Analysis & PDE

Volume 8 No. 7 2015

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msp

ANALYSIS AND PDE
Vol. 8, No. 7, 2015
dx.doi.org/10.2140/apde.2015.8.1757

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BLOW-UP ANALYSIS OF A NONLOCAL LIOUVILLE-TYPE EQUATION

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We establish an equivalence between the *Nirenberg problem* on the circle and the boundary of holomorphic immersions of the disk into the plane. More precisely we study the nonlocal Liouville-type equation

$$(-\Delta)^{\frac{1}{2}}u = \kappa e^u - 1 \quad \text{in } S^1, \tag{1}$$

where $(-\Delta)^{\frac{1}{2}}$ stands for the fractional Laplacian and κ is a bounded function. The equation (1) can actually be interpreted as the prescribed curvature equation for a curve in conformal parametrization. Thanks to this geometric interpretation we perform a subtle blow-up and quantization analysis of (1). We also show a relation between (1) and the analogous equation in $\mathbb{R}$,

$$(-\Delta)^{\frac{1}{2}}u = Ke^u \quad \text{in } \mathbb{R} \tag{2}$$

with K bounded on $\mathbb{R}$.

1. Introduction

A famous problem posed by Louis Nirenberg is the question of for which positive functions K on the standard sphere (S^n, g_{S^n}) there exists a function u on S^n such that the scalar curvature (Gauss curvature in dimension $n = 2$) of the conformal metric $g = e^{2u}g_{S^n}$ is equal to K. This problem, prescribing the scalar curvature within a conformal class of manifolds, has stimulated a lot of works in geometry and analysis. In dimension $n = 2$ it consists in solving the so-called Liouville equation. More precisely, if (Σ, g_0) is a smooth, closed Riemann surface with Gauss curvature K_{g_0}, an easy computation shows that a function $K(x)$ is the Gauss curvature for some metric $g = e^{2u}g_0$ conformally equivalent to the metric g_0 with $u : \Sigma \to \mathbb{R}$ if and only if there exists a solution $u = u(x)$ of

$$-\Delta_{g_0}u = Ke^{2u} - K_{g_0} \quad \text{on } \Sigma, \tag{3}$$

Martinazzi is supported by the Swiss National Foundation, project no. PP00P2-144669.
MSC2010: primary 30C20, 35B65, 58E20, 35B44, 35R11; secondary 30C62.
Keywords: nonlocal Liouville equation, Nirenberg problem, fractional harmonic maps, blow-up analysis of solutions, regularity of solutions, conformal variational problems, quasiconformal mappings in the plane.

where Δ_{g_0} is the Laplace–Beltrami operator on (Σ, g_0) (see, e.g., [Chang 2005] for more details).

In particular, when $\Sigma = \mathbb{R}^2$ or $\Sigma = S^2$, (3) reads, respectively,

$$-\Delta u = K e^{2u} \quad \text{on } \mathbb{R}^2 \tag{4}$$

and

$$-\Delta_{S^2} u = K e^{2u} - 1 \quad \text{on } S^2. \tag{5}$$

Singular Liouville equations of the form

$$-\Delta_{g_0} u = K e^{2u} - K_{g_0} - 2\pi \sum_{i=1}^{m} \alpha_i \delta_{p_i} \quad \text{on } \Sigma \tag{6}$$

have a role in fluid dynamics — see [Tur and Yanovsky 2004] — as well as in the study of electroweak theory or abelian Chern–Simons vortices; see, e.g., [Tarantello 2008]. For the latter cases, singular points represent zeroes of the scalar wave function involved in the model.

Equations (4), (5) and also (6) have been largely studied in the literature. Here we would like to recall the famous blow-up result of Brezis and Merle [1991] concerning (4):

Theorem 1.1 [Brezis and Merle 1991, Theorem 3]. *Assume that $(u_k) \subset L^1(\Omega)$, Ω an open subset of $\mathbb{R}^2$, is a sequence of solutions to (4) satisfying $K_k \geq 0$, $\|K_k\|_{L^p} \leq C_1$, and $\|e^{u_k}\|_{L^{p'}} \leq C_2$ for some $1 < p \leq \infty$. Then, up to subsequences, one of the following alternatives holds: either (u_k) is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$, or $u_k(x) \to -\infty$ uniformly on compact subsets of Ω, or there is a finite nonempty (blow-up) set $B = \{a_1, \dots, a_N\} \subset \Omega$ such that $u_k(x) \to -\infty$ on compact subsets of $\Omega \setminus B$. In addition, in this last case, $K_k e^{2u_k}$ converges in the sense of measure on Ω to $\sum_{i=1}^N \alpha_i \delta_{a_i}$, with $\alpha_i \geq 2\pi/p'$.*

The purpose of this work is to investigate an analogous prescribed curvature problem in dimension 1. Even if this is a classical problem, it has never been studied so far (to our knowledge) from the point of view of conformal geometry. In the case, for instance, of a planar Jordan curve (namely, a continuous closed and simple curve) there is the possibility to parametrize it through the trace of the Riemann mapping between the disk D^2 and the simply connected domain enclosed by the curve. The equation corresponding to such a parametrization is

$$(-\Delta)^{\frac{1}{2}} \lambda = \kappa e^{\lambda} - 1 \quad \text{in } S^1, \tag{7}$$

where $e^{\lambda} d\theta$ and $\kappa e^{\lambda} d\theta$ are the length form and the curvature density, respectively, of the curve in this parametrization. The definition and relevant properties of the operator $(-\Delta)^{\frac{1}{2}}$ will be given in Appendix A.

One of the main results of this paper is the one-to-one correspondence between the solutions to the Nirenberg problem (7) in S^1 and the space of holomorphic immersions of the disk D^2 (see Theorem 1.4 below). This correspondence can be seen as a sort of generalized Riemann mapping theorem.

This permits us to perform a complete blow-up analysis of (7) in the spirit of Theorem 1.1, even if we do not get exactly the same dichotomy. More precisely, our first main result is the following theorem:

Theorem 1.2. *Let $(\lambda_k) \subset L^1(S^1, \mathbb{R})$ be a sequence with*

$$L_k := \|e^{\lambda_k}\|_{L^1(S^1)} \leq \bar{L} \tag{8}$$

satisfying

$$(-\Delta)^{\frac{1}{2}}\lambda_k = \kappa_k e^{\lambda_k} - 1 \quad in \;\; S^1, \tag{9}$$

where $\kappa_k \in L^\infty(S^1, \mathbb{R})$ *satisfies*

$$\|\kappa_k\|_{L^\infty(S^1)} \le \bar{\kappa}. \tag{10}$$

Then up to subsequence we have $\kappa_k e^{\lambda_k} \rightharpoonup \mu$ *weakly in* $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$ *for every* $p < \infty$, *where* μ *is a Radon measure,* $B := \{a_1, \dots, a_N\}$ *is a (possibly empty) subset of* S^1 *and* $\kappa_k \overset{*}{\rightharpoonup} \kappa_\infty$ *in* $L^\infty(S^1)$. *Set* $\bar{\lambda}_k := (1/2\pi)\int_{S^1} \lambda_k \, d\theta$. *Then one of the following alternatives holds*:

(i) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 1$ *and* $\mu = 2\pi\delta_{a_1}$. *In this case,*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad in \;\; W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1\}) \;\; for\ every \;\; p < \infty,$$

where $v_\infty(e^{i\theta}) = -\log\bigl(2(1-\cos(\theta-\theta_1))\bigr)$ *for* $a_1 = e^{i\theta_1}$, *solving*

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + 2\pi\delta_{a_1} \quad in \;\; S^1. \tag{11}$$

(ii) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 2$ *and* $\mu = \pi(\delta_{a_1} + \delta_{a_2})$. *In this case,*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad in \;\; W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1, a_2\}) \;\; for\ every \;\; p < \infty,$$

where

$$v_\infty(e^{i\theta}) = -\tfrac{1}{2}\log\bigl(2(1-\cos(\theta-\theta_1))\bigr) - \tfrac{1}{2}\log\bigl(2(1-\cos(\theta-\theta_2))\bigr), \qquad a_1 = e^{i\theta_1},\ a_2 = e^{i\theta_2},$$

solves

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + \pi\delta_{a_1} + \pi\delta_{a_2} \quad in \;\; S^1. \tag{12}$$

(iii) $|\bar{\lambda}_k| \le C$ *and* $\mu = \kappa_\infty e^{\lambda_\infty} + \pi(\delta_{a_1} + \cdots + \delta_{a_N})$ *for some* $\lambda_\infty \in W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$, *with* $\lambda_\infty, e^{\lambda_\infty} \in L^1(S^1)$ *and*

$$(-\Delta)^{\frac{1}{2}}\lambda_\infty = \kappa_\infty e^{\lambda_\infty} - 1 + \sum_{i=1}^{N} \pi\delta_{a_i} \quad in \;\; S^1. \tag{13}$$

We would like to stress that we obtain a *quantization-type* result, namely the curvature concentrating at each blow-up point is precisely π, without any assumption on the sign of the curvature (this hypothesis is crucial in [Brezis and Merle 1991]) and on the convergence of the κ_k. Actually, several works on equations (4) and (5) have extended the result of Brezis and Merle, showing that, under the crucial assumption that the prescribed curvatures K_k converge in C^0, the amount of curvature concentrating at each point is a multiple of 4π, i.e., a multiple of the total Gaussian curvature of S^2; see, e.g., [Li and Shafrir 1994]. (Also, higher-dimensional extensions were studied under the same strong assumptions of convergence of K_k in C^0 or even C^1; see, e.g., [Druet and Robert 2006; Malchiodi 2006; Martinazzi 2009b].) In [Brezis and Merle 1991] the functions K_k can belong to $L^p(\mathbb{R})$, with $1 < p \le +\infty$. We believe that in the case of the nonlocal Liouville equation (7) the quantization result by π does not hold once we replace $\kappa \in L^\infty$ by $\kappa \in L^p$ with $1 < p < +\infty$.

The fact that we are able to get a quantization result only under the minimal (and geometrically meaningful) bounds (8) and (10) is better understood through the above-mentioned one-to-one correspondence between the solutions to (7) and the space of holomorphic immersions of the disk D^2. Precisely, given a solution λ to (7) with $\kappa \in L^\infty(S^1)$, the function e^λ provides a "conformal" parametrization of a closed curve $\gamma : S^1 \to \mathbb{C}$ in normal parametrization whose curvature at the point $\gamma(z)$ is exactly $\kappa(z)$.

Definition 1.3. A function $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ is called a holomorphic immersion if Φ is holomorphic in D^2 and $\Phi'(z) := \partial_z \Phi(z) \neq 0$ for every $z \in \bar{D}^2$.

A curve $\gamma \in C^1(S^1, \mathbb{C})$ is said to be in normal parametrization if $|\dot{\gamma}|$ is constant, and is in conformal parametrization if there exists a holomorphic immersion $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ with $\Phi|_{S^1} = \gamma$.

Then we have the following characterization:

Theorem 1.4. *A function $\lambda \in L^1(S^1, \mathbb{C})$ with $L := \|e^\lambda\|_{L^1(S^1)} < \infty$ satisfies*

$$(-\Delta)^{\frac{1}{2}}\lambda = \kappa e^\lambda - 1 \quad in\ S^1 \tag{14}$$

for some function $\kappa : S^1 \to \mathbb{R}$, $\kappa \in L^\infty(S^1)$, if and only if there exists a closed curve $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$ with $|\dot{\gamma}| \equiv L/(2\pi)$, a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that, for all $z \in S^1$, we have $\Phi \circ \sigma(z) = \gamma(z)$,

$$|\Phi'(z)| = e^{\lambda(z)} \tag{15}$$

and the curvature of $\Phi(S^1)$ is κ. While Φ uniquely determines λ via (15), λ determines Φ up to a rotation and a translation. Moreover,

$$|\Phi'(z)| = e^{\tilde{\lambda}(z)}, \quad z \in \bar{D}^2, \tag{16}$$

where $\tilde{\lambda} : D^2 \to \mathbb{R}$ is the harmonic extension of λ.

Figures 1, 2 and 5 provide some examples of curves satisfying the assumptions of Theorem 1.4. Theorem 1.4 allows us to interpret and reformulate Theorem 1.2 from the point of view of the behavior

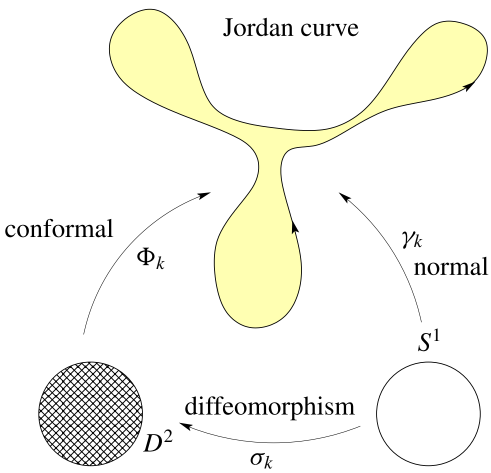

Figure 1. A domain bounded by a Jordan curve γ_k and biholomorphic to the unit disk D^2 via a map $\Phi_k : \bar{D}^2 \to \mathbb{C}$.

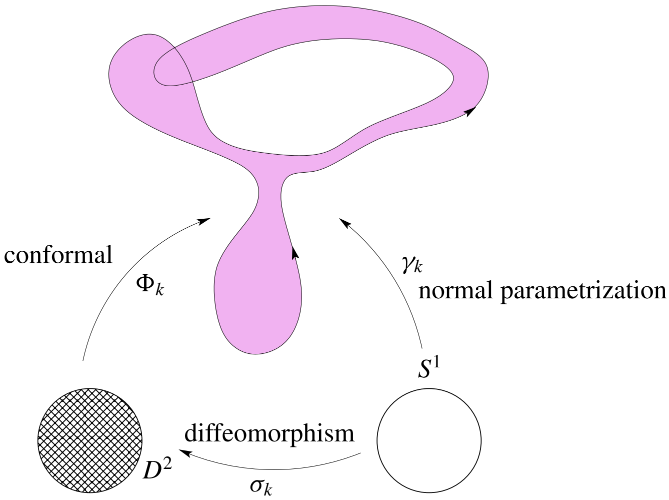

Figure 2. The curve γ_k can have self-intersections. In this case, $\Phi_k : \overline{D}^2 \to \mathbb{C}$ is a holomorphic immersion but it is not injective.

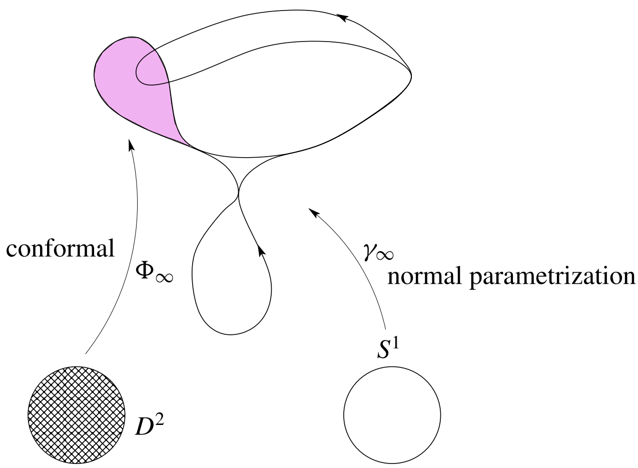

Figure 3. As $k \to \infty$ the curves γ_k can generate a pinching phenomenon. In this case, Φ_k can converge to a constant or, as in this figure, to a holomorphic immersion Φ_∞ (singular at finitely many points of ∂D^2) whose image "selects" one of the "components" bounded by γ_∞.

of the sequences of the curves γ_k (in normal parametrization) and of the immersions Φ_k corresponding to a sequence of solutions to (9); see Figures 3 and 4.

Theorem 1.5. *Let a sequence* $(\lambda_k) \subset L^1(S^1, \mathbb{R})$ *satisfy* (8)–(10), *let* $\Phi_k : \overline{D}^2 \to \mathbb{C}$ *be a holomorphic immersion satisfying* (15), *and let* σ_k *and* γ_k *with* $\gamma_k = \Phi_k \circ \sigma_k$ *be as given by Theorem 1.4. Then, up to extracting a subsequence, there exists an at most countable family* J *such that for every* $j \in J$ *there exist a sequence of Möbius transformations* $f_k^j : \overline{D}^2 \to \overline{D}^2$ *and a finite set of finitely many points*

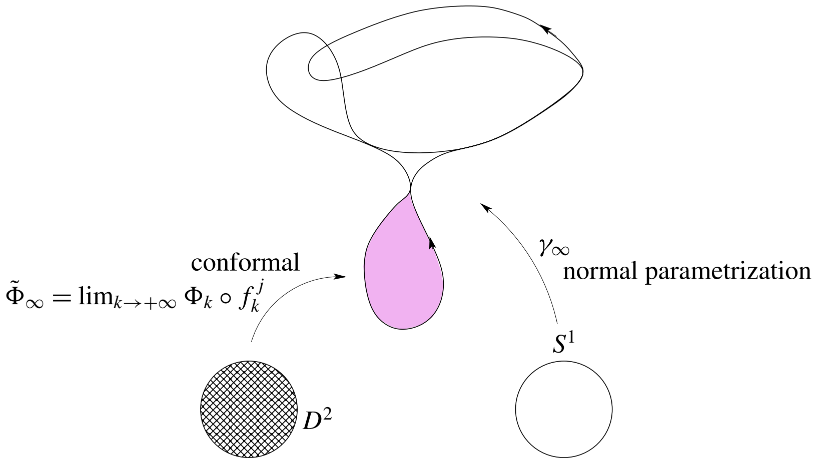

Figure 4. Composing Φ_k as in Figure 3 with suitable Möbius transformations, one can have Φ_∞ cover a different "component" bounded by γ_∞. In this figure one can choose among 4 different components, or choose Φ_∞ to be constant.

$B_j = \{a_1^j, \dots a_{N_j}^j\} \subset S^1$ such that

$$\gamma_k \rightharpoonup \gamma_\infty \quad \text{in } W^{2,p}(S^1) \qquad \text{and} \qquad \tilde\Phi_k^j := \Phi_k \circ f_k^j \rightharpoonup \tilde\Phi_\infty^j \quad \text{in } W^{2,p}_{\mathrm{loc}}(\overline{D}^2 \setminus B_j),$$

where $p < \infty$, the $\tilde\Phi_\infty^j : \overline{D}^2 \setminus B_j \to \mathbb{C}$ are holomorphic immersions satisfying

$$(\gamma_\infty)_*[S^1] = \sum_{j\in J} (\tilde\Phi_\infty^j)_*[S^1 \setminus B_j], \tag{17}$$

where, for any $\phi : S^1 \to \mathbb{C}$ and differential form ω on $\mathbb{C}$,

$$\langle \phi_*[S^1], \omega\rangle := \int_{S^1} \phi^*\omega.$$

If $\lambda_k^j := \log\big|(\tilde\Phi_k^j)'|_{S^1}\big|$ then, up to a subsequence, $\lambda_k^j \rightharpoonup \lambda_\infty^j$ in $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B_j)$, where

$$(-\Delta)^{\frac12}\lambda_\infty^j = \kappa_\infty^j e^{\lambda_\infty^j} - 1 - \sum_{i=1}^{N_j} \pi\delta_{a_i^j} \tag{18}$$

and $\kappa_k \circ f_k^j \overset{*}{\rightharpoonup} \kappa_\infty^j$ in $L^\infty(S^1, \mathbb{R})$ as $k \to +\infty$.

Theorem 1.5 says that it is always possible, up to the action of sequences of Möbius transformations, to recover all the connected components enclosed by the limiting curve γ_∞ (see in particular (17)). We will also see that these components are separated by what we call *pinched points* (see Definition 3.7), namely (roughly speaking) a pair of points $p \neq p' \in S^1$ such that $\gamma_\infty(p) = \gamma_\infty(p')$. The angle between the tangent vectors in these pairs of points is shown to necessarily be π. This also explains the coefficient π in front of each δ_{a_i} in (18).

It would be interesting to compare Theorems 1.2 and 1.5 to the blow-up analysis obtained recently by Mondino and Rivière [2014] in the case of sequences of weak conformal immersions from S^2 into $\mathbb{R}^m$; they study the possible limit of the Liouville equation

$$-\Delta_{g_0} u = K e^{2u} - 1 \quad \text{on } S^2 \tag{19}$$

satisfied by the conformal factor of the immersion Φ ($g_\Phi = e^{2u} g_0$) under the assumption that the second fundamental form is bounded in L^2. Also in their case, a sort of bubbling phenomenon occurs and the choice of different sequences of Möbius transformations of S^2 permits them to detect all the limiting enclosed currents. However, the 2-dimensional blow-up analysis differs substantially from the 1-dimensional case: in the 2-dimensional case the area is quantized, namely there is no production of area in the neck region between the different bubbles, whereas in the 1-dimensional case the quantization of the length does not hold. Precisely, Mondino and Rivière [2014] show that

$$\sum_{\text{“bubbles”}} \int_{S^2} e^{2u_\infty}\, dv = \liminf_{k\to+\infty} \int_{S^2} e^{2u_k}\, dv,$$

whereas in the present situation one can produce examples such that

$$\sum_{\text{“bubbles”}} \int_{S^1} e^{\lambda_\infty}\, d\theta < \liminf_{k\to+\infty} \int_{S^1} e^{\lambda_k}\, d\theta.$$

We insist on the fact that “conformal” parametrizations of planar curves are relevant in different applications. For instance, they should be one of the main tools of the Willmore plateau problem, of the analysis of the renormalizing area of surfaces in the hyperbolic space $\mathcal{H}^2$ and of the free boundaries problem. In particular, for the latter, Da Lio [2015] has observed that there is a one-to-one correspondence between free boundaries and $\frac{1}{2}$-harmonic maps and here we show that the holomorphic immersion ϕ for which $e^{\lambda(z)} = |\partial\phi/\partial\theta(z)|$, $z \in S^1$, is a $\frac{1}{2}$-harmonic map into $\phi(S^1)$.

In forthcoming work, we are going to investigate the topological and differential structure of the subspace of $C^{1,\alpha}(S^1) \times C^{0,\alpha}(S^1)$ made of solutions (u, κ) of the Nirenberg problem in S^1 (the Nirenberg moduli space). The present work should be interpreted as an attempt to describe the “boundary of the Nirenberg moduli space”. We mention that a nonlocal version of the Nirenberg problem in dimension $n \geq 2$ has recently been studied in [Jin et al. 2014; 2015a].

We finally prove a link between (7) and the analogous nonlocal equation in $\mathbb{R}$. Precisely, if $u \in L_{\frac{1}{2}}(\mathbb{R})$ (see (130)), $e^u \in L^1(\mathbb{R})$ and u satisfies

$$(-\Delta)^{\frac{1}{2}} u = K e^u \quad \text{in } \mathbb{R} \tag{20}$$

for some $K \in L^\infty(\mathbb{R})$, then $\lambda(z) := u(\Pi(z)) - \log(1+\sin z)$ (where $\Pi : S^1 \setminus \{-i\} \to \mathbb{R}$ is the stereographic projection) satisfies

$$(-\Delta)^{\frac{1}{2}} \lambda = K \circ \Pi e^\lambda - 1 + (2\pi - \|(-\Delta)^{\frac{1}{2}} u\|_{L^1})\delta_{-i} \quad \text{in } S^1. \tag{21}$$

Owing to this correspondence from Theorem 1.2, we can deduce the following compactness result in $\mathbb{R}$:

Theorem 1.6. *Let $u_k \in L_{\frac{1}{2}}(\mathbb{R})$ be a sequence of solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k = K_k e^{u_k} \quad \text{in } \mathbb{R}$$

with $\|K_k\|_{L^\infty} \leq C$ and $\|e^{u_k}\|_{L^1} \leq C$. Then, up to subsequence, we have $K_k e^{u_k} \rightharpoonup \mu$ weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R} \setminus B)$ for every $p < \infty$, where μ is a finite Radon measure in $\mathbb{R}$, $B := \{a_1, \dots, a_N\}$ is a (possibly empty) subset of $\mathbb{R}$ and $K_k \overset{*}{\rightharpoonup} K_\infty$ in $L^\infty(\mathbb{R})$. Moreover, one of the following alternatives holds:

(i) *$\mu|_{\mathbb{R}\setminus B} = K_\infty e^{u_\infty}$ for some $u_\infty \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \setminus B)$ satisfying*

$$(-\Delta)^{\frac{1}{2}} u_\infty = K_\infty e^{u_\infty} + \sum_{i=1}^{N} \pi \delta_{a_i} \quad \text{in } \mathbb{R}. \tag{22}$$

(ii) *$\mu|_{\mathbb{R}\setminus B} \equiv 0$, $N \leq 2$ and $u_k \to -\infty$ locally uniformly in $\mathbb{R} \setminus B$.*

In particular, we can deduce the following:

Corollary 1.7. *Under the hypotheses of Theorem 1.6, if $K_k \geq 0$ and*

$$\int_{\mathbb{R}} K_k e^{u_k}\, dx \leq 2\pi,$$

then either $N = 1$ and $u_k \to -\infty$ locally uniformly $\mathbb{R} \setminus \{a_1\}$, or $N = 0$ and $u_k \rightharpoonup u_\infty$ in $W^{1,p}(\mathbb{R})$ as $k \to +\infty$, where u_∞ solves

$$(-\Delta)^{\frac{1}{2}} u_\infty = K_\infty e^{u_\infty}. \tag{23}$$

We will give the proof of Theorem 1.6 and Corollary 1.7 in a forthcoming paper.

An interesting consequence of Theorem 1.4 is a proof of the classification of the solutions to the nonlocal equation

$$(-\Delta)^{\frac{1}{2}} u = e^u \quad \text{in } \mathbb{R} \tag{24}$$

under the integrability condition

$$L := \int_{\mathbb{R}} e^u\, dx < \infty. \tag{25}$$

Equation (24) is a special case of the problem

$$(-\Delta)^{n/2} u = (n-1)!\, e^{nu} \quad \text{in } \mathbb{R}^n, \qquad V := \int_{\mathbb{R}^n} e^{nu}\, dx < \infty, \tag{26}$$

which has been studied by several authors in the last decades (see, e.g., [Chen and Li 1991; Chang and Yang 1997; Lin 1998; Jin et al. 2015b; Martinazzi 2009a]). Geometrically, if u solves (26) and $n \geq 2$, then the metric $e^{2u}|dx|^2$ on $\mathbb{R}^n$ has constant Q-curvature $(n-1)!$ and volume V; see, e.g., [Chang 2004]. All the above-mentioned works rely on the application of a moving-plane technique, in order to show that under certain growth conditions at infinity (needed only when $n \geq 3$) the solutions to (26) have the form

$$u_{\mu,x_0}(x) := \log \frac{2\mu}{1 + \mu^2 |x - x_0|^2}, \quad x \in \mathbb{R}^n, \tag{27}$$

for some $\mu > 0$ and $x_0 \in \mathbb{R}^n$. For the case $n = 1$, instead of using the moving-plane technique, we will use stereographic projection to transform (24) into (14), and use the geometric interpretation of the latter (Theorem 1.4) to compute all its solutions (Corollary 2.3 below). This will yield:

Theorem 1.8. *Every function $u \in L_{\frac{1}{2}}(\mathbb{R})$ solving (24)–(25) is of the form (27) for some $\mu > 0$ and $x_0 \in \mathbb{R}$.*

We also remark that, by changing the sign of the nonlinearity in (24), the problem has no solutions. More precisely:

Proposition 1.9. *Given a function $K \in L^\infty(\mathbb{R})$ with $K \leq 0$, the equation*

$$(-\Delta)^{\frac{1}{2}} u = K e^u \quad \text{in } \mathbb{R}$$

has no solution satisfying (25).

The proof of Proposition 1.9 is a simple application of the maximum principle for the operator $(-\Delta)^{\frac{1}{2}}$, but it is worth remarking that, for $n \geq 4$, even solutions to (26) with $(n-1)!$ replaced by $-(n-1)!$ (or any negative constant) do exist, as shown in [Martinazzi 2008].

The paper is organized as follows. In Section 2 we introduce the nonlocal Liouville equation (7) in S^1 and we explain its geometric interpretation. In Section 3 we perform the blow-up and quantization analysis of (7) and in particular we prove Theorems 1.2 and 1.5. Section 4 is devoted to the description of the relation between equations (7) and (20). Finally, in Section 5 we prove Theorem 1.8 and Proposition 1.9.

Notations. We denote by $\langle x, y\rangle$ the scalar product of $x, y \in \mathbb{R}^n$. Let $h : \Omega \subset \mathbb{C} \to \mathbb{R}$ and let $\gamma : S^1 \to \mathbb{C}$ be a curve. We denote by $\int_\gamma h(z)\,|dz|$ or $\int_\gamma h(z)\,d\theta$ the line integral of h along γ. Given $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary part, respectively.

2. Nonlocal Liouville equation in S^1

In this section we study the nonlocal Liouville-type equation

$$(-\Delta)^{\frac{1}{2}} u = \kappa e^u - 1 \quad \text{in } S^1,$$

where $u \in L^1(S^1)$, $(-\Delta)^{\frac{1}{2}} u$ stands for the fractional Laplacian and $\kappa : S^1 \to \mathbb{R}$ is a bounded function. In Appendix A we recall the definition and some properties of the fractional Laplacian in S^1.

Geometric interpretation of the Liouville equation in S^1. The first key step in our analysis is the geometric interpretation of (7). Roughly speaking, such an equation prescribes the curvature of a closed curve in conformal parametrization.

It is easy to verify that for $\phi \in L^1(S^1)$ we have

$$(-\Delta)^{\frac{1}{2}} \phi(\theta) = \sum_{n\in\mathbb{Z}} |n| \hat{\phi}(n) e^{in\theta} = \mathcal{H}\left(\frac{\partial \phi}{\partial \theta}\right) = \frac{\partial \mathcal{H}(\phi)}{\partial \theta}, \tag{28}$$

where $\mathcal{H}$ is the Hilbert transform on S^1 defined by

$$\mathcal{H}(f)(\theta) := \sum_{n\in\mathbb{Z}} -i \operatorname{sign}(n) \hat{f}(n) e^{in\theta}, \quad f \in \mathcal{D}'(S^1).$$

We recall that the Hilbert transform has the following property, a proof of which can be found, e.g., in [Katznelson 2004, Chapter III].

Lemma 2.1. *The Hilbert transform $\mathcal{H}$ is bounded from $L^p(S^1)$ into itself for $1 < p < +\infty$ and it is of weak type $(1, 1)$. A function $f := u + iv$ with $u, v \in L^1(S^1, \mathbb{R})$ can be extended to a holomorphic function in D^2 if and only if $v = \mathcal{H}(u) + a$ for some $a \in \mathbb{C}$.*

Proof of Theorem 1.4. (1) Let $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ be a holomorphic immersion. Set $\lambda := (\log|\Phi'|)|_{S^1}$. Since $\Phi' : D^2 \to \mathbb{C} \setminus \{0\}$ is holomorphic, $\Phi'|_{S^1} = e^{\lambda+i\rho+i\theta_0}$ for some $\theta_0 \in [0, 2\pi)$, where $\rho := \mathcal{H}(\lambda)$ is the Hilbert transform of λ. Indeed, by Lemma 2.1, the function $f := \lambda + i\rho$ has a holomorphic extension $\tilde{f}$ to D^2; hence, $e^{\tilde{f}}$ is holomorphic in D^2 and $e^{\tilde{f}}|_{S^1} = e^f = e^{\lambda+i\rho}$. But $|e^f| = e^\lambda = (|\Phi'|)|_{S^1}$, so that by Lemma B.1 we have $\Phi'/e^{\tilde{f}} = e^{i\theta_0}$ for some constant θ_0. Up to a rotation of Φ we can assume that $\theta_0 = 0$. Up to such a rotation and a translation, Φ is determined by λ, and we have

$$\frac{\partial\Phi(z)}{\partial\theta}(z) = ie^{\lambda(z)+i\rho(z)+i\theta}. \tag{29}$$

Now let

$$s(\theta) := \int_0^\theta \left|\frac{\partial\Phi(e^{i\theta'})}{\partial\theta'}\right| d\theta'.$$

We have $s : [0, 2\pi] \to [0, L]$, where $L = \|\partial\Phi/\partial\theta\|_{L^1(S^1)}$ is the length of the curve $\Phi(S^1)$, and up to a scaling we will assume that $L = 2\pi$. Let $\theta := s^{-1} : [0, 2\pi] \to [0, 2\pi]$. One can also easily see that $\theta \in C^1([0, 2\pi], [0, 2\pi])$. Then, using (29) and that

$$\dot{s}(\theta) = |\Phi'(e^{i\theta})| = e^{\lambda(e^{i\theta})} > 0, \quad \dot{\theta}(s) = e^{-\lambda(e^{i\theta(s)})},$$

we compute

$$\tau(s) := \frac{d}{ds}\Phi(e^{i\theta(s)}) = \Phi'(e^{i\theta(s)})ie^{i\theta(s)}\dot{\theta}(s) = \frac{\partial\Phi}{\partial\theta}(e^{i\theta(s)})e^{-\lambda(e^{i\theta(s)})}.$$

Notice that $|\tau| \equiv 1$, i.e., the curve $\gamma : e^{is} \mapsto \Phi(e^{i\theta(s)})$ is parametrized by arc-length and τ is its unit tangent vector. Using (28), (29) and identifying s with e^{is}, the curvature of γ is given by

$$\begin{aligned}\kappa(s) = \langle i\tau(s), \dot{\tau}(s)\rangle &= \left\langle i\tau(s), \frac{d}{ds}\big(ie^{i\rho(e^{i\theta(s)})+i\theta(s)}\big)\right\rangle \\ &= \left(\frac{d\rho(e^{i\theta(s)})}{d\theta} + 1\right)\dot{\theta}(s) \\ &= ((-\Delta)^{\frac{1}{2}}\lambda(e^{i\theta(s)}) + 1)e^{-\lambda(e^{i\theta(s)})}.\end{aligned} \tag{30}$$

From (30) it follows that λ satisfies (14) with $\kappa(e^{is(\theta)}) := \langle i\tau(s(\theta)), \dot{\tau}(s(\theta))\rangle$. Since $|\kappa(e^{is})| = |\ddot{\gamma}(e^{is})|$ is in $L^\infty(S^1)$, we also have $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$.

(2) Conversely, let us assume that $\lambda \in L^1(S^1)$ with $e^\lambda \in L^1(S^1)$ weakly satisfies (14) for some $\kappa \in L^\infty(S^1)$. By regularity theory, $\lambda \in W^{1,p}(S^1)$ for any $p < \infty$. We set $\rho := \mathcal{H}(\lambda)$. Let $\phi \in W^{1,p}(\bar{D}^2, \mathbb{C})$ be the holomorphic extension of the function $e^{\lambda+i\rho} \in W^{1,p}(S^1)$ and set

$$\Phi(z) := \int_{\Sigma_{0,z}} \phi(w)\, dw, \quad z \in \bar{D}^2, \tag{31}$$

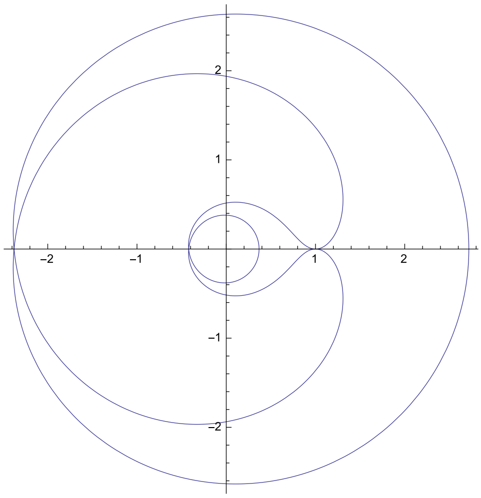

Figure 5. Plot of the curve $e^{\cos\theta}(\cos(2\pi \sin\theta) + i \sin(2\pi \sin\theta))$, $\theta \in [0, 2\pi]$. It has the same kind of self-intersections as the curve $\Phi(e^{i\theta}) = e^{2\pi e^{i\theta}}$, whose plot is difficult to inspect, since $|\Phi(z)|$ oscillates between $e^{2\pi}$ and $e^{-2\pi}$.

where $\Sigma_{0,z}$ is any path in $\bar{D}^2$ connecting 0 and z. Then $\Phi \in W^{2,p}(\bar{D}^2, \mathbb{C})$ satisfies (29). From part (1) we see that κ is the curvature of the curve $\Phi(S^1)$ in normal parametrization.

Let $\hat{\Phi} : \bar{D}^2 \to \mathbb{C}$ be another holomorphic immersion such that $|\hat{\Phi}'(z)| = e^{\lambda(z)}$, $z \in S^1$. We claim that

$$\Phi = e^{i\theta_0}\hat{\Phi} + a \quad \text{in } \bar{D}^2 \text{ for some } \theta_0 \in \mathbb{R},\ a \in \mathbb{C}. \tag{32}$$

Indeed, the function $h := \Phi'/\hat{\Phi}'$ never vanishes in $\bar{D}^2$ and satisfies

$$|h(z)| = \frac{|\Phi'(z)|}{|\hat{\Phi}'(z)|} = \frac{e^{\lambda(z)}}{e^{\lambda(z)}} = 1, \quad z \in S^1.$$

It follows from Lemma B.1 that h is a constant of modulus 1, say $h \equiv e^{i\theta_0}$, and (32) follows at once. □

Remark 2.2. In Theorem 1.4, we cannot expect that Φ is a biholomorphism from $\bar{D}^2$ onto $\Phi(\bar{D}^2)$. For instance, the function $\Phi(z) := e^{az}$ for any $a > 0$ is an immersion and $\Phi(S^1)$ has self-intersections whenever $a \geq \pi$, as is easily seen by writing (see Figure 5)

$$\Phi(e^{i\theta}) = e^{a\cos\theta}(\cos(a \sin\theta) + i \sin(a \sin\theta)).$$

Corollary 2.3. *All functions $\lambda \in L^1(S^1)$ with $e^\lambda \in L^1(S^1)$ that are solutions to*

$$(-\Delta)^{\frac{1}{2}}\lambda = C_0 e^\lambda - 1 \quad \text{on } S^1, \tag{33}$$

where C_0 is an arbitrary positive constant, are given by

$$\lambda(\theta) = \log\left|\frac{\partial}{\partial\theta}\frac{z - a_1}{1 - \bar{a}_1 z}\right| - \log C_0 \tag{34}$$

for some a_1 in D^2.

Proof. Up to the translation $\tilde{\lambda} = \lambda + \log C_0$ we can assume $C_0 = 1$. By Theorem 1.4, the function λ determines a holomorphic immersion $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ such that $\Phi(S^1)$ is a curve of curvature 1; hence, up to a translation, $\Phi(S^1) \subseteq S^1$, and therefore it is a Möbius transformation of the disk. From (15) we infer that $\lambda = \log\left|\Phi'|_{S^1}\right|$, and we conclude. □

The following corollary is an easy consequence of Theorem 1.4 and Corollary 2.3:

Corollary 2.4. *Let Φ, λ and κ be as in Theorem 1.4 and let $f : \bar{D}^2 \to \bar{D}^2$ be a Möbius diffeomorphism. Set $\tilde{\Phi} := \Phi \circ f$, $\tilde{\lambda} := \log\left|\tilde{\Phi}'|_{S^1}\right|$ and $\tilde{\kappa} := \kappa \circ f|_{S^1}$. Then*

$$\tilde{\lambda} = \lambda \circ f|_{S^1} + \log\left|f'|_{S^1}\right| \quad \text{and} \quad (-\Delta)^{\frac{1}{2}}\tilde{\lambda} = \tilde{\kappa} e^{\tilde{\lambda}} - 1.$$

Remark 2.5. One can also give an analogous geometric characterization for an equation of the type

$$(-\Delta)^{\frac{1}{2}}\lambda = \kappa e^{\lambda} - n \quad \text{in } S^1 \tag{35}$$

with $n > 1$. In this case there is a correspondence between the solutions of (35) and holomorphic functions $\Phi : D^2 \to \mathbb{C}$ of the form $\Phi'(z) = \Psi(z) h(z)$, where Ψ is the Blaschke product

$$\Psi(z) := \prod_{k=1}^{n-1} \frac{z - a_k}{1 - \bar{a}_k z}, \qquad a_1, \dots, a_{n-1} \in D^2,$$

and $h(z) \neq 0$ for every $z \in \bar{D}^2$. In this case, $n - 1 = i\Psi \cdot \partial\Psi/\partial\theta = \deg \Psi$.

Next, we show that the existence of a holomorphic immersion of the disk $\bar{D}^2$ is equivalent to the existence of a positive diffeomorphism of the disc $\bar{D}^2$. Such a result can be seen as a sort of generalized Riemann mapping theorem in the case of closed curves which are not necessarily injective. We start with the following lemma, giving better regularity up to the boundary of a holomorphic immersion $u : D^2 \to \mathbb{C}$ under the assumption that the curve $u|_{S^1}$ has a $W^{2,\infty}$-constant-speed parametrization.

Lemma 2.6. *Let $u \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $\partial_z u \neq 0$ in D^2 and suppose there is $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$ with $|\dot{\gamma}|$ constant and a homeomorphism $\sigma : S^1 \to S^1$ such that $\gamma = u \circ \sigma$. Then $u \in W^{2,p}(\bar{D}^2, \mathbb{C})$ for every $p < +\infty$ and $\partial_z u(z) \neq 0$ for all $z \in S^1$.*

Proof. Let $z_0 \in S^1$. Since $\dot{\gamma}(z_0) \neq 0$, we can find some $\rho > 0$ such that $\gamma(S^1 \cap B(z_0, \rho))$ coincides up to a rotation with a piece of the graph of a function $\varphi \in C^{1,\alpha}(\mathbb{R})$ that satisfies $\varphi'(u_1(x_0)) = 0$. We may also assume that $u = u_1 + iu_2$ takes values in the set $\{(\xi, \eta) \in \mathbb{R}^2 \mid \eta \geq \varphi(\xi)\}$. Define

$$\hat{u} = \hat{u}_1 + i\hat{u}_2 \quad \text{with} \quad \hat{u}_1 := u_1, \quad \hat{u}_2 := u_2 - \varphi(u_1).$$

Claim. *The function $\hat{u}_2$ satisfies*

$$\begin{cases} \partial_{x_i}(a_{ij}\partial_{x_j}\hat{u}_2) = 0 & \text{in } B(x_0, \rho) \cap D^2, \\ \hat{u}_2 = 0 & \text{in } B(x_0, \rho) \cap S^1, \end{cases} \tag{36}$$

where the matrix

$$(a_{ij}) = \begin{pmatrix} 1 - \dfrac{1}{1 + (\varphi')^2(u_1)} & \dfrac{\varphi'(u_1)}{1 + (\varphi')^2(u_1)} \\ -\dfrac{\varphi'(u_1)}{1 + (\varphi')^2(u_1)} & 1 - \dfrac{1}{1 + (\varphi')^2(u_1)} \end{pmatrix} \tag{37}$$

is in $L^\infty(\bar{D}^2)$ and uniformly elliptic.

Proof. We can write $u = \hat{u} + i\varphi(u_1)$. Since, by hypothesis, $\partial_{\bar z} u(z) = 0$ for all $z \in D^2$, the following estimates hold:

$$\begin{aligned}
\partial_{\bar z} u_1 &= -i\,\partial_{\bar z} u_2,\\
\partial_{\bar z}\hat{u}(z) &= -i\varphi'(u_1)\partial_{\bar z} u_1 = -\varphi'(u_1)\partial_{\bar z} u_2,\\
\partial_{\bar z} u_1 + i\,\partial_{\bar z}\hat{u}_2(z) &= -i\varphi'(u_1)\partial_{\bar z} u_1,\\
\partial_{\bar z} u_1 &= -\frac{i}{1+i\varphi'(u_1)}\partial_{\bar z}\hat{u}_2(z),\\
\partial_{\bar z}\hat{u} &= -\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar z}\hat{u}_2(z).
\end{aligned}$$

Therefore,

$$\Delta\hat{u}_2 = 4\Im(\partial_z\partial_{\bar z}\hat{u}) = -4\Im\left[\partial_z\left[\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar z}\hat{u}_2(z)\right]\right]. \tag{38}$$

Writing

$$\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar z}\hat{u}_2(z) = \frac{\varphi'(u_1)}{1+(\varphi')^2(u_1)}\,\frac{\partial_{x_1}\hat{u}_2 + \varphi'(u_1)\partial_{x_2}\hat{u}_2 + i(\partial_{x_2}\hat{u}_2 - \varphi'(u_1)\partial_{x_1}\hat{u}_2)}{2},$$

we compute the right-hand side of (38) and get

$$\Delta\hat{u}_2 = -\Im\left[(\partial_{x_1} - i\,\partial_{x_2})\frac{\varphi'(u_1)}{1+(\varphi')^2(u_1)}[(\partial_{x_1}\hat{u}_2 + \varphi'(u_1)\partial_{x_2}\hat{u}_2) + i(\partial_{x_2}\hat{u}_2 - \varphi'(u_1)\partial_{x_1}\hat{u}_2)]\right].$$

Therefore $\hat{u}_2$ satisfies (36)–(37) and the claim is proven. □

Elliptic estimates imply that $\hat{u}_2 \in W^{2,p}(\bar{B}(z_0, r/4) \cap \bar{D}^2)$ for every $p < +\infty$; in particular, it is in $C^{1,\alpha}(\bar{B}(z_0, r/4) \cap \bar{D}^2)$ for every $\alpha \in (0,1)$. Now, since $\hat{u}_2 \geq 0$ in $\bar{D}^2$ and $\hat{u}_2(z_0) = 0$, Hopf's lemma yields that $\partial_r\hat{u}_2(z_0) \neq 0$. Since $u = \hat{u} + i\varphi(u_1)$, it follows that

$$\partial_r u(z_0) = \partial_r\hat{u}_1(z_0) + i\,\partial_r\hat{u}_2(z_0) + i\,\underbrace{\varphi'(u_1(z_0))}_{=0}\,\partial_r\hat{u}_1(z_0) \neq 0$$

and, since $z_0 \in S^1$ was arbitrary, we conclude that $\partial_r u \neq 0$ everywhere on S^1. Then, since u is conformal up to the boundary, we also have $\partial_z u \neq 0$ on S^1. □

We introduce the set

$$\mathcal{T} := \Big\{\gamma : S^1 \to \mathbb{C} \,\Big|\, \gamma \in W^{2,\infty},\ |\dot\gamma| \text{ constant, and there is } \Psi \in C^1(\bar{D}^2, \mathbb{C}) \text{ with } \det\mathrm{Jac}(\Psi(z)) > 0,\ z \in D^2,$$
$$\text{and } (\Psi\circ\sigma)(z) = \gamma(z),\ z \in S^1, \text{ for some diffeomorphism } \sigma : S^1 \to S^1\Big\}.$$

Theorem 2.7 (generalized Riemann mapping theorem). *A curve γ is in $\mathcal{T}$ if and only if there exists a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that $\Phi\circ\sigma = \gamma$.*

Proof. (1) Suppose that there exists a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that $\Phi\circ\sigma = \gamma$. Then one can take $\Psi = \Phi$. Therefore, $\gamma \in \mathcal{T}$.

(2) Conversely, let $\Psi \in C^1(\bar{D}^2, \mathbb{C})$ with $\Psi|_{S^1} = \gamma$ and $\det\mathrm{Jac}(\Psi) > 0$ in D^2.

(2i) Consider the pull-back of the Euclidean metric g on $\mathbb{R}^2$ by Ψ,

$$h_{ij} := \langle \partial_{x_i}\Psi, \partial_{x_j}\Psi \rangle.$$

Since $\det \mathrm{Jac}(\psi) > 0$, we have

$$c^{-1}\delta_{ij} \le (h_{ij}) \le c\delta_{ij}.$$

We can write

$$h = h_{11}\, dx^2 + 2h_{12}\, dx\, dy + h_{22}\, dy^2. \tag{39}$$

Setting $z = x + iy$, one can write h in the form

$$h = \nu\, |dz + \mu\, d\bar{z}|^2,$$

where ν is a positive continuous function on U and μ is a complex-valued continuous function with $\|\mu\|_{L^\infty(\bar{D}^2)} < 1$ on U. Actually, ν and μ are given by

$$\nu = \tfrac{1}{4}\big(h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}\big),$$
$$\mu = \frac{h_{11} - h_{22} + 2ih_{12}}{h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}}.$$

Moreover, Ψ solves the equation

$$\frac{\partial_{\bar{w}}\Psi(w)}{\partial_w\Psi(w)} = \mu(w) \quad \text{in } D^2. \tag{40}$$

The function μ is the so-called Beltrami coefficient associated to the metric h. Now we extend μ by 0 outside $\bar{D}^2$ (we still denote this extension by μ). Then there exists a unique homeomorphism $\xi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ (here $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$) which satisfies, in a distributional sense,

$$\partial_{\bar{z}}\xi = \mu(z)\, \partial_z\xi \quad \text{in } \mathbb{C}$$

and the normalization conditions

$$\xi(0) = 0, \quad \xi(1) = 1, \quad \xi(\infty) = \infty.$$

Moreover, $\xi \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$ and $\partial_z\xi \neq 0$ a.e. in $\mathbb{C}$. The function ξ is called a quasiconformal map with dilation coefficient μ (see, e.g., Theorem 4.30 in [Imayoshi and Taniguchi 1992]).

Since ξ is a homeomorphism, $\xi(S^1)$ is a Jordan curve.

(2ii) Consider now $\tilde{\Psi} := \Psi \circ \xi^{-1} : \xi(\bar{D}^2) \to \mathbb{C}$. From [Imayoshi and Taniguchi 1992, Proposition 4.13] it follows that the complex dilatation of $\tilde{\Psi}$ is 0 in $\xi(D^2)$; therefore, $\partial_{\bar{z}}\tilde{\Psi} = 0$ and $\tilde{\Psi}$ is holomorphic in $\xi(D^2)$; see [Imayoshi and Taniguchi 1992, Lemma 4.6].

(2iii) Now we apply the Riemann mapping theorem: there exists a biholomorphic map u from D^2 onto $\xi(D^2)$. In particular, $\partial_z u \neq 0$ in D^2. Take $\Phi := \Psi \circ \xi^{-1} \circ u$. We observe that $\det \mathrm{Jac}(\Psi) > 0$ implies $\partial_z\Psi \neq 0$ in $\bar{D}^2$. Therefore,

$$\partial_z\Phi = \partial_w(\Psi \circ \xi^{-1})\partial_z u + \partial_{\bar{w}}(\Psi \circ \xi^{-1})\partial_z\bar{u} = \partial_w(\Psi \circ \xi^{-1})\partial_z u + \partial_{\bar{w}}(\Psi \circ \xi^{-1})\overline{\partial_{\bar{z}}u} = \partial_w(\Psi \circ \xi^{-1})\partial_z u.$$

We observe that Φ is holomorphic in D^2 because it is the composition of two holomorphic maps and $\partial_z \Phi \neq 0$ in D^2. From Lemma 2.6, it follows that $\partial_z \Phi \neq 0$ in $\bar{D}^2$ and we conclude the proof of Theorem 2.7. $\square$

From the next lemma we can deduce that if $\gamma \in \mathcal{T}$ then the winding number (or equivalently the degree) of γ is 1.

Lemma 2.8. *Let $\Phi \in W^{2,p}(\bar{D}^2, \mathbb{C})$ for some $1 < p \le +\infty$ be a holomorphic function such that $\partial_z \Phi \neq 0$ in $\bar{D}^2$. Then*

$$\deg \Phi = \frac{1}{2\pi} \int_0^{2\pi} \frac{\langle i\partial_\theta \Phi, \partial_\theta^2 \Phi \rangle}{|\partial_\theta \Phi|^2} \, d\theta = 1 + \frac{1}{2\pi i} \int_{S^1} \frac{f'(z)}{f(z)} \, dz = 1, \tag{41}$$

where $f(z) = \Phi'(z)$.

We note that Lemma 2.8 is a direct corollary of Theorem 1.4. Indeed, $\deg \Phi|_{S^1} = (1/2\pi) \int_{S^1} \kappa |\Phi'| \, d\theta = (1/2\pi) \int_{S^1} \kappa e^{\lambda} \, d\theta$ but, since $(-\Delta)^{\frac{1}{2}} \lambda = \kappa e^{\lambda} - 1$, integrating gives $\int_{S^1} \kappa e^{\lambda} \, d\theta = 2\pi$.

Anyway, we provide a direct proof for the reader's convenience:

Proof. We recall that

$$\Phi'(z) = \frac{1}{2} e^{-i\theta} \left(\frac{\partial \Phi}{\partial r} - \frac{i}{r} \frac{\partial \Phi}{\partial \theta} \right) =: f(z).$$

Since Φ is holomorphic, we have

$$\frac{\partial \Phi}{\partial r} = -\frac{i}{r} \frac{\partial \Phi}{\partial \theta}. \tag{42}$$

Hence,

$$\begin{aligned}
\int_{S^1} \frac{f'(z)}{f(z)} \, dz &= \int_{S^1} \frac{\frac{e^{-i\theta}}{2} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \frac{e^{-i\theta}}{2} \left(\frac{\partial \Phi}{\partial r} - \frac{i}{r} \frac{\partial \Phi}{\partial \theta} \right)}{\frac{e^{-i\theta}}{2} \left(\frac{\partial \Phi}{\partial r} - \frac{i}{r} \frac{\partial \Phi}{\partial \theta} \right)} \, dz \\
&= \int_{S^1} \frac{\left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) \left(-\frac{i}{r} e^{-i\theta} \frac{\partial \Phi}{\partial \theta} \right)}{\frac{\partial \Phi}{\partial r} - \frac{i}{r} \frac{\partial \Phi}{\partial \theta}} \, dz & \text{(by (42))} \\
&= \int_{S^1} e^{-i\theta} \frac{\frac{2i}{r^2} \frac{\partial \Phi}{\partial \theta} - \frac{i}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial^2 \theta}}{\frac{-2i}{r} \frac{\partial \Phi}{\partial \theta}} \, dz \\
&= -\int_{S^1} e^{-i\theta} \, dz + \int_{S^1} e^{-i\theta} \frac{\frac{\partial^2 \Phi}{\partial r \partial \theta}}{-2i \frac{\partial \Phi}{\partial \theta}} \, dz \int_{S^1} e^{-i\theta} \frac{\frac{\partial^2 \Phi}{\partial^2 \theta}}{-2i \frac{\partial \Phi}{\partial \theta}} \, dz & \text{(since } r = 1 \text{ on } S^1) \\
&= -2\pi i - \frac{i}{2} \int_0^{2\pi} \frac{\frac{\partial^2 \Phi}{\partial r \partial \theta}}{\frac{\partial \Phi}{\partial \theta}} \, d\theta - \frac{1}{2} \int_0^{2\pi} \frac{\frac{\partial^2 \Phi}{\partial \theta \partial \theta}}{\frac{\partial \Phi}{\partial \theta}} \, d\theta \\
&= -2\pi i - \int_0^{2\pi} \frac{\frac{\partial^2 \Phi}{\partial \theta \partial \theta}}{\frac{\partial \Phi}{\partial \theta}} \, d\theta & \text{(by (42)).}
\end{aligned} \tag{43}$$

On the other hand, we have

$$\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{-i\overline{\partial_\theta\Phi}\partial^2_{\theta^2}\Phi}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\, d\theta + \frac{1}{2}\int_0^{2\pi} \frac{i\partial_\theta\Phi\overline{\partial^2_{\theta^2}\Phi}}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\, d\theta. \tag{44}$$

We observe that

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} \frac{i\partial_\theta\Phi\overline{\partial^2_{\theta^2}\Phi}}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\, d\theta &= -\frac{i}{2}\int_0^{2\pi} \overline{\partial_\theta\Phi}\frac{\partial^2_{\theta^2}\Phi}{|\partial_\theta\Phi|^2}\, d\theta - \frac{i}{2}\int_0^{2\pi} |\partial_\theta\Phi|^2\partial_\theta(|\partial_\theta\Phi|^{-2})\, d\theta \\ &= -\frac{i}{2}\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\, d\theta. \end{aligned} \tag{45}$$

It follows that

$$\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\, d\theta = -i\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\, d\theta. \tag{46}$$

By combining the estimates (43)–(46), we get

$$\int_{S^1} \frac{1}{2\pi i}\frac{f'(z)}{f(z)}\, dz = -1 - \frac{1}{2\pi i}\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\, d\theta = -1 + \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\, d\theta. \qquad \square$$

Connection with half-harmonic maps. In this subsection we show an interesting connection between the solutions of (7) and the half-harmonic maps into a given curve Γ.

Let $\tilde\phi = \Phi \in C^1(\overline{D}^2, \mathbb{C})$ be the map given by Theorem 2.7 and set $\phi := \Phi|_{S^1}$. Then Φ is conformal up to the boundary, i.e., $\partial\phi/\partial\theta \cdot \partial\tilde\phi/\partial r = 0$ on S^1. Since $\partial\tilde\phi/\partial r\big|_{r=1} = (-\Delta)^{\frac12}\phi$, we deduce

$$(-\Delta)^{\frac12}\phi \perp T_\phi\Gamma, \quad \text{i.e.,} \quad \frac{\partial\phi}{\partial\theta}\cdot(-\Delta)^{\frac12}\phi = 0 \quad \text{on } \mathcal{D}'(S^1). \tag{47}$$

Equation (47) says that ϕ is a $\frac12$-harmonic map into Γ (see [Da Lio and Rivière 2011]).

We would like to recall a characterization of $\frac12$-harmonic maps of S^1 into submanifolds of $\mathbb{R}^n$, which has been already observed in [Da Lio 2015] and then in [Millot and Sire 2015].

Theorem 2.9 [Da Lio et al. $\geq$ 2015]. *Let $u \in H^{\frac12}(S^1, \mathcal{N})$, where $\mathcal{N}$ is a k-dimensional smooth submanifold of $\mathbb{R}^m$ without boundary. Then u is a weak $\frac12$-harmonic map, i.e., $(-\Delta)^{\frac12}u \perp T_u\mathcal{N}$, if and only if its harmonic extension $\tilde u \in W^{1,2}(D^2, \mathbb{R}^m)$ is conformal, in which case*

$$\partial_r\tilde u \perp T_u\mathcal{N} \quad \text{in } \mathcal{D}'(S^1). \tag{48}$$

Proof. Let $u \in H^{\frac12}(S^1, \mathcal{N})$ be a weak $\frac12$-harmonic map and let $\tilde u \in W^{1,2}(D, \mathbb{R}^m)$ be the harmonic extension of u. Then

$$E(u) := \int_{S^1} |(-\Delta)^{\frac14}u|^2\, |dz| = \int_{D^2} |\nabla\tilde u|^2\, |dz|.$$

Claim. *For every $\tilde X \in C^\infty(\overline{D}^2, \mathbb{R}^2)$ such that $\tilde X(z)\cdot z = 0$ for $z \in S^1$,*

$$\left(\frac{d}{dt}\int_{D^2} \left|\nabla\tilde u(z + t\tilde X(z))\right|^2 |dz|\right)\Bigg|_{t=0} = 0. \tag{49}$$

Proof of the claim. It has been proved in [Da Lio and Rivière 2011] that, if u is $\frac{1}{2}$-harmonic, then $u \in C^\infty(S^1)$; in particular, u satisfies

$$\left(\frac{d}{dt}\int_{S^1}\left|(-\Delta)^{\frac{1}{4}}u(z+tX(z))\right|^2|dz|\right)\bigg|_{t=0}=0 \tag{50}$$

for every $X \in C^\infty(S^1)$.

Let $\tilde{X} \in C^\infty(\bar{D}^2, \mathbb{R}^2)$ be such that $\tilde{X}(z)\cdot z = 0$ for $z \in S^1$. We observe that, for all $z \in S^1$, $Y := d\tilde{u}\cdot\tilde{X} = du\cdot\tilde{X} \in T_u\mathcal{N}$ and

$$\left(\frac{d}{dt}\int_{D^2}|\nabla\tilde{u}(z+t\tilde{X}(z))|^2|dz|\right)\bigg|_{t=0}=\int_{D^2}\nabla\tilde{u}\cdot\nabla Y\,|dz| = \int_{S^1}\partial_r\tilde{u}\cdot Y\,|dz| = -\int_{S^1}(-\Delta)^{\frac{1}{2}}u\cdot Y\,|dz| = 0,$$

where the last equality follows from (50). □

From Proposition 2.10 below and the regularity of $\tilde{u}$ up to the boundary, it follows that $\tilde{u}$ is also conformal in $\bar{D}^2$, i.e.,

$$|\partial_{x_1}\tilde{u}| = |\partial_{x_2}\tilde{u}|, \quad \partial_{x_1}\tilde{u}\cdot\partial_{x_2}\tilde{u} = 0.$$

Conversely, suppose the harmonic extension $\tilde{u}$ of u is conformal and satisfies (48). Since $\partial_r\tilde{u} = -(-\Delta)^{\frac{1}{2}}u$, we deduce that u is $\frac{1}{2}$-harmonic. □

Proposition 2.10 [Rivière 2012, Proposition II.2]. *Let $\tilde{u}$ be a map in $W^{1,2}(D^2, \mathbb{R}^m)$ satisfying*

$$\left(\frac{d}{dt}\int_{D^2}|\nabla\tilde{u}_t|^2|dz|\right)\bigg|_{t=0}=0, \quad u_t(x) := u(x+tX(x)),$$

for every $X \in C^\infty(\bar{D}^2, \mathbb{R}^2)$ such that $\langle X(x), x\rangle = 0$ for $x \in S^1$. Then $\tilde{u}$ is conformal in D^2.

In the case of $\frac{1}{2}$-harmonic maps $u : S^1 \to S^1$, we deduce from Theorem 2.9 the following:

Corollary 2.11. *Let $u \in H^{\frac{1}{2}}(S^1, S^1)$ with* $\deg u = 1$. *Then u is a weak $\frac{1}{2}$-harmonic map if and only if its harmonic extension $\tilde{u} : \bar{D}^2 \to \bar{D}^2$ is a Möbius map, namely it has the form*

$$\tilde{u}(z) = e^{i\theta_0}\frac{z-a}{1-\bar{a}z}$$

for some $|a| < 1$ and $\theta_0 \in [0, 2\pi)$.

3. Compactness of the Liouville equation in S^1

In this section we analyze the asymptotics of solutions to (7).

The ε-regularity lemma and first compactness result. A key point in the proof of Theorem 1.2 is an *ε-regularity lemma*, asserting, roughly speaking, that if the L^1 norm in conformal parametrization of the curvature $(\kappa_k e^{\lambda_k})$ is small (less than π) in a neighborhood of a point, then $\lambda_k - C_k$ is uniformly bounded in the same neighborhood for some constant C_k. This result (Lemma 3.3) depends on Theorem 3.2 below.

Lemma 3.1 (fundamental solution of $(-\Delta)^{\frac{1}{2}}$ on S^1). *The function*

$$G(\theta) := -\frac{1}{2\pi}\log(2(1-\cos\theta))$$

belongs to $\mathrm{BMO}(S^1)$, *can be decomposed as*

$$G(\theta)=\frac{1}{\pi}\log\frac{\pi}{|\theta|}+H(\theta),\quad \theta\in[-\pi,\pi]\sim S^1,\qquad \textit{with } H\in C^0(S^1), \tag{51}$$

and satisfies

$$(-\Delta)^{\frac{1}{2}}G=\delta_1-\frac{1}{2\pi}\quad \textit{in } S^1,\qquad \int_{S^1}G(\theta)\,d\theta=0, \tag{52}$$

and, for every function $u\in L^1(S^1)$ *with* $(-\Delta)^{\frac{1}{2}}u\in L^1(S^1)$, *one has*

$$u-\bar u=G*(-\Delta)^{\frac{1}{2}}u:=\int_{S^1}G(\cdot-\theta)(-\Delta)^{\frac{1}{2}}u(\theta)\,d\theta\quad \textit{for almost every } t\in S^1. \tag{53}$$

Proof. The identity (52) follows at once from Lemma 4.3. That $G\in\mathrm{BMO}(S^1)$ follows from parametrizing $S^1=[-\pi,\pi]/\{\pi\sim-\pi\}$, writing $1-\cos\theta=\frac{1}{2}\theta^2+O(\theta^4)$ as $\theta\to 0$ and therefore

$$G(\theta)=-\frac{1}{2\pi}\Big(\log\big(\tfrac{1}{2}\theta^2\big)+\log(1+O(\theta^2))\Big)$$

as $\theta\to 0$. Similarly, (51) follows from the explicit expression of G, since

$$H(\theta):=G(\theta)-\frac{1}{\pi}\log\frac{\pi}{|\theta|}=C+\log(1+O(\theta)^2)\to C\quad \text{as } \theta\to 0$$

and $H(\theta)\to-(\log 2)/(2\pi)$ as $|\theta|\to\pi$, so that $H\in C^0(S^1)$.

To prove (53) for $u\in C^\infty$, we write

$$u(0)-\bar u=\Big\langle\delta_1-\frac{1}{2\pi},u\Big\rangle=\langle(-\Delta)^{\frac{1}{2}}G,u\rangle:=\int_{S^1}G(\theta)(-\Delta)^{\frac{1}{2}}u(\theta)\,d\theta$$

and, translating, one gets (53) also for $t\neq 0$. For a general function $u\in H^{1,1}_{\Delta}(S^1)$, take a sequence $(u_k)\subset C^\infty(S^1)$ with

$$u_k\to u,\quad (-\Delta)^{\frac{1}{2}}u_k\to(-\Delta)^{\frac{1}{2}}u\quad \text{in } L^1(S^1),$$

which can be easily obtained by convolution. Then

$$u\xleftarrow{L^1(S^1)}u_k=\int_{S^1}G(\cdot-\theta)(-\Delta)^s u_k(\theta)\,dy\xrightarrow{L^1(S^1)}\int_{S^1}G(\cdot-\theta)(-\Delta)^s(\theta)\,d\theta,$$

the convergence on the right following from (51) and Fubini's theorem:

$$\int_{S^1}\bigg|\int_{S^1}G(t-\theta)[(-\Delta)^s u_k(\theta)-(-\Delta)^s u(\theta)]\,d\theta\bigg|\,dt\le\|G\|_{L^1(S^1)}\|(-\Delta)^s u_k-(-\Delta)^s u\|_{L^1(S^1)}\to 0$$

as $k\to\infty$. Since the convergence in L^1 implies a.e. convergence (up to a subsequence), (53) follows. The last claim follows at once from the explicit expression of G. □

The following theorem, which is a generalization of Theorem I in [Brezis and Merle 1991], is a sort of Moser–Trudinger inequality and it is crucial for proving Lemma 3.3.

Theorem 3.2. *There exist constants $C_1, C_2 > 0$ such that, for any $\varepsilon \in (0, \pi)$, one has*

$$C_1 \leq \sup_{\substack{u=G*f \\ \|f\|_{L^1(S^1)}\leq 1}} \varepsilon \int_{S^1} e^{(\pi-\varepsilon)|u|}\, d\theta \leq C_2 \tag{54}$$

and, in particular,

$$C_1 \leq \sup_{\substack{u\in L^1(S^1): \\ \|(-\Delta)^{\frac{1}{2}}u-\alpha\|_{L^1(S^1)}\leq 1 \\ \text{for some } \alpha\in\mathbb{R}}} \varepsilon \int_{S^1} e^{(\pi-\varepsilon)|u-\bar{u}|}\, d\theta \leq C_2. \tag{55}$$

Proof. Clearly the second inequality in (55) follows from the second inequality in (54) and (52). Let us now prove (54). Given f with $\|f\|_{L^1(S^1)} \leq 1$ and setting $u = G * f$, we get

$$\begin{aligned} |u(t)| &= \left| \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f(\theta)\, d\theta + \int_{t-\pi}^{t+\pi} H(\theta - t) f(\theta)\, d\theta \right| \\ &\leq \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta + C. \end{aligned}$$

With Jensen's inequality and Fubini's theorem, and using that $\|f\|_{L^1(S^1)} \leq 1$, it follows that

$$\begin{aligned} \int_{-\pi}^{\pi} e^{(\pi-\varepsilon)|u(t)-\bar{u}|}\, dt &\leq C \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta\right) dt \\ &\leq C \int_{-\pi}^{\pi} \int_{t-\pi}^{t+\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \log \frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta\, dt \\ &= C \int_{t-\pi}^{t+\pi} |f(\theta)| \int_{-\pi}^{\pi} \left(\frac{\pi}{|\theta - t|}\right)^{1-\frac{\varepsilon}{\pi}} dt\, d\theta \leq \frac{C_2}{\varepsilon}. \end{aligned} \tag{56}$$

This proves the second inequality in (54).

To prove the first inequalities in (54) and in (55), fix $\varepsilon \in (0, \pi)$, choose $(f_k) \subset C^\infty(S^1)$ nonnegative such that $f_k \to \delta_0$ weakly in the sense of measures with $\|f_k\|_{L^1(S^1)} = 1$, and let u_k solve

$$(-\Delta)^{\frac{1}{2}} u_k = f_k - \frac{1}{2\pi} \qquad \text{in } S^1, \quad \bar{u}_k = 0.$$

Such u_k can easily be constructed using the Fourier formula for $(-\Delta)^{\frac{1}{2}}$; see (123). Then, by Lemma 3.1,

$$|u_k(t)| \geq \int_{S^1} G(t-\theta) f_k(\theta)\, d\theta \geq \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f_k(\theta)\, d\theta - C.$$

Multiplying by $\pi - \varepsilon$, exponentiating, integrating on S^1 and taking the limit as $k \to \infty$, one gets

$$\begin{aligned} \lim_{k\to\infty} \int_{S^1} e^{(\pi-\varepsilon)|u_k(t)|}\, dt &\geq \lim_{k\to\infty} \frac{1}{C} \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f_k(\theta)\, d\theta\right) dt \\ &= \frac{1}{C} \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \log \frac{\pi}{|t|}\right) dt = \frac{1}{C} \int_{-\pi}^{\pi} \left(\frac{\pi}{|t|}\right)^{1-\frac{\varepsilon}{\pi}} dt = \frac{C_1}{\varepsilon}, \end{aligned}$$

which proves (54) and also (55), since $\bar{u}_k = 0$. ∎

Lemma 3.3 (ε-regularity lemma). *Let $u \in L^1(S^1)$ be a solution of*

$$(-\Delta)^{\frac{1}{2}} u = \kappa e^u - 1 \tag{57}$$

with $\kappa \in L^\infty(S^1)$, $e^u \in L^1(S^1)$ and $\Lambda := \|\kappa e^u\|_{L^1}$. Assume that, for some arc $A \subset S^1$,

$$\int_A |\kappa| e^u \, d\theta \le \pi - \varepsilon \tag{58}$$

for some $\varepsilon > 0$. Then, for every arc $A' \Subset A$ with $\operatorname{dist}(A^c, A') = \delta$,

$$\|u - \bar{u}\|_{L^\infty(A')} \le C(\delta, \varepsilon, \Lambda). \tag{59}$$

Proof. Set $f := (-\Delta)^{\frac{1}{2}} u$. We split $f = f_1 + f_2$, where

$$f_1 = \kappa e^u \chi_A, \qquad f_2 = \kappa e^u \chi_{A^c}.$$

Let us now define

$$u_i(t) := G * f_i(t) = \int_{S^1} G(t-\theta) f_i(\theta) \, d\theta, \quad i = 1, 2,$$

where G is as in Lemma 3.1. From (52) and (53) it follows that

$$u - \bar{u} = G * (\kappa e^u - 1) = G * (\kappa e^u) = u_1 + u_2.$$

Choose an arc A'' with $A' \Subset A'' \Subset A$ and $\operatorname{dist}(A'', A^c) = \operatorname{dist}(A', (A'')^c) = \frac{1}{2}\delta$. With (51) we easily bound

$$\|u_2\|_{L^\infty(A'')} \le C_1 = C_1(\Lambda, \delta). \tag{60}$$

It follows from (58) and Theorem 3.2 that $\|e^{|u_1|}\|_{L^p(S^1)} \le C_{p,\varepsilon}$ for some $p > 1$ and, consequently, also $e^{\bar{u}} \le C$. Then, for $t \in A'$ we have

$$\begin{aligned}
u_1(t) &\le \int_A G(t-\theta)(|\kappa| e^{u_1(\theta)} e^{u_2(\theta)+\bar{u}} - 1) \, d\theta \\
&\le \|\kappa\|_{L^\infty} \Bigg(e^{C_1 + \bar{u}} \underbrace{\int_{A''} G(t-\theta) e^{u_1(\theta)} \, d\theta}_{(1)} + \underbrace{\int_{A \setminus A''} G(t-\theta) e^{u(\theta)} \, d\theta + C}_{(2)} \Bigg) \\
&\le C,
\end{aligned}$$

where in (1) we use that $G \in L^q(S^1)$ for $q \in [1, \infty)$ and in (2) we use that $G \in L^\infty(A' \times (A \setminus A''))$. $\square$

Lemma 3.4. *Let $\lambda : S^1 \to S^1$ satisfy $(-\Delta)^{\frac{1}{2}} \lambda \in L^1(S^1)$ and let $\tilde{\lambda}$ be the harmonic extension of λ to D^2. Then*

$$\|\nabla \tilde{\lambda}\|_{L^{(2,\infty)}(D^2)} \le C \|(-\Delta)^{\frac{1}{2}} \lambda\|_{L^1(S^1)} \tag{61}$$

and, for any ball $B_r(x_0)$,

$$\frac{1}{r} \int_{B_r(x_0) \cap D^2} |\nabla \tilde{\lambda}| \, dx \le C \|\nabla \tilde{\lambda}\|_{L^{(2,\infty)}(B_r(x_0) \cap D^2)}. \tag{62}$$

Proof. Let $\lambda : S^1 \to S^1$ satisfy $(-\Delta)^{\frac{1}{2}}\lambda \in L^1(S^1)$ and let $\tilde{\lambda}$ be the harmonic extension of λ to D^2. Then we can write

$$\tilde{\lambda}(x) = \int_{S^1} G(x, y)\frac{\partial \tilde{\lambda}}{\partial \nu}(y)\, dy = \int_{S^1} G(x, y)(-\Delta)^{\frac{1}{2}}\lambda(y)\, dy, \tag{63}$$

where G is the Green function associated to the Neumann problem. It is known that $\nabla_x(G(x, y))$ is in $L^{(2,\infty)}(S^1)$ (see, e.g., [Kenig 1994]). Therefore, $\nabla\tilde{\lambda}(x) \in L^{(2,\infty)}(D^2)$ as well and (61) holds.

The proof of (62) follows from O'Neil's inequality [1963]

$$\int_A |\nabla\tilde{\lambda}|\, dx \le \|\chi_A\|_{L^{(2,1)}(A)}\|\nabla\tilde{\lambda}\|_{L^{(2,\infty)}(A)} = \sqrt{|A|}\|\nabla\tilde{\lambda}\|_{L^{(2,\infty)}(A)}$$

for any $A \subset D^2$. □

Theorem 3.5. *Let (λ_k) be a sequence as in Theorem 1.2 and let $(\Phi_k) \subset C^1(\bar{D}^2, \mathbb{C})$ be holomorphic immersions with $\lambda_k(z) = \log|\Phi_k'(z)|$ for $z \in S^1$ and $\Phi_k(1) = 0$ (compare to Theorem 1.4) Then, up to extracting a subsequence, the set*

$$B := \left\{ a \in S^1 \,\middle|\, \lim_{r\to 0^+} \limsup_{k\to\infty} \int_{B(a,r)\cap S^1} |\kappa_k| e^{\lambda_k}\, d\theta \ge \pi \right\} = \{a_1, \ldots, a_N\} \tag{64}$$

is finite and, for functions $v_\infty \in L^1(S^1, \mathbb{R})$ and $\Phi_\infty \in W^{1,2}(D^2, \mathbb{C})$ we have, for $1 \le p < \infty$,

$$\lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\text{loc}}(S^1 \setminus B), \qquad \bar{\lambda}_k := \frac{1}{2\pi}\int_{S^1} \lambda_k\, d\theta, \tag{65}$$

and

$$\Phi_k \rightharpoonup \Phi_\infty \quad \text{in } W^{2,p}_{\text{loc}}(\bar{D}^2 \setminus B, \mathbb{C}) \text{ and in } W^{1,2}(D^2, \mathbb{C}). \tag{66}$$

Moreover, one of the following alternatives holds:

(1) *The sequence $(\lambda_k) \subset \mathbb{R}$ is bounded and Φ_∞ is a holomorphic immersion of $\bar{D}^2 \setminus B$ (i.e., it is holomorphic in D^2 and $\partial_z\Phi_\infty \neq 0$ for $z \in \bar{D}^2 \setminus B$).*

(2) *$\lambda_k \to -\infty$ locally uniformly as $k \to +\infty$ and $\Phi_\infty \equiv Q$ for some constant $Q \in \mathbb{C}$.*

Proof. The sequence of measures $|\kappa_k|e^{\lambda_k}\, d\theta$ on S^1 is bounded (for the total variation norm); hence, up to extracting a subsequence, we have $|\kappa_k|e^{\lambda_k}\, dx \overset{*}{\rightharpoonup} \mu$ weakly in the sense of measures for a Radon measure $\mu \in \mathcal{M}(S^1)$. Let $B := \{a \in S^1 \mid \mu(\{a\}) \ge \pi\}$. Then B is clearly finite, say $B = \{a_1, \ldots, a_N\}$, and is characterized by the first identity in (64). Indeed, if $\mu(\{a\}) \ge \pi$, then for every $r > 0$ and $\varphi \in C^0(S^1)$ supported in $B(a, r) \cap S^1$ such that $0 \le \varphi \le 1 = \varphi(a)$ one has

$$\limsup_{k\to\infty} \int_{B(a,r)\cap S^1} |\kappa_k|e^{\lambda_k}\, d\theta \ge \limsup_{k\to\infty} \int_{S^1} |\kappa_k|e^{\lambda_k}\varphi\, d\theta = \int_{S^1} \varphi\, d\mu \ge \pi\varphi(a) = \pi,$$

and, conversely, if $\mu(\{a\}) < \pi$, then $\mu(B(a, r_0) \cap S^1) < \pi$ for some $r_0 > 0$; hence, taking $\varphi \in C^0(S^1)$ supported in $B(a, r_0) \cap S^1$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $B(a, r_0/2) \cap S^1$, one gets

$$\limsup_{k\to\infty} \int_{B(a,r_0/2)\cap S^1} |\kappa_k|e^{\lambda_k}\, d\theta \le \limsup_{k\to\infty} \int_{S^1} |\kappa_k|e^{\lambda_k}\varphi\, d\theta = \int_{S^1} \varphi\, d\mu \le \mu(B(a, r_0)) < \pi.$$

We now show that for every compact $K \subset S^1 \setminus B$ there exists a constant c_K depending on $\bar{L}$ and $\bar{\kappa}$ in (8)–(10) such that

$$\|e^{\lambda_k}\|_{L^\infty(K)} \le c_K \tag{67}$$

and

$$\|\lambda_k - \bar{\lambda}_k\|_{L^\infty(K)} \le c_K. \tag{68}$$

Indeed, cover K with finitely many arcs $A_i \cap S_1$ such that

$$\int_{A_i \cap S^1} |\kappa_k| e^{\lambda_k}\, d\theta < \pi.$$

From Lemma 3.3 it follows that $\lambda_k - \bar{\lambda}_k$ is bounded in each A_i, and (68) follows. Moreover, considering that $\|e^{\lambda_k}\|_{L^1(S^1)} = L_k \le \bar{L}$, it follows that $\bar{\lambda}_k$ and λ_k are bounded above, and this proves (67). Now, writing $\lambda_k - \bar{\lambda}_k = G * (\kappa_k e^{\lambda_k} - 1)$ as in (53) of Lemma 3.1, we can bootstrap regularity and obtain that $\lambda_k - \bar{\lambda}_k$ is bounded in $W^{1,p}(K)$ for every $p < \infty$, and (65) follows from weak compactness.

Let $\tilde{\lambda}_k$ be the harmonic extension of λ_k. From (68), (61) and (62) we get

$$\|\tilde{\lambda}_k - \bar{\lambda}_k\|_{L^\infty(\partial(D^2 \setminus \cup_{i=1}^N B(a_i,\delta)))} \le C_\delta \quad \text{for every } \delta > 0;$$

hence,

$$(\tilde{\lambda}_k - \bar{\lambda}_k) \quad \text{is bounded in } W^{1,p}_{\mathrm{loc}}(\bar{D}^2 \setminus B). \tag{69}$$

Since Φ_k is harmonic and conformal,

$$\int_{D^2} |\nabla \Phi_k(z)|^2 \le \tfrac{1}{2} L_k^2. \tag{70}$$

Since $\Phi_k(1) = 0$, it follows that the sequence (Φ_k) is bounded in $W^{1,2}(D^2)$ and, up to a subsequence, $\Phi_k \rightharpoonup \Phi_\infty$ weakly in $W^{1,2}(D^2)$, where Φ_∞ is holomorphic.

From (16) it follows that $|\nabla \Phi_k|$ is bounded in $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$, so Φ_k is bounded in $W^{2,p}_{\mathrm{loc}}(S^1 \setminus B)$ and up to a subsequence one gets $\Phi_k \rightharpoonup \Phi_\infty$ in $W^{2,p}_{\mathrm{loc}}(D^1 \setminus B)$, as desired.

Further, if $\bar{\lambda}_k \to -\infty$ then (69) yields $\nabla \Phi_k \to 0$ uniformly locally in $\bar{D}^2 \setminus B$; hence, Φ_∞ is constant. Similarly, if $\lambda_k \ge -C$ then $|\nabla \Phi_k|$ is locally uniformly lower bounded on $D^2 \setminus B$; hence, $\nabla \Phi_\infty \ne 0$ in $D^2 \setminus B$. □

Blow-up analysis. In this section we associate to a sequence (λ_k) satisfying (8)–(10) a sequence of curves $(\gamma_k) \subset W^{2,\infty}(S^1, \mathbb{C})$ with bounded lengths $L_k \le \bar{L}$, curvatures bounded by $\bar{\kappa}$, and $|\dot{\gamma}_k| \equiv L_k/(2\pi)$; a sequence $(\Phi_k) \subset C^1(\bar{D}^2, \mathbb{C})$ of holomorphic immersions such that $|(\Phi_k')|_{S^1}| = e^{\lambda_k}$; and a sequence of diffeomorphisms $\sigma_k : S^1 \to S^1$ such that $\Phi_k \circ \sigma_k = \gamma_k$. Up to a translation we can assume that $\Phi_k(1) = 0$ and, by the Arzelà–Ascoli theorem, $\gamma_k \to \gamma_\infty$ in $C^1(S^1, \mathbb{C})$ for a curve $\gamma_\infty \in W^{2,\infty}(S^1, \mathbb{C})$.

Notice that (Φ_k) and (λ_k) satisfy the hypothesis of Theorem 3.5 and, up to a subsequence, we can assume that (65) and (66) hold for a finite set $B = \{a_1, \ldots, a_N\}$ and functions $v_\infty \in L^1(S^1, \mathbb{R})$ and $\Phi_\infty \in W^{1,2}(D^2, \mathbb{C})$. Moreover, either (1) or (2) in Theorem 3.5 holds.

We introduce the following distance function $D_k : S^1 \times S^1 \to \mathbb{R}^+$:

$$D_k(q, q') = \inf\left\{ \left(\int_0^1 |\Phi_k'(\Delta_k(t))|^2 |\Delta_k'(t)|^2 \, dt \right)^{\frac{1}{2}} \,\middle|\, \Delta_k \in W^{1,2}([0,1], \bar{D}^2),\ \Delta_k(0) = \sigma_k(q),\ \Delta_k(1) = \sigma_k(q') \right\}, \quad (71)$$

It is well known that the infimum in (71) is attained by a path Δ_k such that $|\Phi_k'(\Delta_k(t))||\Delta_k'(t)|$ is constant. For such a path we then have

$$\left(\int_0^1 |\Phi_k'(\Delta_k(t))|^2 |\Delta'(t)|^2 \, dt \right)^{\frac{1}{2}} = \int_0^1 |\Phi_k'(\Delta_k(t))||\Delta_k'(t)| \, dt =: \int_{\Delta_k} |\Phi_k'(z)| \, |dz|.$$

In the sequel we sometimes identify the parametrization of a curve Δ with its image.

Proposition 3.6. (1) *The function D_k is Lipschitz continuous with $\|\nabla D_k\|_{L^\infty} \le 1$ and it converges uniformly.*

(2) *The infimum in (71) is attained by a curve Δ_k in normal parametrization such that the curvature of $\Phi_k \circ \Delta_k$ is bounded by $\|\kappa_k\|_{L^\infty}$.*

Proof. (1) Let $q, q', \tilde{q}, \tilde{q}' \in S^1$. The following estimate holds:

$$D_k(q, q') \le D_k(\tilde{q}, \tilde{q}') + |\mathrm{arc}\,(\gamma_k(q), \gamma_k(\tilde{q}))| + |\mathrm{arc}\,(\gamma_k(q'), \gamma_k(\tilde{q}'))| \le D_k(\tilde{q}, \tilde{q}') + |q - \tilde{q}| + |q' - \tilde{q}'|,$$

where $\mathrm{arc}(\,\cdot\,,\,\cdot\,)$ is the shortest arc between two points. By exchanging (q, q') and $(\tilde{q}, \tilde{q}')$, we get that

$$|D_k(q, q') - D_k(\tilde{q}, \tilde{q}')| \le |q - \tilde{q}| + |q' - \tilde{q}'|,$$

and we conclude.

(2) For a geodesic Δ with respect to D_k, the curve $\Phi_k \circ \Delta$ is a geodesic in $\mathbb{C}$ under the constraint that $\Phi_k \circ \Delta \subset \Phi_k(\bar{D}^2)$. This must be a union of segments (contained in $\Phi_k(D^2)$) and arcs of the curve γ_k, where the segments touch the curve γ_k tangentially. Hence the curvature of $\Phi_k \circ \Delta$ is bounded by $\|\kappa_k\|_{L^\infty}$.

This completes the proof of Proposition 3.6. □

We give next the definition of a pinched point for the curve γ_∞.

Definition 3.7. A point $p \in S^1$ is called a *pinched point* for the sequence (γ_k) if there exists $p' \in S^1$, $p \neq p'$, such that $\lim_{k \to +\infty} D_k(p, p') = 0$. We call p' the "dual" of p and we will show in Lemma 3.12 below that this dual is unique. We denote by $\mathcal{P}$ the set of the pinched points of γ_∞.

Remark 3.8. The definition of pinched point is independent of Φ_k and σ_k in the sense that if $\tilde{\Phi}_k = \Phi_k \circ f_k$, where $f_k : \bar{D}^2 \to \bar{D}^2$ is a Möbius transformation, and if $\tilde{\sigma}_k = f_k^{-1} \circ \sigma_k$, then

$$\lim_{k \to +\infty} \int_0^1 |\Phi_k'(\Delta(t))||\Delta'(t)| \, dt = 0 \iff \lim_{k \to +\infty} \int_0^1 |\tilde{\Phi}_k'(\tilde{\Delta}(t))||\tilde{\Delta}'(t)| \, dt = 0.$$

Proposition 3.9. *Assume that we are in case* (2) *of Theorem 3.5, i.e., $\Phi_k \to Q$ in $C^{1,\alpha}_{\mathrm{loc}}(\bar{D}^2 \setminus \{a_1, \dots, a_N\})$ for a constant $Q \in \mathbb{C}$. Then $N \in \{1, 2\}$. If $N = 2$, let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the connected components of $S^1 \setminus \{a_1, a_2\}$. Then $\sigma_k^{-1} \to p^\pm$ locally uniformly on $\mathcal{C}_\pm$, where $p^+, p^- \in \mathcal{P}$ are dual. Moreover,*

$Q = \gamma_\infty(p^+) = \gamma_\infty(p^-)$ and $\dot\gamma_\infty(p^+) = -\dot\gamma_\infty(p^-)$, and $\kappa_k e^{\lambda_k} \overset{*}{\rightharpoonup} \pi(\delta_{a_1} + \delta_{a_2})$ and $v_k := \lambda_k - \bar\lambda_k \rightharpoonup v_\infty$ in $W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1, a_2\})$, where v_∞ solves (12). If $N = 1$ then $v_k \to v_\infty$ that solves (11).

Proof. By Theorem 3.5 we have $\bar\lambda_k \to -\infty$ and $\lambda_k \to -\infty$ uniformly locally in $S^1 \setminus B = \{a_1, \dots, a_N\}$. In particular, since the signed Radon measures $\kappa_k e^{\lambda_k}\,dx$ are uniformly bounded, we have $\mu_k \overset{*}{\rightharpoonup} \mu$ for a Radon measure supported in B, which we can then write as $\mu = \sum_{i=1}^N \alpha_i \delta_{a_i}$. Moreover, since

$$\int_{S_1} \kappa_k e^{\lambda_k}\, d\theta = 2\pi,$$

we infer that $\sum_{i=1}^N \alpha_i = 2\pi$.

Let us assume that $N \geq 2$. We want to prove that $\alpha_i = \pi$ for every i, so necessarily $N = 2$. In order to prove that $\alpha_i = \pi$, up to a rotation we can reduce to proving that $\alpha_1 = \pi$ and assume that $a_1 = i$. We can also assume that $N = 2$ and $a_2 = -i$. If this is not the case, it suffices to compose Φ_k with Möbius diffeomorphisms $f_k(z) = (z - it_k)/(1 + it_k z)$ with $t_k \uparrow 1$ slowly enough that $\tilde\Phi_k := \Phi_k \circ f_k$ is still as in case (2) of Theorem 3.5, with $B = \{a_1 = i, a_2 = -i\}$.

Then let Φ_k be as above, with $\Phi_k \rightharpoonup Q$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2 \setminus \{i, -i\})$. Set

$$V_k(z) = e^{-\bar\lambda_k}(\Phi_k(z) - \Phi_k(0)), \quad v_k = \log|V_k'|_{S^1}| = \lambda_k - \bar\lambda_k.$$

By Theorem 3.5 we have

$$v_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{i, -i\}) \text{ and in } \mathscr{D}'(S^1),$$

where v_∞ solves

$$(-\Delta)^{\frac12} v_\infty = \alpha\delta_i + (2\pi - \alpha)\delta_{-i} - 1 \tag{72}$$

for some $\alpha \in \mathbb{R}$. Similarly, $V_k \rightharpoonup V_\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2 \setminus \{i, -i\})$. Solutions to (72) can be computed explicitly using Lemma 3.1, so that

$$v_\infty(e^{i\theta}) = -\frac{\alpha}{2\pi}\log(2(1 - \sin\theta)) - \frac{2\pi - \alpha}{2\pi}\log(2(1 + \sin\theta)).$$

Notice that, writing $z = x + iy$, for $z = e^{i\theta} \in S^1$ we have

$$2(1 - \sin\theta) = x^2 + y^2 - 2y + 1 = |z - i|^2$$

and, similarly, $2(1 + \sin\theta) = |z + i|^2$. In particular, v_∞ can be extended to a holomorphic function

$$\tilde v_\infty(z) := -\frac{\alpha}{2\pi}\log(|z - i|^2) - \frac{2\pi - \alpha}{2\pi}\log(|z + i|^2), \quad z \in \bar D^2 \setminus \{i, -i\}. \tag{73}$$

The estimate (69) together with (16) implies that

$$c_\delta^{-1} \leq |V_k'| \leq c_\delta \quad \text{on } \bar D^2 \setminus (B(i, \delta) \cup B(-i, \delta)) \text{ for every } \delta > 0.$$

Therefore, $V_k \rightharpoonup V_\infty$ as $k \to +\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2 \setminus \{i, -i\})$, where V_∞ is a conformal immersion of $\bar D^2 \setminus \{i, -i\}$. Moreover, still using (16), from (73) we obtain

$$|V_\infty'(z)| = \frac{1}{|z - i|^{\alpha/\pi}|z + i|^{2 - \alpha/\pi}}.$$

Since V'_∞ is holomorphic in D^2, up to a rotation (i.e., multiplication by a constant $e^{i\theta_0}$) we obtain

$$V'_\infty(z) = \frac{1}{(z-i)^{\alpha/\pi}(z+i)^{2-\alpha/\pi}}, \qquad V_\infty(z) = \int_0^z \frac{dz}{(z-i)^{\alpha/\pi}(z+i)^{2-\alpha/\pi}}.$$

Up to possibly switching i with $-i$, we may assume that $\alpha \le \pi$. The function V_∞ is also known as the Schwarz–Christoffel mapping[1] and sends the two arcs $\mathscr{C}_+, \mathscr{C}_- \subset S^1$ joining i and $-i$ (chosen so that $\pm 1 \in \mathscr{C}_\pm$) into two parallel straight lines if $\alpha = \pi$ and into two half-lines meeting at $V_\infty(i)$, forming an angle of $\pi - \alpha$ there if $\alpha < \pi$.

Claim 1. *As $k \to +\infty$ we have $\sigma_k^{-1} \to p^\pm$ in $L^\infty_{\mathrm{loc}}(\mathscr{C}_\pm)$, where $p^+, p^- \in S^1$ with $p^+ \neq p^-$.*

Proof. Notice that $\Phi_k \rightharpoonup Q$ in $W^{2,p}_{\mathrm{loc}}(\bar{D}^2 \setminus \{i, -i\})$ implies that

$$\frac{\partial \sigma_k^{-1}}{\partial \theta} \to 0 \quad \text{uniformly locally in } S^1 \setminus \{i, -i\} \text{ as } k \to +\infty.$$

This proves the first part of the claim. Assume for contradiction that $p^+ = p^-$. Set $p_k^\pm = \sigma_k^{-1}(\pm 1) \to p^\pm$. By assumption, $|\mathrm{arc}(p_k^+, p_k^-)| \to 0$ (here $\mathrm{arc}(p_k^+, p_k^-)$ denotes the shortest arc connecting p_k^+ to p_k^-). Since σ_k is a diffeomorphism, $\sigma_k(\mathrm{arc}(p_k^+, p_k^-))$ contains either $S^1 \cap B(i,\delta)$ or $S^1 \cap B(-i,\delta)$ for small $\delta > 0$. Suppose it contains $S^1 \cap B(i,\delta)$. Then

$$\int_{S^1 \cap B(i,\delta)} e^{\lambda_k}\, d\theta = \int_{S^1 \cap B(i,\delta)} |\Phi'_k(e^{i\theta})|\, d\theta \le \int_{\mathrm{arc}(p_k^+, p_k^-)} |\dot{\gamma}_k|\, d\theta = \frac{L_k}{2\pi} |\mathrm{arc}(p_k^+, p_k^-)| \to 0 \tag{74}$$

as $k \to \infty$. This contradicts that $i \in B$ and concludes the proof of Claim 1. □

Claim 2. *p^+ is a pinched point and p^- is dual to it.*

Proof. Let $p_k^\pm = \sigma_k^{-1}(\pm 1)$ be as above. Consider the path

$$\Delta_k = \mathrm{arc}(\sigma_k(p^+), 1) \cup \mathrm{arc}(\sigma_k(p^-), -1) \cup [-1, 1],$$

where $[-1, 1]$ is the segment in $\bar{D}^2$ joining -1 to 1. Since, as $k \to \infty$, we have

$$\int_{\mathrm{arc}(\sigma_k(p^\pm), \pm 1)} |\Phi'_k(e^{i\theta})|\, d\theta = \int_{\mathrm{arc}(p_k^\pm, p^\pm)} |\dot{\gamma}_k|\, d\theta = \frac{L_k |\mathrm{arc}(p_k^\pm, p^\pm)|}{2\pi} \to 0 \tag{75}$$

and

$$\int_{[-1,1]} |\Phi'_k|\, |dz| \le 2 \sup_{[-1,1]} |\Phi'_k|\, |dz| \to 0,$$

we immediately infer that

$$\int_{\Delta_k} |\Phi'_k|\, |dz| \to 0;$$

hence, p^+ is dual to p^-. This proves Claim 2. □

[1] Up to composition with a conformal transformation, since Schwarz–Christoffel maps are usually defined on the half-plane $\{z \in \mathbb{C} : \Re z > 0\}$ instead of the unit disk.

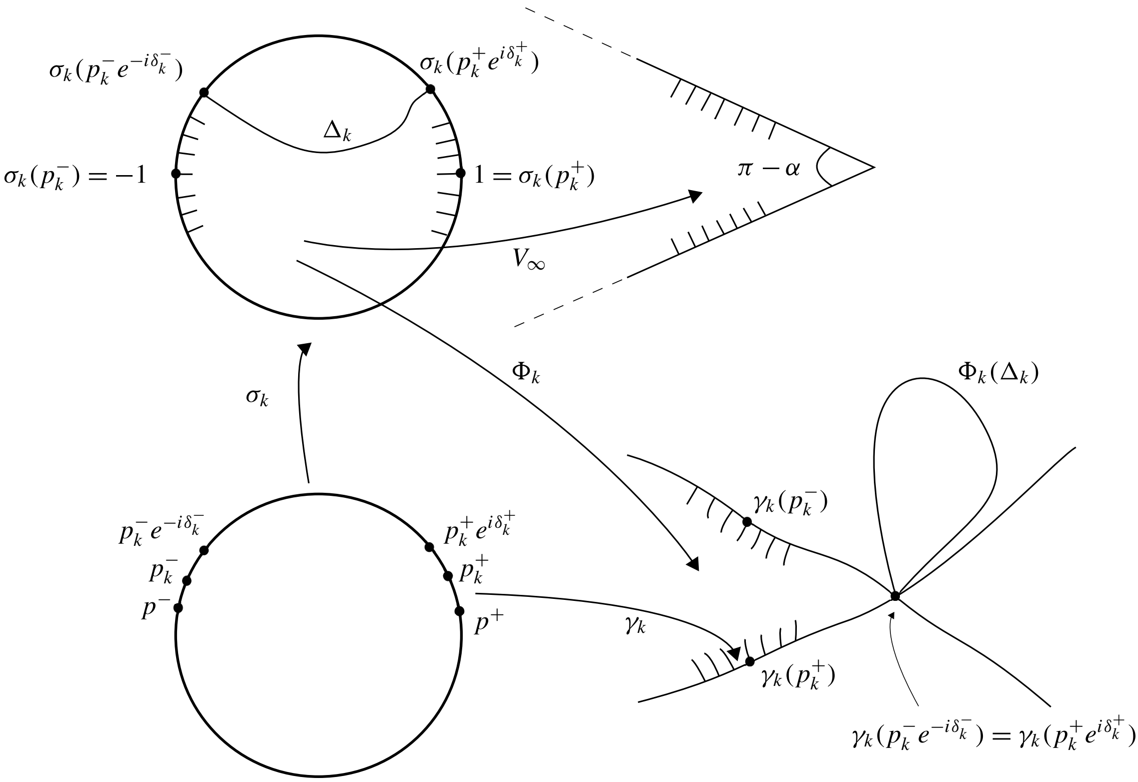

Figure 6. Case 1 in the proof of Proposition 3.9.

Now,

$$\frac{2\pi}{L_k}\dot\gamma_k(p_k^\pm) = \frac{\frac{\partial \Phi_k(\pm 1)}{\partial\theta}}{\left|\frac{\partial \Phi_k(\pm 1)}{\partial\theta}\right|} = \frac{\frac{\partial \Phi_k(\pm 1)}{\partial\theta}}{e^{\bar\lambda_k} e^{\lambda_k(\pm 1)-\bar\lambda_k}} = \frac{\partial V_\infty(\pm 1)}{\partial\theta} e^{\bar\lambda_k - \lambda_k(\pm 1)} + o(1) \quad \text{as } k\to\infty. \tag{76}$$

In particular, denoting by $(v, w)^\wedge$ the angle between two vectors, we have

$$(\dot\gamma_k(p_k^+), \dot\gamma_k(p_k^-))^\wedge \to \left(\frac{\partial V_\infty(1)}{\partial\theta}, \frac{\partial V_\infty(-1)}{\partial\theta}\right)^\wedge = \alpha. \tag{77}$$

We consider different cases:

Case 1: $0<\alpha<\pi$. Since $p_k^\pm \to p^\pm$ and p^+ is pinched to p^-, and since

$$|\gamma_k(p_k^+) - \gamma_k(p_k^-)| \le D_k(p_k^+, p_k^-) \le D_k(p^+, p^-) + \frac{L_k}{2\pi}\big(|\mathrm{arc}(p^+, p_k^+)| + |\mathrm{arc}(p^-, p_k^-)|\big) \to 0 \quad \text{as } k\to\infty,$$

taking (77) and the bound $\bar\kappa$ on the curvature of γ_k into account we see that for positive numbers $\delta_k^\pm \to 0$ (as $k\to\infty$) we have

$$\gamma_k(p_k^+ e^{i\delta_k^+}) = \gamma_k(p_k^- e^{-i\delta_k^-}), \tag{78}$$

i.e., the two curves $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ cross in short time (see Figure 6).

Because $\delta_k^\pm \to 0$, we have

$$D_k(p_k^+ e^{i\delta_k^+}, p_k^- e^{-i\delta_k^-}) \le D_k(p_k^+, p_k^-) + \frac{L_k(\delta_k^+ + \delta_k^-)}{2\pi} \to 0 \quad \text{as } k \to \infty. \tag{79}$$

Now let $\Delta_k : [0, 1] \to \bar{D}^2$ be a geodesic realizing the distance on the left-hand side of (79). Then (78) implies that $\Phi_k \circ \Delta_k$ is a closed curve (nonconstant, since $p_k^+ e^{i\delta_k^+} \neq p_k^- e^{-i\delta_k^-}$ for k large), so that the integral of its curvature is at least π (see Lemma 3.10 below). On the other hand, Proposition 3.6 implies that the curvature of $\Phi_k \circ \Delta_k$ is bounded by $\bar{\kappa}$ and, since the length of this geodesic is going to 0 according to (79), we get a contradiction.

Case 2: $\alpha = 0$. Similarly to case 1, if the curves $\gamma_k(p_k^\pm e^{\pm it})$ cross for small times $\delta_k^\pm \to 0$, we conclude as before. If not, we can at least say that, up to a rotation of the axis,

$$V_\infty(D^2) = \{x + iy : y < 0\} \tag{80}$$

and that, for small times $\delta_k^\pm \to 0$,

$$\Re(\gamma_k(p_k^+ e^{i\delta_k^+})) = \Re(\gamma_k(p_k^- e^{-i\delta_k^-})) \tag{81}$$

and, without loss of generality,

$$\Im(\gamma_k(p_k^+ e^{i\delta_k^+})) > \Im(\gamma_k(p_k^- e^{-i\delta_k^-})), \tag{82}$$

where, for $x, y \in \mathbb{R}$, we use the notation $\Re(x + iy) = x$, $\Im(x + iy) = y$ (see Figure 7). Moreover, since the curvature of γ_k is uniformly bounded and $\delta_k^\pm \to 0$, using (76) and (80) we infer[2]

$$\frac{\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})}{|\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})|} = \frac{\dot{\gamma}_k(p_k^\pm)}{|\dot{\gamma}_k(p_k^\pm)|} + o(1) = -1 + o(1), \tag{83}$$

i.e., the curves $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ at the time $t = \delta_k^\pm$ are almost horizontal and pointing into opposite directions (notice the change of orientation between the curves $t \mapsto \gamma_k(e^{it})$ and $t \mapsto \gamma_k(p_k^- e^{-it})$). As before, (79) holds, so let $\Delta_k : [0, 1] \to \bar{D}^2$ be a geodesic realizing the distance in (79), with $\Delta_k(0) = \gamma_k(p_k^+ e^{i\delta_k^+})$ and $\Delta_k(1) = \gamma_k(p_k^- e^{-i\delta_k^-})$. Up to a reparametrization we can assume that $\tilde{\Delta}_k := \Phi_k \circ \Delta_k : [0, L] \to \mathbb{C}$ satisfies $|\dot{\tilde{\Delta}}_k(t)| \equiv 1$. Since the map Φ_k preserves the orientation, from (83) we infer

$$\Im(\dot{\tilde{\Delta}}_k(0)) \le 0 + o(1), \quad \Im(\dot{\tilde{\Delta}}_k(1)) \ge 0 + o(1),$$

i.e., up to $o(1) \to 0$ as $k \to \infty$ we have that $\dot{\tilde{\Delta}}_k(0)$ points downwards, while $\dot{\tilde{\Delta}}_k(1)$ points upwards. Now using (81) we see that the curve $\tilde{\Delta}_k$ has total curvature at least $\frac{1}{2}\pi - o(1)$ (see Lemma 3.11 below), again contradicting Proposition 3.6 and (79).

Case 3: $\alpha < 0$. Let Δ be the straight segment in $\bar{D}^2$ (seen as a smooth path) joining -1 to 1. Since $\Delta \subset \bar{D}^2 \setminus \{i, -i\}$ we have that $V_k \circ \Delta \to V_\infty \circ \Delta$ and, by the explicit form of V_∞, we deduce that the unit tangent vector of the curve $V_\infty \circ \Delta$ describes an arc in S^1 of length at least $|\alpha| + \pi$ (we are using

[2]The symbol $\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})$ denotes the derivative of the curve $t \mapsto \gamma_k(e^{it})$ evaluated for $e^{it} = p_k^\pm e^{\pm i\delta_k^\pm}$ and *not* the derivative of the curve $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ evaluated for $t = \delta_k^\pm$.

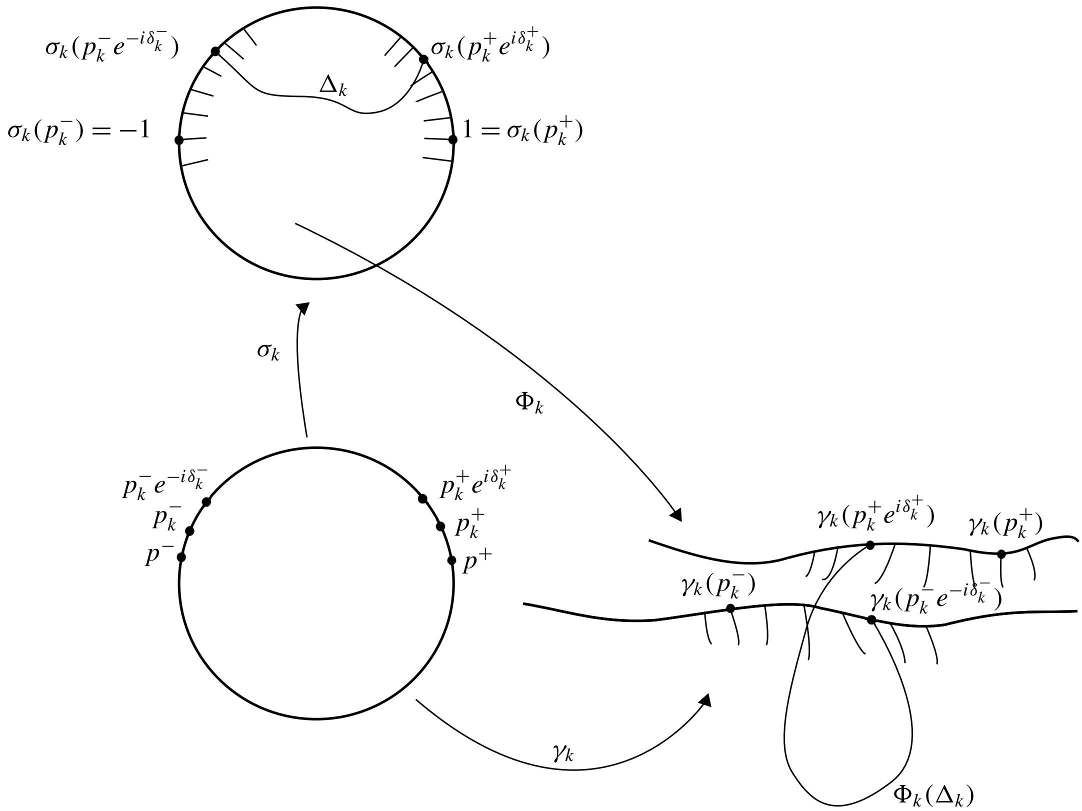

Figure 7. Case 2 in the proof of Proposition 3.9.

that Δ touches S^1 perpendicularly and V_∞ is conformal). This implies that, for k large enough, any C^1 curve of the form $\Phi_k \circ \tilde{\Delta}$ for a curve $\tilde{\Delta} \in C^1([0,1], \bar{D}^2)$ with $\tilde{\Delta}(0) = -1$ and $\tilde{\Delta}(1) = 1$ has a unit tangent vector describing an arc of length no less than $|\alpha| - o(1)$. If such a curve minimizes D_k, since, by Proposition 3.6, its curvature is bounded by $\bar{\kappa}$, its length cannot go to zero as $k \to \infty$. But this contradicts that p^+ and p^- are pinched points, since, if Δ_k is a geodesic minimizing $D_k(\sigma_k(p^+), \sigma_k(p^-))$ (with length going to 0 since p^+ and p^- are pinched), then joining Δ_k with the two arcs $\operatorname{arc}(\sigma_k(p^\pm), \pm 1)$ and using (75) one would obtain paths joining -1 to 1 of D_k-length going to 0.

The only case left is $\alpha = \pi$, which completes the proof of Proposition 3.9. □

In the proof of Proposition 3.9 we have used the following:

Lemma 3.10. *Let $\Delta \in W^{2,\infty}([0, L], \mathbb{C})$ be a curve satisfying $|\dot{\Delta}(t)| = 1$ for every $t \in [0, L]$ and $\Delta(0) = \Delta(L)$. Then*

$$\int_0^L |\kappa(t)|\, dt > \pi,$$

where κ is the curvature of Δ.

Proof. Let $\theta : [0, L] \to \mathbb{R}$ be a continuous function such that $\dot\Delta(t) = e^{i\theta(t)}$ for $t \in [0, L]$. Then it is easy to see that $\dot\theta = \kappa$. We have $\theta([0, L]) = [\theta_-, \theta_+] \subset \mathbb{R}$ for some $\theta_-, \theta_+ \in \mathbb{R}$. Assume now that

$$\theta_+ - \theta_- \leq \pi \tag{84}$$

and set

$$\bar\theta := \tfrac{1}{2}(\theta_+ - \theta_-), \qquad v := e^{i\bar\theta}.$$

Then, since $|\theta(t) - \bar\theta| \leq \frac{1}{2}\pi$ for every $t \in [0, L]$, we have

$$\frac{d}{dt}\langle \Delta(t), v\rangle = \langle \dot\Delta(t), v\rangle = \langle e^{i\theta(t)}, e^{i\bar\theta}\rangle \geq 0,$$

with equality possible only for a proper subset of $[0, L]$, where $|\theta(t) - \bar\theta| = \frac{1}{2}\pi$. But this contradicts that $\Delta(0) = \Delta(L)$. In particular, (84) cannot hold, and we get

$$\int_0^L |\kappa(t)|\,dt = \int_0^L |\dot\theta(t)|\,dt \geq \operatorname{osc}\theta = \theta_+ - \theta_- > \pi. \qquad \square$$

Lemma 3.11. *Let $\Delta \in W^{2,\infty}([0, L], \mathbb{C})$ be a curve satisfying $|\dot\Delta(t) = 1|$ for every $t \in [0, L]$. Assume that*

$$\Re(\Delta(0)) = \Re(\Delta(L)), \qquad \Im(\Delta(0)) < \Im(\Delta(L)), \tag{85}$$

and that for some (small) $\varepsilon > 0$ one has

$$\Im(\dot\Delta(0)) < \varepsilon \quad \text{and} \quad \Im(\dot\Delta(L)) > -\varepsilon. \tag{86}$$

Then

$$\int_0^L |\kappa(t)|\,dt > \frac{\pi}{2} - C\varepsilon,$$

where κ is the curvature of Δ and C is a universal constant.

Proof. Let $\theta \in W^{1,\infty}([0, L], \mathbb{R})$ be as in the proof of Lemma 3.10. Then (85) implies that for some $t_1, t_2 \in [0, L]$ one has $\Re(e^{i\theta(t_1)}) \leq 0$ and $\Re(e^{i\theta(t_2)}) \geq 0$ (otherwise $\dot\Delta$ would always be pointing right, or always left). Condition (86) implies that $\Im(e^{i\theta(0)}) \leq \varepsilon$ and $\Im(e^{i\theta(L)}) > -\varepsilon$. Then we immediately infer that the oscillation of θ is at least $\frac{1}{2}\pi - C\varepsilon$ and we conclude as in the proof of Lemma 3.10, using that $\kappa = \dot\theta$. $\square$

Next we prove some properties concerning the set $\mathcal{P}$:

Lemma 3.12. *Let p^+ and p^- be dual pinched points and assume that $\sigma_k(p^\pm) = \pm 1$. Then Φ_k is as in case (2) of Theorem 3.5, $B = \{a_1, a_2\}$ and $\pm 1 \notin B$. Moreover, every pinched point p has only one dual p' and $|\operatorname{arc}(p, p')| \geq C/\bar\kappa$.*

Proof. Let us start with the first claim. If Φ_k is as in case (1) of Theorem 3.5, then

$$\int_{\Delta_k} |\Phi_k'(z)|\,|dz| \geq C \quad \text{for every } \Delta_k \text{ with } \Delta_k(0) = -1,\ \Delta_k(1) = 1, \tag{87}$$

in contrast with the fact that p^+ and p^- are pinched. Thus we are in case (2) of Theorem 3.5 and, by Proposition 3.9, we have $N \in \{1, 2\}$. Assume now that $a_1 = 1 = \sigma_k(p^+)$ (the reasoning is similar if $a_1 = -1$). Then we compose Φ_k with the Möbius diffeomorphism $f_k(z) = (z - t_k)/(1 - t_k z)$, where $t_k \uparrow 1$ is chosen so that for a fixed small $\delta > 0$ we have, for k large enough,

$$\int_{S^1 \cap B_\delta(1)} |(\Phi_k \circ f_k)'(z)|\,|dz| = \frac{\pi}{2\bar{\kappa}}. \tag{88}$$

In other words, the effect of f_k is to stretch the disk to remove the concentration at the point $a_1 = 1$, concentrating the disk towards -1. Then $\tilde{\Phi}_k := \Phi_k \circ f_k$ is necessarily as in case (1) of Theorem 3.5. Moreover, the corresponding $\tilde{\sigma}_k := f_k^{-1} \circ \sigma_k$ still satisfies $\tilde{\sigma}_k(p^\pm) = \pm 1$, since f_k leaves ± 1 fixed. This, together with (88), contradicts that p^+ and p^- are pinched, since, by conformality and convergence of $\tilde{\Phi}_k$, in a neighborhood $B_{\delta/2}(1)$ we have $|\tilde{\Phi}_k'| \geq C$; hence, (87) holds with $\tilde{\Phi}_k$ instead of Φ_k. Therefore, going back to the original maps Φ_k, we have proven that $\pm 1 \notin B$.

To rule out the case $N = 1$ it suffices to observe that in this case $\sigma_k(p^+)$ and $\sigma_k(p^-)$ would belong to the same connected component of $S^1 \setminus B$; hence, since Φ_k is as in case (2) of Theorem 3.5, we would get $|\mathrm{arc}(\sigma_k^{-1}(1), \sigma_k^{-1}(1))| \to 0$, which is absurd, since $\sigma_k^{-1}(\pm 1) = p^\pm$ and $p^+ \neq p^-$.

Claim 1. *Every pinched point p has a unique dual p'.*

Proof. It suffices to prove that, given any pinched points p^+ and p^- dual to each other, $\dot{\gamma}_\infty(p^+) = -\dot{\gamma}_\infty(p^-)$ (since then a third point $\tilde{p}$ dual to p^+ would be also dual to p^-, whence $\dot{\gamma}_\infty(\tilde{p})$ would have to coincide both with $\dot{\gamma}_\infty(p^+)$ and its opposite, which is impossible). Let us therefore consider two pinched points p^+ and p^-, dual to each other. By considering $\tilde{\Phi}_k := \Phi_k \circ f_k$ and $\tilde{\sigma}_k = f_k^{-1} \circ \sigma_k$ for suitable Möbius transformations f_k, we can assume that $\tilde{\sigma}_k(p^\pm) = \pm 1$. Then, by the previous part of the lemma, $\tilde{\Phi}_k$ blows up at two points a_1 and a_2 different from ± 1. To such a $\tilde{\Phi}_k$ we can then apply Proposition 3.9 with $\mathscr{C}_\pm$ being the connected component of $S^1 \setminus \{a_1, a_2\}$ containing ± 1. We then infer that $\dot{\gamma}_\infty(p^+) = -\dot{\gamma}_\infty(p^-)$. □

Claim 2. *We have* $|\mathrm{arc}(p, p')| \geq C/\bar{\kappa}$.

Proof. This follows from the fact that both arcs $\mathscr{A}_1$ and $\mathscr{A}_1$ joining $\tilde{\sigma}_k(p^\pm) = \pm 1$ contain a blow-up point, a_1 or a_2, so that

$$\int_{\mathscr{A}_i} |\tilde{\kappa}_k| e^{\tilde{\lambda}_k}\,|dz| = \int_{f_k(\mathscr{A}_i)} |\kappa_k| e^{\lambda_k}\,|dz| \geq \pi - o(1). \qquad \square$$

This concludes the proof of Lemma 3.12. □

Lemma 3.13. *The set $\mathscr{P}$ is closed.*

Proof. Let $\{p_n\}$ and $\{p_n'\}$ be a sequence of pinched points and their duals, respectively, with $p_n \to p_\infty$ and $p_n' \to p_\infty'$ as $k \to +\infty$.

We first observe that $|p_n - p_n'| \geq C > 0$ for all $n \geq 0$, so $p_\infty \neq p_\infty'$.

For each p_n there exists a curve $\Delta_{n,k} \subseteq \bar{D}^2$ with $\partial \Delta_{n,k} = \{\sigma_k(p_n), \sigma(p_n')\}$ and

$$\lim_{k \to +\infty} \int_{\Delta_{n,k}} |\Phi_k'(z)|\,|dz| = 0.$$

Since $\gamma_k \to \gamma_\infty$ in $C^1(S^1)$ as $k \to +\infty$, we have

$$\begin{aligned} \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{\operatorname{arc}(p_n,p_\infty)} |\dot{\gamma}_k(t)|\,dt = 0, \\ \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{\operatorname{arc}(p'_n,p'_\infty)} |\dot{\gamma}_k(t)|\,dt = 0. \end{aligned} \tag{89}$$

We set

$$\tilde{\Delta}_{n,k} := \Delta_{n,k} \cup \operatorname{arc}(\sigma_k(p_n), \sigma_k(p_\infty)) \cup \operatorname{arc}(\sigma_k(p_n), \sigma_k(p_\infty)).$$

For all k, we have $\tilde{\Delta}_{n,k} \to \tilde{\Delta}_{\infty,k}$ as $n \to +\infty$ with $\partial\tilde{\Delta}_{k,\infty} = \{\sigma_k(p_\infty), \sigma_k(p'_\infty)\}$ and, since $\Phi_k \circ \sigma_k = \gamma_k$ on S^1 from (89), we have

$$\lim_{k\to+\infty} \int_{\tilde{\Delta}_{k,\infty}} |\Phi'_k(z)|\,|dz| = \lim_{k\to+\infty} \lim_{n\to+\infty} \int_{\tilde{\Delta}_{n,k}} |\Phi'_k(z)|\,|dz| = 0.$$

Hence p_∞ is by definition a pinched point and p'_∞ is its dual. □

We now introduce the following equivalence relation on the set $S^1 \setminus \{\mathcal{P}\}$:

Definition 3.14. Given $p, q \in S^1 \setminus \{\mathcal{P}\}$, we say that $p \sim q$ if and only if there exists a sequence of paths $\Delta_k : [0, 1] \to \bar{D}^2$ with $\Delta_k(0) = \sigma_k(p)$ and $\Delta_k(1) = \sigma_k(q)$ such that

$$\liminf_{k\to+\infty} d_k(\Delta_k, \sigma_k(\mathcal{P})) > 0, \tag{90}$$

where $d_k : \bar{D}^2 \times \bar{D}^2 \to \mathbb{R}^+$ is the distance defined as

$$d_k(z, w) = \inf\left\{ \left(\int_0^1 |\Phi'_k(\Delta(t))|^2 |\dot{\Delta}(t)|^2\,dt \right)^{\frac{1}{2}} \,\middle|\, \Delta \in W^{1,2}([0, 1], \bar{D}^2),\ \Delta(0) = z,\ \Delta(1) = w \right\}.$$

Proposition 3.15. *Let $q \in S^1 \setminus \{\mathcal{P}\}$, and let $\mathcal{A}_q$ and $\mathcal{B}_q$ be the equivalence class and the connected component containing q, respectively. Then $\mathcal{B}_q \subseteq \mathcal{A}_q$.*

Proof. Let $q \in S^1 \setminus \{\mathcal{P}\}$. We show that $\mathcal{A}_q \cap \mathcal{B}_q$ is open and closed in $\mathcal{B}_q$.

(1) $\mathcal{A}_q \cap \mathcal{B}_q$ is open in $\mathcal{B}_q$: Choose $\delta > 0$ small enough so that $e^{it}q \in S^1 \setminus \{\mathcal{P}\}$ for $t \in [-2\delta, 2\delta]$ and

$$\int_{\sigma_k(\operatorname{arc}(e^{-2\delta i}q, e^{2\delta i}q))} |\Phi'_k(z)|\,|dz| < \frac{\pi}{2\bar{\kappa}}. \tag{91}$$

Now set $q_0 = e^{-i\delta}q$, $q_1 = q$ and $q_2 = e^{i\delta}q$. Let f_k be the sequence of Möbius transformations of $\bar{D}^2$ such that $\tilde{\sigma}_k(q_0) = 1$, $\tilde{\sigma}_k(q_1) = e^{2\pi i/3}$ and $\tilde{\sigma}_k(q_2) = e^{4\pi i/3}$. We apply Theorem 3.5 to $\tilde{\Phi}_k := \Phi_k \circ f_k$ and notice that if we are in case (2) of Theorem 3.5, then there are one or two blow-up points. In the latter case, away from the blow-up points $\{a_1, a_2\}$, we have that σ_k^{-1} locally converges to two pinched points, which implies that one of the q_i lies in $\mathcal{P}$, a contradiction. In the former case, for one pair of points, say q_1 and q_2, one has

$$\int_{\operatorname{arc}(q_1,q_2)} |\dot{\gamma}(t)|\,dt = \int_{\operatorname{arc}(\tilde{\sigma}_k(q_1),\tilde{\sigma}_k(q_2))} |\tilde{\Phi}'_k(z)|\,|dz| \to 0,$$

contradicting that $|\dot{\gamma}_k|$ is bounded away from 0 and $|\operatorname{arc}(q_1, q_2)| = \delta$.

Therefore we are in case (1) of Theorem 3.5 and $\tilde{\Phi}_k \rightharpoonup \tilde{\Phi}_\infty$ in $W^{1,2}(\bar{D}^2)$ and in $W^{2,p}_{\mathrm{loc}}(\bar{D}^2 \setminus B)$, where $\tilde{\Phi}_\infty$ is a holomorphic immersion in $\bar{D}^2 \setminus B$, $B = \{a_1, \dots, a_N\}$ and $e^{2j\pi i/3} \notin B$ for $j = 0, 1, 2$. Since $|\tilde{\Phi}'_\infty| > C_\delta > 0$ in $\bar{D}^2 \setminus \bigcup_{i=1}^N B_\delta(a_i)$, for every $p \in \operatorname{arc}(q_0, q_2)$, choosing as Δ_k the segment joining $\sigma_k(p)$ to $\sigma_k(q)$ that satisfies (90) shows that $B_\delta(q) \cap S^1 \subset \mathscr{A}_q$.

(2) $\mathscr{A}_q \cap \mathscr{B}_q$ is closed in $\mathscr{B}_q$: Let $q_n \in \mathscr{A}_q \cap \mathscr{B}_q$ be such that $q_n \to q_\infty \in \mathscr{B}_q$. For every n there exists Δ^k_n with $\Delta^k_n(0) = \sigma_k(q_n)$ and $\Delta^k_n(1) = \sigma_k(q)$, and

$$\liminf_{k \to +\infty} d_k(\Delta^k_n, \sigma_k(\mathscr{P})) > 0. \tag{92}$$

Consider now the path $\Sigma^k_n = \operatorname{arc}(\sigma_k(q_\infty), \sigma_k(q_n)) \cup \Delta^k_n$ joining $\sigma_k(q_\infty)$ to $\sigma_k(q)$. We claim that

$$\liminf_{k \to +\infty} d_k(\Sigma^k_n, \sigma_k(\mathscr{P})) > 0.$$

Indeed, considering (92), it suffices to prove that, for n sufficiently large,

$$\liminf_{k \to +\infty} d_k(\operatorname{arc}(\sigma_k(q_\infty), \sigma_k(q_n)), \sigma_k(\mathscr{P})) > 0. \tag{93}$$

Assume for contradiction that the lim inf in (93) is zero.

For every k and n, let $q^k_n \in \operatorname{arc}(q_\infty, q_n)$ and $p^k_n \in \mathscr{P}$ be such that

$$\liminf_{k \to +\infty} D_k(q^k_n, p^k_n) = 0.$$

Up to a subsequence, $q^k_n \to q_\infty$ and $p^k_n \to p_\infty \in \mathscr{P}$ as $n, k \to \infty$, and

$$\lim_{k \to +\infty} \lim_{n \to +\infty} D_k(q^k_n, p^k_n) = \lim_{k \to +\infty} D_k(q_\infty, p_\infty) = 0,$$

but this contradicts that $q_\infty \notin \mathscr{P}$. This contradiction proves that $q_\infty \in \mathscr{A}_q \cap \mathscr{B}_q$; hence, $\mathscr{A}_q \cap \mathscr{B}_q$ is closed in $\mathscr{B}_q$. □

Proposition 3.16. *Let $\mathscr{A}$ be an equivalence class in $S^1 \setminus \{\mathscr{P}\}$. Then there exists a sequence $f_k : \bar{D}^2 \to \bar{D}^2$ of Möbius transformations such that $\tilde{\Phi}_k := \Phi_k \circ f_k \rightharpoonup \tilde{\Phi}_\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar{D}^2 \setminus B)$, $B = \{a_1, \dots, a_N\}$, and, as usual letting $\tilde{\sigma}_k$ be such that $\gamma_k = \tilde{\Phi}_k \circ \tilde{\sigma}_k$, one has $\tilde{\sigma}_k^{-1} \rightharpoonup \psi_\infty$ in $W^{2,p}_{\mathrm{loc}}(S^1 \setminus B)$,*

$$\psi_\infty(S^1 \setminus B) = \mathscr{A} \tag{94}$$

and $\gamma_\infty(\mathscr{A}) = \tilde{\Phi}_\infty(S^1 \setminus B)$. In fact, $(\gamma_\infty)_*[\mathscr{A}] = (\tilde{\Phi}_\infty)_*[S^1 \setminus B]$.

Proof. Given $q \in \mathscr{A}$, take f_k as in the proof of Proposition 3.15 and set $\tilde{\Phi}_k := \Phi_k \circ f_k$. We have shown that $\tilde{\Phi}_k \rightharpoonup \tilde{\Phi}_\infty$ in $W^{1,2}(\bar{D}^2)$ and in $W^{2,p}_{\mathrm{loc}}(\bar{D}^2 \setminus B)$ for a finite set $B = \{a_1, \dots, a_N\}$, where $\tilde{\Phi}_\infty$ is a holomorphic immersion (Theorem 3.5, case (1)). In particular, this implies that $\psi_k := \tilde{\sigma}_k^{-1}$ is bounded in $W^{2,p}_{\mathrm{loc}}(S^1 \setminus B)$ and, up to a subsequence, $\psi_k \rightharpoonup \psi_\infty$ in $W^{2,p}_{\mathrm{loc}}(S^1 \setminus B)$. Clearly,

$$\psi_\infty(S^1 \setminus B) \subset \mathscr{A}.$$

Conversely, given $p \notin \psi_\infty(S^1 \setminus B)$, we want to show that $p \notin \mathscr{A}$. Given such a p we have $\tilde{\sigma}_k(p) \to a_i$ for some $a_i \in B$, since otherwise we would have $p = \psi_k \circ \tilde{\sigma}_k(p) \to \psi_\infty(p_*)$ for $p_* \in S^1 \setminus B$. Since

$\nabla\tilde{\Phi}_\infty \in L^2(D^2)$, from Fubini's theorem we can find a sequence $\delta_n^i \to 0$ such that

$$\lim_{n\to+\infty}\int_{\partial B(a_i,\delta_n^i)\cap\bar{D}^2} |\nabla\tilde{\Phi}_\infty(z)|^2\,|dz| = 0. \tag{95}$$

For every a_i, set $\{p_{k,n}^{i,-}, p_{k,n}^{i,+}\} = \tilde{\sigma}_k^{-1}(\partial B(a_i,\delta_n^i)\cap S^1)$. We have $|p_{k,n}^{i,-} - p_{k,n}^{i,+}| > C_0$ for any n and k large enough, since by definition of the blow-up points one has, for k large enough,

$$\int_{\mathrm{arc}(p_{k,n}^{i,-},p_{k,n}^{i,+})} |\dot{\gamma}_k(t)|\,dt = \int_{B(a_i,\delta_n^i)\cap S^1} e^{\lambda_k(z)}\,|dz| > \frac{\pi}{2}.$$

Therefore, up to a subsequence, $p_{k,n}^{i,-} \to p_\infty^{i,-}$ and $p_{k,n}^{i,+} \to p_\infty^{i,+}$ with $p_\infty^{i,+} \neq p_\infty^{i,-}$ and

$$\lim_{k\to\infty} D_k(\tilde{\sigma}_k(p_\infty^{i,-}), \tilde{\sigma}_k(p_\infty^{i,+})) = 0$$

In particular, $p_\infty^{i,-}$ and $p_\infty^{i,+}$ are pinched. Then condition (95) implies that any path Δ_k joining $\tilde{\sigma}_k(q)$ and $\tilde{\sigma}_k(p)$ for k large enough is close to $\tilde{\sigma}_k(p_\infty^{i,-}) \in \tilde{\sigma}_k(\mathcal{P})$, so $p \in S^1 \setminus \mathcal{A}$.

Finally,

$$\begin{aligned}
(\gamma_\infty)_*[\mathcal{A}] &= \lim_{\delta\to 0}(\gamma_\infty)_*\Big[\psi_\infty\Big(S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big)\Big],\\
&= \lim_{\delta\to 0}\lim_{k\to\infty}(\gamma_k)_*\Big[\tilde{\sigma}_k^{-1}\Big(S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big)\Big],\\
&= \lim_{\delta\to 0}\lim_{k\to\infty}(\tilde{\Phi}_k)_*\Big[S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big],\\
&= \lim_{\delta\to 0}(\tilde{\Phi}_\infty)_*\Big[S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big],\\
&= (\tilde{\Phi}_\infty)_*[S^1 \setminus B]. \qquad \square
\end{aligned}$$

Quantization result: proof of Theorems 1.2 and 1.5. In this section we prove Theorems 1.2 and 1.5. In Theorem 1.2 we will show that, under the hypothesis of Theorem 3.5, $\kappa_k e^{\lambda_k} \rightharpoonup \mu$ weakly in the sense of Radon measures, where μ is a Radon measure which is the sum of a locally bounded (possibly vanishing) function and a (possibly empty) sum of Dirac masses. We also give precise estimates on the coefficients of the Dirac masses. In Theorem 1.5, we show that up to a suitable choice of Möbius transformations we can "detect" all the connected components arising in the limit.

Proof of Theorem 1.2. From Theorem 3.5 there is a (possibly empty) set $B = \{a_1, \ldots, a_N\} \subset S^1$ such that (65) holds. Moreover, from (8) and (10) it follows that $\|(-\Delta)^{\frac{1}{2}}\lambda_k\|_{L^1(S^1)} \le C$. Therefore, (53) implies

$$\|\lambda_k - \bar{\lambda}_k\|_{L^q(S^1)} \le C \quad \text{for every } q < +\infty.$$

Up to extracting a further subsequence, we have $v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty$ in $L^q(S^1)$ and

$$\kappa_k e^{\lambda_k} \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad (-\Delta)^{\frac{1}{2}} v_k \overset{*}{\rightharpoonup} (-\Delta)^{\frac{1}{2}} v_\infty = \mu - 1 \quad \text{in } \mathcal{M}(S^1), \tag{96}$$

where $\mathcal{M}(S^1)$ denotes the space of finite signed measures on S^1. Up to a subsequence we also have $\kappa_k \overset{*}{\rightharpoonup} \kappa_\infty$ in $L^\infty(S^1)$. We now distinguish three cases.

Case 1: Suppose that we are in case (2) of Theorem 3.5 and $N=1$, i.e., $\lambda_k \to -\infty$ locally uniformly in $S^1 \setminus \{a_1\}$. Then $\mu = c_1\delta_{a_1}$ and, since

$$\int_{S^1} \kappa_k e^{\lambda_k}\, d\theta = 2\pi,$$

it follows at once that $c_1 = 2\pi$. The explicit form of v_∞ follows from Lemma 3.1.

Case 2: Suppose that we are in case (2) of Theorem 3.5 and $N>1$. Then we conclude by applying Proposition 3.9, which in particular implies that $N=2$ and $\mu = \pi\delta_{a_1} + \pi\delta_{a_2}$. Again, the explicit form of v_∞ follows from Lemma 3.1.

Case 3: Suppose that we are in case (1) of Theorem 3.5, i.e., $\lambda_k \ge -C$. Then $\lambda_k \rightharpoonup \lambda_\infty$ weakly in $W^{1,p}_{\mathrm{loc}}(S^1\setminus B)$ and for every $\varphi \in C^\infty_c(S^1\setminus B)$ we have

$$0 = \lim_{k\to\infty}\int_{S^1} (\lambda_k(-\Delta)^{\frac12}\varphi - (\kappa_k e^{\lambda_k}-1)\varphi)\, d\theta = \lim_{k\to\infty}\int_{S^1} (\lambda_\infty(-\Delta)^{\frac12}\varphi - (\mu-1)\varphi)\, d\theta.$$

In particular, the distribution

$$T_\infty := (-\Delta)^{\frac12}\lambda_\infty - \mu + 1$$

is supported in B and, since, by (96), $T_\infty \in \mathcal{M}(S^1)$, the order of T_∞ (as a distribution) is 0; hence,

$$T_\infty = \sum_{j=1}^N c_j\delta_{a_j}.$$

In order to compute the coefficients c_j, let $\chi_\delta : S^1 \to \mathbb{R}$ be 1 on $S^1 \cap \bigcup_{j=1}^n B(a_j,\delta)$ and 0 otherwise. We rewrite (9) as follows:

$$(-\Delta)^{\frac12}\lambda_k = (1-\chi_\delta)\kappa_k e^{\lambda_k} + \chi_\delta\kappa_k e^{\lambda_k} - 1. \tag{97}$$

Since

$$\lim_{k\to\infty}(1-\chi_\delta)\kappa_k e^{\lambda_k} = (1-\chi_\delta)\kappa_\infty e^{\lambda_\infty} \quad \text{in } \mathcal{D}'(S^1),$$

testing (97) with $\varphi \in C^\infty(S^1)$ and letting $k\to\infty$ we get

$$\int_{S^1} (\lambda_\infty(-\Delta)^{\frac12}\varphi - (1-\chi_\delta)\kappa_\infty e^{\lambda_\infty}\varphi + \varphi)\, d\theta = \lim_{k\to\infty}\int_{S^1} \chi_\delta\kappa_k e^{\lambda_k}\varphi\, d\theta$$

and, letting $\delta\to 0$, we infer

$$\langle T_\infty, \varphi\rangle = \lim_{\delta\to 0}\lim_{k\to\infty}\int_{S^1} \chi_\delta\kappa_k e^{\lambda_k}\varphi\, d\theta.$$

By choosing $\varphi = 1$ in a neighborhood of a_j for a fixed j and $\varphi = 0$ in a neighborhood of $B\setminus\{a_j\}$, we get

$$c_j = \lim_{\delta\to 0}\lim_{k\to\infty}\int_{S^1\cap B(a_j,\delta)} \kappa_k e^{\lambda_k}\, d\theta.$$

We now want to compute c_j for a fixed $j \in \{1, \dots, N\}$. Consider the Möbius transformation $f_k(z) = (z - t_k a_j)/(1 - t_k \bar{a}_j z)$, and $\tilde{\Phi}_k := \Phi_k \circ f_k$, for a sequence $t_k \uparrow 1$ to be chosen. By Corollary 2.4 we have

$$\tilde{\lambda}_k := \log |\tilde{\Phi}_k'| = \lambda_k \circ f_k + \log |f_k'|, \quad \tilde{\kappa}_k := \kappa_k \circ f_k,$$

and

$$(-\Delta)^{\frac{1}{2}} \tilde{\lambda}_k = \tilde{\kappa}_k e^{\tilde{\lambda}_k} - 1.$$

Since $\log |f_k'| \to -\infty$ locally uniformly in $\bar{D}^2 \setminus \{a_j\}$ and $\log |f_k'(a_j)| \to \infty$, it is not difficult to see that, if $t_k \uparrow 1$ slowly enough, then $\tilde{\lambda}_k \to -\infty$ uniformly locally in $\bar{D}^2 \setminus \{a_j, -a_j\}$ and we can apply Proposition 3.9 to $\tilde{\Phi}_k$ and obtain that

$$\tilde{\kappa}_k e^{\tilde{\lambda}_k} \overset{*}{\rightharpoonup} \pi(\delta_{a_j} + \delta_{-a_j}).$$

With a change of variable we then get

$$\pi = \lim_{\delta \to 0} \lim_{k \to \infty} \int_{S^1 \cap B(a_j, \delta)} \tilde{\kappa}_k e^{\tilde{\lambda}_k} \, d\theta = \lim_{\delta \to 0} \lim_{k \to \infty} \int_{f_k(S^1 \cap B(a_j, \delta))} \kappa_k e^{\lambda_k} \, d\theta = c_j,$$

where the last identity holds up to having $t_k \uparrow 1$ slowly enough. □

Proof of Theorem 1.5. From Proposition 3.15 it follows that $S^1 \setminus \{\mathscr{P}\} = \bigcup_{j \in J} \mathscr{A}_i$, where J is an at most countable set and $\mathscr{A}_j$ is an equivalence class generated by the relation in Definition 3.14. From Proposition 3.16 it follows that for every class $\mathscr{A}_j$ there is a sequence of Möbius transformations $f_k^j(z)$ such that

$$\tilde{\Phi}_k^j := \Phi_k \circ f_k^j \rightharpoonup \tilde{\Phi}_\infty^j \quad \text{in } W^{2,p}_{\text{loc}}(\bar{D}^2 \setminus B_j), \qquad B_j = \{b_1^j, \dots b_{N_j}^j\},$$

where $\tilde{\Phi}_\infty^j : \bar{D}^2 \setminus B_j \to \mathbb{R}^2$ is a conformal immersion and $\gamma_\infty(\mathscr{A}_j) = \tilde{\Phi}_\infty^j(S^1 \setminus B_j)$. Moreover, we have

$$(\gamma_\infty)_*[S^1 \setminus \mathscr{P}] = \sum_{j \in J} (\tilde{\Phi}_\infty^j)_*[S^1 \setminus B_j].$$

We have

$$\sum_{j \in J} (\gamma_\infty)_*[\mathscr{A}_j] = \sum_{j \in J} (\tilde{\Phi}_\infty^j)_*[S^1 \setminus B_j]$$

and it remains to prove that

$$(\gamma_\infty)_*[\mathscr{P}] = 0.$$

In order to do that, let $\tau : \mathscr{P} \to \mathscr{P}$ be the bijection which, to a pinched point p, associates its dual. For a differential form $\phi : \mathbb{C} \to L(\mathbb{C}, \mathbb{C})$, we have

$$(\gamma_\infty)_*[\mathscr{P}](\phi) = \int_{\mathscr{P}} \phi(\gamma_\infty(t)) \dot{\gamma}_\infty(t) \, dt. \tag{98}$$

Now recall that

$$\gamma_\infty(t) = \gamma_\infty(\tau(t)), \quad \dot{\gamma}_\infty(t) = -\dot{\gamma}_\infty(\tau(t)). \tag{99}$$

For a sequence $t_n \in \mathcal{P}$ with $t_n \to t \in \mathcal{P}$ as $n \to \infty$, we have

$$\begin{aligned} \gamma_\infty(t_n) &= \gamma_\infty(t) + \dot\gamma_\infty(t)(t_n - t) + o(t_n - t), \\ \gamma_\infty(\tau(t_n)) &= \gamma_\infty(\tau(t)) + \dot\gamma_\infty(\tau(t))(\tau(t_n) - \tau(t)) + o(\tau(t_n) - \tau(t)), \end{aligned} \tag{100}$$

where for simplicity of notation we identified S^1 with the interval $[0, 2\pi]$, with zero corresponding to a point in $S^1 \setminus \mathcal{P}$. Using (99) and (100) we infer that

$$\lim_{n\to\infty} \frac{\tau(t_n) - \tau(t)}{t_n - t} = -1.$$

Then, at a density point of $\mathcal{P}$, we have $d\tau/dt = -1$ in the sense of approximate differentials (if the density of $\mathcal{P}$ is everywhere 0 then $|\mathcal{P}| = 0$ and we are done). Therefore,

$$\int_{\mathcal{P}} \phi(\gamma_\infty(t))\dot\gamma_\infty(t)\,dt = -\int_{\mathcal{P}} \phi\big(\gamma_\infty(\tau(t))\big)\dot\gamma_\infty(\tau(t))\,dt = -\int_{\tau(\mathcal{P})=\mathcal{P}} \phi(\gamma_\infty(t))\dot\gamma_\infty(t)\,dt,$$

where in the first identity we used (99) and in the second identity we made a change of variable. This proves that the integral in (98) vanishes for every differential form ϕ; hence, $(\gamma_\infty)_*[\mathcal{P}] = 0$.

Since, for every $j \in J$, the sequence $(\tilde\Phi_k^j)$ is as in case (1) of Theorem 3.5, i.e., setting $\lambda_k^j := \log\big|(\tilde\Phi_k^j)'|_{S^1}\big|$ we have $|\bar\lambda_k^j| \le C$, we can apply Theorem 1.2(iii) and it follows at once that the blow-up set of λ_k^j is B_j. □

4. Relation between the Liouville equations in $\mathbb{R}$ and S^1

Consider the conformal map $G : D^2 \to \mathbb{R}^2$ given by

$$G(z) = \frac{iz+1}{z+i} = \frac{z+\bar z + i(|z|^2-1)}{1+|z|^2+i(\bar z - z)}.$$

We will use on the domain D^2 the coordinate $z = \xi + i\eta$ and on the target $\mathbb{R}^2$ the coordinates (x, y) or $x + iy$. Writing G in components,

$$G^1(z) = \Re G(z) = \frac{2\xi}{(1+\eta)^2+\xi^2}, \qquad G^2(z) = \Im G(z) = \frac{\xi^2+\eta^2-1}{(1+\eta)^2+\xi^2},$$

and using the polar coordinates (r, θ) on D^2 one easily verifies

$$\frac{\partial G^1}{\partial r}\bigg|_{r=1} = 0, \qquad \frac{\partial G^2}{\partial r}\bigg|_{r=1} = \frac{1}{1+\eta}, \qquad \frac{\partial G^1}{\partial \theta}\bigg|_{r=1} = -\frac{1}{1+\eta}, \qquad \frac{\partial G^2}{\partial \theta}\bigg|_{r=1} = 0.$$

Notice that $G|_{S^1}(\xi + i\eta) = \xi/(1+\eta)$, i.e., $\Pi := G^1|_{S^1}$ is the classical stereographic projection from $S^1 \setminus \{-i\}$ onto $\mathbb{R}$. Its inverse is

$$\Pi^{-1}(x) = \frac{2x}{1+x^2} + i\left(-1 + \frac{2}{1+x^2}\right). \tag{101}$$

If we write $\Pi^{-1}(x) = e^{i\theta(x)}$, we get the useful relation

$$1 + \sin(\theta(x)) = \frac{2}{1+x^2}, \qquad \frac{2}{1+\Pi(\theta)^2} = 1 + \sin\theta, \tag{102}$$

which follows easily from $\sin(\theta(x)) = \Im(\Pi^{-1}(x)) = (1-x^2)/(1+x^2)$.

Proposition 4.1. *Given $u : \mathbb{R} \to \mathbb{R}$, set $v := u \circ \Pi : S^1 \to \mathbb{R}$, where $\Pi := G^1|_{S^1}$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})$ if and only if $v \in L^1(S^1)$. In this case,*

$$(-\Delta)^{\frac{1}{2}} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}} u)(\Pi(e^{i\theta}))}{1+\sin\theta} \quad \text{in } \mathscr{D}'(S^1 \setminus \{-i\}), \tag{103}$$

that is,

$$\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle = \langle (-\Delta)^{\frac{1}{2}} u, \varphi \circ \Pi^{-1} \rangle \quad \text{for every } \varphi \in C_0^\infty(S^1 \setminus \{-i\}).$$

Further, if $(-\Delta)^{\frac{1}{2}} u \in L^1(\mathbb{R})$ or, equivalently, $(-\Delta)^{\frac{1}{2}} v|_{S^1 \setminus \{-i\}} \in L^1(S^1)$, then

$$(-\Delta)^{\frac{1}{2}} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}} u)(\Pi(e^{i\theta}))}{1+\sin\theta} - \gamma \delta_{-i} \quad \text{in } \mathscr{D}'(S^1), \qquad \gamma = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u \, dx. \tag{104}$$

Proof. Since

$$\int_{S^1} |v| \, d\theta = \int_{\mathbb{R}} \frac{2|v(\Pi^{-1}(x))|}{1+x^2} \, dx,$$

it is clear that $v \in L^1(S^1)$ if and only if $u \in L_{\frac{1}{2}}(\mathbb{R})$.

Given now $\varphi \in C_c^\infty(S^1 \setminus \{-1\})$, set $\psi := \varphi \circ \Pi^{-1} \in C_c^\infty(\mathbb{R})$ and let $\tilde{\varphi} \in C^\infty(\bar{D}^2)$ and $\tilde{\psi} \in C^\infty \cap L^\infty(\bar{R}^2_+)$ be the harmonic extensions of φ and ψ given by the Poisson formulas (125) and (132), respectively. It is not difficult to see that, setting $\bar{G} = (G^1, -G^2)$, $\tilde{\psi} \circ \bar{G}|_{\bar{D}^2}$ is continuous, harmonic in D^2 and it coincides with $\tilde{\varphi}$ on S^1. Then, by the maximum principle, $\tilde{\varphi} = \tilde{\psi} \circ \bar{G}$ in $\bar{D}^2 \setminus \{-i\}$.

Using polar coordinates we compute

$$\frac{\partial \tilde{\varphi}}{\partial r}\bigg|_{r=1} \circ \Pi^{-1} = \frac{\partial(\tilde{\varphi} \circ G^{-1})}{\partial x} \frac{\partial G^1}{\partial r}\bigg|_{r=1} + \frac{\partial(\tilde{\varphi} \circ G^{-1})}{\partial y} \frac{\partial G^2}{\partial r}\bigg|_{r=1} = -\frac{\partial \tilde{\psi}}{\partial y}\bigg|_{y=0} \frac{1+x^2}{2}.$$

Then, using Propositions A.1 and A.3, we get

$$\begin{aligned}
\langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle &= \int_{S^1} v \frac{\partial \tilde{\varphi}}{\partial r}\bigg|_{r=1} d\theta \\
&= \int_{\mathbb{R}} (v \circ \Pi^{-1}(x)) \left(\frac{\partial \tilde{\varphi}}{\partial r}\bigg|_{r=1} \circ \Pi^{-1}(x) \right) \frac{2}{1+x^2} \, dx \\
&= -\int_{\mathbb{R}} u \frac{\partial \tilde{\psi}}{\partial y}\bigg|_{y=0} dx \\
&= \langle (-\Delta)^{\frac{1}{2}} u, \psi \rangle,
\end{aligned}$$

so that (103) is proven.

In order to prove (104), set $f := ((-\Delta)^{\frac{1}{2}} v)|_{S^1 \setminus \{-i\}} \in \mathscr{D}'(S^1 \setminus \{-i\})$ and notice that

$$\|f\|_{L^1(S^1)} = \|(-\Delta)^{\frac{1}{2}} u\|_{L^1(\mathbb{R})} = \gamma.$$

Since $f \in L^1(S^1) \subset \mathscr{D}'(S^1)$, we have

$$T := (-\Delta)^{\frac{1}{2}} v - f \in \mathscr{D}'(S^1) \tag{105}$$

and $\operatorname{supp}(T) \subset \{-i\}$. We claim that $T = c\delta_{-i}$ for some constant c. By a rotation of S^1, it is convenient to assume that T is supported at $\{1\}$. In this case, we can write

$$T = \sum_{k=0}^{N} c_k D^k \delta_0$$

for some $N \in \mathbb{N}$ and $c_0, \dots, c_N \in \mathbb{C}$, which leads to

$$\langle T, \varphi \rangle = \sum_{k=0}^{N} c_k (-1)^k D^k \varphi_0 = \sum_{k=0}^{N} c_k \sum_{n \in \mathbb{Z}} (-in)^k \overline{\hat{\varphi}(n)} \quad \text{for } \varphi \in \mathfrak{D}(S^1). \tag{106}$$

On the other hand, according to (124) we have, for $\varphi \in \mathfrak{D}(S^1)$,

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}} v, \varphi \rangle &= \int_{S^1} v(\theta) \sum_{n \in \mathbb{N}} |n| \overline{\hat{\varphi}(n)} e^{-in\theta}\, d\theta \\ &= \sum_{n \in \mathbb{N}} |n| \overline{\hat{\varphi}(n)} \int_{S^1} v(\theta) e^{-in\theta}\, d\theta \\ &= 2\pi \sum_{n \in \mathbb{N}} |n| \hat{v}(n) \overline{\hat{\varphi}(n)}, \end{aligned} \tag{107}$$

where the sum can be moved outside the integral because $\sum_{n \in \mathbb{N}} |n| |\hat{\varphi}(n)| < \infty$. Similarly,

$$\langle f, \varphi \rangle = 2\pi \sum_{n \in \mathbb{N}} \hat{f}(n) \overline{\hat{\varphi}(n)} \quad \text{for } \varphi \in \mathfrak{D}(S^1). \tag{108}$$

Clearly (105), (106), (107) and (108) are compatible only if $c_k = 0$ for $k = 1, \dots, N$, hence proving (up to rotating back) that $T = c_0 \delta_{-i}$, as claimed. Finally, testing with $\varphi = 1$ we obtain

$$0 = \langle (-\Delta)^{\frac{1}{2}} v, 1 \rangle = \langle f, 1 \rangle + \langle T, 1 \rangle = \|(-\Delta)^{\frac{1}{2}} u\|_{L^1} + c_0,$$

which implies that $c_0 = -\|(-\Delta)^{\frac{1}{2}} u\|_{L^1}$. □

Now, given $u \in L_{\frac{1}{2}}(\mathbb{R})$ we want to define a function $\lambda \in L^1(S^1)$ such that

$$\Pi^*(e^{2u} |dx|^2) = e^{2\lambda} |d\theta|^2,$$

where Π^* denotes the pull-back of the stereographic projection, while $|dx|^2$ and $|d\theta|^2$ are the standard metrics on $\mathbb{R}$ and S^1, respectively. Since

$$\Pi^*(e^{2u} |dx|^2) = \left(\frac{\partial \Pi}{\partial \theta} \right)^2 e^{2u(\Pi(\theta))} |d\theta|^2,$$

we find

$$\lambda(\theta) = u(\Pi(\theta)) + \log \left| \frac{\partial \Pi}{\partial \theta} \right| = u(\Pi(\theta)) - \log(1 + \sin\theta) \tag{109}$$

or equivalently, using (102),

$$u(x) = \lambda(\Pi^{-1}(x)) + \log \frac{2}{1 + x^2}. \tag{110}$$

Using Proposition 4.1 we can now easily relate $(-\Delta)^{\frac{1}{2}}u$ and $(-\Delta)^{\frac{1}{2}}\lambda$.

Proposition 4.2. *Given $u:\mathbb{R}\to\mathbb{R}$, set λ as in (109). Then $u\in L_{\frac{1}{2}}(\mathbb{R})$ if and only if $\lambda\in L^1(S^1)$, and $(-\Delta)^{\frac{1}{2}}u\in L^1(\mathbb{R})$ if and only if $(-\Delta)^{\frac{1}{2}}\lambda\in L^1(S^1\setminus\{-i\})$. In this case, u solves (20) if and only if λ solves*

$$(-\Delta)^{\frac{1}{2}}\lambda=\kappa e^{\lambda}-1+(2\pi-c)\delta_{-i}\quad \text{in } S^1 \tag{111}$$

with $\kappa=V\circ\Pi$ and $c=\|(-\Delta)^{\frac{1}{2}}u\|_{L^1(\mathbb{R})}$.

Proof. This follows at once from Proposition 4.2 and Lemma 4.3, below. □

Lemma 4.3. *We have*

$$(-\Delta)^{\frac{1}{2}}\log(1+\sin\theta)=1-2\pi\delta_{-i}.$$

Proof. Notice that by (102) we can write

$$\log(1+\sin\theta)=u_{1,0}(\Pi(\theta)),\qquad u_{1,0}(x)=\log\frac{2}{1+x^2}.$$

Then Propositions 5.1 and 4.1 imply

$$\begin{aligned}(-\Delta)^{\frac{1}{2}}\log(1+\sin\theta)=\frac{(-\Delta)^{\frac{1}{2}}u(\Pi(\theta))}{1+\sin\theta}-\|(-\Delta)^{\frac{1}{2}}u\|_{L^1}\delta_{-i}&=\frac{e^{u_{1,0}(\Pi(\theta))}}{1+\sin\theta}-\delta_i\int_{\mathbb{R}}e^{u_{1,0}(x)}\,dx\\&=1-2\pi\delta_{-i}.\end{aligned}$$

□

5. Proof of Theorem 1.8 and Proposition 1.9

Before proving Theorem 1.8, we show that the functions defined in (27) are indeed solutions of (24)–(25).

Proposition 5.1. *For every $\mu>0$ and $x_0\in\mathbb{R}$, the function u_{μ,x_0} defined in (27) belongs to $L_{\frac{1}{2}}(\mathbb{R})$, satisfies (25) with $L=2\pi$, and solves (24).*

Proof. That $u_{\lambda,x_0}\in L_{\frac{1}{2}}(\mathbb{R})$ and $\int_{\mathbb{R}}e^{u_{\lambda,x_0}}\,dx=2\pi$ is elementary. The equation is invariant under translations and dilations in the sense that, for all $x_0\in\mathbb{R}$ and $\lambda>0$, if u is a solution of (24) then $u(\lambda(x+x_0))+\log(\lambda)$ is a solution of (24) as well; hence, it suffices to prove that $u_{1,0}(x)=\log(2/(1+x^2))$ is a solution. From Proposition A.3 we get, integrating by parts,

$$\begin{aligned}\pi(-\Delta)^{\frac{1}{2}}u_{1,0}(x)&=\lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}\frac{\log\frac{1+y^2}{1+x^2}}{(x-y)^2}\,dy\\&=\lim_{\varepsilon\to0}\left\{-\frac{\log\frac{1+y^2}{1+x^2}}{y-x}\Big|_{-\infty}^{x-\varepsilon}-\frac{\log\frac{1+y^2}{1+x^2}}{y-x}\Big|_{x+\varepsilon}^{\infty}+\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}\frac{2y}{(y-x)(1+y^2)}\,dy\right\}\\&=\lim_{\varepsilon\to0}\left\{\frac{2\arctan y+x\log\frac{(y-x)^2}{1+y^2}}{1+x^2}\Big|_{-\infty}^{x-\varepsilon}+\frac{2\arctan y+x\log\frac{(y-x)^2}{1+y^2}}{1+x^2}\Big|_{x+\varepsilon}^{\infty}\right\}\\&=\frac{2\pi}{1+x^2}=\pi e^{u_{1,0}(x)}.\end{aligned}$$

□

Theorem 5.2. *There exist constants $C_1, C_2 > 0$ such that for any $\varepsilon \in (0, \pi)$ one has*

$$C_1 \le \sup_{\substack{u \in \tilde{H}^{1,1}_{\Delta}(I) \\ \|(-\Delta)^{\frac{1}{2}} u\|_{L^1(I)} \le 1}} \frac{\varepsilon}{|I|} \int_I e^{(\pi-\varepsilon)|u|}\, d\theta \le C_2, \tag{112}$$

where $\tilde{H}^{1,1}_{\Delta}(I) := \{u \in L^1(\mathbb{R}) : \operatorname{supp}(u) \subset \bar{I},\ (-\Delta)^{\frac{1}{2}} u \in L^1(\mathbb{R})\}$.

Lemma 5.3. *The Green function of $(-\Delta)^{\frac{1}{2}}$ on the interval $I = (-1, 1)$ can be decomposed as*

$$G_{\frac{1}{2}}(x, y) = F_{\frac{1}{2}}(|x-y|) + H_{\frac{1}{2}}(x, y),$$

where $F_{\frac{1}{2}}(x) := (1/\pi)\log(1/|x|)$ and $H_{\frac{1}{2}}$ is bounded above.

Proof. This follows from the explicit expression of $G(x, y)$ (see, e.g., [Blumenthal et al. 1961; Bucur 2015]), namely

$$G(x, y) = \frac{1}{2\pi}\int_0^{r_0(x,y)} \frac{1}{\sqrt{r(r+1)}}\, dr = \frac{1}{\pi}\log\big(\sqrt{r_0(x, y)} + \sqrt{r_0(x, y) + 1}\big),$$

where

$$r_0(x, y) := \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}. \qquad \square$$

Proof of Theorem 5.2. Up to a translation and dilation we can assume that $I = (-1, 1)$. With Lemma 5.3 we write, for $u \in \tilde{H}^{1,1}_{\Delta}(I)$ and $f := (-\Delta)^{\frac{1}{2}} u$,

$$|u(x)| = \left|\int_I G(x, y) f(y)\, dy\right|,$$

and we bound

$$G(x, y) \le \frac{1}{\pi}\log\left(\frac{2}{|x-y|}\right) + C, \qquad x, y, \in I,$$

hence

$$|u(x)| \le \frac{1}{\pi}\int_I \log\left(\frac{2}{|x-y|}\right)|f(y)|\, dy + C \tag{113}$$

and, exactly as in (56), one gets

$$\int_I e^{(\pi-\varepsilon)|u(x)|}\, dx \le C\int_I |f(y)| \int_I \left(\frac{2}{|x-y|}\right)^{1-\frac{\varepsilon}{\pi}} dx\, dy \le \frac{C}{\varepsilon}.$$

The rest of the proof is also similar to the proof of Theorem 3.2. $\square$

Remark 5.4. A slight modification of (112) is

$$C_1 \le \sup_{\substack{u = F_{1/2} * f \\ \operatorname{supp}(f) \subset \bar{I},\ \|f\|_{L^1(I)} \le 1}} \frac{\varepsilon}{|I|} \int_I e^{(\pi-\varepsilon)|u|}\, d\theta \le C_2, \tag{114}$$

where $F_{\frac{1}{2}}$ is as in Lemma 5.3. The proof of (114) is similar to the proof of (112), since $u = F_{\frac{1}{2}} * f$ obviously satisfies (113). An alternative proof of a nonsharp version of (114), namely

$$\sup_{\substack{u=F_{\frac{1}{2}}*f \\ \operatorname{supp}(f)\subset \bar{I},\, \|f\|_{L^1(I)}\le 1}} \int_I e^{\delta|u-\bar{u}|}\,d\theta \le C_2 \quad \text{for some } \delta>0 \text{ and } \bar{u} := \fint_I u\,dx,$$

can be obtained noticing that, for $u = F_{\frac{1}{2}} * f$, one has $[u]_{\mathrm{BMO}(I)} \le C[F_{\frac{1}{2}}]_{\mathrm{BMO}(\mathbb{R})}\|f\|_{L^1(I)}$, and one can apply the John–Nirenberg inequality.

Proposition 5.5. *Let $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy (24)–(25). Then there is a constant $C_0 \in \mathbb{R}$ such that*

$$u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) e^{u(y)}\,dy + C_0. \tag{115}$$

In the proof of Proposition 5.5 we use two lemmata.

Lemma 5.6. *For any $f \in L^1(\mathbb{R})$ the function*

$$w(x) := \mathcal{I}[f](x) := \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\,dy \tag{116}$$

is well defined, belongs to $L_{\frac{1}{2}}(\mathbb{R})$ and satisfies

$$(-\Delta)^{\frac{1}{2}} w = f \quad \text{in } \mathcal{S}'. \tag{117}$$

Proof of Lemma 5.6. Let us first assume that f belongs to the Schwartz space $\mathcal{S}$. Remember that, for $F(x) := (1/\pi)\log(1/|x|)$, we have (see, e.g., [Vladimirov 1971, p. 132])

$$\hat{F}(\xi) = \mathcal{P}\frac{1}{|\xi|} + C\delta_0 \quad \text{in } \mathcal{S}', \tag{118}$$

where $\mathcal{P}(1/|\xi|) \in \mathcal{S}'$ is the tempered distribution defined by

$$\left\langle \mathcal{P}\frac{1}{|\xi|}, \varphi \right\rangle = \int_{|\xi|\le 1} \frac{\varphi(\xi)-\varphi(0)}{|\xi|}\,d\xi + \int_{|\xi|>1} \frac{\varphi(\xi)}{|\xi|}\,d\xi, \qquad \varphi \in \mathcal{S}. \tag{119}$$

For every $f \in C_c^\infty(\mathbb{R})$ one easily sees that $F * f \in C^\infty(\mathbb{R})$ and $F * f \in L_{\frac{1}{2}}(\mathbb{R})$. Then

$$\begin{aligned}
\langle (-\Delta)^{\frac{1}{2}}(F*f), \varphi\rangle &:= \int_{\mathbb{R}} (F*f)\mathcal{F}^{-1}(|\xi|\hat{\varphi})\,dx \\
&= \int_{\mathbb{R}} F(\tilde{f} * \mathcal{F}^{-1}(|\xi|\hat{\varphi}))\,dx \\
&= \int_{\mathbb{R}} F\mathcal{F}\big(\mathcal{F}^{-1}(\tilde{f} * \mathcal{F}^{-1}(|\xi|^{2\sigma}\hat{\varphi}))\big)\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} F\mathcal{F}(\hat{f}|\xi|\hat{\varphi})\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}\hat{\hat{\varphi}}\,d\xi = \int_{\mathbb{R}} f\varphi\,dx,
\end{aligned} \tag{120}$$

where in order to apply (119) in the fifth identity can approximate the function $\psi(\xi) = \hat{f}|\xi|\hat{\varphi}$ by a sequence of functions $\psi_\varepsilon = \hat{f}\eta_\varepsilon\hat{\hat{\varphi}} \in \mathcal{S}(\mathbb{R})$ with $\eta_\varepsilon \in C^\infty(\mathbb{R})$ suitably chosen (see, for instance, [Jin et al. 2015b]). Hence, $(-\Delta)^{\frac{1}{2}}(F * f) = f$ in $\mathcal{D}'(\mathbb{R})$ and, since $f \in \mathcal{D}(\mathbb{R})$, the identity also holds in a strong sense. Moreover, since obviously

$$(-\Delta)^{\frac{1}{2}}\left(\frac{1}{\pi}\int_{\mathbb{R}} \log(1+|y|) f(y)\,dy\right) = 0,$$

we see that (117) is satisfied when $f \in \mathcal{D}(\mathbb{R})$.

For a general function $f \in L^1(\mathbb{R})$ we can find a sequence $(f_k) \subset \mathcal{D}(\mathbb{R})$ with $f_k \to f$ in $L^1(\mathbb{R})$ and take $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$(I)_k := \langle (-\Delta)^{\frac{1}{2}} \mathcal{I}[f_k], \varphi\rangle = \langle f_k, \varphi\rangle \to \langle f, \varphi\rangle$$

as $k \to \infty$, while

$$(I)_k = \langle \mathcal{I}[f_k], (-\Delta)^{\frac{1}{2}}\varphi\rangle = \int_{\mathbb{R}} \mathcal{I}[f_k](x)\psi(x)\,dx,$$

where $\psi := (-\Delta)^{\frac{1}{2}}\varphi$ satisfies

$$|\psi(x)| \le C(1+|x|^2). \tag{121}$$

It remains to show that

$$\int_{\mathbb{R}} \mathcal{I}[f_k - f](x)\psi(x)\,dx \to 0 \quad \text{as } k \to \infty.$$

Define $g_k := f_k - f \to 0$ in $L^1(\mathbb{R})$. Then, from $\|h_1 * h_2\|_{L^1} \le \|h_1\|_{L^1}\|h_2\|_{L^1}$, we get

$$\left|\int_{B(x,1)} \log\left(\frac{1+|y|}{|x-y|}\right) g_k(y)\,dy\right| \le \log(2+|x|)\|g_k\|_{L^1(\mathbb{R})} + C\|g_k\|_{L^1}$$

and, using that for $|x-y| \ge 1$ we have $\log((1+|y|)/|x-y|) \le C(1+\log(|x|))$,

$$\left|\int_{\mathbb{R}\setminus B(x,1)} \log\left(\frac{1+|y|}{|x-y|}\right) g_k(y)\,dy\right| \le C(1+\log|x|)\|g_k\|_{L^1}.$$

Therefore, taking (121) into account, we see that

$$(I)_k \to \langle \mathcal{I}[f], (-\Delta)^{\frac{1}{2}}\varphi\rangle \quad \text{as } k \to \infty;$$

hence, we conclude that $(-\Delta)^{\frac{1}{2}} w = f$ in $\mathcal{S}'(\mathbb{R})$. □

Lemma 5.7. *Let $f \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy $(-\Delta)^{\frac{1}{2}} f = 0$. Then f is constant.*

Proof. This is identical to the proof of Lemma 14 in [Jin et al. 2015b]. □

Proof of Proposition 5.5. Set $w(x)$ as in (116) with $f(y) := e^{u(y)}$. Then $(-\Delta)^{\frac{1}{2}}(u-w) = 0$ by Lemma 5.6; hence, by Lemma 5.7, $u - w \equiv C_0$ for some $C_0 \in \mathbb{R}$. □

Proposition 5.8. *Let $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy (24)–(25). Then $u \in C^\infty(\mathbb{R})$.*

Proof. Up to scaling, assume that

$$\int_{-1}^{1} e^{u(x)}\,dx < \varepsilon,$$

where ε will be fixed later.

Let us split $u = u_1 + u_2$, where

$$u_1(x) = \frac{1}{\pi}\int_{-1}^{1} \log\left(\frac{1+|y|}{|x-y|}\right) e^{u(y)}\,dy + C_0 = \frac{1}{\pi}\int_{-1}^{1} \log\left(\frac{1}{|x-y|}\right) e^{u(y)}\,dy + C_1. \tag{122}$$

Then (115) implies that u_2 is defined by the same formula, integrating over $\mathbb{R}\setminus[-1,1]$ instead of $\mathbb{R}$. It is easy to see that

$$\|u_2\|_{L^\infty([-\frac{1}{2},\frac{1}{2}])} \le C\int_{\mathbb{R}} e^{u(x)}\,dx < \infty.$$

From (114) if follows that, given $p < \infty$, choosing $\varepsilon > 0$ small enough (depending on p) we have $e^{|u_1|} \in L^p([-1,1])$, so $e^u \in L^p\big[-\frac{1}{2}, \frac{1}{2}\big]$.

The same argument, together with translations and dilations, can be performed in a neighborhood of every point in $\mathbb{R}$, giving $e^u \in L^p_{\mathrm{loc}}(\mathbb{R})$ for $1 < p < \infty$. Going back to (115) it is easy to bootstrap regularity and prove that u is actually smooth. □

Corollary 5.9. *Every function $\lambda \in L^1(S^1)$ solving (33) with $(-\Delta)^{\frac{1}{2}}\lambda \in L^1(S^1)$ is smooth.*

Proof. By Proposition 4.2 the function $u:\mathbb{R}\to\mathbb{R}$ given by (110) is in $L_{\frac{1}{2}}(\mathbb{R})$ and it solves (24). Then, by Proposition 5.8, u is smooth; hence, $\lambda \in C^\infty(S^1\setminus\{-i\})$. Since (33) is invariant under rotations, we have that actually $\lambda \in C^\infty(S^1)$. □

Lemma 5.10. *For $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^1(\mathbb{R})$ solving (24)–(25), set*

$$\alpha := \int_{\mathbb{R}} e^{u(x)}\,dx.$$

Then $\alpha = 2\pi$.

Proof. This argument is taken from [Xu 2005] and is based on a Pohozaev-type identity. Differentiating (115) (for instance, by splitting the domain of integration into $[-a,a]$ and $\mathbb{R}\setminus[-a,a]$ for some $a > |x|$ and using elementary calculus) we obtain

$$x\frac{\partial u}{\partial x} = -\frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{x}{x-y} e^{u(y)}\,dy.$$

Multiplying by $e^{u(x)}$ and integrating with respect to x on the interval $[-R,R]$, we get

$$(I) := \int_{-R}^{R} x\frac{\partial u}{\partial x} e^{u(x)}\,dx = -\frac{1}{\pi}\int_{-R}^{R} \mathrm{PV}\int_{\mathbb{R}} \frac{x}{x-y} e^{u(y)}\,dy\, e^{u(x)}\,dx =: (II).$$

Integrating by parts we find

$$(I) = \int_{-R}^{R} x\frac{\partial e^{u(x)}}{\partial x}\,dx = R(e^{u(R)} + e^{u(-R)}) - \int_{-R}^{R} e^{u(x)}\,dx \to -\alpha \quad \text{as } R\to\infty,$$

where we used that, at least on a subsequence, $R(e^{u(R)}-e^{u(-R)})\to 0$ as $R\to\infty$, otherwise (25) would be violated. As for (II), we compute

$$(II)=-\frac{1}{2\pi}\int_{-R}^{R}\int_{\mathbb{R}}e^{u(y)}\,dy\,e^{u(x)}\,dx-\frac{1}{2\pi}\int_{-R}^{R}\mathrm{PV}\int_{\mathbb{R}}\frac{x+y}{x-y}e^{u(y)}\,dy\,e^{u(x)}\,dx\to-\frac{\alpha^2}{2\pi}+0$$

as $R\to\infty$. Therefore, from $(I)=(II)$ we infer $\alpha=\alpha^2/(2\pi)$, i.e., $\alpha=2\pi$. □

Proof of Theorem 1.8. Given $u\in L_{\frac12}(\mathbb{R})$ satisfying (24)–(25), by Proposition 4.2 the function $\lambda(\theta):=u(\Pi(\theta))-\log(1+\sin\theta)$ solves

$$(-\Delta)^{\frac12}\lambda=e^{\lambda}-1+(2\pi-\alpha)\delta_{-i}\quad\text{in } S^1$$

and, by Lemma 5.10, $\alpha=2\pi$; hence,

$$(-\Delta)^{\frac12}\lambda=e^{\lambda}-1\quad\text{in } S^1.$$

By Corollary 2.3, λ is of the form given by (34) for some $a\in D^2$.

To complete the proof, write $a=\alpha e^{i\theta_0}=\alpha(t+is)$ with $\alpha,t,s\in\mathbb{R}$. We have

$$u(x)=\lambda\circ\Pi^{-1}(x)+\log\frac{2}{1+x^2}=\log\frac{2(1-\alpha^2)}{|1-\alpha(t+is)\Pi^{-1}(x)|^2(1+x^2)}.$$

The right-hand side can be computed using (101):

$$\begin{aligned}u(x)&=\log\frac{2(1-\alpha^2)}{\left|1+\alpha\frac{-2tx+s(1-x^2)}{1+x^2}-i\alpha\frac{2sx+t(1-x^2)}{1+x^2}\right|^2(1+x^2)}\\&=\log\frac{2(1-\alpha^2)}{x^2(1-2\alpha s+\alpha^2)-4\alpha tx+1+2\alpha s+\alpha^2}.\end{aligned}$$

Completing the square in the denominator on the right-hand side, we get

$$u(x)=\log\frac{2(1-\alpha^2)}{(1-2\alpha s+\alpha^2)\left(x-\frac{2\alpha t}{1-2\alpha s+\alpha^2}\right)^2+\frac{(1-\alpha^2)^2}{1-2\alpha s+\alpha^2}}=\log\frac{2\mu}{1+\mu^2(x-x_0)^2}$$

with

$$x_0=\frac{2\alpha t}{1-2\alpha s+\alpha^2},\qquad\mu=\frac{1-2\alpha s+\alpha^2}{1-\alpha^2}.$$ □

The following can been seen as a nonlocal version of the classical mean value property of harmonic functions. It appears in [Silvestre 2007, Proposition 2.2.6] in a slightly different case, but with a proof which readily extends to the following case.

Proposition 5.11. *There exists a positive function $\gamma_1\in C^{1,1}(\mathbb{R})$ with $\int_{\mathbb{R}}\gamma_1\,dx=1$ such that, setting $\gamma_\lambda(x):=(1/\lambda)\gamma_1(x/\lambda)$, we have*

$$u(x_0)\ge u*\gamma_\lambda(x_0)$$

for every $\lambda>0$ and every $u\in L_{\frac12}(\mathbb{R})$ satisfying $(-\Delta)^{\frac12}u\ge 0$.

Proof of Proposition 1.9. Since $(-\Delta)^{\frac{1}{2}} u \le 0$, we have, by Proposition 5.11 below,

$$u(0) \le u * \gamma_\lambda(0) \quad \text{for every } \lambda > 0,$$

where γ_λ is as in Proposition 5.11. Since $d\mu_\lambda(x) := \gamma_\lambda(-x)\,dx$ satisfies $\int_{\mathbb{R}} d\mu_\lambda = 1$, from Jensen's inequality we get

$$\int_{\mathbb{R}} e^u \, d\mu_\lambda \ge \exp\biggl(\int_{\mathbb{R}} u \, d\mu_\lambda\biggr) = e^{u*\gamma_\lambda(0)} \ge e^{u(0)}.$$

On the other hand, since $d\mu_\lambda \le (C/\lambda)\,dx$, we estimate

$$\int_{\mathbb{R}} e^u \, dx \ge \frac{\lambda}{C} \int_{\mathbb{R}} e^u \, d\mu_\lambda \ge \frac{\lambda}{C} e^{u(0)} \to \infty \quad \text{as } \lambda \to \infty,$$

contradicting (25). □

Appendix A: The fractional Laplacian

The half-Laplacian on S^1. Given $u \in L^1(S^1)$, we define its Fourier coefficients as

$$\hat{u}(n) = \frac{1}{2\pi} \int_{S^1} u(\theta) e^{-in\theta} \, d\theta, \qquad n \in \mathbb{Z}.$$

If u is smooth, we can define

$$(-\Delta)^{\frac{1}{2}} u(\theta) = \sum_{n\in\mathbb{Z}} |n| \hat{u}(n) e^{in\theta}. \tag{123}$$

For $u \in L^1(S^1)$, we can define $(-\Delta)^{\frac{1}{2}} u \in \mathcal{D}'(S^1)$ as a distribution as

$$\langle (-\Delta)^{\frac{1}{2}} u, \varphi \rangle := \int_{S^1} u (-\Delta)^{\frac{1}{2}} \varphi \, d\theta, \qquad \varphi \in C^\infty(S^1). \tag{124}$$

Notice that $\varphi \in C^\infty(S^1)$ implies that $(-\Delta)^{\frac{1}{2}} \varphi \in C^\infty(S^1)$ (here, $(-\Delta)^{\frac{1}{2}} \varphi$ is defined as in (123)). In fact, given $\varphi \in L^1(S^1)$, we have $\varphi \in C^\infty(S^1)$ if and only if $\hat{\varphi}(n) = o(|n|^{-k})$ for every $k \ge 0$.

We can also give a definition of $(-\Delta)^{\frac{1}{2}} u$ in terms of harmonic extensions. If $u \in L^1(S^1)$, let $\tilde{u}(r, \theta)$ be its harmonic extension in D^2, explicitly given by the Poisson formula

$$\tilde{u}(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) u(t) \, dt, \qquad P(r,\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \tag{125}$$

Then one can define (using polar coordinates)

$$(-\Delta)^{\frac{1}{2}} u = \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1} \quad \text{in } \mathcal{D}'(S^1), \tag{126}$$

where the distribution $\partial\tilde{u}/\partial r\big|_{r=1}$ is defined as

$$\biggl\langle \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1}, \varphi \biggr\rangle := \int_{S^1} u \frac{\partial \tilde{\varphi}}{\partial r}\bigg|_{r=1} d\theta,$$

where $\varphi \in C^\infty(S^1)$ and $\tilde{\varphi}$ is the harmonic extension of φ in D^2.

Notice that, if $u \in C^\infty(S^1)$, the equivalence of (123), (124) and, in fact, (126) is elementary, and (126) holds pointwise. For instance, the equivalence of (123) and (126) follows at once from

$$\tilde{u}(r,\theta) = \sum_{n\in\mathbb{Z}} \hat{u}(n) r^{|n|} e^{in\theta}.$$

Proposition A.1. *The definitions (124) and (126) are equivalent.*

Proof. Since (126) holds pointwise for smooth functions, one has, for $u \in L^1(S^1)$ and $\varphi \in C^\infty(S^1)$,

$$\langle (-\Delta)^{\frac{1}{2}} u, \varphi\rangle := \int_{S^1} u(-\Delta)^{\frac{1}{2}}\varphi\, dx = \int_{S^1} u \frac{\partial\tilde{\varphi}}{\partial\theta}\, d\theta =: \left\langle \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1}, \varphi \right\rangle. \qquad \square$$

For $u \in C^{1,\alpha}(S^1)$, there is also the following pointwise definition of $(-\Delta)^{\frac{1}{2}} u$:

Proposition A.2. *If $u \in C^{1,\alpha}(S^1)$ for some $\alpha \in (0,1]$, then $(-\Delta)^{\frac{1}{2}} u \in C^{0,\alpha}(S^1)$ and*

$$(-\Delta)^{\frac{1}{2}} u(e^{i\theta}) = \frac{1}{\pi}\,\mathrm{PV}\int_0^{2\pi} \frac{u(e^{i\theta}) - u(e^{it})}{2-2\cos(\theta - t)}\, dt, \tag{127}$$

where the principal value is well defined because $2 - 2r\cos(\theta - t) = (\theta - t)^2 + O((\theta - t)^4)$ as $t \to \theta$.

Proof. Considering Proposition A.1, it suffices to show the equivalence of (126) and (127). Set $\tilde{u}$ as in (125). Then

$$\begin{aligned}
\frac{\partial \tilde{u}(r,\theta)}{\partial r}\bigg|_{r=1} &= \lim_{r\uparrow 1} \frac{\tilde{u}(r,\theta) - u(e^{i\theta})}{r-1} \\
&= \lim_{r\uparrow 1} \frac{1}{2\pi(r-1)} \int_0^{2\pi} \frac{(1-r^2)(u(e^{i\theta}) - u(e^{it}))}{1 - 2r\cos(\theta - t) + r^2}\, dt \\
&= \lim_{r\uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1+r)(u(e^{i\theta}) - u(e^{it}))}{1 - 2r\cos(\theta - t) + r^2}\, dt \\
&= \frac{1}{\pi}\,\mathrm{PV}\int_0^{2\pi} \frac{u(e^{i\theta}) - u(e^{it})}{2 - 2r\cos(\theta - t)}\, dt. \qquad \square
\end{aligned}$$

The half-Laplacian on $\mathbb{R}$. For $u \in \mathcal{S}$ (the Schwarz space of rapidly decaying functions), we set

$$\widehat{(-\Delta)^{\frac{1}{2}} u}(\xi) = |\xi|\hat{u}(\xi), \qquad \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx. \tag{128}$$

One can prove that

$$(-\Delta)^{\frac{1}{2}} u(x) = \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{u(x) - u(y)}{(x-y)^2}\, dy := \frac{1}{\pi}\lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus[-\varepsilon+x,x+\varepsilon]} \frac{u(x)-u(y)}{(x-y)^2}\, dy, \tag{129}$$

from which it follows that

$$\sup_{x\in\mathbb{R}} |(1+x^2)(-\Delta)^{\frac{1}{2}}\varphi(x)| < \infty \quad \text{for every } \varphi \in \mathcal{S}.$$

Then one can set

$$L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2}\, dx < \infty \right\}, \tag{130}$$

and, for every $u \in L_{\frac{1}{2}}(\mathbb{R})$, one defines the tempered distribution $(-\Delta)^{\frac{1}{2}}u$ as

$$\langle(-\Delta)^{\frac{1}{2}}u, \varphi\rangle := \int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}}\varphi\,dx = \int_{\mathbb{R}} u\mathscr{F}^{-1}(|\xi|\hat{\varphi}(\xi))\,dx \quad \text{for every } \varphi \in \mathscr{S}. \tag{131}$$

An alternative definition of $(-\Delta)^{\frac{1}{2}}$ can be given via the Poisson integral. For $u \in L_{\frac{1}{2}}(\mathbb{R})$, define the Poisson integral

$$\tilde{u}(x, y) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{yu(y)}{(y^2 + (x-\xi)^2)}\,d\xi, \qquad y > 0, \tag{132}$$

which is harmonic in $\mathbb{R} \times (0, \infty)$ and whose trace on $\mathbb{R} \times \{0\}$ is u. Then we have

$$(-\Delta)^{\frac{1}{2}}u = -\frac{\partial\tilde{u}}{\partial y}\bigg|_{y=0}, \tag{133}$$

where the identity is pointwise if u is regular enough (for instance, $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R})$), and has to be read in the sense of distributions in general, with

$$\left\langle -\frac{\partial\tilde{u}}{\partial y}\bigg|_{y=0}, \varphi\right\rangle := \left\langle u, -\frac{\partial\tilde{\varphi}}{\partial y}\bigg|_{y=0}\right\rangle, \quad \varphi \in \mathscr{S}, \quad \tilde{\varphi} \text{ as in (132)}. \tag{134}$$

More precisely:

Proposition A.3. *If $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^{1,\alpha}_{\mathrm{loc}}((a, b))$ for some interval $(a, b) \subset \mathbb{R}$ and some $\alpha \in (0, 1)$, then $(-\Delta)^{\frac{1}{2}}u$, the tempered distribution defined in (131), coincides on the interval (a, b) with the functions given by (129) and (133). For general $u \in L_{\frac{1}{2}}(\mathbb{R})$, the definitions (131) and (133) are equivalent, where the right-hand side of (133) is defined by (134).*

Proof. Assume that $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^{1,\alpha}_{\mathrm{loc}}((a, b))$. Following [Caffarelli and Silvestre 2007], we have, for $x \in (a, b)$,

$$\frac{\partial\tilde{u}(x, y)}{\partial y}\bigg|_{y=0} = \lim_{y\to 0}\frac{\tilde{u}(x, y) - \tilde{u}(x, 0)}{y} = \lim_{y\to 0}\frac{1}{\pi}\int_{\mathbb{R}}\frac{u(\xi) - u(x)}{y^2 + (\xi - x)^2}\,d\xi = \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}}\frac{u(\xi) - u(x)}{(\xi - x)^2}\,d\xi,$$

where the last convergence follows from dominated convergence outside $B_1(x)$ and by a Taylor expansion in a neighborhood of x. This proves the equivalence of (129) and (133). The equivalence between (129) and (131) amounts to showing that

$$\int_{\mathbb{R}} u\mathscr{F}^{-1}(|\xi|\hat{\varphi}(\xi))\,dx = \frac{1}{\pi}\int_{\mathbb{R}}\mathrm{PV}\int_{\mathbb{R}}\frac{u(x) - u(y)}{(x - y)^2}\,dy\,\varphi(x)\,dx \tag{135}$$

whenever $\varphi \in \mathscr{S}$ is supported in (a, b). When $u \in \mathscr{S}$, the equivalence is shown, e.g., in [Caffarelli and Silvestre 2007] (passing through the definition given in (128)). In the general case, one approximates u with functions $u_k \in \mathscr{S}$ converging to u uniformly locally in (a, b) and in $L_{\frac{1}{2}}(\mathbb{R})$, as shown in Proposition 2.1.4 of [Silvestre 2007] (in order to have convergence in (135) as $u_k \to u$, it is convenient to consider φ compactly supported first, in case (a, b) is not bounded).

The last statement follows at once by noticing that, applying (133) to $\varphi \in \mathscr{S}$, one gets

$$\left\langle u, -\frac{\partial\tilde{\varphi}}{\partial y}\bigg|_{y=0}\right\rangle = \langle u, (-\Delta)^{\frac{1}{2}}\varphi\rangle. \qquad \square$$

Appendix B: Useful results from complex analysis

Lemma B.1. *Let $h \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $h(S^1) \subset S^1$ and $0 \notin h(D^2)$. Then h is constant.*

Proof. Since h never vanishes, $\log |h|$ is well defined, harmonic and vanishes on S^1, hence everywhere. This implies that $|h| \equiv 1$ and, from the conformality of h, it follows that h is constant. □

The following is a generalization of Lemma B.1:

Lemma B.2 [Burckel 1979]. *If $h \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $h(S^1) \subset S^1$ and $\deg h|_{S^1} = n \geq 0$, then h is a Blaschke product of degree n, i.e.,*

$$h(z) = e^{i\theta_0} \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}, \qquad a_1, \dots, a_n \in D^2,\ \theta_0 \in \mathbb{R}.$$

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Received 30 Apr 2015. Revised 2 Jul 2015. Accepted 29 Jul 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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nonprofit scientific publishing

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ANALYSIS & PDE

Volume 8 No. 7 2015