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ASYMPTOTICS OF HADAMARD TYPE FOR EIGENVALUES OF THE NEUMANN PROBLEM ON C^1-DOMAINS FOR ELLIPTIC OPERATORS

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This article investigates how the eigenvalues of the Neumann problem for an elliptic operator depend on the domain in the case when the domains involved are of class C^1. We consider the Laplacian and use results developed previously for the corresponding Lipschitz case. In contrast with the Lipschitz case, however, in the C^1-case we derive an asymptotic formula for the eigenvalues when the domains are of class C^1. Moreover, as an application we consider the case of a C^1-perturbation when the reference domain is of class $C^{1,\alpha}$.

1. Introduction

The results presented in this article are based on an abstract framework for eigenvalues of the Neumann problem previously developed by Kozlov and Thim [2014], where we considered applications to Lipschitz- and $C^{1,\alpha}$-domains. However, the corresponding result for C^1-domains was omitted. In this study we present an asymptotic formula of Hadamard type for perturbations in the case when the domains are of class C^1. We also apply this theorem to the case when the reference domain is $C^{1,\alpha}$, which simplifies the expressions involved.

Partial differential equations are typically not easily solvable when the domain is merely C^1. Indeed, the existence result for Laplace's equation on a general C^1-domain with L^p-data on the boundary was only finally resolved by [Fabes et al. 1978]. This problem was difficult due to the fact that proving that the layer potentials define compact operators (so Fredholm theory is applicable, similar to the $C^{1,\alpha}$-case) was rather technical. The results are based on estimates for the Cauchy integral on Lipschitz curves and we only obtain L^p-estimates for the gradient. As a consequence, the problem of eigenvalue dependence on a C^1-domain becomes difficult.

Hadamard [1908] — see also [Maz′ya and Shaposhnikova 1998] — studied a special type of perturbations of domains with smooth boundary in the early twentieth century, where the perturbed domain Ω_ε is represented by $x_\nu = h(x')$ with $x' \in \partial\Omega_0$, x_ν the signed distance to the boundary ($x_\nu < 0$ for $x \in \Omega_0$), and h a smooth function bounded by a small parameter ε. Hadamard considered the Dirichlet problem, but a formula of Hadamard type for the first nonzero eigenvalue of the Neumann Laplacian is given by

$$\Lambda(\Omega_\varepsilon) = \Lambda(\Omega_0) + \int_{\partial\Omega_0} h(|\nabla\varphi|^2 - \Lambda(\Omega_0)\varphi^2)\, dS + o(\varepsilon),$$

where dS is the surface measure on $\partial\Omega_0$ and φ is an eigenfunction corresponding to $\Lambda(\Omega_0)$ such that $\|\varphi\|_{L^2(\Omega_0)} = 1$; compare with [Grinfeld 2010]. In more general terms, eigenvalue dependence on domain perturbations is a classical and important problem going far back. Moreover, these problems are closely related to shape optimization; see, e.g., [Henrot 2006; Sokołowski and Zolésio 1992], and references found therein.

Specifically, let Ω_1 and Ω_2 be domains in $\mathbb{R}^n$, $n \geq 2$, and consider the spectral problems

$$\begin{cases} -\Delta u = \Lambda(\Omega_1)u & \text{in } \Omega_1, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_1 \end{cases} \tag{1-1}$$

and

$$\begin{cases} -\Delta v = \Lambda(\Omega_2)v & \text{in } \Omega_2, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{1-2}$$

where ∂_ν is the normal derivative with respect to the outward normal. In the case of nonsmooth boundary, we consider the corresponding weak formulations. The analogous Dirichlet problems have previously been considered [Kozlov 2006; 2013; Kozlov and Nazarov 2010; 2012], however the Neumann problem requires a different approach as regards what one can use as a proximity quantity between the two domains and the operators involved.

We will require that the domains are close, in the sense that the Hausdorff distance between the sets Ω_1 and Ω_2, namely

$$d = \max\Big\{ \sup_{x\in\Omega_1} \inf_{y\in\Omega_2} |x - y|, \ \sup_{y\in\Omega_2} \inf_{x\in\Omega_1} |x - y| \Big\}, \tag{1-3}$$

is small. For example, if the problem in (1-1) has a discrete spectrum and the two domains Ω_1 and Ω_2 are close, then the problem in (1-2) has precisely J_m eigenvalues $\Lambda_k(\Omega_2)$ close to $\Lambda_m(\Omega_1)$; see, for instance, Lemma 3.1 in [Kozlov and Thim 2014]. Here, J_m is the dimension of the eigenspace X_m corresponding to $\Lambda_m(\Omega_1)$. The aim is to characterize the difference $\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)$ for $k = 1, 2, \ldots, J_m$.

In a previous study [Kozlov and Thim 2014], we considered the cases when the domains are Lipschitz or $C^{1,\alpha}$, with $0 < \alpha < 1$, as applications of an abstract framework. The main result is an asymptotic result for $C^{1,\alpha}$-domains, where Ω_1 is a $C^{1,\alpha}$-domain and Ω_2 is a Lipschitz perturbation of Ω_1, in the sense that the perturbed domain Ω_2 can be characterized by a function h defined on the boundary $\partial\Omega_1$ such that every point $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$ and x_ν is the signed distance to $\partial\Omega_1$ as defined above. Moreover, the function h is assumed to be Lipschitz continuous and satisfy $|\nabla h| \leq Cd^\alpha$. We proved that, if the problem in (1-1) has a discrete spectrum and m is fixed, then there exists a constant $d_0 > 0$ such that, if $d \leq d_0$, then

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + O(d^{1+\alpha}) \tag{1-4}$$

for every $k = 1, 2, \ldots, J_m$. Here, $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\kappa(\varphi, \psi) = \int_{\partial\Omega_1} h(x')(\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi)\, dS(x') \quad \text{for all } \psi \in X_m, \tag{1-5}$$

where $\varphi \in X_m$. Moreover, $\kappa_1, \kappa_2, \ldots, \kappa_{J_m}$ in (1-4) run through all eigenvalues of (1-5), counting their multiplicities; see Theorem 1.1 in [Kozlov and Thim 2014].

In the case when the domains are merely Lipschitz, we only obtain that there exists a constant C, independent of d, such that $|\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)| \leq Cd$ for every $k = 1, 2, \ldots, J_m$; see Corollary 6.11 in [Kozlov and Thim 2014]. Furthermore, in Section 6.7 there, we provide an example which shows that we can not get an asymptotic result of the type above for the Lipschitz case.

1A. New results. The main result of this article is proved in Section 4B, where an asymptotic formula for $\Lambda_m(\Omega_2) - \Lambda_k(\Omega_1)$ in the case of C^1-domains is derived. The main term consists of extensions of eigenfunctions to (1-1) and the remainder is of order $o(d)$; see Theorem 4.4. We suppose that Ω_2 is a Lipschitz perturbation of a C^1-domain Ω_1 such that the Hausdorff distance d between Ω_1 and Ω_2 is small and the outward normals n_1 and n_2 — taken at the corresponding points of Ω_1 and Ω_2, respectively — are comparable in the sense that $n_1 - n_2 = o(1)$ as $d \to 0$ (uniformly). If we also require that $\Omega_2 \subset \Omega_1$ to avoid the need for extension theorems, we obtain the following result:

Theorem 1.1. *Suppose that Ω_1 is a C^1-domain, that Ω_2 is as described above, and that $\Omega_2 \subset \Omega_1$. In addition, assume that the problem in (1-1) has a discrete spectrum and that m is fixed. Then there exists a constant $d_0 > 0$ such that, if $d \leq d_0$, then*

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \tau_k + o(d) \quad \text{for } k = 1, 2, \ldots, J_m. \tag{1-6}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \int_{\Omega_1 \setminus \Omega_2} (\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi) \, dx \quad \text{for all } \psi \in X_m, \tag{1-7}$$

where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \ldots, \tau_{J_m}$ in (1-6) run through all eigenvalues of (1-7), counting their multiplicities.

Note that the main term is of order d and that the remainder is strictly smaller as $d \to 0$.

As an application, in Section 5 we consider the case when the perturbation is of Hadamard type and we assume that the reference domain Ω_1 is a $C^{1,\alpha}$-domain. Indeed, if Ω_2 is a perturbation of Ω_1 in the sense that the perturbed domain Ω_2 can be characterized by a Lipschitz function h defined on the boundary $\partial\Omega_1$ such that $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$, x_ν is the signed distance to $\partial\Omega_1$ as defined above, and $\nabla h = o(1)$ as $d \to 0$ (uniformly), we obtain the following result; see Theorem 5.1.

Theorem 1.2. *Suppose that Ω_1 is a $C^{1,\alpha}$-domain, that Ω_2 is a perturbation as described above, that the problem in (1-1) has a discrete spectrum, and that m is fixed. Then there exists a constant $d_0 > 0$ such that, if $d \leq d_0$, then*

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + o(d) \tag{1-8}$$

for every $k = 1, 2, \ldots, J_m$. Here, $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\kappa(\varphi, \psi) = \int_{\partial\Omega_1} h(x')(\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi) \, dS(x') \quad \text{for all } \psi \in X_m, \tag{1-9}$$

where $\varphi \in X_m$. Moreover, $\kappa_1, \kappa_2, \ldots, \kappa_{J_m}$ in (1-8) run through all eigenvalues of (1-9) counting their multiplicities.

We also note here that Theorem 1.2 is sharp. Indeed, the main term in (1-9) is of order d and the example given in Section 6.7 in [Kozlov and Thim 2014] shows that this cannot be improved.

2. Notation and definitions

We will use the same abstract setting and notation that was used in [Kozlov and Thim 2014]. Let us summarize the notation. We consider the operator $1 - \Delta$; a number λ is an eigenvalue of the operator $1 - \Delta$ if and only if $\lambda - 1$ is an eigenvalue of $-\Delta$. The reason for considering $1 - \Delta$ is to avoid technical difficulties due to the eigenvalue zero. Enumerate the eigenvalues $\Lambda_k(\Omega_1) = \lambda_k - 1$ for $k = 1, 2, \ldots$ of (1-1) according to $0 < \lambda_1 < \lambda_2 < \cdots$. Similarly, we let $\Lambda_k(\Omega_2) = \mu - 1$ be the eigenvalues of (1-2). Suppose that H_1 and H_2 are infinite-dimensional subspaces of a Hilbert space H. We denote the inner product on H by $(\cdot, \cdot)$. Let the operators $K_j \colon H_j \to H_j$ be positive definite and self-adjoint for $j = 1, 2$. Furthermore, let K_1 be compact. We consider the spectral problems

$$K_1 \varphi = \lambda^{-1} \varphi, \quad \varphi \in H_1, \tag{2-1}$$

and

$$K_2 U = \mu^{-1} U, \quad U \in H_2, \tag{2-2}$$

and denote by λ_k^{-1} for $k = 1, 2, \ldots$ the eigenvalues of K_1. Let $X_k \subset H_1$ be the eigenspace corresponding to the eigenvalue λ_k^{-1}. Moreover, we denote the dimension of X_k by J_k and define $\mathscr{X}_m = X_1 + X_2 + \cdots X_m$, where $m \geq 1$ is any integer. In this article we study eigenvalues of (2-2) located in a neighborhood of λ_m^{-1}, where m is fixed. Note that it is known that there are precisely J_m eigenvalues of (1-2) near λ_m^{-1}; see, e.g., Lemma 3.1 in [Kozlov and Thim 2014]. We wish to describe how close they are in the case of C^1-domains.

Let $S_1 \colon H \to H_1$ and $S_2 \colon H \to H_2$ be orthogonal projectors and define S as the restriction of S_2 to H_1. To compare K_1 and K_2, we define the operator $B \colon H_1 \to H_2$ as $B = K_2 S - S K_1$. For $\varphi \in \mathscr{X}_m$, $B\varphi$ is typically small in applications. Furthermore, we use the convention that C is a generic constant that can change from line to line, but always depend only on the parameters. We also use the notation κ for a generic function $\kappa \colon [0, \infty) \mapsto [0, \infty)$ such that $\kappa(\delta) = o(1)$ as $\delta \to 0$.

2A. Domains in $\mathbb{R}^n$. Let Ω_1 be the reference domain, which will be fixed throughout. We will assume that Ω_1 and Ω_2 are at least Lipschitz domains. Then there exists a positive constant M such that the boundary $\partial \Omega_1$ can be covered by a finite number of balls B_k, $k = 1, 2, \ldots, N$, where there exist orthogonal coordinate systems in which

$$\Omega_1 \cap B_k = \{ y = (y', y_n) : y_n > h_k^{(1)}(y') \} \cap B_k,$$

where the center of B_k is at the origin and $h_k^{(1)}$ are Lipschitz functions, i.e.,

$$|h_k^{(1)}(y') - h_k^{(1)}(x')| \leq M |y' - x'|,$$

such that $h_k^{(1)}(0) = 0$. We assume that Ω_2 belongs to the class of domains where Ω_2 is close to Ω_1 in the sense that Ω_2 can be described by

$$\Omega_2 \cap B_k = \{y = (y', y_n) : y_n > h_k^{(2)}(y')\} \cap B_k,$$

where $h_k^{(2)}$ are also Lipschitz continuous with Lipschitz constant M.

The case when Ω_1 is a C^1- or $C^{1,\alpha}$-domain is defined analogously, with the addition that $h_k^{(1)} \in C^1(\mathbb{R}^{n-1})$ (or $C^{1,\alpha}(\mathbb{R}^{n-1})$) such that

$$h_k^{(1)}(0) = \partial_{x_i} h_k^{(1)}(0) = 0, \quad i = 1, 2, \ldots, n-1.$$

Note that when Ω_1 is a C^1-domain we obtain that, for $P, Q \in \partial\Omega_1$, the outward normal n_1 of Ω_1 satisfies

$$n_1(P) - n_1(Q) = o(1) \quad \text{as } |P - Q| \to 0$$

uniformly.

2B. *Perturbations of C^1-domains.* The situation we consider is the case when the reference domain Ω_1 is a C^1-domain and the perturbed domain Ω_2 is close in the sense of Section 2A. We require that Ω_2 is a Lipschitz domain such that

$$|\nabla(h_k^{(1)} - h_k^{(2)})| = o(1) \quad \text{as } d \to 0 \tag{2-3}$$

uniformly. This condition can be compared to the one we used in [Kozlov and Thim 2014] for perturbations of $C^{1,\alpha}$-domains:

$$|\nabla(h_k^{(1)} - h_k^{(2)})| \le Cd^\alpha. \tag{2-4}$$

Note that $h_k^{(2)}$ are only assumed to be Lipschitz continuous and satisfy (2-3) and (2-4), respectively.

3. Definition of the operators K_j

Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^n$ ($\Omega_1 \cap \Omega_2 \ne \varnothing$) and put $H = L^2(\mathbb{R}^n)$ and $H_j = L^2(\Omega_j)$ for $j = 1, 2$, where functions in H_j are extended to $\mathbb{R}^n$ by zero outside of Ω_j if necessary. For $f \in L^2(\Omega_j)$, the weak solution to the Neumann problem $(1 - \Delta)W_j = f$ in Ω_j and $\partial_\nu W_j = 0$ on $\partial\Omega_j$ for $j = 1, 2$ satisfies

$$\int_{\Omega_j} (\nabla W_j \cdot \nabla v + W_j v) \, dx = \int_{\Omega_j} f v \, dx \quad \text{for every } v \in H^1(\Omega_j),$$

and the Cauchy–Schwarz inequality implies that

$$\|\nabla W_j\|_{L^2(\Omega_j)} + \|W_j\|_{L^2(\Omega_j)} \le \|f\|_{L^2(\Omega_j)} \quad \text{for all } f \in L^2(\Omega_j).$$

We define the operators K_j on $L^2(\Omega_j)$, $j = 1, 2$, as the solution operators corresponding to the domains Ω_j, i.e., $K_j f = W_j$. The operators K_j are self-adjoint and positive definite and, if Ω_j are, e.g., Lipschitz, also compact.

3A. *Results for Lipschitz domains.* We will work with results for Lipschitz domains and then refine estimates using the additional smoothness of the C^1-case. Let Ω be a Lipschitz domain. The truncated cones $\Gamma(x')$ at $x' \in \partial\Omega$ are given by, e.g.,

$$\Gamma(x') = \{x \in \Omega : |x - x'| < 2 \operatorname{dist}(x, \partial\Omega)\}$$

and the nontangential maximal function is defined on the boundary $\partial\Omega$ by

$$N(u)(x') = \max_{k=1,2,\dots,N} \sup\{|u(x)| : x \in \Gamma(x') \cap B_k\}.$$

For the case when Ω_1 and Ω_2 are Lipschitz, one can show that

$$\|N(K_j u)\|_{L^2(\partial\Omega_j)} + \|N(\nabla K_j u)\|_{L^2(\partial\Omega_j)} \le C\|u\|_{L^2(\Omega_j)}, \quad j = 1, 2, \tag{3-1}$$

where the constant C depends only on the Lipschitz constant M and $B_1, B_2, \dots, B_N$. We interpret $\partial_\nu K_j u = 0$ on $\partial\Omega_j$ in the sense that $n \cdot \nabla K_j u \to 0$ nontangentially (with limits taken inside cones $\Gamma(x')$) at almost every point on $\partial\Omega$, where n is the outward normal. These results are discussed further in Section 6.2 of [Kozlov and Thim 2014]. Let us summarize that reference's Lemmas 6.2 and 6.3 for convenience.

Lemma 3.1. *Let Ω be a Lipschitz domain. Then*:

(i) *If $g \in L^2(\partial\Omega)$, then there exists a unique (up to constants) function u in $H^1(\Omega)$ such that $(1-\Delta)u = 0$ in Ω and $\partial_\nu u = g$ a.e. on $\partial\Omega$ in the nontangential sense and, moreover,*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

(ii) *If $f \in L^2(\Omega)$, then there exists a unique function u in $H^1(\Omega)$ such that $(1-\Delta)u = f$ in Ω and $\partial_\nu u = 0$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Here, the constant C depends only on M and $B_1, B_2, \dots, B_N$.

The corresponding lemma for the Dirichlet case is also known and one can prove it using an argument similar to the one used to prove Lemmas 6.2 and 6.3 in [Kozlov and Thim 2014].

Lemma 3.2. *Let Ω be a Lipschitz domain. Then*:

(i) *If $g \in L^2(\partial\Omega)$, then there exists a unique function $u \in H^1(\Omega)$ such that $(1-\Delta)u = 0$ in Ω, $u = g$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

(ii) *If $f \in L^2(\Omega)$, then there exists a unique function $u \in H^1(\Omega)$ such that $(1-\Delta)u = f$ in Ω, $u = 0$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Here, the constant C depends only on M and $B_1, B_2, \dots, B_N$.

We conclude with an extension result for Lipschitz domains; see, e.g., [Kozlov and Thim 2014, Lemma 6.4(i)] for a proof.

Lemma 3.3. *Suppose that $f \in H^1(\partial\Omega)$ and $g \in L^2(\partial\Omega)$, where Ω is a Lipschitz domain. Then there exists a function $u \in H^1(\Omega^c)$ such that $u \to f$ and $n \cdot \nabla u \to g$ nontangentially at almost every point on $\partial\Omega$, where n is the outward normal of Ω, and there exists a constant C such that*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C(\|f\|_{H^1(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}),$$

where C depends only on M and $B_1, B_2, \ldots, B_N$.

4. Main results

Let us proceed to prove the main results. In Section 4A, we prove a key lemma concerning an estimate for $\partial_\nu K_j S_j \varphi$ on $\partial(\Omega_1 \cap \Omega_2)$. Using this estimate, we can refine results for Lipschitz domains that were previously developed in [Kozlov and Thim 2014] and, as a result, obtain an asymptotic formula describing the difference between λ_m^{-1} and μ_m^{-1} in terms of eigenfunctions of K_1.

4A. *Boundary estimates for C^1-domains.* Since $\partial_\nu \varphi = 0$ on $\partial\Omega_1$, we would expect that $\partial_\nu \varphi$ is small also on Ω_2 if the domains are close. However, since in the C^1-case we only obtain solutions with derivatives in L^p, this problem becomes more difficult than the corresponding issue in the $C^{1,\alpha}$-case (which was solved in [Kozlov and Thim 2014]). To this end, we will exploit that, locally on the boundaries $\partial\Omega_j$, the normal vectors can be approximated by constant unit vectors e_n (with respect to the local coordinate system). That is, we approximate the surface by its tangent plane at a specific point. We obtain the following result:

Lemma 4.1. *Let $P \in \partial(\Omega_1 \cap \Omega_2)$ and $\delta > 0$ such that $B(P, 2\delta) \subset B_k$ for some k, where B_k are the balls covering $\Omega_1 \cap \Omega_2$ given in Section 2A. Then there exists a function $\kappa(\delta)$ such that*

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap B(P,\delta)} |\partial_\nu K_j S_j \varphi|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx, \quad j = 1, 2, \tag{4-1}$$

for every $\varphi \in \mathscr{X}_m$, where $\kappa(\delta) = o(1)$ as $\delta \to 0$.

Proof. Let $B = B(P, 2\delta)$. We wish to consider $\partial_\nu K_j S_j \varphi$ on $\partial(\Omega_1 \cap \Omega_2)$. However, since $\nabla K_j S_j \varphi$ only exist in the sense of L^2, it is nontrivial to exploit the fact that $\partial_\nu K_j S_j \varphi$ is zero on $\partial\Omega_j$. Therefore, let us instead consider $\partial_{x_n} K_j S_j \varphi$ (with respect to the coordinate system in B_k). The outward normal of Ω_j is comparable to e_n in B_k and $\partial_\nu K_j S_j \varphi = 0$ on $\partial\Omega_j$, so we expect $\partial_{x_n} K_j S_j \varphi$ to be small on $\partial\Omega_j \cap B_k$. Indeed, since $\nabla K_j S_j \varphi \cdot n_j \to 0$ nontangentially on $\partial\Omega_j$ and $n_j = e_n + o(1)$ as $\delta \to 0$, we obtain that

$$\int_{\partial\Omega_j \cap B} |\partial_{x_n} K_j S_j \varphi|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-2}$$

However, we cannot expect $\partial_{x_n} K_j S_j \varphi$ to be small on all of Ω_j. The idea is to use the fact that ∂_{x_n} commutes with $(1 - \lambda_m - \Delta)$. Indeed, we see that if $\Phi = \partial_{x_n} K_1 S_1 \varphi$, then $(1 - \lambda_m - \Delta)\Phi = 0$ in Ω_1 and $\Phi = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1$. The case when $j = 2$ will be treated similarly but requires some additional

steps. Let us consider the equation $(1 - \lambda_m - \Delta)\Phi = 0$ in Ω_1 and $\Phi = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1$. We split this equation in two separate parts.

Part 1. Let Φ_p be the solution to $(1 - \lambda_m - \Delta)\Phi_p = 0$ in Ω_1, $\Phi_p = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1 \cap B$ and, on $\partial\Omega_1 \cap B^c$, we let $\Phi_p = 0$. Lemma 3.2 implies that Φ_p satisfies

$$\int_{\partial\Omega_1} |N(\Phi_p)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-3}$$

Then it follows that

$$\int_{\Omega_1 \cap \partial\Omega_2 \cap B} |\Phi_p|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2. \tag{4-4}$$

Part 2. Let Φ_h be the solution to $(1 - \lambda_m - \Delta)\Phi_h = 0$ in Ω_1, $\Phi_h = 0$ on $\partial\Omega_1 \cap B$, and $\Phi_h = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1 \cap B^c$. To prove an estimate for Φ_h on $\partial\Omega_1 \cap B$ similar to the one given for Φ_p in (4-4), we use a local estimate for solutions to the Dirichlet problem, where we exploit that the boundary data is zero on $\Omega_1 \cap B$. Indeed, let $\frac{1}{2}B$ be the ball with the same center as B but half the radius. Then Theorem 5.24 in [Kenig and Pipher 1993] (for example) implies that

$$\int_{\partial\Omega_1 \cap \frac{1}{2}B} |N(\nabla\Phi_h)|^2 \, dS(x') \le C \int_{\Omega_1 \cap B} |\nabla\Phi_h|^2 \, dx \tag{4-5}$$

since the tangential gradient of Φ_h is zero on the boundary. This, in turn, implies that the left-hand side in (4-5) is finite and, furthermore, since also $\Phi_h = 0$ on $\Omega_1 \cap B$, it follows that

$$\int_{\Omega_1 \cap \partial\Omega_2 \cap \frac{1}{2}B} |\Phi_h|^2 \, dS(x') \le Cd \int_{\Omega_1} |\varphi|^2 \, dx, \tag{4-6}$$

where d is the Hausdorff distance between Ω_1 and Ω_2.

Equations (4-4) and (4-6) are sufficient to obtain that

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap \frac{1}{2}B} |N(\partial_{x_n} K_1 S_1 \varphi)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx$$

since $\Phi = \Phi_p + \Phi_h$.

Turning our attention to when $j = 2$, we see that $(1 - \Delta) K_2 S_2 \varphi = S_2 \varphi$ and that this equation is not homogeneous. Moreover, the right-hand side is not necessarily small. However, since $S\varphi = \lambda_m K_2 S\varphi - \lambda_m B\varphi$ and $B\varphi$ is small, we can consider

$$(1 - \lambda_m - \Delta) K_2 S_2 \varphi = -\lambda_m B\varphi. \tag{4-7}$$

Let Ψ be the weak solution to $(1 - \lambda_m - \Delta)\Psi = -\lambda_m B\varphi$ in Ω_2 and $\Psi = 0$ on $\partial\Omega_2$. Then

$$\|\Psi\|_{H^1(\Omega_2)} \le C\|B\varphi\|_{L^2(\Omega_2)}$$

and the trace of Ψ is defined on $\partial\Omega$. Moreover, from Lemma 3.2 we obtain that

$$\|N(\Psi)\|_{L^2(\partial\Omega_2)} \le C\|B\varphi\|_{L^2(\Omega_2)}. \tag{4-8}$$

Now, put $\Phi = \Psi + W$. Then $(1 - \lambda_m - \Delta)W = 0$ and $W = \partial_{x_n} K_2 S_2 \varphi$ on $\partial\Omega_2$. It is now possible to carry out steps 1 and 2 for W in Ω_2 analogously to Φ in Ω_1, exchanging the roles of Ω_1 and Ω_2. Thus, using the same notation, we obtain that

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap \frac{1}{2}B} |N(W)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-9}$$

Finally, Lemma 6.6 in [Kozlov and Thim 2014] states that $\|B\varphi\|_{L^2(\Omega_2)}^2 \le Cd\|\varphi\|_{L^2(\Omega_1)}^2$, so this fact and equations (4-8) and (4-9) prove that

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap \frac{1}{2}B} |N(\partial_{x_n} K_2 S_2 \varphi)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-10}$$

We can now conclude the proof by observing that the outward normal on $\partial(\Omega_1 \cap \Omega_2)$ is given by n_1 or n_2 at almost every point, and $n_j = e_n + r_j$ with $r_j = \kappa(\delta)$, $j = 1, 2$, so we obtain that

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap \frac{1}{2}B} |\partial_\nu K_j S_j \varphi|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \qquad \square$$

The previous lemma is local in nature, but due to compactness we can prove the following corollary:

Corollary 4.2. *There exists a constant $d_0 > 0$ such that, if $d \le d_0$, then*

$$\int_{\partial(\Omega_1 \cap \Omega_2)} |\partial_\nu K_j S_j \varphi|^2 \, dS(x') \le \kappa(d) \int_{\Omega_1} |\varphi|^2 \, dx, \quad j = 1, 2, \tag{4-11}$$

for every $\varphi \in \mathscr{X}_m$, where $\kappa(d) = o(1)$ as $d \to 0$.

Proof. By compactness, if d is small we can cover $\partial(\Omega_1 \cap \Omega_2)$ by a finite number of balls $B(P, d)$ such that $B(P, 2d) \subset B_k$ for some k, where B_k are the covering balls from Section 2A. By choosing d_0 small enough and letting $\delta = d$ in the previous lemma, the result in the corollary now follows. $\square$

4B. *Proof of Theorem 1.1.* The following proposition is a reformulation of Proposition 6.10 in [Kozlov and Thim 2014], where the proof can also be found. The expressions with tildes are the extensions of the corresponding functions provided by Lemma 3.3. We will use this result and Corollary 4.2 to prove Theorem 1.1.

Proposition 4.3. *Suppose that Ω_1 and Ω_2 are Lipschitz domains in the sense of Section 2A. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{3/2}) \quad \text{for } k = 1, 2, \dots, J_m. \tag{4-12}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \lambda_m^{-1} \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m)\widetilde{K_2 S \varphi}\psi + \nabla \widetilde{K_2 S \varphi} \cdot \nabla \psi) \, dx$$

$$- \lambda_m^{-1} \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m)(K_2 S \varphi)\widetilde{\psi} + \nabla K_2 S \varphi \cdot \nabla \widetilde{\psi}) \, dx \tag{4-13}$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (4-12) run through all eigenvalues of (4-13), counting their multiplicities.

Let us now prove a version of this proposition that holds specifically for C^1-domains. We will show the following result:

Theorem 4.4. *Suppose that Ω_1 is a C^1-domain and that Ω_2 is a perturbation in the sense of Section 2B satisfying (2-3). Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + o(d) \quad \text{for } k = 1, 2, \ldots, J_m. \tag{4-14}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \lambda_m^{-1} \int_{\Omega_1 \setminus \Omega_2} ((1 - \lambda_m)\varphi\psi + \nabla\varphi \cdot \nabla\psi) \, dx - \lambda_m^{-1} \int_{\Omega_2 \setminus \Omega_1} ((1 - \lambda_m)\widetilde{\varphi}\widetilde{\psi} + \nabla\widetilde{\varphi} \cdot \nabla\widetilde{\psi}) \, dx \tag{4-15}$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \ldots, \tau_{J_m}$ in (4-14) run through all eigenvalues of (4-15), counting their multiplicities.

Proof. We need to prove that (4-13) can be expressed as (4-15) up to a term of order $o(d)$. Since $K_2 S\varphi = B\varphi + \lambda_m^{-1} S\varphi$, we let

$$\widetilde{K_2 S\varphi} = \widetilde{B\varphi} + \lambda_m^{-1}\widetilde{\varphi},$$

where $\widetilde{B\varphi}$ is the extension of $B\varphi$ from $\Omega_1 \cap \Omega_2$ and $\widetilde{\varphi}$ is the extension of φ from Ω_1, both provided by Lemma 3.3. We show that $\widetilde{B\varphi}$ is small and that $\lambda_m^{-1}\widetilde{\varphi}$ gives the main term. To this end, let $V = B\varphi$ in $\Omega_1 \cap \Omega_2$. Then $(1 - \Delta)V = 0$ in $\Omega_1 \cap \Omega_2$, $\partial_\nu V = \partial_\nu K_2 S\varphi$ on $\partial\Omega_1 \cap \Omega_2$, and $\partial_\nu V = -\partial_\nu K_1\varphi$ on $\Omega_1 \cap \partial\Omega_2$. Using Corollary 4.2 and Lemma 3.1, we then obtain that

$$\|N(V)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} + \|N(\nabla V)\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \le \kappa(d)\|\varphi\|_{L^2(\Omega_1)},$$

where $\kappa(d) = o(1)$ as $d \to 0$, and thus

$$\|N(\widetilde{B\varphi})\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} + \|N(\nabla\widetilde{B\varphi})\|_{L^2(\partial(\Omega_1 \cap \Omega_2))} \le \kappa(d)\|\varphi\|_{L^2(\Omega_1)}^2.$$

Now, the Cauchy–Schwarz inequality implies that

$$\begin{aligned}
\int_{\Omega_1 \setminus \Omega_2} |\nabla\widetilde{B\varphi} \cdot \nabla\psi| \, dx &\le \left(\int_{\Omega_1 \setminus \Omega_2} |\nabla\widetilde{B\varphi}|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_1 \setminus \Omega_2} |\nabla\psi|^2 \, dx\right)^{\frac{1}{2}} \\
&\le Cd\left(\int_{\partial(\Omega_1 \cap \Omega_2)} N(\nabla\widetilde{B\varphi})^2 \, dS(x')\right)^{\frac{1}{2}} \left(\int_{\Omega_1 \setminus \Omega_2} |\nabla\psi|^2 \, dx\right)^{\frac{1}{2}} \\
&= o(d)
\end{aligned}$$

and, similarly,

$$\int_{\Omega_1 \setminus \Omega_2} |\widetilde{B\varphi}\psi| \, dx \le Cd\left(\int_{\partial(\Omega_1 \cap \Omega_2)} N(\widetilde{B\varphi})^2 \, dS(x')\right)^{\frac{1}{2}} \left(\int_{\Omega_1 \setminus \Omega_2} |\psi|^2 \, dx\right)^{\frac{1}{2}} = o(d).$$

Analogously, one can show that the corresponding expressions involving $B\varphi$ on $\Omega_2 \setminus \Omega_1$ are also of order $o(d)$. $\quad\square$

To pass from $\lambda_m^{-1} - \mu_m^{-1}$ to $\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)$, observe that

$$\lambda_m^{-1} - \mu_k^{-1} = \lambda_m^{-2}\left(\frac{\lambda_m}{\mu_k}(\mu_k - \lambda_m)\right) = \lambda_m^{-2}\left(\mu_k - \lambda_m - \frac{(\mu_k - \lambda_m)^2}{\mu_k}\right),$$

where $(\mu_k - \lambda_m)^2 = O(d^2)$ since Ω_1 and Ω_2 are at least Lipschitz; see Corollary 6.11 in [Kozlov and Thim 2014]. Note also that, if it is the case that $\Omega_2 \subset \Omega_1$, we can simplify the previous theorem by removing the second integral in (4-15) and avoid the use of extensions of eigenfunctions; compare with the statement of Theorem 1.1 in the introduction.

5. C^1-perturbations of $C^{1,\alpha}$-domains

Suppose that Ω_1 is a $C^{1,\alpha}$-domain and that it is possible to characterize the perturbed domain Ω_2 by a Lipschitz function h defined on the boundary $\partial\Omega_1$ such that $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$ and x_ν is the signed distance to the boundary $\partial\Omega_1$ (with $x_\nu < 0$ when $x \in \Omega_1$). We assume that $\nabla h = o(1)$ as $d \to 0$ (uniformly). In this case, we can simplify the expression given in Theorem 4.4 and avoid the use of extensions by stating the formula (4-14) as a boundary integral.

Theorem 5.1. *Suppose that Ω_1 is a $C^{1,\alpha}$-domain and that Ω_2 is as described above. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + o(d) \tag{5-1}$$

for $k = 1, 2, \ldots, J_m$. Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \lambda_m^{-2} \int_{\partial\Omega_1} h(x')((1 - \lambda_m)\varphi\psi + \nabla\varphi \cdot \nabla\psi)\, dS(x') \quad \text{for all } \psi \in X_m, \tag{5-2}$$

where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \ldots, \tau_{J_m}$ in (5-1) run through all eigenvalues of (5-2), counting their multiplicities.

Proof. Since Ω_1 is a $C^{1,\alpha}$-domain, we can use results from the proof of Corollary 6.17 in [Kozlov and Thim 2014]. In that proof, we showed that $\varphi \in C^{1,\alpha}(\Omega_1)$ and also that φ can be extended to a function $\widetilde{\varphi} \in C^{1,\alpha}(\mathbb{R}^n)$ such that

$$\int_{\Omega_1 \setminus \Omega_2} \left(|\varphi(x) - \varphi(x', 0)|^2 + |\nabla\varphi(x) - \nabla\varphi(x', 0)|^2\right) dx \leq Cd^{1+\alpha}\|\varphi\|_{L^2(\Omega_1)}^2,$$

with the corresponding estimate holding for $\widetilde{\varphi}$ on $\Omega_2 \setminus \Omega_1$. Hence, Theorem 4.4 implies that $\lambda_m^{-1} - \mu_k^{-1}$ is given by

$$\lambda_m^{-2}\Bigg(\int_{\partial\Omega_1 \cap \Omega_2^c} \int_0^{h(x')} \left((1 - \lambda_m)\varphi(x', 0)\psi(x', 0) + \nabla\varphi(x', 0) \cdot \nabla\psi(x', 0)\right) dx_\nu\, dS(x')$$

$$- \int_{\partial\Omega_1 \cap \Omega_2} \int_0^{-h(x')} \left((1 - \lambda_m)\widetilde{\varphi}(x', 0)\widetilde{\psi}(x', 0) + \nabla\widetilde{\varphi}(x', 0) \cdot \nabla\widetilde{\psi}(x', 0)\right) dx_\nu\, dS(x')\Bigg) + o(d).$$

The desired conclusion follows from this statement. □

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