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SEMILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY DE SITTER, KERR–DE SITTER AND MINKOWSKI SPACETIMES

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We show the small data solvability of suitable semilinear wave and Klein–Gordon equations on geometric classes of spaces, which include so-called asymptotically de Sitter and Kerr–de Sitter spaces as well as asymptotically Minkowski spaces. These spaces allow general infinities, called conformal infinity in the asymptotically de Sitter setting; the Minkowski-type setting is that of nontrapping Lorentzian scattering metrics introduced by Baskin, Vasy and Wunsch. Our results are obtained by showing the *global* Fredholm property, and indeed invertibility, of the underlying linear operator on suitable L^2-based function spaces, which also possess appropriate algebra or more complicated multiplicative properties. The linear framework is based on the b-analysis, in the sense of Melrose, introduced in this context by Vasy to describe the asymptotic behavior of solutions of linear equations. An interesting feature of the analysis is that *resonances*, namely poles of the inverse of the Mellin-transformed b-normal operator, which are "quantum" (not purely symbolic) objects, play an important role.

1. Introduction

In this paper we consider semilinear wave equations in contexts such as asymptotically de Sitter and Kerr–de Sitter spaces as well as asymptotically Minkowski spaces. The word "asymptotically" here does *not* mean that the asymptotic behavior has to be that of exact de Sitter, etc., spaces, or even a perturbation of these at infinity; much more general infinities, which nonetheless possess a similar structure as far as the underlying analysis is concerned, are allowed. Recent progress [Vasy 2013a; Baskin et al. 2014] allows one to set up the analysis of the associated linear problem *globally* as a Fredholm problem, concretely using

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the framework of Melrose's [1993] b-pseudodifferential operators on appropriate compactifications M of these spaces. (The b-analysis itself originates in Melrose's work on the propagation of singularities for the wave equation on manifolds with smooth boundary, and Melrose described a systematic framework for elliptic b-equations. Here "b" refers to analysis based on vector fields tangent to the boundary of the space; we give some details later in the introduction and further details in Section 2A, where we recall the setting of [Vasy 2013a].) This allows one to use the contraction mapping theorem to solve semilinear equations with small data in many cases, since typically the semilinear terms can be considered perturbations of the linear problem. That is, as opposed to solving an evolution equation on time intervals of some length, possibly controlling this length in some manner, and iterating the solution using (almost) conservation laws, we solve the equation globally in one step.

As Fredholm analysis means that one has to control the linear operator L modulo compact errors, which in these settings means modulo terms which are *both* smoother and more decaying, the underlying linear analysis involves both arguments based on the principal symbol of the wave operator and on its so-called (b-)normal operator family, which is a holomorphic family $\widehat{N}(L)(\sigma)$ of operators on ∂M. In settings in which there is an $\mathbb{R}^+$-action in the normal variable and the operator is dilation invariant, this simply means Mellin-transforming in the normal variable. Replacing the normal variable by its logarithm, this is equivalent to a Fourier transform.

At the principal symbol level, one encounters real-principal-type phenomena as well as radial points of the Hamilton flow at the boundary of the compactified underlying space M; these allow for the usual (for wave equations) loss of one (b-)derivative relative to elliptic problems. Physically, in the de Sitter and Kerr–de Sitter-type settings, radial points correspond to a red shift effect. In Kerr–de Sitter spaces there is an additional loss of derivatives due to trapping. On the other hand, the b-normal operator family enters via the poles σ_j of the meromorphic inverse $\widehat{N}(L)(\sigma)^{-1}$; these poles, called *resonances*, determine the decay and growth rates of solutions of the linear problem at ∂M, namely $\Im\sigma_j > 0$ means growing while $\Im\sigma_j < 0$ means decaying solutions. Translated into the nonlinear setting, taking powers of solutions of the linear equation means that growing linear solutions become even more growing, thus the nonlinear problem is uncontrollable; while decaying linear solutions become even more decaying, thus the nonlinear effects become negligible at infinity. Correspondingly, the location of these resonances becomes crucial for nonlinear problems. We note that, in addition to providing solvability of semilinear problems, our results can also be used to obtain the *asymptotic expansion* of the solution.

In short, we present a *systematic approach* to the analysis of semilinear wave and Klein–Gordon equations: Given an appropriate structure of the space at infinity and given that the location of the resonances fits well with the nonlinear terms — see the discussion below — one can solve (suitable) semilinear equations. Thus, the main purpose of this paper is to present the first step towards a general theory for the global study of linear and nonlinear wave-type equations; the semilinear applications we give are meant to show how far we can get in the nonlinear regime using relatively simple means and lend themselves to meaningful comparisons with existing literature; see the discussion below. In particular, our approach readily generalizes to the analysis of *quasilinear* equations, provided one understands the necessary (b-)analysis for nonsmooth metrics. Since the first version of this paper, we described such generalizations in detail in the context of asymptotically de Sitter [Hintz 2013] and asymptotically Kerr–de Sitter [Hintz and Vasy 2014a] spaces.

We now describe our setting in more detail. We consider semilinear wave equations of the form

$$(\Box_g - \lambda)u = f + q(u, du)$$

on a manifold M, where q is (typically, though more general functions are also considered) a polynomial vanishing at least quadratically at $(0, 0)$ (so contains no constant or linear terms, which should be included either in f or in the operator on the left-hand side). The derivative du is measured relative to the metric structure (e.g., when constructing polynomials in it). Here, g and λ fit in one of the following scenarios, which we state slightly informally, with references to the precise theorems. We discuss the terminology afterwards in more detail, but the reader unfamiliar with the terms could drop the word "asymptotically" and "even" to obtain specific examples.

(1) *A neighborhood of the backward light cone from future infinity in an asymptotically de Sitter space*: (This may be called a static region or patch of an asymptotically de Sitter space, even when there is no timelike Killing vector field.) In order to solve the semilinear equation, if $\lambda > 0$ one can let q be an arbitrary polynomial with quadratic vanishing at the origin, or indeed a more general function. If $\lambda = 0$ and q depends on du only, the same conclusion holds. Further, in either case, one obtains an expansion of the solution at infinity. See Theorems 2.25 and 2.37 and Corollary 2.28.

(2) *Kerr–de Sitter space, including a neighborhood of the event horizon, or more general spaces with normally hyperbolic trapping, discussed below*: In the main part of the section we assume $\lambda > 0$ and allow $q = q(u)$ with quadratic vanishing at the origin. We also obtain an expansion at infinity. See Theorems 3.7 and 3.11 and Corollary 3.10. However, in Section 3C we briefly discuss nonlinearities involving derivatives which are appropriately behaved at the trapped set.

(3) *Global* even *asymptotically de Sitter spaces*: These are in some sense the easiest examples as they correspond, via extension across the conformal boundary, to working on a manifold without boundary. Here, $\lambda = \frac{1}{4}(n-1)^2 + \sigma^2$. While the equation is unchanged if one replaces σ by $-\sigma$, the process of extending across the boundary breaks this symmetry, and in Section 4 we mostly consider $\Im\sigma \leq 0$. If $\Im\sigma < 0$ is sufficiently small and the dimension satisfies $n \geq 6$, quadratic vanishing of q suffices; if $n \geq 4$ then cubic vanishing is sufficient. If q does not involve derivatives, then $\Im\sigma \geq 0$ small also works, and if $\Im\sigma > 0$ and $n \geq 5$, or $\Im\sigma = 0$ and $n \geq 6$, then quadratic vanishing of q is sufficient. See Theorems 4.10, 4.12 and 4.15. Using the results from "static" asymptotically de Sitter spaces, quadratic vanishing of q in fact suffices for all $\lambda > 0$, and indeed $\lambda \geq 0$ if $q = q(du)$, but the decay estimates for solutions are lossy relative to the *decay* of the forcing. See Theorem 4.17.

(4) *Nontrapping Lorentzian scattering (generalized asymptotically Minkowski) spaces,* $\lambda = 0$: If $q = q(du)$, we allow q with quadratic vanishing at 0 if $n \geq 5$; and cubic if $n \geq 4$. If $q = q(u)$, we allow q with quadratic vanishing if $n \geq 6$; and cubic if $n \geq 4$. Further, for $q = q(du)$ quadratic satisfying a null condition, $n = 4$ also works. See Theorems 5.12, 5.14 and 5.20.

We now recall these settings in more detail. First — see [Vasy 2010] — an asymptotically de Sitter space is an appropriate generalization of the Riemannian conformally compact spaces of Mazzeo and Melrose [1987], namely a smooth manifold with boundary, $\widetilde{M}$, with interior $\widetilde{M}^\circ$ equipped with a Lorentzian metric $\tilde{g}$, which we take to be of signature $(1, n-1)$ for the sake of definiteness, and with a boundary defining function ρ such that $\hat{g} = \rho^2 \tilde{g}$ is a smooth, symmetric 2-cotensor of signature $(1, n-1)$ up to the boundary of $\widetilde{M}$ and $\hat{g}(d\rho, d\rho) = 1$ (thus, the boundary defining function is timelike, and thus the boundary is spacelike; the last equality makes the curvature asymptotically constant). In addition, $\partial\widetilde{M}$ has two components, $\widetilde{X}_\pm$ (each of which may be a union of connected components), with all null-geodesics $c = c(s)$ of $\tilde{g}$ tending to $\widetilde{X}_+$ as $s \to +\infty$ and to $\widetilde{X}_-$ as $s \to -\infty$, or vice versa. Notice that in the interior of $\widetilde{M}$ the conformal factor ρ^{-2} simply reparameterizes the null-geodesics, so equivalently one can require that null-geodesics of $\hat{g}$ reach $\widetilde{X}_\pm$ at finite parameter values. Analogously to asymptotically hyperbolic spaces, where this was shown by Graham and Lee [1991], on such a space one can always introduce a product decomposition $(\partial\widetilde{M})_z \times [0, \delta)_\rho$ near $\partial\widetilde{M}$ (possibly changing ρ) such that the metric has a warped product structure $\hat{g} = d\rho^2 - h(\rho, z, dz)$, $\tilde{g} = \rho^{-2}\hat{g}$; the metric is called even if h can be taken even in ρ, i.e., a smooth function of ρ^2. We refer to [Guillarmou 2005] for the introduction of even metrics in the asymptotically hyperbolic context and to [Vasy 2010; 2013a; 2014] for further discussion.

Blowing up a point p at $\widetilde{X}_+$, which essentially means introducing spherical coordinates around it, we obtain a manifold with corners $[\widetilde{M}; p]$ with a blow-down map $\beta : [\widetilde{M}; p] \to \widetilde{M}$ that is a diffeomorphism away from the *front face*, which gets mapped to p by β. Just like blowing up the origin in Minkowski space desingularizes the future (or past) light cone, this blow-up desingularizes the backward light cone from p on $\widetilde{M}$, which lifts to a smooth submanifold transversal to the front face on $[\widetilde{M}; p]$ which intersects the front face in a sphere Y. The interior of this lifted backward light cone, at least near the front face, is a generalization of the static patch in de Sitter space, and we refer to a neighborhood M_δ, $\delta > 0$, of the closure of the interior M_+ of the lifted backward light cone in $[\widetilde{M}; p]$ which only intersects the boundary of $[\widetilde{M}; p]$ in the interior of the front face (so M_δ is a noncompact manifold with boundary X_δ and, say, boundary defining function τ) as the "static" asymptotically de Sitter problem. See Figure 1. Via a doubling process, X_δ can be replaced by a compact manifold without boundary, X, and M_δ by $M = X \times [0, \tau_0)_\tau$, an approach taken in [Vasy 2013a], where complex absorption was used; or, indeed, one can instead work

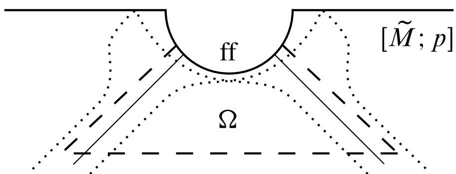

Figure 1. Setup of the "static" asymptotically de Sitter problem. Indicated are the blow-up of $\widetilde{M}$ at p and the front face, the lift of the backward light cone to $[\widetilde{M}; p]$ (solid), and lifts of backward light cones from points near to p (dotted); moreover, $\Omega \subset M$ (dashed boundary) is a submanifold with corners within M (which is not drawn here; see [Vasy 2013a] for a description of M using a doubling procedure in a similar context). The role of Ω is explained in Section 2A.

in a compact region $\Omega \subset M_\delta$ by adding artificial, spacelike boundaries, as we do here in Section 2A. With such an Ω, the distinction between M and M_δ is irrelevant, and we simply write M below.

See [Vasy 2010; 2013a] for relating the "global" and the "static" problems. We note that the lift of $\tilde{g}$ to M in the static region is a Lorentzian b-metric, that is, a smooth symmetric section of signature $(1, n-1)$ of the second tensor power of the b-cotangent bundle, ${}^{\mathrm{b}}T^*M$. The latter is the dual of ${}^{\mathrm{b}}TM$, whose smooth sections are smooth vector fields on M tangent to ∂M; sections of ${}^{\mathrm{b}}T^*M$ are smooth combinations of $d\tau/\tau$ and smooth one-forms on X, relative to a product decomposition $X \times [0, \delta)_\tau$ near $X = \partial M$. See also Section 2A.

As mentioned earlier, the methods of [Vasy 2013a] work in a rather general b-setting, including generalizations of "static" asymptotically de Sitter spaces. Kerr–de Sitter space, described from this perspective in [Vasy 2013a, §6], can be thought of as such a generalization. In particular, it still carries a Lorentzian b-metric, but with a somewhat more complicated structure, of which the only important part for us is that it has trapped rays. More concretely, it is best to consider the bicharacteristic flow in the b-cosphere bundle, ${}^{\mathrm{b}}S^*M$ (projections of null-bicharacteristics being just the null-geodesics), quotienting out by the $\mathbb{R}^+$-action on the fibers of ${}^{\mathrm{b}}T^*M \setminus o$. On the "static" asymptotically de Sitter space, each half of the spherical b-conormal bundle ${}^{\mathrm{b}}SN^*Y$ consists of (a family of) saddle points of the null-bicharacteristic flow (these are called *radial sets*); the stable and unstable directions are normal to ${}^{\mathrm{b}}SN^*Y$ itself, with one of the stable or unstable manifolds being the conormal bundle of the lifted light cone (which plays the role of the *event horizon* in black hole settings), and the other being the characteristic set within the boundary X (so, within the boundary, the radial sets ${}^{\mathrm{b}}SN^*Y$ are actually sources or sinks). Then, on asymptotically de Sitter spaces, all null-bicharacteristics over $\overline{M_+} \setminus X$ either leave Ω in finite time or (if they lie on the conormal bundle of the event horizon) tend to ${}^{\mathrm{b}}SN^*Y$ as the parameter goes to $\pm\infty$, with each bicharacteristic tending to ${}^{\mathrm{b}}SN^*Y$ in at most one direction. The main difference for Kerr–de Sitter space is that there are null-bicharacteristics which do not leave $\overline{M_+} \setminus X$ and do not tend to ${}^{\mathrm{b}}SN^*Y$. On de Sitter–Schwarzschild space (nonrotating black holes) these future-trapped rays project to a sphere, called the photon sphere, times $[0, \delta)_\tau$; on general Kerr–de Sitter space the trapped set deforms, but is still *normally hyperbolic*, a setting studied by Wunsch and Zworski [2011] and Dyatlov [2015].

We refer to [Baskin et al. 2014, §3] and to Section 5A here for a definition of asymptotically Minkowski spaces, but roughly they are manifolds with boundary M with Lorentzian metrics g on the interior M° conformal to a b-metric $\hat{g}$ as $g = \tau^{-2}\hat{g}$, with τ a boundary defining function[1] (so these are Lorentzian *scattering* metrics in the sense of [Melrose 1994], i.e., symmetric cotensors in the second power of the scattering cotangent bundle, and of signature $(1, n-1)$), with a real C^∞ function v defined on M with dv and $d\tau$ linearly independent at $S = \{v = 0, \tau = 0\}$, and with a specific behavior of the metric at S which reflects that of Minkowski space on its radial compactification near the boundary of the light cone at infinity (so S is the light cone at infinity in this greater generality). Concretely, the specific form is

$$\tau^2 g = \hat{g} = v\,\frac{d\tau^2}{\tau^2} - \left(\frac{d\tau}{\tau} \otimes \alpha + \alpha \otimes \frac{d\tau}{\tau}\right) - \tilde{h},$$

[1] In Section 5 we switch to ρ as the boundary defining function for consistency with [Baskin et al. 2014].

where α is a smooth one-form on M, equal to $\frac{1}{2}\,dv$ at S, and $\tilde{h}$ is a smooth 2-cotensor on M that is positive definite on the annihilator of $d\tau$ and dv (which is a codimension 2 space).[2] The difference between the de Sitter-type and Minkowski settings is in part this conformal factor, τ^{-2}, but more importantly, as this conformal factor again does not affect the behavior of the null-bicharacteristics, so one can consider those of $\hat{g}$ on ${}^{\mathrm{b}}S^*M$, at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S (see Section 2) the nature of the radial points is source or sink rather than a saddle point of the flow. (One also makes a nontrapping assumption in the asymptotically Minkowski setting.)

Now we comment on the specific way these settings fit into the b-framework, and the way the various restrictions described above arise:

(1) *Asymptotically "static" de Sitter*: Due to a zero resonance for the linear problem when $\lambda = 0$, which moves to the lower half plane for $\lambda > 0$, in this setting $\lambda > 0$ works in general; $\lambda = 0$ works if q depends on du but not on u. The relevant function spaces are L^2-based b-Sobolev spaces (see Section 2) on the bordification (partial compactification) of the space, or analogous spaces plus a finite expansion. Further, the semilinear terms involving du have coefficients corresponding to the b-structure, i.e., b-objects are used to create functions from the differential forms or, equivalently, b-derivatives of u are used.

(2) *Kerr–de Sitter space*: This is an extension of (1); the framework is essentially the same, with the difference being that there is now trapping corresponding to the "photon sphere". This makes first-order terms in the nonlinearity nonperturbative, unless they are well adapted to the trapping. Thus, we assume $\lambda > 0$. The relevant function spaces are as in the asymptotically de Sitter setting.

(3) *Global* even *asymptotically de Sitter spaces*: In order to get reasonable results, one needs to measure regularity relatively finely, using the module of vector fields tangent to what used to be the conformal boundary in the extension. The relevant function spaces are thus Sobolev spaces with additional (finite) conormal regularity. Further, du has coefficients corresponding to the 0-structure of Mazzeo and Melrose, in the same sense the b-structure was used in (1). The range of λ here is limited by the process of extension across the boundary; for nonlinearities involving u only, the restriction amounts to (at least very slowly) decaying solutions for the linear problem (without extension across the conformal boundary).

Another possibility is to view global de Sitter space as a union of static patches. Here, the b-Sobolev spaces on the static parts translate into 0-Sobolev spaces on the global space, which have weights that are shifted by a dimension-dependent amount relative to the weights of the b-spaces. This approach allows many of the nonlinearities that we can deal with on static parts; however, the resulting decay estimates on u are quite lossy relative to the decay of the forcing term f.

(4) *Nontrapping Lorentzian scattering spaces (generalized asymptotically Minkowski spaces),* $\lambda = 0$: Note that if $\lambda > 0$, the type of the equation changes drastically; it naturally fits into Melrose's scattering algebra[3]

[2] More general, "long-range" scattering metrics also work for the purposes of this paper without any significant changes; the analysis of these is currently being completed by Baskin, Vasy and Wunsch. The difference is the presence of smooth multiples of $\tau\,d\tau^2/\tau^2$ in the metric near $\tau = 0$, $v = 0$. These do not affect the normal operator, but slightly change the dynamics in ${}^{\mathrm{b}}S^*M$. This, however, does not affect the function spaces to be used for our semilinear problem.

[3] In many ways the scattering algebra is actually much better behaved than the b-algebra, in particular it is symbolic in the sense of weights/decay. Thus, with numerical modifications, our methods should extend directly.

rather than the b-algebra which can be used for $\lambda = 0$. While the results here are quite robust and there are no issues with trapping, they are more involved as one needs to keep track of regularity relative to the module of vector fields tangent to the light cone at infinity. The relevant function spaces are b-Sobolev spaces with additional b-conormal regularity corresponding to the aforementioned module. Further, du has coefficients corresponding to Melrose's scattering structure. These spaces, in the special case of Minkowski space, are related to the spaces used by Klainerman [1985], using the infinitesimal generators of the Lorentz group, but, while Klainerman works in an $L^\infty L^2$ setting, we remain purely in a (weighted) L^2-based setting, as the latter is more amenable to the tools of microlocal analysis.

We reiterate that, while the way the four types of spaces fit into it differs somewhat, the underlying linear framework is that of L^2-based b-analysis on manifolds with boundary, except that in the global view of asymptotically de Sitter spaces one can eliminate the boundary altogether.

In order to underline the generality of the method, we emphasize that, corresponding to cases (1) and (2), in b-settings in which one can work on standard b-Sobolev spaces the restrictions on the solvability of the semilinear equations are simply given by the presence of resonances for the Mellin-transformed normal operator in $\Im\sigma \geq 0$, which would allow growing solutions to the equation (with the exception of $\Im\sigma = 0$, in which case the nonlinear iterative arguments produce growth unless the nonlinearity has a special structure), making the nonlinearity nonperturbative and the losses at high energy estimates for this Mellin-transformed operator and the closely related b-principal symbol estimates when one has trapping. (It is these losses that cause the difference in the trapping setting between nonlinearities with or without derivatives.) In particular, the results are necessarily optimal in the nontrapping setting of (1), as shown even by an ODE; see Remark 2.31. In the trapping setting it is not clear precisely what improvements are possible for nonlinearities with derivatives, though, when there are no derivatives in the nonlinearity, we already have no restrictions on the nonlinearity and to this extent the result is optimal.

On Lorentzian scattering spaces, more general function spaces are used and it is not in principle clear whether the results are optimal, but at least comparison with the work of Klainerman [1985; 1986] and Christodoulou [1986] for perturbations of Minkowski space gives consistent results; see the comments below. On global asymptotically de Sitter spaces, the framework of [Vasy 2013a; 2013b] is very convenient for the linear analysis, but it is not clear to what extent it gives optimal results in the nonlinear setting. The reason why more precise function spaces become necessary is the following: There are two basic properties of spaces of functions on manifolds with boundaries, namely differentiability and decay. Whether one can have both at the same time for the linear analysis depends on the (Hamiltonian) dynamical nature of radial points: when defining functions of the corresponding boundaries of the compactified cotangent bundle have opposite character (stable vs. unstable) one can have both at the same time, otherwise not; see Propositions 2.1 and 5.2 for details. For nonlinear purposes, the most convenient setting, in which we are in (1), is if one can work with spaces of arbitrarily high regularity and fast decay, and corresponds to saddle points of the flow in the above sense. In (4), however, working in higher regularity spaces, which is necessary in order to be able to make sense of the nonlinearity, requires using faster-growing (or at least less decaying) weights, which is problematic when dealing with nonlinearities (e.g., polynomials) since multiplication gives even worse growth properties then. Thus, to make the nonlinear analysis work, the function spaces we use need to have more structure; it is a module regularity that is used to capture some weaker regularity in order to enable work in spaces with acceptable weights.

While all results are stated for the scalar equation, analogous results hold in many cases for operators on natural vector bundles, such as the d'Alembertian (or Klein–Gordon operator) on differential forms, since the linear arguments work in general for operators with scalar principal symbol whose subprincipal symbol satisfies appropriate estimates at radial sets — see [Vasy 2013a, Remark 2.1] — though of course for semilinear applications the presence of resonances in the closed upper half plane has to be checked. This already suffices to obtain the well-posedness of the semilinear equations on asymptotically de Sitter spaces that we consider in this paper; for this purpose one needs to know the poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space only. On asymptotically Minkowski spaces, the absence of poles of an asymptotically hyperbolic resolvent in a region has to be checked in addition — see Theorem 5.3 — and the situation depends crucially on the delicate balance of weights and regularity, as alluded to above. Note that, on *perturbations* of Minkowski space, this absence of poles follows from the appropriate behavior of the poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space.

The degree to which these nonlinear problems have been studied differs, with the Minkowski problem (on perturbations of Minkowski space, as opposed to our more general setting) being the most studied. There semilinear and indeed even quasilinear equations are well understood due to the work of Christodoulou [1986] and Klainerman [1985; 1986], with their book [1993] on the global stability of Einstein's equation being one of the main achievements. (We also refer to the work of Lindblad and Rodnianski [2005; 2010] simplifying some of the arguments, of Bieri [2009] relaxing some of the decay conditions, of Wang [2010] obtaining asymptotic expansions, and of Lindblad [2008] for results on a class of quasilinear equations. Hörmander's [1997] book provides further references in the general area. There are numerous works on the *linear* problem, and estimates this yields for the nonlinear problems, such as Strichartz estimates; here we refer to the recent work of Metcalfe and Tataru [2012] for a parametrix construction in low regularity, and references therein.) We obtain results comparable to these (when restricted to the semilinear setting), on a larger class of manifolds; see Remark 5.17. For nonlinearities which do not involve derivatives, slightly stronger results have been obtained, in a slightly different setting, in [Chruściel and Łęski 2006]; see Remark 5.18.

On the other hand, there is little (nonlinear) work on the asymptotically de Sitter and Kerr–de Sitter settings; indeed the only paper the authors are aware of is [Baskin 2013] in roughly comparable generality in terms of the setting, though in *exact* de Sitter space Yagdjian [2009; 2012] has studied a large class of semilinear equations with no derivatives. Baskin's result is for a semilinear equation with no derivatives and a single exponent, using his [2010] parametrix construction, namely u^p with[4] $p = 1 + 4/(n-2)$, and for $\lambda > \frac{1}{4}(n-1)^2$. In the same setting, $p > 1 + 4/(n-1)$ works for us, and thus Baskin's setting is in particular included. Yagdjian works with the explicit solution operator (derived using special functions) in exact de Sitter space, again with no derivatives in the nonlinearity. While there are some exponents that his results cover (for $\lambda > \frac{1}{4}(n-1)^2$, all $p > 1$ work for him) that ours do not directly (but indirectly, via the static model, we in fact obtain such results), the range $\left(\frac{1}{4}(n-1)^2 - \frac{1}{4}, \frac{1}{4}(n-1)^2\right)$ is excluded by

[4]The dimension of the spacetime in Baskin's paper is $n+1$; we continue using our notation above.

him while covered by our work for sufficiently large p. In the (asymptotically) Kerr–de Sitter setting, to our knowledge, there has been no similar semilinear work, however Luk [2013] and Tohaneanu [2012] studied semilinear waves on Kerr spacetimes. We recall finally that there is more work on the linear problem in de Sitter, de Sitter–Schwarzschild and Kerr–de Sitter spaces. We refer to [Vasy 2013a] for more detail; some references are [Polarski 1989; Yagdjian and Galstian 2009; Sá Barreto and Zworski 1997; Bony and Häfner 2008; Vasy 2010; Baskin 2010; Dafermos and Rodnianski 2007; Dyatlov 2011a; 2011b] and Melrose, Sá Barreto and Vasy [Melrose et al. 2014]. Also, while it received more attention, the linear problem on Kerr space does not fit directly into our setting; see the introduction of [Vasy 2013a] for an explanation and for further references, and [Dafermos and Rodnianski 2013] for more background and additional references.

While the basic ingredients of the necessary linear b-analysis were analyzed in [Vasy 2013a], the solvability framework was only discussed in the dilation-invariant setting, and in general the asymptotic expansion results were slightly lossy in terms of derivatives in the non-dilation-invariant case. We remedy these issues in this paper, providing a full Fredholm framework. The key technical tools are the propagation of b-singularities at b-radial points which are saddle points of the flow in ${}^{\mathrm{b}}S^*M$ — see Proposition 2.1 — as well as the b-normally hyperbolic versions, proved in [Hintz and Vasy 2014b], of the semiclassical normally hyperbolic trapping estimates of Wunsch and Zworski [2011]; the rest of the Fredholm setup is discussed in Section 2A in the nontrapping and Section 3A in the normally hyperbolic trapping setting. The analogue of Proposition 2.1 for sources and sinks was already proved in [Baskin et al. 2014, §4]; our Lorentzian scattering metric Fredholm discussion, which relies on this, is in Section 5A.

We emphasize that our analysis would be significantly less cumbersome in terms of technicalities if we were not including Cauchy hypersurfaces and solved a globally well-behaved problem by imposing sufficiently rapid decay at past infinity instead (it is standard to convert a Cauchy problem into a forward solution problem). Cauchy hypersurfaces are only necessary for us if we deal with a problem ill-behaved in the past because complex absorption does not force appropriate forward supports even though it does so at the level of singularities; otherwise we can work with appropriate (weighted) Sobolev spaces. The latter is the case with Lorentzian scattering spaces, which thus provide an ideal example for our setting. It can also be done in the global setting of asymptotically de Sitter spaces, as in setting (3) above, essentially by realizing these as the boundary of the appropriate compactification of a Lorentzian scattering space; see [Vasy 2014]. In the case of Kerr–de Sitter black holes, in the presence of dilation invariance, one has access to a similar luxury: complex absorption does the job, as in [Vasy 2013a]; the key aspect is that it needs to be imposed *outside* the static region we consider. For a general Lorentzian b-metric with a normally hyperbolic trapped set, this may not be easy to arrange, and we do work by adding Cauchy hypersurfaces, even at the cost of the resulting technical complications, which are rather artificial in terms of PDE theory. For perturbations of Kerr–de Sitter space, however, it is possible to forego the latter for well-posedness by an appropriate gluing to complete the space with actual Kerr–de Sitter space in the past for the purposes of functional analysis. We remark that Cauchy hypersurfaces are somewhat ill-behaved for L^2-based estimates, which we use, but match $L^\infty L^2$ estimates quite well, which explains the large role they play in existing hyperbolic theory, such as [Klainerman 1985] or [Hörmander 1985a, Chapter 23.2]. We hope that adopting this more commonly used form of "truncation" of hyperbolic problems will aid the readability of the paper.

We also explain the role that the energy estimates (as opposed to microlocal energy estimates) play. These mostly arise to deal with the artificially introduced boundaries; if other methods are used to truncate the flow, their role reduces to checking that, in certain cases, when the microlocal machinery only guarantees Fredholm properties of the underlying linear operators, the potential finite-dimensional kernel and cokernel are indeed trivial. Asymptotically Minkowski spaces illustrate this best, as the Hamilton flow is globally well behaved there; see Section 5A.

The other key technical tool is the algebra property of b-Sobolev spaces and other spaces with additional conormal regularity. These are stated in the respective sections; the case of the standard b-Sobolev spaces reduces to the algebra property of the standard Sobolev spaces on $\mathbb{R}^n$. Given the algebra properties, the results are proved by applying the contraction mapping theorem to the linear operator.

In summary, the plan of this paper is the following. In each of the sections below we consider one of these settings, and first describe the Sobolev spaces on which one has invertibility for the linear problems of interest, then analyze the algebra properties of these Sobolev spaces, finally proving the solvability of the semilinear equations by checking that the hypotheses of the contraction mapping theorem are satisfied.

2. Asymptotically de Sitter spaces: generalized static model

In this section we discuss solving semilinear wave equations on asymptotically de Sitter spaces from the "static perspective", i.e., in neighborhoods (in a blown-up space) of the backward light cone from a fixed point at future conformal infinity; see Figure 1. The main ingredient is extending the linear theory from that of [Vasy 2013a] in various ways, which is the subject of Section 2A. In the following parts of this section we use this extension to solve semilinear equations and to obtain their asymptotic behavior.

First, however, we recall some of the basics of b-analysis. As a general reference, we refer the reader to [Melrose 1993]. Thus, let M be an n-dimensional manifold with boundary X and denote by $\mathcal{V}_b(M)$ the space of *b-vector fields*, which consists of all vector fields on M which are tangent to X. Elements of $\mathcal{V}_b(M)$ are sections of a natural vector bundle over M, the *b-tangent bundle* ${}^bTM$. Its dual, the *b-cotangent bundle*, is denoted ${}^bT^*M$. In local coordinates, $(\tau, z) \in [0,\infty) \times \mathbb{R}^{n-1}$ near the boundary, the fibers of ${}^bTM$ are spanned by $\tau\partial_\tau, \partial_{z_1}, \dots, \partial_{z_{n-1}}$, with $\tau\partial_\tau$ being a nontrivial b-vector field up to and including $\tau = 0$ (even though it degenerates as an ordinary vector field), while the fibers of ${}^bT^*M$ are spanned by $d\tau/\tau, dz_1, \dots, dz_{n-1}$. A *b-metric* g on M is then simply a nondegenerate section of the second symmetric tensor power of ${}^bT^*M$, that is, of the form

$$g = g_{00}(\tau,z)\,\frac{d\tau^2}{\tau^2} + \sum_{i=1}^{n-1} g_{0i}(\tau,z)\left(\frac{d\tau}{\tau}\otimes dz_i + dz_i \otimes \frac{d\tau}{\tau}\right) + \sum_{i,j=1}^{n-1} g_{ij}(\tau,z)\,dz_i\otimes dz_j, \quad g_{ij} = g_{ji},$$

with smooth coefficients $g_{k\ell}$. In terms of the coordinate $t = -\log\tau \in \mathbb{R}$ — thus $d\tau/\tau = -dt$ — the b-metric g therefore approaches a stationary (t-independent in the local coordinate system) metric exponentially fast as $\tau = e^{-t}$.

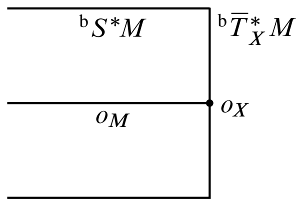

Figure 2. The radially compactified cotangent bundle ${}^{\mathrm{b}}\overline{T}^*M$ near ${}^{\mathrm{b}}\overline{T}^*_X M$; the cosphere bundle ${}^{\mathrm{b}}S^*M$, viewed as the boundary at fiber infinity of ${}^{\mathrm{b}}\overline{T}^*M$, is also shown, as well as the zero section $o_M \subset {}^{\mathrm{b}}\overline{T}^*M$ and the zero section over the boundary $o_X \subset {}^{\mathrm{b}}\overline{T}^*_X M$.

The *b-conormal bundle* ${}^{\mathrm{b}}N^*Y$ of a boundary submanifold $Y \subset X$ of M is the subbundle of ${}^{\mathrm{b}}T^*_Y M$ whose fiber over $p \in Y$ is the annihilator of vector fields on M tangent to Y and X. In local coordinates (τ, z', z''), where Y is defined by $z' = 0$ in X, these vector fields are smooth linear combinations of $\tau\partial_\tau$, $\partial_{z''_j}$, $z'_i \partial_{z'_j}$ and $\tau\partial_{z'_k}$, whose span in ${}^{\mathrm{b}}T_p M$ is that of $\tau\partial_\tau$ and $\partial_{z''_j}$, and thus the fiber of the b-conormal bundle is spanned by the dz'_j, i.e., has the same dimension as the codimension of Y in X (and *not* that in M, corresponding to $d\tau/\tau$ not annihilating $\tau\partial_\tau$).

We define the *b-cosphere bundle* ${}^{\mathrm{b}}S^*M$ to be the quotient of ${}^{\mathrm{b}}T^*M \setminus o$ by the $\mathbb{R}^+$-action; here o is the zero section. Likewise, we define the spherical b-conormal bundle of a boundary submanifold $Y \subset X$ as the quotient of ${}^{\mathrm{b}}N^*Y \setminus o$ by the $\mathbb{R}^+$-action; it is a submanifold of ${}^{\mathrm{b}}S^*M$. A better way to view ${}^{\mathrm{b}}S^*M$ is as the boundary at fiber infinity of the fiber-radial compactification ${}^{\mathrm{b}}\overline{T}^*M$ of ${}^{\mathrm{b}}T^*M$, where the fibers are replaced by their radial compactification; see [Vasy 2013a, §2] and also Section 5A. The b-cosphere bundle ${}^{\mathrm{b}}S^*M \subset {}^{\mathrm{b}}\overline{T}^*M$ still contains the boundary of the compactification of the "old" boundary ${}^{\mathrm{b}}\overline{T}^*_X M$; see Figure 2.

Next, the algebra $\mathrm{Diff}_{\mathrm{b}}(M)$ of *b-differential operators* generated by $\mathcal{V}_{\mathrm{b}}(M)$ consists of operators of the form

$$\mathcal{P} = \sum_{|\alpha|+j\le m} a_\alpha(\tau, z)(\tau D_\tau)^j D_z^\alpha$$

with $a_\alpha \in C^\infty(M)$, writing $D = \frac{1}{i}\partial$ as usual. (With $t = -\log\tau$ as above, the coefficients of $\mathcal{P}$ are thus constant up to exponentially decaying remainders as $t \to \infty$.) Writing elements of ${}^{\mathrm{b}}T^*M$ as

$$\sigma\,\frac{d\tau}{\tau} + \sum_j \zeta_j\, dz_j, \tag{2-1}$$

we have the principal symbol

$$\sigma_{\mathrm{b},m}(\mathcal{P}) = \sum_{|\alpha|+j=m} a_\alpha(\tau, z)\sigma^j \zeta^\alpha,$$

which is a homogeneous degree-m function in ${}^{\mathrm{b}}T^*M \setminus o$. Principal symbols are multiplicative, i.e., $\sigma_{\mathrm{b},m+m'}(\mathcal{P} \circ \mathcal{P}') = \sigma_{\mathrm{b},m}(\mathcal{P})\sigma_{\mathrm{b},m'}(\mathcal{P}')$, and one has a connection between operator commutators and Poisson brackets, to wit

$$\sigma_{\mathrm{b},m+m'-1}(i[\mathcal{P}, \mathcal{P}']) = \mathsf{H}_p p', \quad p = \sigma_{\mathrm{b},m}(\mathcal{P}),\ p' = \sigma_{\mathrm{b},m'}(\mathcal{P}'),$$

where H_p is the extension of the Hamilton vector field from $T^*M^\circ \setminus o$ to ${}^{\mathrm{b}}T^*M \setminus o$, which is thus a homogeneous degree-$(m-1)$ vector field on ${}^{\mathrm{b}}T^*M \setminus o$ tangent to the boundary ${}^{\mathrm{b}}T^*_X M$. In local coordinates (τ, z) on M near X, with b-dual coordinates (σ, ζ) as in (2-1), this has the form

$$\mathsf{H}_p = (\partial_\sigma p)(\tau\partial_\tau) - (\tau\partial_\tau p)\partial_\sigma + \sum_j ((\partial_{\zeta_j} p)\partial_{z_j} - (\partial_{z_j} p)\partial_{\zeta_j}); \tag{2-2}$$

see [Baskin et al. 2014, Equation (3.20)], where a somewhat different notation is used, given by [Baskin et al. 2014, Equation (3.19)].

While elements of $\mathrm{Diff}_{\mathrm{b}}(M)$ commute to leading order in the symbolic sense, they do not commute in the sense of the order of decay of their coefficients. (This is in contrast to the scattering algebra; see [Melrose 1994].) The *normal operator* captures the leading-order part of $\mathcal{P} \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ in the latter sense, namely

$$N(\mathcal{P}) = \sum_{j+|\alpha|\le m} a_\alpha(0,z)(\tau D_\tau)^j D_z^\alpha.$$

One can define $N(\mathcal{P})$ invariantly as an operator on the model space $M_I := [0,\infty)_\tau \times X$ by fixing a boundary defining function of M; see [Vasy 2013a, §3]. Identifying a collar neighborhood of $X \subset M$ with a neighborhood of $\{0\}\times X$ in M_I, we then have $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_{\mathrm{b}}^m(M)$ (near ∂M). Since $N(\mathcal{P})$ is dilation-invariant (equivalently, translation-invariant in $t = -\log\tau$), it is naturally studied via the Mellin transform in τ (equivalently, Fourier transform in $-t$), which leads to the *(Mellin-transformed) normal operator family*

$$\widehat{N}(\mathcal{P})(\sigma) \equiv \widehat{\mathcal{P}}(\sigma) = \sum_{j+|\alpha|\le m} a_\alpha(0,z)\sigma^j D_z^\alpha,$$

which is a holomorphic family of operators $\widehat{\mathcal{P}}(\sigma) \in \mathrm{Diff}^m(X)$.

Passing from $\mathrm{Diff}_{\mathrm{b}}(M)$ to the algebra of *b-pseudodifferential operators* $\Psi_{\mathrm{b}}(M)$ amounts to allowing symbols to be more general functions than polynomials; apart from symbols being smooth functions on ${}^{\mathrm{b}}T^*M$ rather than on T^*M if M was boundaryless, this is entirely analogous to the way one passes from differential to pseudodifferential operators, with the technical details being a bit more involved. One can have a rather accurate picture of b-pseudodifferential operators, however, by considering the following: For $a \in C^\infty({}^{\mathrm{b}}T^*M)$, we say $a \in S^m({}^{\mathrm{b}}T^*M)$ if a satisfies

$$|\partial_w^\alpha \partial_\xi^\beta a(w,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all multiindices } \alpha,\ \beta$$

in any coordinate chart, where w are coordinates in the base and ξ coordinates in the fiber; more precisely, in local coordinates (τ, z) near X, we take $\xi = (\sigma, \zeta)$ as above. We define the quantization $\mathrm{Op}(a)$ of a, acting on smooth functions u supported in a coordinate chart, by

$$\mathrm{Op}(a)u(\tau,z) = (2\pi)^{-n}\int e^{i(\tau-\tau')\tilde\sigma + i(z-z')\zeta}\phi\Big(\frac{\tau-\tau'}{\tau}\Big)a(\tau,z,\tau\tilde\sigma,\zeta)u(\tau',z')\,d\tau'\,dz'\,d\tilde\sigma\,d\zeta,$$

where the τ'-integral is over $[0,\infty)$, and $\phi \in C_c^\infty\big(\big(-\frac{1}{2},\frac{1}{2}\big)\big)$ is identically 1 near 0. The cutoff ϕ ensures that these operators lie in the "small b-calculus" of Melrose, in particular that such quantizations act on weighted b-Sobolev spaces, defined below. For general u, define $\mathrm{Op}(a)u$ using a partition of unity. We write $\mathrm{Op}(a) \in \Psi_{\mathrm{b}}^m(M)$; every element of $\Psi_{\mathrm{b}}^m(M)$ is of the form $\mathrm{Op}(a)$ for some $a \in S^m({}^{\mathrm{b}}T^*M)$ modulo the set $\Psi_{\mathrm{b}}^{-\infty}(M)$ of smoothing operators. We say that a is a *symbol* of $\mathrm{Op}(a)$. The equivalence class of a in $S^m({}^{\mathrm{b}}T^*M)/S^{m-1}({}^{\mathrm{b}}T^*M)$ is invariantly defined on ${}^{\mathrm{b}}T^*M$ and is called the *principal symbol* of $\mathrm{Op}(a)$.

If $A \in \Psi_{\mathrm{b}}^{m_1}(M)$ and $B \in \Psi_{\mathrm{b}}^{m_2}(M)$, then AB, $BA \in \Psi_{\mathrm{b}}^{m_1+m_2}(M)$, while $[A, B] \in \Psi_{\mathrm{b}}^{m_1+m_2-1}(M)$, and its principal symbol is $\frac{1}{i}\mathsf{H}_a b \equiv \frac{1}{i}\{a, b\}$, with H_a as above.

Lastly, we recall the notion of *b-Sobolev spaces*: Fixing a volume b-density ν on M, which locally is a positive multiple of $|(1/\tau)\, d\tau\, dz|$, we define, for $s \in \mathbb{N}$,

$$H_{\mathrm{b}}^s(M) = \{u \in L^2(M, \nu) : V_1 \cdots V_j u \in L^2(M, \nu),\ V_i \in \mathcal{V}_{\mathrm{b}}(M),\ 1 \le i \le j \le s\},$$

which one can extend to $s \in \mathbb{R}$ by duality and interpolation. *Weighted b-Sobolev spaces* are denoted $H_{\mathrm{b}}^{s,\alpha}(M) = \tau^\alpha H_{\mathrm{b}}^s(M)$, that is, their elements are of the form $\tau^\alpha u$ with $u \in H_{\mathrm{b}}^s(M)$. Any b-pseudodifferential operator $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$ defines a bounded linear map $\mathcal{P} : H_{\mathrm{b}}^{s,\alpha}(M) \to H_{\mathrm{b}}^{s-m,\alpha}(M)$ for all $s, \alpha \in \mathbb{R}$. Correspondingly, there is a notion of wave front set $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u) \subset {}^{\mathrm{b}}S^*M$ for a distribution $u \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$, defined analogously to the wave front set of distributions on $\mathbb{R}^n$ or closed manifolds. A point $\varpi \in {}^{\mathrm{b}}S^*M$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u)$ if and only if there exists $\mathcal{P} \in \Psi_{\mathrm{b}}^0(M)$, elliptic at ϖ (i.e., with principal symbol nonvanishing on the ray corresponding to ϖ) such that $\mathcal{P}u \in H_{\mathrm{b}}^{s,\alpha}(M)$. Notice however that we *do* need to have a priori control on the weight α (we are *assuming* $u \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$), which again reflects the lack of commutativity of $\Psi_{\mathrm{b}}(M)$ even to leading order in the sense of decay of coefficients at ∂M.

2A. ***The linear Fredholm framework.*** The goal of this section is to fully extend the results of [Vasy 2013a] on linear estimates for wave equations for b-metrics to non-dilation-invariant settings, and to explicitly discuss Cauchy hypersurfaces, since that work concentrated on complex absorption. Namely, while the results there on linear estimates for wave equations for b-metrics are optimally stated when the metrics and thus the corresponding operators are dilation-invariant, that is, when near $\tau = 0$ the normal operator can be identified with the operator itself — see Vasy's Lemma 3.1 — the estimates for Sobolev derivatives are lossy for general b-metrics in [Vasy 2013a, Proposition 3.5], essentially because one should not treat the difference between the normal operator and the actual operator purely as a perturbation. Therefore, we first strengthen the linear results of Vasy in the non-dilation-invariant setting by analyzing b-radial points which are saddle points of the Hamilton flow. This is similar to [Baskin et al. 2014, §4], where the analogous result was proved when the b-radial points are sources or sinks. This is then used to set up a Fredholm framework for the linear problem. If one is mainly interested in the dilation-invariant case, one can use [Vasy 2013a, Lemma 3.1] in place of Theorems 2.18–2.21 below, either adding the boundary corresponding to H_2 below, or still using complex absorption as was done in [Vasy 2013a].

So suppose $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$ with M a manifold with boundary. (The dilation-invariant analysis of [Vasy 2013a, §2] applies to the Mellin-transformed normal operator $\widehat{\mathcal{P}}(\sigma)$.) Let p be the principal symbol of $\mathcal{P}$, which we assume to be *real-valued*, and let H_p be the Hamilton vector field of p. Let $\tilde{\rho}$ denote a homogeneous defining function of ${}^{\mathrm{b}}S^*M$ of degree -1. Then the rescaled Hamilton vector field

$$V = \tilde{\rho}^{m-1}\mathsf{H}_p$$

is a C^∞ vector field on ${}^{\mathrm{b}}\overline{T}^*M$ away from the 0-section, and it is tangent to all boundary faces. The characteristic set Σ is the zero-set of the smooth function $\tilde{\rho}^m p$ in ${}^{\mathrm{b}}S^*M$. We refer to the flow of V in $\Sigma \subset {}^{\mathrm{b}}S^*M$ as the Hamilton, or (null-)bicharacteristic flow; its integral curves, the (null-)bicharacteristics, are reparameterizations of those of the Hamilton vector field H_p, projected by the quotient map ${}^{\mathrm{b}}T^*M \setminus o \to {}^{\mathrm{b}}S^*M$.

2A1. *Generalized b-radial sets.* The standard propagation of singularities theorem in the characteristic set Σ in the b-setting is that, for $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, within Σ, $\mathrm{WF}_{\mathrm{b}}^{s,r}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u)$ is a union of maximally extended integral curves (i.e., null-bicharacteristics) of $\mathcal{P}$. This is vacuous at points where V vanishes (as a smooth vector field); these points are called *radial points*, since, at such a point, H_p itself (on ${}^{\mathrm{b}}T^*M \setminus o$) is radial, that is, is a multiple of the generator of the dilations of the fiber of the b-cotangent bundle. At a radial point α, V acts on the ideal $\mathcal{I}$ of C^∞ functions vanishing at α, and thus on $T_\alpha^*{}^{\mathrm{b}}\overline{T}^*M$, which can be identified with $\mathcal{I}/\mathcal{I}^2$. Since V is tangent to both boundary hypersurfaces, given by $\tau = 0$ and $\tilde{\rho} = 0$, $d\tau$ and $d\tilde{\rho}$ are automatically eigenvectors of the linearization of V. We are interested in a generalization of the situation, in which we have a smooth submanifold L of ${}^{\mathrm{b}}S_X^*M$ consisting of radial points which are a source or sink for V within ${}^{\mathrm{b}}T_X^*M$ but, if a source — so in particular $d\tilde{\rho}$ is in an unstable eigenspace — then $d\tau$ is in the (necessarily one-dimensional) stable eigenspace, and vice versa. Thus, L is a saddle point of the Hamilton flow.

In view of the bicharacteristic flow on Kerr–de Sitter space (which, unlike the nonrotating de Sitter–Schwarzschild black holes, does not have this precise radial point structure), it is important to be slightly more general, as in [Vasy 2013a, §2.2]. Thus, we assume that dp does not vanish where p does, namely, at Σ, and is linearly independent of $d\tau$ at $\{\tau = 0, p = 0\} = \Sigma \cap {}^{\mathrm{b}}S_X^*M$, so Σ is a smooth submanifold of ${}^{\mathrm{b}}S^*M$ transversal to ${}^{\mathrm{b}}S_X^*M$. For L, assume simply that $L = L_+ \cup L_-$, where $L_\pm = \mathcal{L}_\pm \cap {}^{\mathrm{b}}S_X^*M$ are smooth disjoint submanifolds of ${}^{\mathrm{b}}S_X^*M$ and $\mathcal{L}_\pm$ are smooth disjoint submanifolds of Σ transversal to ${}^{\mathrm{b}}S_X^*M$ (these play the role of the two halves of the conormal bundles of event horizons), defined locally near ${}^{\mathrm{b}}S_X^*M$, with $\tilde{\rho}^{m-1}\mathsf{H}_p$ tangent to $\mathcal{L}_\pm$, with a homogeneous degree-zero quadratic defining function ρ_0 (explained below) of $\mathcal{L}$ within Σ such that

$$\tilde{\rho}^{m-2}\mathsf{H}_p\tilde{\rho}|_{L_\pm} = \mp\beta_0 \quad \text{and} \quad -\tilde{\rho}^{m-1}\tau^{-1}\mathsf{H}_p\tau|_{L_\pm} = \mp\tilde{\beta}\beta_0, \quad \beta_0, \tilde{\beta} \in C^\infty(L_\pm) \text{ with } \beta_0, \tilde{\beta} > 0, \tag{2-3}$$

and, with $\beta_1 > 0$,

$$\mp\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0 - \beta_1\rho_0 \tag{2-4}$$

is nonnegative modulo cubic vanishing terms at $L_\pm$. Here, the phrase "quadratic defining function ρ_0" means that ρ_0 vanishes quadratically at $\mathcal{L}$ (and vanishes only at $\mathcal{L}$), with the vanishing nondegenerate, in the sense that the Hessian is positive definite, corresponding to ρ_0 being a sum of squares of linear defining functions whose differentials span the conormal bundle of $\mathcal{L}$ within Σ.

Under these assumptions, L_- is a source and L_+ is a sink within ${}^{\mathrm{b}}S_X^*M$, in the sense that nearby bicharacteristics *within* ${}^{\mathrm{b}}S_X^*M$ all tend to $L_\pm$ as the parameter along them goes to $\pm\infty$, but at L_- there is also a stable, and at L_+ an unstable, manifold, namely $\mathscr{L}_-$ and $\mathscr{L}_+$. Indeed, bicharacteristics in $\mathscr{L}_\pm$ remain there by the tangency of $\tilde\rho^{m-1}\mathsf{H}_p$ to $\mathscr{L}_\pm$; further, $\tau \to 0$ along them as the parameter goes to $\mp\infty$ by (2-3), at least sufficiently close to $\tau = 0$, since $L_\pm$ are defined in $\mathscr{L}_\pm$ by $\tau = 0$.

In order to simplify the statements, we assume that

$$\tilde\beta \ \text{ is constant on } \ L_\pm, \quad \tilde\beta = \beta > 0;$$

we refer the reader to [Vasy 2013a, Equations (2.5)–(2.6)] and the discussion throughout that paper, where a general $\tilde\beta$ is allowed, at the cost of either $\sup\tilde\beta$ or $\inf\tilde\beta$ playing a role in various statements depending on signs. Finally, we assume that $\mathcal{P} - \mathcal{P}^* \in \Psi_{\mathrm{b}}^{m-2}(M)$ for convenience (with respect to some b-metric), as this is the case for the Klein–Gordon equation.[5]

Proposition 2.1. *Suppose $\mathcal{P}$ is as above.*

If $s \geq s'$, $s' - \frac{1}{2}(m-1) > \beta r$ and $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u) = \varnothing$ and $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s',r}(u) = \varnothing$, and, in a neighborhood of $L_\pm$, $\mathscr{L}_\pm \cap \{\tau > 0\}$ are disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

On the other hand, if $s - \frac{1}{2}(m-1) < \beta r$ and $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u) = \varnothing$ and a punctured neighborhood of $L_\pm$ in $\Sigma \cap {}^{\mathrm{b}}S^*_X M$, with $L_\pm$ removed, is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

Remark 2.2. The decay order r plays the role of $-\Im\sigma$ in [Vasy 2013a] in view of the Mellin transform in the dilation-invariant setting identifying weighted b-Sobolev spaces of weight r with semiclassical Sobolev spaces on the boundary on the line $\Im\sigma = -r$; see [ibid., Equation (3.8)–(3.9)]. Thus, the threshold regularity in this proposition is a direct translation of that in Vasy's Propositions 2.3–2.4.

Proof. We remark first that $\tilde\rho^{m-1}\mathsf{H}_p\rho_0$ vanishes quadratically on $\mathscr{L}_\pm$, since $\tilde\rho^{m-1}\mathsf{H}_p$ is tangent to $\mathscr{L}_\pm$ and ρ_0 itself vanishes there quadratically. Further, this quadratic expression is positive definite near $\tau = 0$ since it is so at $\tau = 0$. Correspondingly, we can strengthen (2-4) to

$$\mp\tilde\rho^{m-1}\mathsf{H}_p\rho_0 - \tfrac{1}{2}\beta_1\rho_0 \tag{2-5}$$

being nonnegative modulo cubic terms vanishing at $\mathscr{L}_\pm$ in a neighborhood of $\tau = 0$.

Notice next that, using (2-5) in the first case and (2-3) in the second, and that $L_\pm$ is defined in Σ by $\tau = 0$ and $\rho_0 = 0$, there exist $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\alpha \in \Sigma, \quad \rho_0(\alpha) < \delta_0, \quad \tau(\alpha) < \delta_1 \quad \text{and} \quad \rho_0(\alpha) \neq 0 \quad \Longrightarrow \quad (\mp\tilde\rho^{m-1}\mathsf{H}_p\rho_0)(\alpha) > 0$$

and

$$\alpha \in \Sigma, \quad \rho_0(\alpha) < \delta_0 \quad \text{and} \quad \tau(\alpha) < \delta_1 \quad \Longrightarrow \quad (\pm\tilde\rho^{m-1}\tau^{-1}\mathsf{H}_p\tau)(\alpha) > 0.$$

[5] The natural assumption is that the principal symbol of $\frac{1}{2i}(\mathcal{P} - \mathcal{P}^*) \in \Psi_{\mathrm{b}}^{m-1}(M)$ at $L_\pm$ is

$$\pm\hat\beta\beta_0\tilde\rho^{-m+1}, \quad \hat\beta \in C^\infty(L_\pm).$$

If $\hat\beta$ vanishes, Proposition 2.1 is valid without a change; otherwise, it shifts the threshold quantity $s - \frac{1}{2}(m-1) - \beta r$ below in Proposition 2.1 to $s - \frac{1}{2}(m-1) - \beta r + \hat\beta$ if $\hat\beta$ is constant, with modifications as in [Vasy 2013a, Proof of Propositions 2.3–2.4] otherwise.

Similarly to [Vasy 2013a, Proof of Propositions 2.3–2.4], which is not in the b-setting, and [Baskin et al. 2014, Proof of Proposition 4.4], which is, but concerns only sources and sinks (corresponding to Minkowski-type spaces), we consider commutants

$$C \in \tau^{-r}\Psi_{\mathrm{b}}^{s-(m-1)/2}(M) = \Psi_{\mathrm{b}}^{s-(m-1)/2,-r}(M)$$

with principal symbol

$$c = \phi(\rho_0)\phi_0(p_0)\phi_1(\tau)\tilde{\rho}^{-s+(m-1)/2}\tau^{-r}, \quad p_0 = \tilde{\rho}^m p,$$

where $\phi_0 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi' \le 0$ in $[0,\infty)$ and ϕ supported in $(-\delta_0,\delta_0)$, while $\phi_1 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi_1' \le 0$ in $[0,\infty)$ and ϕ_1 supported in $(-\delta_1,\delta_1)$, so that

$$\alpha \in \operatorname{supp} d(\phi\circ\rho_0) \cap \operatorname{supp}(\phi_1\circ\tau)\cap\Sigma \quad\Longrightarrow\quad \mp(\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0)(\alpha) > 0$$

and

$$\pm\tilde{\rho}^{m-1}\tau^{-1}\mathsf{H}_p\tau$$

remains positive on $\operatorname{supp}(\phi_1\circ\tau)\cap\operatorname{supp}(\phi\circ\rho_0)$.

The main contribution then comes from the weights, which give

$$\tilde{\rho}^{m-1}\mathsf{H}_p(\tilde{\rho}^{-s+(m-1)/2}\tau^{-r}) = \mp\bigl(-s+\tfrac{1}{2}(m-1)+\beta r\bigr)\beta_0\tilde{\rho}^{-s+(m-1)/2}\tau^{-r},$$

where the sign of the factor in parentheses on the right-hand side being negative (resp. positive) gives the first (resp. second) case of the statement of the proposition. Further, the sign of the term in which $\phi_1(\tau)$ (resp. $\phi(\rho_0)$) gets differentiated, yielding $\pm\tau\tilde{\beta}\beta_0\phi_1'(\tau)$ (resp. $\phi'(\rho_0)\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0$) is, when $s-\frac{1}{2}(m-1)-\beta r > 0$, the opposite of (resp. same as) these terms, while when $s-\frac{1}{2}(m-1)-\beta r < 0$, it is the same as (resp. opposite of) these terms. Correspondingly,

$$\begin{aligned}\sigma_{2s}(i[\mathscr{P},C^*C]) = \mp 2\bigl(&-\beta_0\bigl(s-\tfrac{1}{2}(m-1)-\beta r\bigr)\phi\phi_0\phi_1 - \beta_0\tilde{\beta}\tau\phi\phi_0\phi_1' \\ &\mp(\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0)\phi'\phi_0\phi_1 + m\beta_0 p_0\phi\phi_0'\phi_1\bigr)\phi\phi_0\phi_1\tilde{\rho}^{-2s}\tau^{-2r}.\end{aligned}$$

We can regularize using $S_\epsilon \in \Psi_{\mathrm{b}}^{-\delta}(M)$ for $\epsilon > 0$, uniformly bounded in $\Psi_{\mathrm{b}}^0(M)$, converging to Id in $\Psi_{\mathrm{b}}^{\delta'}(M)$ for $\delta' > 0$, with principal symbol $(1+\epsilon\tilde{\rho}^{-1})^{-\delta}$, as in [Vasy 2013a, Proof of Propositions 2.3–2.4], where the only difference was that the calculation was on $X = \partial M$, and thus the pseudodifferential operators were standard ones, rather than b-pseudodifferential operators. The a priori regularity assumption on $\mathrm{WF}_{\mathrm{b}}^{s',r}(u)$ arises as the regularizer has the opposite sign as compared to the contribution of the weights, thus the amount of regularization one can do is limited. The positive commutator argument then proceeds completely analogously to [Vasy 2013a, Proof of Propositions 2.3–2.4], except that, as in that reference, one has to assume a priori bounds on the term with the sign opposite to that of $s-\frac{1}{2}(m-1)-\beta r$, of which there is exactly one for either sign (unlike in [Vasy 2013a], in which only $s-\frac{1}{2}(m-1)+\beta\Im\sigma < 0$ has such a term), thus on $\Sigma\cap\operatorname{supp}(\phi_1'\circ\tau)\cap\operatorname{supp}(\phi\circ\rho_0)$ when $s-\frac{1}{2}(m-1)-\beta r > 0$ and on $\Sigma\cap\operatorname{supp}(\phi_1\circ\tau)\cap\operatorname{supp}(\phi'\circ\rho_0)$ when $s-\frac{1}{2}(m-1)-\beta r < 0$.

Using the openness of the complement of the wave front set, we can finally choose ϕ and ϕ_1 (satisfying the support conditions, among others) so that the a priori assumptions are satisfied, choosing ϕ_1 first and then shrinking the support of ϕ in the first case, with the choice being made in the opposite order in the second case, completing the proof of the proposition. □

2A2. *Complex absorption.* In order to have good Fredholm properties we either need a complete Hamilton flow, or need to "stop it" in a manner that gives suitable estimates; one may want to do the latter to avoid global assumptions on the flow on the ambient space. The microlocally best-behaved version is given by complex absorption; it is microlocal, works easily with Sobolev spaces of arbitrary order, and makes the operator elliptic in the absorbing region, giving rise to very convenient analysis. The main downside of complex absorption is that it does not automatically give forward mapping properties for the support of solutions in settings like the wave equation, even though at the level of singularities it does have the desired forward property. It was used extensively in [Vasy 2013a] — in the dilation-invariant setting, the bicharacteristics on $X \times (0, \infty)_\tau$ are controlled (by the invariance) as $\tau \to \infty$ as well as when $\tau \to 0$, and thus one need not use complex absorption there but, instead, decay as $\tau \to \infty$ (corresponding to growth as $\tau \to 0$ on these dilation-invariant spaces) gives the desired forward property; complex absorption was only used to cut off the flow within X. Here we want to localize in τ as well and, while complex absorption can achieve this, it loses the forward *support* character of the problem. Thus, complex absorption will not be of use to us when solving semilinear forward problems later on; however, as it is conceptually much cleaner, we discuss Fredholm properties using it first before turning to adding artificial (spacelike) boundary hypersurfaces in the next section, which allow for the solution of forward problems but require additional technicalities.

Thus, we now consider $\mathcal{P} - i\mathcal{Q} \in \Psi_b^m(M)$ and $\mathcal{Q} \in \Psi_b^m(M)$, with real principal symbol q, being the complex absorption similar to [Vasy 2013a, §§2.2 and 2.8]; we assume that $\mathrm{WF}_b'(\mathcal{Q}) \cap L = \emptyset$. Here the semiclassical version, discussed in the above work with further references there, is a close parallel to our b-setting; it is essentially equivalent to the b-setting in the special case that $\mathcal{P}$ and $\mathcal{Q}$ are dilation-invariant, for then the Mellin transform gives rise exactly to the semiclassical problem as the Mellin-dual parameter goes to infinity. Thus, we assume that the characteristic set Σ of $\mathcal{P}$ has the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,$$

with each of $\Sigma_\pm$ being a union of connected components and

$$\mp q \geq 0 \quad \text{near} \quad \Sigma_\pm.$$

Recall from [Vasy 2013a, §2.5], which in turn is a simple modification of the semiclassical results of Nonnenmacher and Zworski [2009], and Datchev and Vasy [2012], that, under these sign conditions on q, estimates can be propagated in the backward direction along the Hamilton flow on Σ_+ and in the forward direction for Σ_-, or, phrased as a wave front set statement (the property of being singular propagates in the opposite direction as the property of being regular!), $\mathrm{WF}^s(u)$ is invariant in $(\Sigma_+ \setminus {}^{\mathrm{b}}S_X^*M) \setminus \mathrm{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow, and is invariant in $(\Sigma_- \setminus {}^{\mathrm{b}}S_X^*M) \setminus \mathrm{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow. (That is, the invariance is away from the boundary X; we address the behavior at the boundary in the rest of the paragraph.) Since this is a principal symbol argument, given in [Vasy 2013a, §2.5; Datchev and Vasy 2012, Lemma 5.1], its extension to the b-setting only requires minimal changes. Namely, assuming one is away from radial points, as one may (since at these the statement is vacuous), one constructs the principal symbol c of the commutant on ${}^{\mathrm{b}}T^*M \setminus o$ as a C^∞ function c_0 on ${}^{\mathrm{b}}S^*M$ with derivative of a fixed sign along the Hamilton flow in the region where one wants to obtain the estimate (exactly the same way as for real-principal-type proofs) multiplied by weights in τ and $\tilde{\rho}$, making the Hamilton derivative of c_0 large relative to c_0 to control the error terms from the weights, and computes $\langle u, -i[C^*C, \widetilde{\mathcal{P}}]u\rangle$, where $\widetilde{\mathcal{P}}$ is the symmetric part of $\mathcal{P} - i\mathcal{Q}$ (so has principal symbol p) and $\widetilde{\mathcal{Q}}$ is the antisymmetric part. This gives

$$-2\Re\langle u, iC^*C(\mathcal{P} - i\mathcal{Q})u\rangle - 2\Re\langle u, C^*C\widetilde{\mathcal{Q}}u\rangle.$$

The issue here is that the second term on the right-hand side involves $C^*C\widetilde{\mathcal{Q}}$, which is one order higher than $[C^*C, \widetilde{\mathcal{P}}]$, so, while it itself has a desirable sign, one needs to be concerned about subprincipal terms.[6] However, one rewrites

$$2\Re\langle u, C^*C\widetilde{\mathcal{Q}}u\rangle = 2\Re\langle u, C^*\widetilde{\mathcal{Q}}Cu\rangle + 2\Re\langle u, C^*[C, \widetilde{\mathcal{Q}}]u\rangle.$$

Now, the first term is positive modulo a controllable error by the sharp Gårding inequality or if one arranges that q is the square of a symbol. This controllability claim uses the derivative of c, arising in the symbol of the commutator with $\widetilde{\mathcal{P}}$, to provide the control: since $\widetilde{\mathcal{Q}}$ is positive modulo an operator one order lower and in the term involving this operator, the principal symbol c of C is not differentiated, writing c as c_0 times a weight, where c_0 is homogeneous of degree zero, and taking the derivative of c_0 large relative to c_0, as is already used to control weights, etc., controls this error term (modulo which we have positivity) as well. On the other hand, the second can be rewritten in terms of $[C, [C, \widetilde{\mathcal{Q}}]]$, $(C^* - C)[C, \widetilde{\mathcal{Q}}]$, etc., which are all controllable as they drop *two* orders relative to the product $C^*C\widetilde{\mathcal{Q}}$. This gives rise to the result, namely that, for $u \in H_{\mathrm{b}}^{-\infty,r}$, $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ is invariant in $\Sigma_+ \setminus \mathrm{WF}^{s-m+1,r}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow and in $\Sigma_- \setminus \mathrm{WF}^{s-m+1,r}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow.

In analogy with [Vasy 2013a, Definition 2.12], we say that $\mathcal{P} - i\mathcal{Q}$ is *nontrapping* if all bicharacteristics in Σ from any point in $\Sigma \setminus (L_+ \cup L_-)$ flow to $\mathrm{Ell}(q) \cup L_+ \cup L_-$ in both the forward and backward directions (i.e., either enter $\mathrm{Ell}(q)$ in finite time or tend to $L_+ \cup L_-$). Notice that, as $\Sigma_\pm$ are closed under the Hamilton flow, bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward, in Σ_+ (resp. backward, in Σ_-), direction. Indeed, by the nontrapping hypothesis, these bicharacteristics have to reach the elliptic set of $\mathcal{Q}$ as they cannot tend to L_+ (resp. L_-): $\mathcal{L}_+$ and $\mathcal{L}_-$ are unstable (resp. stable) manifolds and these bicharacteristics cannot enter the boundary — which is preserved by the flow — so cannot lie in the stable (resp. unstable) manifolds of $L_+ \cup L_-$, which are within ${}^{\mathrm{b}}S^*_X M$. Similarly, bicharacteristics in $(\Sigma \cap {}^{\mathrm{b}}S^*_X M) \setminus (L_+ \cup L_-)$ necessarily reach the elliptic set

[6] In fact, as the principal symbol of $C^*C\widetilde{\mathcal{Q}}$ is real, the real part of its subprincipal symbol is well defined and is the real part of c^2q, where c and q include the real parts of their subprincipal terms, and is all that matters for this argument, so one could proceed symbolically.

of $\mathcal{Q}$ in the backward, in Σ_+ (resp. forward, in Σ_-), direction. Then, for s and r satisfying

$$s - \tfrac{1}{2}(m-1) > \beta r,$$

one has an estimate

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \le C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s',r}} \tag{2-6}$$

provided one assumes $s' < s$ and

$$s' - \tfrac{1}{2}(m-1) > \beta r, \quad u \in H_{\mathrm{b}}^{s',r}.$$

Indeed, this is a simple consequence of $u \in H_{\mathrm{b}}^{s',r}$ and $(\mathcal{P} - i\mathcal{Q})u \in H_{\mathrm{b}}^{s-m+1,r}$ implying $u \in H_{\mathrm{b}}^{s,r}$ via the closed graph theorem; see [Hörmander 1985b, Proof of Theorem 26.1.7; Vasy 2013b, §4.3]. This implication in turn holds as, on the elliptic set of $\mathcal{Q}$, one has the stronger statement $u \in H_{\mathrm{b}}^{s+1,r}$ under these conditions, and then, using real-principal-type propagation of regularity in the *backward* direction on Σ_+ and the *forward* direction on Σ_-, one can propagate the microlocal membership of $H_{\mathrm{b}}^{s,r}$ (i.e., the absence of the corresponding wave front set) in the backward (resp. forward) direction on Σ_+ (resp. Σ_-). Since bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward (resp. backward) direction, and thus one has $H_{\mathrm{b}}^{s,r}$ membership along them by what we have shown, Proposition 2.1 extends this membership to $L_\pm$, and hence to a neighborhood of these, and by our nontrapping assumption every bicharacteristic enters either this neighborhood of $L_\pm$ or the elliptic set of $\mathcal{Q}$ in finite time in the backward (resp. forward) direction, so by the real-principal-type propagation of singularities we have the claimed microlocal membership everywhere.

Reversing the direction in which one propagates estimates, one also has a similar estimate for the adjoint $\mathcal{P}^* + i\mathcal{Q}^*$, except now one needs to have

$$s - \tfrac{1}{2}(m-1) < \beta r$$

in order to propagate through the saddle points in the opposite direction, that is, from within ${}^{\mathrm{b}}S^*_X M$ to $\mathcal{L}_\pm$. Then, for $s' < s$,

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \le C\|(\mathcal{P}^* + i\mathcal{Q}^*)u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s',r}}. \tag{2-7}$$

The issue with these estimates is that $H_{\mathrm{b}}^{s,r}$ does not include compactly into the error term $H_{\mathrm{b}}^{s',r}$ on the right-hand side, due to the lack of additional decay. Thus, these estimates are insufficient to show Fredholm properties, which in fact do not hold in general.

We thus further assume that there are no poles of the inverse of the Mellin conjugate $\widehat{(\mathcal{P} - i\mathcal{Q})}(\sigma)$ of the normal operator $N(\mathcal{P} - i\mathcal{Q})$ on the line $\Im\sigma = -r$. Here we refer to [Vasy 2013a, §3.1] for a brief discussion of the normal operator and the Mellin transform; this cited section also contains more detailed references to [Melrose 1993]. Then, using the Mellin transform, which is an isomorphism between weighted b-Sobolev spaces and semiclassical Sobolev spaces (see Equations (3.8)–(3.9) in [Vasy 2013a]) and the estimates for $(\mathcal{P}-i\mathcal{Q})\widehat{\ }(\sigma)$ (including the high-energy, i.e., semiclassical, estimates,[7] all of which is discussed in detail in [Vasy 2013a, §2] — the high energy assumptions of [Vasy 2013a, §2] hold by our assumptions on the b-flow at ${}^{\mathrm{b}}S^*_X M$ — and which imply that, for all but a discrete set of r, the aforementioned lines do not contain such poles), we obtain that, on $\mathbb{R}^+_\rho \times \partial M$,

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \le C\|N(\mathcal{P}-i\mathcal{Q})v\|_{H_{\mathrm{b}}^{s-m+1,r}} \tag{2-8}$$

when

$$s-\tfrac{1}{2}(m-1) > \beta r.$$

Again, we have an analogous estimate for $N(\mathcal{P}^*+i\mathcal{Q}^*)$:

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \le C\|N(\mathcal{P}^*+i\mathcal{Q}^*)v\|_{H_{\mathrm{b}}^{s-m+1,r}} \tag{2-9}$$

provided $-r$ is not the imaginary part of a pole of the inverse of $(\mathcal{P}^*+i\mathcal{Q}^*)\widehat{\ }$ and provided

$$s-\tfrac{1}{2}(m-1) < \beta r.$$

As $(\mathcal{P}^*+i\mathcal{Q}^*)\widehat{\ }(\sigma) = (\widehat{\mathcal{P}}-i\widehat{\mathcal{Q}})^*(\bar\sigma)$ — see the discussion in [Vasy 2013a] preceding Equation (3.25) — the requirement on $-r$ is the same as r not being the imaginary part of a pole of the inverse of $\widehat{\mathcal{P}}-i\widehat{\mathcal{Q}}$.

We apply these results by first letting $\chi \in C_c^\infty(M)$ be identically 1 near ∂M supported in a collar neighborhood of ∂M, which we identify with $(0,\epsilon)_\tau \times \partial M$ of the normal operator space. Then, *assuming* $s'-\frac{1}{2}(m-1) > \beta r$,

$$\|u\|_{H_{\mathrm{b}}^{s',r}} \le \|\chi u\|_{H_{\mathrm{b}}^{s',r}} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}} \le C\|N(\mathcal{P}-i\mathcal{Q})\chi u\|_{H_{\mathrm{b}}^{s'-m+1,r}} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}}. \tag{2-10}$$

Now, if $K = \operatorname{supp}(1-\chi)$, then

$$\|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}} \le C\|u\|_{H^{s'}(K)} \le C'\|u\|_{H_{\mathrm{b}}^{s',\tilde r}} \le C''\|u\|_{H_{\mathrm{b}}^{s'+1,\tilde r}}$$

for any $\tilde r$. On the other hand, $N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}) \in \tau\Psi_{\mathrm{b}}^m([0,\epsilon)\times\partial M)$, so

$$\begin{aligned} N(\mathcal{P}-i\mathcal{Q})\chi u &= (\mathcal{P}-i\mathcal{Q})\chi u + (N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}))\chi u \\ &= \chi(\mathcal{P}-i\mathcal{Q})u + [\mathcal{P}-i\mathcal{Q},\chi]u + (N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}))\chi u \end{aligned}$$

plus the fact that $[\mathcal{P}-i\mathcal{Q},\chi]$ is supported in K and $\|\chi(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}} \le \|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}}$ show that, for all $\tilde r$,

$$\|N(\mathcal{P}-i\mathcal{Q})\chi u\|_{H_{\mathrm{b}}^{s'-m+1,r}} \le \|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,\tilde r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,r-1}}. \tag{2-11}$$

Combining (2-6), (2-10) and (2-11), we deduce that (with new constants, and taking s' sufficiently small and $\tilde r = r-1$)

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \le C\|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,r-1}}, \tag{2-12}$$

[7] The high-energy estimates are actually implied by b-principal symbol-based estimates on the normal operator space $M_\infty = X\times\overline{\mathbb{R}^+}$, $X=\partial M$, on spaces $\tau^r H_{\mathrm{b}}^s(M_\infty)$ corresponding to $\Im\sigma = -r$, but we do not explicitly discuss this here.

where now the inclusion $H_b^{s,r} \to H_b^{s'+1,r-1}$ is compact when we choose, as we may, $s' < s-1$, requiring, however, $s' - \frac{1}{2}(m-1) > \beta r$. Recall that this argument required that s, r and s' satisfied the requirements preceding (2-6) and that $-r$ was not the imaginary part of any pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$.

Analogous estimates hold for $(\mathcal{P} - i\mathcal{Q})^*$, where now we write $\tilde{s}$, $\tilde{r}$ and $\tilde{s}'$ for the Sobolev orders for the eventual application:

$$\|u\|_{H_b^{\tilde{s},\tilde{r}}} \leq C\|(\mathcal{P} - i\mathcal{Q})^* u\|_{H_b^{\tilde{s}-m+1,\tilde{r}}} + C\|u\|_{H_b^{\tilde{s}'+1,\tilde{r}-1}} \tag{2-13}$$

provided $\tilde{s}$ and $\tilde{r}$ in place of s and r satisfy the requirements stated before (2-7), and provided $-\tilde{r}$ is not the imaginary part of a pole of $(\mathcal{P}^* + i\mathcal{Q}^*)\widehat{\ }$ (i.e., $\tilde{r}$ of $\widehat{\mathcal{P}} - i\widehat{\mathcal{Q}}$). Note that we *do not* have a stronger requirement for $\tilde{s}'$, unlike for s' above, since upper bounds for s imply those for $s' \leq s$.

Via a standard functional analytic argument — see [Hörmander 1985b, Proof of Theorem 26.1.7] and also [Vasy 2013a, §2.6] in the present context — we thus obtain Fredholm properties of $\mathcal{P} - i\mathcal{Q}$, in particular solvability, modulo a (possible) finite-dimensional obstruction in $H_b^{s,r}$ if

$$s - \tfrac{1}{2}(m-1) - 1 > \beta r. \tag{2-14}$$

Concretely, we take $\tilde{s} = m-1-s$, $\tilde{r} = -r$, and $s' < s-1$ sufficiently close to $s-1$ that $s' - \frac{1}{2}(m-1) > \beta r$ (which is possible by (2-14)). Thus, $s - \frac{1}{2}(m-1) > \beta r$ means $\tilde{s} - \frac{1}{2}(m-1) = \frac{1}{2}(m-1) - s < -\beta r = \beta\tilde{r}$, so the space on the left-hand side of (2-12) is dual to that in the first term on the right-hand side of (2-13), and the same for the equations interchanged, and notice that the condition on the poles of the inverse of the Mellin-transformed normal operators is the same for both $\mathcal{P} - i\mathcal{Q}$ and $\mathcal{P}^* + i\mathcal{Q}^*$: $-r$ is not the imaginary part of a pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$. Let

$$\mathcal{Y}^{s,r} = H_b^{s,r}(M), \quad \mathcal{X}^{s,r} = \{u \in H_b^{s,r}(M) : (\mathcal{P} - i\mathcal{Q})u \in H_b^{s-1,r}(M)\},$$

and note that $\mathcal{Y}^{s,r}$ and $\mathcal{X}^{s,r}$ are complete, where, in the case of $\mathcal{X}^{s,r}$, the natural norm is

$$\|u\|^2_{\mathcal{X}^{s,r}} = \|u\|^2_{H_b^{s,r}(M)} + \|(\mathcal{P} - i\mathcal{Q})u\|^2_{H_b^{s-1,r}(M)};$$

see Remark 2.19. Our discussion thus far yields:

Proposition 2.3. *Suppose that $\mathcal{P}$ is nontrapping. Suppose $s, r \in \mathbb{R}$, $s - \frac{1}{2}(m-1) - 1 > \beta r$, and $-r$ is not the imaginary part of a pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$, where $\mathcal{P} - i\mathcal{Q}$ is a priori a map*

$$\mathcal{P} - i\mathcal{Q} : H_b^{s,r}(M) \to H_b^{s-2,r}(M).$$

Then

$$\mathcal{P} - i\mathcal{Q} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$$

is Fredholm.

2A3. *Initial value problems.* As already mentioned, the main issue with the argument using complex absorption that it does not guarantee the forward nature (in terms of supports) of the solution for a wave-like equation, although it does guarantee the correct microlocal structure. So now we assume that $\mathcal{P} \in \mathrm{Diff}_b^2(M)$ and that there is a Lorentzian b-metric g such that

$$\mathcal{P} - \Box_g \in \mathrm{Diff}_b^1(M), \quad \mathcal{P} - \mathcal{P}^* \in \mathrm{Diff}_b^0(M). \tag{2-15}$$

Then one can run a completely analogous argument using energy-type estimates by restricting the domain we consider to be a manifold with corners, where the new boundary hypersurfaces are spacelike with respect to g, i.e., given by level sets of timelike functions. Such a possibility was mentioned in [Vasy 2013a, Remark 2.6], though it was not described in detail as it was not needed there, essentially because the existence and uniqueness argument for forward solutions was given only for dilation-invariant operators. The main difference between using complex absorption and adding boundary hypersurfaces is that the latter limit the Sobolev regularity one can use, with the most natural choice coming from energy estimates. However, a posteriori one can improve the result to better Sobolev spaces using propagation of singularities results.

So assume now that $U \subset M$ is open and we have two functions $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in $C^\infty(M)$, both of which, restricted to U, are timelike (in particular have nonzero differential) near their respective 0-level sets H_1 and H_2, and

$$\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty)) \subset U.$$

Notice that the timelike assumption forces $d\mathfrak{t}_j$ to not lie in $N^*X = N^*\partial M$ (for its image in the b-cosphere bundle would be zero) and thus, if the H_j intersect X, they do so transversally. We assume that the H_j intersect only away from X and that they do so transversally, that is, the differentials of $\mathfrak{t}_j$ are independent at the intersection. Then Ω is a manifold with corners with boundary hypersurfaces H_1, H_2 and X (all intersected with Ω). We, however, keep thinking of Ω as a domain in M. The role of the elliptic set of $\mathcal{Q}$ is now played by ${}^{\mathrm{b}}S^*_{H_j}M$, $j = 1, 2$. The *nontrapping* assumption becomes (see Figure 3) that:

(1) All bicharacteristics in $\Sigma_\Omega = \Sigma \cap {}^{\mathrm{b}}S^*_\Omega M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_+$ in the forward direction (i.e., either enter ${}^{\mathrm{b}}S^*_{H_1}M$ in finite time or tend to L_+) and to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_+$ in the backward direction.

(2) From any point in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ the bicharacteristics flow to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_-$ in the forward direction and to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_-$ in the backward direction.

In particular, orienting the characteristic set by letting Σ_- be the *future-oriented* and Σ_+ the *past-oriented* part, $d\mathfrak{t}_1$ is future-oriented, while $d\mathfrak{t}_2$ is past-oriented.

On a manifold with corners, such as Ω, one can consider supported and extendible distributions; see [Hörmander 1985a, Appendix B.2] for the smooth boundary setting, with simple changes needed only for the corners setting, which is discussed in [Vasy 2008, §3], for example. Here we consider Ω as a domain in M, and thus its boundary face $X \cap \Omega$ is regarded as having a different character from the $H_j \cap \Omega$, that is, the support and extendibility considerations do not arise at X — all distributions are regarded as acting on a subspace of C^∞ functions on Ω vanishing at X to infinite order, i.e., they are automatically extendible distributions at X. On the other hand, at H_j we consider both extendible distributions, acting on C^∞ functions vanishing to infinite order at H_j, and supported distributions, which act on all C^∞

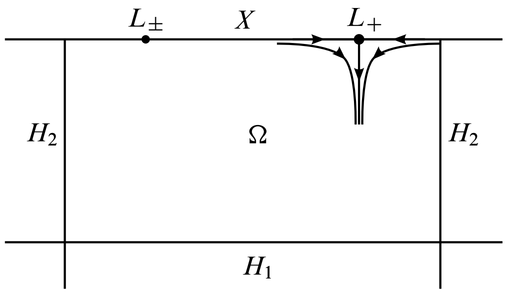

Figure 3. Setup for the discussion of the forward problem. Near the spacelike hypersurfaces H_1 and H_2, which are the replacement for the complex absorbing operator $\mathcal{Q}$, we use standard (nonmicrolocal) energy estimates, and away from them, we use b-microlocal propagation results, including at the radial sets $L_\pm$. The bicharacteristic flow — in fact, its projection to the base — is only indicated near L_+; near L_-, the directions of the flowlines are reversed.

functions (as far as conditions at H_j are concerned). For example, the space of supported distributions at H_1 extendible at H_2 (and at X, as we always tacitly assume) is the dual space of the subspace of $C^\infty(\Omega)$ consisting of functions vanishing to infinite order at H_2 and X (but not necessarily at H_1). An equivalent way of characterizing this space of distributions is that they are restrictions of elements of the dual of $\dot{C}^\infty(M)$ (consisting of C^∞ functions on M vanishing to infinite order at X) with support in $\mathfrak{t}_1 \geq 0$ to C^∞ functions on Ω which vanish to infinite order at X and H_2, thus, in the terminology of [Hörmander 1985a], restrictions to $\Omega \setminus (H_2 \cup X)$.

The main interest is in spaces induced by the Sobolev spaces $H_{\mathrm{b}}^{s,r}(M)$. Notice that the Sobolev norm is of a completely different nature at X than at the H_j, namely the derivatives are based on complete, rather than incomplete, vector fields: $\mathcal{V}_{\mathrm{b}}(M)$ is being restricted to Ω, so one obtains vector fields tangent to X but not to the H_j. As for supported and extendible distributions corresponding to $H_{\mathrm{b}}^{s,r}(M)$, we have, for instance,

$$H_{\mathrm{b}}^{s,r}(M)^{\bullet,-},$$

with the first superscript on the right denoting whether supported ($\bullet$) or extendible ($-$) distributions are discussed at H_1, and the second the analogous property at H_2, which consists of restrictions of elements of $H_{\mathrm{b}}^{s,r}(M)$ with support in $\mathfrak{t}_1 \geq 0$ to $\Omega \setminus (H_2 \cup X)$. Then elements of $C^\infty(\Omega)$ with the analogous vanishing conditions, so in the example vanishing to infinite order at H_1 and X, are dense in $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$; further, the dual of $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$ is $H_{\mathrm{b}}^{-s,-r}(M)^{-,\bullet}$ with respect to the L^2 (sesquilinear) pairing.

First we work locally. For this purpose it is convenient to introduce another timelike function $\tilde{\mathfrak{t}}_j$, not necessarily timelike, and consider

$$\Omega_{[t_0,t_1]} = \mathfrak{t}_j^{-1}([t_0,\infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1]) \quad \text{and} \quad \Omega_{(t_0,t_1)} = \mathfrak{t}_j^{-1}((t_0,\infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1)),$$

and similarly on half-open, half-closed intervals. Thus, $\Omega_{[t_0,t_1]}$ becomes smaller as t_0 becomes larger or t_1 becomes smaller.

We then consider energy estimates on $\Omega_{[T_0,T_1]}$. In order to set up the following arguments, choose

$$T_- < T_-' < T_0 \quad \text{and} \quad T_1 < T_+' < T_+,$$

and assume that $\Omega_{[T_-,T_+]}$ is compact, $\Omega_{[T_0,T_1]}$ is nonempty, and $\mathfrak{t}_j$ is timelike on $\Omega_{[T_-,T_+]}$. The energy estimates propagate estimates in the direction of either increasing or decreasing $\mathfrak{t}_j$. With the extendible or supported character of distributions at $\tilde{\mathfrak{t}}_j = T_+$ being irrelevant for this matter in the case being considered and thus dropped from the notation (so $(-)$ refers to extendibility at $\mathfrak{t}_j = T_0$), consider

$$\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_+]})^- \to H_{\mathrm{b}}^{s-2,r}(\Omega_{[T_0,T_+]})^-, \quad s,\ r \in \mathbb{R}.$$

The energy estimate, with backward propagation in $\mathfrak{t}_j$, from $\tilde{\mathfrak{t}}_j^{-1}([T_+', T_+])$, in this setting takes the form:

Lemma 2.4. *Let $r \in \mathbb{R}$. There is $C > 0$ such that, for $u \in H_{\mathrm{b}}^{2,r}(\Omega_{[T_0,T_+]})^-$,*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^-} \le C\big(\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_0,T_+]})^-} + \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_+]}\cap\tilde{\mathfrak{t}}_j^{-1}([T_+',T_+]))^-}\big). \tag{2-16}$$

This also holds with $\mathcal{P}$ replaced by $\mathcal{P}^*$, acting on the same spaces.

Remark 2.5. The lemma is also valid if one has several boundary hypersurfaces, that is, if one replaces $\mathfrak{t}_j^{-1}([t_0,\infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0},\infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0},\infty))$ in the definition of $\Omega_{[t_0,t_1]}$, and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{j,1}])\cap\tilde{\mathfrak{t}}_k^{-1}((-\infty,t_{k,1}])$, i.e., regarding $\mathfrak{t}_j$ and/or $\tilde{\mathfrak{t}}_j$ as vector-valued, and propagating backwards in $\mathfrak{t}_{j_0}$ for some fixed j_0, under the additional hypothesis that $\mathfrak{t}_{j_0}$ is timelike in $\Omega_{[t_0,t_1]}$, and all $\mathfrak{t}_j$, $j \neq j_0$, are timelike near their respective zero sets, with the same timelike character at $\mathfrak{t}_{j_0}$. (One can also have more than two such functions.) To see this, replace $\chi(\mathfrak{t}_j)$ by $\chi_{j_0}(\mathfrak{t}_{j_0})\chi_k(\mathfrak{t}_k)$ and analogously for $\tilde{\chi}$ in the definition of V in (2-17), where χ_k is the characteristic function of $[t_{k,0},\infty)$, while letting $W = G({}^{\mathrm{b}}d\mathfrak{t}_{j_0},\cdot)$. Then $\chi'\tilde{\chi}\tau^\alpha A^\sharp$ is replaced by $\chi_j'\chi_k\tilde{\chi}_j\tilde{\chi}_k\tau^\alpha A^\sharp + \chi_j\chi_k'\tilde{\chi}_j\tilde{\chi}_k\tau^\alpha A^\sharp$, etc., and our additional hypothesis guarantees that the matrix $A^\sharp$ is indeed positive definite: The contribution from differentiating χ_{j_0} is positive definite by the timelike nature of $d\mathfrak{t}_{j_0}$, while the contribution from differentiating χ_j, $j \neq j_0$, giving δ-distributions at the hypersurfaces $\mathfrak{t}_j^{-1}(t_{j,0})$, is positive definite by the second part of the above additional hypothesis and can therefore be dropped as in the proof of Lemma 2.4 below. Thus χ_{j_0}' can still be used to dominate χ_{j_0}; the terms in which $\tilde{\chi}_j$ is differentiated have support where $\tilde{\mathfrak{t}}_j$ is in $(T_{+,j}', T_{+,j})$, so the control region on the right-hand side of (2-16) is the union of these sets.

In our application this situation arises as we need the estimates on $\mathfrak{t}_1^{-1}([T_0,T_1]) \cap \mathfrak{t}_2^{-1}([0,\infty))$ and $\mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([T_0,T_1])$, with $T_0 = 0$ and $T_1 > 0$ small. For instance, in the latter case $\mathfrak{t}_2$ plays the role of $\mathfrak{t}_j$ above, while $-\mathfrak{t}_1$ and $\mathfrak{t}_2$ play the role of $\tilde{\mathfrak{t}}_j$ and $\tilde{\mathfrak{t}}_k$; see Figure 4.

Proof of Lemma 2.4. To see (2-16), one proceeds as in [Vasy 2013a, §3.3] and considers

$$V = -i\chi(\mathfrak{t}_j)\tilde{\chi}(\tilde{\mathfrak{t}}_j)\tau^\alpha W \tag{2-17}$$

with $W = G(d\mathfrak{t}_j,\cdot)$ a timelike vector field and with χ, $\tilde{\chi} \in C^\infty(\mathbb{R})$, both nonnegative, to be specified. Then, choosing a *Riemannian* b-metric $\tilde{g}$,

$$-i(V^*\Box_g - \Box_g^* V) = {}^{\mathrm{b}}d_{\tilde{g}}^* C^{\mathrm{b}}\,{}^{\mathrm{b}}d,$$

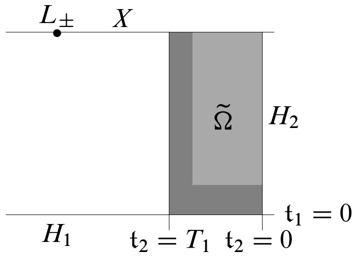

Figure 4. A domain $\tilde\Omega = \mathfrak{t}_2^{-1}([0,\infty)) \cap \big((-\mathfrak{t}_1)^{-1}((-\infty,0]) \cap \mathfrak{t}_2^{-1}((-\infty,T_1])\big)$ on which we will apply the energy estimate (2-16). The a priori control region is indicated in dark gray.

with the subscript on the adjoint on the right-hand side denoting the metric with respect to which it is taken, ${}^{\mathrm{b}}d : C^\infty(M) \to C^\infty(M; {}^{\mathrm{b}}T^*M)$ being the b-differential, and with

$$C^{\flat} = \chi'\tilde\chi\tau^\alpha A^\sharp + \chi\tilde\chi'\tau^\alpha \tilde A^\sharp + \chi\tilde\chi\tau^\alpha R^{\flat},$$

where $A^\sharp$, $\tilde A^\sharp$ and $R^\flat$ are bundle endomorphisms of ${}^{\mathbb{C}\mathrm{b}}T^*M$, and $A^\sharp$ and $\tilde A^\sharp$ are positive definite. Proceeding further, replacing $\Box_g$ by $\mathcal{P}$ one has

$$\begin{aligned} -i(V^*\mathcal{P} - \mathcal{P}^*V) &= {}^{\mathrm{b}}d_{\tilde g}^* C^\sharp\, {}^{\mathrm{b}}d + (\tilde E_1)_{\tilde g}^* \tau^\alpha \chi\tilde\chi\, {}^{\mathrm{b}}d + {}^{\mathrm{b}}d_{\tilde g}^* \tau^\alpha \chi\tilde\chi \tilde E_2, \\ C^\sharp &= \chi'\tilde\chi\tau^\alpha A^\sharp + \chi\tilde\chi'\tau^\alpha \tilde A^\sharp + \chi\tilde\chi\tau^\alpha \tilde R^\sharp \end{aligned} \tag{2-18}$$

with $\tilde E_j$ bundle maps from the trivial bundle over M to ${}^{\mathbb{C}\mathrm{b}}T^*M$, $A^\sharp$ and $\tilde A^\sharp$ as before, and $\tilde R^\sharp$ a bundle endomorphism of ${}^{\mathbb{C}\mathrm{b}}T^*M$, as follows by expanding

$$-i(V^*(\mathcal{P} - \Box_g) - (\mathcal{P} - \Box_g)^*V),$$

using that $\mathcal{P} - \Box_g \in \mathrm{Diff}_{\mathrm{b}}^1(M)$. We regard the second term on the right-hand side of (2-18) as the one requiring a priori control by $\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_+]} \cap \tilde{\mathfrak{t}}_j^{-1}([T'_+,T_+]))^-}$; we achieve this by making $\tilde\chi$ supported in $(-\infty, T_+)$, identically 1 near $(-\infty, T'_+]$, so $d\tilde\chi$ is supported in (T'_+, T_+). Now, making $\chi' \geq 0$ large relative to χ on $\mathrm{supp}(\chi\tilde\chi)$, as in[8] [Vasy 2013a, Equation (3.27)], allows one to dominate all terms without derivatives of χ. In order to obtain a nondegenerate estimate up to $\mathfrak{t}_j = T_0$, one cuts off χ at $\mathfrak{t}_j = T_0$ using the Heaviside function, so χ' gives a (positive!) δ-distribution there. Applying (2-18) to v, pairing with v and integrating by parts, the δ-distributions have the same sign as $\chi' A^\sharp$ and can thus be dropped. Put differently, without the sharp cutoff, one again computes the same pairing, but this time on the domain $\Omega_{[T_0,T_+]}$, thus picking up boundary terms with the correct sign in the integration by parts, so these terms can be dropped. This proves the energy estimate (2-16) when one takes $\alpha = -2r$. ☐

Propagating in the forward direction, from $\mathfrak{t}_j^{-1}([T_-, T'_-])$, where now $-$ denotes the character of the space at T_1 (so $-$ refers to extendibility at $\mathfrak{t}_j = T_1$),

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^-} \leq C\big(\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_-,T_1]})^-} + \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_-,T_1]} \cap \mathfrak{t}_j^{-1}([T_-,T'_-]))^-}\big). \tag{2-19}$$

[8] Though, there, the sign of χ' is opposite, as the estimate is propagated in the opposite direction.

In particular, for u supported in $\mathfrak{t}_j \geq T_0$, the last estimate becomes, with the first superscript on the right denoting whether supported ($\bullet$) or extendible ($-$) distributions are discussed at $\mathfrak{t} = T_0$ and the second superscript the same at $\mathfrak{t} = T_1$,

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-20}$$

when

$$\mathcal{P} : H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-} \to H_b^{s-2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$$

and $u \in H_b^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. To summarize, we state both this and (2-16) in terms of these supported spaces:

Corollary 2.6. *Let $r, \tilde{r} \in \mathbb{R}$. For $u \in H_b^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, one has*

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-21}$$

while, for $v \in H_b^{2,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$, the estimate

$$\|v\|_{H_b^{1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_b^{0,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \tag{2-22}$$

holds.

A duality argument, combined with propagation of singularities, thus gives:

Lemma 2.7. *Let $s \geq 0$, $r \in \mathbb{R}$. Then there is $C > 0$ with the following property: If $f \in H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$ and*

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|f\|_{H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Remark 2.8. As in Remark 2.5, the lemma remains valid in more generality, namely, if one replaces $\mathfrak{t}_j^{-1}([t_0,\infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0},\infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0},\infty))$ and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{j,1}]) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{k,1}])$ in the definition of $\Omega_{[t_0,t_1]}$, provided that the $\mathfrak{t}_j$ have linearly independent differentials on their joint zero set, and similarly for the $\tilde{\mathfrak{t}}_j$. The place where this linear independence is used (the energy estimate above does not need this) is for the continuous Sobolev extension map, valid on manifolds with corners; see [Vasy 2008, §3].

Proof. We work on the slightly bigger region $\Omega_{[T'_-,T'_+]}$, applying the energy estimates with T_0 replaced by T'_-, T_1 replaced by T'_+. First, by the supported property at $\mathfrak{t}_j = T_0$, one can regard f as an element of $H_b^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}$ with support in $\Omega_{[T_0,T_1]}$. Let

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-} \subset H_b^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

be an extension of f, so $\tilde{f}$ is supported in $\Omega_{[T_0,T'_+]}$ and restricts to f; by the definition of spaces of extendible distributions as quotients of spaces of distributions on a larger space — see [Hörmander 1985a, Appendix B.2] — we can assume

$$\|\tilde{f}\|_{H_b^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \leq 2\|f\|_{H_b^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}}. \tag{2-23}$$

By (2-16) applied with $\mathcal{P}$ replaced by $\mathcal{P}^*$ and $\tilde{r} = -r$,

$$\|\phi\|_{H_b^{1,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \le C\|\mathcal{P}^*\phi\|_{H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \tag{2-24}$$

for $\phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$. Correspondingly, by the Hahn–Banach theorem, there exists

$$\tilde{u} \in (H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet})^* = H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

such that

$$\langle \mathcal{P}\tilde{u}, \phi\rangle = \langle \tilde{u}, \mathcal{P}^*\phi\rangle = \langle \tilde{f}, \phi\rangle, \quad \phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$$

and

$$\|\tilde{u}\|_{H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \le C\|\tilde{f}\|_{H_b^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}}. \tag{2-25}$$

One can regard $\tilde{u}$ as an element of $H_b^{0,r}(\Omega_{[T_-,T'_+]})^{\bullet,-}$ with support in $\Omega_{[T'_-,T'_+]}$, with $\tilde{f}$ similarly extended; then $\langle \mathcal{P}\tilde{u}, \phi\rangle = \langle \tilde{f}, \phi\rangle$ for $\phi \in \dot{C}_c^\infty(\Omega_{(T_-,T'_+)})$ (here the dot over C^∞ refers to infinite-order vanishing at $X = \partial M$!), so $\mathcal{P}\tilde{u} = \tilde{f}$ in distributions. Since $\tilde{u}$ vanishes on $\Omega_{(T_-,T'_-)}$ and

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T_-,T'_+]})^{\bullet,-},$$

propagation of singularities applied on $\Omega_{(T_-,T'_+)}$ (which has only the boundary $\partial M = X$) gives that $\tilde{u} \in H_{b,\mathrm{loc}}^{s,r}(\Omega_{(T_-,T'_+)})$ (here we are ignoring the two boundaries, $\mathfrak{t}_j = T_-$, T'_+, not making a uniform statement there, but we are not ignoring $\partial M = X$). In addition, for χ, $\tilde{\chi} \in C_c^\infty(\Omega_{(T_-,T'_+)})$ with $\tilde{\chi} \equiv 1$ on supp χ, we have the estimate

$$\|\chi\tilde{u}\|_{H_b^{s,r}(\Omega_{[T_-,T'_+]})} \le C\big(\|\tilde{\chi}\mathcal{P}\tilde{u}\|_{H_b^{s-1,r}(\Omega_{[T_-,T'_+]})} + \|\tilde{\chi}\tilde{u}\|_{H_b^{0,r}(\Omega_{[T_-,T'_+]})}\big). \tag{2-26}$$

In view of the support property of $\tilde{u}$, this gives that, restricting to $\Omega_{(T_-,T_1]}$, we obtain an element of $H_b^{s,r}(\Omega_{(T_-,T_1]})^-$ with support in $\Omega_{[T_0,T_1]}$, i.e., an element of $H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. The desired estimate follows from (2-25), controlling the second term of the right-hand side of (2-26), and (2-23) as well as using $\mathcal{P}\tilde{u} = \tilde{f}$. ☐

At this point, u given by Lemma 2.7 is not necessarily unique. However:

Lemma 2.9. *Let $s, r \in \mathbb{R}$. If $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ is such that $\mathcal{P}u = 0$, then $u = 0$.*

Proof. Propagation of singularities, as in the proof of Lemma 2.7, regarding u as a distribution on (T_-, T_1) with support in $[T_0, T_1)$ gives that $u \in H_{b,\mathrm{loc}}^{\infty,r}(\Omega_{(T_-,T_1)})$. Taking $T_0 < T'_1 < T_1$, letting $u' = u|_{[T_0,T'_1]}$, (2-21) shows that $u' = 0$. Since T'_1 is arbitrary, this shows $u = 0$. ☐

Corollary 2.10. *Let $s \ge 0$ and $r \in \mathbb{R}$. Then there is $C > 0$ with the following property*:

If $f \in H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists a unique $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$. Further, this unique u satisfies

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \le C\|f\|_{H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Proof. Existence is Lemma 2.7; uniqueness is linearity plus Lemma 2.9, which, together with the estimate in Lemma 2.7, prove the corollary. ☐

Corollary 2.11. *Let $s \geq 0$ and $r, \tilde{r} \in \mathbb{R}$. For $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$,*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-27}$$

while, for $v \in H_{\mathrm{b}}^{s,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$ with $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$,

$$\|v\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}}. \tag{2-28}$$

Remark 2.12. Again, this estimate remains valid for vector-valued $\mathfrak{t}_j$ and $\tilde{\mathfrak{t}}_j$ — see Remarks 2.5 and 2.8 — under the linear independence condition of the latter.

Proof. It suffices to consider (2-27). Let $f = \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ and let $u' \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be given by Corollary 2.10. In view of the uniqueness statement of Corollary 2.10, $u = u'$. Then the estimate of Corollary 2.10 proves the claim. □

This yields the following kind of propagation of singularities result:

Proposition 2.13. *Let $s \geq 0$ and $r \in \mathbb{R}$. If $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$.*

If instead $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{-,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$ and, for some $\widetilde{T}_0 > T_0$, $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]})^{-,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$.

Remark 2.14. One can "mix and match" the two parts of the proposition in the setting of Remark 2.5, with, say, a supportedness condition at $\tilde{\mathfrak{t}}_j$ and only an extendibility assumption at $\tilde{\mathfrak{t}}_k$, but with an $H_{\mathrm{b}}^{s,r}$ membership assumption on u in $\Omega_{[T_0,T_1]} \setminus \tilde{\mathfrak{t}}_k^{-1}((-\infty, \widetilde{T}_1))$, $\widetilde{T}_1 < T_1$, with a completely analogous argument. For instance, in the setting of Figure 4, one gets the regularity under supportedness assumptions at H_1, just extendibility at $\mathfrak{t}_2 = T_1$, but a priori regularity for $\mathfrak{t}_2 \in (\widetilde{T}_1, T_1)$.

Proof. Let $u' \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be the unique solution in $H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ of $\mathcal{P}u' = f$ where $f = \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$; we obtain u' by applying the existence part of Corollary 2.10. Then $u, u' \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ and $\mathcal{P}(u-u') = 0$. Applying Lemma 2.9, we conclude that $u = u'$, which completes the proof of the first part.

For the second part, let $\chi \in C^\infty(\mathbb{R})$ be supported in (T_0, ∞), identically 1 near $[\widetilde{T}_0, \infty)$, and consider $u' = (\chi \circ \mathfrak{t}_j)u \in H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, with the support property arising from the vanishing of χ near T_0. Then $\mathcal{P}u' = [\mathcal{P}, (\chi \circ \mathfrak{t}_j)]u + (\chi \circ \mathfrak{t}_j)\mathcal{P}u$. Now the first term on the right-hand side is in $H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ because, on the support of $d\chi$, which is in $\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]}$, u is in $H_{\mathrm{b}}^{s,r}$ and the commutator is first order, while the second term is in the desired space since $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$, and, as for u itself, the cutoff improves the support property. Thus, the first part of the lemma is applicable, giving that $\chi u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. Since $(1-\chi)u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$ by the a priori assumption, the conclusion follows. □

We take $T_0 = 0$ and thus consider, for $s \geq 0$,

$$\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-} \tag{2-29}$$

$$\text{and}\quad \mathcal{P}^* : H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{-,\bullet}. \tag{2-30}$$

In combination with the real-principal-type propagation results and Proposition 2.1, this yields, under the nontrapping assumptions, much as in the complex absorbing case, that[9]

$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \le C\|\mathcal{P}u\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}} + C\|u\|_{H_b^{0,r}(\Omega)^{\bullet,-}}, \quad \beta r < -\tfrac{1}{2},\ s > 0, \tag{2-31}$$

and

$$\|u\|_{H_b^{s,\tilde r}(\Omega)^{-,\bullet}} \le C\|\mathcal{P}^* u\|_{H_b^{s-1,\tilde r}(\Omega)^{-,\bullet}} + C\|u\|_{H_b^{0,\tilde r}(\Omega)^{-,\bullet}}, \quad \beta \tilde r > s-\tfrac{1}{2},\ s > 0. \tag{2-32}$$

We could proceed as in the complex absorption case to make the space on the left-hand side include compactly into the "error term" on the right using the normal operators. As this imposes some constraints — see (2-14) — which, together with the requirements of the energy estimates, namely that the Sobolev order is nonnegative, mean that we would get slightly too strong restrictions on s — see Remark 2.20 — we proceed instead with a direct energy estimate. We thus assume that Ω is sufficiently small that there is a boundary defining function τ of M with $d\tau/\tau$ timelike on Ω, of the same timelike character as $\mathfrak{t}_2$, opposite to $\mathfrak{t}_1$. As explained in [Vasy 2013a, §7], in this case there is $C > 0$ such that, for $\Im\sigma > C$, $\widehat{P}(\sigma)$ is necessarily invertible.

The energy estimate is:

Lemma 2.15. *There exists $r_0 < 0$ such that, for $r \le r_0$ and $-\tilde r \le r_0$, there is $C > 0$ such that, for $u \in H_b^{2,r}(\Omega)^{\bullet,-}$ and $v \in H_b^{2,\tilde r}(\Omega)^{-,\bullet}$, one has*

$$\begin{aligned} \|u\|_{H_b^{1,r}(\Omega)^{\bullet,-}} &\le C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega)^{\bullet,-}}, \\ \|v\|_{H_b^{1,\tilde r}(\Omega)^{-,\bullet}} &\le C\|\mathcal{P}^* v\|_{H_b^{0,\tilde r}(\Omega)^{-,\bullet}}. \end{aligned} \tag{2-33}$$

Proof. We run the argument of Lemma 2.4 globally on Ω using a timelike vector field (e.g., starting with $W = G(d\tau/\tau, \cdot)$) that has, as a multiplier, a sufficiently large positive power $\alpha = -2r$ of τ, that is, replacing (2-17) by

$$V = -i\tau^\alpha W.$$

Then the term with τ^α differentiated (which in (2-18) is included in the $\widetilde{R}^\sharp$ term), and thus possessing a factor of α, is used to dominate the other, "error", terms in (2-18), completing the proof of the lemma as in Lemma 2.4. □

This can be used as in Lemma 2.7 to provide solvability and, using the propagation of singularities — which in this case includes the use of Proposition 2.1, noting that $s - \frac{1}{2} > \beta r$ is automatically satisfied — improved regularity. In particular, we obtain the following analogues of Corollaries 2.10–2.11:

Corollary 2.16. *There is $r_0 < 0$ such that, for $r \le r_0$ and $s \ge 0$, there is $C > 0$ with the following property: If $f \in H_b^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_b^{s,r}(\Omega)^{\bullet,-}$ such that $\mathcal{P}u = f$.*

Further, this unique u satisfies

$$\|u\|_{H_b^{s,r}(\Omega)^{\bullet,-}} \le C\|f\|_{H_b^{s-1,r}(\Omega)^{\bullet,-}}.$$

[9] In fact, the error term on the right-hand side can be taken to be supported in a smaller region, since, at H_1 in the first case and at H_2 in the second, there are no error terms due to the energy estimates (2-21), applied with $\mathcal{P}^*$ in place of $\mathcal{P}$ in the second case.

Corollary 2.17. *There is $r_0 < 0$ such that, if $r < r_0$, $-\tilde{r} < r_0$ and $s \geq 0$, then there is $C > 0$ such that the following holds*:

For $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, one has

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} \tag{2-34}$$

while, for $v \in H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}$ with $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}$, one has

$$\|v\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}}. \tag{2-35}$$

We restate Corollary 2.16 as an invertibility statement.

Theorem 2.18. *There is $r_0 < 0$ with the following property. Suppose $s \geq 0$, $r \leq r_0$, and let*

$$\mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}, \quad \mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} : \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}\},$$

where $\mathcal{P}$ is a priori a map $\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$. Then

$$\mathcal{P} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$$

is a continuous, invertible map, with continuous inverse.

Remark 2.19. Both $\mathcal{Y}^{s,r}$ and $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being

$$\|u\|^2_{\mathcal{X}^{s,r}} = \|u\|^2_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} + \|\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}},$$

as follows by the continuity of $\mathcal{P}$ as a map $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$ and the completeness of the b-Sobolev spaces $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$.

Remark 2.20. Using normal operators as in the discussion leading to Proposition 2.3, one would get the following statement: Suppose $s > 1$ and $s - \frac{3}{2} > \beta r$. Then, with $\mathcal{X}^{s,r}$ and $\mathcal{Y}^{s,r}$ as above, $\mathcal{P} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s,r}$ is Fredholm. Here the main loss is that one needs to assume $s > 1$; this is done since, in the argument, one needs to take s' with $s' + 1 < s$ in order to transition the normal operator estimates from $N(\mathcal{P})u$ to $\mathcal{P}u$ and still have a compact inclusion, but the normal operator estimates need $s' \geq 0$ as, due to the boundary H_2, they are again based on energy estimates. Using the direct global energy estimate eliminates this loss, which is an artifact of combining local energy estimates with the b-theory. In particular, in the complex absorption setting, this problem does not arise, but, on the other hand, one need not have the forward support property of the solution.

The results of [Vasy 2013a] then are immediately applicable to obtain an expansion of the solutions; the main point of the following theorem being the elimination of the losses in differentiability in Vasy's Proposition 3.5 due to Proposition 2.1.

Theorem 2.21 (strengthened version of [Vasy 2013a, Proposition 3.5]). *Let M be a manifold with a nontrapping b-metric g as above, with boundary X and let τ be a boundary defining function, $\mathcal{P}$ as in (2-15). Suppose the domain Ω is as defined above and $d\tau/\tau$ is timelike.*

Let σ_j be the poles of $\widehat{\mathcal{P}}^{-1}$ and let ℓ be such that $\Im\sigma_j + \ell \notin \mathbb{N}$ for all j. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}\phi \subset (0,\infty)$ *and $\phi \circ \mathfrak{t}_1 \equiv 1$ near $X \cap \Omega$. Then, for $s > \frac{3}{2} + \beta\ell$, there are $m_{jl} \in \mathbb{N}$ such that solutions of $\mathcal{P}u = f$ with $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ and $u \in H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}$, $s \ge s_0 \ge 1$, $s_0 - \frac{1}{2} > \beta r_0$, satisfy that, for some $a_{jl\kappa} \in C^\infty(X \cap \Omega)$,*

$$u' = u - \sum_j \sum_{l \in \mathbb{N}} \sum_{\kappa \le m_{jl}} \tau^{i\sigma_j + l} (\log\tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{jl\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}, \tag{2-36}$$

where the sum is understood to be over a finite set with $-\Im\sigma_j + l < \ell$.

Here the (semi)norms of both $a_{jl\kappa}$ in $C^\infty(X \cap \Omega)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$.

The analogous result also holds if f possesses an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, namely

$$f = f' + \sum_j \sum_{\kappa \le m'_j} \tau^{\alpha_j} (\log\tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa}$$

with $f' \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ and $a_{j\kappa} \in C^\infty(X \cap \Omega)$, where terms corresponding to the expansion of f are added to (2-36) in the sense of the extended union of index sets [Melrose 1993, §5.18], *recalled below in Definition 2.32.*

Remark 2.22. Here the factor $\phi \circ \mathfrak{t}_1$ is added to cut off the expansion away from H_1, thus assuring that u' is in the indicated space (a supported distribution).

Also, the sum over l is generated by the lack of dilation invariance of $\mathcal{P}$. If we take ℓ such that $-\Im\sigma_j > \ell - 1$ for all j then all the terms in the expansion arise directly from the resonances, thus $l = 0$ and $m_{j0} + 1$ is the order of the pole of $\widehat{\mathcal{P}}^{-1}$ at σ_j, with the $a_{j0\kappa}$ being resonant states.

Proof. First assume that $-\Im\sigma_j > \ell$ for every j; thus there are no terms subtracted from u in (2-36). We proceed as in [Vasy 2013a, Proposition 3.5], but use the propagation of singularities, in particular Propositions 2.1 and 2.13, to eliminate the losses. First, by the propagation of singularities, using $s_0 - \frac{1}{2} > \beta r_0$ and $s \ge s_0$, $s \ge 0$,

$$u \in H_{\mathrm{b}}^{s,r_0}(\Omega)^{\bullet,-}.$$

Thus, as $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_{\mathrm{b}}^2(M)$,

$$N(\mathcal{P})u = f - \tilde{f}, \quad \text{where} \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-}. \tag{2-37}$$

We apply [Vasy 2013a, Lemma 3.1] (using $s \ge s_0 \ge 1$), which is the lossless version of Vasy's Proposition 3.5 in the dilation-invariant case. Note that the lemma is stated on the normal operator space M_∞, which does not have a boundary face corresponding to H_2, i.e., $S_2 \times (0,\infty)$, with complex absorption being used instead. However, given the analysis on $X \cap \Omega$ discussed above, all the arguments go through essentially unchanged: this is a Mellin transform and contour deformation argument.

One thus obtains (2-36) with ℓ replaced by $\ell' = \min(\ell, r_0 + 1)$, except that $u = u' \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$, corresponding to the $\tilde{f}$ term in $N(\mathcal{P})u$ rather than $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$, as desired. However, using $\mathcal{P}u = f \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$, we deduce by the propagation of singularities, using $s - 1 > \beta\ell' + \frac{1}{2}$, $s \ge 0$, that $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \le r_0 + 1$, we have proved (2-36). Otherwise we iterate, replacing r_0 by $r_0 + 1$. We thus reach the conclusion, (2-36), in finitely many steps.

If there are j such that $-\Im\sigma_j < \ell$ then, in the first step, when using [Vasy 2013a, Lemma 3.1], we obtain the partial expansion u_1 corresponding to $\ell' = \min(\ell, r_0 + 1)$ in place of ℓ; here we may need to decrease ℓ' by an arbitrarily small amount to make sure that ℓ' is not $-\Im\sigma_j$ for any j. Further, the terms of the partial expansion are annihilated by $N(\mathcal{P})$, so u' satisfies

$$\mathcal{P}u' = \mathcal{P}u - N(\mathcal{P})u_1 - (\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$$

as $(\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{\infty,r_0+1}(\Omega)^{\bullet,-}$ in fact, due to the conormality of u_1 and $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_{\mathrm{b}}^2(M)$. Correspondingly, the propagation of singularities result is applicable as above to conclude that $u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \le r_0 + 1$ we are done. Otherwise, we have better information on $\tilde{f}$ in the next step, namely

$$\tilde{f} = (\mathcal{P} - N(\mathcal{P}))u = (\mathcal{P} - N(\mathcal{P}))u' + (\mathcal{P} - N(\mathcal{P}))u_1$$

with the first term in $H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-}$ (same as in the case first considered above, without relevant resonances), while the expansion of u_1 shows that $(\mathcal{P} - N(\mathcal{P}))u_1$ has a similar expansion, but with an extra power of τ (i.e., $\tau^{i\sigma_j}$ is shifted to $\tau^{i\sigma_j+1}$). We can now apply Vasy's Lemma 3.1 again; in the case of the terms arising from the partial expansion, u_1, there are now new terms corresponding to shifting the powers $\tau^{i\sigma_j}$ to $\tau^{i\sigma_j+1}$, as stated in the referred lemma, and possibly causing logarithmic terms if $\sigma_j - i$ is also a pole of $\widehat{\mathcal{P}}^{-1}$. Iterating in the same manner proves the theorem when $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$. When f has an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, the same argument works; [Vasy 2013a, Lemma 3.1] gives the terms with the extended union, which then further generate additional terms due to $\mathcal{P} - N(\mathcal{P})$, just as the resonance terms did. $\square$

There is one problem with this theorem for the purposes of semilinear equations: the resonant terms with $\Im\sigma_j \ge 0$ which give rise to unbounded, or at most just bounded, terms in the expansion which become larger when one takes powers of these, or when one iteratively applies $\mathcal{P}^{-1}$ (with the latter being the only issue if $\Im\sigma_j = 0$ and the pole is simple).

Concretely, we now consider an asymptotically de Sitter space $(\widetilde{M}, \tilde{g})$. We then blow up a point p at the future boundary $\widetilde{X}_+$, as discussed in the introduction (see p. 1810), to obtain the analogue of the static model of de Sitter space $M = [\widetilde{M}; p]$ with the pulled back metric g, which is a b-metric near the front face (but away from the side face); let $\mathcal{P} = \Box_g - \lambda$. If $\widetilde{M}$ is actual de Sitter space, then M is the actual static model; otherwise, the metric of the asymptotically de Sitter space, frozen at p, induces a de Sitter metric, g_0, which is well defined at the front face of the blow-up M (but away from its side faces) as a b-metric. In particular, the resonances in the "static region" of any asymptotically de Sitter space are the same as in the static model of actual de Sitter space.

On actual de Sitter space, the poles of $\widehat{\mathcal{P}}^{-1}$ are those on the hyperbolic space in the interior of the light cone equipped with a potential, as described in [Vasy 2010, Lemma 7.11], or indeed in [Vasy 2013a, Proposition 4.2], where essentially the present notation is used.[10] As shown in [Vasy 2010,

[10]In [Vasy 2010, Lemma 7.11] $-\sigma^2$ plays the same role as σ^2 here or in [Vasy 2013a, Proposition 4.2].

Corollary 7.18], converted to our notation, the only possible poles are at

$$i\hat{s}_\pm(\lambda) - i\mathbb{N}, \quad \hat{s}_\pm(\lambda) = -\tfrac{1}{2}(n-1) \pm \sqrt{\tfrac{1}{4}(n-1)^2 - \lambda}. \tag{2-38}$$

In particular, when $\lambda = m^2$, $m > 0$, we conclude:

Lemma 2.23. *For $m > 0$, $\mathcal{P} = \Box_g - m^2$, with g induced by an asymptotically de Sitter metric as above, all poles of $\widehat{\mathcal{P}}^{-1}$ have strictly negative imaginary part.*

In other words, for small mass $m > 0$, there are no resonances σ of the Klein–Gordon operator with $\Im\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (2-36) no longer has a constant term. Let us fix such $m > 0$ and $\epsilon_0 > 0$, which ensures that, for $0 < \epsilon < \epsilon_0$, the only term in the asymptotic expansion (2-36), when $s > \frac{1}{2} + \epsilon$ and $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, is the "remainder" term $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$. Here we use that $\beta = 1$ in de Sitter space, hence also on an asymptotically de Sitter space; see [Vasy 2013a, §4.4], in particular the second displayed equation after Equation (4.16) there, which computes β in accordance with Remark 2.2.

Being interested in finding *forward solutions* to (nonlinear) wave equations on *asymptotically de Sitter spaces*, we now define the forward solution operator

$$S_{\mathrm{KG}} : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{2-39}$$

using Theorems 2.18 and 2.21.

Remark 2.24. If $\widetilde{M} \subset M$ is an asymptotically de Sitter space with global time function t, $\tau = e^{-t}$ is the defining function for future infinity, and the domain Ω is such that $\Omega \cap \widetilde{M} = \{\tau < \tau_0\}$, then S_{KG} in fact restricts to a forward solution operator on $\widetilde{M}$ itself; indeed, if $E : H_{\mathrm{b}}^{s-1,\epsilon}(\{\tau < \tau_0\}) \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is an extension operator, then the forward solution operator on $\{\tau < \tau_0\}$ is given by extending $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\{\tau < \tau_0\})$ using E, finding the forward solution on Ω using S_{KG}, and restricting back to $\{\tau < \tau_0\}$. The result is independent of the extension operator, as is easily seen from standard energy estimates; see in particular [Vasy 2013a, Proposition 3.9].

2B. ***A class of semilinear equations.*** Let us fix $m > 0$ and $\epsilon_0 > 0$ as above for statements about semilinear equations involving the Klein–Gordon operator; for equations involving the wave operator only, let $-\epsilon_0$ be equal to the largest imaginary part of all nonzero resonances of $\Box_g$. In Theorem 2.25 and further in the subsequent sections, bundles like ${}^{\mathrm{b}}T^*\Omega$ refer to ${}^{\mathrm{b}}T^*_\Omega M$; the boundaries H_j of Ω are regarded as artificial and do not affect the cotangent bundle or the corresponding vector fields.

Theorem 2.25. *Let $0 \leq \epsilon < \epsilon_0$ and $s > \frac{3}{2} + \epsilon$. Moreover, let*

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R,$$

where we use the norms corresponding to the map q. Then there is a constant $C_L > 0$ such that the following holds: If $L(0) < C_L$, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{2-40}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\| \le R$, that depends continuously on f.

More generally, suppose

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0,0,0) = 0$ and

$$\|q(u, {}^{\mathrm{b}}du, w) - q(u', {}^{\mathrm{b}}du', w')\| \le L(R)(\|u - u'\| + \|w - w'\|)$$

provided $\|u\| + \|w\|$, $\|u'\| + \|w'\| \le R$, where we use the norms corresponding to the map q, for a continuous nondecreasing function $L : \mathbb{R}_{\ge 0} \to \mathbb{R}$. Then there is a constant $C_L > 0$ such that the following holds: If $L(0) < C_L$, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du, \Box_g u) \tag{2-41}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \le R$, that depends continuously on f.

Further, if $\epsilon > 0$ and the nonlinearity is of the form $q({}^{\mathrm{b}}du)$, with

$$q : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

having a small Lipschitz constant near 0, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution u with $u - (\phi \circ \mathfrak{t}_1)c = u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, where $c \in \mathbb{C}$, that depends continuously on f, i.e., $c \in \mathbb{C}$ and $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ depend continuously on f. Here, $\phi \in C^\infty(\mathbb{R})$ with support in $(0,\infty)$ and $\mathfrak{t}_1$ are as in Theorem 2.21. In fact, the statement even holds for nonlinearities $q(u, {}^{\mathrm{b}}du)$ provided

$$q : (\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)) \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

has a small Lipschitz constant near 0.

Proof. To prove the first part, let S_{KG} be the forward solution operator for $\Box_g - m^2$ as in (2-39). We want to apply the Banach fixed point theorem to the operator $T_{\mathrm{KG}} : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $T_{\mathrm{KG}}u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$.

Let $C_L = \|S_{\mathrm{KG}}\|^{-1}$; then we have the estimate

$$\|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\| \le \|S_{\mathrm{KG}}\| L(R') \|u - v\| \le C_0 \|u - v\| \tag{2-42}$$

for $\|u\|, \|v\| \le R$ and a constant $C_0 < 1$, granted that $L(R) \le C_0 \|S_{\mathrm{KG}}\|^{-1}$, which holds for small $R > 0$ by assumption on L. Then, T_{KG} maps the R-ball in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ into itself if $\|S_{\mathrm{KG}}\|(\|f\| + L(R)R) \le R$, i.e., if $\|f\| \le R(\|S_{\mathrm{KG}}\|^{-1} - L(R))$. Put

$$C = R(\|S_{\mathrm{KG}}\|^{-1} - L(R)).$$

Then the existence of a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with $\|u\| \le R$ to the PDE (2-40) with $\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \le C$ follows from the Banach fixed point theorem.

To prove the continuous dependence of u on f, suppose we are given $u_j \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $j = 1, 2$, with $\|u_j\| \le R$, and $f_j \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f_j\| \le C$, such that

$$(\Box_g - m^2)u_j = f_j + q(u_j, {}^{\mathrm{b}}du_j), \quad j = 1, 2.$$

Then

$$(\Box_g - m^2)(u_1 - u_2) = f_1 - f_2 + q(u_1, {}^{\mathrm{b}}du_1) - q(u_2, {}^{\mathrm{b}}du_2),$$

hence

$$\|u_1 - u_2\| \le \|S_{\mathrm{KG}}\| \big(\|f_1 - f_2\| + L(R)\|u_1 - u_2\|\big),$$

which in turn gives

$$\|u_1 - u_2\| \le \frac{\|f_1 - f_2\|}{1 - C_0}.$$

This completes the proof of the first part.

For the more general statement, we use the fact that one can think of $\Box_g$ in the nonlinearity as a first-order operator. Concretely, we work on the coisotropic space

$$\mathscr{X} = \{u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathscr{X}} = \|u\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}}.$$

This is a Banach space: if (u_k) is a Cauchy sequence in $\mathscr{X}$, then $u_k \to u$ in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and $\Box_g u_k \to v$ in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$; in particular,

$$\Box_g u_k \to \Box_g u \quad \text{and} \quad \Box_g u_k \to v \quad \text{in } \tau^{\epsilon} H_{\mathrm{b}}^{s-2}(\Omega)^{\bullet,-},$$

thus $\Box_g u = v \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, which was to be shown. We then define $T_{\mathrm{KG}} : \mathscr{X} \to \mathscr{X}$ by $T_{\mathrm{KG}} u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du, \Box_g u))$ and obtain the estimate

$$\begin{aligned}\|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\|_{\mathscr{X}} &= \|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|q(u, {}^{\mathrm{b}}du, \Box_g u) - q(v, {}^{\mathrm{b}}dv, \Box_g v)\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \\ &\le (\|S_{\mathrm{KG}}\| + 1)L(R)(\|u - v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u - \Box_g v\|_{H_{\mathrm{b}}^{s-1,\epsilon}}) \\ &= (\|S_{\mathrm{KG}}\| + 1)L(R)\|u - v\|_{\mathscr{X}} \le C_0 \|u - v\|_{\mathscr{X}}\end{aligned}$$

for $u, v \in \mathscr{X}$ with norms bounded by R, with $C_0 < 1$ if $R > 0$ is small enough, provided we require $L(0) < C_L := (\|S_{\mathrm{KG}}\| + 1)^{-1}$. Then, for $u \in \mathscr{X}$ with $\|u\| \le R$,

$$\|T_{\mathrm{KG}}u\|_{\mathscr{X}} \le (\|S_{\mathrm{KG}}\| + 1)(\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} + L(R)R) \le R$$

if $\|f\| \le C$ with $C > 0$ small. Thus, T_{KG} is a contraction on $\mathscr{X}$, and we obtain the solvability of (2-41). The continuous dependence of the solution on the forcing term f is proved as above.

For the third part, we use the forward solution operator $S : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to \mathscr{Y} := \mathbb{C} \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ for $\Box_g$; note that $\mathscr{Y}$ is a Banach space with norm $\|(c, u')\|_{\mathscr{Y}} = |c| + \|u'\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}}$. (See Section 2C for related, more general statements.) We will apply the Banach fixed point theorem to the operator $T : \mathscr{Y} \to \mathscr{Y}$, $Tu = S(f + q(u, {}^{\mathrm{b}}du))$: we again have an estimate like (2-42), since ${}^{\mathrm{b}}du \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-}$ for $u \in \mathscr{Y}$ and, for small $R > 0$, T maps the R-ball around 0 in $\mathscr{Y}$ into itself if the norm of f in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is small, as above. The continuous dependence of the solution on the forcing term is proved as above. $\square$

The following basic statement ensures that there are interesting nonlinearities q that satisfy the requirements of the theorem; see also Section 2C.

Lemma 2.26. *Let $s > \frac{1}{2}n$; then $H_{\mathrm{b}}^s(\mathbb{R}_+^n)$ is an algebra. In particular, $H_{\mathrm{b}}^s(N)$ is an algebra on any compact n-dimensional manifold N with boundary which is equipped with a b-metric.*

Proof. The first statement is the special case $k = 0$ of Lemma 4.4 after a logarithmic change of coordinates, which gives an isomorphism $H_{\mathrm{b}}^s(\mathbb{R}_+^n) \cong H^s(\mathbb{R}^n)$; the lemma is well known in this case (see, e.g., [Taylor 1997, Chapter 13.3]). The second statement follows by localization and from the coordinate invariance of H_{b}^s. $\square$

More, related statements will be given in Section 4B.

Remark 2.27. The algebra property of $H_{\mathrm{b}}^s(N)$ for $s > \frac{1}{2}\dim(N)$ is a special case of the fact that, for any $F \in C^\infty(\mathbb{R})$ (for real-valued u) or $F \in C^\infty(\mathbb{C})$ (for complex-valued u) with $F(0) = 0$, the composition map $H_{\mathrm{b}}^s(N) \to H_{\mathrm{b}}^s(N)$, $u \mapsto F \circ u$, is well defined and continuous; see, for example, [Taylor 1997, Chapter 13.10]. In the real-valued u case, if $F(0) \ne 0$ then writing $F(t) = F(0) + tF_1(t)$ shows that $F \circ u \in \mathbb{C} + H_{\mathrm{b}}^s(N)$. If $r > 0$, then $H_{\mathrm{b}}^{s,r}(N) \subset H_{\mathrm{b}}^s(N)$ shows that $F_1(u) \in H_{\mathrm{b}}^s(N)$, thus $F \circ u = F(0) + uF_1(u) \in \mathbb{C} + H_{\mathrm{b}}^{s,r}(N)$; and, if F vanishes to order k at 0, then $F(t) = t^k F_k(t)$, so $F \circ u = u^k(F_k \circ u)$, and the multiplicative properties of $H_{\mathrm{b}}^{s,r}(N)$ show that $F \circ u \in H_{\mathrm{b}}^{s,kr}(N)$. The argument is analogous for complex-valued u, indeed for $\mathbb{R}^L$-valued u, using Taylor's theorem on F at the origin.

Corollary 2.28. *If $s > \frac{1}{2}n$, the hypotheses of Theorem 2.25 hold for nonlinearities $q(u) = cu^p$, $p \ge 2$ an integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_{\mathrm{b}}^s(M)$.*

If $s - 1 > \frac{1}{2}n$, the hypotheses of Theorem 2.25 hold for nonlinearities

$$q(u, {}^{\mathrm{b}}du) = \sum_{2 \le j + |\alpha| \le d} q_{j\alpha} u^j \prod_{k \le |\alpha|} X_{\alpha,k} u, \tag{2-43}$$

where $q_{j,\alpha} \in \mathbb{C} + H_{\mathrm{b}}^s(M)$, $X_{\alpha,k} \in \mathcal{V}_{\mathrm{b}}(M)$.

Thus, in either case, for $m > 0$, $0 \le \epsilon < \epsilon_0$, $s > \frac{3}{2} + \epsilon$ and for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{2-44}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, *with* $\|u\| \leq R$, *that depends continuously on* f.

The analogous conclusion also holds for $\Box_g u = f + q(u, {}^{\mathrm{b}}du)$ *provided* $\epsilon > 0$ *and*

$$q(u, {}^{\mathrm{b}}du) = \sum_{\substack{2 \leq j+|\alpha| \leq d \\ |\alpha| \geq 1}} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \tag{2-45}$$

with the solution being in $\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $\phi \circ \mathfrak{t}_1$ *identically* 1 *near* $X \cap \Omega$ *and vanishing near* H_1.

Remark 2.29. For such polynomial nonlinearities, the Lipschitz constant $L(R)$ in the statement of Theorem 2.25 satisfies $L(0) = 0$.

Remark 2.30. In this paper, we do not prove that one obtains smooth (i.e., conormal) solutions if the forcing term is smooth (conormal); see [Hintz 2013] for such a result in the quasilinear setting.

Since in Theorem 2.25 we allow q to depend on $\Box_g u$, we can in particular solve certain quasilinear equations if $s > \max\left(\frac{1}{2} + \epsilon, \frac{1}{2}n + 1\right)$: Suppose for example that $q' : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1}(\Omega)^{\bullet,-}$ is continuous with $\|q'(u) - q'(v)\| \leq L'(R)\|u - v\|$ for $u, v \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with norms bounded by R, where $L' : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally bounded; then we can solve the equation

$$(1 + q'(u))(\Box_g - m^2)u = f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

provided the norm of f is small. Indeed, if we put $q(u, w) = -q'(u)(w - m^2 u)$, then $q(u, \Box_g u) = -q'(u)(\Box_g - m^2)u$ and the PDE becomes

$$(\Box_g - m^2)u = f + q(u, \Box_g u),$$

which is solvable by Theorem 2.25, since, with $\|\cdot\| = \|\cdot\|_{H_{\mathrm{b}}^{s-1,\epsilon}}$, for u, $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and w, $w' \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|u\| + \|w\|$, $\|u'\| + \|w'\| \leq R$, we have

$$\begin{aligned}\|q(u,w) - q(u',w')\| &\leq \|q'(u) - q'(u')\| \, \|w - m^2 u\| + \|q'(u')\| \, \|w - w' - m^2(u - u')\| \\ &\leq L'(R)((1+m^2)R + m^2 R)\|u - u'\| + L'(R)R\|w - w'\| \\ &\leq L(R)(\|u - u'\| + \|w - w'\|)\end{aligned}$$

with $L(R) \to 0$ as $R \to 0$.

By a similar argument, one can also allow q' to depend on ${}^{\mathrm{b}}du$ and $\Box_g u$.

Remark 2.31. Recalling the discussion following Theorem 2.21, let us emphasize the importance of $\widehat{P}(\sigma)^{-1}$ having no poles in the closed upper half plane by looking at the explicit example of the operator $\mathcal{P} = \partial_x$ in 1 dimension. In terms of $\tau = e^{-x}$, we have $\mathcal{P} = -\tau\partial_\tau$, thus $\widehat{P}(\sigma) = -i\sigma$, considered as an operator on the boundary (which is a single point) at $+\infty$ of the radial compactification of $\mathbb{R}$; hence $\widehat{P}(\sigma)^{-1}$ has a simple pole at $\sigma = 0$, corresponding to constants being annihilated by $\mathcal{P}$. Now suppose we want to find a forward solution of $u' = u^2 + f$, where $f \in C_c^\infty(\mathbb{R})$. In the first step of the iterative procedure described above, we will obtain a constant term; the next step gives a term that is linear in x (x being the antiderivative of 1), i.e., in $\log \tau$, then we get quadratic terms and so on, therefore the iteration does not converge (for general f), which is of course to be expected, since solutions to $u' = u^2 + f$ in general blow up in finite time. On the other hand, if $\mathcal{P} = \partial_x + 1$ then $\widehat{P}(\sigma)^{-1} = (1 - i\sigma)^{-1}$, which has a simple pole at $\sigma = -i$, which means that forward solutions u of $u' + u = u^2 + f$ with f as above can be constructed iteratively and the first term of the expansion of u at $+\infty$ is $c\tau^{i(-i)} = ce^{-x}$, $c \in \mathbb{C}$.

2C. ***Semilinear equations with polynomial nonlinearity.*** With polynomial nonlinearities as in (2-43), we can use the second part of Theorem 2.21 to obtain an asymptotic expansion for the solution; see Remark 2.38 and, in a slightly different setting, Section 3B for details on this. Here, we instead define a space that encodes asymptotic expansions directly in such a way that we can run a fixed point argument directly.

To describe the exponents appearing in the expansion, we use index sets, as introduced by Melrose [1993].

Definition 2.32. (1) An *index set* is a discrete subset $\mathcal{E}$ of $\mathbb{C} \times \mathbb{N}_0$ satisfying the conditions

   (a) if $(z, k) \in \mathcal{E}$ then $(z, j) \in \mathcal{E}$ for $0 \le j \le k$, and
   (b) if (z_j, k_j) is a sequence of elements of $\mathcal{E}$ with $|z_j| + k_j \to \infty$, then $\Re z_j \to \infty$.

(2) For any index set $\mathcal{E}$, define

$$w_{\mathcal{E}}(z) = \begin{cases} \max\{k \in \mathbb{N}_0 : (z, k) \in \mathcal{E}\} & \text{if } (z, 0) \in \mathcal{E}, \\ -\infty & \text{otherwise.} \end{cases}$$

(3) For two index sets $\mathcal{E}$ and $\mathcal{E}'$, define their extended union by

$$\mathcal{E} \mathbin{\overline{\cup}} \mathcal{E}' = \mathcal{E} \cup \mathcal{E}' \cup \{(z, l + l' + 1) : (z, l) \in \mathcal{E},\ (z, l') \in \mathcal{E}'\}$$

and their product by $\mathcal{E}\mathcal{E}' = \{(z + z', l + l') : (z, l) \in \mathcal{E},\ (z', l') \in \mathcal{E}'\}$. We shall write $\mathcal{E}^k$ for the k-fold product of $\mathcal{E}$ with itself.

(4) A *positive index set* is an index set $\mathcal{E}$ with the property that $\Re z > 0$ for all $z \in \mathbb{C}$ with $(z, 0) \in \mathcal{E}$.

Remark 2.33. To ensure that the class of polyhomogeneous conormal distributions with a given index set $\mathcal{E}$ is invariantly defined, [Melrose 1993] in addition requires that $(z, k) \in \mathcal{E}$ implies $(z + j, k) \in \mathcal{E}$ for all $j \in \mathbb{N}_0$. In particular, this is a natural condition in non-dilation-invariant settings, as in Theorem 2.21. A convenient way to enforce this condition in all relevant situations is to enlarge the index set corresponding to the poles of the inverse of the normal operator accordingly; see the statement of Theorem 2.37.

Observe though that this condition is not needed in the dilation-invariant cases of the solvability statements below.

Since we want to capture the asymptotic behavior of solutions near $X \cap \Omega$, we fix a cutoff $\phi \in C^\infty(\mathbb{R})$ with support in $(0, \infty)$ such that $\phi \circ \mathfrak{t}_1 \equiv 1$ near $X \cap \Omega$ (we already used such a cutoff in Theorem 2.21), and make the following definition:

Definition 2.34. Let $\mathcal{E}$ be an index set, and let s, $r \in \mathbb{R}$. For $\epsilon > 0$ with the property that there is no $(z, 0) \in \mathcal{E}$ with $\Re z = \epsilon$, define the space $\mathcal{X}_{\mathcal{E}}^{s,r,\epsilon}$ to consist of all tempered distributions v on M with support in $\overline{\Omega}$ such that

$$v' = v - \sum_{\substack{(z,k)\in\mathscr{E} \\ \Re z<\epsilon}} \tau^z (\log\tau)^k (\phi\circ \mathrm{t}_1) v_{z,k} \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{2-46}$$

for certain $v_{z,k} \in H^r(X\cap\Omega)$.

Observe that the terms $v_{z,k}$ in the expansion (2-46) are uniquely determined by v, since $\epsilon > \Re z$ for all $z\in\mathbb{C}$ for which $(z,0)$ appears in the sum (2-46); then also v' is uniquely determined by v. Therefore, we can use the isomorphism

$$\mathscr{X}_{\mathscr{E}}^{s,r,\epsilon} \cong \Biggl(\bigoplus_{\substack{(z,k)\in\mathscr{E} \\ \Re z<\epsilon}} H^r(X\cap\Omega)\Biggr) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$$

to give $\mathscr{X}_{\mathscr{E}}^{s,r,\epsilon}$ the structure of a Banach space.

Lemma 2.35. *Let $\mathscr{P}$ and $\mathscr{F}$ be positive index sets, and let $\epsilon>0$. Define $\mathscr{E}'_0 = \mathscr{P}\,\overline{\cup}\,\mathscr{F}$ and, recursively, $\mathscr{E}'_{N+1} = \mathscr{P}\,\overline{\cup}\,\bigl(\mathscr{F}\cup\bigcup_{k\geq 2}(\mathscr{E}'_N)^k\bigr)$; put $\mathscr{E}_N = \{(z,k)\in\mathscr{E}'_N : 0<\Re z\leq\epsilon\}$. Then there exists $N_0\in\mathbb{N}$ such that $\mathscr{E}_N = \mathscr{E}_{N_0}$ for all $N\geq N_0$; moreover, the limiting index set $\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon) := \mathscr{E}_{N_0}$ is finite.*

Proof. Writing $\pi_1 : \mathbb{C}\times\mathbb{N}_0 \to \mathbb{C}$ for the projection, one has

$$\pi_1\mathscr{E}_1 = \Biggl\{ z = \sum_{j=1}^k z_j \;:\; 0<\Re z\leq\epsilon,\ k\geq 1,\ z_j\in\pi_1\mathscr{E}_0 \Biggr\},$$

and it is then clear that $\pi_1\mathscr{E}_N = \pi_1\mathscr{E}_1$ for all $N\geq 1$. Since $\mathscr{E}_0$ is a positive index set, there exists $\delta>0$ such that $\Re z\geq\delta$ for all $z\in\mathscr{E}_0$; hence, $\pi_1\mathscr{E}_\infty = \pi_1\mathscr{E}_1$ is finite.

To finish the proof, we need to show that, for all $z\in\mathbb{C}$, the number $w_{\mathscr{E}_N}(z)$ stabilizes. Defining $p(z) = w_{\mathscr{P}}(z)+1$ for $z\in\pi_1\mathscr{P}$ and $p(z)=0$ otherwise, we have a recursion relation

$$w_{\mathscr{E}_N}(z) = p(z) + \max\Biggl\{ w_{\mathscr{F}}(z), \max_{\substack{z=z_1+\cdots+z_k \\ k\geq 2,\, z_j\in\pi_1\mathscr{E}_\infty}} \Biggl\{\sum_{j=1}^k w_{\mathscr{E}_{N-1}}(z_j)\Biggr\}\Biggr\}, \quad N\geq 1. \tag{2-47}$$

For each z_j appearing in the sum, we have $\Im z_j \leq \Im z - \delta$. Thus, we can use (2-47) with z replaced by such z_j and N replaced by $N-1$ to express $w_{\mathscr{E}_N}(z)$ in terms of a finite number of $p(z_\alpha)$ and $w_{\mathscr{F}}(z_\alpha)$, $\Im z_\alpha\leq\Im z$, and a finite number of $w_{\mathscr{E}_{N-2}}(z_\beta)$, $z_\beta\leq\Im z - 2\delta$. Continuing in this way, after $N_0 = \lfloor(\Im z)/\delta\rfloor + 1$ steps we have expressed $w_{\mathscr{E}_N(z)}$ in terms of a finite number of $p(z_\gamma)$ and $w_{\mathscr{F}}(z_\gamma)$, $\Im z_\gamma\leq\Im z$, only, and this expression is independent of N as long as $N\geq N_0$. □

Definition 2.36. Let $\mathscr{P}$ and $\mathscr{F}$ be positive index sets and let $\epsilon>0$ be such that there is no $(z,0)$ in $\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)$ with $\Re z=\epsilon$, with $\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)$ as defined in the statement of Lemma 2.35. Then, for s, $r\in\mathbb{R}$, define the Banach spaces

$$\mathscr{X}_{\mathscr{P},\mathscr{F}}^{s,r,\epsilon} := \mathscr{X}_{\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)}^{s,r,\epsilon},$$
$${}^0\mathscr{X}_{\mathscr{P},\mathscr{F}}^{s,r,\epsilon} := \mathscr{X}_{\mathscr{E}_\infty(\mathscr{P},\mathscr{F},\epsilon)\cup\{(0,0)\}}^{s,r,\epsilon}.$$

Note that the spaces ${}^{(0)}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ are Banach algebras for $s > \frac{1}{2}n$ in the sense that there is a constant $C > 0$ such that $\|uv\| \le C\|u\|\,\|v\|$ for all $u, v \in {}^{(0)}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$. Moreover, $\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ interacts well with the forward solution operator S_{KG} of $\Box_g - m^2$ in the sense that $u \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ and $k \ge 2$ — with $\mathscr{P}$ being related to the poles of $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, as will be made precise in the statement of Theorem 2.37 below — implies $S_{\mathrm{KG}}(u^k) \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$.

We can now state the result giving an asymptotic expansion of the solution of $(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ for polynomial nonlinearities q.

Theorem 2.37. *Let $\epsilon > 0$, $s > \max\bigl(\frac{3}{2} + \epsilon, \frac{1}{2}n + 1\bigr)$, and q as in (2-43). Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, and $m_j + 1$ is the order of the pole of $\widehat{\mathcal{P}}(\sigma)^{-1}$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) : 0 \le \ell \le m_j,\ k \in \mathbb{N}_0\}$. Assume that $\epsilon \ne \Re(i\sigma_j)$ for all j and that, moreover, $m > 0$, which implies that $\mathscr{P}$ is a positive index set; see Lemma 2.23. Finally, let $\mathscr{F}$ be a positive index set.*

Then, for small enough $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, with $\|u\| \le R$, that depends continuously on f; in particular, u has an asymptotic expansion with remainder term in $H^{s,\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-}$.

Further, if the polynomial nonlinearity is of the form $q({}^{\mathrm{b}}du)$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ with $\|f\| \le C$, the equation

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution $u \in {}^{0}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, with $\|u\| \le R$, that depends continuously on f.

Proof. By Theorem 2.21 and the definition of the space $\mathscr{X} = \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S_{\mathrm{KG}} : \mathscr{X} \to \mathscr{X}$ of $\Box_g - m^2$. Thus, we can apply the Banach fixed point theorem to the operator $T : \mathscr{X} \to \mathscr{X}$, $Tu = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$, where we note that $q : \mathscr{X} \to \mathscr{X}$, which follows from the definition of $\mathscr{X}$ and the fact that q is a polynomial only involving terms of the form $u^j \prod_{k \le |\alpha|} X_{\alpha,k} u$ for $j + |\alpha| \ge 2$. This condition on q also ensures that T is a contraction on a sufficiently small ball in $\mathscr{X}_+$.

For the second part, writing ${}^{0}\mathscr{X} = {}^{0}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S : \mathscr{X} \to {}^{0}\mathscr{X}$. But $q({}^{\mathrm{b}}du) : {}^{0}\mathscr{X} \to \mathscr{X}$, since ${}^{\mathrm{b}}d$ annihilates constants, and we can thus finish the proof as above.

The continuous dependence of the solution on the right-hand side is proved as in Theorem 2.25. □

Note that $\epsilon > 0$ is (almost) unrestricted here, and thus we can get arbitrarily many terms in the asymptotic expansion if we work with arbitrarily high Sobolev spaces.

The condition that the polynomial $q(u, {}^{\mathrm{b}}du)$ does not involve a linear term is very important as it prevents logarithmic terms from stacking up in the iterative process used to solve the equation. Also, adding a term νu to $q(u, {}^{\mathrm{b}}du)$ effectively changes the Klein–Gordon parameter from $-m^2$ to $\nu - m^2$, which changes the location of the poles of $\widehat{P}(\sigma)^{-1}$; in the worst case, if $\nu > m^2$, this would even cause a pole to move to $\Im\sigma > 0$, corresponding to a resonant state that blows up exponentially in time. Lastly, let us remark that the form (2-45) of the nonlinearity is not sufficient to obtain an expansion beyond leading order, since, in the iterative procedure, logarithmic terms would stack up in the next-to-leading-order term of the expansion.

Remark 2.38. Instead of working with the spaces ${}^{(0)}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, which have the expansion built in, one could alternatively first prove the existence of a solution u in a (slightly) decaying b-Sobolev space, which then allows one to regard the polynomial nonlinearity as a perturbation of the linear operator $\Box_g - m^2$; then an iterative application of the dilation-invariant result [Vasy 2013a, Lemma 3.1] gives an expansion of the solution to the nonlinear equation. We will follow this idea in the discussion of polynomial nonlinearities on asymptotically Kerr-de Sitter spaces in the next section.

3. Kerr–de Sitter space

In this section we analyze semilinear waves on Kerr–de Sitter space and, more generally, on spaces with normally hyperbolic trapping, discussed below. The effect of the latter is a loss of derivatives for the linear estimates in general, but we show that at least derivatives with principal symbol vanishing on the trapped set are well behaved. We then use these results to solve semilinear equations in the rest of the section.

3A. *Linear Fredholm theory.* The linear theorem in the case of normally hyperbolic trapping for dilation-invariant operators $\mathcal{P} = \Box_g - \lambda$ is the following:

Theorem 3.1 (see [Vasy 2013a, Theorem 1.4]). *Let M be a manifold with a b-metric g as above, with boundary X, and let τ be the boundary defining function with $\mathcal{P}$ as in (2-15). If g has normally hyperbolic trapping, $\mathfrak{t}_1$ and Ω are as above, and $\phi \in C^\infty(\mathbb{R})$ is as in Theorem 2.21, then there exist $C' > 0$, $\varkappa > 0$ and $\beta \in \mathbb{R}$ such that, for $0 \leq \ell < C'$ and $s > \frac{1}{2} + \beta\ell$, $s \geq 0$, solutions $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega)^{\bullet,-}$ of $(\Box_g - \lambda)u = f$ with $f \in H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$ satisfy that, for some $a_{j\kappa} \in C^\infty(\Omega \cap X)$ (which are the resonant states) and $\sigma_j \in \mathbb{C}$ (which are the resonances),*

$$u' = u - \sum_j \sum_{\kappa \leq m_j} \tau^{i\sigma_j} (\log \tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}. \tag{3-1}$$

Here the (semi)norms of both $a_{j\kappa}$ in $C^\infty(\Omega \cap X)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$. The same conclusion holds for sufficiently small perturbations of the metric as a symmetric bilinear form on ${}^{\mathrm{b}}TM$ provided the trapping is normally hyperbolic.

In order to state the analogue of Theorems 2.18 and 2.21 when one has *normally hyperbolic trapping* at $\Gamma \subset {}^{\mathrm{b}}S^*_X M$, we will employ nontrapping estimates in certain so-called normally isotropic functions spaces, established in [Hintz and Vasy 2014b]. To put our problem into the context of [Hintz and Vasy 2014b], we need some notation in addition to that in Section 2; in the setting of Section 2, as leading up to Theorem 2.18 — see the discussion above Figure 3 — we define

(1) the *forward trapped set in* Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ through which bicharacteristics do not flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_+$ in the forward direction (i.e., they do not reach ${}^{\mathrm{b}}S^*_{H_1} M$ in finite time and they do not tend to L_+),

(2) the *backward trapped set in* Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_+$ in the backward direction,

(3) the *forward trapped set in* Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_-$ in the forward direction, and

(4) the *backward trapped set in* Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_-$ in the backward direction.

The *forward trapped set* Γ_- is the union of the forward trapped sets in $\Sigma_\pm$, and analogously for the *backward trapped set* Γ_+. The *trapped set* Γ is the intersection of the forward and backward trapped sets. We say that $\mathcal{P}$ is *normally hyperbolically trapping*, or has normally hyperbolic trapping, if $\Gamma \subset {}^{\mathrm{b}}S^*_X M$ is b-normally hyperbolic in the sense discussed in [Hintz and Vasy 2014b, §3.2].

Following [Hintz and Vasy 2014b], we introduce replacements for the b-Sobolev spaces used in Section 2, which are called *normally isotropic at* Γ; these spaces $\mathcal{H}^s_{\mathrm{b},\Gamma}$ — see also (3-2) — and dual spaces $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ are just the standard b-Sobolev spaces $H^s_{\mathrm{b}}(M)$ and $H^{-s}_{\mathrm{b}}(M)$, respectively, microlocally away from Γ.

Concretely, suppose Γ is locally (in a neighborhood U_0 of Γ) defined by $\tau = 0$, $\phi_+ = \phi_- = 0$, $\hat{p} = 0$ in ${}^{\mathrm{b}}S^*M$, with $d\tau$, $d\phi_+$, $d\phi_-$, $d\hat{p}$ and $\hat{p} = \tilde{\rho}^m p$, linearly independent at Γ. Here, one should think of ϕ_- as being a defining function of $\Gamma_+ \cap \Sigma_+$ or $\Gamma_- \cap \Sigma_-$ within ${}^{\mathrm{b}}S^*M$, and ϕ_+ of $\Gamma_\pm \cap \Sigma_\mp$ within ${}^{\mathrm{b}}S^*_X M$. Then, taking any $Q_\pm \in \Psi^0_{\mathrm{b}}(M)$ with principal symbol $\phi_\pm$, $\widehat{P} \in \Psi^0_{\mathrm{b}}(M)$ with principal symbol $\hat{p}$, and $Q_0 \in \Psi^0_{\mathrm{b}}(M)$ elliptic on U_0^c with $\mathrm{WF}'_{\mathrm{b}}(Q_0) \cap \Gamma = \varnothing$, we define the (global) b-normally isotropic spaces at Γ of order s, $\mathcal{H}^s_{\mathrm{b},\Gamma} = \mathcal{H}^s_{\mathrm{b},\Gamma}(M)$, by the norm

$$\|u\|^2_{\mathcal{H}^s_{\mathrm{b},\Gamma}} = \|Q_0 u\|^2_{H^s_{\mathrm{b}}} + \|Q_+ u\|^2_{H^s_{\mathrm{b}}} + \|Q_- u\|^2_{H^s_{\mathrm{b}}} + \|\tau^{1/2} u\|^2_{H^s_{\mathrm{b}}} + \|\widehat{P} u\|^2_{H^s_{\mathrm{b}}} + \|u\|^2_{H^{s-1/2}_{\mathrm{b}}}, \tag{3-2}$$

and let $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ be the dual space relative to L^2, which is

$$Q_0 H^{-s}_{\mathrm{b}} + Q_+ H^{-s}_{\mathrm{b}} + Q_- H^{-s}_{\mathrm{b}} + \tau^{1/2} H^{-s}_{\mathrm{b}} + \widehat{P} H^{-s}_{\mathrm{b}} + H^{-s+1/2}_{\mathrm{b}}.$$

In particular,

$$\begin{aligned} H^s_{\mathrm{b}}(M) \subset \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \subset H^{s-1/2}_{\mathrm{b}}(M) \cap H^{s,-1/2}_{\mathrm{b}}(M), \\ H^{s+1/2}_{\mathrm{b}}(M) + H^{s,1/2}_{\mathrm{b}}(M) \subset \mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(M) \subset H^s_{\mathrm{b}}(M). \end{aligned} \tag{3-3}$$

Microlocally away from Γ, $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ is indeed just the standard H^s_{b} space, while $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ is H^{-s}_{b}, since at least one of Q_0, $Q_\pm$, τ and $\widehat{P}$ is elliptic; the space is independent of the choice of Q_0 satisfying the criteria, since at least one of $Q_\pm$, τ and $\widehat{P}$ is elliptic on $U_0 \setminus \Gamma$. Moreover, every operator in $\Psi^k_{\mathrm{b}}(M)$ defines a continuous map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to \mathcal{H}^{s-k}_{\mathrm{b},\Gamma}(M)$ because, for $A \in \Psi^k_{\mathrm{b}}(M)$, $Q_+ A u = A Q_+ u + [Q_+, A] u$ and $[Q_+, A] \in \Psi^{k-1}_{\mathrm{b}}(M)$; the analogous statement also holds for the dual spaces.

The nontrapping estimates then are:

Proposition 3.2 (see [Hintz and Vasy 2014b, Theorem 3]). *With* $\mathcal{P}$, $\mathcal{H}^s_{\mathrm{b},\Gamma}$ *and* $\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}$ *as above, for any neighborhood* U *of* Γ *and any* N, *there exist* $B_0 \in \Psi^0_{\mathrm{b}}(M)$ *elliptic at* Γ *and* $B_1, B_2 \in \Psi^0_{\mathrm{b}}(M)$ *with* $\mathrm{WF}'_{\mathrm{b}}(B_j) \subset U$, $j = 0, 1, 2$, $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_+ = \varnothing$, *and* $C > 0$, *such that*

$$\|B_0 u\|_{\mathcal{H}^s_{\mathrm{b},\Gamma}} \le \|B_1 \mathcal{P} u\|_{\mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}} + \|B_2 u\|_{H^s_{\mathrm{b}}} + C\|u\|_{H^{-N}_{\mathrm{b}}}, \tag{3-4}$$

i.e., if all the functions on the right-hand side are in the indicated spaces ($B_1\mathcal{P}u \in \mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}$, *etc.*) *then* $B_0u \in \mathcal{H}^s_{\mathrm{b},\Gamma}$, *and the inequality holds.*

The same conclusion also holds if we assume $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_- = \varnothing$ *instead of* $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_+ = \varnothing$.

Finally, if $r < 0$ *then, with* $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_+ = \varnothing$, (3-4) *becomes*

$$\|B_0u\|_{H^{s,r}_{\mathrm{b}}} \le \|B_1\mathcal{P}u\|_{H^{s-m+1,r}_{\mathrm{b}}} + \|B_2u\|_{H^{s,r}_{\mathrm{b}}} + C\|u\|_{H^{-N,r}_{\mathrm{b}}} \tag{3-5}$$

while, if $r > 0$ *then, with* $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_- = \varnothing$,

$$\|B_0u\|_{H^{s,r}_{\mathrm{b}}} \le \|B_1\mathcal{P}u\|_{H^{s-m+1,r}_{\mathrm{b}}} + \|B_2u\|_{H^{s,r}_{\mathrm{b}}} + C\|u\|_{H^{-N,r}_{\mathrm{b}}}. \tag{3-6}$$

Remark 3.3. Note that the weighted versions (3-5)–(3-6) use *standard* weighted b-Sobolev spaces.

Next, if $\Omega \subset M$, as in Section 2, is such that ${}^{\mathrm{b}}S^*_{H_j}\Omega \cap \Gamma = \varnothing$, $j = 1, 2$, then spaces such as

$$\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$$

are not only well defined but are standard H^s_{b}-spaces near the H_j. The inclusions analogous to (3-3) also hold for the corresponding spaces over Ω.

Notice that elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$, with the issues being at Γ corresponding to (3-3) (thus there is no distinction between the behavior on the Ω vs. the M-based spaces). However, if $A \in \Psi^p_{\mathrm{b}}(M)$ has principal symbol vanishing on Γ then

$$A : \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to H^{s-p}_{\mathrm{b}}(M) \quad \text{and} \quad A : H^s_{\mathrm{b}}(M) \to \mathcal{H}^{*,s-p}_{\mathrm{b},\Gamma}(M), \tag{3-7}$$

as A can be expressed as $A_+Q_+ + A_-Q_- + A_\partial\tau + \hat{A}\hat{P} + A_0Q_0 + R$ with $A_\pm$, A_0, A_∂, $\hat{A} \in \Psi^0_{\mathrm{b}}(M)$ and $R \in \Psi^{-1}_{\mathrm{b}}(M)$, with the second mapping property following by duality as $\Psi^p_{\mathrm{b}}(M)$ is closed under adjoints and the principal symbol of the adjoint vanishes wherever that of the original operator does. Correspondingly, if $A_j \in \Psi^{m_j}_{\mathrm{b}}(M)$, $j = 1, 2$, have principal symbol vanishing at Γ then $A_1A_2u : \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to \mathcal{H}^{*,s-m_1-m_2}_{\mathrm{b},\Gamma}(M)$.

We consider $\mathcal{P}$ as a map

$$\mathcal{P} : \mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-} \to \mathcal{H}^{s-2}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$$

and let

$$\mathcal{Y}^s_\Gamma = \mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}, \quad \mathcal{X}^s_\Gamma = \{u \in \mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-} : \mathcal{P}u \in \mathcal{Y}^{s-1}_\Gamma\}.$$

While $\mathcal{X}^s_\Gamma$ is complete,[11] it is a slightly exotic space, unlike $\mathcal{X}^s$ in Theorem 2.18, which is a coisotropic space depending on Σ (and thus the principal symbol of $\mathcal{P}$) only, since elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$, as remarked earlier. In fact, $\mathcal{X}^s_\Gamma$ actually depends on $\mathcal{P}$ modulo $\Psi^0_{\mathrm{b}}(M)$ plus

[11] Also, elements of $C^\infty(\Omega)$ vanishing to infinite order at H_1 and $X \cap \Omega$ are dense in $\mathcal{X}^s_\Gamma$. Indeed, in view of [Melrose et al. 2013, Lemma A.3] the only possible issue is at Γ, thus the distinction between Ω and M may be dropped. To complete the argument, one proceeds as in the quoted lemma, using the ellipticity of σ at Γ, letting $\Lambda_n \in \Psi^{-\infty}_{\mathrm{b}}(M)$, $n \in \mathbb{N}$, be a quantization of $\phi(\sigma/n)a$ with $a \in C^\infty({}^{\mathrm{b}}S^*M)$ supported in a neighborhood of Γ and identically 1 near Γ, and $\phi \in C^\infty_c(\mathbb{R})$, noting that $[\Lambda_n, \mathcal{P}] \in \Psi^{-\infty}_{\mathrm{b}}(M)$ is uniformly bounded in $\Psi^0_{\mathrm{b}}(M) + \tau\Psi^1_{\mathrm{b}}(M)$ in view of (2-2), and thus, for $u \in \mathcal{X}^s_\Gamma$, $\mathcal{P}\Lambda_nu = \Lambda_n\mathcal{P}u + [\mathcal{P}, \Lambda_n]u \to \mathcal{P}u$ in $\mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ since $[\mathcal{P}, \Lambda_n]$ is uniformly bounded, so $H^{s-1/2}_{\mathrm{b}} \cap H^{s,-1/2}_{\mathrm{b}} \to H^{s-1/2}_{\mathrm{b}} \cap H^{s-1,1/2}_{\mathrm{b}}$, and thus $\mathcal{H}^s_{\mathrm{b},\Gamma} \to \mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ by (3-3).

first-order pseudodifferential operators of the form $A_1 A_2$, $A_1 \in \Psi_b^0(M)$ with $A_2 \in \Psi_b^1(M)$, both with principal symbol vanishing at Γ. Here, the operators should have Schwartz kernels supported away from the H_j; near H_j (but away from Γ), one should say $\mathcal{P}$ matters modulo $\mathrm{Diff}_b^1(M)$, i.e., only the principal symbol of $\mathcal{P}$ matters.

We then have:

Theorem 3.4. *Suppose $s \geq \frac{3}{2}$ and that the inverse of the Mellin-transformed normal operator $\widehat{\mathcal{P}}(\sigma)^{-1}$ has no poles with $\Im\sigma \geq 0$. Then*

$$\mathcal{P} : \mathcal{X}_\Gamma^s \to \mathcal{Y}_\Gamma^{s-1}$$

is invertible, giving the forward solution operator.

Proof. First, with $r < -\frac{1}{2}$, so with dual spaces having weight $\tilde{r} > \frac{1}{2}$, Theorem 2.18 holds without changes, as Proposition 3.2 gives nontrapping estimates in this case on the standard b-Sobolev spaces. In particular, if $r \ll 0$, $\operatorname{Ker}\mathcal{P}$ is trivial even on $H_b^{s-1/2,r}(\Omega)^{\bullet,-}$, hence certainly on its subspace $\mathcal{H}_{b,\Gamma}^s(\Omega)^{\bullet,-}$. Similarly, $\operatorname{Ker}\mathcal{P}^*$ is trivial on $H_b^{s,\tilde{r}}(\Omega)^{-,\bullet}$ for $\tilde{r} \gg 0$, and thus, with $r < -\frac{1}{2}$, for $f \in H_b^{-1,r}(\Omega)^{\bullet,-}$ there exists $u \in H_b^{0,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u = f$. Further, making use of the nontrapping estimates in Proposition 3.2, if $r < 0$ and $f \in H_b^{s-1,r}(\Omega)^{\bullet,-}$ then the argument of Theorem 2.21 improves this statement to $u \in H_b^{s,r}(\Omega)^{\bullet,-}$.

In particular, if $f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_b^{s-1,0}(\Omega)^{\bullet,-}$, then $u \in H_b^{s,r}(\Omega)^{\bullet,-}$ for $r < 0$. This can be improved using the argument of Theorem 2.21. Indeed, with $-1 \leq r < 0$ arbitrary, $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_b^2(M)$ implies, as in (2-37), that

$$N(\mathcal{P})u = f - \tilde{f}, \quad \text{where} \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_b^{s-2,r+1}(\Omega)^{\bullet,-}. \tag{3-8}$$

But $f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_b^{s-1,0}(\Omega)^{\bullet,-}$, hence the right-hand side is in $H_b^{s-2,0}(\Omega)^{\bullet,-}$; thus the dilation-invariant result, [Vasy 2013a, Lemma 3.1], gives $u \in H_b^{s-1,0}(\Omega)^{\bullet,-}$. This can then be improved further since, in view of $\mathcal{P}u = f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-}$, propagation of singularities, most crucially Proposition 3.2, yields $u \in \mathcal{H}_{b,\Gamma}^s(\Omega)^{\bullet,-}$. This completes the proof of the theorem. ☐

This result shows the importance of controlling the resonances in $\Im\sigma \geq 0$. For the wave operator on exact Kerr–de Sitter space, Dyatlov's [2011a; 2011b] analysis shows that the zero resonance of $\Box_g$ is the only one in $\Im\sigma \geq 0$, the residue at 0 having constant functions as its range. For the Klein–Gordon operator $\Box_g - m^2$, the statement is even better from our perspective as there are no resonances in $\Im\sigma \geq 0$ for $m > 0$ small. This is pointed out in [Dyatlov 2011a]; we give a direct proof based on perturbation theory.

Lemma 3.5. *Let $\mathcal{P} = \Box_g$ on exact Kerr–de Sitter space. Then, for small $m > 0$, all poles of $(\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$ have strictly negative imaginary part.*

Proof. By perturbation theory, the inverse family of $\widehat{\mathcal{P}}(\sigma) - \lambda$ has a simple pole at $\sigma(\lambda)$ coming with a single resonant state $\phi(\lambda)$ and a dual state $\psi(\lambda)$, with analytic dependence on λ, where $\sigma(0) = 0$, $\phi(0) \equiv 1$, and $\psi(0) = 1_{\{\mu>0\}}$, where we use the notation of [Vasy 2013a, §6]. Differentiating $\widehat{\mathcal{P}}(\sigma(\lambda))\phi(\lambda) = \lambda\phi(\lambda)$ with respect to λ and evaluating at $\lambda = 0$ gives

$$\sigma'(0)\widehat{\mathcal{P}}'(0)\phi(0) + \widehat{\mathcal{P}}(0)\phi'(0) = \phi(0).$$

Pairing this with $\psi(0)$, which is orthogonal to $\operatorname{Ran}\widehat{\mathcal{P}}(0)$, yields

$$\sigma'(0) = \frac{\langle \psi(0), \phi(0)\rangle}{\langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle}.$$

Since $\phi(0) = 1$ and $\psi(0) = 1_{\{\mu>0\}}$, this implies

$$\operatorname{sgn}\Im\sigma'(0) = -\operatorname{sgn}\Im\langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle. \tag{3-9}$$

To find the latter quantity, we note that the only terms in the general form of the d'Alembertian that could possibly yield a nonzero contribution here are terms involving τD_τ and either D_r, D_ϕ or D_θ. Concretely, using the explicit form of the dual metric G — see Equation (6.1) in [Vasy 2013a] — in the new coordinates $t = \tilde{t} + h(r)$, $\phi = \tilde{\phi} + P(r)$ and $\tau = e^{-t}$, with $h(r)$ and $P(r)$ as in Vasy's Equation (6.5),

$$G = -\rho^{-2}\biggl(\tilde{\mu}(\partial_r - h'(r)\tau\partial_\tau + P'(r)\partial_\phi)^2 + \frac{(1+\gamma)^2}{\varkappa\sin^2\theta}(-a\sin^2\theta\,\tau\partial_\tau + \partial_\phi)^2 + \varkappa\partial_\theta^2 \\ - \frac{(1+\gamma)^2}{\tilde{\mu}}(-(r^2+a^2)\tau\partial_\tau + a\partial_\phi)^2\biggr),$$

and its determinant is $|\det G|^{1/2} = (1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}$, so we see that the only nonzero contribution to the right-hand side of (3-9) comes from the term

$$(1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}D_r\bigl((1+\gamma)^{-2}\rho^2\sin\theta\,\rho^{-2}\tilde{\mu}h'(r)\bigr)\tau D_\tau = -i\rho^{-2}\partial_r(\tilde{\mu}h'(r))\tau D_\tau$$

of the d'Alembertian. Mellin-transforming this amounts to replacing τD_τ by σ; then differentiating the result with respect to σ gives

$$\begin{aligned}\langle \psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle &= -i\int_{\tilde{\mu}>0}\rho^{-2}\partial_r(\tilde{\mu}h'(r))\,d\mathrm{vol} \\ &= -i\int_0^\pi\int_0^{2\pi}\int_{r_-}^{r_+}(1+\gamma)^{-2}\sin\theta\,\partial_r(\tilde{\mu}h'(r))\,dr\,d\phi\,d\theta \\ &= -\frac{4\pi i}{(1+\gamma)^2}\bigl[(\tilde{\mu}h'(r))\big|_{r_+} - (\tilde{\mu}h'(r))\big|_{r_-}\bigr].\end{aligned} \tag{3-10}$$

Since the singular part of $h'(r)$ at $r_\pm$ (which are the roots of $\tilde{\mu}$) is $h'(r) = \mp(1+\gamma)(r^2+a^2)/\tilde{\mu}$, the right-hand side of (3-10) is positive up to a factor of i; thus $\Im\sigma'(0) < 0$, as claimed. □

In other words, for small mass $m > 0$, there are no resonances σ of the Klein–Gordon operator with $\Im\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (3-1) no longer has a constant term. Correspondingly, for $\epsilon \in \mathbb{R}$, $\epsilon \leq \epsilon_0$, Theorem 3.1 gives the forward solution operator

$$S_{\mathrm{KG,I}} : H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{3-11}$$

in the dilation-invariant case.

Further, Theorem 3.4 is applicable and gives the forward solution operator

$$S_{\mathrm{KG}} : \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s}(\Omega)^{\bullet,-} \tag{3-12}$$

on the normally isotropic spaces.

For the semilinear application, for nonlinearities without derivatives, it is important that the loss of derivatives $\varkappa$ in the space $H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}$ is at most 1. This is not explicitly specified in [Wunsch and Zworski 2011], though their proof directly gives (see especially the part before their Section 4.4) that, for small $\epsilon > 0$, $\varkappa$ can be taken proportional to ϵ and there is $\epsilon_0' > 0$ such that $\varkappa \in (0,1]$ for $\epsilon < \epsilon_0'$. We reduce $\epsilon_0 > 0$ above if needed so that $\epsilon_0 \le \epsilon_0'$; then (3-11) holds with $\varkappa = c\epsilon \in (0,1]$ if $\epsilon < \epsilon_0$, where $c > 0$.

In fact, one does not need to go through Wunsch and Zworski's proof, as the Phragmén–Lindelöf theorem allows one to obtain the same conclusion from their final result:

Lemma 3.6. *Let $h : U \to E$ be a holomorphic function on the half strip $U = \{z \in \mathbb{C} : 0 \le \Im z \le c,\ \Re z \ge 1\}$ that is continuous on $\bar{U}$ with values in a Banach space E and suppose, moreover, that there are constants $A, C > 0$ such that*

$$\|h(z)\| \le \begin{cases} C|z|^{k_1} & \text{if } \Im z = 0, \\ C|z|^{k_2} & \text{if } \Im z = c, \\ C\exp(A|z|) & \text{if } z \in \bar{U}. \end{cases}$$

Then there is a constant $C' > 0$ such that

$$\|h(z)\| \le C'|z|^{k_1(1-(\Im z)/c)+k_2(\Im z)/c}$$

for all $z \in \bar{U}$.

Proof. Consider the function $f(z) = z^{k_1 - i(k_2-k_1)z/c}$, which is holomorphic on a neighborhood of $\bar{U}$. Writing $z \in \bar{U}$ as $z = x + iy$ with $x, y \in \mathbb{R}$, one has

$$|f(z)| = |z|^{k_1} \exp\left(\Im\left(\frac{k_2-k_1}{c} z \log z\right)\right) = |z|^{k_1}|z|^{(k_2-k_1)\Im z/c} \exp\left(\frac{k_2-k_1}{c} x \arctan\left(\frac{y}{x}\right)\right).$$

Noting that $|x\arctan(y/x)| = y|(x/y)\arctan(y/x)|$ is bounded by c for all $x + iy \in \bar{U}$, we conclude that

$$e^{-|k_2-k_1|}|z|^{k_1(1-\Im z/c)+k_2\Im z/c} \le |f(z)| \le e^{|k_2-k_1|}|z|^{k_1(1-\Im z/c)+k_2\Im z/c}.$$

Therefore, $f(z)^{-1}h(z)$ is bounded by a constant C' on $\partial\bar{U}$, and satisfies an exponential bound for $z \in U$. By the Phragmén–Lindelöf theorem, $\|f(z)^{-1}h(z)\|_E \le C'$, and the claim follows. $\square$

Since, for any $\delta > 0$, we can bound $|\log z| \le C_\delta |z|^\delta$ for $|\Re z| \ge 1$, we obtain that the inverse family $R(\sigma) = \widehat{\mathcal{P}}(\sigma)^{-1}$ of the normal operator of $\Box_g$ on (asymptotically) Kerr–de Sitter spaces — as in [Vasy 2013a] but here in the setting of artificial boundaries, as opposed to complex absorption — satisfies a bound

$$\|R(\sigma)\|_{|\sigma|^{-(s-1)}H^{s-1}_{|\sigma|^{-1}}(X\cap\Omega)\to|\sigma|^{-s}H^{s}_{|\sigma|^{-1}}(X\cap\Omega)} \le C_\delta|\sigma|^{-1+\varkappa'+\delta} \tag{3-13}$$

for any $\delta > 0$, $\Im\sigma \ge -c\varkappa'$ and $|\Re\sigma|$ large. Therefore, as mentioned above, by the proof of Theorem 3.1, in particular using [Vasy 2013a, Lemma 3.1], we can assume $\varkappa \in (0,1]$ in the dilation-invariant result, Theorem 3.1, if we take $C' > 0$ small enough, i.e., if we do not go too far into the lower half plane $\Im\sigma < 0$, which amounts to only taking terms in the expansion (3-1) which decay to at most some fixed order, which we may assume to be less than $-\Im\sigma_j$ for all resonances σ_j.

3B. ***A class of semilinear equations; equations with polynomial nonlinearity.*** In the following semilinear applications, let us fix $\varkappa \in (0, 1]$ and ϵ_0 as explained before Lemma 3.6, so that we have the forward solution operator $S_{\mathrm{KG,I}}$ as in (3-11).

We then have statements paralleling Theorems 2.25 and 2.37 and Corollary 2.28, namely Theorems 3.7 and 3.11 and Corollary 3.10, respectively.

Theorem 3.7. *Suppose (M, g) is dilation invariant. Let $-\infty < \epsilon < \epsilon_0$, $s > \frac{1}{2} + \beta\epsilon$, $s \geq 1$, and let $q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ be a continuous function with $q(0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u)-q(v)\| \leq L(R)\|u-v\|, \quad \|u\|,\ \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

More generally, suppose

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0,0) = 0$ and

$$\|q(u,w)-q(u',w')\| \leq L(R)(\|u-u'\| + \|w-w'\|)$$

provided $\|u\| + \|w\|$, $\|u'\| + \|w'\| \leq R$, where we use the norms corresponding to the map q, for a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, \Box_g u)$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}} \leq R$, that depends continuously on f.

Proof. We use the proof of the first part of Theorem 2.25, where, in the current setting, the solution operator $S_{\mathrm{KG,I}}$ maps $H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and the contraction map is $T : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $Tu = S_{\mathrm{KG,I}}(f + q(u))$.

For the general statement, we follow the proof of the second part of Theorem 2.25, where we now instead use the Banach space

$$\mathscr{X} = \{u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{\tau^\epsilon H_{\mathrm{b}}^{s-1+\varkappa}},$$

which is a Banach space by the same argument as in the proof of Theorem 2.25. □

We have a weaker statement in the general, non-dilation-invariant case, where we work in unweighted spaces.

Theorem 3.8. *Let $s \geq 1$ and suppose $q : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ is a continuous function with $q(0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

An analogous statement holds for nonlinearities $q = q(u, \Box_g u)$ which are continuous maps

$$q : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^s(\Omega)^{\bullet,-},$$

vanish at $(0, 0)$, and have a small Lipschitz constant near 0.

Proof. Since

$$S_{\mathrm{KG}} : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \subset \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1/2}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s+1/2}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$$

by (3-3) and (3-12), this follows again from the Banach fixed point theorem. □

Remark 3.9. The proof of Theorem 3.4 shows that equations on function spaces with negative weights (i.e., growing near infinity) behave as nicely as equations on the static part of asymptotically de Sitter spaces, discussed in Section 2. However, naturally occurring nonlinearities (e.g., polynomials) will not be continuous nonlinear operators on such growing spaces.

Corollary 3.10. *If $s > \frac{1}{2}n$, the hypotheses of Theorem 3.8 hold for nonlinearities $q(u) = cu^p$, $p \geq 2$ an integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_{\mathrm{b}}^s(M)$.*

Thus, for small $m > 0$ and $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

If f satisfies stronger decay assumptions, then u does as well. More precisely, denoting the inverse family of the normal operator of the Klein–Gordon operator with (small) mass m by $R_m(\sigma) = (\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$, which has poles only in $\Im\sigma < 0$ (see Lemma 3.5 and [Dyatlov 2011a; Vasy 2013a]) and, moreover, defining the spaces $\mathcal{X}_{\mathcal{F}}^{s,r,\epsilon}$ and $\mathcal{X}_{\mathcal{P},\mathcal{F}}^{s,r,\epsilon}$ analogously to the corresponding spaces in Section 2C, we have the following result:

Theorem 3.11. *Fix $0 < \epsilon < \min\{C', \frac{1}{2}\}$ and let $s \gg s' \geq \max(\frac{1}{2} + \beta\epsilon, \frac{1}{2}n, 1 + \varkappa)$. (A concrete bound for s will be given in the course of the proof; see Equation (3-15).) Let*

$$q(u) = \sum_{p=2}^{d} q_p u^p, \quad q_p \in H_{\mathrm{b}}^s(M).$$

Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $R_m(\sigma)$, and $m_j + 1$ is the order of the pole of $R_m(\sigma)$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) : 0 \leq \ell \leq m_j,\ k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \Re(i\sigma_j)$ for all j, and that $m > 0$ is so small that $\mathscr{P}$ is a positive index set. Finally, let $\mathscr{F}$ be a positive index set.

Then, for small enough $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}_{\mathscr{F}}^{s,s,\epsilon}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u) \tag{3-14}$$

has a unique solution $u \in \mathscr{X}_{\mathscr{P},\mathscr{F}}^{s',s',\epsilon}$, with $\|u\| \leq R$, that depends continuously on f; in particular, u has an asymptotic expansion with remainder in $H_{\mathrm{b}}^{s',\epsilon}(\Omega)^{\bullet,-}$.

Proof. Let us write $\mathscr{P} = \Box_g - m^2$. Let $\delta < \frac{1}{2}$ be such that $0 < 2\delta < \Re z$ for all $(z, 0) \in \mathscr{F}$; then $f \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-}$. Now, for $u \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$, consider $Tu := S_{\mathrm{KG}}(f + q(u))$. First of all, $f + q(u) \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s}(\Omega)^{\bullet,-}$, thus the proof of Theorem 3.4 shows that $Tu \in H_{\mathrm{b}}^{s+1,r}(\Omega)^{\bullet,-}$ with $r < 0$ arbitrary. Therefore,

$$N(\mathscr{P})u = f + q(u) + (N(\mathscr{P}) - \mathscr{P})u \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-} + H_{\mathrm{b}}^{s-1,r+1}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s-1,2\delta}(\Omega)^{\bullet,-},$$

and thus, if $\delta > 0$ is sufficiently small, namely, $\delta < \frac{1}{2}\inf\{-\Im\sigma_j\}$, Theorem 3.1 implies $u \in H_{\mathrm{b}}^{s-\varkappa,2\delta}(\Omega)^{\bullet,-}$. Since we can choose $\varkappa = c\delta$ for some constant $c > 0$, we obtain

$$Tu \in \bigcap_{r>0} H_{\mathrm{b}}^{s+1,r}(\Omega)^{\bullet,-} \cap H_{\mathrm{b}}^{s-c\delta,2\delta}(\Omega)^{\bullet,-} \subset \bigcap_{r'>0} H_{\mathrm{b}}^{s,2\delta-2c\delta^2/(1+c\delta)-r'}(\Omega)^{\bullet,-}$$

by interpolation. In particular, choosing $\delta > 0$ even smaller if necessary, we obtain $Tu \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$. Applying the Banach fixed point theorem to the map T thus gives a solution $u \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$ to (3-14).

For this solution u, we obtain

$$N(\mathscr{P})u = \mathscr{P}u + (N(\mathscr{P}) - \mathscr{P})u \in H_{\mathrm{b}}^{s,2\delta} + H_{\mathrm{b}}^{s-2,\delta+1} \subset H_{\mathrm{b}}^{s-2,2\delta}$$

since q only has quadratic and higher terms. Hence Theorem 3.1 implies that $u = u_1 + u'$, where u_1 is an expansion with terms coming from poles of $\widehat{\mathscr{P}}^{-1}$ whose decay order lies between δ and 2δ, and $u' \in H_{\mathrm{b}}^{s-1-\varkappa,2\delta}(\Omega)^{\bullet,-}$. This in turn implies that $f + q(u)$ has an expansion with remainder term in $H_{\mathrm{b}}^{s-1-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-}$; thus

$$N(\mathscr{P})u \in H_{\mathrm{b}}^{s-3-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-} \text{ plus an expansion,}$$

and we proceed iteratively, until, after k more steps, we have $4 \cdot 2^k\delta \geq \epsilon$, and then u has an expansion with remainder term $H_{\mathrm{b}}^{s-3-2k-\varkappa,\epsilon}(\Omega)^{\bullet,-}$ provided we can apply Theorem 3.1 in the iterative procedure, i.e., provided $s-3-2k-\varkappa =: s' > \max\bigl(\frac{1}{2}+\beta\epsilon, \frac{1}{2}n, 1+\varkappa\bigr)$. This is satisfied if

$$s > \max\bigl(\tfrac{1}{2}+\beta\epsilon, \tfrac{1}{2}n, 1+\varkappa\bigr) + 2\lceil \log_2(\epsilon/\delta)\rceil + \varkappa - 1. \tag{3-15}$$

This concludes the proof. □

3C. ***Semilinear equations with derivatives in the nonlinearities.*** Theorem 3.4 allows one to solve even semilinear equations with derivatives in some cases. For instance, in the case of de Sitter–Schwarzschild space, within $\Sigma \cap {}^{\mathrm{b}}S^*_X M$, Γ is given by $r = r_c$, $\sigma_1(D_r) = 0$, where $r_c = \frac{3}{2} r_s$ is the radius of the photon sphere; see, e.g., [Vasy 2013a, §6.4]. Thus, nonlinear terms such as $(r-r_c)(\partial_r u)^2$ are allowed for $s > \frac{1}{2}n + 1$ since $\partial_r : \mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to H^{s-1}_{\mathrm{b}}(M)$, with the latter space being an algebra, while multiplication by $r - r_c$ maps this space to $\mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$, by (3-7). Thus, a straightforward modification of Theorem 3.8, applying the fixed point theorem on the normally isotropic spaces directly, gives well-posedness.

4. Asymptotically de Sitter spaces: global approach

We can approach the problem of solving nonlinear wave equations on global asymptotically de Sitter spaces in two ways: either we proceed as in the previous two sections, first showing invertibility of the linear operator on suitable spaces and then applying the contraction mapping principle to solve the nonlinear problem; or we use the solvability results from Section 2 for backward light cones from points at future conformal infinity and glue the solutions on all these "static" parts together to obtain a global solution. The first approach, which we will follow in Section 4A–4D, has the disadvantage that the conditions on the nonlinearity that guarantee the existence of solutions are quite restrictive, however, if the conditions are met, one has good decay estimates for solutions. The second approach, on the other hand, detailed in Section 4E, allows many of the nonlinearities, suitably reinterpreted, that work on "static parts" of asymptotically de Sitter spaces (i.e., backward light cones), but the decay estimates for solutions are quite weak relative to the decay of the forcing term because of the gluing process.

4A. ***The linear framework.*** Let g be the metric on an n-dimensional asymptotically de Sitter space X with global time function t [Vasy 2010]. Then, following [Vasy 2013a, Section 4], the operator[12]

$$P_\sigma = \mu^{-1/2}\mu^{i\sigma/2-(n+1)/4}\bigl(\Box_g - \bigl(\tfrac{1}{2}(n-1)\bigr)^2 - \sigma^2\bigr)\mu^{-i\sigma/2+(n+1)/4}\mu^{-1/2} \tag{4-1}$$

extends nondegenerately to an operator on a closed manifold $\widetilde{X}$ which contains the compactification $\overline{X}$ of the asymptotically de Sitter space as a submanifold with boundary Y, where $Y = Y_- \cup Y_+$ has two connected components, which we call the boundary of X at past and future infinity, respectively. The expression "nondegenerately" here means that, near $Y_\pm$, P_σ fits into the framework of [Vasy 2013a]. Here, $\mu = 0$ is the defining function of Y and $\mu > 0$ is the interior of the asymptotically de Sitter space. Moreover, null-bicharacteristics of P_σ tend to $Y_\pm$ as $t \to \pm\infty$.

Following [Vasy 2014], let us in fact assume that $\widetilde{X} = \overline{C_-} \cup \overline{X} \cup \overline{C_+}$ is the union of the compactifications of asymptotically de Sitter space X and two asymptotically hyperbolic caps $C_\pm$; as Vasy explains, one

12 P_σ in our notation corresponds to $P^*_{\bar\sigma}$ in [Vasy 2013a], the latter operator being the one for which one solves the forward problem.

might need to take two copies of X to construct $\widetilde{X}$. For the purposes of the next statement, we recall that variable-order Sobolev spaces $H^s(\widetilde{X})$ were discussed in [Baskin et al. 2014, Appendix A]. Then P_σ is the restriction to X of an operator $\widetilde{P}_\sigma \in \mathrm{Diff}^2(\widetilde{X})$, which is Fredholm as a map

$$\widetilde{P}_\sigma : \widetilde{\mathscr{X}}^s \to \widetilde{\mathscr{Y}}^{s-1}, \quad \widetilde{\mathscr{X}}^s = \{u \in H^s : \widetilde{P}_\sigma u \in H^{s-1}\}, \quad \widetilde{\mathscr{Y}}^{s-1} = H^{s-1},$$

where $s \in C^\infty(S^*\widetilde{X})$, monotone along the bicharacteristic flow, is such that $s|_{N^*Y_-} > \frac{1}{2} - \Im\sigma$, $s|_{N^*Y_+} < \frac{1}{2} - \Im\sigma$, and s is constant near $S^*Y_\pm$. Note that the choice of signs here is opposite to the one in [Vasy 2014], since here we are going to construct the forward solution operator on X.

Restricting our attention to X, we define the space $H^s(X)^{\bullet,-}$ to be the completion in $H^s(X)$ of the space of C^∞ functions that vanish to infinite order at Y_-; thus, the superscripts indicate that distributions in $H^s(X)^{\bullet,-}$ are supported distributions near Y_- and extendible distributions near Y_+. Then, define the spaces

$$\mathscr{X}^s = \{u \in H^s(X)^{\bullet,-} : P_\sigma u \in H^{s-1}(X)^{\bullet,-}\}, \quad \mathscr{Y}^{s-1} = H^{s-1}(X)^{\bullet,-}.$$

Theorem 4.1. *Fix $\sigma \in \mathbb{C}$ and $s \in C^\infty(S^*\overline{X})$ as above. Then $P_\sigma : \mathscr{X}^s \to \mathscr{Y}^{s-1}$ is invertible and $P_\sigma^{-1} : H^{s-1}(X)^{\bullet,-} \to H^s(X)^{\bullet,-}$ is the forward solution operator of P_σ.*

Proof. First, let us assume $\Re\sigma \gg 0$, so semiclassical and large parameter estimates are applicable to $\widetilde{P}_\sigma$, and let $T_0 \in \mathbb{R}$ be such that s is constant in $\{t \le T_0\}$. Then, for any $T_1 \le T_0$, we can paste together microlocal energy estimates for $\widetilde{P}_\sigma$ near $\overline{C_-}$ and standard energy estimates for the wave equation in $\{t \le T_1\}$ away from Y_-, as in the derivation of Equation (3.29) of [Vasy 2013a], and thereby obtain

$$\|u\|_{H^1(\{t\le T_1\})} \lesssim \|\widetilde{P}_\sigma u\|_{H^0(\{t\le T_1\})}; \tag{4-2}$$

thus, for $f \in C^\infty(\widetilde{X})$, $\operatorname{supp} f \subset \{t \ge T_1\}$ implies $\operatorname{supp} \widetilde{P}_\sigma^{-1} f \subset \{t \ge T_1\}$. Choosing $\phi \in C_c^\infty(X)$ with support in $\{t \ge T_1\}$ and $\psi \in C^\infty(\widetilde{X})$ with support in $\{t \le T_1\}$, we therefore obtain $\psi \widetilde{P}_\sigma^{-1}\phi = 0$. Since $\widetilde{P}_\sigma^{-1}$ is meromorphic, this continues to hold for all $\sigma \in \mathbb{C}$ such that $\Im\sigma > \frac{1}{2} - s$. Since $T_1 \le T_0$ is arbitrary, this, together with standard energy estimates on the asymptotically de Sitter space X, proves that P_σ^{-1} propagates supports forward, provided P_σ is invertible. Moreover, elements of $\ker \widetilde{P}_\sigma$ are supported in $\overline{C_+}$.

The invertibility of P_σ is a consequence of [Baskin et al. 2014, Lemma 8.3] (also see Footnote 15 there): let $E : H^{s-1}(X)^{\bullet,-} \to H^{s-1}(\widetilde{X})$ be a continuous extension operator that extends by 0 in $\overline{C_-}$ and $R : H^s(\widetilde{X}) \to H^s(X)^{-,-}$ the restriction; then $R \circ \widetilde{P}_\sigma^{-1} \circ E$ does not have poles, and, since

$$\bigcup_{T_1 \le T_0} H^s(\{t > T_1\})^{\bullet,-} \subset H^s(X)^{\bullet,-}$$

(where $\bullet$ denotes supported distributions at $\{t = T_1\}$ and Y_-, respectively) is dense, $R \circ \widetilde{P}_\sigma^{-1} \circ E$ in fact maps into $H^s(X)^{\bullet,-}$; thus $P_\sigma^{-1} = R \circ \widetilde{P}_\sigma^{-1} \circ E$ indeed exists and has the claimed properties. □

In our quest for forward solutions of semilinear equations, we restrict ourselves to a submanifold with boundary $\Omega \subset \overline{X}$ containing and localized near future infinity, so that we can work in fixed-order Sobolev spaces; moreover, it will be useful to measure the conormal regularity of solutions to the linear equation at the conormal bundle of the boundary of X at future infinity more precisely. So let $H^{s,k}(\widetilde{X}, Y_+)$ be the subspace of $H^s(\widetilde{X})$ with k-fold regularity with respect to the $\Psi^0(\widetilde{X})$-module $\mathcal{M}$ of first-order pseudodifferential operators with principal symbol vanishing on N^*Y_+. A result of Haber and Vasy [2013, Theorem 6.3], with $s_0 = \frac{1}{2} - \Im\sigma$ in our case, shows that $f \in H^{s-1,k}(\widetilde{X}, Y_+)$, $\widetilde{P}_\sigma u = f$ with u a distribution, in fact imply that $u \in H^{s,k}(\widetilde{X}, Y_+)$. So, if we let $H^{s,k}(\Omega)^{\bullet,-}$ denote the space of all $u \in H^s(X)^{\bullet,-}$ which are restrictions to Ω of functions in $H^{s,k}(\widetilde{X}, Y_+)$, supported in $\Omega \cup \overline{C_+}$, the argument of Theorem 4.1 shows that we have a forward solution operator $S_\sigma : H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}$ provided

$$s < \tfrac{1}{2} - \Im\sigma. \tag{4-3}$$

4A1. *The backward problem.* Another problem that we will briefly consider below is the backward problem, i.e., where one solves the equation on X backward from Y_+, which is the same, up to relabelling, as solving the equation forward from Y_-. Thus, we have a backward solution operator $S_\sigma^- : H^{s-1,k}(\Omega)^{-,\bullet} \to H^{s,k}(\Omega)^{-,\bullet}$ (where Ω is chosen as above so that we can use fixed-order Sobolev spaces) provided $s > \frac{1}{2} - \Im\sigma$. Similarly to the above, $-$ denotes extendible distributions at $\partial\Omega \cap X^\circ$ and $\bullet$ denotes supported distributions at Y_+; the module regularity is measured at Y_+.

4B. ***Algebra properties of $H^{s,k}(\Omega)^{\bullet,-}$.*** Let us call a polynomially bounded, measurable function $w : \mathbb{R}^n \to (0, \infty)$ a *weight function*. For such a weight function w, we define

$$H^{(w)}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : w\hat{u} \in L^2(\mathbb{R}^n)\}.$$

The following lemma is similar in spirit to, but different from, Strichartz's [1971] result on Sobolev algebras; it is the basis for the multiplicative properties of the more delicate spaces considered below.

Lemma 4.2. *Let w_1, w_2 and w be weight functions such that one of the quantities*

$$\begin{aligned} M_+ &:= \sup_{\xi\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta, \\ M_- &:= \sup_{\eta\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\xi \end{aligned} \tag{4-4}$$

is finite. Then $H^{(w_1)}(\mathbb{R}^n) \cdot H^{(w_2)}(\mathbb{R}^n) \subset H^{(w)}(\mathbb{R}^n)$.

Proof. For $u, v \in S(\mathbb{R}^n)$, we use Cauchy–Schwarz to estimate

$$\begin{aligned} \|uv\|^2_{H^{(w)}} &= \int w(\xi)^2 |\widehat{uv}(\xi)|^2 \, d\xi \\ &= \int w(\xi)^2 \left(\int w_1(\eta)|\hat{u}(\eta)| w_2(\xi-\eta)|\hat{v}(\xi-\eta)| w_1(\eta)^{-1} w_2(\xi-\eta)^{-1} \, d\eta\right)^2 d\xi \\ &\le \int \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta\right)\left(\int w_1(\eta)^2|\hat{u}(\eta)|^2 w_2(\xi-\eta)^2|\hat{v}(\xi-\eta)|^2 \, d\eta\right) d\xi \\ &\le M_+ \|u\|^2_{H^{(w_1)}} \|v\|^2_{H^{(w_2)}} \end{aligned}$$

as well as

$$\begin{aligned}\|uv\|_{H^{(w)}}^2 &\le \int\Bigl(\int w_2(\xi-\eta)^2|\hat{v}(\xi-\eta)|^2\,d\eta\Bigr)\Bigl(\int\Bigl(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\Bigr)^2 w_1(\eta)^2|\hat{u}(\eta)|^2\,d\eta\Bigr)\,d\xi\\ &= \|v\|_{H^{(w_2)}}^2\int w_1(\eta)^2|\hat{u}(\eta)|^2\Bigl(\int\Bigl(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\Bigr)^2\,d\xi\Bigr)\,d\eta\\ &\le M_-\|u\|_{H^{(w_1)}}^2\|v\|_{H^{(w_2)}}^2.\end{aligned}$$

Since $S(\mathbb{R}^n)$ is dense in $H^{(w_1)}(\mathbb{R}^n)$ and $H^{(w_2)}(\mathbb{R}^n)$, the lemma follows. □

In particular, if

$$\Bigl\|\frac{w(\xi)}{w(\eta)w(\xi-\eta)}\Bigr\|_{L^\infty_\xi L^2_\eta} < \infty, \tag{4-5}$$

then $H^{(w)}$ is an algebra.

For example, the weight function $w(\xi)=\langle\xi\rangle^s$ for $s>\frac{1}{2}n$ satisfies (4-5), as we will check below, which implies that $H^s(\mathbb{R}^n)$ is an algebra for $s>\frac{1}{2}n$; this is the special case $k=0$ of Lemma 4.4 below and is well known; see, e.g., [Taylor 1997, Chapter 13.3]. Also, product-type weight functions $w_d(\xi)=\langle\xi'\rangle^s\langle\xi''\rangle^k$ (where $\xi=(\xi',\xi'')\in\mathbb{R}^{d+(n-d)}$) for $s>\frac{1}{2}d$ and $k>\frac{1}{2}(n-d)$ satisfy (4-5).

The following lemma, together with the triangle inequality $\langle\xi\rangle^\alpha\lesssim\langle\eta\rangle^\alpha+\langle\xi-\eta\rangle^\alpha$ for $\alpha\ge 0$, will often be used to check conditions like (4-4).

Lemma 4.3. *Suppose $\alpha,\beta\ge 0$ are such that $\alpha+\beta>n$. Then*

$$\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta}\in L^\infty(\mathbb{R}^n_\xi).$$

Proof. Splitting the domain of integration into the two regions $\{\langle\eta\rangle<\langle\xi-\eta\rangle\}$ and $\{\langle\eta\rangle\ge\langle\xi-\eta\rangle\}$, we obtain the bound

$$\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta}\le 2\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^{\alpha+\beta}},$$

which is finite in view of $\alpha+\beta>n$. □

Another important consequence of Lemma 4.2 is that $H^{s'}(\mathbb{R}^n)$ is an $H^s(\mathbb{R}^n)$-module provided $|s'|\le s$ and $s>\frac{1}{2}n$, which follows for $s'\ge 0$ from $M_+<\infty$, and for $s'<0$ either by duality or from $M_-<\infty$ (with $M_\pm$ as in the statement of the lemma, with the corresponding weight functions).

Lemma 4.4. *Write $x\in\mathbb{R}^n$ as $x=(x',x'')\in\mathbb{R}^{d+(n-d)}$. For $s\in\mathbb{R}$ and $k\in\mathbb{N}_0$, let*

$$\mathcal{Y}^{s,k}_d(\mathbb{R}^n)=\{u\in H^s(\mathbb{R}^n): D^k_{x''}u\in H^s(\mathbb{R}^n)\}.$$

Then, for $s>\frac{1}{2}d$ and $s+k>\frac{1}{2}n$, $\mathcal{Y}^{s,k}_d(\mathbb{R}^n)$ is an algebra.

Proof. Using the Leibniz rule, we see that it suffices to show that if $u,v\in\mathcal{Y}^{s,k}_d$ then $D^\alpha_{x''}uD^\beta_{x''}v\in H^s$, provided $|\alpha|+|\beta|\le k$. Since $D^\alpha_{x''}u\in\mathcal{Y}^{s,k-|\alpha|}_d$ and $D^\beta_{x''}v\in\mathcal{Y}^{s,k-|\beta|}_d$, this amounts to showing that

$$\mathcal{Y}^{s,a}_d\cdot\mathcal{Y}^{s,b}_d\subset H^s\quad\text{if } a+b\ge k. \tag{4-6}$$

Using the characterization $\mathcal{Y}^{s,a}_d = H^{(w)}$ for $w(\xi) = \langle\xi\rangle^s \langle\xi''\rangle^k$, Lemma 4.2 in turn reduces this to the estimate

$$\begin{aligned}\int \frac{\langle\xi\rangle^{2s}}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}}\,d\eta \\ &\lesssim \int \frac{d\eta}{\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}} + \int \frac{d\eta}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b}} \\ &\le \int \frac{d\eta'}{\langle\xi'-\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b+2(s-s')}} + \int \frac{d\eta'}{\langle\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a+2(s-s')}\langle\xi''-\eta''\rangle^{2b}},\end{aligned}$$

where we choose $\frac{1}{2}d < s' < s$ such that $a+b+s-s' > \frac{1}{2}(n-d)$, which holds if $k+s > \frac{1}{2}(n-d)+s'$, which is possible by our assumptions on s and k. The integrals are uniformly bounded in ξ: for the η'-integrals, this follows from $s' > \frac{1}{2}d$; for the η''-integrals, we use Lemma 4.3. □

We shall now use this (noninvariant) result to prove algebra properties of spaces with iterated module regularity: Consider a compact manifold without boundary X and a submanifold Y. Let $\mathcal{M} \supset \Psi^0(X)$ be the $\Psi^0(X)$-module of first-order pseudodifferential operators whose principal symbol vanishes on N^*Y. For $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$, define

$$H^{s,k}(X,Y) = \{u \in H^s(X) : \mathcal{M}^k u \in H^s(X)\}.$$

Proposition 4.5. *Suppose* $\dim(X) = n$ *and* $\operatorname{codim}(Y) = d$. *Assume that* $s > \frac{1}{2}d$ *and* $s+k > \frac{1}{2}n$. *Then* $H^{s,k}(X,Y)$ *is an algebra.*

Proof. Away from Y, $H^{s,k}(X,Y)$ is just $H^{s+k}(X)$, which is an algebra since $s+k > \frac{1}{2}\dim(X)$. Thus, since the statement is local, we may assume that we have a product decomposition near Y, namely $X = \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''}$, $Y = \{x' = 0\}$, and that we are given arbitrary $u, v \in H^{s,k}(X,Y)$ with compact support close to $(0,0)$ for which we have to show $uv \in H^{s,k}(X,Y)$. Notice that, for $f \in H^s(X)$ with such small support, $f \in H^{s,k}(X,Y)$ is equivalent to $\mathcal{M}'^k f \in H^s(X)$, where $\mathcal{M}'$ is the $C^\infty(M)$-module of differential operators generated by Id, $\partial_{x''_i}$ and $x'_j \partial_{x'_k}$, where $1 \le i \le n-d$ and $1 \le j,k \le d$.

Thus the proposition follows from the following statement: for s and k as in the statement of the proposition,

$$H^{s,k}(\mathbb{R}^n, \mathbb{R}^{n-d}) := \{u \in H^s(\mathbb{R}^n) : (x')^{\tilde\alpha} D^\alpha_{x'} D^\beta_{x''} u \in H^s(\mathbb{R}^n),\ |\tilde\alpha| = |\alpha|,\ |\alpha|+|\beta| \le k\}$$

is an algebra. Using the Leibniz rule, we thus have to show that

$$((x')^{\tilde\alpha} D^\alpha_{x'} D^\beta_{x''} u)((x')^{\tilde\gamma} D^\gamma_{x'} D^\delta_{x''} v) \in H^s \tag{4-7}$$

provided $|\tilde\alpha| = |\alpha|$, $|\tilde\gamma| = |\gamma|$ and $|\alpha|+|\beta|+|\gamma|+|\delta| \le k$. Since the two factors in (4-7) lie in $H^{s,k-|\alpha|-|\beta|}$ and $H^{s,k-|\gamma|-|\delta|}$, respectively, this amounts to showing that $H^{s,a} \cdot H^{s,b} \subset H^s$ for $a+b \ge k$. This, however, is easy to see, since $H^{s,c} \subset \mathcal{Y}^{s,c}_d$ for all $c \in \mathbb{N}_0$ and $\mathcal{Y}^{s,a}_d \cdot \mathcal{Y}^{s,b}_d \subset H^s$ was proved in (4-6). □

In order to be able to obtain sharper results for particular nonlinear equations in Section 4C, we will now prove further results in the case $\operatorname{codim}(Y) = 1$, which we will assume to hold from now on; also, we fix $n = \dim(X)$.

Proposition 4.6. *Assume that $s > \frac{1}{2}$ and $k > \frac{1}{2}(n-1)$. Then $H^{s,k}(X,Y) \cdot H^{s-1,k}(X,Y) \subset H^{s-1,k}(X,Y)$.*

Proof. Using the Leibniz rule, this follows from $\mathcal{Y}_1^{s,a} \cdot \mathcal{Y}_1^{s-1,b} \subset H^{s-1}$ for $a+b \geq k$. This, as before, can be reduced to the local statement on $\mathbb{R}^n = \mathbb{R}_{x_1} \times \mathbb{R}^{n-1}_{x'}$ with $Y = \{x_1 = 0\}$. We write $\xi = (\xi_1, \xi') \in \mathbb{R}^{1+(n-1)}$ and $\eta = (\eta_1, \eta') \in \mathbb{R}^{1+(n-1)}$. By Lemma 4.2, the case $s \geq 1$ follows from the estimate

$$\begin{aligned}
\int \frac{\langle \xi \rangle^{2(s-1)}}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} \, d\eta& \\
&\lesssim \int \frac{d\eta}{\langle \eta \rangle^{2} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} + \int \frac{d\eta}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \\
&\leq 2 \int \frac{d\eta_1}{\langle \eta_1 \rangle^{2s}} \int \frac{d\eta'}{\langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \in L^\infty_\xi
\end{aligned}$$

by Lemma 4.3.

If $\frac{1}{2} < s \leq 1$, then ξ_1 and ξ' play different roles. Indeed, the background regularity to be proved is H^{s-1}, $s-1 \leq 0$, thus the continuity of multiplication in the conormal direction to Y is proved by "duality" (using Lemma 4.2 with $M_- < \infty$), whereas the continuity in the tangential (to Y) directions, where both factors have $k > \frac{1}{2}(n-1)$ derivatives, is proved directly (using Lemma 4.2 with $M_+ < \infty$). So, let $u \in \mathcal{Y}_1^{s,a}$ and $v \in \mathcal{Y}_1^{s-1,b}$, and put

$$u_0(\xi) = \langle \xi \rangle^s \langle \xi' \rangle^a u(\xi) \in L^2(\mathbb{R}^n), \quad v_0(\xi) = \langle \xi \rangle^{s-1} \langle \xi' \rangle^b v(\xi) \in L^2(\mathbb{R}^n).$$

Then

$$\langle \xi \rangle^{s-1} \widehat{uv}(\xi) = \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \eta' \rangle^b \langle \xi - \eta \rangle^s \langle \xi' - \eta' \rangle^a} u_0(\xi - \eta) v_0(\eta) \, d\eta,$$

hence, by Cauchy–Schwarz and Lemma 4.3,

$$\begin{aligned}
\int \langle \xi \rangle^{2(s-1)} |\widehat{uv}(\xi)|^2 \, d\xi& \\
&\leq \int \left(\int \frac{d\eta'}{\langle \eta' \rangle^{2b} \langle \xi' - \eta' \rangle^{2a}} \right) \left(\int \left| \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \xi - \eta \rangle^s} u_0(\xi - \eta) v_0(\eta) \, d\eta_1 \right|^2 d\eta' \right) d\xi \\
&\lesssim \iint \left(\int |u_0(\xi - \eta)|^2 \, d\eta_1 \right) \left(\int \frac{\langle \eta \rangle^{2(1-s)}}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2s}} |v_0(\eta)|^2 \, d\eta_1 \right) d\eta' \, d\xi \\
&\lesssim \iint \|u_0(\cdot, \xi' - \eta')\|^2_{L^2} |v_0(\eta)|^2 \left(\int \frac{1}{\langle \xi - \eta \rangle^{2s}} + \frac{1}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2(2s-1)}} \, d\xi_1 \right) d\xi' \, d\eta \\
&\lesssim \|u\|^2_{\mathcal{Y}_1^{s,a}} \|v\|^2_{\mathcal{Y}_1^{s-1,b}},
\end{aligned}$$

since $\frac{1}{2} < s \leq 1$, so $1-s \geq 0$ and $2s-1 > 0$, and the ξ_1-integral is thus bounded from above by

$$\int \frac{1}{\langle \xi_1 - \eta_1 \rangle^{2s}} + \frac{1}{\langle \xi_1 \rangle^{2(1-s)} \langle \xi_1 - \eta_1 \rangle^{2(2s-1)}} \, d\xi_1 \in L^\infty_{\eta_1}.$$

The proof is complete. □

For semilinear equations whose nonlinearity does not involve any derivatives, one can afford to lose derivatives in multiplication statements. We give two useful results in this context, the first being a consequence of Proposition 4.6.

Corollary 4.7. *Let $\mu \in C^\infty(X)$ be a defining function for Y, i.e., $\mu|_Y \equiv 0$, $d\mu \neq 0$ on Y, and μ vanishes on Y only. Suppose $s > \frac{1}{2}$ and $\ell \in \mathbb{C}$ are such that $\Re\ell + \frac{3}{2} > s$. Then multiplication by μ_+^ℓ defines a continuous map $H^{s,k}(X,Y) \to H^{s-1,k}(X,Y)$ for all $k \in \mathbb{N}_0$.*

Proof. By the Leibniz rule it suffices to prove the statement for $k = 0$. We have $\mu_+^\ell \in H^{\Re\ell+1/2-\epsilon;\infty}(X,Y)$ for all $\epsilon > 0$: indeed, the Fourier transform of $\chi(x)x_+^\ell$ on $\mathbb{R}$, with $\chi \in C_c^\infty(\mathbb{R})$, is bounded by a constant multiple of $\langle\xi\rangle^{-\Re\ell-1}$, which is an element of $\langle\xi\rangle^{-r}L^2_\xi$ if and only if $r - \Re\ell - 1 < -\frac{1}{2}$, that is, if $\Re\ell + \frac{1}{2} > r$. Hence, the corollary follows from Proposition 4.6, since one has $\Re\ell + \frac{1}{2} - \epsilon \geq s - 1$ for some $\epsilon > 0$ provided $\Re\ell + \frac{3}{2} > s$. □

Proposition 4.8. *Let $0 \leq s', s_1, s_2 < \frac{1}{2}$ be such that $s' < s_1 + s_2 - \frac{1}{2}$, and let $k > \frac{1}{2}(n-1)$. Then $H^{s_1,k}(X,Y) \cdot H^{s_2,k}(X,Y) \subset H^{s',k}(X,Y)$.*

Proof. Using the Leibniz rule, this reduces to the statement that $\mathcal{Y}_1^{s_1,a} \cdot \mathcal{Y}_1^{s_2,b} \subset H^{s'}$ if $a + b \geq k$. Splitting variables $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, Lemma 4.2 in turn reduces this to the observation that

$$\int \frac{\langle\xi\rangle^{2s'}}{\langle\eta\rangle^{2s_1}\langle\eta'\rangle^{2a}\langle\xi-\eta\rangle^{2s_2}\langle\xi'-\eta'\rangle^{2b}}\,d\eta \lesssim \left(\int \frac{d\eta_1}{\langle\eta_1\rangle^{2(s_1-s')}\langle\xi_1-\eta_1\rangle^{2s_2}} + \int \frac{d\eta_1}{\langle\eta_1\rangle^{2s_1}\langle\xi_1-\eta_1\rangle^{2(s_2-s')}}\right)\int \frac{d\eta'}{\langle\eta'\rangle^{2a}\langle\xi'-\eta'\rangle^{2b}}$$

is uniformly bounded in ξ by Lemma 4.3, in view of $s' < s_1 + s_2 - \frac{1}{2} < \min\{s_1, s_2\}$, thus $s_1 - s' > 0$ and $s_2 - s' > 0$, and $s_1 + s_2 - s' > \frac{1}{2}$, as well as $a + b > \frac{1}{2}(n-1)$. □

Corollary 4.9. *Let $p \in \mathbb{N}$ and $s = \frac{1}{2} - \epsilon$ with $0 \leq \epsilon < 1/(2p)$, and let $k > \frac{1}{2}(n-1)$. Then $u \in H^{s,k}(X,Y)$ implies $u^p \in H^{0,k}(X,Y)$.*

Proof. Proposition 4.8 gives $u^2 \in H^{1/2-2\epsilon-\epsilon_2',k}$ for all $\epsilon_2' > 0$, thus $u^3 \in H^{1/2-3\epsilon-\epsilon_3',k}$ for all $\epsilon_3' > 0$, since $\epsilon_2' > 0$ is arbitrary; continuing in this way gives $u^p \in H^{1/2-p\epsilon-\epsilon_p',k}$ for all $\epsilon_p' > 0$, and the claim follows. □

4C. *A class of semilinear equations.* Recall that, provided $s < \frac{1}{2} - \Im\sigma$, we have a forward solution operator $S_\sigma : H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}$ of P_σ, defined in (4-1). Let us fix such $s \in \mathbb{R}$ and $\sigma \in \mathbb{C}$. Undoing the conjugation, we obtain a forward solution operator

$$S = \mu^{-1/2}\mu^{-i\sigma/2+(n+1)/4}S_\sigma\mu^{i\sigma/2-(n+1)/4}\mu^{-1/2},$$
$$S : \mu^{(n+3)/4+\Im\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-} \to \mu^{(n-1)/4+\Im\sigma/2}H^{s,k}(\Omega)^{\bullet,-},$$

of $\Box_g - \frac{1}{4}(n-1)^2 - \sigma^2$. Since g is a 0-metric, the natural vector fields to appear in a nonlinear equation are 0-vector fields; see Section 4E for a brief discussion of these concepts. However, since the analysis is based on ordinary Sobolev spaces relative to which one has b-regularity (regularity with respect to the module $\mathcal{M}$), we consider b-vector fields in the nonlinearities. In case one does use 0-vector fields, the solvability conditions can be relaxed; see Section 4D.

Theorem 4.10. *Suppose $s < \frac{1}{2} - \Im\sigma$. Let*

$$q : \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-} \times \mu^{(n-1)/4+\Im\sigma/2} H^{s,k-1}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$\left(\Box_g - \left(\tfrac{1}{2}(n-1)\right)^2 - \sigma^2\right)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.25. □

Remark 4.11. As in Theorem 2.25, we can also allow nonlinearities $q(u, {}^{\mathrm{b}}du, \Box_g u)$, provided

$$q : \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-} \times \mu^{(n-1)/4+\Im\sigma/2} H^{s-1,k}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \times \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} \\ \to \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$$

is continuous, $q(0,0,0) = 0$ and q has a small Lipschitz constant near 0.

4D. ***Semilinear equations with polynomial nonlinearity.*** Next, we want to find a forward solution of the semilinear PDE

$$\left(\Box_g - \left(\tfrac{1}{2}(n-1)\right)^2 - \sigma^2\right)u = f + c\mu^A u^p X(u), \tag{4-8}$$

where $c \in C^\infty(\tilde{X})$ and $X(u) = \prod_{j=1}^q X_j u$ is a q-fold product of derivatives of u along vector fields $X_j \in \mathcal{M}$. The purpose of the following computations is to obtain conditions on A, p and q which guarantee that the map $u \mapsto c\mu^A u^p X(u)$ satisfies the conditions of the map q in Theorem 4.10. Note that the derivatives in the nonlinearity lie in the module $\mathcal{M}$ (in coordinates: $\mu\partial_\mu$ and ∂_y), whereas, as mentioned above, the natural vector fields are 0-derivatives (in coordinates: $x\partial_x = 2\mu\partial_\mu$ and $x\partial_y = \mu^{1/2}\partial_y$) but, since it does not make the computation more difficult, we consider module instead of 0-derivatives and compensate this by allowing any weight μ^A in front of the nonlinearity.

Rephrasing the PDE in terms of P_σ using $\tilde{u} = \mu^{i\sigma/2-(n+1)/4+1/2}u$ and $\tilde{f} = \mu^{-1/2+i\sigma/2-(n+1)/4} f$, we obtain

$$\begin{aligned} P_\sigma \tilde{u} &= \tilde{f} + c\mu^A \mu^{-1/2+i\sigma/2-(n+1)/4} \mu^{(p+q)(-i\sigma/2+(n-1)/4)} \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}) \\ &= \tilde{f} + c\mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}), \end{aligned}$$

where $f_j \in C^\infty(\widetilde{X})$ and

$$\ell = A + (p+q-1)\bigl(-\tfrac{1}{2}i\sigma + \tfrac{1}{4}(n-1)\bigr) - 1. \tag{4-9}$$

Therefore, if $\tilde{u} \in H^{s,k}(\Omega)^{\bullet,-}$, we obtain that the right-hand side of the equation lies in $H^{s,k-1}(\Omega)^{\bullet,-}$ if $\tilde{f} \in H^{s,k-1}(\Omega)^{\bullet,-}$, $s > \frac{1}{2}$, $k > \frac{1}{2}(n+1)$ — which, by Proposition 4.5, implies that $H^{s,k-1}(\Omega)^{\bullet,-}$ is an algebra — and

$$\Re\ell + \tfrac{1}{2} = A + (p+q-1)\bigl(\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1)\bigr) - \tfrac{1}{2} > s, \tag{4-10}$$

since this condition ensures that $\mu^\ell \in H^{s,\infty}(X)$, which implies that multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \to H^{s,k-1}(\Omega)^{\bullet,-}$.[13] Given the restriction (4-3) on s and $\Im\sigma$, we see that, by choosing $s > \frac{1}{2}$ close to $\frac{1}{2}$ and $\Im\sigma < 0$ close to 0, we obtain the condition

$$p + q > 1 + \frac{4(1-A)}{n-1}. \tag{4-11}$$

If these conditions are satisfied, the right side of the rewritten PDE lies in $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$, so Theorem 4.10 is applicable, and thus (4-8) is well posed in these spaces.

From (4-11) with $A = 0$, we see that quadratic nonlinearities are fine for $n \geq 6$, and cubic ones for $n \geq 4$.

To sum this up, we revert back to $u = \mu^{(n-1)/4 - i\sigma/2}\tilde{u}$ and $f = \mu^{(n+3)/4 - i\sigma/2}\tilde{f}$:

Theorem 4.12. *Let $s > \frac{1}{2}$ and $k > \frac{1}{2}(n+1)$, and assume $A \in \mathbb{R}$ and $p, q \in \mathbb{N}_0$, $p + q \geq 2$, satisfy condition (4-10). Moreover, suppose $\sigma \in \mathbb{C}$ satisfies (4-3), i.e., $\Im\sigma < \frac{1}{2} - s$. Finally, let $c \in C^\infty(\widetilde{M})$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j are vector fields in $\mathcal{M}$. Then, for small enough $R > 0$, there exists a constant $C > 0$ such that, for all $f \in \mu^{(n+3)/4 + \Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the PDE*

$$\bigl(\Box_g - \bigl(\tfrac{1}{2}(n-1)\bigr)^2 - \sigma^2\bigr)u = f + c\mu^A u^p X(u)$$

has a unique solution $u \in \mu^{(n-1)/4 + \Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

The same conclusion holds if the nonlinearity is a finite sum of terms of the form $c\mu^A u^p X(u)$ provided each such term separately satisfies (4-3).

Proof. Reformulating the PDE in terms of $\tilde{u}$ and $\tilde{f}$ as above, this follows from an application of the Banach fixed point theorem to the map

$$H^{s,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}, \quad \tilde{u} \mapsto S_\sigma\Bigl(\tilde{f} + \mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j\tilde{u})\Bigr),$$

with ℓ given by (4-9) and $f_j \in C^\infty(\widetilde{X})$. Here, $p + q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H^{s,k}(\Omega)^{\bullet,-}$. $\square$

[13] If one works in higher regularity spaces, $s \geq \frac{3}{2}$, we in fact only need $\Re\ell + \frac{3}{2} > s$, since then multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-} \to \dot{H}^{s-1,k}(\Omega)^{\bullet,-}$. However, the solvability criterion (4-11) would be weaker, namely the role of the dimension n shifts by 2, since in order to use $s \geq \frac{3}{2}$ we need $\Im\sigma < -1$.

Remark 4.13. Even though the above conditions force $\Im\sigma < 0$, let us remark that the conditions of the theorem, most importantly (4-10), can be satisfied if $m^2 = \frac{1}{4}(n-1)^2 + \sigma^2 > 0$ is real, which thus means that we are in fact considering a nonlinear equation involving the Klein–Gordon operator $\Box_g - m^2$. Indeed, let $\sigma = i\tilde{\sigma}$ with $\tilde{\sigma} < 0$; then condition (4-10) with $A = 0$ and $p+q = 2$ becomes $\tilde{\sigma} > 2 - \frac{1}{2}(n-1)$ (where we accordingly have to choose $s > \frac{1}{2}$ close to $\frac{1}{2}$, depending on $\tilde{\sigma}$), and the requirement $\tilde{\sigma} < 0$ forces $n \geq 6$. On the other hand, we want $\frac{1}{4}(n-1)^2 - \tilde{\sigma}^2 = m^2 > 0$; we thus obtain the condition

$$0 < m^2 < \left(\tfrac{1}{2}(n-1)\right)^2 - \left(2 - \tfrac{1}{2}(n-1)\right)^2$$

for masses m that Theorem 4.12 can handle, which does give a nontrivial range of allowed m for $n \geq 6$.

Remark 4.14. Let us compare the situation in Theorem 4.12 with the situation for the static model of an asymptotically de Sitter space in Section 2. First, we can solve fewer equations globally on asymptotically de Sitter spaces and, second, we need stronger regularity assumptions in order to make an iterative argument work: In the static model, we needed to be in a b-Sobolev space of order greater than $\frac{1}{2}(n+2)$, which in the non-blown-up picture corresponds to 0-regularity of order greater than $\frac{1}{2}(n+2)$, whereas, in the global version, we need a background Sobolev regularity greater than $\frac{1}{2}$, relative to which we have "b-regularity" (i.e., regularity with respect to the module $\mathcal{M}$) of order greater than $\frac{1}{2}(n+1)$. This comparison is of course only a qualitative one, though, since the underlying geometries in the two cases are different.

Using Proposition 4.6 and Corollary 4.7, one can often improve this result. Thus, let us consider the most natural case of (4-8), in which we use 0-derivatives X_j, corresponding to the 0-structure on the *not* even-ified manifold X, and no additional weight. The only difference this makes is if there are tangential 0-derivatives (in coordinates: $\mu^{1/2}\partial_y$). For simplicity of notation, let us therefore assume that $X_j = \mu^{1/2}\tilde{X}_j$, $1 \leq j \leq \alpha$, and $X_j = \tilde{X}_j$, $\alpha < j \leq q$, where the $\tilde{X}_j$ are vector fields in $\mathcal{M}$. Then the PDE (4-8), rewritten in terms of P_σ, $\tilde{u}$ and $\tilde{f}$, becomes

$$P_\sigma \tilde{u} = \tilde{f} + c\mu^\ell \tilde{u}^p \prod_{j=1}^{q} (\tilde{f}_j + \tilde{X}_j \tilde{u}) \tag{4-12}$$

with $\tilde{f}_j \in C^\infty(\tilde{X})$, where

$$\ell = \tfrac{1}{2}\alpha + (p+q-1)\left(-\tfrac{1}{2}(i\sigma) + \tfrac{1}{4}(n-1)\right) - 1.$$

First, suppose that there are no derivatives in the nonlinearity, so that $p \geq 2$ and $q = \alpha = 0$. Then $\mu^\ell \tilde{u}^p \in H^{s-1,k}(\Omega)^{\bullet,-}$ provided $\Re\ell + \frac{3}{2} > s > \frac{1}{2}$ by Corollary 4.7; choosing s arbitrarily close to $\frac{1}{2}$, this is equivalent to

$$\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1) > 0. \tag{4-13}$$

This is a very natural condition: the solution operator for the linear wave equation produces solutions with asymptotics $\mu^{(n-1)/4 \pm i\sigma/2}$; see (2-38), and recall that we are working with the even-ified manifold with boundary defining function $\mu = x^2$. The nonlinear equation (4-8) should therefore only be well behaved if solutions to the linear equation decay at infinity, i.e., if $\pm\Im\sigma + \frac{1}{4}(n-1) \geq 0$. Since we need

$\Im\sigma < 0$ to be allowed to take $s > \frac{1}{2}$, condition (4-13) is equivalent to the (small) decay of solutions to the linear equation at infinity (where $\mu = 0$).

Next, let us assume that $q > 0$. Then the nonlinear term in (4-12) is an element of

$$\mu^\ell H^{s,k}(\Omega)^{\bullet,-} \cdot H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s,k-1}(\Omega)^{\bullet,-}$$

by Proposition 4.6, provided $\Re\ell + \frac{1}{2} > s > \frac{1}{2}$, which gives the condition

$$\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1) > 1 - \tfrac{1}{2}\alpha,$$

where we again choose $s > \frac{1}{2}$ arbitrarily close to $\frac{1}{2}$, so for $\alpha = 2$ we again get condition (4-13) and for $\alpha > 2$ we get an even weaker one.

Finally, let us discuss a nonlinear term of the form $c\mu^A u^p$, $p \geq 2$, in the setting of even lower regularity $0 \leq s < \frac{1}{2}$, the technical tool here being Corollary 4.9: Rewriting the PDE (4-8) with this nonlinearity in terms of P_σ, $\tilde{u}$ and $\tilde{f}$, we get

$$P_\sigma \tilde{u} = \tilde{f} + c\mu^\ell \tilde{u}^p, \quad \ell = A + (p-1)\big(-\tfrac{1}{2}i\sigma + \tfrac{1}{4}(n-1)\big) - 1.$$

Let $s = \frac{1}{2} - \epsilon$ with $0 \leq \epsilon < 1/(2p)$. Then, if $\tilde{u} \in H^{1/2-\epsilon,k}(\Omega)^{\bullet,-}$ with $k > \frac{1}{2}(n-1)$, Corollary 4.9 yields $\tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-}$; thus

$$\mu^\ell \tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$$

provided $\Re\ell \geq 0$, that is,

$$n > 1 + \frac{4(1-A)}{p-1} - 2\Im\sigma, \tag{4-14}$$

where we still require $\Im\sigma < \frac{1}{2} - s = \epsilon$, which in particular allows σ to be real if $\epsilon > 0$.

In summary:

Theorem 4.15. *Let $p \geq 2$ be an integer, $\frac{1}{2} - 1/(2p) < s \leq \frac{1}{2}$, $k > \frac{1}{2}(n-1)$, and suppose $\sigma \in \mathbb{C}$ is such that $\Im\sigma < \frac{1}{2} - s$. Moreover, assume $A \in \mathbb{R}$ and the dimension n satisfy condition (4-14). Then, for small enough $R > 0$, there exists a constant $C > 0$ such that, for all $f \in \mu^{(n+3)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the PDE*

$$\big(\Box_g - \big(\tfrac{1}{2}(n-1)\big)^2 - \sigma^2\big)u = f + c\mu^A u^p$$

has a unique solution $u \in \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

In particular, if $\frac{1}{4} < s < \frac{1}{2}$, $0 < \Im\sigma < \frac{1}{2} - s$ and $A = 0$, then quadratic nonlinearities are fine for $n \geq 5$; if $\Im\sigma = 0$ and $A = 0$, then they work for $n \geq 6$.

4D1. *Backward solutions to semilinear equations with polynomial nonlinearity.* Recalling the setting of Section 4A1, let us briefly turn to the backward problem for (4-8), which we rephrase in terms of P_σ as above. For simplicity, let us only consider the "least sophisticated" conditions, namely $s > \frac{1}{2}$, $k > \frac{1}{2}(n+1)$,

$$A + (p+q-1)\big(\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1)\big) - \tfrac{1}{2} > s, \tag{4-15}$$

and — this is the important change compared to the forward problem — $s > \frac{1}{2} - \Im\sigma$, where the latter guarantees the existence of the backward solution operator S_σ^-. Thus, if $\Im\sigma > 0$ is large enough and $s > \frac{1}{2}$ satisfies (4-15), then (4-8) is solvable in any dimension.

In the special case that we only consider 0-derivatives and no extra weight, which corresponds to putting $A = q + \frac{1}{2}\alpha$, we obtain the condition

$$\Im\sigma > \frac{4\left(1 - q - \frac{1}{2}\alpha\right) - (p + q - 1)(n - 1)}{2(p + q + 1)}$$

if we choose $s > \frac{1}{2} - \Im\sigma$ close to $\frac{1}{2}$, which in particular allows $\Im\sigma \geq 0$, and thus σ^2 *arbitrary*, if $p > 1 + 4/(n-1)$ (so $p \geq 2$ is acceptable if $n \geq 6$) or $q + \frac{1}{2}\alpha \geq 1$.

4E. ***From static parts to global asymptotically de Sitter spaces.*** Let us consider the equation

$$(\Box_g - m^2)u = f + q(u, {}^0du), \tag{4-16}$$

where the reason for using the 0-differential ${}^0d$ (see below) will be given momentarily. The idea is that every point in X lies in the interior of the backward light cone from some point p at future infinity Y_+, denoted S_p; that is, the blow-up of $\overline{X}$ at p contains the static part S_p of an asymptotically de Sitter space, where the solvability statements have been explained in Section 2. Consider a suitable neighborhood $\Omega_p \subset [\overline{X}; p]$ of the static patch as in Section 2, so the boundary of Ω_p is the union of ∂S_p and an "artificial" spacelike boundary, which on the non-blown-up space $\overline{X}$ all meet at the point p, and a Cauchy surface. In fact, we may choose the Ω_p in a fashion that is uniform in p. We then solve (4-16) on Ω_p, thereby obtaining a forward solution u_p and, by local uniqueness for $\Box_g - m^2$ in X, all such solutions agree on their overlap, i.e., $u_p \equiv u_q$ on $\Omega_p \cap \Omega_q$. Therefore, we can define a function u by setting $u = u_p$ on Ω_p, $p \in Y_+$, which then is a solution of (4-16) on X. To make this precise, we need to analyze the relationships between the function spaces on the Ω_p, $p \in Y_+$, and X. As we will see in Lemma 4.16, b-Sobolev spaces on the blow-ups Ω_p of $\overline{X}$ at boundary points are closely related to 0-Sobolev spaces on X.

Recall the definition of 0-Sobolev spaces on a manifold with boundary M (for us, $M = \overline{X}$) with a 0-metric, that is, a metric of the form $x^{-2}\hat{g}$ with x a boundary defining function, where $\hat{g}$ extends nondegenerately to the boundary: If $\mathcal{V}_0(M) = x\mathcal{V}(M)$ denotes the Lie algebra of 0-vector fields, where $\mathcal{V}(M)$ are smooth vector fields on M, and $\mathrm{Diff}_0^*(M)$ the enveloping algebra of 0-differential operators, then

$$H_0^k(M) = \{u \in L^2(M, d\mathrm{vol}) : Pu \in L^2(M, d\mathrm{vol}),\ P \in \mathrm{Diff}_0^k(M)\}$$

and $H_0^{k,\ell}(M) = x^\ell H_0^k(M)$. For clarity, we shall write $L_0^2(M) = L^2(M, d\mathrm{vol})$. We also recall the definition of the 0-(co)tangent spaces: if $\mathcal{I}_p$ denotes the ideal of $C^\infty(M)$ functions vanishing at $p \in M$, then the 0-tangent space at p is defined as ${}^0T_pM = \mathcal{V}_0(M)/\mathcal{I}_p \cdot \mathcal{V}_0(M)$, and the 0-cotangent space at p, ${}^0T_p^*M$, as the dual of ${}^0T_pM$. In local coordinates $(x, y) \in \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ near the boundary of M, we have $d\mathrm{vol} = f(x, y)(dx/x)(dy/x^{n-1})$ with f smooth and nonvanishing, and $\mathcal{V}_0(M)$ is spanned by $x\partial_x$ and $x\partial_y$; also, $x\partial_x$ and $x\partial_{y_j}$, $j = 2, \ldots, n$, form a basis of ${}^0T_pM$ (for $p \in \partial M$, which is the only place where 0-spaces differ from the standard spaces), and dx/x and dy_j/x, $j = 2, \dots, n$, form a basis of ${}^0T^*_pM$. The exterior derivative d induces the first-order 0-differential operator ${}^0d$ on sections of Λ^0TM; this follows from

$$df = (\partial_x f)\,dx + (\partial_y f)\,dy = (x\partial_x f)\,\frac{dx}{x} + (x\partial_y f)\,\frac{dy}{x}.$$

Now, let $\Omega \subset \overline{X}$ be a domain as in Section 4A. Moreover, let $\beta_p : \Omega_p \to X$ be the blow-down map. We then have:

Lemma 4.16. *Let $k \in \mathbb{N}_0$ and $\ell \in \mathbb{R}$. Then there are constants $C > 0$ and $C_\delta > 0$ such that, for all $\delta > 0$,*

$$\|f\|_{H_0^{k,\ell-(n-1)/2-\delta}(\Omega)^\bullet} \le C_\delta \sup_{p\in Y_+} \|\beta_p^* f\|_{H_b^{k,\ell}(\Omega_p)^{\bullet,-}} \le CC_\delta \|f\|_{H_0^{k,\ell}(\Omega)^\bullet}. \tag{4-17}$$

Here, $\bullet$ indicates supported distributions at the "artificial" boundary and $-$ extendible distributions at all other boundary hypersurfaces.

Proof. Let us work locally near a point $p \in Y_+$; since $Y_+ \cong \mathbb{S}^{n-1}$ is compact, all constructions below can be made uniformly in p. The only possible issues are near the boundary $Y_+ = \{x = 0\}$, with x a boundary defining function; hence, let us work in a product neighborhood $Y_+ \times [0, 2\epsilon)_x$, $\epsilon > 0$, of Y_+, and let us assume u is supported is $Y_+ \times [0, \epsilon]$.

We use coordinates x, $y_2, \dots, y_n$ such that $y_j = 0$ at p. Coordinates on S_p are then x, $z_2, \dots, z_n$ with $z_j = y_j/x$, that is, $\beta_p(x, z) = (x, xz)$, with the restriction $\sum_{j=2}^n |z_j|^2 \le 1$. Therefore,

$$\|\beta_p^* f\|^2_{L^2_b} \approx \int_{S_p} |\beta_p^* f(x,z)|^2\,\frac{dx}{x}\,dz = \int_{\beta_p(S_p)} |f(x,xz)|^2\,\frac{dx}{x}\,dz \le \int |f(x,y)|^2\,\frac{dx}{x}\,\frac{dy}{x^{n-1}} \approx \|f\|^2_{L^2_0}.$$

Adding weights to this estimate is straightforward. Next, we observe

$$\begin{aligned} x\partial_x(\beta_p^* f)(x,z) &= x\partial_x f(x,xz) + zx\partial_y f(x,xz), \\ \partial_z(\beta_p^* f)(x,z) &= x\partial_y f(x,xz) \end{aligned} \tag{4-18}$$

and, since $|z| \le 1$, we conclude that $\beta_p^* f \in H_b^1(S_p)$ is equivalent to f, $x\partial_x f$, $x\partial_y f \in L^2_0(\beta_p(S_p))$, which proves the second inequality in (4-17) in the case $k = 1$; the general case is similar.

For the first inequality in (4-17), we first note that the additional weight comes from the number of static parts, i.e., interiors of backward light cones from points in Y_+, that one needs to cover any fixed half space $\{x \ge x_0\}$. Namely, for $0 < x_0 \le \epsilon$, let $\mathscr{B}(x_0) \subset Y_+$ be a set of points such that every point in $\{x \ge x_0\}$ lies in S_p for some $p \in \mathscr{B}(x_0)$; then we can choose $\mathscr{B}(x_0)$ such that $|\mathscr{B}(x_0)| \le Cx_0^{-(n-1)}$, where $|\cdot|$ denotes the number of elements in a set. This follows from the observation that the area of the slice $x = x_0$ of S_p within $Y_+ \cong \mathbb{S}^{n-1}$ (keeping in mind that we are working in a product neighborhood of Y_+) is bounded from below by cx_0^{n-1} for some p-independent constant $c > 0$. Indeed, note that null-geodesics of the 0-metric g are, up to reparametrization, the same as null-geodesics of the conformally related metric x^2g, which is a nondegenerate Lorentzian metric up to Y_+. See also Figure 5 below.

Thus, putting $\alpha = \frac{1}{2}(n-1)+\delta$, $\delta > 0$, we estimate

$$\begin{aligned}\int_{x\le\epsilon} |x^\alpha f(x,y)| \frac{dx}{x}\frac{dy}{x^{n-1}} &= \sum_{j=0}^\infty \int_{2^{-j-1}\epsilon<x\le 2^{-j}\epsilon} |x^\alpha f(x,y)|^2 \frac{dx}{x}\frac{dy}{x^{n-1}} \\ &\lesssim \sum_{j=0}^\infty 2^{-2\alpha j} \sum_{p\in\mathscr{B}(2^{-j-1}\epsilon)} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}} \\ &\lesssim \sum_{j=0}^\infty 2^{-2\alpha j}(2^{-j-1}\epsilon)^{-n+1} \sup_{p\in Y_+} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}} \\ &\lesssim \sum_{j=0}^\infty 2^{-j(2\alpha-n+1)} \sup_{p\in Y_+} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}},\end{aligned}$$

with the sum converging since $2\alpha - n + 1 = 2\delta > 0$. Weights and higher-order Sobolev spaces are handled similarly, using (4-18). □

In particular, this explains why in (4-16) we take $d = {}^0d : H_0^{k,\ell}(X) \to H_0^{k-1,\ell}(X; {}^0T^*X)$, namely this is necessary in order to make the global equation interact well with the static patches.

Since we want to consider local problems to solve the global one, the nonlinearity q must be local in the sense that $q(u, {}^0du)(p)$ for $p \in X$ only depends on p and its arguments evaluated at p; let us, for simplicity, assume that q is in fact a polynomial, as in (2-43).

Using Corollary 2.28, we then obtain:

Theorem 4.17. *Let $0 \le \epsilon < \epsilon_0$ with ϵ_0 as in Section 2B, and $s > \max\big(\frac{3}{2}+\epsilon, \frac{1}{2}n+1\big)$, $s \in \mathbb{N}$. Let*

$$q(u, {}^0du) = \sum_{2\le j+|\alpha|\le d} q_{j\alpha} u^j \prod_{k\le|\alpha|} X_{\alpha,k}u,$$

where $q_{j,\alpha} \in \mathbb{C} + H_0^s(\overline{X})$, $X_{\alpha,k} \in \mathcal{V}_0(M)$. Then there exists $C > 0$ such that, for all $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^0du)$$

has a unique solution $u \in \bigcap_{\delta>0} H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ that depends continuously on f. Here, we allow $m = 0$ if every summand of q contains at least one 0-derivative, and require $m > 0$ if this is not the case, e.g., if $q = q(u)$ is simply the sum of (multiples of) powers of u.

The analogous conclusion also holds for $\Box_g u = f + q({}^0du)$ provided $\epsilon > 0$, with the solution u being in $\bigcap_{\delta>0} H_0^{s,-(n-1)/2-\delta}(\Omega)^\bullet$. Moreover, for all $p \in Y_+$, the limit $u_\partial(p) := \lim_{p'\to p, p'\in X} u(p')$ exists, $u_\partial \in C^{0,\epsilon}(Y_+)$, and $u - u_\partial(\phi\circ \mathrm{t}_1) \in x^\epsilon C^0(\overline{X})$, where $\phi \circ \mathrm{t}_1$ is identically 1 near Y_+ and vanishes near the "artificial" boundary of Ω.

Proof. We start by proving the first part: If $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ then $f_p = \beta_p^* f \in H_{\mathrm{b}}^{s-1,\epsilon}(S_p)$ is a uniformly bounded family in the respective norms, by Lemma 4.16. We can then use Corollary 2.28 to solve

$$(\Box_g - m^2)u_p = f_p + q(u_p, {}^{\mathrm{b}}du_p)$$

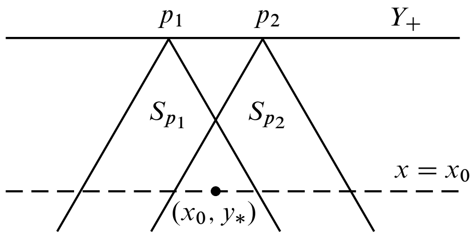

Figure 5. Setup for the proof of $u_\partial \in C^{0,\epsilon}(Y_+)$; shown are the backward light cones from two nearby points $p_1,\ p_2 \in Y_+$ that intersect within the slice $\{x = x_0\}$ at a point (x_0, y_*).

in the static part S_p, where we use that q is a polynomial and the fact that ${}^{\mathrm{b}}T^*_{p'}S_p$ naturally injects into ${}^0T^*_{\beta_p(p')}\Omega$ for $p' \in S_p$ to make sense of the nonlinearity; we thus obtain a uniformly bounded family $u_p = \tilde{u}_p|_{S_p} \in H_{\mathrm{b}}^{s,\epsilon}(S_p)^{\bullet,-}$. By local uniqueness and since f vanishes near Y_-, we see that the function u, defined by $u(\beta_p(p')) = u_p(p')$ for $p \in Y_+$ and $p' \in S_p$, is well defined and, by Lemma 4.16, we indeed have $u \in H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ for all $\delta > 0$.

For the second part, we follow the same strategy, obtaining solutions $u_p = c_p(\phi \circ \mathfrak{t}_1) + u'_p$ of

$$\Box_g u_p = f_p + q({}^{\mathrm{b}}du_p),$$

where $c_p \in \mathbb{C}$ and $u'_p \in H_{\mathrm{b}}^{s,\epsilon}(S_p)^{\bullet,-}$ are uniformly bounded, thus u_p is uniformly bounded in $H_{\mathrm{b}}^{s,-\delta}(\Omega)^\bullet$ for every fixed $\delta > 0$ and, therefore, the existence of a unique solution u follows as before. Put $u_\partial(p) := c_p$; then $u_\partial(p) = \lim_{p' \to p, p' \in S_p} u(p')$, since $u'_p \in x^\epsilon C^0(S_p)$ by the Sobolev embedding theorem. We first prove that u_∂ so defined is ϵ-Hölder continuous. Let us work in local coordinates (x, y) near a point $(0, y_0)$ in Y_+. Now, u'_p is uniformly bounded in $x^\epsilon C^0(S_p)$ and since, for $x_0 > 0$ arbitrary, we have $c_{p_1} + u'_{p_1}(x_0, y_*) = c_{p_2} + u'_{p_2}(x_0, y_*)$ for all $p_1, p_2 \in Y_+$ provided $|p_1 - p_2| \le c x_0$ for some constant $c > 0$, which ensures that $S_{p_1} \cap S_{p_2} \cap \{x = x_0\}$ is nonempty and thus contains a point (x_0, y_*) (see Figure 5), we obtain

$$|c_{p_1} - c_{p_2}| = |u'_{p_1}(x_0, y_*) - u'_{p_2}(x_0, y_*)| \le C x_0^\epsilon \quad \text{when} \quad |p_1 - p_2| \le c x_0$$

for all x_0, thus

$$\frac{|u_\partial(p_1) - u_\partial(p_2)|}{|p_1 - p_2|^\epsilon} \le C, \qquad p_1,\ p_2 \in Y_+.$$

This in particular implies that

$$\begin{aligned} |u(x,y) - u_\partial(0,y_0)| &\le |u(x,y) - u_\partial(0,y)| + |u_\partial(0,y) - u_\partial(0,y_0)| \\ &\le C(|y - y_0|^\epsilon + x^\epsilon) \to 0 \quad \text{as } x \to 0,\ y \to y_0, \end{aligned} \tag{4-19}$$

hence we in fact have $u_\partial(p) = \lim_{p' \to p,\, p' \in X} u(p')$. Finally, putting $y = y_0$ in (4-19) proves that $u - u_\partial(\phi \circ \mathfrak{t}_1) \in x^\epsilon C^0(\overline{X})$. $\square$

The major lossy part of the argument is the conversion from f to the family $\beta_p^* f$: even though the second inequality in Lemma 4.16 is optimal (e.g., for functions which are supported in a single static patch), one loses $\frac{1}{2}(n-1)$ orders of decay relative to the gluing estimate, i.e., the first inequality in Lemma 4.16, which is used to pass from the family u_p to u.

Observe on the other hand that the decay properties of u, without regard to those of f, in the first part of the theorem are very natural, since the constant function 1 is an element of $\bigcap_{\delta>0} H_0^{\infty,-(n-1)/2-\delta}(X)$, thus u has an additional decay of ϵ relative to constants.

Remark 4.18. For the proof of Theorem 4.17 it is irrelevant whether certain 0-Sobolev spaces are algebras, since the main analysis, Corollary 2.28, is carried out on b-Sobolev spaces.

5. Lorentzian scattering spaces

5A. ***The linear Fredholm framework.*** We now consider n-dimensional nontrapping asymptotically Minkowski spacetimes (M, g), a notion which includes the radial compactification of Minkowski spacetime. This notion was briefly recalled in the introduction (see p. 1811); here we restate this in the notation of [Baskin et al. 2014, §3], where this notion was introduced.

Thus, M is compact with smooth boundary, with a boundary defining function ρ (we switch the notation from τ mainly to emphasize that ρ is not everywhere timelike) and *scattering vector fields* $V \in \mathcal{V}_{\mathrm{sc}}(M)$, introduced by Melrose [1994], are smooth vector fields of the form $\rho V'$, $V' \in \mathcal{V}_{\mathrm{b}}(M)$. Hence, if the z_j are local coordinates on ∂M extended to a neighborhood in M, then a local basis of these vector fields over $C^\infty(M)$ is $\rho^2\partial_\rho$, $\rho\partial_{z_j}$. Correspondingly, $\mathcal{V}_{\mathrm{sc}}(M)$ is the set of smooth sections of a vector bundle ${}^{\mathrm{sc}}TM$, which is therefore, roughly speaking, $\rho\,{}^{\mathrm{b}}TM$. The vector field $\rho^2\partial_\rho$ is well defined up to a positive factor at $\rho = 0$ and is called the *scattering normal vector field* of ∂M. The dual bundle of ${}^{\mathrm{sc}}TM$, called the *scattering cotangent bundle*, is denoted by ${}^{\mathrm{sc}}T^*M$. If M is the radial compactification of $\mathbb{R}^n$, obtained by gluing a sphere at infinity via the reciprocal polar coordinate map $(r, \omega) \mapsto (r^{-1}, \omega) \in (0, 1)_\rho \times \mathbb{S}^{n-1}_\omega$, that is, adding $\rho = 0$ to the right-hand side (corresponding to "$r = \infty$"), then $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by (the lifts of) the translation-invariant vector fields over $C^\infty(M)$.

A *Lorentzian scattering metric* g is a Lorentzian signature, taken to be $(1, n-1)$, metric on ${}^{\mathrm{sc}}TM$, i.e., a smooth symmetric section of ${}^{\mathrm{sc}}T^*M \otimes {}^{\mathrm{sc}}T^*M$ with this signature with the following additional properties:

(1) There is a real C^∞ function v defined on M with dv and $d\rho$ linearly independent at "the light cone at infinity", $S = \{v = 0,\ \rho = 0\}$.

(2) $g(\rho^2\partial_\rho, \rho^2\partial_\rho)$ has the same sign as v at $\rho = 0$, i.e., $\rho^2\partial_\rho$ is timelike in $v > 0$ and spacelike in $v < 0$.

(3) Near S,
$$g = v\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \frac{\tilde{h}}{\rho^2},$$
where α is a smooth one-form on M,
$$\alpha = \tfrac{1}{2}\,dv + \mathcal{O}(v) + \mathcal{O}(\rho),$$
and $\tilde{h}$ is a smooth 2-cotensor on M, which is positive definite on the (codimension-two) annihilator of $d\rho$ and dv.

A Lorentzian scattering metric is *nontrapping* if:

(1) $S = S_+ \cup S_-$ (each a disjoint union of connected components) and in $X = \partial M$ the open set $\{v > 0\} \cap X$ decomposes as $C_+ \cup C_-$ (disjoint union), with $\partial C_+ = S_+$ and $\partial C_- = S_-$; we write $C_0 = \{v < 0\} \cap X$.

(2) The projections of all null-bicharacteristics in ${}^{\mathrm{sc}}T^*M \setminus o$ to M tend to $S_\pm$ as their parameter tends to $\pm\infty$ or vice versa.

Since a conformal factor only reparameterizes bicharacteristics, this means that, with $\hat{g} = \rho^2 g$, which is a b-metric on M, the projections of all null-bicharacteristics of $\hat{g}$ in ${}^{\mathrm{b}}T^*M \setminus o$ tend to $S_\pm$. As already pointed out in the introduction (see p. 1812), the difference between the de Sitter-type and Minkowski settings is that at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S the nature of the radial points is source or sink rather than a saddle point of the flow at $L_\pm$ discussed in Section 2A.

We first state solvability properties, namely we show that, under the assumptions of [Baskin et al. 2014, §3], the problem of finding a tempered solution to $\Box_g w = f$ is a Fredholm problem in suitable weighted Sobolev spaces. In particular, there is only a finite-dimensional obstruction to existence. Then we strengthen the assumptions somewhat and show actual solvability in the strong sense that, in these spaces, the solution w satisfies that, if f is vanishing to infinite order near $\overline{C_-}$, then so is w.

Let

$$L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$$

be the "conjugated" b-wave operator (as in [Baskin et al. 2014, §4]), which is formally self-adjoint with respect to the density of the Lorentzian b-metric $\hat{g} = \rho^2 g$; further, $L = \Box_{\hat{g}} - \gamma$, where $\gamma \in C^\infty(M)$ is real-valued. Choose

$$m \in C^\infty({}^{\mathrm{b}}S^*M) \text{ a variable (Sobolev) order function, decreasing along the direction of the Hamilton flow oriented to the future, i.e., towards } S_+. \tag{5-1}$$

Remark 5.1. In the actual application of asymptotically Minkowski spaces, one can take m to be a function on M rather than ${}^{\mathrm{b}}S^*M$ by making it take constant values near $\overline{C_+}$ (resp. $\overline{C_-}$) corresponding to the requirements at $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) below, and transitioning in between using a time function as in the discussion preceding Theorem 5.3, i.e., making m of the form $F \circ \tilde{\mathfrak{t}}$ for appropriate F. Since this simplifies some arguments below, we assume this whenever it is convenient.

With

$$\mathcal{R}_+ = {}^{\mathrm{b}}SN^*S_+ \quad (\text{resp. } \mathcal{R}_- = {}^{\mathrm{b}}SN^*S_-)$$

the future (resp. past) radial sets in ${}^{\mathrm{b}}S^*M$ — see [Baskin et al. 2014, §3.6] — and with

$$m + l < \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad m + l > \tfrac{1}{2} \quad \text{at } \mathcal{R}_-,$$

and m constant near $\mathcal{R}_+ \cup \mathcal{R}_-$, one has an estimate

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \le C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + C\|u\|_{H_{\mathrm{b}}^{m',l}} \tag{5-2}$$

provided one assumes $m' < m$,

$$m' + l > \tfrac{1}{2} \quad \text{at } \mathcal{R}_- \qquad \text{and} \qquad u \in H_{\mathrm{b}}^{m',l}.$$

To see this, we recall and record a slight improvement of [Baskin et al. 2014, Proposition 4.4]:

Proposition 5.2. *Suppose L is as above.*

If $m+l < \frac{1}{2}$ and $u \in H_{\mathrm{b}}^{-\infty,l}(M)$, then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu) = \varnothing$ and a punctured neighborhood of $\mathcal{R}_\pm$, with $\mathcal{R}_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*M$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$.

On the other hand, if $m'+l > \frac{1}{2}$, $m \geq m'$, $u \in H_{\mathrm{b}}^{-\infty,l}(M)$ and $\mathrm{WF}_{\mathrm{b}}^{m',l}(u) \cap \mathcal{R}_\pm = \varnothing$, then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu) = \varnothing$.

Proof. The first statement is proved in [Baskin et al. 2014, Proposition 4.4]. The second statement follows the same way, but in that case the product of the required powers of the boundary defining functions, $\rho^{-2l}\tilde{\rho}^{-2m+1}$, with $\tilde{\rho}$ the defining function of fiber infinity[14] as in Section 2A, in the commutant of [Baskin et al. 2014, Proposition 4.4] provides a favorable sign, thus [Baskin et al. 2014, Equation (4.1)] holds without the E term. However, when regularizing, the regularizer contributes a term with the opposite sign, exactly as in [Vasy 2013a, Proof of Propositions 2.3–2.4]; this forces the requirement on the a priori regularity, namely $\mathrm{WF}_{\mathrm{b}}^{m',l}(u) \cap \mathcal{R}_\pm = \varnothing$, exactly as in those propositions; see also Proposition 2.1 above. □

Indeed, due to the closed graph theorem, (5-2) follows immediately from the b-radial point regularity statements of Proposition 5.2 for sources and sinks, and the propagation of b-singularities for variable-order Sobolev spaces, which is not proved in [Baskin et al. 2014], but whose analogue in standard Sobolev spaces is proved there in [Baskin et al. 2014, Proposition A.1] (with additional references given to related results in the literature) and, as it is a purely symbolic argument, the extension to the b-setting is straightforward. (We refer to Proposition 2.1 here and [Baskin et al. 2014, Proposition 4.4] extending the radial point results, Propositions 2.3–2.4, of [Vasy 2013a], from the boundaryless setting to the b-setting.)

One also has a similar estimate for L when one replaces m by a weight $\tilde{m}$ which is increasing along the direction of the Hamilton flow oriented towards the past,

$$\tilde{m} + \tilde{l} > \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad \tilde{m} + \tilde{l} < \tfrac{1}{2} \quad \text{at } \mathcal{R}_-,$$

provided one assumes $\tilde{m}' < \tilde{m}$,

$$\tilde{m}' + \tilde{l} > \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad u \in H_{\mathrm{b}}^{\tilde{m}',\tilde{l}}.$$

Further, L can be replaced by L^*. Thus,

$$\|u\|_{H_{\mathrm{b}}^{\tilde{m},\tilde{l}}} \leq C\|L^*u\|_{H_{\mathrm{b}}^{\tilde{m}-1,\tilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{\tilde{m}',\tilde{l}}}. \tag{5-3}$$

Just as in the asymptotically de Sitter and Kerr–de Sitter settings, one wants to improve these estimates so that the space $H_{\mathrm{b}}^{m,l}$ and, respectively, $H_{\mathrm{b}}^{\tilde{m},\tilde{l}}$ on the left-hand side includes compactly into the error term

[14] This defining function is denoted by ν in [Baskin et al. 2014].

on the right-hand side. This argument is completely analogous to Section 2A using the Mellin-transformed normal operator estimates obtained in [Baskin et al. 2014, §5]. We thus further assume that there are no poles of the Mellin conjugate $\hat{L}(\sigma)$ on the line $\Im\sigma = -l$. Then, using the Mellin transform and the estimates for $\hat{L}(\sigma)$ (including the high-energy estimates, which imply that for all but a discrete set of l the aforementioned lines do not contain such poles), as in Section 2A we obtain that, on $\mathbb{R}^+_\rho \times \partial M$,

$$\|v\|_{H_{\mathrm{b}}^{\hat{m},l}} \le C\|N(L)v\|_{H_{\mathrm{b}}^{\hat{m}-1,l}} \tag{5-4}$$

when $\hat{m} \in C^\infty(S^*\partial M)$ is a variable-order function decreasing along the direction of the Hamilton flow oriented to the future, Λ_+ (resp. Λ_-) is the future (resp. past) radial set in $S^*\partial M$, and with

$$\hat{m} + l < \tfrac{1}{2} \quad \text{at } \Lambda_+, \qquad \hat{m} + l > \tfrac{1}{2} \quad \text{at } \Lambda_-.$$

One can take

$$\hat{m} = m|_{T^*\partial M},$$

for instance, under the identification of $T^*\partial M$ as a subspace of ${}^{\mathrm{b}}T^*_{\partial M}M$, taking into account that homogeneous degree-zero functions on $T^*\partial M \setminus o$ are exactly functions on $S^*\partial M$, and analogously on ${}^{\mathrm{b}}T^*_{\partial M}M$. However, in the limit $\sigma \to \infty$, one should use norms depending on σ, reflecting the dependence of the semiclassical norm on h. We recall from Remark 5.1 that in the main case of interest one can take m to be a pullback from M and thus the Mellin-transformed operator norms are independent of σ. In either case, we simply write m in place of $\hat{m}$.

Again, we have an analogous estimate for $N(L^*)$:

$$\|v\|_{H_{\mathrm{b}}^{\tilde{m},\tilde{l}}} \le C\|N(L^*)v\|_{H_{\mathrm{b}}^{\tilde{m}-1,\tilde{l}}} \tag{5-5}$$

provided $-\tilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$, and provided $\tilde{m}$ satisfies the requirements above. As $\widehat{L^*}(\sigma) = (\hat{L})^*(\bar{\sigma})$, the requirement on $-\tilde{l}$ is the same as $\tilde{l}$ not being the imaginary part of a pole of $\hat{L}$.

At this point, the argument of the paragraph of (2-10) in Section 2A can be repeated verbatim to yield that, for m with $m + l > \frac{3}{2}$ at $\mathcal{R}_-$ (with the stronger restriction coming from the requirements on m' at $\mathcal{R}_-$, $\tilde{m}'$ at $\mathcal{R}_+$, and $m' < m - 1$, $\tilde{m}' < \tilde{m} - 1$; recall that one needs to estimate the normal operator on these primed spaces) and $m + l < \frac{1}{2}$ at $\mathcal{R}_+$,

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \le C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,l-1}}, \tag{5-6}$$

where now the inclusion $H_{\mathrm{b}}^{m,l} \to H_{\mathrm{b}}^{m'+1,l-1}$ is compact (as we choose $m' < m - 1$); this argument required m, l and m' satisfied the requirements preceding (5-2), and that $-l$ is not the imaginary part of any pole of $\hat{L}$.

Analogous estimates hold for L^*:

$$\|u\|_{H_{\mathrm{b}}^{\tilde{m},\tilde{l}}} \le C\|L^*u\|_{H_{\mathrm{b}}^{\tilde{m}-1,\tilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,\tilde{l}-1}} \tag{5-7}$$

provided $\tilde{m}$, $\tilde{l}$ and $\tilde{m}'$ satisfy the requirements stated before (5-3), $\tilde{m}' < \tilde{m} - 1$, and $-\tilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$ (i.e., $\tilde{l}$ of $\hat{L}$).

Via the same functional analytic argument as in Section 2A, we thus obtain Fredholm properties of L, in particular solvability, modulo a (possible) finite-dimensional obstruction, in $H_{\mathrm{b}}^{m,l}$ if

$$m+l>\tfrac{3}{2}\quad\text{at }\mathcal{R}_-,\qquad m+l<-\tfrac{1}{2}\quad\text{at }\mathcal{R}_+.$$

More precisely, we take $\tilde{m}=1-m$ and $\tilde{l}=-l$, so $m+l<-\frac{1}{2}$ at $\mathcal{R}_+$ means $\tilde{m}+\tilde{l}=1-(m+l)>\frac{3}{2}$, so the space on the left-hand side of (5-6) is dual to that in the first term on the right-hand side of (5-7), and the same for the equations interchanged. Then the Fredholm statement is for

$$L:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l},$$

with

$$\mathcal{Y}^{s,r}=H_{\mathrm{b}}^{s,r},\quad \mathcal{X}^{s,r}=\{u\in H_{\mathrm{b}}^{s,r}:Lu\in H_{\mathrm{b}}^{s-1,r}\}.$$

Note that, by propagation of singularities, i.e., most importantly using Proposition 5.2, with $\operatorname{Ker} L\subset H_{\mathrm{b}}^{m,l}$ and $\operatorname{Ker} L^*\subset H_{\mathrm{b}}^{1-m,-l}$ a priori,

$$\operatorname{Ker} L\subset H_{\mathrm{b}}^{m^\flat,l}\ \text{ and }\ \operatorname{Ker} L^*\subset H_{\mathrm{b}}^{1-m^\flat,-l}\quad\text{if}\quad m^\flat+l>\tfrac{1}{2}\ \text{at }\mathcal{R}_-\ \text{and}\ m^\flat+l<\tfrac{1}{2}\ \text{at }\mathcal{R}_+. \tag{5-8}$$

We can improve this further using the propagation of singularities. Namely, suppose one merely has

$$m+l>\tfrac{3}{2}\quad\text{at }\mathcal{R}_-,\qquad m+l<\tfrac{1}{2}\quad\text{at }\mathcal{R}_+, \tag{5-9}$$

so the requirement at $\mathcal{R}_+$ is weakened. Then let $m^\sharp=m-1$ near $\mathcal{R}_+$ and $m^\sharp\le m$ everywhere, but still satisfying the requirements for the order function along the Hamilton flow, so the Fredholm result is applicable with $m^\sharp$ in place of m. Now, if $u\in\mathcal{X}^{m^\sharp,l}$, $Lu=f$ and $f\in\mathcal{Y}^{m-1,l}\subset\mathcal{Y}^{m^\sharp-1,l}$, then Proposition 5.2 gives $u\in\mathcal{X}^{m,l}$. Further, if $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ are trivial, this gives that, for m and l as in (5-9) satisfying also the conditions along the Hamilton flow, $L:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l}$ is invertible.

Now, as invertibility (the absence of kernel and cokernel) is preserved under sufficiently small perturbations, it holds in particular for perturbations of the Minkowski metric which are Lorentzian scattering metrics in our sense, with closeness measured in smooth sections of the second symmetric power of ${}^{\mathrm{b}}T^*M$. (Note that nontrapping is also preserved under such perturbations.)

For more general asymptotically Minkowski metrics we note that, due to Theorem 2.21 (which does not have any requirements for the timelike nature of the boundary defining function, and which works locally near $\overline{C_-}$ either by working on (extendible) function spaces or by using the localization given by wave propagation as in §3.3 of [Vasy 2013a] or Section 4A here), elements of $\operatorname{Ker} L$ on $H_{\mathrm{b}}^{m,l}$, with m and l as above, lie in $\dot{C}^\infty(M)$ locally near $\overline{C_-}$ provided all resonances, i.e., poles of $\hat{L}(\sigma)$, in $\Im\sigma<-l$ have polar parts (coefficients of the Laurent series) that map into distributions supported on $\overline{C_+}$. As shown in [Vasy 2014, Remark 4.17], when $\hat{L}(\sigma)$ arises from a Lorentzian conic metric as in[15] [Vasy 2014, Equation (3.5)], but with the arguments applicable without significant changes in our more general

[15] In [Vasy 2014], the boundary defining function used to define the Mellin transform is replaced by its reciprocal, which effectively switches the sign of σ in the operator, but also the backward propagator is considered (propagating toward the past light cone), which reverses the role of σ and $-\sigma$ again, so in fact, the signs in [Vasy 2014] and [Baskin et al. 2014] agree for the formulae connecting the asymptotically hyperbolic resolvents and the global operator, $\hat{L}(\sigma)$.

case, see also [Baskin et al. 2014, §7] for our general setting, and [Vasy 2013a, Remark 4.6] for a related discussion with complex absorption, the resonances of $\hat{L}(\sigma)$ consist of the resonances of the asymptotically hyperbolic resolvents on the caps, namely $\mathcal{R}_{C_+}(\sigma)$ and $\mathcal{R}_{C_-}(-\sigma)$, as well as possibly imaginary integers $\sigma \in i\mathbb{Z}\setminus\{0\}$, with resonant states when $\Im\sigma < 0$ being differentiated delta distributions at $S_+ = \partial C_+$ while the dual states are differentiated delta distributions at $S_- = \partial C_-$ when $\Im\sigma > 0$; the latter arise, e.g., as poles on even-dimensional Minkowski space. More generally, when composed with extension of $C_c^\infty(\overline{C_-}\cup C_0)$ by zero to $C^\infty(X)$ from the right and with restriction to $\overline{C_-}\cup C_0$ from the left, the only poles of $\hat{L}(\sigma)$ are those of $\mathcal{R}_{C_-}(-\sigma)$ as well as the possible $\sigma \in i\mathbb{N}_+$. Thus, fixing $l > -1$, one can conclude that elements of $\operatorname{Ker} L$ are in $\dot{C}^\infty(M)$ locally near $\overline{C_-}$ provided $\mathcal{R}_{C_-}(\tilde\sigma)$ has no poles in $\Im\tilde\sigma > l$. (The only change for $l \le -1$ is that one needs to exclude the potential pure imaginary integer poles as well.) The analogous statement for $\operatorname{Ker} L^*$ on $H_{\mathrm{b}}^{\tilde m,\tilde l}$ is that, fixing $\tilde l > -1$, elements are in $\dot{C}^\infty(M)$ near $\overline{C_+}$ provided $\mathcal{R}_{C_+}(\tilde\sigma)$ has no poles in $\Im\tilde\sigma > \tilde l$. As $\tilde l = -l$ for our duality arguments, the weakest symmetric assumption (in terms of strength at C_+ and C_-) is that $\mathcal{R}_{C_\pm}$ do not have any poles in the closed upper half plane; here the closure is added to make sure L is actually Fredholm on $H_{\mathrm{b}}^{m,l}$ with $l = 0$. In general, if one wants to use other values of l, one needs to assume the absence of poles in $\Im\sigma \ge -|l|$ (if one wants to keep the hypotheses symmetric).

Note that, assuming $d\rho/\rho$ is timelike (with respect to $\hat g$) near $\overline{C_-}$, one *automatically* has the absence of poles of $\mathcal{R}_{C_-}$ in an upper half plane, and the finiteness (with multiplicity) of the number of poles in any upper half plane, by the semiclassical estimates of [Vasy 2013a, §§3.2 and 7.2] (one can ignore the complex absorption discussion there), so in this case the issue is that of a possible finite number of resonances. There is an analogous statement if $d\rho/\rho$ is timelike near $\overline{C_+}$ for $\mathcal{R}_{C_+}$.

Now, assuming still that $d\rho/\rho$ is timelike at, and hence near, $\overline{C_-}$, it is easy to construct a function $\mathfrak{t}$ which has a timelike differential near $\overline{C_-}$, and appropriate sublevel sets are small neighborhoods of $\overline{C_-}$. Once one has such a function $\mathfrak{t}$, energy estimates can be used to conclude that, in such a neighborhood, rapidly vanishing solutions of $Lu = 0$ actually vanish in this neighborhood, so elements of $\operatorname{Ker} L$ have support disjoint from $\overline{C_-}$; similarly, elements of $\operatorname{Ker} L^*$ have support disjoint from $\overline{C_+}$.

Concretely, with $\widehat{G}$ the dual b-metric of $\hat g$, let U_- be a neighborhood of $\overline{C_-}$ and let $0 < \epsilon_0 < \epsilon_1$, $\tilde\epsilon > 0$ and $\delta > 0$ be such that $\{\rho \le \tilde\epsilon,\ v \ge -\epsilon_1\} \cap U_-$ is a compact subset of U_- and, on U_-,

$$\rho < \tilde\epsilon \quad\text{and}\quad v > -\epsilon_1 \quad\Longrightarrow\quad \widehat{G}\Big(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\Big) > \delta,$$

$$\rho < \tilde\epsilon, \quad\text{and}\quad -\epsilon_1 < v < -\epsilon_0 \quad\Longrightarrow\quad \widehat{G}\Big(\frac{d\rho}{\rho}, dv\Big) < 0 \quad\text{and}\quad \widehat{G}(dv, dv) > 0.$$

Such U_- and constants indeed exist. First, there is U_- and $\tilde\epsilon' > 0$, $\epsilon_1' > 0$ such that $\{\rho \le \tilde\epsilon',\ v \ge -\epsilon_1'\} \cap U_-$ is a compact subset of U_- since $\overline{C_-}$ is defined by $\{\rho = 0,\ v \ge 0\}$ in a neighborhood of $\overline{C_-}$ with $d\rho \ne 0$ there and $dv \ne 0$ near $v = 0$; we then consider $\tilde\epsilon < \tilde\epsilon'$ and $\epsilon_1 < \epsilon_1'$ below. Next, since $\widehat{G}(d\rho/\rho, d\rho/\rho)$ is positive on a neighborhood of $\overline{C_-}$ by assumption (thus, for any sufficiently small ϵ_1 and $\tilde\epsilon$ there is a desired δ such that the first inequality is satisfied) and $\widehat{G}(d\rho/\rho, dv)|_{S_-} = -2$, any sufficiently small ϵ_1 and $\tilde\epsilon$ give $\widehat{G}(d\rho/\rho, dv) < 0$ in the desired region, and finally $\widehat{G}(dv, dv) > 0$ on C_0 near S_- (as

$\widehat{G}(dv, dv) = -4v + \mathcal{O}(v^2)$ there), so, choosing ϵ_1 sufficiently small, $\epsilon_0 < \epsilon_1$, and then $\tilde{\epsilon}$ sufficiently small we satisfy all criteria.

Now let ϵ_- and ϵ_+ be such that $0 < \epsilon_- < \epsilon_+ < \tilde{\epsilon}$, and let $\phi \in C^\infty(\mathbb{R})$ have $\phi' \le 0$, $\phi = 0$ near $[-\epsilon_0, \infty)$, $\phi > \tilde{\epsilon}$ near $(-\infty, -\epsilon_1]$ and $\phi' < 0$ when ϕ takes values in $[\epsilon_-, \epsilon_+]$. Then $\mathfrak{t} = \rho + \phi(v)$ has the property that, on U_-,

$$\mathfrak{t} \le \epsilon_+ \quad \Longrightarrow \quad \rho, \phi(v) \le \epsilon_+ \quad \Longrightarrow \quad \rho < \tilde{\epsilon} \quad \text{and} \quad v > -\epsilon_1,$$

and

$$v \ge -\epsilon_0 \quad \Longrightarrow \quad \mathfrak{t} = \rho.$$

Thus, on U_-, if $v \ge -\epsilon_0$ and $\mathfrak{t} \le \epsilon_+$ then $d\mathfrak{t}$ is timelike as $d\rho$ is such, while if $v < -\epsilon_0$ and $\mathfrak{t} \le \epsilon_+$ then

$$\widehat{G}(d\mathfrak{t}, d\mathfrak{t}) = \rho^2 \widehat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right) + 2\phi'(v)\rho\widehat{G}\left(\frac{d\rho}{\rho}, dv\right) + (\phi'(v))^2 \widehat{G}(dv, dv)$$

and all terms are nonnegative in view of $-\epsilon_1 < v < -\epsilon_0$ and $\rho \le \tilde{\epsilon}$, with the inequality being strict when $\mathfrak{t} \in [\epsilon_-, \epsilon_+]$ (as well as in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+])$). Thus, near $\mathfrak{t}^{-1}([\epsilon_-, \epsilon_+]) \cap U_-$, $\mathfrak{t}$ is a timelike function; the same is true on $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$. Choose $\chi \in C^\infty(\mathbb{R})$ with $\chi' \le 0$, $\chi = 1$ near $(-\infty, \epsilon_-]$ and $\chi = 0$ near $[\epsilon_+, \infty)$, and let $\chi \circ \mathfrak{t}$, defined by this formula in U_-, be extended to M as 0 outside U_-; since $\mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$ is a compact subset of U_-, this gives a C^∞ function. Further, ρ is also timelike, with $d\rho/\rho$ and $d\mathfrak{t}$ in the same component of the timelike cone; see Figure 6. Correspondingly, one can apply energy estimates using the timelike vector field $V = (\chi \circ \mathfrak{t})\rho^{-\ell}\widehat{G}(d\rho/\rho, \cdot)$; see [Vasy 2013a, §3.3] leading up to Equation (3.24) and the subsequent discussion, which in turn is based on [Vasy 2012, §§3–4]. Here one needs to make both $-\chi'$ large relative to χ and $\ell > 0$ large (making the b-derivative of $\rho^{-\ell}$ large relative to $\rho^{-\ell}$), as discussed in the Mellin-transformed setting in [Vasy 2013a, §3.3], in [Vasy 2012, §§3–4], as well as in Section 2A here (with τ in place of ρ, but with the sign of ℓ reversed due to the difference between b-saddle points and b-sinks/sources). Notice that taking ℓ large is exactly where the rapid decay near $\overline{C_-}$ is used.

We have seen that the existence of appropriate timelike functions, such as $\mathfrak{t}$, in a neighborhood of $\overline{C_+}$ and $\overline{C_-}$ is automatic (in a slightly degenerate sense at $\overline{C_\pm}$ themselves) when $d\rho/\rho$ is timelike in these regions; indeed these functions could be extended to a neighborhood of C_0 if v is appropriately chosen.

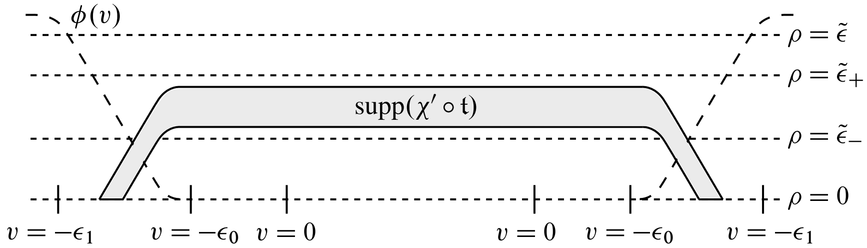

Figure 6. Setup for energy estimates near $\overline{C_-}$; the shaded region is the support of $\chi' \circ \mathfrak{t}$, where $-\chi'$ is used to dominate χ to give positivity in the energy estimate; near $\rho = 0$ and on supp$(\chi \circ \mathfrak{t})$, i.e., in the region between $\rho = 0$ and the shaded region, a sufficiently large weight $\rho^{-\ell}$ gives positivity.

In order to conclude that elements of $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ vanish globally, however, we need to control *all* of the interior of M. This can be accomplished by showing global hyperbolicity of M°, which in turn can be seen by applying a result due to Geroch.[16] Namely, by [Geroch 1970, Theorem 11] it suffices to show that a suitable $\mathscr{S}$ is a Cauchy surface, which, by [ibid., Property 6], follows if we show that $\mathscr{S}$ is achronal, closed, and every null-geodesic intersects and then reemerges from $\mathscr{S}$. In order to define $\mathscr{S}$, it is useful to define $\hat{\mathfrak{t}} = \psi \circ \mathfrak{t}$ in U_-, where $\psi \in C^\infty(\mathbb{R})$, $\psi' \geq 0$, $\psi(t) = t$ near $t \leq \epsilon_-$, $\psi'(t) > 0$ for $t < \epsilon_+$ and $\psi'(t) = 0$ for $t \geq \epsilon_+$; let $T = \psi(\epsilon_+) > \epsilon_-$. Further, extend $\hat{\mathfrak{t}}$ to M as equal to T outside U_-; since $U_- \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+])$ is compact, this gives a C^∞ function on M. Thus, $\hat{\mathfrak{t}} \in C^\infty(M)$ is a globally weakly timelike function, in that $\widehat{G}(d\hat{\mathfrak{t}}, d\hat{\mathfrak{t}}) \geq 0$, and it is strictly timelike in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$. In particular, it is monotone along all null-geodesics. Further, $\hat{\mathfrak{t}} = 0$ at S_- and $\hat{\mathfrak{t}} = T > 0$ at S_+, and indeed near S_+. Then we claim that $\mathscr{S} = \hat{\mathfrak{t}}^{-1}(\epsilon_-) \cap M^\circ$ is a Cauchy surface.

Now, $\mathscr{S}$ is closed in M° since $\overline{\mathscr{S}}$ is closed in M; indeed, it is a closed embedded submanifold. By our nontrapping assumption, every null-geodesic in M° tends to S_+ in one direction and S_- in the other direction, so on future-oriented null-geodesics (ones tending to S_+), $\hat{\mathfrak{t}}$ is monotone increasing, attaining all values in $(0, T]$. Since at the ϵ_- level set of $\mathfrak{t}$, and hence of $\hat{\mathfrak{t}}$, $d\hat{\mathfrak{t}}$ is strictly timelike, the value ϵ_- is attained exactly once for $\hat{\mathfrak{t}}$ along null-geodesics. Thus, every null-geodesic intersects $\mathscr{S}$ and then reemerges from it. Finally, $\mathscr{S}$ is achronal, i.e., there exist no timelike curves connecting two points on $\mathscr{S}$: any future-oriented timelike curve (meaning with tangent vector in the timelike cone whose boundary is the future light cone) in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$ has $\hat{\mathfrak{t}}$ monotone increasing, with the increase being strict near $\mathscr{S}$, so again the value ϵ_- can be attained at most once on such a curve. In summary, this proves that M° is globally hyperbolic, so every solution of $Lu = 0$ with vanishing Cauchy data on $\mathscr{S}$ vanishes identically; in particular, by what we have observed, $\operatorname{Ker} L$ and $\operatorname{Ker} L^*$ are trivial on the indicated spaces.

In summary:

Theorem 5.3. *If (M, g) is a nontrapping Lorentzian scattering metric in the sense of* [Baskin et al. 2014], *$|l| < 1$, and*

(1) *the induced asymptotically hyperbolic resolvents $\mathcal{R}_{C_\pm}$ have no poles in $\Im\sigma \geq -|l|$, and*

(2) *$d\rho/\rho$ is timelike near $\overline{C_+} \cup \overline{C_-}$,*

then, for order functions $m \in C^\infty({}^{\mathrm{b}}S^*M)$ satisfying (5-1) *and* (5-9), *the forward problem for the conjugated wave operator L, that is, with L considered as a map*

$$L : \mathscr{X}^{m,l} \to \mathscr{Y}^{m-1,l},$$

is invertible.

Extending the notation of [Baskin et al. 2014], especially §4, for $m, l \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we denote by $H_{\mathrm{b}}^{m,l,k}(M)$ the space of all $u \in H_{\mathrm{b}}^{m,l}(M)$ (i.e., $u \in \rho^l H_{\mathrm{b}}^m(M)$, where ρ is the boundary defining function of M) such that $\mathcal{M}^j u \in H_{\mathrm{b}}^{m,l}(M)$ for all $0 \leq j \leq k$. Here, $\mathcal{M} \subset \Psi_{\mathrm{b}}^1(M)$ is the $\Psi_{\mathrm{b}}^0(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R}_+$ of the operator $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2}$; in the coordinates ρ, v, y as in [Baskin et al. 2014] (ρ being as above, v

[16] In Geroch's notation, our M° is M.

a defining function of the light cone at infinity within ∂M, and y coordinates within in the light cone at infinity), $\mathcal{M}$ has local generators $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_y. Then Baskin's results extend our theorem to the spaces with module regularity.

Namely, [Baskin et al. 2014, Proposition 4.4], guarantees the module regularity $u \in H_{\mathrm{b}}^{m,l,k}(M)$ of a solution u of $Lu = f$ if f has matching module regularity $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ and if u is in $H_{\mathrm{b}}^{m+k,l}(M)$ near $\overline{C_-}$. To be precise, that proposition is stated making the stronger assumption, $f \in H_{\mathrm{b}}^{m-1+k,l}(M)$. However, the proof goes through for just $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ in a completely analogous manner to the result of Haber and Vasy [2013, Theorem 6.3], where (in the boundaryless setting, for a Lagrangian radial set) the result is stated in this generality.

If $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ then, in particular, f is locally in $H_{\mathrm{b}}^{m+k-1,l}$ near $\overline{C_-}$, thus, taking into account that $m + l > \frac{1}{2}$ already there, u is in $H_{\mathrm{b}}^{m+k,l}$ in that region by Proposition 5.2 (by the first case there, that is, in the high-regularity regime). Thus, an application of the closed graph theorem gives the following boundedness result:

Theorem 5.4. *Under the assumptions of Theorem 5.3, L^{-1} has the property that it restricts to*

$$L^{-1} : H_{\mathrm{b}}^{m-1,l,k} \to H_{\mathrm{b}}^{m,l,k}, \quad k \geq 0,$$

as a bounded map.

In particular, letting $\Omega = \{\tilde{\mathfrak{t}} \geq 0\}$, where $\tilde{\mathfrak{t}} = \hat{\mathfrak{t}} - \epsilon_-$ so that it attains the value 0 within $M \setminus (\overline{C_+} \cup \overline{C_-})$, we have a forward solution operator S of L which maps $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ into $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$, given that $m + l < \frac{1}{2}$; let us assume that m is constant in Ω. Here, $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ consists of supported distributions at $\partial\Omega \cap C_0^\circ = \{\tilde{\mathfrak{t}} = 0\}$.

Remark 5.5. Using the arguments leading to Theorem 5.3 in the current, forward problem, setting, but now also using standard energy estimates near the artificial boundary $\tilde{\mathfrak{t}} = 0$ of Ω, we see that it suffices to control the resonances of the asymptotically hyperbolic resolvent in the upper cap C_+ in order to ensure the invertibility of the forward problem.

5B. ***Algebra properties of $H_{\mathrm{b}}^{m,-\infty,k}$.*** In order to discuss nonlinear wave equations on an asymptotically Minkowski space, we need to discuss the algebra properties of $H_{\mathrm{b}}^{m,-\infty,k} = \bigcup_{l \in \mathbb{R}} H_{\mathrm{b}}^{m,l,k}$. Even though we are only interested in the space $H_{\mathrm{b}}^{m,-\infty,k}(\Omega)^\bullet$, we consider $H_{\mathrm{b}}^{m,-\infty,k}(M)$, where m is constant on M for notational simplicity, and the results we prove below are valid for $H_{\mathrm{b}}^{m,-\infty,k}(\Omega)^\bullet$ by the same proofs.

We start with the following lemma:

Lemma 5.6. *Let $l_1, l_2 \in \mathbb{R}$ and $k > \frac{1}{2}n$. Then $H_{\mathrm{b}}^{0,l_1,k} \cdot H_{\mathrm{b}}^{0,l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$.*

Proof. The generators $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_y of $\mathcal{M}$ take on a simpler form if we blow up the point $(\rho, v) = (0, 0)$. It is most convenient to use projective coordinates on the blown-up space, namely:

(1) Near the interior of the front face, we use the coordinates $\tilde{\rho} = \rho \geq 0$ and $s = v/\rho \in \mathbb{R}$. We compute $\rho\partial_\rho = \tilde{\rho}\partial_{\tilde{\rho}} - s\partial_s$, $v\partial_v = s\partial_s$ and $\rho\partial_v = \partial_s$; since $(d\rho/\rho)\, dv\, dy = d\tilde{\rho}\, ds\, dy$ (this is the b-density

from $H_{\mathrm{b}}^{0,l,k}$), the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$A^{l,k} := \{u \in \tilde{\rho}^l L^2(d\tilde{\rho}\, ds\, dy) : \mathscr{A}^j u \in \tilde{\rho}^l L^2(d\tilde{\rho}\, ds\, dy),\ 0 \le j \le k\},$$

where $\mathscr{A}$ is the C^∞-module of differential operators generated by ∂_s, $\tilde{\rho}\partial_{\tilde{\rho}}$ and ∂_y.

Now, observe that $\tilde{\rho}^l L^2(d\tilde{\rho}\, ds\, dy) = \tilde{\rho}^{l-1/2} L^2((d\tilde{\rho}/\rho)\, ds\, dy)$; therefore, we can rewrite

$$A^{l,k} = \Big\{u \in \tilde{\rho}^{l-1/2} L^2\Big(\frac{d\tilde{\rho}}{\rho}\, ds\, dy\Big) : \mathscr{A}^j u \in \tilde{\rho}^{l-1/2} L^2\Big(\frac{d\tilde{\rho}}{\rho}\, ds\, dy\Big),\ 0 \le j \le k\Big\} = \tilde{\rho}^{l-1/2} H_{\mathrm{b}}^k\Big(\frac{d\tilde{\rho}}{\rho}\, ds\, dy\Big).$$

In particular, by the Sobolev algebra property, Lemma 2.26, and the locality of the multiplication, choosing $k > \frac{1}{2}n$ ensures that $\tilde{\rho}^{l_1-1/2} H_{\mathrm{b}}^k \cdot \tilde{\rho}^{l_2-1/2} H_{\mathrm{b}}^k \subset \tilde{\rho}^{l_1+l_2-1} H_{\mathrm{b}}^k$, which is to say $A^{l_1,k} \cdot A^{l_2,k} \subset A^{l_1+l_2-1/2,k}$.

(2) Near either corner of the blown-up space, we use $\tilde{v} = v$ and $t = \rho/v$ (say, $\tilde{v} \ge 0$, $t \ge 0$). We compute $\rho\partial_\rho = t\partial_t$, $v\partial_v = \tilde{v}\partial_{\tilde{v}} - t\partial_t$, $\rho\partial_v = t\tilde{v}\partial_{\tilde{v}} - t^2\partial_t$; and, since $(d\rho/\rho)\, dv\, dy = (dt/t)\, d\tilde{v}\, dy$, the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$B^{l,k} := \Big\{u \in (t\tilde{v})^l L^2\Big(\frac{dt}{t}\, d\tilde{v}\, dy\Big) : \mathscr{B}^j u \in (t\tilde{v})^l L^2\Big(\frac{dt}{t}\, d\tilde{v}\, dy\Big),\ 0 \le j \le k\Big\},$$

where $\mathscr{B}$ is the C^∞-module of differential operators generated by $t\partial t$, $\tilde{v}\partial_{\tilde{v}}$ and ∂_y. Again, we can rewrite this as

$$B^{l,k} = t^l \tilde{v}^{l-1/2} H_{\mathrm{b}}^k\Big(\frac{dt}{t}\, \frac{d\tilde{v}}{\tilde{v}}\, dy\Big),$$

which implies that, for $k > \frac{1}{2}n$,

$$B^{l_1,k} \cdot B^{l_2,k} \subset t^{l_1+l_2} v^{l_1+l_2-1} H_{\mathrm{b}}^k\Big(\frac{dt}{t}\, \frac{d\tilde{v}}{\tilde{v}}\, dy\Big) \subset B^{l_1+l_2-1/2,k}.$$

To relate these two statements to the statement of the lemma, we use cutoff functions χ_A and χ_B to localize within the two coordinate systems. More precisely, choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}_s)$ such that $\chi(s) \equiv 1$ near $s = 0$, $\chi(s) = 0$ for $|s| \ge 2$, and $\chi^{1/2} \in C_c^\infty(\mathbb{R}_s)$. Then multiplication with $\chi_A(\rho, v) := \chi(v/\rho)$ is a continuous map $H_{\mathrm{b}}^{0,l,k} \to A^{l,k}$. Indeed, to check this, one simply observes that $\mathcal{M}^j \chi_A \in L^\infty$ for all $j \in \mathbb{N}_0$. Similarly, letting $\chi_B(\rho, v) := 1 - \chi_A(\rho, v)$, multiplication with χ_B is a continuous map $H_{\mathrm{b}}^{0,l,k} \to B^{l,k}$. Finally, note that we have $A^{l,k}, B^{l,k} \subset H_{\mathrm{b}}^{0,l,k}$.

To put everything together, take $u_j \in H_{\mathrm{b}}^{0,l_j,k}$ ($j = 1, 2$); then

$$u_1 u_2 = (\chi_A u_1)(\chi_A u_2) + (\chi_B u_1)(\chi_B u_2) + (\chi_A u_1)(\chi_B u_2) + (\chi_B u_1)(\chi_A u_2).$$

The first two terms then lie in $H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$. To deal with the third term, write

$$(\chi_A u_1)(\chi_B u_2) = (\chi_A^{1/2} u_1)(\chi_A^{1/2} \chi_B u_2) \in A^{l_1,k} \cdot A^{l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k},$$

and likewise for the fourth term. Thus, $u_1 u_2 \in H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$, as claimed. □

Remark 5.7. The proof actually shows more, namely that

$$H_{\mathrm{b}}^{0,l,k} H_{\mathrm{b}}^{0,l',k} \subset \rho_{\mathrm{ff}}^{-1/2} H_{\mathrm{b}}^{0,l+l',k}, \tag{5-10}$$

where ρ_{ff} is the defining function of the front face $\rho = v = 0$, e.g., $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$. The reason that (5-10) is a natural statement is that module- and b-derivatives are the same away from $\rho = v = 0$; hence, regularity with respect to the module $\mathcal{M}$ is, up to a weight that is a power of ρ_{ff}, the same as b-regularity.

More abstractly speaking, the above proof shows the following: if ρ_b denotes a boundary defining function of the other boundary hypersurface $\partial[M; S_+] \setminus \mathrm{ff}$ of $[M; S_+]$, then

$$H_{\mathrm{b}}^{0,l,k} \cong \rho_{\mathrm{ff}}^{-1/2} (\rho_{\mathrm{ff}} \rho_b)^l H_{\mathrm{b}}^k([M; S_+]).$$

Note that one can also show this in one step, introducing the coordinates $\rho_{\mathrm{ff}} \geq 0$ and $s = v/(\rho + \rho_{\mathrm{ff}}) \in [-1, 1]$ on $[M; S_+]$ in a neighborhood of ff, and mimicking the above proof, which, however, is computationally less convenient.

Remark 5.8. We can extend the lemma to $H_{\mathrm{b}}^{m,l,k} H_{\mathrm{b}}^{m,l',k} \subset H_{\mathrm{b}}^{m,l+l'-1/2,k}$ for $m \in \mathbb{N}_0$ using the Leibniz rule to distribute the m b-derivatives among the two factors and then using the lemma for the case $m = 0$.

The following corollary, which will play an important role in Section 5E, improves Lemma 5.6 if we have higher b-regularity.

Corollary 5.9. *Let $k > \frac{1}{2}n$, $0 \leq \delta < 1/n$ and $l, l' \in \mathbb{R}$. Then*:

(1) $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$.

(2) $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{1,l',k} \subset H_{\mathrm{b}}^{1,l+l'-1/2+\delta,k}$.

Proof. If $s = 1/(2\delta) > \frac{1}{2}n$, then

$$H_{\mathrm{b}}^{s,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l',k}; \tag{5-11}$$

indeed, using the Leibniz rule to distribute the k module-derivatives among the two factors and cancelling the weights, this amounts to showing that $H_{\mathrm{b}}^{s,0,k_1} H_{\mathrm{b}}^{0,0,k_2} \subset H_{\mathrm{b}}^{0,0,0}$ for $k_1 + k_2 \geq k$; but this is true even for $k_1 = k_2 = 0$, since H_{b}^s is a multiplier on H_{b}^0 provided $s > \frac{1}{2}n$.

On the other hand, the lemma gives

$$H_{\mathrm{b}}^{0,l,k} H_{\mathrm{b}}^{0,l',k} \subset \rho^{-1/2} H_{\mathrm{b}}^{0,l+l',k}. \tag{5-12}$$

Interpolating in the first factor between (5-11) and (5-12) thus gives the first statement.

For the second statement, use the Leibniz rule to distribute the one b-derivative to either factor; then one has to show $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$ and the same inclusion with l and l' switched, which is what we just proved. ☐

Lemma 5.6 and Remark 5.7 imply that, for $u \in H_{\mathrm{b}}^{m,l,k}$, $p \geq 1$, with $m \geq 0$ and $k > \frac{1}{2}n$, we have $u^p \in H_{\mathrm{b}}^{m,pl-(p-1)/2,k}$; in fact, $u^p \in \rho_{\mathrm{ff}}^{-(p-1)/2} H_{\mathrm{b}}^{m,pl,k}$; see Remark 5.7. Using Corollary 5.9, we can improve this to the statement that $u \in H_{\mathrm{b}}^{m,l,k}$ implies $u^p \in H_{\mathrm{b}}^{m,pl-(p-1)/2+(p-1)\delta,k}$ for $m \geq 1$.

For nonlinearities that only involve powers u^p, we can afford to lose differentiability, as at the end of Section 4B, and gain decay in return, as the following lemma shows.

Lemma 5.10. *Let $\alpha > \frac{1}{2}$, $l \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Then $\rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^{0,l,k} \subset \rho^{1/2-\alpha} H_{\mathrm{b}}^{-1,l,k}$, where $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$.*

Proof. We may assume $l = 0$ and that u is supported in $|v| < 1$, $\rho < 1$. First, consider the case $k = 0$. Let $u \in \rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^{0}$ and put

$$\tilde{u}(\rho, v, y) = \int_{-\infty}^{v} u(\rho, w, y)\, dw,$$

so $\partial_v \tilde{u} = u$. We have to prove $\chi\tilde{u} \in \rho^{1/2-\alpha} H_{\mathrm{b}}^{0}$ if $\chi \equiv 1$ near $\operatorname{supp} u$, which implies $u \in H_{\mathrm{b}}^{-1}$, as $\partial_v : H_{\mathrm{b}}^{0} \to H_{\mathrm{b}}^{-1}$ and the b-Sobolev space are local spaces. But

$$|\tilde{u}(\rho, v, y)|^2 \le \left(\int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{2\alpha} |u(\rho, w, y)|^2\, dw \right) \int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{-2\alpha}\, dw; \tag{5-13}$$

now,

$$\int_{-1}^{1} \rho_{\mathrm{ff}}^{-2\alpha}\, dw = \rho^{1-2\alpha} \int_{-1/\rho}^{1/\rho} \frac{dz}{(1+|z|^2)^{\alpha}} \lesssim \rho^{1-2\alpha}$$

for $\alpha > \frac{1}{2}$, so, with the v integral considered on a fixed interval, say $|v| < 2$ (notice that the right-hand side in (5-13) is independent of v!),

$$\iiint \rho^{2\alpha-1} |\tilde{u}(\rho, v, y)|^2 \frac{d\rho}{\rho}\, dv\, dy \lesssim \iiint \rho_{\mathrm{ff}}^{2\alpha} |u(\rho, w, y)|^2 \frac{d\rho}{\rho}\, dw\, dy,$$

proving the claim for $k = 0$. Now, $\rho\partial_\rho$ and ∂_y just commute with this calculation, so the corresponding derivatives are certainly well behaved. On the other hand, $\partial_v \tilde{u} = u$, so the estimates involving at least one v-derivative are just those for u itself. $\square$

Corollary 5.11. *Let $k,\ p \in \mathbb{N}$ be such that $k > \frac{1}{2}n$ and $p \ge 2$. Let $l \in \mathbb{R}$ and $u \in H_{\mathrm{b}}^{0,l,k}$. Then $u^p \in H_{\mathrm{b}}^{-1,lp-(p-1)/2+1/2-\delta,k}$ with $\delta = 0$ if $p \ge 3$ and $\delta > 0$ if $p = 2$.*

Proof. This follows from $u^p \in \rho_{\mathrm{ff}}^{-(p-1)/2-\delta} H_{\mathrm{b}}^{0,lp,k}$ and the previous lemma, using that $\frac{1}{2}(p-1)+\delta > \frac{1}{2}$ with δ as stated. $\square$

In other words, we gain the decay $\rho^{1/2-\delta}$ if we give up one derivative.

5C. ***A class of semilinear equations.*** We are now set to discuss solutions to nonlinear wave equations on an asymptotically Minkowski space. Under the assumptions of Theorem 5.3, we obtain a forward solution operator $S : H_{\mathrm{b}}^{m-1,l,k}(\Omega)^{\bullet} \to H_{\mathrm{b}}^{m,l,k}(\Omega)^{\bullet}$ of $P = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2}$ provided $|l| < 1$, $m + l < \frac{1}{2}$ and $k \ge 0$.

Undoing the conjugation, we obtain a forward solution operator

$$\widetilde{S} = \rho^{(n-2)/2} S \rho^{-2} \rho^{-(n-2)/2}, \quad \widetilde{S} : H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^{\bullet} \to H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^{\bullet},$$

of $\Box_g$.

Since g is a Lorentzian scattering metric, the natural vector fields to appear in a nonlinear equation are scattering vector fields; more generally, since the analysis is carried out on b-spaces, we indeed allow b-vector fields in the following statement:

Theorem 5.12. *Let*

$$q : H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^{\bullet} \times H_{\mathrm{b}}^{m-1,l+(n-2)/2,k}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet} \to H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^{\bullet}$$

be a continuous function with $q(0,0)=0$ *such that there exists a continuous nondecreasing function* $L:\mathbb{R}_{\geq 0}\to\mathbb{R}$ *satisfying*

$$\|q(u,{}^{\mathrm{b}}du)-q(v,{}^{\mathrm{b}}dv)\|\leq L(R)\|u-v\|,\quad \|u\|,\ \|v\|\leq R.$$

Then there is a constant $C_L>0$ *such that the following holds: if* $L(0)<C_L$ *then, for small* $R>0$, *there exists* $C>0$ *such that, for all* $f\in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ *with* $\|f\|\leq C$, *the equation*

$$\Box_g u=f+q(u,{}^{\mathrm{b}}du)$$

has a unique solution $u\in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, *with* $\|u\|\leq R$, *that depends continuously on* f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.25. □

Remark 5.13. Here, just as in Theorem 4.10, we can also allow q to depend on $\Box_g u$.

5D. *Semilinear equations with polynomial nonlinearity.*

Next, we want to find a forward solution of the semilinear PDE

$$\Box_g u=f+cu^pX(u),$$

where $c\in C^\infty(M)$, $p\in\mathbb{N}_0$, and $X(u)=\prod_{j=1}^q\rho V_j(u)$ is a q-fold product of derivatives of u along scattering vector fields; here, V_j are b-vector fields. Let us assume $p+q\geq 2$ in order for the equation to be genuinely nonlinear. We rewrite the PDE as

$$L(\rho^{-(n-2)/2}u)=\rho^{-(n-2)/2-2}f+c\rho^{-2}\rho^{(p-1)(n-2)/2}(\rho^{-(n-2)/2}u)^p\prod_{j=1}^q\rho V_j(\rho^{(n-2)/2}\rho^{-(n-2)/2}u).$$

Introducing $\tilde{u}=\rho^{-(n-2)/2}u$ and $\tilde{f}=\rho^{-(n-2)/2-2}f$ yields the equation

$$\begin{aligned}L\tilde{u}&=\tilde{f}+c\rho^{(p-1)(n-2)/2-2}\tilde{u}^p\prod_{j=1}^q\rho^{n/2}(f_j\tilde{u}+V_j\tilde{u})\\&=\tilde{f}+c\rho^{(p-1)(n-2)/2+qn/2-2}\tilde{u}^p\prod_{j=1}^q(f_j\tilde{u}+V_j\tilde{u}),\end{aligned}\tag{5-14}$$

where the f_j are smooth functions. Now suppose that $\tilde{u}\in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ with $m+l<\frac{1}{2}$, $m\geq 1$ and $k>\frac{1}{2}n$ (so that $H_{\mathrm{b}}^{m-1,-\infty,k}(\Omega)^\bullet$ is an algebra); then the second summand of the right-hand side of (5-14) lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^\bullet$, where

$$\ell=\tfrac{1}{2}(p-1)(n-2)+\tfrac{1}{2}qn-2+pl-\tfrac{1}{2}(p-1)+ql-\tfrac{1}{2}(q-1)-\tfrac{1}{2}.$$

For this space to lie in $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ (which we want in order to be able to apply the solution operator S and land in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$, so that a fixed point argument as in Section 2 can be applied), we thus need $\ell\geq l$, which can be rewritten as

$$\tfrac{1}{2}(p-1)(l+(n-3))+q\big(l+\tfrac{1}{2}(n-1)\big)\geq 2.\tag{5-15}$$

For $m=1$ and $l<\frac{1}{2}-m$ less than, but close to, $-\frac{1}{2}$, we thus get the condition

$$(p-1)(n-4)+q(n-2)>4.$$

If there are only nonlinearities involving derivatives of u, i.e., $p=0$, we get the condition $q>1+2/(n-2)$, that is, quadratic nonlinearities are fine for $n\geq 5$, and cubic ones for $n\geq 4$.

Note that, if $q=0$, we can actually choose $m=0$ and $l<\frac{1}{2}$ close to $\frac{1}{2}$, and we have Corollary 5.11 at hand. Thus we can improve (5-15) to $(p-1)\big(\frac{1}{2}+\frac{1}{2}(n-3)\big)>2-\frac{1}{2}$, i.e., $p>1+3/(n-2)$, hence quadratic nonlinearities can be dealt with if $n\geq 6$, whereas cubic nonlinearities are fine as long as $n\geq 4$. Observe that this condition on p always implies $p>1$, which is a natural condition, since $p=1$ would amount to changing $\Box_g$ into $\Box_g-m^2$ (if one chooses the sign appropriately). But the Klein–Gordon operator naturally fits into a scattering framework, as mentioned in the introduction (see p. 1812), therefore requires a different analysis; we will not pursue this further in this paper.

To summarize the general case, note that $\tilde u\in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ is equivalent to $u\in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, and $\tilde f\in H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ to $f\in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$; thus:

Theorem 5.14. *Let $|l|<1$, $m+l<\frac{1}{2}$, $k>\frac{1}{2}n$, and assume that $p,q\in\mathbb{N}_0$ with $p+q\geq 2$ satisfy condition (5-15) or the weaker conditions given above in the cases where $p=0$ or $q=0$; let $m\geq 0$ if $q=0$, otherwise let $m\geq 1$. Moreover, let $c\in C^\infty(M)$ and $X(u)=\prod_{j=1}^q X_j u$, where X_j is a scattering vector field on M. Then, for small enough $R>0$, there exists a constant $C>0$ such that, for all $f\in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ with $\|f\|\leq C$, the equation*

$$\Box_g u=f+cu^pX(u)$$

has a unique solution $u\in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, with $\|u\|\leq R$, that depends continuously on f.

The same conclusion holds if the nonlinearity is a finite sum of terms of the form $cu^pX(u)$ provided each such term separately satisfies (5-15).

Proof. Reformulating the PDE in terms of $\tilde u$ and $\tilde f$ as above, this follows from an application of the Banach fixed point theorem to the map

$$H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet\to H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet,\quad \tilde u\mapsto S\bigg(\tilde f+c\rho^{(p-1)(n-2)/2+qn/2-2}\tilde u^p\prod_{j=1}^q(f_j\tilde u+V_j\tilde u)\bigg)$$

with m, l and k as in the statement of the theorem. Here, $p+q\geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$. $\square$

Remark 5.15. If the derivatives in the nonlinearity only involve module-derivatives, we get a slightly better result, since we can work with $\tilde u\in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$. Indeed, a module-derivative falling on $\tilde u$ gives an element of $H_{\mathrm{b}}^{0,l,k-1}(\Omega)^\bullet$, applied to which the forward solution operator produces an element of $H_{\mathrm{b}}^{1,l,k-1}(\Omega)^\bullet\subset H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$.

The numbers work out as follows: In condition (5-15), we now take $l<\frac{1}{2}$ close to $\frac{1}{2}$, thus obtaining

$$(p-1)(n-2)+qn>4.$$

Thus, in the case that there are only derivatives in the nonlinearity, i.e., $p=0$, we get $q>1+2/n$, which allows for quadratic nonlinearities provided $n\geq 3$.

Remark 5.16. Observe that we can improve (5-15) in the case $p\geq 1$, $q\geq 1$ and $m\geq 1$ by using the δ-improvement from Corollary 5.9, namely, the right-hand side of (5-14) actually lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^\bullet$, where now

$$\ell=\tfrac{1}{2}(p-1)(n-2)+\tfrac{1}{2}qn-2+pl-\tfrac{1}{2}(p-1)+(p-1)\delta+ql-\tfrac{1}{2}(q-1)-\tfrac{1}{2}+\delta,$$

which satisfies $\ell\geq l$ if

$$\tfrac{1}{2}(p-1)(l+(n-3)+\delta)+q\big(l+\tfrac{1}{2}(n-1)\big)+\delta\geq 2,$$

which for $l<-\frac{1}{2}$ close to $-\frac{1}{2}$ means $(p-1)(n-4+2\delta)+q(n-2)+2\delta>4$, where $0<\delta<1/n$.

Remark 5.17. Let us compare the above result with Christodoulou's [1986]. A special case of his theorem states[17] that the Cauchy problem for the wave equation on Minkowski space with small initial data in $H_{k,k-1}(\mathbb{R}^{n-1})$ admits a global solution $u\in H^k_{\mathrm{loc}}(\mathbb{R}^n)$ with decay $|u(x)|\lesssim(1+(v/\rho)^2)^{-(n-2)/2}$; here, $k=\frac{1}{2}n+2$, and n is assumed to be at least 4 and even; when $n=4$, the nonlinearity is moreover assumed to satisfy the null condition. The only polynomial nonlinearity that we cannot deal with using the above argument is thus the null-form nonlinearity in 4 dimensions.

To make a further comparison possible, we express $H_{k,\delta}(\mathbb{R}^{n-1})$ as a b-Sobolev space on the radial compactification of $\mathbb{R}^{n-1}$: Note that $u\in H_{k,\delta}(\mathbb{R}^{n-1})$ is equivalent to $(\langle x\rangle D_x)^\alpha u\in\langle x\rangle^{-\delta}L^2(\mathbb{R}^{n-1})$, $|\alpha|\leq k$. In terms of the boundary defining function ρ of $\partial\overline{\mathbb{R}^{n-1}}$ and the standard measure $d\omega$ on the unit sphere $\mathbb{S}^{n-2}\subset\mathbb{R}^{n-1}$, we have $L^2(\mathbb{R}^{n-1})=L^2((d\rho/\rho^2)\,(dy/\rho^{n-2}))=\rho^{(n-1)/2}L^2((d\rho/\rho)\,dy)$, and thus $H_{k,\delta}(\mathbb{R}^{n-1})=\rho^{(n-1)/2+\delta}H_{\mathrm{b}}^k(\tilde{\mathfrak{t}}=0)$. Therefore, converting the Cauchy problem into a forward problem, the forcing lies in $H_{\mathrm{b}}^{k,(n-1)/2+k-1,0}(\Omega)^\bullet=H_{\mathrm{b}}^{n/2+2,n+1/2,0}(\Omega)^\bullet$. Comparing this with the space $H_{\mathrm{b}}^{0,l+(n-2)/2+2,n/2+1}$, with $l<\frac{1}{2}$, needed for our argument, we see that Christodoulou's result applies to a regime of fast decay which is disjoint from our slow decay (or even mild growth) regime.

Remark 5.18. In the case of nonlinearities u^p, the result of [Christodoulou 1986] implies the existence of global solutions to $\Box_g u=f+u^p$ if the spacetime dimension n is *even* and $n\geq 4$ if $p\geq 3$; in even dimensions $n\geq 6$, $p\geq 2$ suffices; the above result extends this to all dimensions satisfying the respective inequalities. In a somewhat similar context — see the work of Chruściel and Łęski [2006] — it has been proved that $p\geq 2$ in fact works in all dimensions $n\geq 5$.

5E. *Semilinear equations with null condition.* With g the Lorentzian scattering metric on an asymptotically Minkowski space satisfying the assumptions of Theorem 5.3 as before, define the null-form $Q({}^{\mathrm{sc}}du,{}^{\mathrm{sc}}dv)=g^{jk}\partial_j u\partial_k v$ and write $Q({}^{\mathrm{sc}}du)$ for $Q({}^{\mathrm{sc}}du,{}^{\mathrm{sc}}du)$. We are interested in solving the PDE

$$\Box_g u=Q({}^{\mathrm{sc}}du)+f.$$

[17] Note that n is the dimension of Minkowski space here, whereas Christodoulou uses $n+1$.

The previous discussion solves this for $n \geq 5$; thus, let us from now on assume $n = 4$. To make the computations more transparent, we will keep the n in the notation and only substitute $n = 4$ when needed. Rewriting the PDE in terms of the operator $L = \rho^{-2}\rho^{-(n-2)/2}\Box_g\rho^{(n-2)/2}$ as above, we get

$$L\tilde{u} = \tilde{f} + \rho^{-(n-2)/2-2}Q({}^{\mathrm{sc}}d(\rho^{(n-2)/2}\tilde{u})),$$

where $\tilde{u} = \rho^{-(n-2)/2}u$ and $\tilde{f} = \rho^{-(n-2)/2-2}f$. We can write $Q({}^{\mathrm{sc}}du) = \frac{1}{2}\Box_g(u^2) - u\Box_g u$, so the PDE becomes

$$\begin{aligned} L\tilde{u} &= \tilde{f} + \rho^{-(n-2)/2-2}\bigl(\tfrac{1}{2}\Box_g(\rho^{n-2}\tilde{u}^2) - \rho^{(n-2)/2}\tilde{u}\Box_g(\rho^{(n-2)/2}\tilde{u})\bigr) \\ &= \tilde{f} + \tfrac{1}{2}L(\rho^{(n-2)/2}\tilde{u}^2) - \rho^{(n-2)/2}\tilde{u}L\tilde{u}. \end{aligned}$$

Since the results of Section 5B give small improvements on the decay of products of $H_{\mathrm{b}}^{1,*,*}$ functions with $H_{\mathrm{b}}^{m,*,*}$ functions ($m \geq 0$), one wants to solve this PDE on a function space that keeps track of these small improvements.

Definition 5.19. For $l \in \mathbb{R}$, $k \in \mathbb{N}_0$ and $\alpha \geq 0$, define the space

$$\mathscr{X}^{l,k,\alpha} := \{v \in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet : Lv \in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet\}$$

with norm

$$\|v\|_{\mathscr{X}^{l,k,\alpha}} = \|v\|_{H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet} + \|Lv\|_{H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet}. \tag{5-16}$$

By an argument similar to the one used in the proof of Theorem 2.25, we see that $\mathscr{X}^{l,k,\alpha}$ is a Banach space.

On $\mathscr{X}^{l,k,\alpha}$, with $\alpha > 0$ chosen below, we want to run an iteration argument: Start by defining the operator $T : \mathscr{X}^{l,k,\alpha} \to H_{\mathrm{b}}^{1,-\infty,k}(\Omega)^\bullet$ by

$$T : \tilde{u} \mapsto S(\tilde{f} - \rho^{(n-2)/2}\tilde{u}L\tilde{u}) + \tfrac{1}{2}\rho^{(n-2)/2}\tilde{u}^2.$$

Note that $\tilde{u} \in \mathscr{X}^{l,k,\alpha}$ implies, using Corollary 5.9 with $\delta < 1/n$,

$$\begin{aligned} \rho^{(n-2)/2}\tilde{u}^2 &\in \rho^{(n-2)/2}H_{\mathrm{b}}^{1,2(l+\alpha)-1/2+\delta,k}(\Omega)^\bullet = H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet, \\ \rho^{(n-2)/2}\tilde{u}L\tilde{u} &\in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \\ S(\rho^{(n-2)/2}\tilde{u}L\tilde{u}) &\in H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \end{aligned} \tag{5-17}$$

where in the last inclusion we need to require $1 + \bigl(2l + \alpha + \frac{1}{2}(n-3) + \delta\bigr) < \frac{1}{2}$, which for $n = 4$ means

$$l < -\tfrac{1}{2} - \tfrac{1}{2}(\alpha + \delta); \tag{5-18}$$

let us assume from now on that this condition holds. Furthermore, (5-17) implies that $T\tilde{u}$ is in $H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet$. Finally, we analyze

$$L(T\tilde{u}) \in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + \tfrac{1}{2}L(\rho^{(n-2)/2}\tilde{u}^2).$$

Using that L is a second-order b-differential operator, we have

$$\begin{aligned}\rho^{(n-2)/2}L(\tilde{u}^2) &\in 2\rho^{(n-2)/2}\tilde{u}L\tilde{u}+\rho^{(n-2)/2}H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet\\ &\subset H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet+H_{\mathrm{b}}^{0,2(l+\alpha)+(n-3)/2,k}(\Omega)^\bullet\\ &=H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet,\end{aligned}$$

which gives

$$\begin{aligned}L(\rho^{(n-2)/2}\tilde{u}^2) &\in L(\rho^{(n-2)/2})\tilde{u}^2+\rho^{(n-2)/2}L(\tilde{u}^2)+\rho^{(n-2)/2}H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet\\ &\subset H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet+H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet\\ &\qquad\qquad+H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta+\alpha}(\Omega)^\bullet\\ &=H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet.\end{aligned}$$

Hence, putting everything together,

$$L(T\tilde{u})\in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^\bullet.$$

Therefore, we have $T\tilde{u}\in\mathscr{X}^{l,k,\alpha}$ provided

$$\begin{aligned}2l+\alpha+\tfrac{1}{2}(n-3)+\delta &\geq l+\alpha,\\ 2l+\alpha+\tfrac{1}{2}(n-3)+\min\{\alpha,\delta\} &\geq l,\end{aligned}$$

which for $0<\alpha<\delta$ and $n=4$ is equivalent to

$$l\geq-\tfrac{1}{2}-\delta,\quad l\geq-\tfrac{1}{2}-2\alpha. \tag{5-19}$$

This is consistent with condition (5-18) if $-\frac{1}{2}-\frac{1}{2}(\alpha+\delta)>-\frac{1}{2}-2\alpha$, that is, if $\alpha>\frac{1}{3}\delta$.

Finally, for the map T to be well defined, we need $S\tilde{f}\in\mathscr{X}^{l,k,\alpha}$, hence $\tilde{f}\in\mathrm{Ran}_{\mathscr{X}^{l,k,\alpha}}L$, which is in particular satisfied if $\tilde{f}\in H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^\bullet$. Indeed, since $1+l+\alpha<1-\frac{1}{2}-\frac{1}{2}(\delta-\alpha)<\frac{1}{2}$ by condition (5-18), the element $S\tilde{f}\in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet$ is well defined.

We have proved:

Theorem 5.20. *Let $c\in\mathbb{C}$, $0<\delta<\frac{1}{4}$, $\frac{1}{3}\delta<\alpha<\delta$, and let $-\frac{1}{2}-2\alpha\leq l<-\frac{1}{2}-\frac{1}{2}(\alpha+\delta)$. Then, for small enough $R>0$, there exists a constant $C>0$ such that, for all $f\in H_{\mathrm{b}}^{0,l+3+\alpha,k}(\Omega)^\bullet$ with $\|f\|\leq C$, the equation*

$$\Box_g u=f+cQ({}^{\mathrm{sc}}du)$$

has a unique solution $u\in\mathscr{X}^{l+1,k,\alpha}$, with $\|u\|\leq R$, that depends continuously on f.

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References

[Baskin 2010] D. Baskin, "A parametrix for the fundamental solution of the Klein–Gordon equation on asymptotically de Sitter spaces", *J. Funct. Anal.* **259**:7 (2010), 1673–1719. MR 2011h:58044 Zbl 1200.35247

[Baskin 2013] D. Baskin, "Strichartz estimates on asymptotically de Sitter spaces", *Ann. Henri Poincaré* **14**:2 (2013), 221–252. MR 3028039 Zbl 1262.83020

[Baskin et al. 2014] D. Baskin, A. Vasy, and J. Wunsch, "Asymptotics of radiation fields in asymptotically Minkowski space", preprint, 2014. To appear in *Amer. J. Math.* arXiv 1212.5141

[Bieri 2009] L. Bieri, "Part I: Solutions of the Einstein vacuum equations", pp. 1–295 in *Extensions of the stability theorem of the Minkowski space in general relativity*, AMS/IP Stud. Adv. Math. **45**, Amer. Math. Soc., Providence, RI, 2009. MR 2537047 Zbl 1172.83001

[Bony and Häfner 2008] J.-F. Bony and D. Häfner, "Decay and non-decay of the local energy for the wave equation on the de Sitter–Schwarzschild metric", *Comm. Math. Phys.* **282**:3 (2008), 697–719. MR 2010h:58041 Zbl 1159.35007

[Christodoulou 1986] D. Christodoulou, "Global solutions of nonlinear hyperbolic equations for small initial data", *Comm. Pure Appl. Math.* **39**:2 (1986), 267–282. MR 87c:35111 Zbl 0612.35090

[Christodoulou and Klainerman 1993] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series **41**, Princeton University Press, NJ, 1993. MR 95k:83006 Zbl 0827.53055

[Chruściel and Łęski 2006] P. T. Chruściel and S. Łęski, "Polyhomogeneous solutions of nonlinear wave equations without corner conditions", *J. Hyperbolic Differ. Equ.* **3**:1 (2006), 81–141. MR 2007g:58031 Zbl 1099.35154

[Dafermos and Rodnianski 2007] M. Dafermos and I. Rodnianski, "The wave equation on Schwarzschild–de Sitter space times", preprint, 2007. arXiv 0709.2766

[Dafermos and Rodnianski 2013] M. Dafermos and I. Rodnianski, "Lectures on black holes and linear waves", pp. 97–205 in *Evolution equations*, edited by D. Ellwood et al., Clay Math. Proc. **17**, Amer. Math. Soc., Providence, RI, 2013. MR 3098640 Zbl 1300.83004

[Datchev and Vasy 2012] K. Datchev and A. Vasy, "Gluing semiclassical resolvent estimates via propagation of singularities", *Int. Math. Res. Not.* **2012**:23 (2012), 5409–5443. MR 2999147 Zbl 1262.58019

[Dyatlov 2011a] S. Dyatlov, "Quasi-normal modes and exponential energy decay for the Kerr–de Sitter black hole", *Comm. Math. Phys.* **306**:1 (2011), 119–163. MR 2012g:58055 Zbl 1223.83029

[Dyatlov 2011b] S. Dyatlov, "Exponential energy decay for Kerr–de Sitter black holes beyond event horizons", *Math. Res. Lett.* **18**:5 (2011), 1023–1035. MR 2875874 Zbl 1253.83020

[Dyatlov 2015] S. Dyatlov, "Resonance projectors and asymptotics for r-normally hyperbolic trapped sets", *J. Amer. Math. Soc.* **28**:2 (2015), 311–381. MR 3300697 Zbl 06394348

[Geroch 1970] R. Geroch, "Domain of dependence", *J. Mathematical Phys.* **11** (1970), 437–449. MR 42 #5585 Zbl 0189.27602

[Graham and Lee 1991] C. R. Graham and J. M. Lee, "Einstein metrics with prescribed conformal infinity on the ball", *Adv. Math.* **87**:2 (1991), 186–225. MR 92i:53041 Zbl 0765.53034

[Guillarmou 2005] C. Guillarmou, "Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds", *Duke Math. J.* **129**:1 (2005), 1–37. MR 2006k:58051 Zbl 1099.58011

[Haber and Vasy 2013] N. Haber and A. Vasy, "Propagation of singularities around a Lagrangian submanifold of radial points", preprint, 2013. To appear in *Bulletin de la SMF*. arXiv 1110.1419

[Hintz 2013] P. Hintz, "Global well-posedness for quasilinear wave equations on asymptotically de Sitter spaces", preprint, 2013. arXiv 1311.6859

[Hintz and Vasy 2014a] P. Hintz and A. Vasy, "Global analysis of quasilinear wave equations on asymptotically Kerr–de Sitter spaces", preprint, 2014. arXiv 1404.1348

[Hintz and Vasy 2014b] P. Hintz and A. Vasy, "Non-trapping estimates near normally hyperbolic trapping", *Math. Res. Lett.* **21**:6 (2014), 1277–1304. MR 3335848 Zbl 06438826

[Hörmander 1985a] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren der Math. Wissenschaften **274**, Springer, Berlin, 1985. MR 87d:35002a Zbl 0601.35001

[Hörmander 1985b] L. Hörmander, *The analysis of linear partial differential operators, IV: Fourier integral operators*, Grundlehren der Math. Wissenschaften **275**, Springer, Berlin, 1985. MR 87d:35002a Zbl 0612.35001

[Hörmander 1997] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques et Applications **26**, Springer, Berlin, 1997. MR 98e:35103 Zbl 0881.35001

[Klainerman 1985] S. Klainerman, "Uniform decay estimates and the Lorentz invariance of the classical wave equation", *Comm. Pure Appl. Math.* **38**:3 (1985), 321–332. MR 86i:35091 Zbl 0635.35059

[Klainerman 1986] S. Klainerman, "The null condition and global existence to nonlinear wave equations", pp. 293–326 in *Nonlinear systems of partial differential equations in applied mathematics, I* (Santa Fe, NM, 1984), edited by B. Nicolaenko et al., Lectures in Appl. Math. **23**, Amer. Math. Soc., Providence, RI, 1986. MR 87h:35217 Zbl 0599.35105

[Lindblad 2008] H. Lindblad, "Global solutions of quasilinear wave equations", *Amer. J. Math.* **130**:1 (2008), 115–157. MR 2009b:58062 Zbl 1144.35035

[Lindblad and Rodnianski 2005] H. Lindblad and I. Rodnianski, "Global existence for the Einstein vacuum equations in wave coordinates", *Comm. Math. Phys.* **256**:1 (2005), 43–110. MR 2006b:83020 Zbl 1081.83003

[Lindblad and Rodnianski 2010] H. Lindblad and I. Rodnianski, "The global stability of Minkowski space-time in harmonic gauge", *Ann. of Math.* (2) **171**:3 (2010), 1401–1477. MR 2011k:58042 Zbl 1192.53066

[Luk 2013] J. Luk, "The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes", *J. Eur. Math. Soc.* (*JEMS*) **15**:5 (2013), 1629–1700. MR 3082240 Zbl 1280.35154

[Mazzeo and Melrose 1987] R. R. Mazzeo and R. B. Melrose, "Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature", *J. Funct. Anal.* **75**:2 (1987), 260–310. MR 89c:58133 Zbl 0636.58034

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics **4**, AK Peters, Wellesley, MA, 1993. MR 96g:58180 Zbl 0796.58050

[Melrose 1994] R. B. Melrose, "Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces", pp. 85–130 in *Spectral and scattering theory* (Sanda, 1992), edited by M. Ikawa, Lecture Notes in Pure and Appl. Math. **161**, Dekker, New York, 1994. MR 95k:58168 Zbl 0837.35107

[Melrose et al. 2013] R. Melrose, A. Vasy, and J. Wunsch, *Diffraction of singularities for the wave equation on manifolds with corners*, Astérisque **351**, Société Mathématique de France, Paris, 2013. MR 3100155 Zbl 1277.35004

[Melrose et al. 2014] R. Melrose, A. Sá Barreto, and A. Vasy, "Asymptotics of solutions of the wave equation on de Sitter–Schwarzschild space", *Comm. Partial Differential Equations* **39**:3 (2014), 512–529. MR 3169793 Zbl 1286.35145

[Metcalfe and Tataru 2012] J. Metcalfe and D. Tataru, "Global parametrices and dispersive estimates for variable coefficient wave equations", *Math. Ann.* **353**:4 (2012), 1183–1237. MR 2944027 Zbl 1259.35006

[Nonnenmacher and Zworski 2009] S. Nonnenmacher and M. Zworski, "Quantum decay rates in chaotic scattering", *Acta Math.* **203**:2 (2009), 149–233. MR 2011c:58063 Zbl 1226.35061

[Polarski 1989] D. Polarski, "On the Hawking effect in de Sitter space", *Classical Quantum Gravity* **6**:5 (1989), 717–722. MR 90j:83076

[Sá Barreto and Zworski 1997] A. Sá Barreto and M. Zworski, "Distribution of resonances for spherical black holes", *Math. Res. Lett.* **4**:1 (1997), 103–121. MR 97m:83063 Zbl 0883.35120

[Strichartz 1971] R. S. Strichartz, "A note on Sobolev algebras", *Proc. Amer. Math. Soc.* **29** (1971), 205–207. MR 43 #905 Zbl 0251.46061

[Taylor 1997] M. E. Taylor, *Partial differential equations, III: Nonlinear equations*, Applied Mathematical Sciences **117**, Springer, New York, 1997. MR 98k:35001 Zbl 0869.35004

[Tohaneanu 2012] M. Tohaneanu, "Strichartz estimates on Kerr black hole backgrounds", *Trans. Amer. Math. Soc.* **364**:2 (2012), 689–702. MR 2012i:35397 Zbl 1234.35275

[Vasy 2008] A. Vasy, "Propagation of singularities for the wave equation on manifolds with corners", *Ann. of Math.* (2) **168**:3 (2008), 749–812. MR 2009i:58037 Zbl 1171.58007

[Vasy 2010] A. Vasy, "The wave equation on asymptotically de Sitter-like spaces", *Adv. Math.* **223**:1 (2010), 49–97. MR 2011i:58046 Zbl 1191.35064

[Vasy 2012] A. Vasy, "The wave equation on asymptotically anti de Sitter spaces", *Anal. PDE* **5**:1 (2012), 81–144. MR 2957552 Zbl 1277.35014

[Vasy 2013a] A. Vasy, "Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces (with an appendix by Semyon Dyatlov)", *Invent. Math.* **194**:2 (2013), 381–513. MR 3117526 Zbl 1315.35015

[Vasy 2013b] A. Vasy, "Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates", pp. 487–528 in *Inverse problems and applications: inside out, II*, edited by G. Uhlmann, Math. Sci. Res. Inst. Publ. **60**, Cambridge Univ. Press, 2013. MR 3135765 Zbl 1316.58016

[Vasy 2014] A. Vasy, "Resolvents, Poisson operators and scattering matrices on asymptotically hyperbolic and de Sitter spaces", *J. Spectr. Theory* **4**:4 (2014), 643–673. MR 3299810 Zbl 06424787

[Wang 2010] F. Wang, *Radiation field for vacuum Einstein equation*, Ph.D. thesis, Massachusetts Institute of Technology, 2010.

[Wunsch and Zworski 2011] J. Wunsch and M. Zworski, "Resolvent estimates for normally hyperbolic trapped sets", *Ann. Henri Poincaré* **12**:7 (2011), 1349–1385. MR 2846671 Zbl 1228.81170

[Yagdjian 2009] K. Yagdjian, "The semilinear Klein–Gordon equation in de Sitter spacetime", *Discrete Contin. Dyn. Syst. Ser. S* **2**:3 (2009), 679–696. MR 2010k:35318 Zbl 1194.35271

[Yagdjian 2012] K. Yagdjian, "Global existence of the scalar field in de Sitter spacetime", *J. Math. Anal. Appl.* **396**:1 (2012), 323–344. MR 2956966 Zbl 1252.83020

[Yagdjian and Galstian 2009] K. Yagdjian and A. Galstian, "Fundamental solutions for the Klein–Gordon equation in de Sitter spacetime", *Comm. Math. Phys.* **285**:1 (2009), 293–344. MR 2009j:58041 Zbl 1228.81180

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