{R}_b$ (see (2-16) and (4-31)) can be considered as the principal part of the symbol $w_{q,b}$, while the difference between the anti-Wick and the Weyl quantization is negligible. Then $(1/2\pi)\Phi_{\mathcal{T}_{q,b}}(\lambda) = (b/2\pi)\Phi_{\mathcal{T}_q}(\lambda)$ is just the main semiclassical asymptotic term for the eigenvalue counting function for a compact pseudodifferential operator with Weyl symbol $\mathcal{T}_{q,b}$.

(ii) There exists an extensive family of alternative sets of assumptions for Theorem 2.3 (see, e.g., [Ivrii 1998; Dauge and Robert 1987]). We have chosen here hypotheses which, for certain, are not the most general ones, but are quite explicit and, hopefully, easy to absorb.

Let us comment briefly on our results. Nowadays, there exists a relatively wide literature on the local spectral asymptotics for various magnetic quantum Hamiltonians. Let us concentrate here on three types of perturbations of H_0 which are considered to be of particular interest (see, e.g., [Ivrii 1998; Mao 2012]):

- Electric perturbations $H_0 + Q$ where $Q : \mathbb{R}^2 \to \mathbb{R}$ plays the role of the perturbative *electric potential*.
- Magnetic perturbations $(-i\nabla - A_0 - A)^2$, where $A = (A_1, A_2)$, and $B := \partial A_2/\partial x_1 - \partial A_1/\partial x_2$ is the perturbative *magnetic field*.
- Metric perturbations $\sum_{j,k=1,2} \Pi_j(\delta_{jk} + m_{jk})\Pi_k$, where $m = \{m_{jk}\}_{j,k=1,2}$ is an appropriate perturbative matrix-valued function.

Typically, the perturbations Q, B, or m are supposed to decay in a suitable sense at infinity. Slowly decaying Q, for example $Q \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ with $\rho > 0$, were considered in [Raĭkov 1990], and the main asymptotic terms of the corresponding counting functions $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found, utilizing, in particular, anti-Wick pseudodifferential operators. In [Ivrii 1998, Theorem 11.3.17], the case of *combined* electric, magnetic, and metric slowly decaying perturbations was investigated; the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$, as well as certain remainder estimates were obtained. The semiclassical microlocal analysis applied in [Ivrii 1998] imposed restrictions on the symbols involved, which, in some sense or another, had to decay at infinity less rapidly than their derivatives. These restrictions excluded some rapidly decaying perturbations, e.g., those of compact support, or of exponential decay with $\beta \geq \frac{1}{2}$ (see (2-5)).

Raikov and Warzel [2002] used a different approach based on the spectral analysis of Berezin–Toeplitz operators and obtained the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ in the case of potential perturbations Q of exponential decay or of compact support. In particular, in [Raikov and Warzel 2002], formulas of the type (2-4) or (2-14) appeared for the first time. Here, we essentially improve the methods developed in [Raikov and Warzel 2002]. These improvements lead also to more precise results for certain rapidly decaying electric perturbations. Namely, assume that $Q \geq 0$ admits a decay at infinity which is compatible in a suitable sense with the decay of m. Then the results of the article extend quite easily to operators of the form

$$H_{\pm} \pm Q, \tag{2-20}$$

so that $H_{\pm} \pm Q$ are perturbations of H_0 having a definite sign. We do not include these generalizations just in order to avoid an unreasonable increase of the size of the article due to results which do not require any really new arguments.

Combined perturbations of H_0 by compactly supported B and Q were considered in [Rozenblum and Tashchiyan 2008], where the main asymptotic terms of $\mathcal{N}_q^{\pm}(\lambda)$ as $\lambda \downarrow 0$ were found. Note that the magnetic perturbations of H_0 are never of fixed sign, which creates specific difficulties, successfully overcome in [Rozenblum and Tashchiyan 2008].

To our best knowledge, no results on the spectral asymptotics for rapidly decaying *metric* perturbations of H_0 appeared before in the literature. We also included in the article our result on slowly decaying metric perturbations (see Theorem 2.3), since it is coherent with the unified approach of the article and is proved by methods quite different from those in [Ivrii 1998].

Finally, let us discuss briefly the relation of $H_\pm$ to the Bochner Laplacians. Assume that the elements of m are real. In $\mathbb{R}^2$ introduce a Riemannian metric generated by the inverse of $g^\pm$, and the connection 1-form $\sum_{j=1,2} A_{0,j}\, dx_j$. Set $\gamma_\pm := (\det g^\pm)^{-1/2}$. Then the standard positive-definite Bochner Laplacian, self-adjoint in $L^2(\mathbb{R}^2; \gamma_\pm\, dx)$, is written in local coordinates as

$$\mathscr{L}_\pm := \gamma_\pm^{-1} \sum_{j,k=1,2} \Pi_j g^\pm_{jk} \gamma_\pm \Pi_k.$$

Let $U_\pm : L^2(\mathbb{R}^2; \gamma_\pm dx) \to L^2(\mathbb{R}^2; dx)$ be the unitary operator defined by $U_\pm f := \gamma_\pm^{1/2} f$. Then we have

$$U_\pm \mathscr{L}_\pm U_\pm^* = H_\pm + Q_\pm, \tag{2-21}$$

where

$$Q_\pm := \frac{1}{4} \sum_{j,k=1,2} \left(g^\pm_{jk} \frac{\partial \ln \gamma_\pm}{\partial x_k} \frac{\partial \ln \gamma_\pm}{\partial x_j} + 2 \frac{\partial}{\partial x_j} \left(g^\pm_{jk} \frac{\partial \ln \gamma_\pm}{\partial x_k} \right) \right).$$

Generally speaking, the functions $Q_\pm$ do not have a definite sign coinciding with the sign of the operators $H_\pm - H_0$; hence, the operators on the right-hand side of (2-21) are not exactly of the form of (2-20). The fact that the symbol of a Toeplitz operator does not have a definite sign may cause considerable difficulties in the study of the spectral asymptotics of this operator if the symbol decays rapidly, and, in particular, when its support is compact (see, e.g., [Pushnitski and Rozenblum 2011]). Hopefully, we will overcome these difficulties in a future work, where we would consider the local spectral asymptotics of $\mathscr{L}_\pm$.

3. Reduction to Berezin–Toeplitz operators

In this section we reduce the analysis of the functions $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the spectral asymptotics for certain compact operators of Berezin–Toeplitz type. To this end, we will need some more notations, and several auxiliary results from the abstract theory of compact operators in Hilbert space.

In what follows, we denote by $\mathbb{1}_M$ the characteristic function of the set M. Let T be a self-adjoint operator in a Hilbert space,[1] and $\mathcal{I} \subset \mathbb{R}$ be an interval. Set

$$N_{\mathcal{I}}(T) := \operatorname{rank} \mathbb{1}_{\mathcal{I}}(T),$$

where, in accordance with our general notations, $\mathbb{1}_{\mathcal{I}}(T)$ is the spectral projection of T corresponding to $\mathcal{I}$. Thus, if $\mathcal{I} \cap \sigma_{\mathrm{ess}}(T) = \varnothing$, then $N_{\mathcal{I}}(T)$ is just the number of the eigenvalues of T lying on $\mathcal{I}$ and counted with their multiplicities. In particular,

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_{q-1}, \Lambda_q - \lambda)}(H_-), \quad q \in \mathbb{Z}_+,\ \lambda \in (0, 2b), \tag{3-1}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q + \lambda, \Lambda_{q+1})}(H_+), \quad q \in \mathbb{Z}_+,\ \lambda \in (0, 2b), \tag{3-2}$$

the functions $\mathcal{N}_q^\pm$ being defined in (2-3). Let $T = T^*$ be a linear compact operator in a Hilbert space. For $s > 0$, set

$$n_\pm(s; T) := N_{(s,\infty)}(\pm T);$$

[1] All the Hilbert spaces considered in the article are assumed to be separable.

thus, $n_+(s; T)$ (resp. $n_-(s; T)$) is just the number of the eigenvalues of the operator T larger than s (resp. smaller than $-s$), counted with multiplicities. If $T_j = T_j^*$, $j = 1, 2$, are two linear compact operators acting in a given Hilbert space, then the Weyl inequalities

$$n_\pm(s_1 + s_2; T_1 + T_2) \le n_\pm(s_1; T_1) + n_\pm(s_2; T_2) \tag{3-3}$$

hold for $s_j > 0$ (see, e.g., [Birman and Solomjak 1987, Section 9.2, Theorem 9]).

Fix $q \in \mathbb{Z}_+$ and denote by P_q the orthogonal projection onto $\operatorname{Ker}(H_0 - \Lambda_q)$. Since the operator $H_0^{-1} W H_0^{-1}$ is compact, the operator $P_q W P_q = \Lambda_q^2 P_q H_0^{-1} W H_0^{-1} P_q$ is compact as well. Similarly, the operators $H_0^{-1} W H_\pm^{-1/2}$ are compact, and hence the operators

$$P_q W H_\pm^{-1} W P_q = \Lambda_q^2 P_q (H_0^{-1} W H_\pm^{-1/2})(H_\pm^{-1/2} W H_0^{-1}) P_q$$

are compact as well.

Proposition 3.1. *Under the general assumptions of the article we have*

$$\begin{aligned} &n_+((1+\varepsilon)\lambda; P_q W P_q \mp P_q W H_\pm^{-1} W P_q) + O(1) \\ &\qquad \le \mathcal{N}_q^\pm(\lambda) \le n_+((1-\varepsilon)\lambda; P_q W P_q \mp P_q W H_\pm^{-1} W P_q) + O(1), \quad \lambda \downarrow 0, \end{aligned} \tag{3-4}$$

for each $\varepsilon \in (0, 1)$.

Proof. The argument is close in spirit to the proof of [Raikov and Warzel 2002, Proposition 4.1], and is based again on the (generalized) Birman–Schwinger principle. However, since the operator $H_0^{-1/2} W H_0^{-1/2}$ is only bounded but not compact, we cannot apply the Birman–Schwinger principle to the operator pair $(H_0, H_\pm)$, and apply it instead to the resolvent pair $(H_0^{-1}, H_\pm^{-1})$. First of all, note that there exist Λ_- and Λ_+ with $\Lambda_- \in (0, \Lambda_0)$ if $q = 0$, $\Lambda_- \in (\Lambda_{q-1}, \Lambda_q)$ if $q \in \mathbb{N}$, and $\Lambda_+ \in (\Lambda_q, \Lambda_{q+1})$ if $q \in \mathbb{Z}_+$, such that

$$\mathcal{N}_q^-(\lambda) = N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-), \quad \lambda \in (0, \Lambda_q - \Lambda_-), \tag{3-5}$$

$$\mathcal{N}_q^+(\lambda) = N_{(\Lambda_q + \lambda, \Lambda_+)}(H_+), \quad \lambda \in (0, \Lambda_+ - \Lambda_q). \tag{3-6}$$

Further, evidently,

$$N_{(\Lambda_-, \Lambda_q - \lambda)}(H_-) = N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_-^{-1}) = N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-), \tag{3-7}$$

$$N_{(\Lambda_q + \lambda, \Lambda_+)}(H_+) = N_{(\Lambda_+^{-1}, (\Lambda_q + \lambda)^{-1})}(H_+^{-1}) = N_{(\Lambda_+^{-1}, (\Lambda_q + \lambda)^{-1})}(H_0^{-1} - T_+), \tag{3-8}$$

with $T_- := H_-^{-1} - H_0^{-1}$ and $T_+ := H_0^{-1} - H_+^{-1}$. Note that the operators $T_\pm$ are nonnegative and compact. By the generalized Birman–Schwinger principle (see, e.g., [Alama et al. 1989, Theorem 1.3]) we have

$$\begin{aligned} &N_{((\Lambda_q - \lambda)^{-1}, \Lambda_-^{-1})}(H_0^{-1} + T_-) \\ &= n_+\big(1; T_-^{1/2}((\Lambda_q - \lambda)^{-1} - H_0^{-1})^{-1} T_-^{1/2}\big) - n_+(1; T_-^{1/2}(\Lambda_-^{-1} - H_0^{-1})^{-1} T_-^{1/2}) - \dim \operatorname{Ker}(H_- - \Lambda_-), \end{aligned} \tag{3-9}$$

and

$$N_{(\Lambda_+^{-1},(\Lambda_q+\lambda)^{-1})}(H_0^{-1}-T_+)$$
$$=n_+\big(1;\,T_+^{1/2}(H_0^{-1}-(\Lambda_q+\lambda)^{-1})^{-1}T_+^{1/2}\big)-n_+(1;\,T_+^{1/2}(H_0^{-1}-\Lambda_+^{-1})^{-1}T_+^{1/2})-\dim\operatorname{Ker}(H_+-\Lambda_+). \quad (3\text{-}10)$$

Since the operators $T_\pm$ are compact and $\Lambda_\pm\notin\sigma(H_0)$, we find that the two last terms on the right-hand side of (3-9) and (3-10), which are independent of λ, are finite. Next, the Weyl inequalities (3-3) imply

$$\begin{aligned}
n_+\big(1+\varepsilon;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big)-n_-\big(\varepsilon;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}\big)&\\
\le n_+\big(1;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}T_-^{1/2}\big)&\\
\le n_+\big(1-\varepsilon;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big)&\\
+n_+\big(\varepsilon;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}\big)& \quad (3\text{-}11)
\end{aligned}$$

for any $\varepsilon\in(0,1)$. The operator $T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2}$ tends in norm as $\lambda\downarrow 0$ to the compact operator

$$T_-^{1/2}\Big(\sum_{j\in\mathbb{Z}_+\setminus\{q\}}(\Lambda_q^{-1}-\Lambda_j^{-1})^{-1}P_j\Big)T_-^{1/2}.$$

Therefore,

$$n_\pm(\varepsilon;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}(I-P_q)T_-^{1/2})=O(1),\quad \lambda\downarrow 0, \quad (3\text{-}12)$$

for any $\varepsilon>0$. Next, for any $s>0$ we have

$$\begin{aligned}
n_+\big(s;\,T_-^{1/2}((\Lambda_q-\lambda)^{-1}-H_0^{-1})^{-1}P_qT_-^{1/2}\big)&=n_+\big(s;\,((\Lambda_q-\lambda)^{-1}-\Lambda_q^{-1})^{-1}T_-^{1/2}P_qT_-^{1/2}\big)\\
&=n_+(s\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\,P_qT_-P_q). \quad (3\text{-}13)
\end{aligned}$$

Hence, (3-9) and (3-11)–(3-13) yield

$$\begin{aligned}
&n_+((1+\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\,P_qT_-P_q)+O(1)\\
&\quad\le N_{((\Lambda_q-\lambda)^{-1},\Lambda_-^{-1})}(H_0^{-1}+T_-)\le n_+((1-\varepsilon)\lambda(\Lambda_q-\lambda)^{-1}\Lambda_q^{-1};\,P_qT_-P_q)+O(1),\quad\lambda\downarrow 0, \quad (3\text{-}14)
\end{aligned}$$

for any $\varepsilon\in(0,1)$. Similarly, (3-10) and the analogues of (3-11)–(3-13) for positive perturbations imply

$$\begin{aligned}
&n_+((1+\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};\,P_qT_+P_q)+O(1)\\
&\quad\le N_{(\Lambda_+^{-1},(\Lambda_q+\lambda)^{-1})}(H_0^{-1}-T_+)\le n_+((1-\varepsilon)\lambda(\Lambda_q+\lambda)^{-1}\Lambda_q^{-1};\,P_qT_+P_q)+O(1),\quad\lambda\downarrow 0. \quad (3\text{-}15)
\end{aligned}$$

By the resolvent identity, we have $T_\pm=H_0^{-1}WH_0^{-1}\mp H_0^{-1}WH_\pm^{-1}WH_0^{-1}$, so that

$$P_qT_\pm P_q=\Lambda_q^{-2}(P_qWP_q\mp P_qWH_\pm^{-1}WP_q).$$

Thus,

$$n_+(s;\,P_qT_\pm P_q)=n_+(s\Lambda_q^2;\,P_qWP_q\mp P_qWH_\pm^{-1}WP_q),\quad s>0. \quad (3\text{-}16)$$

Putting together (3-5)–(3-8) and (3-14)–(3-16), we easily obtain (3-4). □

4. Unitary equivalence for Berezin–Toeplitz operators

Our first goal in this section is to show that, under certain regularity conditions on the matrix m, the operator $P_q W P_q$, $q \in \mathbb{Z}_+$, with domain $P_q L^2(\mathbb{R}^2)$, is unitarily equivalent to $P_0 w_q P_0$ with domain $P_0 L^2(\mathbb{R}^2)$, where w_q is the multiplier by a suitable function $w_q : \mathbb{R}^2 \to \mathbb{C}$. In fact, we will need a slightly more general result, and that is why we introduce first the appropriate notations.

As usual, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $z := x_1 + i x_2$ and $\bar{z} := x_1 - i x_2$, so that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Introduce the magnetic annihilation operator

$$a := -2i e^{-b|x|^2/4} \frac{\partial}{\partial \bar{z}} e^{b|x|^2/4} = -2i\left(\frac{\partial}{\partial \bar{z}} + \frac{bz}{4}\right)$$

and the magnetic creation operator

$$a^* := -2i e^{b|x|^2/4} \frac{\partial}{\partial z} e^{-b|x|^2/4} = -2i\left(\frac{\partial}{\partial z} - \frac{b\bar{z}}{4}\right)$$

with common domain $\operatorname{Dom} a = \operatorname{Dom} a^* = \operatorname{Dom} H_0^{1/2}$. The operators a and a^* are closed and mutually adjoint in $L^2(\mathbb{R}^2)$. On $\operatorname{Dom} H_0$ we have $[a, a^*] = 2b$ and

$$H_0 = a^* a + b = a a^* - b = \frac{1}{2}(a a^* + a^* a). \tag{4-1}$$

Moreover, on $\operatorname{Dom} H_0^{1/2}$ we have

$$\Pi_1 = \frac{1}{2}(a + a^*), \qquad \Pi_2 = \frac{1}{2i}(a - a^*), \tag{4-2}$$

the operators Π_j, $j = 1, 2$, being introduced in (1-1). Next, define the operator $\mathbb{A} : \operatorname{Dom} H_0^{1/2} \to L^2(\mathbb{R}^2; \mathbb{C}^2)$ by

$$\mathbb{A} u := \begin{pmatrix} a^* u \\ a u \end{pmatrix}, \qquad u \in \operatorname{Dom} H_0^{1/2}.$$

Then, (4-1) implies that $H_0 = \frac{1}{2}\mathbb{A}^*\mathbb{A}$. Further, introduce the Hermitian matrix-valued function

$$\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

with $\omega_{jk} \in L^\infty(\mathbb{R}^2)$, $j, k = 1, 2$. Fix $q \in \mathbb{Z}_+$ and define the operator

$$P_q \mathbb{A}^* \Omega \mathbb{A} P_q = \Lambda_q P_q H_0^{-1/2} \mathbb{A}^* \Omega \mathbb{A} H_0^{-1/2} P_q, \tag{4-3}$$

which is bounded and self-adjoint in $P_q L^2(\mathbb{R}^2)$. Utilizing (4-2), we easily find that

$$P_q W P_q = \tfrac{1}{2} P_q \mathbb{A}^* U \mathbb{A} P_q, \tag{4-4}$$

where

$$U := \mathbb{O}^* m \mathbb{O}, \quad \mathbb{O} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{4-5}$$

so that

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \quad \text{with} \quad \begin{aligned} u_{11} &:= \tfrac{1}{2}(\operatorname{Tr} m - 2 \operatorname{Im} m_{12}), \\ u_{22} &:= \tfrac{1}{2}(\operatorname{Tr} m + 2 \operatorname{Im} m_{12}), \\ u_{12} = \bar{u}_{21} &:= \tfrac{1}{2}(m_{11} - m_{22} - 2i \operatorname{Re} m_{12}). \end{aligned}$$

Introduce the Laguerre polynomials

$$\mathrm{L}_q^{(m)} := \sum_{j=0}^{q} \binom{q+m}{q-j} \frac{(-t)^j}{j!}, \quad t \in \mathbb{R}, \ q \in \mathbb{Z}_+, \ m \in \mathbb{Z}_+; \tag{4-6}$$

as usual, we write $\mathrm{L}_q^{(0)} = \mathrm{L}_q$, and for notational convenience we set $q\mathrm{L}_{q-1} = 0$ for $q = 0$. By [Gradshteyn and Ryzhik 1965, Equation 8.974.3] we have

$$\sum_{j=0}^{q} \mathrm{L}_j^{(m)}(t) = \mathrm{L}_q^{(m+1)}(t), \quad t \in \mathbb{R}, \ q \in \mathbb{Z}_+, \ m \in \mathbb{Z}_+. \tag{4-7}$$

Proposition 4.1. *Let Ω be a Hermitian 2×2 matrix-valued function with entries $\omega_{jk} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $j, k = 1, 2$. Fix $q \in \mathbb{Z}_+$. Then the operator $P_q \mathbb{A}^* \Omega \mathbb{A} P_q$ with domain $P_q L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator $P_0 w_q P_0$ with domain $P_0 L^2(\mathbb{R}^2)$, where*

$$w_q = w_q(\Omega) := \begin{cases} 2b(q+1)\mathrm{L}_{q+1}\Big(-\dfrac{\Delta}{2b}\Big)\omega_{11} + 2bq\mathrm{L}_{q-1}\Big(-\dfrac{\Delta}{2b}\Big)\omega_{22} - 8 \operatorname{Re} \mathrm{L}_{q-1}^{(2)}\Big(-\dfrac{\Delta}{2b}\Big)\dfrac{\partial^2 \omega_{12}}{\partial \bar{z}^2} & \text{if } q \geq 1, \\ 2b\mathrm{L}_1\Big(-\dfrac{\Delta}{2b}\Big)\omega_{11} & \text{if } q = 0, \end{cases} \tag{4-8}$$

and $\Delta = \sum_{j=1,2} \partial^2/\partial x_j^2$, so that, in accordance with (4-6), $\mathrm{L}_s^{(m)}(-\Delta/(2b))$ with $s \in \mathbb{Z}_+$ and $m \in \mathbb{Z}_+$ is just the differential operator $\sum_{j=0}^{s} \binom{s+m}{s-j} \Delta^j/(j!(2b)^j)$ of order $2s$ with constant coefficients.

Proof. Set

$$\varphi_{0,k}(x) := \sqrt{\frac{b}{2\pi k!}} \Big(\frac{b}{2}\Big)^{k/2} z^k e^{-b|x|^2/4}, \quad x \in \mathbb{R}^2, \ k \in \mathbb{Z}_+,$$

$$\varphi_{q,k}(x) := \sqrt{\frac{1}{(2b)^q q!}} (a^*)^q \varphi_{0,k}(x), \quad x \in \mathbb{R}^2, \ k \in \mathbb{Z}_+, \ q \in \mathbb{N}.$$

Then $\{\varphi_{q,k}\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis of $P_q L^2(\mathbb{R}^2)$, sometimes called *the angular momentum basis* (see, e.g., [Raikov and Warzel 2002] or [Bruneau et al. 2004, Section 9.1]). Evidently, for $k \in \mathbb{Z}_+$ we have

$$a^* \varphi_{q,k} = \sqrt{2b(q+1)} \varphi_{q+1,k}, \quad q \in \mathbb{Z}_+, \qquad a \varphi_{q,k} = \begin{cases} \sqrt{2bq}\, \varphi_{q-1,k}, & q \geq 1, \\ 0, & q = 0. \end{cases} \tag{4-9}$$

Define the unitary operator $\mathscr{W} : P_q L^2(\mathbb{R}^2) \to P_0 L^2(\mathbb{R}^2)$ by $\mathscr{W} : u \mapsto v$, where

$$u = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{q,k}, \quad v = \sum_{k\in\mathbb{Z}_+} c_k \varphi_{0,k}, \quad \{c_k\}_{k\in\mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+). \tag{4-10}$$

We will show that

$$P_q \mathbb{A}^* \Omega \mathbb{A} P_q = \mathscr{W}^* P_0 w_q P_0 \mathscr{W}. \tag{4-11}$$

For $V \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, $m, s \in \mathbb{Z}_+$, and $k, \ell \in \mathbb{Z}_+$, set

$$\Xi_{m,s}(V; k, \ell) := \langle V\varphi_{m,k}, \varphi_{s,\ell}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$. Taking into account (4-9) and (4-10), we easily find that

$$\begin{aligned}\langle P_q \mathbb{A}^* \Omega \mathbb{A} P_q u, u\rangle = 2b \sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+} ((q+1)\,\Xi_{q+1,q+1}(\omega_{11}; k, \ell) + q\,\Xi_{q-1,q-1}(\omega_{22}; k, \ell)) c_k \bar{c}_\ell \\ + 2b\sqrt{q(q+1)}\,2\,\mathrm{Re}\sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+} \Xi_{q+1,q-1}(\omega_{21}; k, \ell) c_k \bar{c}_\ell\end{aligned} \tag{4-12}$$

if $q \ge 1$, and

$$\langle P_0 \mathbb{A}^* \Omega \mathbb{A} P_0 u, u\rangle = 2b \sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+} \Xi_{1,1}(\omega_{11}; k, \ell) c_k \bar{c}_\ell. \tag{4-13}$$

Moreover,

$$\langle P_0 w_q P_0 v, v\rangle = \sum_{k\in\mathbb{Z}_+}\sum_{\ell\in\mathbb{Z}_+} \Xi_{0,0}(w_q; k, \ell) c_k \bar{c}_\ell, \quad q \in \mathbb{Z}_+. \tag{4-14}$$

In [Bruneau et al. 2004, Lemma 9.2] (see also the remark after Equation (2.2) in [Bony et al. 2014]), it was shown that

$$\Xi_{m,m}(V; k, \ell) = \Xi_{0,0}\Big(\mathrm{L}_m\Big(-\frac{\Delta}{2b}\Big)V; k, \ell\Big), \quad m \in \mathbb{Z}_+. \tag{4-15}$$

Now (4-13), (4-15) with $m = 1$ and $V = \omega_{11}$, and (4-14) with $q = 0$ imply (4-11) in the case $q = 0$. Assume $q \ge 1$. By (4-15), we have

$$\Xi_{q+1,q+1}(\omega_{11}; k, \ell) = \Xi_{0,0}\Big(\mathrm{L}_{q+1}\Big(-\frac{\Delta}{2b}\Big)\omega_{11}; k, \ell\Big), \tag{4-16}$$

$$\Xi_{q-1,q-1}(\omega_{22}; k, \ell) = \Xi_{0,0}\Big(\mathrm{L}_{q-1}\Big(-\frac{\Delta}{2b}\Big)\omega_{22}; k, \ell\Big). \tag{4-17}$$

Let us now consider the quantity $\Xi_{q+1,q-1}(V; k, \ell)$. Using (4-9), we easily find that, for $q \ge 2$, we have

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{\sqrt{2b(q+1)}}\Xi_{q,q-1}([V, a^*]; k, \ell) + \sqrt{\frac{q-1}{q+1}}\,\Xi_{q,q-2}(V; k, \ell), \tag{4-18}$$

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}}\Xi_{q-1,q-1}\big([[V, a^*], a^*]; k, \ell\big) + \sqrt{\frac{q-1}{q}}\,\Xi_{q-1,q-2}([V, a^*]; k, \ell). \tag{4-19}$$

Moreover, $[V, a^*] = 2i\,\partial V/\partial z$, and

$$[[V, a^*], a^*] = -4\frac{\partial^2 V}{\partial z^2}. \tag{4-20}$$

Using (4-19), it is not difficult to prove by induction that

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \sum_{j=0}^{q-1} \Xi_{j,j}\big([[V, a^*], a^*]; k, \ell\big), \quad q \geq 1. \tag{4-21}$$

Now (4-15), (4-20), and (4-7) imply

$$\begin{aligned}\sum_{j=0}^{q-1} \Xi_{j,j}\big([[V, a^*], a^*]; k, \ell\big) &= \sum_{j=0}^{q-1} \Xi_{0,0}\left(-4\mathrm{L}_j\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 V}{\partial z^2}; k, \ell\right) \\ &= \Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 V}{\partial z^2}; k, \ell\right).\end{aligned} \tag{4-22}$$

Setting

$$\mathscr{D}_q := -4\mathrm{L}_{q-1}^{(1)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}, \quad q \in \mathbb{N}, \tag{4-23}$$

we find that (4-21) and (4-22) imply

$$\Xi_{q,q-1}([V, a^*]; k, \ell) = \frac{1}{\sqrt{2bq}} \Xi_{0,0}(\mathscr{D}_q V; k, \ell). \tag{4-24}$$

Bearing in mind (4-18), (4-15), and (4-24), it is not difficult to prove by induction that

$$\Xi_{q+1,q-1}(V; k, \ell) = \frac{1}{2b\sqrt{q(q+1)}} \sum_{s=1}^{q} \Xi_{0,0}(\mathscr{D}_s V; k, \ell). \tag{4-25}$$

Note that (4-7) and (4-25) imply

$$\sum_{s=1}^{q} \mathscr{D}_s = -4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2}{\partial z^2}. \tag{4-26}$$

Now, (4-25) and (4-26) entail

$$2b\sqrt{q(q+1)}\,\Xi_{q+1,q-1}(\omega_{21}; k, \ell) = \Xi_{0,0}\left(-4\mathrm{L}_{q-1}^{(2)}\left(-\frac{\Delta}{2b}\right)\frac{\partial^2 \omega_{21}}{\partial z^2}; k, \ell\right). \tag{4-27}$$

Finally, (4-12) and (4-14) combined with (4-16), (4-17), and (4-27) yield (4-11) with $q \geq 1$. $\square$

In the rest of the section we establish two other suitable representations for the operators $P_q V P_q$, $q \in \mathbb{Z}_+$, with $V : \mathbb{R}^2 \to \mathbb{C}$.

Proposition 4.2. (i) [Fernández and Raikov 2004, Lemma 3.1; Bony et al. 2014, Section 2.3] *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ satisfy $\lim_{|x|\to\infty} V(x) = 0$. Then, for each $q \in \mathbb{Z}_+$, the operator $P_q V P_q$ is compact.*

(ii) [Raikov and Warzel 2002, Lemma 3.3] *Assume in addition that V is radially symmetric, i.e., there exists $v : [0,\infty) \to \mathbb{C}$ such that $V(x) = v(|x|)$, $x \in \mathbb{R}^2$. Then the eigenvalues of the operator $P_q V P_q$ with domain $P_q L^2(\mathbb{R}^2)$, counted with multiplicities, coincide with the set*

$$\{\langle V\varphi_{q,k}, \varphi_{q,k}\rangle\}_{k\in\mathbb{Z}_+}. \tag{4-28}$$

In particular, the eigenvalues of $P_0 V P_0$ coincide with

$$\frac{1}{k!}\int_0^\infty v\Big(\Big(\frac{2t}{b}\Big)^{\frac{1}{2}}\Big)e^{-t}t^k\,dt, \quad k \in \mathbb{Z}_+. \tag{4-29}$$

Remarks. (i) Let us recall that, if f is, say, a bounded function of exponential decay, then

$$(\mathcal{M}f)(z) := \int_0^\infty f(t)t^{z-1}\,dt, \quad z \in \mathbb{C},\ \operatorname{Re} z > 0,$$

is sometimes called *the Mellin transform* of f. Some of the asymptotic properties as $k \to \infty$ of the integrals (4-29), which we will later obtain and use in the proofs of Theorems 2.1 and 2.2, could possibly be deduced from the general theory of the Mellin transform.

(ii) Combining Propositions 4.1 and 4.2, we find that, if the matrix-valued function Ω is radially symmetric and diagonal, then the operator $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ acting in $P_qL^2(\mathbb{R}^2)$ is unitarily equivalent to a *diagonal* operator in $\ell^2(\mathbb{Z}_+)$. If Ω is just radially symmetric, then $P_q\mathbb{A}^*\Omega\mathbb{A}P_q$ is unitarily equivalent to a *tridiagonal* operator acting in $\ell^2(\mathbb{Z}_+)$.

The last proposition in this section concerns the unitary equivalence between the Berezin–Toeplitz operator P_0WP_0 and a certain Weyl pseudodifferential operator. Let us recall the definition of Weyl pseudodifferential operators acting in $L^2(\mathbb{R})$. Denote by $\Gamma(\mathbb{R}^2)$ the set of functions $\psi : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\|\psi\|_{\Gamma(\mathbb{R}^2)} := \sup_{(y,\eta)\in\mathbb{R}^2}\ \sup_{\ell,m=0,1}\left|\frac{\partial^{\ell+m}\psi(y,\eta)}{\partial y^\ell\partial\eta^m}\right| < \infty.$$

Then the operator $\mathrm{Op}^{\mathrm{w}}(\psi)$, defined initially as a mapping between the Schwartz class $\mathcal{S}(\mathbb{R})$ and its dual class $\mathcal{S}'(\mathbb{R})$ by

$$(\mathrm{Op}^{\mathrm{w}}(\psi)u)(y) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi\Big(\frac{y+y'}{2},\eta\Big)e^{i(y-y')\eta}u(y')\,dy'\,d\eta, \quad y \in \mathbb{R},$$

extends uniquely to an operator bounded in $L^2(\mathbb{R})$. Moreover, there exists a constant c such that

$$\|\mathrm{Op}^{\mathrm{w}}(\psi)\| \le c\|\psi\|_{\Gamma(\mathbb{R}^2)} \tag{4-30}$$

(see, e.g., [Boulkhemair 1999, Corollary 2.5(i)]).

Remark. Inequalities of the type (4-30) are known as *Calderón–Vaillancourt* estimates.

Put

$$\mathcal{R}_b := -b^{-1/2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4-31}$$

and, for $V : \mathbb{R}^2 \to \mathbb{C}$, define

$$V_b(x) := V(\mathcal{R}_b x), \quad x \in \mathbb{R}^2,\ b > 0.$$

Moreover, set $\mathcal{G}(x) := e^{-|x|^2}/\pi$, $x \in \mathbb{R}^2$.

Proposition 4.3 [Pushnitski et al. 2013, Theorem 2.11, Corollary 2.8]. *Let $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. Then the operator $P_0 V P_0$ with domain $P_0 L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator $\mathrm{Op}^{\mathrm{w}}(V_b * \mathcal{G})$.*

Remark. The operator $\mathrm{Op}^{\mathrm{aw}}(\psi) := \mathrm{Op}^{\mathrm{w}}(\psi * \mathcal{G})$ is called a pseudodifferential operator with *anti-Wick symbol* ψ (see, e.g., [Shubin 2001, Section 24]).

5. Proofs of Theorems 2.1 and 2.2

In this section we complete the proofs of Theorems 2.1 and 2.2, concerning perturbations of compact support and of exponential decay.

Let $T = T^*$ be a compact operator in a Hilbert space such that $\operatorname{rank} \mathbb{1}_{(0,\infty)}(T) = \infty$. Denote by $\{\nu_k(T)\}_{k=0}^\infty$ the nonincreasing sequence of the positive eigenvalues of T, counted with multiplicities.

Recall that $m_<(x) \le m_>(x)$ are the eigenvalues of the matrix $m(x)$ for $x \in \mathbb{R}^2$. Since the matrix U (see (4-5)) is unitarily equivalent to m, $m_\gtrless$ are also the eigenvalues of U. Next, we check that Proposition 3.1 implies the following:

Corollary 5.1. *Under the general assumptions of the article, there exist constants $0 < c_<^\pm \le c_>^\pm < \infty$ and $k_0 \in \mathbb{Z}_+$ such that*

$$c_<^\pm \nu_{k+k_0}(P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le \pm(\lambda_{k,q}^\pm - \Lambda_q) \le c_>^\pm \nu_{k-k_0}(P_q \mathbb{A}^* m_> \mathbb{A} P_q) \tag{5-1}$$

for sufficiently large $k \in \mathbb{N}$.

Proof. It is easy to see that

$$0 \le P_q W H_\pm^{-1} W P_q \le c_\pm P_q W P_q \tag{5-2}$$

with

$$c_\pm := \|H_\pm^{-1/2} W H_\pm^{-1/2}\| \le \sup_{x\in\mathbb{R}^2} \left| m(x)(I \pm m(x))^{-1} \right|.$$

Note that $0 \le c_- < \infty$ and $0 \le c_+ < 1$. Moreover, by (4-4) and the mini-max principle,

$$n_+(2s;\ P_q \mathbb{A}^* m_< \mathbb{A} P_q) \le n_+(s;\ P_q W P_q) \le n_+(2s;\ P_q \mathbb{A}^* m_> \mathbb{A} P_q), \quad s > 0. \tag{5-3}$$

Now, (3-4), (5-2), and (5-3) imply that, for any $\varepsilon \in (0, 1)$, we have

$$n_+(2\lambda(1+\varepsilon);\ P_q \mathbb{A}^* m_< \mathbb{A} P_q) + O(1) \le \mathcal{N}_q^-(\lambda) \le n_+(2\lambda(1-\varepsilon);\ (1+c_-) P_q \mathbb{A}^* m_> \mathbb{A} P_q) + O(1), \tag{5-4}$$

$$n_+(2\lambda(1-\varepsilon);\ P_q \mathbb{A}^* m_> \mathbb{A} P_q) + O(1) \ge \mathcal{N}_q^+(\lambda) \ge n_+(2\lambda(1+\varepsilon);\ (1-c_+) P_q \mathbb{A}^* m_< \mathbb{A} P_q) + O(1) \tag{5-5}$$

as $\lambda \downarrow 0$, and estimates (5-4)–(5-5) yield (5-1) with

$$c_<^- = \frac{1}{2(1+\varepsilon)}, \quad c_>^- = \frac{1+c_-}{2(1-\varepsilon)}, \quad c_<^+ = \frac{1-c_+}{2(1+\varepsilon)}, \quad c_>^+ = \frac{1}{2(1-\varepsilon)},$$

and sufficiently large $k_0 \in \mathbb{N}$. $\square$

Let us now complete the proof of Theorem 2.1. Let $\zeta_1 \in C_0^\infty(\mathbb{R}^2)$, $\zeta_1 \geq 0$, $\zeta_1 = 1$ on $\operatorname{supp} m_>$. Set $\zeta_2(x) := (\max_{y\in\mathbb{R}^2} m_>(y))\zeta_1(x)$, $x \in \mathbb{R}^2$. Evidently, $m_> \leq \zeta_2$ on $\mathbb{R}^2$, so that

$$\nu_k(P_q\mathbb{A}^*m_>\mathbb{A}P_q) \leq \nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q), \quad k \in \mathbb{Z}_+. \tag{5-6}$$

Further, by Proposition 4.1, the operator $P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q$ is unitarily equivalent to the operator $P_0\zeta_3P_0$, where

$$\zeta_3 := 2b\left((q+1)\mathrm{L}_{q+1}\left(-\frac{\Delta}{2b}\right) + q\mathrm{L}_{q-1}\left(-\frac{\Delta}{2b}\right)\right)\zeta_2.$$

Therefore,

$$\nu_k(P_q\mathbb{A}^*\zeta_2\mathbb{A}P_q) = \nu_k(P_0\zeta_3P_0), \quad k \in \mathbb{Z}_+. \tag{5-7}$$

Let $R_> > 0$ be so large that the disk $B_{R_>}(0)$ of radius $R_>$ centered at the origin contains the support of ζ_3. Then,

$$\nu_k(P_0\zeta_3P_0) \leq \max_{x\in\mathbb{R}^2} |\zeta_3(x)|\, \nu_k(P_0\mathbb{1}_{B_{R_>}(0)}P_0), \quad k \in \mathbb{Z}_+. \tag{5-8}$$

Putting together (5-6), (5-7), and (5-8), we find that there exists a constant $K_> < \infty$ such that

$$\nu_k(P_q\mathbb{A}^*m_>\mathbb{A}P_q) \leq K_>\nu_k(P_0\mathbb{1}_{B_{R_>}(0)}P_0), \quad k \in \mathbb{Z}_+. \tag{5-9}$$

On the other hand,

$$\nu_k(P_q\mathbb{A}^*m_<\mathbb{A}P_q) \geq \nu_k(P_q a m_< a^* P_q). \tag{5-10}$$

Applying (4-9), we easily find that the operators $P_q a m_< a^* P_q$ and $2b(q+1)P_{q+1}m_<P_{q+1}$ are unitarily equivalent. Hence,

$$\nu_k(P_q a m_< a^* P_q) = 2b(q+1)\nu_k(P_{q+1}m_<P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-11}$$

Further, since $m_<$ is nonnegative, continuous, and does not vanish identically, there exist $c_0 > 0$, $R_< \in (0,\infty)$, and $x_0 \in \mathbb{R}^2$ such that $m_<(x) \geq c_0\mathbb{1}_{B_{R_<}(x_0)}(x)$, $x \in \mathbb{R}^2$. Therefore,

$$\nu_k(P_{q+1}m_<P_{q+1}) \geq c_0\nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-12}$$

The operators $P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}$ and $P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}$ are unitarily equivalent under the magnetic translation which maps x_0 into 0 (see, e.g., [Raikov and Warzel 2002, Equation (4.21)]). Therefore,

$$\nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(x_0)}P_{q+1}) = \nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}), \quad k \in \mathbb{Z}_+. \tag{5-13}$$

Combining (5-10)–(5-13), we find that there exists a constant $K_<$ such that

$$K_<\nu_k(P_{q+1}\mathbb{1}_{B_{R_<}(0)}P_{q+1}) \leq \nu_k(P_q\mathbb{A}^*m_<\mathbb{A}P_q), \quad k \in \mathbb{Z}_+. \tag{5-14}$$

By (5-9) and (5-14), it remains to study the asymptotic behavior as $k \to \infty$ of $\nu_k(P_m\mathbb{1}_{B_R(0)}P_m)$, with $m \in \mathbb{Z}_+$ and $R \in (0,\infty)$ fixed. This asymptotic analysis relies on the representation (4-28), and results sufficient for our purposes are available in the literature. Namely, we have:

Lemma 5.2 [Combes et al. 2004, Section 4, Corollary 2]. *Let $m \in \mathbb{Z}_+$, $R \in (0, \infty)$, $b \in (0, \infty)$. Set $\varrho := bR^2/2$. Then*

$$\nu_k(P_m \mathbb{1}_{B_R(0)} P_m) = \frac{e^{-\varrho} \varrho^{-m+1} k^{2m-1} \varrho^k}{m! k!} (1 + o(1)), \quad k \to \infty. \tag{5-15}$$

Now, asymptotic relation (2-1) follows from (5-1), (5-9), (5-14), (5-15), and the elementary fact that $\ln k! = k \ln k + O(k)$ as $k \to \infty$.

In the remaining part of this section we prove Theorem 2.2 concerning perturbations m of exponential decay. Assume that m satisfies (2-5). Then there exist $\delta_{\gtrless} \in \mathbb{R}$, $\delta_< \le \delta_>$, and $r > 1$ such that

$$|x|^{\delta_<} e^{-\gamma |x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) \le m_<(x) \le m_>(x) \le |x|^{\delta_>} e^{-\gamma |x|^{2\beta}} \mathbb{1}_{\mathbb{R}^2 \setminus B_r(0)}(x) + \max_{y \in \mathbb{R}^2} m_>(y) \mathbb{1}_{B_r(0)}(x), \quad x \in \mathbb{R}^2. \tag{5-16}$$

Let $\eta_{\gtrless,0} \in C^\infty(\mathbb{R}^2; [0, 1])$ be two radially symmetric functions such that $\eta_{<,0} = 1$ on $\mathbb{R}^2 \setminus B_{r+1}(0)$, $\eta_{<,0} = 0$ on $B_r(0)$, and $\eta_{>,0} = 1$ on $\mathbb{R}^2 \setminus B_r(0)$, $\eta_{>,0} = 0$ on $B_{r-1}(0)$. For $x \in \mathbb{R}^2$ set

$$\begin{aligned} \eta_{<,1}(x) &:= |x|^{\delta_<} e^{-\gamma |x|^{2\beta}} \eta_{<,0}(x), \\ \eta_{>,1}(x) &:= |x|^{\delta_>} e^{-\gamma |x|^{2\beta}} \eta_{>,0}(x) + \max_{y \in \mathbb{R}^2} m_>(y)(1 - \eta_{<,0}(x)). \end{aligned}$$

Evidently, $\eta_{\gtrless,1} \in C_{\mathrm{b}}^\infty(\mathbb{R}^2)$, and by (5-16),

$$\eta_{<,1}(x) \le m_<(x), \quad m_>(x) \le \eta_{>,1}(x), \quad x \in \mathbb{R}^2.$$

Therefore, for $k \in \mathbb{Z}_+$, we have

$$\begin{aligned} \nu_k(P_q \mathbb{A}^* m_< \mathbb{A} P_q) &\ge \nu_k(P_q \mathbb{A}^* \eta_{<,1} \mathbb{A} P_q), \\ \nu_k(P_q \mathbb{A}^* m_> \mathbb{A} P_q) &\le \nu_k(P_q \mathbb{A}^* \eta_{>,1} \mathbb{A} P_q). \end{aligned} \tag{5-17}$$

Further, set

$$\eta_{\gtrless,2} := 2b \left((q+1) \mathrm{L}_{q+1} \left(-\frac{\Delta}{2b} \right) + q \mathrm{L}_{q-1} \left(-\frac{\Delta}{2b} \right) \right) \eta_{\gtrless,1}.$$

According to Proposition 4.1, the operators $P_q \mathbb{A}^* \eta_{\gtrless,1} \mathbb{A} P_q$, $q \in \mathbb{Z}_+$, and $P_0 \eta_{\gtrless,2} P_0$ are unitarily equivalent. Therefore,

$$\nu_k(P_q \mathbb{A}^* \eta_{\gtrless,1} \mathbb{A} P_q) = \nu_k(P_0 \eta_{\gtrless,2} P_0), \quad k \in \mathbb{Z}_+. \tag{5-18}$$

Next, a tedious but straightforward calculation shows that

$$\eta_{\gtrless,2}(x) = \eta_{\gtrless,3}(x)(1 + o(1)), \quad |x| \to \infty, \tag{5-19}$$

where

$$\eta_{\gtrless,3}(x) := C_{q,\beta} |x|^{\delta_{\gtrless}} e^{-\gamma |x|^{2\beta}} \begin{cases} 1 & \text{if } \beta \in \left(0, \frac{1}{2}\right], \\ |x|^{2(q+1)(2\beta-1)} & \text{if } \beta \in \left(\frac{1}{2}, \infty\right), \end{cases} \quad x \in \mathbb{R}^2 \setminus \{0\},$$

and $C_{q,\beta} > 0$ are some constants. Even though the exact values of $C_{q,\beta}$ will not play any role in the sequel, we indicate here these values for the sake of the completeness of the exposition:

$$C_{q,\beta} = \begin{cases} 2\Lambda_q & \text{if } \beta \in \left(0, \frac{1}{2}\right), \\ 2b\Big((q+1)\mathrm{L}_{q+1}\Big(-\frac{(2\beta\gamma)^2}{2b}\Big) + q\mathrm{L}_{q-1}\Big(-\frac{(2\beta\gamma)^2}{2b}\Big)\Big) & \text{if } \beta = \frac{1}{2}, \\ \dfrac{(2\beta\gamma)^{2(q+1)}}{(2b)^q q!} & \text{if } \beta \in \left(\frac{1}{2}, \infty\right). \end{cases}$$

Hence, by (5-19), there exists $R \in (0, \infty)$ such that for $x \in \mathbb{R}^2$ we have

$$\eta_{<,2} \geq \tfrac{1}{2}\eta_{<,3}\mathbb{1}_{\mathbb{R}^2 \setminus B_R(0)} - c_< \mathbb{1}_{B_R(0)} =: \eta_{<,4}(x), \tag{5-20}$$

$$\eta_{>,2} \leq \tfrac{3}{2}\eta_{>,3}\mathbb{1}_{\mathbb{R}^2 \setminus B_R(0)} + c_> \mathbb{1}_{B_R(0)} =: \eta_{>,4}(x) \tag{5-21}$$

with $c_{\gtrless} := \max_{y \in \mathbb{R}^2} |\eta_{\gtrless,2}(y)|$. Thus, for any admissible $k \in \mathbb{Z}_+$, we have

$$\nu_k(P_0\eta_{<,2}P_0) \geq \nu_k(P_0\eta_{<,4}P_0), \qquad \nu_k(P_0\eta_{>,2}P_0) \leq \nu_k(P_0\eta_{>,4}P_0). \tag{5-22}$$

In order to complete the proof of Theorem 2.2, we need a couple of auxiliary results. For $\beta > 0$, $\mu > 0$, and $\varrho > 0$, set

$$\mathscr{I}_{\beta,\mu}(k) := \int_0^\infty e^{-\mu t^\beta - t} t^k \, dt, \quad \mathscr{E}_\varrho(k) := \int_0^\varrho e^{-t} t^k \, dt, \quad k > -1, \tag{5-23}$$

and, for $\delta \in \mathbb{R}$, $c_0 > 0$ and $c_1 \in \mathbb{R}$, put

$$\mathscr{L}(k) = \mathscr{L}_{\beta,\mu,\varrho,\delta}(k; c_0, c_1) := \frac{c_0\mathscr{I}_{\beta,\mu}(k+\delta) + c_1\mathscr{E}_\varrho(k-\delta_-)}{\Gamma(k+1)}, \quad k > \max\{-1, -\delta-1\},$$

where $\delta_- := \max\{0, -\delta\}$.

Lemma 5.3. *Let $\beta > 0$, $\mu > 0$, $\varrho > 0$, $c_0 > 0$, and $\delta \in \mathbb{R}$, $c_1 \in \mathbb{R}$.*

(i) *The asymptotic relations*

$$\ln \mathscr{L}(k) = \begin{cases} -\sum_{1 \leq j < 1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k) & \textit{if } \beta \in (0,1), \\ -\ln(1+\mu)k + O(\ln k) & \textit{if } \beta = 1, \\ -\dfrac{\beta-1}{\beta} k \ln k + k\Big(\dfrac{\beta - 1 - \ln(\mu\beta)}{\beta}\Big) & \\ \qquad\qquad - \sum_{1 \leq j < \beta/(\beta-1)} g_j k^{(1/\beta-1)j+1} + O(\ln k) & \textit{if } \beta \in (1, \infty), \end{cases} \tag{5-24}$$

hold true as $k \to \infty$, the coefficients f_j and g_j being introduced in the statement of Theorem 2.2.

(ii) *We have $\mathscr{L}'(k) < 0$ for sufficiently large k.*

Proof. First, let $\delta = 0$. Assume $\beta \in (0, 1)$, $k > 0$, and make the change of variable $t \mapsto ks$ in the first integral in (5-23). Thus we find that

$$\mathscr{I}_{\beta,\mu}(k) = k^{k+1} \int_0^\infty e^{-kF(s;k^{\beta-1})} \, ds. \tag{5-25}$$

The function $F(s;k^{\beta-1})$ defined in (2-9) attains its unique minimum at $s_<(k^{\beta-1})$, and we have $\partial^2 F(s_<(k^{\beta-1});k^{\beta-1})/\partial s^2 = 1+o(1)$, $k\to\infty$. Therefore, applying a standard argument close to the usual Laplace method for asymptotic evaluation of integrals depending on a large parameter, we easily find that

$$\int_0^\infty e^{-kF(s;k^{\beta-1})}\,ds = (2\pi)^{1/2}e^{-kF(s_<(k^{\beta-1});k^{\beta-1})}k^{-1/2}(1+o(1)),\quad k\to\infty. \tag{5-26}$$

Bearing in mind that $F(s_<(k^{\beta-1});k^{\beta-1}) = f(k^{\beta-1})$ (see (2-10)), $f(0)=1$, and

$$\ln\Gamma(k+1) = k\ln k - k + \tfrac{1}{2}\ln k + O(1),\quad k\to\infty, \tag{5-27}$$

(see, e.g., [Abramowitz and Stegun 1964, Equation 6.1.40]), we find that (5-25)–(5-26) imply

$$\begin{aligned}\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= k - kf(k^{\beta-1}) + O(\ln k)\\ &= k - k\sum_{0\le j<1/(1-\beta)}\frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0)k^{(\beta-1)j} + O(\ln k)\\ &= -\sum_{1\le j<1/(1-\beta)}\frac{1}{j!}\frac{d^j f}{d\epsilon^j}(0)k^{(\beta-1)j+1} + O(\ln k)\\ &= -\sum_{1\le j<1/(1-\beta)} f_j k^{(\beta-1)j+1} + O(\ln k),\quad k\to\infty.\end{aligned} \tag{5-28}$$

In the case $\beta=1$, we simply have

$$\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)} = \frac{1}{\Gamma(k+1)}\int_0^\infty e^{-(\mu+1)t}t^k\,dt = (\mu+1)^{-k-1},$$

that is,

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = -(\ln(1+\mu))k + O(1),\quad k\to\infty. \tag{5-29}$$

Now let $\beta\in(1,\infty)$. Making the change of variable $t\mapsto k^{1/\beta}s$ with $k>0$ in (5-23), we find

$$\mathcal{J}_{\beta,\mu}(k) := k^{(k+1)/\beta}\int_0^\infty e^{-kG(s;k^{(1/\beta-1)})}\,ds. \tag{5-30}$$

The function $G(s;k^{1/\beta-1})$ defined in (2-11), attains its unique minimum at $s_>(k^{1/\beta-1})$, and we have

$$\frac{\partial^2 G}{\partial s^2}(s_>(k^{1/\beta-1}),k^{1/\beta-1}) = \beta(\mu\beta)^{2/\beta}(1+o(1)),\quad k\to\infty.$$

Arguing as in the derivation of (5-26), we obtain

$$\int_0^\infty e^{-kG(s;k^{1/\beta-1})}\,ds = \sqrt{2\pi\beta}\,(\mu\beta)^{-1/\beta}e^{-kG(s_>(k^{1/\beta-1});k^{1/\beta-1})}k^{-1/2}(1+o(1)),\quad k\to\infty. \tag{5-31}$$

Bearing in mind that $G(s_>(k^{1/\beta-1}); k^{1/\beta-1}) = g(k^{1/\beta-1})$ (see (2-12)), and $g(0) = (1+\ln(\mu\beta))/\beta$, we find that (5-30), (5-31), and (5-27) imply

$$\begin{aligned} \ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) &= -\frac{\beta-1}{\beta}k\ln k + k - kg(k^{1/\beta-1}) + O(\ln k) \\ &= -\frac{\beta-1}{\beta}k\ln k + k - k\sum_{0\le j<\beta/(\beta-1)} \frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(1/\beta-1)j} + O(\ln k) \\ &= -\frac{\beta-1}{\beta}k\ln k + k(1-g(0)) - \sum_{1\le j<\beta/(\beta-1)} \frac{1}{j!}\frac{d^jg}{d\epsilon^j}(0)k^{(1/\beta-1)j+1} + O(\ln k) \\ &= -\frac{\beta-1}{\beta}k\ln k + k\left(\frac{\beta-1-\ln(\mu\beta)}{\beta}\right) - \sum_{1\le j<\beta/(\beta-1)} g_jk^{(1/\beta-1)j+1} + O(\ln k), \end{aligned} \tag{5-32}$$

as $k\to\infty$. Let us now consider general $\delta\in\mathbb{R}$. By (5-27),

$$\ln\left(\frac{\Gamma(k+\delta+1)}{\Gamma(k+1)}\right) = \delta\ln k + O(1), \quad k\to\infty. \tag{5-33}$$

Putting together (5-28), (5-29), (5-32), and (5-33), we find that

$$\ln\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right) - \ln\left(\frac{\mathcal{J}_{\beta,\mu}(k)}{\Gamma(k+1)}\right) = O(\ln k), \quad k\to\infty. \tag{5-34}$$

Finally, by (5-15), we easily find that, for each fixed $\delta\in\mathbb{R}$, we have

$$\frac{\mathcal{E}_\varrho(k-\delta_-)}{\Gamma(k+1)} = o\left(\frac{\mathcal{J}_{\beta,\mu}(k+\delta)}{\Gamma(k+1)}\right), \quad k\to\infty. \tag{5-35}$$

The combination of (5-28), (5-29), (5-32), (5-34), and (5-35) implies (5-24).

For (ii), we have

$$\mathcal{L}'(k) = c_0\left(\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{J}_{\beta,\mu}(k+\delta)\right) + c_1\left(\frac{\mathcal{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{E}_\varrho(k-\delta_-)\right), \tag{5-36}$$

$$\mathcal{J}'_{\beta,\mu}(k) = \int_0^\infty e^{-\mu t^\beta - t}t^k\ln t\,dt, \quad \mathcal{E}'_\varrho(k) = \int_0^\varrho e^{-t}t^k\ln t\,dt,$$

and

$$\frac{\Gamma'(k+1)}{\Gamma(k+1)} = \ln k + \frac{1}{2k} + O(k^{-2}), \quad k\to\infty,$$

(see, e.g., [Abramowitz and Stegun 1964, Equation 6.3.18]). Performing an asymptotic analysis similar to the one in the proof of the first part of the lemma, we find that there exists a function $\Psi = \Psi_{\beta,\mu,\delta}$ such that $\Psi(k) < 0$ for k large enough, and

$$\frac{\mathcal{J}'_{\beta,\mu}(k+\delta)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{J}_{\beta,\mu}(k+\delta) = \Psi(k)(1+o(1)), \tag{5-37}$$

$$\frac{\mathcal{E}'_\varrho(k-\delta_-)}{\Gamma(k+1)} - \frac{\Gamma'(k+1)}{\Gamma(k+1)^2}\mathcal{E}_\varrho(k-\delta_-) = o(\Psi(k)) \tag{5-38}$$

as $k \to \infty$. Putting together (5-36), (5-37), and (5-38), we conclude that $\mathscr{L}'(k) < 0$ for sufficiently large k. □

Taking into account the definition of the functions $\eta_{\gtrless,4}$ in (5-20)–(5-21), the mini-max principle, representation (4-29), as well as Lemma 5.3(ii), we find that there exist constants $c_{j,\gtrless} > 0$, $j = 0, 1$, $\tilde{\delta}_{\gtrless} \in \mathbb{R}$, and $k_0 \in \mathbb{Z}_+$ such that

$$\begin{aligned} \nu_k(P_0 \eta_{<,4} P_0) &\geq \mathscr{L}_{\beta,\mu,\varrho,\tilde{\delta}_<}(k + k_0; c_{0,<}, -c_{1,<}) \\ \nu_k(P_0 \eta_{>,4} P_0) &\leq \mathscr{L}_{\beta,\mu,\varrho,\tilde{\delta}_>}(k; c_{0,>}, c_{1,>}), \end{aligned} \tag{5-39}$$

for $\mu = \gamma(2/b)^\beta$, $\varrho = bR^2/2$, and sufficiently large $k \in \mathbb{Z}_+$.

Putting together (5-1), (5-17), (5-18), (5-22), (5-39), and (5-24), we obtain (2-6)–(2-8).

6. Proof of Theorem 2.3

Estimates (3-4) combined with the Weyl inequalities (3-3) and the mini-max principle entail

$$\begin{aligned} &n_+(\lambda(1+\varepsilon);\, P_q W P_q) + O(1) \\ &\qquad \leq \mathcal{N}_q^-(\lambda) \leq n_+(\lambda(1-\varepsilon)^2;\, P_q W P_q) + n_+(\lambda\varepsilon(1-\varepsilon);\, P_q W H_-^{-1} W P_q) + O(1), \end{aligned} \tag{6-1}$$

and

$$\begin{aligned} &n_+(\lambda(1+\varepsilon)^2;\, P_q W P_q) - n_+(\lambda\varepsilon(1+\varepsilon);\, P_q W H_+^{-1} W P_q) + O(1) \\ &\qquad \leq \mathcal{N}_q^+(\lambda) \leq n_+(\lambda(1-\varepsilon);\, P_q W P_q) + O(1) \end{aligned} \tag{6-2}$$

as $\lambda \downarrow 0$. It is easy to check that we have

$$P_q W H_\pm^{-1} W P_q \leq C_{1,\pm} P_q \mathbb{A}^* \langle \cdot \rangle^{-2\rho} \mathbb{A} P_q$$

with

$$C_{1,\pm} := \| H_0^{1/2} H_\pm^{-1/2} \|^2 \big(\sup_{x \in \mathbb{R}^2} \langle x \rangle^\rho m_>(x) \big)^2.$$

Therefore, for any $s > 0$,

$$n_+(s;\, P_q W H_\pm^{-1} W P_q) \leq n_+(s;\, C_{1,\pm} P_q \mathbb{A}^* \langle \cdot \rangle^{-2\rho} \mathbb{A} P_q). \tag{6-3}$$

Further, by Proposition 4.1, the operator $P_q W P_q$ (resp. $P_q \mathbb{A}^* \langle \cdot \rangle^{-2\rho} \mathbb{A} P_q$) is unitarily equivalent to $\frac{1}{2} P_0 w_q(U) P_0$ (resp. $P_0 w_q(\langle \cdot \rangle^{-2\rho} I) P_0$). Hence, for any $s > 0$,

$$n_+(s;\, P_q W P_q) = n_+(2s;\, P_0 w_q(U) P_0), \tag{6-4}$$

$$n_+(s;\, P_q \mathbb{A}^* \langle \cdot \rangle^{-2\rho} \mathbb{A} P_q) = n_+(s;\, P_0 w_q(\langle \cdot \rangle^{-2\rho} I) P_0) \leq n_+(s;\, C_2 P_0 \langle \cdot \rangle^{-2\rho} P_0) \tag{6-5}$$

with $C_2 := \sup_{x \in \mathbb{R}^2} \langle x \rangle^{2\rho} |w_q(\langle x \rangle^{-2\rho} I)|$. Now, write

$$\tfrac{1}{2} w_q(U) = \mathscr{T}_q + \tilde{\mathscr{T}}_q,$$

the symbol $\mathcal{T}_q$ being defined in (2-16), and note the crucial circumstance that $\tilde{\mathcal{T}}_q \in \mathcal{S}^{-\rho-2}(\mathbb{R}^2)$. Then the Weyl inequalities (3-3) entail

$$\begin{aligned} n_+(s(1+\varepsilon);\, P_0\mathcal{T}_q P_0) - n_-(s\varepsilon;\, P_0\tilde{\mathcal{T}}_q P_0) &\le n_+(2s;\, P_0 w_q(U) P_0) \\ &\le n_+(s(1-\varepsilon);\, P_0\mathcal{T}_q P_0) + n_+(s\varepsilon;\, P_0\tilde{\mathcal{T}}_q P_0) \end{aligned} \tag{6-6}$$

for any $s > 0$ and $\varepsilon \in (0, 1)$. Evidently,

$$n_\pm(s;\, P_0\tilde{\mathcal{T}}_q P_0) \le n_+(s;\, C_3 P_0\langle\,\cdot\,\rangle^{-\rho-2} P_0), \quad s > 0, \tag{6-7}$$

with $C_3 := \sup_{x\in\mathbb{R}^2}\langle x\rangle^{\rho+2}|\tilde{\mathcal{T}}_q(x)|$. Recalling Proposition 4.3, we find that we have reduced the asymptotic analysis of $\mathcal{N}_q^\pm(\lambda)$ as $\lambda \downarrow 0$ to the eigenvalue asymptotics for a pseudodifferential operator with elliptic anti-Wick symbol of negative order. The spectral asymptotics for operators of this type has been extensively studied in the literature since the 1970s. In particular, we have the following:

Proposition 6.1. *Let $0 \le \psi \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 0$. Assume that there exists $0 < \psi_0 \in C^\infty(\mathbb{S}^1)$ such that $\lim_{|x|\to\infty} |x|^\rho \psi(x) = \psi_0(x/|x|)$. Then we have*

$$n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) = (2\pi)^{-1}\Phi_\psi(\lambda)(1+o(1)), \quad \lambda \downarrow 0, \tag{6-8}$$

which is equivalent to

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) = \mathscr{C}(\psi_0) := \frac{1}{4\pi}\int_0^{2\pi} \psi_0(\cos\theta, \sin\theta)^{2/\rho}\, d\theta.$$

Proof. Evidently, for each $\varepsilon \in (0, 1)$ there exist real functions $\psi_{\pm,\varepsilon} \in C^\infty(\mathbb{R}^2)$ such that

$$\begin{gathered} \psi_{-,\varepsilon}(x) \le \psi(x) \le \psi_{+,\varepsilon}(x), \quad x \in \mathbb{R}^2, \\ \psi_{\pm,\varepsilon}(x) = (1 \mp \varepsilon)^{-1}|x|^{-\rho}\psi_0\Big(\frac{x}{|x|}\Big), \quad x \in \mathbb{R}^2,\ |x| \ge R, \end{gathered}$$

for some $R \in (0, \infty)$. Applying the monotonicity of the anti-Wick quantization with respect to the symbol (see, e.g., [Shubin 2001, Proposition 24.1]), the mini-max principle, and the Weyl inequalities, we obtain

$$\begin{aligned} &n_+((1+\varepsilon)\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon})) - n_-\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{-,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{-,\varepsilon}))\big) \\ &\quad \le n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le n_+((1-\varepsilon)\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon})) + n_+\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{+,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{+,\varepsilon}))\big). \end{aligned} \tag{6-9}$$

By [Dauge and Robert 1987], we have the semiclassical result

$$n_+(\lambda;\, \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})) = (2\pi)^{-1}\Phi_{\psi_{\pm,\varepsilon}}(\lambda)(1+o(1)), \quad \lambda \downarrow 0. \tag{6-10}$$

Further, by [Shubin 2001, Theorem 24.1] the differences $\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$ are pseudodifferential operators of lower order than $\mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon})$, so that we easily obtain

$$\lim_{\lambda\downarrow 0} \lambda^{2/\rho} n_\pm\big(\varepsilon\lambda;\, (\mathrm{Op}^{\mathrm{aw}}(\psi_{\pm,\varepsilon}) - \mathrm{Op}^{\mathrm{w}}(\psi_{\pm,\varepsilon}))\big) = 0, \quad \varepsilon > 0. \tag{6-11}$$

Now, (6-9)–(6-11) imply

$$(1+\varepsilon)^{-4/\rho}\mathscr{C}(\psi_0) \le \liminf_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le \limsup_{\lambda\downarrow 0} \lambda^{2/\rho} n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\psi)) \le (1-\varepsilon)^{-4/\rho}\mathscr{C}(\psi_0)$$

for $\varepsilon \in (0,1)$. Letting $\varepsilon \downarrow 0$, we obtain (6-8). □

By Propositions 4.3 and 6.1, we have

$$n_+(\lambda;\, P_0\mathscr{T}_q P_0) = n_+(\lambda;\, \mathrm{Op}^{\mathrm{aw}}(\mathscr{T}_{q,b})) = \frac{1}{2\pi}\Phi_{\mathscr{T}_{q,b}}(\lambda)(1+o(1)) = \frac{b}{2\pi}\Phi_{\mathscr{T}_q}(\lambda)(1+o(1)), \quad \lambda \downarrow 0, \tag{6-12}$$

with $\mathscr{T}_{q,b} = \mathscr{T}_q \circ \mathscr{R}_b$, $\mathscr{R}_b$ being defined in (4-31). Finally, for $\rho_0 > \rho$, we have

$$n_+(\lambda;\, P_0\langle\,\cdot\,\rangle^{-\rho_0} P_0) = O(\lambda^{-2/\rho_0}) = o(\Phi_{\mathscr{T}_q}(\lambda)), \quad \lambda \downarrow 0. \tag{6-13}$$

Now, (2-17) easily follows from (6-1)–(6-8), (6-12), and (6-13). The equivalence of (2-18) and (2-19) can be checked by arguing as in the proof of [Shubin 2001, Proposition 13.1].

Appendix: Compactness of the resolvent differences

A priori, the operators H_0 and $H_\pm$, self-adjoint in $L^2(\mathbb{R}^2)$, could be defined as the Friedrichs extensions of the operators $\sum_{j=1,2}\Pi_j^2$ and $\sum_{j,k=1,2}\Pi_j g_{jk}^{\pm}\Pi_k$ defined on $C_0^\infty(\mathbb{R}^2)$. Such a definition implies immediately that

$$\mathrm{Dom}\, H_0^{1/2} = \mathrm{Dom}\, H_\pm^{1/2} = \{u \in L^2(\mathbb{R}^2) \mid \Pi_j u \in L^2(\mathbb{R}^2),\ j = 1, 2\},$$

and that the operators $H_\pm^{1/2}H_0^{-1/2}$ and $H_0^{1/2}H_\pm^{-1/2}$ are bounded. By [Gérard et al. 1991, Proposition A.2], the operators H_0 and $H_\pm$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and have a common domain

$$\mathrm{Dom}\, H_0 = \mathrm{Dom}\, H_\pm = \{u \in L^2(\mathbb{R}^2) \mid \Pi_j\Pi_k u \in L^2(\mathbb{R}^2),\ j,k = 1, 2\}.$$

Let us now prove the compactness of the operator $H_0^{-1} - H_\pm^{-1}$ in $L^2(\mathbb{R}^2)$. Since we have

$$H_0^{-1} - H_\pm^{-1} = \pm H_0^{-1} W H_\pm^{-1} = \pm H_0^{-1} W H_0^{-1} H_0 H_\pm^{-1},$$

it suffices to prove the compactness of $H_0^{-1}WH_0^{-1}$. The operators $H_0^{-1}WH_0^{-1} = \frac{1}{2}H_0^{-1}\mathbb{A}^*U\mathbb{A}H_0^{-1}$ and $\frac{1}{2}H_0^{-1}\mathbb{A}^*m_>\mathbb{A}H_0^{-1}$ are bounded, self-adjoint, and positive. Moreover,

$$H_0^{-1}\mathbb{A}^*U\mathbb{A}H_0^{-1} \le H_0^{-1}\mathbb{A}^*m_>\mathbb{A}H_0^{-1}. \tag{A-1}$$

On the other hand,

$$H_0^{-1}\mathbb{A}^*m_>\mathbb{A}H_0^{-1} = H_0^{-1}a^*m_>aH_0^{-1} + H_0^{-1}am_>a^*H_0^{-1}. \tag{A-2}$$

By (A-1) and (A-2), it suffices to prove the compactness of the operator $m_>^{1/2}a^*H_0^{-1}$. We have

$$m_>^{1/2}a^*H_0^{-1} = m_>^{1/2}H_0^{-1/2}(H_0^{-1/2}a^* + 2bH_0^{-1/2}a^*H_0^{-1}).$$

The operator $H_0^{-1/2}a^* + 2bH_0^{-1/2}a^*H_0^{-1}$ is bounded, so that it suffices to prove the compactness of $m_>^{1/2}H_0^{-1/2}$ which follows from $m_> \in L^\infty(\mathbb{R}^2)$, $\lim_{|x|\to\infty} m_>(x) = 0$, and the diamagnetic inequality (see, e.g., [Avron et al. 1978, Theorem 2.5]).

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