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A TOPOLOGICAL JOIN CONSTRUCTION AND THE TODA SYSTEM ON COMPACT SURFACES OF ARBITRARY GENUS

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We consider the *Toda system* of Liouville equations on a compact surface Σ

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right) - \rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right), \\ -\Delta u_2 = 2\rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right) - \rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right), \end{cases}$$

which arises as a model for nonabelian Chern–Simons vortices. Here h_1 and h_2 are smooth positive functions and ρ_1 and ρ_2 two positive parameters.

For the first time, the ranges $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 \in (4\pi, 8\pi)$ are studied with a variational approach on surfaces with arbitrary genus. We provide a general existence result by using a new improved Moser–Trudinger-type inequality and introducing a topological join construction in order to describe the interaction of the two components u_1 and u_2.

1. Introduction

We are interested here in the *Toda system* on a compact surface Σ

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right) - \rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right), \\ -\Delta u_2 = 2\rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right) - \rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right), \end{cases} \tag{1}$$

where Δ is the Laplace–Beltrami operator, ρ_1 and ρ_2 are two nonnegative parameters, $h_1, h_2 : \Sigma \to \mathbb{R}$ are smooth positive functions, and Σ is a compact orientable surface without boundary with a Riemannian metric g. For the sake of simplicity, we normalize the total volume of Σ so that $|\Sigma| = 1$.

The above system has been widely studied in the literature since it is motivated by problems in both differential geometry and mathematical physics. In geometry, it relates to the Frenet frame of holomorphic curves in $\mathbb{CP}^n$ [Bolton and Woodward 1997; Calabi 1953; Chern and Wolfson 1987]. In mathematical physics, it models nonabelian Chern–Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential [Dunne 1995; Tarantello 2008; 2010; Yang 2001].

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Equation (1) is variational, and solutions correspond to critical points of the Euler–Lagrange functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ ($\rho = (\rho_1, \rho_2)$) given by

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i\, dV_g - \log \int_\Sigma h_i e^{u_i}\, dV_g \right), \tag{2}$$

where $Q(u_1, u_2)$ is a quadratic form that has the expression

$$Q(u_1, u_2) = \tfrac{1}{3}(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2). \tag{3}$$

The structure of J_ρ strongly depends on the range of the two parameters ρ_1 and ρ_2. An important tool in treating this kind of functional is the Moser–Trudinger inequality; see (7). For the Toda system, a similar sharp inequality was derived in [Jost and Wang 2001]:

$$4\pi \log \int_\Sigma e^{u_1 - \bar{u}_1}\, dV_g + 4\pi \log \int_\Sigma e^{u_2 - \bar{u}_2}\, dV_g \le \int_\Sigma Q(u_1, u_2)\, dV_g + C_\Sigma, \quad (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma); \tag{4}$$

here $\bar{u}_i$ stands for the average of u_i on Σ.

By means of the latter inequality, we immediately get existence of a critical point provided both ρ_1 and ρ_2 are less than 4π: indeed for these values, one can minimize J_ρ using standards methods of the calculus of variations. The case of larger ρ_i is subtler due to the fact that J_ρ is unbounded from below.

Before describing the main difficulties of (1), we consider its scalar counterpart: the Liouville equation

$$-\Delta u = 2\rho \left(\frac{h e^u}{\int_\Sigma h e^u\, dV_g} - 1 \right), \tag{5}$$

where h is a smooth positive function on Σ and ρ a positive real number.

Equation (5) appears in conformal geometry in the problem of prescribing the Gaussian curvature, whereas in mathematical physics it describes models in abelian Chern–Simons theory. The literature on (5) is broad with many results regarding existence, blow-up analysis, compactness, etc. [Malchiodi 2008b; Tarantello 2010].

As with many geometric problems, (5) presents blow-up phenomena. It was proved in [Brezis and Merle 1991; Li 1999; Li and Shafrir 1994] that, for a sequence of solutions $(u_n)_n$ that blow up around a point p, the following quantization property holds:

$$\lim_{r \to 0} \lim_{n \to +\infty} \int_{B_r(p)} h e^{u_n}\, dV_g = 4\pi.$$

Moreover, the limit function (after rescaling) can be viewed as the logarithm of the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation.

Concerning the Toda system (1), a sequence of solutions can blow up in three different ways: one component blows up and the other stays bounded, one component blows up faster than the other, or both components diverge at the same rate. Jost et al. [2006] proved that the volume quantizations in these scenarios are $(0, 4\pi)$ or $(4\pi, 0)$ in the first case, $(4\pi, 8\pi)$ or $(8\pi, 4\pi)$ for the second one, and $(8\pi, 8\pi)$

for the last situation. Moreover, each alternative may indeed occur [D'Aprile et al. 2015; 2014; del Pino et al. 2005; Esposito et al. 2005; Musso et al. 2013].

With this at hand, with some further analysis, it is possible to obtain a compactness property, namely that the set of solutions to (1) is bounded (in any smoothness norm) for (ρ_1, ρ_2) bounded away from multiples of 4π (see Theorem 2.1). This fact, combined with a monotonicity method from [Struwe 1985], allows one to attack problem (1) via min-max methods.

Let us now discuss the variational strategy for proving existence of solutions and how our result compares to the existing literature. The goal is to introduce min-max schemes based on the study of the sublevels of the Euler–Lagrange functional. Consider the scalar case (5), with Euler–Lagrange energy

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 \, dV_g + 2\rho\left(\int_\Sigma u \, dV_g - \log\int_\Sigma h e^u \, dV_g\right). \tag{6}$$

By the classical Moser–Trudinger inequality

$$8\pi \log\int_\Sigma e^{(u-\bar{u})} \, dV_g \le \frac{1}{2}\int_\Sigma |\nabla u|^2 \, dV_g + C_{\Sigma,g}, \tag{7}$$

the latter energy is coercive if and only if $\rho < 4\pi$. A key result in treating this kind of problem without coercivity conditions (i.e., when $\rho > 4\pi$) is an improved version of (7), usually refereed to as the Chen–Li inequality and obtained in [Chen and Li 2001; Djadli 2008] (see also [Djadli and Malchiodi 2008]); roughly speaking, it states that, if the function e^u is *spread* (in a quantitative sense) among at least $(k+1)$ regions of Σ, $k \in \mathbb{N}$, then the constant in the left-hand side of (7) can be taken nearly $(k+1)$ times larger. This in turn implies that, for such functions u, $I_\rho(u)$ is bounded below even when $\rho < 4(k+1)\pi$. Therefore, if ρ satisfies the latter inequality and if $I_\rho(u)$ attains large negative values (i.e., when the lower bounds fail), e^u should be concentrated near at most k points of Σ; see [Djadli 2008] for a formal proof of this fact.

To describe such low sublevels, it is then natural to introduce the family of unit measures Σ_k that are supported at at most k points of Σ, known as *formal barycenters* of Σ of order k:

$$\Sigma_k = \left\{\sum_{i=1}^{k} t_i\delta_{x_i} : \sum_{i=1}^{k} t_i = 1,\ t_i \ge 0,\, x_i \in \Sigma \text{ for all } i = 1,\dots,k\right\}. \tag{8}$$

Endowed with the weak topology of distributions, Σ_1 is homeomorphic to Σ while, for $k \ge 2$, Σ_k is a stratified set (union of open manifolds of different dimensions). It is possible to show that the homology of Σ_k is always nontrivial and, using suitable test functions, that it injects into that of sufficiently low sublevels of I_ρ: this gives existence of solutions to (5) via suitable min-max schemes for every $\rho \notin 4\pi\mathbb{N}$.

Returning to the Toda system (1), a first existence result was presented in [Malchiodi and Ndiaye 2007] for $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 < 4\pi$ (see also [Jost et al. 2006] for the case $k = 1$). When one of the two parameters is small, the system (1) resembles the scalar case (5) and one can adapt the above argument to this framework as well. When both parameters exceed the value 4π, the description of the low sublevels becomes more involved due to the interaction of the two components u_1 and u_2.

The first variational approach to understand this interaction was given by Malchiodi and Ruiz [2013], who obtained an existence result for $(\rho_1, \rho_2) \in (4\pi, 8\pi)^2$. This was done in particular by showing that, if both components of the system concentrate near the same point and with the same *rate*, then the constants in the left-hand side of (4) can be nearly doubled.

Later, the case of general parameters $(\rho_1, \rho_2) \notin \Lambda$ was considered in [Battaglia et al. 2015] but only for surfaces of positive genus. Using improved inequalities à la Chen and Li, it is possible to prove that, if $\rho_1 < 4(k+1)\pi$ and $\rho_2 < 4(l+1)\pi$, $k, l \in \mathbb{N}$, and if $J_\rho(u_1, u_2)$ is sufficiently low, then either e^{u_1} is close to Σ_k or e^{u_2} is close to Σ_l in the distributional sense. This (not mutually exclusive) alternative can be expressed in terms of the *topological join* of Σ_k and Σ_l. Recall that, given two topological spaces A and B, their join $A * B$ is defined as the family of elements of the form

$$A * B = \frac{\{(a, b, s) : a \in A,\ b \in B,\ s \in [0, 1]\}}{E}, \tag{9}$$

where E is an equivalence relation such that

$$(a_1, b, 1) \overset{E}{\sim} (a_2, b, 1) \quad \text{for all } a_1, a_2 \in A,\ b \in B, \qquad (a, b_1, 0) \overset{E}{\sim} (a, b_2, 0) \quad \text{for all } a \in A,\ b_1, b_2 \in B.$$

This construction allows one to map low sublevels of J_ρ into $\Sigma_k * \Sigma_l$, with the join parameter s expressing whether distributionally e^{u_1} is closer to Σ_k or e^{u_2} is closer to Σ_l.

The hypothesis on the genus of Σ in [Battaglia et al. 2015] was used in the following way: on such surfaces, one can construct two *disjoint* simple noncontractible curves γ_1 and γ_2 such that Σ retracts on each of them through continuous maps Π_1 and Π_2. By means of these retractions, low-energy sublevels may be described in terms of $(\gamma_1)_k$ or $(\gamma_2)_l$ only. On the other hand, one can build test functions modeled on $(\gamma_1)_k * (\gamma_2)_l$ for which each component u_i only concentrates near γ_i, to somehow *minimize* the interaction between the two components u_1 and u_2, due to the fact that γ_1 and γ_2 are disjoint.

We prove here the following result, which for the first time applies to surfaces of arbitrary genus when both parameters ρ_i are supercritical and one of them also arbitrarily large:

Theorem 1.1. *Let h_1 and h_2 be two positive smooth functions, and let Σ be any compact surface. Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 \in (4\pi, 8\pi)$. Then problem (1) has a solution.*

Remark 1.2. Theorem 1.1 is new when Σ is a sphere and $k \geq 3$. As we already discussed, the case of surfaces with positive genus was covered in [Battaglia et al. 2015]. The case of $\Sigma \simeq S^2$ and $k = 1$ was covered in [Malchiodi and Ruiz 2013], while for $k = 2$ it was covered in [Lin et al. 2014]. In the latter paper, the authors indeed computed the Leray–Schauder degree of the equation for the range of ρ_i in Theorem 1.1. It turns out that the degree of (1) is 0 for the sphere when $k \geq 3$: since solutions do exist by Theorem 1.1, it means that either they are degenerate or that degrees of multiple ones cancel, so a global degree counting does not detect them. A similar phenomenon occurs for (5) on the sphere, when $\rho > 12\pi$ [Chen and Lin 2003]. Even for positive genus, we believe that our approach could be useful in computing the degree of the equation, as it happened in [Malchiodi 2008a] for the scalar equation (5). More precisely, we speculate that the degree should be computable as $1 - \chi(Y)$, where the set Y is given in (51). This is satisfied for example in the case of the sphere thanks to Lemma 5.4.

Other results on the degree of the system, but for different ranges of parameters, are available in [Malchiodi and Ruiz 2015].

As described above, in the situation of Theorem 1.1, it is natural to characterize low sublevels of the Euler–Lagrange energy J_ρ by means of the topological join $\Sigma_k * \Sigma_1$ (notice that $\Sigma_1 \simeq \Sigma$). However, differently from [Battaglia et al. 2015], we crucially take into account the interaction between the two components u_1 and u_2. As one can see from (3), the quadratic energy Q penalizes situations in which the gradients of the two components are aligned, and we would like to make a quantitative description of this effect. Our proof uses four new main ingredients.

- A refinement of the projection from low-energy sublevels onto the topological join $\Sigma_k * \Sigma_1$ from [Battaglia et al. 2015] (see Section 3), which uses the scales of concentration of the two components and which extends some construction in [Malchiodi and Ruiz 2013]. Having to deal with arbitrarily high values of ρ_1, differently from [Malchiodi and Ruiz 2013], we also need to take into account the stratified structure of Σ_k and the closeness in measure sense to its substrata.

- A new, scaling-invariant improved Moser–Trudinger inequality for system (1); see Proposition 3.5. This is inspired from another one in [Brezis and Merle 1991] for singular Liouville equations, i.e., of the form (5) but with Dirac masses on the right-hand side. The link between the two problems arises in the situation when one of the two components in (1) is much more concentrated than the other: in this case, the measure associated to its exponential function resembles a Dirac delta compared to the other one. The above improved inequality gives extra constraints to the projection on the topological join; see Proposition 3.7 and Corollary 3.8.

- A new set of test functions showing that the characterization of low-energy levels of J_ρ is sharp, as a subset Y of $\Sigma_k * \Sigma_1$. We need indeed to build test functions modeled on a set that contains $\Sigma_{k-1} * \Sigma_1$, and the stratified nature of Σ_{k-1} makes it hard to obtain uniform upper estimates on such functions.

- A new topological argument showing the noncontractibility of the above set Y, which we use then crucially to develop our min-max scheme. The fact that Y is simply connected and has Euler characteristic equal to 1 forces us to use rather sophisticated tools from algebraic topology.

We expect that our approach might extend to the case of general physical parameters ρ_1 and ρ_2, including the *singular Toda system*, in which Dirac masses (corresponding to ramification or vortex points) appear in the right-hand side of (1); see also [Battaglia 2015] for some results with this approach.

The paper is organized as follows. In Section 2, we recall some improved versions of the Moser–Trudinger inequality: first some that rely on the *macroscopic* spreading of the components u_1 and u_2 and then some refined ones, which are scaling-invariant. In Section 3, we derive a new — still scaling-invariant — improved version of the Moser–Trudinger inequality for systems, and we use it to find a characterization of low-energy levels of J_ρ by means of a subset Y of the topological join $\Sigma_k * \Sigma_1$. In Section 4, we construct then suitable test functions that show the optimality of the above characterization. In Section 5, we finally introduce the variational method to prove the existence of solutions.

Notation. The symbol $B_r(p)$ stands for the open metric ball of radius r and center p, while $A_p(r_1, r_2)$ is the open annulus of radii r_1 and r_2 and center p. For the complement of a set Ω in Σ, we will write Ω^c. Given a function $u \in L^1(\Sigma)$ and $\Omega \subset \Sigma$, the average of u on Ω is denoted by the symbol

$$\fint_\Omega u\, dV_g = \frac{1}{|\Omega|} \int_\Omega u\, dV_g,$$

while $\bar{u}$ stands for the average of u in Σ: since we are assuming $|\Sigma| = 1$, we have

$$\bar{u} = \int_\Sigma u\, dV_g = \fint_\Sigma u\, dV_g.$$

We also write

$$\mathcal{N}(f, D) = \frac{f}{\int_D f\, dV_g}.$$

The sublevels of the functional J_ρ will be denoted by

$$J_\rho^a = \{(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma) : J_\rho(u_1, u_2) \le a\}.$$

Throughout the paper, the letter C will stand for large constants that are allowed to vary among different formulas or even within the same line. To stress the dependence of the constants on some parameter, we add subscripts to C, as C_δ, etc. We will write $o_r(1)$ to denote quantities that tend to 0 as $r \to 0$ or $r \to +\infty$; we will similarly use the symbol $O_r(1)$ for bounded quantities.

2. Preliminaries

We begin by stating a compactness property that is needed in order to run the variational methods. Letting Λ be the set defined as

$$\Lambda = (4\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \cup 4\pi\mathbb{N}) \subseteq \mathbb{R}^2, \tag{10}$$

by the local blow-up in [Jost et al. 2006] and some analysis [Battaglia and Mancini 2015], one deduces:

Theorem 2.1 [Battaglia and Mancini 2015; Jost et al. 2006]. *For (ρ_1, ρ_2) in a fixed compact set of $\mathbb{R}^2 \setminus \Lambda$, the family of solutions to (1) is uniformly bounded in $C^{2,\beta}$ for some $\beta > 0$.*

In the next two subsections, we will discuss some improved versions of the Moser–Trudinger inequality (4) that hold under suitable assumptions on the components of the system. The first type of inequality relies on the spreading of the (exponentials of the) components over the surface [Battaglia et al. 2015]. The second one, from [Malchiodi and Ruiz 2013], relies instead on comparing the scales of concentration of the two components.

2.1. *Macroscopic improved inequalities.* Here comes the first kind of improved inequality: basically, if the *masses* of both e^{u_1} and e^{u_2} are *spread* on at least $k+1$ and $l+1$ different sets, then the logarithms in (4) can be multiplied by $k+1$ and $l+1$, respectively. Notice that this result was given in [Malchiodi and Ndiaye 2007] in the case $l = 0$ and in [Malchiodi and Ruiz 2013] in the case $k = l = 1$. The proof relies on localizing (4) by using cut-off functions near the regions of volume concentration. For (7), this was previously shown in [Chen and Li 1991].

Lemma 2.2 [Battaglia et al. 2015]. *Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$, and $\{\Omega_{1,i}, \Omega_{2,j}\}_{i\in\{0,\dots,k\}, j\in\{0,\dots,l\}} \subset \Sigma$ be such that*

$$d(\Omega_{1,i}, \Omega_{1,i'}) \geq \delta \quad \text{for all } i, i' \in \{0, \dots, k\} \text{ with } i \neq i',$$
$$d(\Omega_{2,j}, \Omega_{2,j'}) \geq \delta \quad \text{for all } j, j' \in \{0, \dots, l\} \text{ with } j \neq j'.$$

Then for any $\varepsilon > 0$, there exists $C = C(\varepsilon, \delta, \theta, k, l, \Sigma)$ such that any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ satisfying

$$\int_{\Omega_{1,i}} e^{u_1}\, dV_g \geq \theta \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i \in \{0, \dots, k\},$$
$$\int_{\Omega_{2,j}} e^{u_2}\, dV_g \geq \theta \int_\Sigma e^{u_2}\, dV_g \quad \text{for all } j \in \{0, \dots, l\}$$

satisfies

$$4\pi(k+1) \log \int_\Sigma e^{u_1 - \bar{u}_1}\, dV_g + 4\pi(l+1) \log \int_\Sigma e^{u_2 - \bar{u}_2}\, dV_g \leq (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C.$$

As one can see, larger constants in the left-hand side of (4) can be helpful in obtaining lower bounds on the functional J_ρ even when the coefficients ρ_1 and ρ_2 exceed the threshold value $(4\pi, 4\pi)$. A consequence of this fact is that, when the energy $J_\rho(u_1, u_2)$ is large and negative, then e^{u_1} and e^{u_2} are forced to *concentrate* near certain points in Σ whose number depends on ρ_1 and ρ_2. To make this description rigorous, it is convenient to introduce some further notation.

We denote by $\mathcal{M}(\Sigma)$ the set of all Radon measures on Σ and introduce a distance on it by using duality versus Lipschitz functions; that is, we set

$$\boldsymbol{d}(\nu_1, \nu_2) = \sup_{\|f\|_{\mathrm{Lip}(\Sigma)} \leq 1} \left| \int_\Sigma f\, d\nu_1 - \int_\Sigma f\, d\nu_2 \right|, \qquad \nu_1, \nu_2 \in \mathcal{M}(\Sigma). \tag{11}$$

This is known as the *Kantorovich–Rubinstein distance*.

The following result was proven using the improved inequality from Lemma 2.2 (see previous page for $\mathcal{N}$):

Proposition 2.3 [Battaglia et al. 2015]. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then, for any $\varepsilon > 0$, there exists $L > 0$ such that any $(u_1, u_2) \in J_\rho^{-L}$ satisfies either*

$$\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < \varepsilon \quad \text{or} \quad \boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l) < \varepsilon.$$

When a measure is $\boldsymbol{d}$-close to an element in Σ_k (see (8)), it is then possible to map it continuously to a nearby element in this set. The next proposition collects some properties of this map from Proposition 2.2 in [Battaglia et al. 2015] and Lemma 2.3 in [Djadli and Malchiodi 2008] (together with the proof of Lemma 3.10).

Proposition 2.4. *Given $l \in \mathbb{N}$, for ε_l sufficiently small, there exists a continuous retraction*

$$\psi_l : \{\nu \in \mathcal{M}(\Sigma) : \boldsymbol{d}(\nu, \Sigma_l) < 2\varepsilon_l\} \to \Sigma_l.$$

Here continuity refers to the distance $\boldsymbol{d}$. In particular, if $\nu_n \rightharpoonup \nu$ in the sense of measures, with $\nu \in \Sigma_l$, then $\psi_l(\nu_n) \to \nu$.

Furthermore, the following property holds: given any $\varepsilon > 0$, there exists $\varepsilon' \ll \varepsilon$ with ε' depending on l and ε such that if $\boldsymbol{d}(\nu, \Sigma_{l-1}) > \varepsilon$ then there exist l points $x_1, \dots, x_l$ such that

$$d(x_i, x_j) > 2\varepsilon' \quad \textit{for } i \neq j, \qquad \int_{B_{\varepsilon'}(x_i)} \nu > \varepsilon' \quad \textit{for all } i = 1, \dots, l.$$

The alternative in Proposition 2.3 can be expressed naturally in terms of the topological join of $\Sigma_k * \Sigma_l$; see also the comments after (9). Indeed, we can define a map from the low sublevels J_ρ^{-L} onto this set.

Proposition 2.5 [Battaglia et al. 2015]. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then for $L > 0$ sufficiently large there exists a continuous map*

$$\Psi : J_\rho^{-L} \to \Sigma_k * \Sigma_l.$$

Proof. The proof is carried out exactly as in Proposition 4.7 of [Battaglia et al. 2015]. We repeat here the argument for the reader's convenience as we will need to suitably modify it later on. By Proposition 2.3, we know that for any $\varepsilon > 0$, taking $L > 0$ sufficiently large, $(u_1, u_2) \in J_\rho^{-L}$ satisfies either $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < \varepsilon$ or $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l) < \varepsilon$ (or both). Using then Proposition 2.4, it follows that either $\psi_k(\mathcal{N}(e^{u_1}, \Sigma))$ or $\psi_l(\mathcal{N}(e^{u_2}, \Sigma))$ is well-defined. We let $d_1 = \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)$ and $d_2 = \boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l)$ and introduce a function $\tilde{s} = \tilde{s}(d_1, d_2)$ in the following way:

$$\tilde{s}(d_1, d_2) = f\left(\frac{d_1}{d_1 + d_2}\right),$$

where f is given by

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{4}], \\ 2z - \frac{1}{2} & \text{if } t \in (\frac{1}{4}, \frac{3}{4}), \\ 1 & \text{if } t \in [\frac{3}{4}, 1]. \end{cases}$$

We finally set

$$\Psi(u_1, u_2) = (1 - \tilde{s})\psi_k(\mathcal{N}(e^{u_1}, \Sigma)) + \tilde{s}\psi_l(\mathcal{N}(e^{u_2}, \Sigma)). \tag{12}$$

One just has to observe that, when one of the two ψ is not defined, the other necessarily is. Therefore, the map is well-defined by the equivalence relation of the topological join; see (9). □

2.2. ***Scaling-invariant improved inequalities.*** Malchiodi and Ruiz [2013] set up a tool to deal with situations to which Lemma 2.2 does not apply, for example in cases when both e^{u_1} and e^{u_2} are concentrated around only one point. They provided a definition of the *center* and the *scale of concentration* of such functions, to obtain new improved inequalities in terms of these. We are interested here in measures concentrated around possibly multiple points. We need therefore a localized version of the argument in [Malchiodi and Ruiz 2013], which applies to measures supported in a ball and sufficiently concentrated around its center.

Given $x_0 \in \Sigma$ and $r > 0$ small, consider the set

$$\mathcal{A}_{x_0,r} = \left\{ f \in L^1(B_r(x_0)) : f > 0 \text{ a.e. and } \int_{B_r(x_0)} f \, dV_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$.

Fix a constant $R > 1$, and let $R_0 = 3R$. Define $\sigma : B_r(x_0) \times \mathscr{A}_{x_0,r} \to (0, +\infty)$ such that

$$\int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g = \int_{(B_{R_0 \sigma(x,f)}(x))^c \cap B_r(x_0)} f \, dV_g. \tag{13}$$

It is easy to check that $\sigma(x, f)$ is uniquely determined and continuous (both in $x \in B_r(x_0)$ and in $f \in L^1$). Moreover (see (3.2) in [Malchiodi and Ruiz 2013]), σ satisfies

$$d(x, y) \le R_0 \max\{\sigma(x, f), \sigma(y, f)\} + \min\{\sigma(x, f), \sigma(y, f)\}. \tag{14}$$

We now define $T : B_r(x_0) \times \mathscr{A}_{x_0,r} \to \mathbb{R}$ as

$$T(x, f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g.$$

Lemma 2.6 ([Malchiodi and Ruiz 2013] with minor changes). *If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x}, f) = \max_{y \in \overline{B_r(x_0)}} T(y, f)$, then $\sigma(\bar{x}, f) < 3\sigma(x, f)$ for any other $x \in \overline{B_r(x_0)}$.*

As a consequence of the previous lemma and of a covering argument, one can obtain the following:

Lemma 2.7 ([Malchiodi and Ruiz 2013] with minor changes). *There exists a fixed $\tau > 0$ such that*

$$\max_{x \in \overline{B_r(x_0)}} T(x, f) > \tau > 0 \quad \textit{for all } f \in \mathscr{A}_{x_0,r}.$$

Let us define $\sigma : \mathscr{A}_{x_0,r} \to \mathbb{R}$ by

$$\sigma(f) = 3 \min\{\sigma(x, f) : x \in \overline{B_r(x_0)}\},$$

which is obviously a continuous function.

Given τ as in Lemma 2.7, consider the set

$$S(f) = \{x \in \overline{B_r(x_0)} : T(x, f) > \tau, \ \sigma(x, f) < \sigma(f)\}. \tag{15}$$

If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x}, f) = \max_{x \in \overline{B_r(x_0)}} T(x, f)$, then Lemmas 2.6 and 2.7 imply that $\bar{x} \in S(f)$. Therefore, $S(f)$ is a nonempty set for any $f \in \mathscr{A}_{x_0,r}$. Moreover, recalling (13) and the notation before it, from (14), we have that

$$\operatorname{diam}(S(f)) \le (R_0 + 1)\sigma(f). \tag{16}$$

We will now restrict ourselves to a class of functions in $L^1(B_r(x_0))$ that are almost entirely concentrated near the center x_0. In this case, one expects $\sigma(f)$ to be small and points in $S(f)$ to be close to x_0: see Remark 2.8 for precise estimates in this spirit. Given $\varepsilon > 0$ small, let us introduce the class of functions

$$\mathscr{C}_{\varepsilon,r}(x_0) = \left\{ f \in \mathscr{A}_{x_0,r} : \int_{B_\varepsilon(x_0)} f \, dV_g > 1 - \varepsilon \right\}. \tag{17}$$

Remark 2.8. For this class of functions, we claim that $T(x, f) \le \varepsilon$ when $d(x, x_0) > 2\varepsilon$. In fact, if $\sigma(x, f) \le d(x, x_0) - \varepsilon$, then we are done since

$$T(x, f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g \le \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f \, dV_g \le \varepsilon.$$

If this is not the case, i.e., $\sigma(x, f) > d(x, x_0) - \varepsilon$, then using $d(x, x_0) > 2\varepsilon$, we obtain

$$R_0\sigma(x, f) > R_0(d(x, x_0) - \varepsilon) > \tfrac{1}{2} R_0\, d(x, x_0) \\ > d(x, x_0) + \varepsilon.$$

Similarly as before, we get

$$T(x, f) = \int_{(B_{R_0\sigma(x,f)}(x))^c \cap B_r(x_0)} f\, dV_g \le \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f\, dV_g \le \varepsilon.$$

Being τ-universal, ε can be taken so small that $(T(x, f) - \tau)^+ = 0$ outside $B_{2\varepsilon}(x_0)$ for all $f \in \mathscr{C}_{\varepsilon,r}(x_0)$.

By the Nash embedding theorem, we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically, $N \in \mathbb{N}$. Take an open tubular neighborhood $\Sigma \subset U \subset \mathbb{R}^N$ of Σ and $\delta > 0$ small enough so that

$$\mathrm{co}[B_x((R_0 + 1)\delta) \cap \Sigma] \subset U \quad \text{for all } x \in \Sigma, \tag{18}$$

where co denotes the convex hull in $\mathbb{R}^N$.

For $f \in \mathscr{C}_{\varepsilon,r}(x_0)$, we define now

$$\eta(f) = \frac{\int_\Sigma (T(x, f) - \tau)^+ (\sigma(f) - \sigma(x, f))^+ x\, dV_g}{\int_\Sigma (T(x, f) - \tau)^+ (\sigma(f) - \sigma(x, f))^+\, dV_g} \in \mathbb{R}^N,$$

which is well-defined; see Remark 2.8. The map η yields a sort of center of mass in $\mathbb{R}^N$ of the measure induced by f. Observe that the integrands become nonzero only on the set $S(f)$. However, whenever $\sigma(f) \le \delta$, (16) and (18) imply that $\eta(f) \in U$, and so we can define

$$\beta : \{f \in \mathscr{A}_{x_0,r} : \sigma(f) \le \delta\} \to \Sigma, \quad \beta(f) = P \circ \eta(f),$$

where $P : U \to \Sigma$ is the orthogonal projection.

We finally define the map $\psi : \mathscr{C}_{\varepsilon,r}(x_0) \to \Sigma \times (0, r)$, which will be the main tool of this subsection:

$$\psi(f) = (\beta, \sigma). \tag{19}$$

Roughly, this map expresses the center of mass of f and its scale of concentration around this point.

Malchiodi and Ruiz [2013] proved that, if both components (u_1, u_2) of the Toda system concentrate around the same point in Σ, with the same scale of concentration, then the constants in the left-hand side of (4) can be nearly doubled.

Remark 2.9. The core of the argument of the improved inequality in [Malchiodi and Ruiz 2013] consists of proving that

$$\psi(\mathscr{N}(e^{u_1}, B_r(x))) = \psi(\mathscr{N}(e^{u_2}, B_r(y)))$$

implies the existence of $\sigma > 0$ and of two balls $B_\sigma(z_1)$ and $B_\sigma(z_2)$ such that

$$\frac{\int_{B_\sigma(z_i)} e^{u_i}\, dV_g}{\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0, \quad \frac{\int_{(B_{R\sigma}(z_i))^c \cap B_r(z_i)} e^{u_i}\, dV_g}{\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0 \quad \text{for } i = 1, 2 \text{ with } d(z_1, z_2) \lesssim \sigma \tag{20}$$

for some fixed positive constant γ_0. Once this is achieved, the improved inequality is obtained by scaling arguments and Kelvin inversions (see Section 3 in [Malchiodi and Ruiz 2013] for full details).

Even when e^{u_1} and e^{u_2} are not necessarily concentrated near a single point, the assumptions of the next proposition still allow us to obtain (20) and hence again nearly double constants in the left-hand side of (4).

Proposition 2.10 ([Malchiodi and Ruiz 2013] with minor changes). *Let $\tilde{\varepsilon} > 0$ and $\delta' > 0$. Then there exist $R = R(\tilde{\varepsilon})$ and ψ as in definition (19) such that, for any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ such that there exist $x, y \in \Sigma$ with*

$$\int_{B_r(x)} e^{u_1}\, dV_g \geq \delta' \int_\Sigma e^{u_1} dV_g, \qquad \int_{B_r(y)} e^{u_2}\, dV_g \geq \delta' \int_\Sigma e^{u_1}\, dV_g,$$
$$\mathcal{N}(e^{u_1}, B_r(x)) \in \mathscr{C}_{\varepsilon,r}(x), \qquad \mathcal{N}(e^{u_2}, B_r(y)) \in \mathscr{C}_{\varepsilon,r}(y)$$

and

$$\psi(\mathcal{N}(e^{u_1}, B_r(x))) = \psi(\mathcal{N}(e^{u_2}, B_r(y))), \tag{21}$$

the following inequality holds:

$$8\pi\left(\log \int_\Sigma e^{u_1 - \bar{u}_1}\, dV_g + \log \int_\Sigma e^{u_2 - \bar{u}_2}\, dV_g\right) \leq (1+\tilde{\varepsilon}) \int_\Sigma Q(u_1, u_2)\, dV_g + C \tag{22}$$

for some $C = C(\tilde{\varepsilon}, \delta', \Sigma)$.

Remark 2.11. (i) Condition (21) can be relaxed. In fact, let $C_1 > 1$ and $C_2 > 0$ be two positive constants and define

$$\psi(\mathcal{N}(e^{u_1}, B_r(x))) = (\beta_1, \sigma_1), \qquad \psi(\mathcal{N}(e^{u_2}, B_r(y))) = (\beta_2, \sigma_2).$$

Then, the result still holds true if

$$\frac{1}{C_1} \leq \frac{\sigma_1}{\sigma_2} \leq C_1, \qquad d(\beta_1, \beta_2) \leq C_2 \sigma_1.$$

In such a case, the constant C would also depend on C_1 and C_2.

(ii) In the right-hand side of (22), one can actually integrate $Q(u_1, u_2)$ only in any set compactly containing $B_r(x) \cup B_r(y)$. This can be seen using suitable cut-off functions; see the comments before Lemma 2.2.

We can now improve this result for situations in which the first component of the system is concentrated around l points of Σ, $l \in \mathbb{N}$. The proof relies on combining the argument for Proposition 2.10 with the macroscopic improved inequality of Lemma 2.2 (see also Remark 2.11(ii)).

Proposition 2.12. *Let $\tilde{\varepsilon} > 0$, $\delta' > 0$, and $k \in \mathbb{N}$. Then there exist $R = R(\tilde{\varepsilon})$ and ψ as in definition (19) such that, for any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with the property that there exist $\{x_i\}_{i \in \{1,\dots,k\}} \subset \Sigma$ and $y \in \Sigma$ with*

$$d(x_i, x_j) > 4\delta' \quad \text{for all } i, j \in \{1, \dots, k\} \text{ with } i \neq j,$$
$$\int_{B_{\delta'}(x_i)} e^{u_1}\, dV_g \geq \delta' \int_\Sigma e^{u_1}\, dV_g \quad \text{for } i = 1, \dots, k, \qquad \int_{B_{\delta'}(y)} e^{u_2}\, dV_g \geq \delta' \int_\Sigma e^{u_2}\, dV_g$$

such that

$$\mathcal{N}(e^{u_1}, B_{\delta'}(x_i)) \in \mathscr{C}_{\varepsilon,\delta'}(x_i) \quad \text{for } i = 1, \dots, k, \qquad \mathcal{N}(e^{u_2}, B_{\delta'}(y)) \in \mathscr{C}_{\varepsilon,\delta'}(y)$$

and

$$\psi(\mathcal{N}(e^{u_1}, B_{\delta'}(x_l))) = \psi(\mathcal{N}(e^{u_2}, B_{\delta'}(y))) \quad \textit{for some } l \in \{1, \dots, k\},$$

the following inequality holds:

$$4\pi(k+1)\log \int_\Sigma e^{u_1-\bar{u}_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2-\bar{u}_2}\, dV_g \le (1+\tilde{\varepsilon}) \int_\Sigma Q(u_1, u_2)\, dV_g + C$$

for some $C = C(\tilde{\varepsilon}, \delta', l, \Sigma)$.

In the next section, we will derive a new improved inequality for the Toda system with scaling-invariant features; see Proposition 3.5. The result is inspired by arguments developed in [Bartolucci and Malchiodi 2013] for the singular Liouville equation where a Dirac delta is involved (see Remark 3.6), and for the first time, this type of inequality is presented for a two-component problem.

3. A refined projection onto the topological join

Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. By Proposition 2.5, we have the existence of a map Ψ from the low sublevels of J_ρ onto the topological join $\Sigma_k * \Sigma_1$; see (8) and (9). However, we will next need to also take into account the fine structure of the measures e^{u_1} and e^{u_2} as described in (19). For this reason, we will modify the map Ψ so that the join parameter s in (9) will depend on the local centers of mass and the local scales defined in (19) and (23). We will see in the sequel that this will provide extra information for describing functions in the low sublevels of J_ρ.

3.1. ***Construction.*** We start by defining the local centers of mass and the local scales of functions that are concentrated around l well-separated points of Σ.

Let $l \ge 2$, consider $0 < \varepsilon_l \ll \varepsilon_{l-1} \ll 1$ as given in Proposition 2.4, and suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_l) < 2\varepsilon_l$ so that ψ_l is well-defined. Assume moreover $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{l-1}) > \varepsilon_{l-1}$. By the second part of Proposition 2.4, there exist $\varepsilon'_{l-1} \ll \varepsilon_{l-1}$ and l points $x_1^l, \dots, x_l^l$ such that

$$d(x_i^l, x_j^l) > 2\varepsilon'_{l-1} \quad \text{for } i \ne j, \qquad \int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\, dV_g > \varepsilon'_{l-1} \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \dots, l.$$

We then localize u_1 around the point x_i^l and define

$$f_{\mathrm{loc}}^{x_i^l}(u_1) = \frac{e^{u_1}\chi_{B_{\varepsilon'_{l-1}}(x_i^l)}}{\int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\, dV_g}.$$

Given $\varepsilon > 0$, by the second assertion of Proposition 2.4, taking ε_l sufficiently small, one gets

$$\int_{B_\varepsilon(x_i^l)} f_{\mathrm{loc}}^{x_i^l}(u_1)\, dV_g > 1 - \varepsilon \quad \text{for } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_l) < 2\varepsilon_l.$$

It follows that $f_{\mathrm{loc}}^{x_i^l}(u_1) \in \mathscr{C}_{\varepsilon, \varepsilon'_{l-1}}(x_i^l)$ (see (17)), and hence, the map ψ in (19) is well-defined on $f_{\mathrm{loc}}^{x_i^l}(u_1)$. We then set

$$\big(\beta_{x_i^l}, \sigma_{x_i^l}\big) := \psi\big(f_{\mathrm{loc}}^{x_i^l}(u_1)\big). \tag{23}$$

In this way, starting from a function with $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_l) < 2\varepsilon_l$ and such that $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{l-1}) > \varepsilon_{l-1}$, we obtain, around each point x_i^l, a notion of local center of mass and scale of concentration.

When $l = 1$, we have to deal with just one point x_1^1 of Σ. We then apply the map ψ to the function $f_{\mathrm{loc}}^{x_1^1}$ directly.

As we discussed above, we would like to map low-energy sublevels of J_ρ into the topological join $\Sigma_k * \Sigma_1$ taking the above scales into account. More precisely, the parameter s in (9) will depend on the local scale $\sigma_{x_i^l}$ only of the points near the center of mass of e^{u_2} (in case of ambiguity, we will define a sort of averaged scale).

To proceed rigorously, let $0 < \varepsilon_k \ll \varepsilon_{k-1} \ll \cdots \ll \varepsilon_1 \ll 1$ be as before. We consider cut-off functions $\mathfrak{f}$, $\mathfrak{g}_l$, and $\mathfrak{h}$ for $l = 1, \dots, k-1$ such that

$$\mathfrak{f}(t) = \begin{cases} 0, & t \geq 2\varepsilon_k, \\ 1, & t \leq \varepsilon_k, \end{cases}$$

$$\mathfrak{g}_l(t) = \begin{cases} 0, & t \geq 2\varepsilon_l, \\ 1, & t \leq \varepsilon_l, \end{cases} \qquad l = 1, \dots, k-1, \tag{24}$$

$$\mathfrak{h}(t) = \begin{cases} 0, & t \geq \frac{1}{8}\varepsilon'_{k-1}, \\ 1, & t \leq \frac{1}{16}\varepsilon'_{k-1}. \end{cases} \tag{25}$$

We define now a *global scale* $\sigma_1(u_1) \in (0, 1]$ for e^{u_1} in three steps. Suppose $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < 2\varepsilon_1$ so that $\psi(f_{\mathrm{loc}}^z(u_2)) = (\beta_z, \sigma_z)$ is well-defined.

First of all, we define an *averaged scale* for e^{u_1} by recurrence in the following way. If we have $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_1) < 2\varepsilon_1$, we set $C_1(u_1) = \sigma_{x_1^1}$. For $l \in \{2, \dots, k-1\}$, we define recursively

$$C_l(u_1) = \mathfrak{g}_{l-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{l-1})\big)C_{l-1}(u_1) + \big(1 - \mathfrak{g}_{l-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{l-1})\big)\big)\frac{1}{l}\sum_{i=1}^{l} \sigma_{x_i}^l.$$

Secondly, we interpolate between $C_{k-1}(u_1)$ and the local scale of the closest point to β_z among the $\beta_{x_i^k}$ (provided they are well-defined), setting

$$B(u_1, u_2) = \mathfrak{h}\big(d\big(\beta_z, \{\beta_{x_1^k}, \dots, \beta_{x_k^k}\}\big)\big)\sigma_x + \big(1 - \mathfrak{h}\big(d\big(\beta_z, \{\beta_{x_1^k}, \dots, \beta_{x_k^k}\}\big)\big)\big)\frac{1}{k}\sum_{i=1}^{k} \sigma_{x_i^k},$$

$$A(u_1, u_2) = \mathfrak{g}_{k-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1})\big)C_{k-1}(u_1) + \big(1 - \mathfrak{g}_{k-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1})\big)\big)B(u_1, u_2),$$

where $x = x_j^k$ was chosen so that it realizes the minimum of $d(\beta_z, \{\beta_{x_1^k}, \dots, \beta_{x_k^k}\})$: notice that, since $d(x_j^k, x_l^k) \geq 2\varepsilon'_{k-1}$ for $j \neq l$, by (25) the point realizing the latter minimum is unique if $\mathfrak{h} \neq 0$.

As a third and final step, to check whether e^{u_1} is $\boldsymbol{d}$-close to Σ_k, we set

$$\sigma_1(u_1) = \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)A(u_1, u_2) + \big(1 - \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)\big).$$

We define next the *global scale* $\sigma_2(u_2) \in (0, 1]$ of e^{u_2}. We will be interested here in functions concentrated near just one point of Σ. Therefore, we just need the single local scale $C_1(u_2) = \sigma_z$ if $\psi(f_{\mathrm{loc}}^z(u_2)) = (\beta_z, \sigma_z)$ is well-defined. Moreover, we have to check the $\boldsymbol{d}$-closeness of e^{u_2} to Σ_1. Hence, the scale reads

$$\sigma_2(u_2) = \mathfrak{g}_1\big(\boldsymbol{d}\big(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1\big)\big)\sigma_z + \big(1 - \mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1)\big)\big).$$

We can now specify the join parameter s in (9). Fix a constant $M \gg 1$, and consider the function

$$F_M(t) = \begin{cases} 0, & t \le 1/M, \\ \dfrac{t}{1+t}, & t \in [2/M, M], \\ 1, & t \ge 2M. \end{cases}$$

We then define

$$s(u_1, u_2) = F_M\left(\frac{\sigma_1(u_1)}{\sigma_2(u_2)}\right). \tag{26}$$

We now pass to considering the maps ψ_k and ψ_1 that are needed in the projection onto the join $\Sigma_k * \Sigma_1$; see (12). As mentioned in the introduction of this section, it is convenient to modify these maps in such a way that they take into account the local centers of mass defined in (19) and (23). More precisely, when e^{u_1} is concentrated in k well-separated points of Σ, we would rather consider the local centers of mass $\beta_{x_i^l}$ in (23) than the supports of the map ψ_k in Proposition 2.4.

Suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < 2\varepsilon_k$ so that ψ_k is well-defined, and suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) > \varepsilon_{k-1}$ so that $\beta_{x_i^k}$ are defined for $i = 1, \dots, k$. Let

$$\psi_k(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) = \sum_{i=1}^{k} t_i \delta_{y_i}, \qquad t_i \in [0, 1],\ \ y_i \in \Sigma.$$

Observe that, by construction and by the second statement in Proposition 2.4, $d(\beta_{x_i^k}, y_i) \to 0$ as $\varepsilon_k \to 0$. Hence, there exists a geodesic γ_i joining y_i and $\beta_{x_i^k}$ in unit time. We then perform an interpolation:

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) = \begin{cases} \sum_{i=1}^{k} t_i \delta_{y_i} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \le \varepsilon_{k-1}, \\ \sum_{i=1}^{k} t_i \delta_{\gamma_i(\varepsilon_{k-1}^{-1}\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) - 1)} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \in (\varepsilon_{k-1}, 2\varepsilon_{k-1}), \\ \sum_{i=1}^{k} t_i \delta_{\beta_{x_i^k}} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \ge 2\varepsilon_{k-1}. \end{cases} \tag{27}$$

For a function u_2 with $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < 2\varepsilon_1$, letting $\psi_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_z$, we let

$$\widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_{\beta_z}. \tag{28}$$

With these maps and this join parameter, we finally define the refined projection $\widetilde{\Psi} : J_\rho^{-L} \to \Sigma_k * \Sigma_1$ as

$$\widetilde{\Psi}(u_1, u_2) = (1 - s)\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) + s\widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)). \tag{29}$$

3.2. ***A new improved Moser–Trudinger inequality.*** Using the improved geometric inequality in [Bartolucci and Malchiodi 2013] for the singular Liouville equation, we can provide a dilation-invariant improved inequality for system (1). Before stating the main result, we prove some auxiliary lemmas; we first recall our notation on annuli at the end of Section 1.

Lemma 3.1. *Let $\gamma_0 > 0$, $\tau_0 > 0$, $z \in \Sigma$, and $r_2 > r_1 > 0$ (both small) be such that*

$$\frac{\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g}{\int_\Sigma e^{u_2}\,dV_g} > \gamma_0 \quad \textit{and} \quad \sup_{y\in A_z(r_1,r_2)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g} < 1-\tau_0. \tag{30}$$

Then for any $\varepsilon > 0$, there exist $C = C(\varepsilon, \tau_0, \gamma_0)$, $\tilde{\tau}_0 = \tilde{\tau}_0(\tau_0, \gamma_0)$, $\tilde{r}_1 \in [r_1/C, r_1/4]$, $\tilde{r}_2 \in [4r_2, Cr_2]$, and $\tilde{u}_2 \in H^1(\Sigma)$ such that

(a) *$\tilde{u}_2$ is constant in $B_{\tilde{r}_1}(z)$ and on $\partial B_{\tilde{r}_2}(z)$,*

(b)
$$\int_{A_z(\tilde{r}_1,\tilde{r}_2)} |\nabla \tilde{u}_2|^2\,dV_g \le \int_{A_z(\tilde{r}_1,\tilde{r}_2)} |\nabla u_2|^2\,dV_g + \varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g,$$

(c)
$$\sup_{y\in A_z(\tilde{r}_1,\tilde{r}_2)} \frac{\int_{B_{\tilde{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\,dV_g}{\int_{A_z(\tilde{r}_1,\tilde{r}_2)} e^{\tilde{u}_2}\,dV_g} < 1-\tilde{\tau}_0.$$

Proof. First of all, we modify u_2 so that it becomes constant in $B_{\tilde{r}_1}(z)$ and on $\partial B_{\tilde{r}_2}(z)$. Take $\varepsilon > 0$: we can find $C = C(\varepsilon)$ and properly chosen $\tilde{r}_1 \in [r_1/C, r_1/4]$ and $\tilde{r}_2 \in [4r_2, Cr_2]$ such that

$$\int_{A_z(\tilde{r}_1,2\tilde{r}_1)} |\nabla u_2|^2\,dV_g \le \varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\tilde{r}_2/2,\tilde{r}_2)} |\nabla u_2|^2\,dV_g \le \varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g.$$

We denote by $\bar{u}_2(\tilde{r}_1)$ and $\bar{u}_2(\tilde{r}_2)$ the averages

$$\bar{u}_2(\tilde{r}_1) = \fint_{A_z(\tilde{r}_1,2\tilde{r}_1)} u_2\,dV_g, \qquad \bar{u}_2(\tilde{r}_2) = \fint_{A_z(\tilde{r}_2/2,\tilde{r}_2)} u_2\,dV_g. \tag{31}$$

Now let χ be a cut-off function, with values in $[0, 1]$, such that

$$\chi = \begin{cases} 0 & \text{in } B_{\tilde{r}_1}(z), \\ 1 & \text{in } A_z(2\tilde{r}_1, \tilde{r}_2/2), \\ 0 & \text{in } (B_{\tilde{r}_2}(z))^c, \end{cases}$$

and define

$$\tilde{u}_2 = \begin{cases} \chi(d(x,z))u_2 + (1-\chi(d(x,z))\bar{u}_2(\tilde{r}_1)) & \text{in } B_{2\tilde{r}_1}(z), \\ u_2 & \text{in } A_z(2\tilde{r}_1, \tilde{r}_2/2), \\ \chi(d(x,z))u_2 + (1-\chi(d(x,z))\bar{u}_2(\tilde{r}_2)) & \text{in } (B_{\tilde{r}_2/2}(z))^c. \end{cases} \tag{32}$$

By Poincaré's inequality, the Dirichlet energy of $\tilde{u}_2$ is bounded by

$$\int_{A_z(\tilde{r}_1,2\tilde{r}_1)} |\nabla \tilde{u}_2|^2\,dV_g \le \widetilde{C}\varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\tilde{r}_2/2,\tilde{r}_2)} |\nabla \tilde{u}_2|^2\,dV_g \le \widetilde{C}\varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g,$$

where $\widetilde{C}$ is a universal constant. Hence, one gets

$$\int_{A_z(\tilde{r}_1,\tilde{r}_2)} |\nabla \tilde{u}_2|^2\,dV_g \le \int_{A_z(\tilde{r}_1,\tilde{r}_2)} |\nabla u_2|^2\,dV_g + 2\widetilde{C}\varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g.$$

We are left with proving that there exists $\tilde{\tau}_0 = \tilde{\tau}_0(\tau_0, \gamma_0)$ such that

$$\sup_{y\in A_z(\tilde{r}_1,\tilde{r}_2)} \frac{\int_{B_{\tilde{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\,dV_g}{\int_{A_z(\tilde{r}_1,\tilde{r}_2)} e^{\tilde{u}_2}\,dV_g} < 1-\tilde{\tau}_0. \tag{33}$$

If this isn't the case, there exist $(u_{2,n})_n \subset H^1(\Sigma)$ satisfying (30), $(\tilde r_{1,n})_n \subset [r_1/C, r_1/4]$, $(\tilde r_{2,n})_n \subset [4r_2, Cr_2]$, and cut-off functions $(\chi_n)_n$ and $(\tilde u_{2,n})_n \subset H^1(\Sigma)$ defined analogously to $\tilde u_2$ in (32) such that

$$\frac{e^{\tilde u_{2,n}}}{\int_{A_z(\tilde r_{1,n},\tilde r_{2,n})} e^{\tilde u_{2,n}}\,dV_g} \rightharpoonup \delta_{\bar x} \tag{34}$$

in the sense of measures for some $\bar x \in A_z(r_1/C, Cr_2)$. We distinguish between three situations.

Case 1. Suppose first that $\bar x \in A_z(r_1, 2r_2)$. By the choices of the cut-off functions and (32), as $\tilde u_{2,n}$ coincides with $u_{2,n}$ on $A_z(r_1/2, 2r_2)$, it follows that

$$\frac{e^{u_{2,n}}}{\int_{A_z(r_1,2r_2)} e^{u_{2,n}}\,dV_g} = \frac{e^{\tilde u_{2,n}}}{\int_{A_z(r_1,2r_2)} e^{\tilde u_{2,n}}\,dV_g} \rightharpoonup \delta_{\bar x}. \tag{35}$$

Case 1.1. Let $\bar x \in A_z(r_1, \frac{3}{2}r_2)$. To get a contradiction to (35), we prove that there exists $\bar\tau_0 = \bar\tau_0(\tau_0, \gamma_0)$ such that

$$\sup_{y\in A_z(r_1,(3/2)r_2)} \int_{B_{\bar\tau_0 d(y,z)}(y)} e^{u_{2,n}}\,dV_g \le (1-\bar\tau_0)\int_{A_z(r_1,2r_2)} e^{u_{2,n}}\,dV_g. \tag{36}$$

Let $\bar\tau_0 = \tau_0/2$. If $B_{\bar\tau_0 d(y,z)}(y) \subseteq A_z(r_1(1-\tau_0), r_2(1+\tau_0))$, we can use directly the second part of the assumption (30) on $u_{2,n}$ to get the bound on the left-hand side of (36) (taking $\bar\tau_0$ sufficiently small). Moreover, by the first part of (30) on $u_{2,n}$, we deduce

$$\int_{A_z(r_1,r_2)} e^{u_{2,n}}\,dV_g \ge \gamma_0 \int_\Sigma e^{u_{2,n}}\,dV_g \ge \gamma_0 \int_{A_z(r_1,2r_2)} e^{u_{2,n}}\,dV_g.$$

Given then $B_r(y) \subseteq A_z(r_2, 2r_2)$, since $B_r(y) \cap A_z(r_1, r_2) = \varnothing$, by the first inequality in (30),

$$\int_{B_r(y)} e^{u_{2,n}}\,dV_g \le (1-\gamma_0)\int_{A_z(r_1,2r_2)} e^{u_{2,n}}\,dV_g \quad \text{for any } B_r(y) \subseteq A_z(r_2, 2r_2). \tag{37}$$

Now if $B_{\bar\tau_0 d(y,z)}(y) \subseteq A_z(r_2, 2r_2)$, we exploit (37) to deduce the bound on the left-hand side of (36) taking a possibly smaller $\bar\tau_0$. This concludes the proof of the claim (36).

Case 1.2. Suppose $\bar x \in A_z(\frac{5}{4}r_2, 2r_2)$. Using again (37), we obtain a contradiction to (35).

Case 2. Consider now $\bar x \in A_z(r_1/2, r_2)$: reasoning exactly as in Case 1, we get a contradiction.

Case 3. We are left with the case $\bar x \in (A_z(r_1/2, 2r_2))^c$: notice that, differently from the previous two cases, the cut-off functions χ_n might not be identically equal to 1 near $\bar x_0$. For this choice of $\bar x$ and by (34),

$$\frac{\int_{A_z(r_1,r_2)} e^{\tilde u_{2,n}}\,dV_g}{\int_{A_z(\tilde r_{1,n},\tilde r_{2,n})} e^{\tilde u_{2,n}}\,dV_g} \to 0. \tag{38}$$

Using the definition of $\tilde u_{2,n}$ in $A_z(\tilde r_{2,n}/2, \tilde r_{2,n})$ given by (32) and applying Young's inequality with $1/p = \chi_n$ and $1/q = 1-\chi_n$, we have

$$e^{\tilde u_{2,n}} = e^{\chi_n u_{2,n}} e^{(1-\chi_n)\bar u_2(\tilde r_{2,n})} \le \chi_n e^{u_{2,n}} + (1-\chi_n)e^{\bar u_{2,n}(\tilde r_{2,n})} \quad \text{in } A_z(\tilde r_{2,n}/2, \tilde r_{2,n}). \tag{39}$$

Recall the notation in (31): by Jensen's inequality, it follows that

$$e^{\bar{u}_{2,n}(\tilde{r}_{2,n})} \leq \fint_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{u_{2,n}}\, dV_g.$$

Therefore, integrating (39), one can show that

$$\int_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\, dV_g \leq 2 \int_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{u_{2,n}}\, dV_g.$$

Similarly, we get

$$\int_{A_z(\tilde{r}_{1,n},2\tilde{r}_{1,n})} e^{\tilde{u}_{2,n}}\, dV_g \leq 2 \int_{A_z(\tilde{r}_{1,n},2\tilde{r}_{1,n})} e^{u_{2,n}}\, dV_g.$$

In conclusion, we have

$$\int_{A_z(\tilde{r}_{1,n},\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\, dV_g \leq 2 \int_{\Sigma} e^{u_{2,n}}\, dV_g.$$

This, together with (38), implies that

$$\frac{\int_{A_z(r_1,r_2)} e^{u_{2,n}}\, dV_g}{\int_{\Sigma} e^{u_{2,n}}\, dV_g} \leq 2 \frac{\int_{A_z(r_1,r_2)} e^{\tilde{u}_{2,n}}\, dV_g}{\int_{A_z(\tilde{r}_{1,n},\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\, dV_g} \to 0,$$

which is in contradiction with (30). Therefore we are done. $\square$

Lemma 3.2. *Under the same assumptions of Lemma 3.1, let $\tilde{u}_2 \in H^1(\Sigma)$ be the function given there. Then property* (c) *can be extended to the following: there exists $\bar{\tau}_0 > 0$ such that*

$$\sup_{y \in B_{\tilde{r}_2}(z),\, y \neq z} \frac{\int_{B_{\bar{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\, dV_g}{\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g} < 1 - \bar{\tau}_0. \tag{40}$$

Proof. By property (c) of Lemma 3.1, we just have to show (40) for $y \in B_{\tilde{r}_1}(z)$. Observe that, by definition, $\tilde{u}_2$ is constant in $B_{\tilde{r}_1}(z)$. Therefore, for any $B_{\tilde{\tau}_0 d(y,z)}(y) \subseteq B_{\tilde{r}_1}(z)$, which implies $d(y,z) \leq \tilde{r}_1$, we have

$$\int_{B_{\tilde{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\, dV_g = \frac{\tilde{\tau}_0^2 d(y,z)^2}{\tilde{r}_1^2} \int_{B_{\tilde{r}_1}(z)} e^{\tilde{u}_2}\, dV_g \leq \tilde{\tau}_0^2 \int_{B_{\tilde{r}_1}(z)} e^{\tilde{u}_2}\, dV_g \leq \tilde{\tau}_0^2 \int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g,$$

and we conclude that (40) holds for $\tilde{\tau}_0$ small enough. For the same choice of $\tilde{\tau}_0$, we are left with the case $B := B_{\tilde{\tau}_0 d(y,z)}(y) \cap (B_{\tilde{r}_1}(z))^c \neq \varnothing$. The integral over B will be bounded by the integral over a larger ball with center shifted onto $\partial B_{\tilde{r}_1}(z)$. Using normal coordinates at z, consider the shift of center $y \mapsto \tilde{r}_1 y/d(y,z)$. Then we have, using the property (c),

$$\int_B e^{\tilde{u}_2}\, dV_g \leq \int_{B_{\tilde{\tau}_0 \tilde{r}_1}(\tilde{r}_1 y/d(y,z))} e^{\tilde{u}_2}\, dV_g \leq (1 - \tilde{\tau}_0) \int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g.$$

Therefore, we get

$$\int_{B_{\tilde{\tau}_0 d(y,z)(y)}} e^{\tilde{u}_2}\, dV_g \leq \tilde{\tau}_0^2 \int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g + \int_B e^{\tilde{u}_2} dV_g \leq \tilde{\tau}_0^2 \int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g + (1 - \tilde{\tau}_0) \int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\, dV_g.$$

Taking $\bar{\tau}_0$ possibly smaller, we obtain the conclusion. $\square$

We recall here an improved geometric inequality with $k = 1$ and $\alpha = 1$.

Proposition 3.3 [Bartolucci and Malchiodi 2013, Proposition 4.1]. *Let $p \in \Sigma$, and let $r > 0$ and $\tau_0 > 0$. Then for any $\varepsilon > 0$, there exists $C = C(\varepsilon, r)$ such that*

$$\log \int_{B_r(p)} d(x, p)^2 e^{2v} \, dV_g \le \frac{1+\varepsilon}{8\pi} \int_{B_r(p)} |\nabla v|^2 \, dV_g + C$$

for every function $v \in H_0^1(B_r(p))$ such that

$$\sup_{y \in B_r(p),\, y \neq p} \frac{\int_{B_{\tau_0 d(y,p)}(y)} d(x, p)^2 e^{2v} \, dV_g}{\int_{B_r(p)} d(x, p)^2 e^{2v} \, dV_g} < 1 - \tau_0.$$

We now state the new improved Moser–Trudinger inequality.

Remark 3.4. In what follows, the number r is supposed to be small but not tending to 0 while σ could be arbitrarily small.

Proposition 3.5. *Let $r > 0$, $\gamma_0 > 0$, and $\tau_0 > 0$. For any $\varepsilon > 0$, there exists $C = C(\varepsilon, r, \tau_0, \gamma_0)$ such that, if for some $\sigma \in (0, r/C^2)$ and $z \in \Sigma$*

$$\frac{\int_{B_{\sigma/2}(z)} e^{u_1} \, dV_g}{\int_\Sigma e^{u_1} \, dV_g} > \gamma_0, \qquad \frac{\int_{A_z(C\sigma, r/C)} e^{u_2} \, dV_g}{\int_\Sigma e^{u_2} \, dV_g} > \gamma_0 \tag{41}$$

and

$$\sup_{y \in A_z(C\sigma, r/C)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2} \, dV_g}{\int_{A_z(C\sigma, r/C)} e^{u_2} \, dV_g} < 1 - \tau_0, \tag{42}$$

then

$$4\pi \log \int_\Sigma e^{u_1 - \bar{u}_1} \, dV_g + 8\pi \log \int_\Sigma e^{u_2 - \bar{u}_2} \, dV_g \le \int_{B_r(z)} Q(u_1, u_2) \, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g + C.$$

Proof. Taking r sufficiently small, we may suppose that we have the Euclidean flat metric in the ball $B_{Cr}(z)$. Suppose for simplicity that $\bar{u}_1 = \bar{u}_2 = 0$ and that $z = 0$. Observe that we can write

$$\log \int_{B_r(0)} e^{u_2} \, dV_g = \log \int_{B_r(0)} |x|^2 e^{2(u_2/2 - \log|x|)} \, dV_g.$$

We wish to apply Proposition 3.3 to $u_2/2 - \log|x|$, so we need to modify this function in such a way that it becomes constant outside a given ball. Moreover, it will be useful to also replace it with a constant inside a smaller ball. In this process, we should not lose the volume-spreading property (42). By Lemma 3.1, this can be done, and we let $C = C(\varepsilon, \tau_0, \gamma_0)$, $\tilde{r}_1 \in [\sigma, C\sigma/4]$, $\tilde{r}_2 \in [4r/C, r]$, and $\tilde{u}_2 \in H^1(\Sigma)$ be as in the statement of the lemma. By property (a) in Lemma 3.1 and by Lemma 3.2, we are in position to apply

Proposition 3.3 to $(\tilde{u}_2 - \tilde{u}_2(\tilde{r}_2)) \in H_0^1(B_{\tilde{r}_2}(0))$ and get

$$\begin{aligned}
\log \int_\Sigma e^{u_2}\,dV_g &\le \log \int_{A_0(C\sigma,r/C)} e^{u_2}\,dV_g + C = \log \int_{A_0(C\sigma,r/C)} |x|^2 e^{2(u_2/2-\log|x|)}\,dV_g + C \\
&\le \log \int_{B_{\tilde{r}_2}(0)} |x|^2 e^{2\tilde{u}_2}\,dV_g + C = \log \int_{B_{\tilde{r}_2}(0)} |x|^2 e^{2(\tilde{u}_2-\tilde{u}_2(\tilde{r}_2))}\,dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1+\varepsilon}{8\pi} \int_{A_0(\tilde{r}_1,\tilde{r}_2)} |\nabla \tilde{u}_2|^2\,dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1+\varepsilon}{8\pi} \int_{A_0(\tilde{r}_1,\tilde{r}_2)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g + \varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1}{8\pi} \int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\,dV_g + \tilde{u}_2(\tilde{r}_2) + C, \qquad (43)
\end{aligned}$$

where in the first row we exploited (41) while in the last one we used the definitions of $\tilde{r}_1$ and $\tilde{r}_2$. Observe that by the definition (32) of $\tilde{u}_2$ we have

$$\tilde{u}_2(\tilde{r}_2) = ⨍_{A_z(\tilde{r}_2/2,\tilde{r}_2)} (\tfrac{1}{2}u_2 - \log|x|)\,dV_g.$$

Applying Hölder's and Poincaré's inequalities, one gets

$$\begin{aligned}
⨍_{A_z(\tilde{r}_2/2,\tilde{r}_2)} (\tfrac{1}{2}u_2 - \log|x|)\,dV_g &\le ⨍_{A_z(\tilde{r}_2/2,\tilde{r}_2)} |u_2|\,dV_g + \widetilde{C}_r \le C_r \|u_2\|_{L^2(\Sigma)} + \widetilde{C}_r \\
&\le C_r \left(\int_\Sigma |\nabla u_2|^2\,dV_g \right)^{1/2} + \widetilde{C}_r \le \varepsilon \int_\Sigma |\nabla u_2|^2\,dV_g + \frac{\widetilde{C}_r C_r}{\varepsilon}. \qquad (44)
\end{aligned}$$

Inserting the latter estimate into (43), we deduce

$$\log \int_\Sigma e^{u_2}\,dV_g \le \frac{1}{8\pi} \int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g + \varepsilon \int_\Sigma Q(u_1,u_2)\,dV_g + C. \qquad (45)$$

Using the fact that

$$\tfrac{1}{4}|\nabla u_2|^2 = Q(u_1,u_2) - \tfrac{1}{12}|\nabla(u_2+2u_1)|^2,$$

we obtain

$$\begin{aligned}
\int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g &= \frac{1}{4} \int_{A_0(\sigma,r)} |\nabla u_2|^2\,dV_g - 2\pi \log\sigma + 2\pi \bar{u}_2(\sigma) + C \\
&= \int_{A_0(\sigma,r)} Q(u_1,u_2)\,dV_g - \frac{1}{12} \int_{A_0(\sigma,r)} |\nabla(u_2+2u_1)|^2\,dV_g \\
&\qquad - 2\pi \log\sigma + 2\pi \bar{u}_2(\sigma) + C, \qquad (46)
\end{aligned}$$

where $\bar{u}_2(\sigma) = ⨍_{B_\sigma(0)} u_2\,dV_g$.

We claim now that for any $\tilde{\varepsilon} > 0$ one has

$$\int_{A_0(\sigma,r)} |\nabla(u_2+2u_1)|^2\,dV_g \ge 2\pi \left(\frac{2}{\tilde{\varepsilon}} (\bar{u}_2(\sigma) + 2\bar{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2} \log\sigma \right) - \varepsilon \int_\Sigma Q(u_1,u_2)\,dV_g - C. \qquad (47)$$

Letting $v(x) = u_2(x) + 2u_1(x)$, we have to prove

$$\int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g \geq 2\pi \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right),$$

where $\bar{v}(\sigma) = \bar{u}_2(\sigma) + 2\bar{u}_1(\sigma)$. Choose $k \in \mathbb{N}$ such that

$$\int_{A_0(2^k\sigma, 2^{k+1}\sigma)} |\nabla v|^2 \, dV_g \leq \varepsilon \int_\Sigma |\nabla v|^2 \, dV_g,$$

and define

$$\begin{cases} \tilde{u}(x) = \bar{v}(\sigma) & \text{if } x \in B_{2^k\sigma}(0), \\ \Delta \tilde{u}(x) = 0 & \text{if } x \in A_0(2^k\sigma, 2^{k+1}\sigma), \\ \tilde{u}(x) = v(x) & \text{if } x \notin B_{2^{k+1}\sigma}(0). \end{cases}$$

Then there exists a universal constant C_0 such that

$$\begin{aligned} \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g &\leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma |\nabla v|^2 \, dV_g \\ &\leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g. \end{aligned}$$

Solving the Dirichlet problem in $A_0(2^k\sigma, r)$ with constant data $\bar{v}(\sigma)$ on $\partial B_{2^k\sigma}(0)$, one gets

$$\begin{cases} w(x) = A \log \sigma & \text{if } |x| > 2^k\sigma, \\ w(2^k\sigma) = A \log(2^k\sigma) = \bar{v}(\sigma) & \text{if } |x| = 2^k\sigma \end{cases}$$

for some constant A. We have that

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g = 2\pi A^2 \log \frac{1}{2^k\sigma} - C = 2\pi \frac{\bar{v}(\sigma)^2}{\log(1/2^k\sigma)} - C.$$

Moreover,

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g \leq \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g.$$

Finally, using Young's inequality

$$\bar{v}(\sigma) \log \frac{1}{\sigma} \leq \frac{1}{2} \left(\tilde{\varepsilon} \bar{v}(\sigma)^2 + \frac{1}{\tilde{\varepsilon}} \left(\log \frac{1}{\sigma} \right)^2 \right),$$

we end up with

$$\frac{\bar{v}(\sigma)^2}{\log(1/\sigma)} \geq \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right).$$

Therefore, we conclude

$$\begin{aligned} 2\pi \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right) - C &\leq 2\pi \frac{\bar{v}(\sigma)^2}{\log(1/\sigma)} - C = \int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g \\ &\leq \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g \leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g, \end{aligned}$$

which proves the claim (47).

Inserting (47) into (46), we have

$$\int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2 \, dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - \tfrac{1}{12}2\pi\left(\frac{2}{\tilde{\varepsilon}}(\bar{u}_2(\sigma)+2\bar{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2}\log\sigma\right) \\ - 2\pi\log\sigma + 2\pi\bar{u}_2(\sigma) + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

Choosing $\tilde{\varepsilon} = \frac{1}{6}$, we obtain

$$\int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2 \, dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - 4\pi\bar{u}_1(\sigma) - 8\pi\log\sigma \\ + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \quad (48)$$

We use then (48) in (45) to get

$$8\pi\log\int_\Sigma e^{u_2}\, dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\, dV_g - 4\pi\bar{u}_1(\sigma) - 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \quad (49)$$

For the first component, we consider the scalar local Moser–Trudinger inequality (see for example Proposition 2.3 of [Malchiodi and Ruiz 2013]), namely

$$\log\int_{B_{r/2}(0)} e^{u_1}\, dV_g \le \frac{1}{16\pi}\int_{B_r(0)} |\nabla u_1|^2\, dV_g + \bar{u}_1(r) + \varepsilon\int_\Sigma |\nabla u_1|^2\, dV_g + C \\ \le \frac{1}{4\pi}\int_{B_r(0)} Q(u_1,u_2)\, dV_g + \bar{u}_1(r) + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

Performing a dilation to $B_\sigma(0)$, one gets

$$4\pi\log\int_{B_{\sigma/2}(0)} e^{u_1}\, dV_g \le \int_{B_\sigma(0)} Q(u_1,u_2)\, dV_g + 4\pi\bar{u}_1(\sigma) + 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C.$$

We then use the assumption (41), and we obtain

$$4\pi\log\int_\Sigma e^{u_1}\, dV_g \le \int_{B_\sigma(0)} Q(u_1,u_2)\, dV_g + 4\pi\bar{u}_1(\sigma) + 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C. \quad (50)$$

Summing equations (49) and (50), we deduce

$$4\pi\log\int_\Sigma e^{u_1}\, dV_g + 8\pi\log\int_\Sigma e^{u_2}\, dV_g \le \int_{B_r(z)} Q(u_1,u_2)\, dV_g + \varepsilon\int_\Sigma Q(u_1,u_2)\, dV_g + C,$$

which concludes the proof. □

Remark 3.6. The above result is inspired by the work [Bartolucci and Malchiodi 2013] (see in particular Proposition 4.1 there), where the singular Liouville equation is considered. The authors derive a geometric inequality by means of the angular distribution of the conformal volume near the singularities. Somehow the singular equation can be seen as the limit case of the regular one. Roughly speaking, when one component is much more concentrated with respect to the other one, its effect resembles that of a Dirac delta.

3.3. ***Lower bounds on the functional J_ρ.*** We are going to exploit the improved inequality stated in Proposition 3.5 to derive new lower bounds of the energy functional J_ρ defined in (2); see Proposition 3.7. This will give us some extra constraints for the map from the low sublevels of J_ρ onto the topological join $\Sigma_k * \Sigma_1$; see (9).

Given a small $\delta > 0$, our aim is to describe the low sublevels of the functional J_ρ by means of the set

$$Y := (\Sigma_k * \Sigma_1) \setminus S \subseteq \Sigma_k * \Sigma_1, \tag{51}$$

where

$$S = \left\{ \left(\nu, \delta_z, \tfrac{1}{2}\right) \in \Sigma_k * \Sigma_1 : \nu = \sum_{i=1}^{k} t_i \delta_{x_i},\ d(x_i, x_j) \geq \delta \text{ for all } i \neq j,\ \delta \leq t_i \leq 1-\delta \text{ for all } i,\ z \in \operatorname{supp}(\nu) \right\}. \tag{52}$$

We will show that there is a lower bound for J_ρ whenever $\widetilde{\Psi}$, which is defined in (29), has image inside S; see Proposition 3.7.

Consider $\mathscr{C}_{\varepsilon,r}(x_0)$ as given in (17), $f \in \mathscr{C}_{\varepsilon,r}(x_0)$, and ψ defined in (19). Before stating the next main result, we recall some properties of the map ψ; see Proposition 3.1 in [Malchiodi and Ruiz 2013] (with minor changes).

Fact. *Let $\psi(f) = (\beta, \sigma)$. Then given $R > 1$, there exists $p \in \Sigma$ with the properties*

$$d(p,\beta) \leq C'\sigma \quad \textit{for some } C' = C'(R), \qquad \int_{B_\sigma(p)\cap B_r(x_0)} f\, dV_g > \tau, \qquad \int_{(B_{R\sigma}(p))^c \cap B_r(x_0)} f\, dV_g > \tau, \tag{53}$$

where τ depends only on R and Σ.

Recall also the distance $\boldsymbol{d}$ between measures in (11), the numbers $\varepsilon_i > 0$ in Proposition 2.4, the projections $\widetilde{\psi}_k$ and $\widetilde{\psi}_1$ in (27)–(28), and the definition of the parameter s in the topological join given by (26).

Proposition 3.7. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$ and that $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < 2\varepsilon_k$ and $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < \varepsilon_1$. Let*

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) = \sum_{i=1}^{k} t_i \delta_{x_i}, \qquad \widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_{\beta_z}.$$

There exist $\delta > 0$ and $L > 0$ such that, if the properties

(1) *$d(x_i, x_j) \geq \delta$ for all $i \neq j$ and $t_i \in [\delta, 1-\delta]$ for all $i = 1, \dots, k$,*

(2) *$s(u_1, u_2) = \frac{1}{2}$, and*

(3) *$\beta_z = x_l$ for some $l \in \{1, \dots, k\}$*

hold true, then

$$J_\rho(u_1, u_2) \geq -L.$$

Proof. Suppose without loss of generality that $\bar{u}_1 = \bar{u}_2 = 0$. We first observe that exploiting the assumption $s(u_1, u_2) = \frac{1}{2}$ we deduce $\sigma_1(u_1) = \sigma_2(u_2)$. Secondly, it is not difficult to show that from property (1) it follows that $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \geq 2\varepsilon_{k-1}$. Therefore, by the definition of $\widetilde{\psi}_k$, we deduce that $x_i = \beta_{x_i^k}$ for $i = 1, \ldots, k$, where the $\beta_{x_i^k}$ are the local centers of mass given by (23). Hence, we get

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) = \sum_{i=1}^{k} t_i \delta_{\beta_{x_i^k}}.$$

Recalling that we have set (see Section 3.1)

$$\sigma_2(u_2) = \mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1)\big)\sigma_z + \big(1 - \mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1)\big)\big),$$

using the fact that $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < \varepsilon_1$, by the definition of $\mathfrak{g}_1$ in (24), $\sigma_2(u_2)$ reduces to σ_z. We recall now also the definition of $\sigma_1(u_1)$, namely

$$\sigma_1(u_1) = \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)A(u_1, u_2) + \big(1 - \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)\big)$$

with $A(u_1, u_2)$ defined in Section 3.1. Assuming $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < 2\varepsilon_k$ implies $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)) > 0$. Again, using property (1), we obtain from $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \geq 2\varepsilon_{k-1}$ that $\mathfrak{g}_{k-1}(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1})) = 0$ and hence $A(u_1, u_2) = B(u_1, u_2)$ (see the notation before (26)). Moreover, the property (3) implies that $\mathfrak{h}(d(\beta_z, \{\beta_{x_1^k}, \ldots, \beta_{x_k^k}\})) = 1$. Therefore, $B(u_1, u_2) = \sigma_{x_l^k}$. Hence, one finds

$$\sigma_{u_1} = \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)\sigma_{x_l^k} + \big(1 - \mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)\big)\big).$$

We distinguish between two cases.

Case 1. Suppose first that $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)) = 1$. In this case, we obtain $\sigma_{x_l^k} = \sigma_1(u_1) = \sigma_2(u_2) = \sigma_z$. By this fact and by property (3), we get $(\beta_{x_l^k}, \sigma_{x_l^k}) = (\beta_z, \sigma_z)$. Let $r = \delta/4$: from (53) and the definitions of β_z and $\beta_{x_i^k}$, there exists $\tilde{\gamma}_0 > 0$ such that

$$\int_{B_r(\beta_{x_i^k})} e^{u_1}\, dV_g \geq \tilde{\gamma}_0 \int_\Sigma e^{u_1}\, dV_g \quad \text{for } i = 1, \ldots, k, \qquad \int_{B_r(\beta_z)} e^{u_2}\, dV_g \geq \tilde{\gamma}_0 \int_\Sigma e^{u_2}\, dV_g. \tag{54}$$

Therefore, we are in position to apply Proposition 2.12 and get

$$4(k+1)\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C_r.$$

The conclusion then follows from the expression of J_ρ and from the upper bounds on ρ_1 and ρ_2.

Case 2. Suppose now $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)) < 1$: we deduce immediately that $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) \in (\varepsilon_k, 2\varepsilon_k)$.

Given $\varepsilon > 0$, let $R = R(\varepsilon)$ be such that Proposition 2.10 holds true. Let $C' = C'(R)$ and $\tau = \tau(R)$ be as in (53). Take $\tau_0 = \tau/100$ and $\gamma_0 = \tilde{\gamma}_0\tau$, where γ_0 is given as in (54), and let $C = C(\varepsilon, r, \tau_0, \gamma_0)$ be the constant obtained in Proposition 3.5. We then define $\widetilde{C} = \max\{C', C\}$. Moreover, observe that by construction $\sigma_{x_l^k} \leq \sigma_1(u_1) = \sigma_2(u_2) = \sigma_z$.

If $\sigma_{x_l^k} \leq \sigma_z \leq \widetilde{C}^8 \sigma_{x_l^k}$, we still can apply Proposition 2.12 as before; see Remark 2.11. Consider now the case $\widetilde{C}^8 \sigma_{x_l^k} \leq \sigma_z$. We distinguish between two situations.

Case 2.1. If r is as in Case 1, suppose that

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g \tag{55}$$

(the right side exceeds $\tilde{\gamma}_0\tau_0 \int_\Sigma e^{u_2}\,dV_g$; see (54)). By the fact that $\widetilde{C}^4\sigma_{x_l^k} \ll \sigma_z$, from (53), we also get

$$\int_{(B_{R\widetilde{C}^4\sigma_{x_l^k}}(\beta_z))^c\cap B_r(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g > \tilde{\gamma}_0\tau_0 \int_\Sigma e^{u_2}\,dV_g. \tag{56}$$

The conditions on the local scale of u_1, given by $(\beta_{x_l^k}, \sigma_{x_l^k}) = \psi(f_{\mathrm{loc}}^{x_l^k}(u_1))$, yield by (53) the existence of $p \in \Sigma$ such that

$$\int_{B_{\sigma_{x_l^k}}(p)} e^{u_1}\,dV_g > \tau \int_{B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g,$$

$$\int_{(B_{R\sigma_{x_l^k}}(p))^c\cap B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tau \int_{B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g.$$

The latter formulas, together with (55) and (56), imply an improved Moser–Trudinger inequality (see Remarks 2.9 and 2.11):

$$8\pi\left(\log \int_\Sigma e^{u_1}\,dV_g + \log \int_\Sigma e^{u_2}\,dV_g\right) \le (1+\varepsilon) \int_{B_r(\beta_z)} Q(u_1,u_2)\,dV_g + C_0(\varepsilon, r, \tau, \tilde{\gamma}_0). \tag{57}$$

Case 2.2. Suppose now that the second situation occurs, namely

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(z)} e^{u_2}\,dV_g \le \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g. \tag{58}$$

The goal is to apply the improved inequality stated in Proposition 3.5. Take $\sigma = (C')^2\sigma_{x_l^k}$ and $A_{\beta_z}(C\sigma, r/C)$ as the annulus on which we will test the conditions (41)–(42). We start by considering (41). Observe that

$$\int_{B_{\sigma/2}(z)} e^{u_1}\,dV_g > \gamma_0 \int_\Sigma e^{u_1}\,dV_g$$

follows from (53) and (54) by the choice of σ and γ_0. Similarly, using the volume concentration of u_2 in $(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)$ in (53) and (recalling the definition of $\widetilde{C}$) $C\sigma \ll R\sigma_z$, we get

$$\int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\,dV_g > \gamma_0 \int_\Sigma e^{u_2}\,dV_g$$

by taking ε_1 sufficiently small in Proposition 3.7. We are left by proving condition (42), i.e.,

$$\sup_{y\in A_{\beta_z}(C\sigma, r/C)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\,dV_g} < 1-\tau_0.$$

If this is not the case, then there exists $y \in A_{\beta_z}(C\sigma, r/C)$ such that

$$\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g \geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g.$$

Using the assumption (58) and $\sigma < \widetilde{C}^4 \sigma_{x_l^k}$, we get

$$\begin{aligned}\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g &\geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g \geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g \\ &= (1-\tau_0) \int_{B_r(\beta_z)} e^{u_2}\, dV_g - (1-\tau_0) \int_{B_{C\sigma}(\beta_z)} e^{u_2}\, dV_g \geq (1-2\tau_0) \int_{B_r(\beta_z)} e^{u_2}\, dV_g.\end{aligned}$$

Moreover, by the property of the local scale of u_2 given by $(\beta_z, \sigma_z) = \psi(f^z_{\mathrm{loc}}(u_2))$ (see (53)), we have

$$\int_{B_{\sigma_z}(p)} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g, \qquad \int_{(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g.$$

Notice that by the choice of τ_0 the three properties above cannot hold simultaneously. Hence, we have a contradiction. Finally, we are in position to apply Proposition 3.5 and deduce that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C.$$

Observe that by the latter formula and by (57), in both Cases 2.1 and 2.2, we can assert that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C. \tag{59}$$

Recall that under Case 2 we have $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) > \varepsilon_k$. By the second part of Proposition 2.4 (applied with $l = k+1$), there exist $\bar{\varepsilon}_k > 0$, depending only on ε_k, and $k+1$ points $\bar{x}_1, \ldots, \bar{x}_{k+1}$ such that

$$d(\bar{x}_i, \bar{x}_j) > 2\bar{\varepsilon}_k \quad \text{for } i \neq j, \qquad \int_{B_{\bar{\varepsilon}_k}(\bar{x}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \ldots, k+1.$$

Without loss of generality, we can assume $\delta < \bar{\varepsilon}_k/8$. By this the choice of δ, there exist k points $\bar{y}_1, \ldots, \bar{y}_k$ such that

$$d(\bar{y}_i, \bar{y}_j) > \bar{\varepsilon}_k \quad \text{for } i \neq j, \qquad d(\bar{y}_i, \beta_{x_i^k}) > \delta \quad \text{for all } i = 1, \ldots, k,$$
$$\int_{B_{\bar{\varepsilon}_k}(\bar{y}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \ldots, k.$$

We perform then a local Moser–Trudinger inequality for u_1 in each region (see (50)), and summing up, we have (recall that $r = \delta/4$)

$$4k\pi \log \int_\Sigma e^{u_1}\, dV_g \leq \int_{(B_r(\beta_{x_l^k}))^c} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C_r, \tag{60}$$

where the average was estimated using Hölder's and Poincaré's inequalities as in (44). By summing (59) and (60), we deduce

$$4(k+1)\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \le (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C,$$

so we conclude as in Case 1. □

By Proposition 3.7, we obtain:

Corollary 3.8. *Let S be as in (52), and let $Y = (\Sigma_k * \Sigma_1) \setminus S$. Then, for $\tilde{L} > 0$ large, $\widetilde{\Psi}$ (defined in (29)) maps the low sublevels $J_\rho^{-\tilde{L}}$ into the set Y.*

4. Test functions

We show that the lower bound in Proposition 3.7 is optimal; see also Corollary 3.8. In fact, we will construct suitable test functions modeled on Y on which J_ρ attains arbitrarily negative values.

To describe our construction, let us recall the test functions employed for the scalar case (5). When $\rho > 4\pi$, as mentioned in Section 1, the energy I_ρ in (6) is unbounded below. One can see that using test functions of the type

$$\varphi_{\lambda,z}(x) = \log\left(\frac{\lambda}{1+\lambda^2 d(x,z)^2}\right)^2, \tag{61}$$

for a given point $z \in \Sigma$ and for $\lambda > 0$, as $\lambda \to +\infty$, these satisfy the properties

$$e^{\varphi_{\lambda,z}} \rightharpoonup \delta_z \quad \text{and} \quad I_\rho(\varphi_{\lambda,z}) \to -\infty \quad (\rho > 4\pi), \tag{62}$$

holding uniformly in $z \in \Sigma$. More generally, if $\rho \in (4k\pi, 4(k+1)\pi)$, a natural family of test functions can be modeled on Σ_k [Djadli 2008; Djadli and Malchiodi 2008]. In fact, setting

$$\varphi_{\lambda,\nu}(x) = \log \sum_{i=1}^k t_i \left(\frac{\lambda}{1+\lambda^2 d(x,x_i)^2}\right)^2, \qquad \nu = \sum_{i=1}^k t_i \delta_{x_i}, \tag{63}$$

similarly to (62), for $\lambda \to +\infty$, one has uniformly in $\nu \in \Sigma_k$

$$\boldsymbol{d}(e^{\varphi_{\lambda,\nu}}, \nu) \to 0 \quad \text{and} \quad I_\rho(\varphi_{\lambda,\nu}) \to -\infty \quad (\rho \in (4k\pi, 4(k+1)\pi)).$$

When dealing with the energy functional J_ρ in (2), one can expect to interpolate between the $\varphi_{\lambda,\nu}$ for the component u_1 and the $\varphi_{\lambda,z}$ for u_2 when $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Therefore, the topological join $\Sigma_k * \Sigma_1$ represents a natural object to globally parametrize this family with the join parameter s playing the role of interpolation parameter. However, as mentioned in Section 1, the cross term in the quadratic energy penalizes gradients pointing in the same direction. By this reason, not all elements in $\Sigma_k * \Sigma_1$ will give rise to test functions with low energy. It will turn out that the subset Y of $\Sigma_k * \Sigma_1$ (see (51)) will be the right one at which to look.

4.1. ***A convenient deformation of*** $Y \cap \{s = \frac{1}{2}\}$**.** We construct here a continuous deformation of $Y \cap \{s = \frac{1}{2}\}$, which is relatively open in the join $\Sigma_k * \Sigma_1$, onto some closed subset: see Corollary 4.6. This will allow us to build test functions depending on a compact space of parameters, which is easier. Before doing this, we recall some facts from Section 3 of [Malchiodi 2008a].

There exists a deformation retract $H_0(\mathfrak{t}, \cdot)$ of a neighborhood (with respect to the metric induced by $\boldsymbol{d}$ in (11)) of Σ_{k-1} in Σ_k onto Σ_{k-1}. To see this, one can take a positive δ_1 small enough and consider a nonincreasing continuous function $F_0 : (0, +\infty) \to (0, +\infty)$ such that

$$F_0(t) = \frac{1}{t} \quad \text{for } t \in (0, \delta_1], \qquad F_0(t) = \frac{1}{2\delta_1} \quad \text{for } t > 2\delta_1. \tag{64}$$

We then define $F : \Sigma_k \setminus \Sigma_{k-1} \to \mathbb{R}$ as

$$F\left(\sum_{i=1}^{k} t_i \delta_{x_i}\right) = \underbrace{\sum_{i \neq j} F_0(d(x_i, x_j))}_{F_1((x_i)_i)} + \underbrace{\sum_{i=1}^{k} \frac{1}{t_i(1 - t_i)}}_{F_2((t_i)_i)}. \tag{65}$$

Notice that F is well-defined on $\Sigma_k \setminus \Sigma_{k-1}$ as it is invariant under permutation of the couples $(t_i, x_i)_{i=1,\dots,k}$. Observe also that it tends to $+\infty$ as its argument approaches Σ_{k-1}. Moreover, the gradient of F with respect to the metric of $\Sigma^k \times T_0$ (where T_0 is the simplex containing the k-tuple $T := (t_i)_i$) tends to $+\infty$ in norm as $\sum_{i=1}^{k} t_i \delta_{x_i}$ tends to Σ_{k-1}. It follows that, sending L to $+\infty$, we get a deformation retract of $F_L := \{F \geq L\} \cup \Sigma_{k-1}$ onto Σ_{k-1} for L sufficiently large. We then obtain H_0 by a reparametrization of the (positive) gradient flow of F.

We introduce now the set $\widetilde{Y}_{1/2} \subseteq Y \cap \{s = \frac{1}{2}\} \subseteq \Sigma_k * \Sigma_1$ defined as

$$\widetilde{Y}_{1/2} = \{(\nu, \delta_z, \tfrac{1}{2}) : \nu \in \Sigma_{k-1}\} \cup \{(\nu, \delta_z, \tfrac{1}{2}) : \nu \in \Sigma_k \setminus \Sigma_{k-1},\ z \notin \operatorname{supp}(\nu)\}.$$

Lemma 4.1. *There exists a continuous deformation $\widetilde{H}(\mathfrak{t}, \cdot)$ of the set $Y \cap \{s = \frac{1}{2}\}$ onto $\widetilde{Y}_{1/2}$.*

Proof. Let $\delta > 0$ be as in (52). Consider $0 < \tilde{\delta} \ll \delta$, and let $\tilde{f} : (0, +\infty) \to (0, +\infty)$ be a nonincreasing continuous function given by

$$\tilde{f}(t) = \begin{cases} 1/t^2 & \text{in } t \leq \tilde{\delta}, \\ 0 & \text{in } t \geq 2\tilde{\delta}. \end{cases}$$

Moreover, recall the deformation retract $H_0(\mathfrak{t}, \cdot)$ of a neighborhood of Σ_{k-1} in Σ_k onto Σ_{k-1} constructed above. To define $\widetilde{H}$, we distinguish among four situations, fixing $\hat{\delta} \ll \tilde{\delta}$ (in particular, we take $\hat{\delta}$ so small that H_0 is well-defined on the $3\hat{\delta}$-neighborhood of Σ_{k-1} in the metric $\boldsymbol{d}$).

(i) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \leq \hat{\delta}$. Recall that elements in $Y \cap \{s = \frac{1}{2}\}$ are triples of the form $(\nu, \delta_z, \frac{1}{2})$ with $\nu \in \Sigma_k$. In this first case, we project ν onto Σ_{k-1} while δ_z remains fixed. If H_0 is the retraction described above, we simply define $\widetilde{H}$ to be

$$\widetilde{H}(\mathfrak{t}, \nu, \delta_z, \tfrac{1}{2}) = (H_0(\mathfrak{t}, \nu), \delta_z, \tfrac{1}{2}).$$

(ii) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \in [\hat{\delta}, 2\hat{\delta}]$. Let

$$\nu_1(\mathrm{t}) = H_0(\mathrm{t}, \nu) = \sum_{i=1}^{k} t_i(\mathrm{t})\delta_{x_i(\mathrm{t})}.$$

If $\tilde{f}$ is as before, we introduce the following flow acting on the support of δ_z:

$$\frac{d}{d\mathrm{t}} z(\mathrm{t}) = \sum_{i=1}^{k} t_i(\mathrm{t}) f\big(d(z(\mathrm{t}), x_i(\mathrm{t}))\big) \nabla_z d(z(\mathrm{t}), x_i(\mathrm{t})). \tag{66}$$

To define $\widetilde{H}$ in this case, we interpolate from a constant motion in z and (66) depending on $\boldsymbol{d}(\nu, \Sigma_{k-1})$:

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu_1(\mathrm{t}), \delta_{z(\mathrm{t}(\boldsymbol{d}(\nu, \Sigma_{k-1}) - \hat{\delta})/\hat{\delta})}, \tfrac{1}{2}\big).$$

Notice that when $\boldsymbol{d}(\nu, \Sigma_{k-1}) = 2\hat{\delta}$ we get $z(\mathrm{t}(\boldsymbol{d}(\nu, \Sigma_{k-1}) - \hat{\delta})/\hat{\delta}) = z(\mathrm{t})$ and this point never intersects the support of $\nu_1(\mathrm{t})$ unless $\nu_1(\mathrm{t}) \in \Sigma_{k-1}$. Therefore, as for case (i), $\widetilde{H}(1, \nu, \delta_z, \frac{1}{2}) \in \widetilde{Y}_{1/2}$.

(iii) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \in [2\hat{\delta}, 3\hat{\delta}]$. In this case, the evolution of ν interpolates between the projection onto Σ_{k-1} and staying fixed; i.e., we set

$$\nu_2(\mathrm{t}) = H_0\bigg(\mathrm{t}\frac{3\hat{\delta} - \boldsymbol{d}(\nu, \Sigma_{k-1})}{\hat{\delta}}, \nu\bigg)$$

and let $z(\mathrm{t})$ evolve according to (66) with $t_i(\mathrm{t})$ and $x_i(\mathrm{t})$ given by $\sum_{i=i}^{k} t_i(\mathrm{t})\delta_{x_i(\mathrm{t})} = \nu_2(\mathrm{t})$, so we define $\widetilde{H}$ as

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu_2(\mathrm{t}), \delta_{z(\mathrm{t})}, \tfrac{1}{2}\big).$$

(iv) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \geq 3\hat{\delta}$. The deformation $\widetilde{H}$ now leaves ν fixed while we let $z(\mathrm{t})$ evolve by (66) with $t_i(\mathrm{t}) \equiv t_i$ and $x_i(\mathrm{t}) \equiv x_i$:

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu, \delta_{z(\mathrm{t})}, \tfrac{1}{2}\big).$$

Observe that in this case, by the definition of $\tilde{f}$ and by the choice of $\tilde{\delta}$, the latter flow of z does not intersect the support of ν and $d(z, z(1)) = O(\tilde{\delta})$. □

We next *slice* the set $\widetilde{Y}_{1/2}$ in the second entry δ_z: for $p \in \Sigma$, we introduce $\widetilde{Y}_{(1/2,p)} \subseteq \Sigma_k$ given by

$$\widetilde{Y}_{(1/2,p)} = \big\{\nu \in \Sigma_k : \big(\nu, \delta_p, \tfrac{1}{2}\big) \in \widetilde{Y}_{1/2}\big\}, \tag{67}$$

so that

$$\widetilde{Y}_{1/2} = \bigcup_{p \in \Sigma} \big(\widetilde{Y}_{(1/2,p)}, \delta_p, \tfrac{1}{2}\big).$$

In Proposition 4.4, we will further deform $\widetilde{Y}_{(1/2,p)}$ to some compact subset of Σ_k (depending on p).

Let $\delta_2 > 0$ be a small number, $p \in \Sigma$, and χ_{δ_2} a cut-off function such that

$$\chi_{\delta_2} = \begin{cases} 0 & \text{in } B_{\delta_2}(p), \\ 1 & \text{in } (B_{2\delta_2}(p))^c. \end{cases} \tag{68}$$

We start by proving the following lemmas (we are extending the notation in (8) to any subset of Σ):

Lemma 4.2. *Let $p \in \Sigma$, and let $\delta_2 > 0$ be as before. There exists $\delta_3 > 0$ sufficiently small such that the above-defined map $H_0(\mathfrak{t}, \cdot)$ is a deformation retract of*

$$\left\{ \nu \in \widetilde{Y}_{(1/2,p)} : \int_\Sigma \chi_{\delta_2}\, d\nu \ge \delta_2,\ \boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2} \right) \in (0, \delta_3) \right\} \cap \{ \boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3 \}$$

onto $(\Sigma \setminus \{p\})_{k-1}$ with the property that for all $\mathfrak{t} \in [0, 1]$ we have $p \notin \operatorname{supp} H_0(\mathfrak{t}, \nu)$.

Proof. Let δ_1 be as in (64). We can assume that $\delta_1 \le \delta_2/16$. We first prove that $H_0(\mathfrak{t}, \cdot)$ has the property that, as the $\boldsymbol{d}$-distance of ν from Σ_{k-1} tends to 0, the support of the measure $H_0(\mathfrak{t}, \nu)$ is contained in a shrinking neighborhood of the support of ν (uniformly in ν). We will then show that H_0 restricted to the particular set considered in the statement gives the desired deformation retract.

To prove the first assertion, we endow Σ^k, to which the k-tuple $X := (x_i)_i$ belongs, with the product metric and the simplex T_0, containing the k-tuple $T := (t_i)_i$, with its standard metric induced from $\mathbb{R}^k$. Then one can notice that, as the singularities of F_1 and F_2 behave like the inverse of the distance from the boundaries of their domains, there exists a constant C such that

$$\frac{1}{C} F_1(X)^2 - C \le |\nabla_X F_1(X)| \le C F_1(X)^2 + C, \qquad \frac{1}{C} F_2(T)^2 - C \le |\nabla_T F_2(T)| \le C F_2(T)^2 + C. \quad (69)$$

We now consider the evolution $\mathfrak{s} \mapsto \zeta(\nu, \mathfrak{s})$ with initial datum ν in a small neighborhood of Σ_{k-1}, where, we recall, F attains large values and its gradient does not vanish. If we evolve by the gradient of F, then X evolves by the gradient of F_1 and T by the gradient of F_2. By the last formula, we then have

$$\left| \frac{dX}{d\mathfrak{s}} \right| = |\nabla_X F_1| \le C F_1(X)^2 + C.$$

On the other hand, still by (69), we have that

$$\frac{dF}{d\mathfrak{s}} = |\nabla_X F_1(X)|^2 + |\nabla_T F_2(T)|^2 \ge \frac{1}{C^2} F_1(X)^4 + \frac{1}{C^2} F_2(T)^4 - 2C.$$

Notice that this quantity is strictly positive if F is large enough (see (65)), which allows us to invert the function $\mathfrak{s} \mapsto F(\zeta(\nu, \mathfrak{s}))$. Therefore, if $\mathfrak{s}_\nu$ is the maximal time of existence for $\zeta(\nu, \mathfrak{s})$, we can write that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} = \int_{F(\nu)}^{\infty} \left| \frac{dX}{d\mathfrak{s}} \right| \frac{1}{dF/d\mathfrak{s}}\, dF.$$

By the above two inequalities, we deduce that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \le \int_{F(\nu)}^{\infty} \frac{C F_1(X)^2 + C}{F_1(X)^4/C^2 + F_2(T)^4/C^2 - 2C}\, dF.$$

By elementary inequalities, recalling that $F = F_1(X) + F_2(T)$, we also find

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \le \widetilde{C} \int_{F(\nu)}^{\infty} \frac{1}{F^2 - \widetilde{C}}\, dF.$$

Therefore, as ν approaches Σ_{k-1}, namely for $F(\nu)$ large, we find that the displacement of X becomes smaller and smaller. This gives us the claim stated at the beginning of the proof.

Next, we observe that, by having $\nu \in \widetilde{Y}_{(1/2,p)}$ and $\boldsymbol{d}(\chi_{\delta_2}\nu/\|\chi_{\delta_2}\nu\|, \Sigma_{k-2}) > 0$ by assumption, it follows that there exists at most one point of the support of ν in the ball $B_{(3/4)\delta_2}(p)$ that does not coincide with p. Moreover, by the above claim, we have that the points outside $B_{\delta_2}(p)$ following the flow induced by F move by a distance of order $o_{\delta_3}(1)$ since $\boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3$. Therefore, choosing δ_3 sufficiently small, we get the existence of at most one point in the ball $B_{(3/4)\delta_2}(p)$, different from p, even while the flow is acting.

By the choice of F_1 (see (64)–(65)) and by the choice $\delta_1 \le \delta_2/16$, we deduce that the point inside $B_{(3/4)\delta_2}(p)$ is not affected by the flow and in particular does not collapse onto p. □

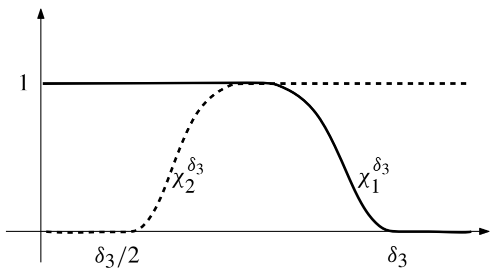
Lemma 4.3. *There exists a deformation retract $H(\mathrm{t}, \cdot)$ of $\{\nu \in \widetilde{Y}_{(1/2,p)} : \int_\Sigma \chi_{\delta_2}\, d\nu \ge \delta_2\}$ to the set*

$$\mathcal{B} := (\Sigma \setminus B_{\delta_2}(p))_k \cup \big\{\mathrm{card}\big((\mathrm{supp}(\nu)) \setminus B_{\delta_2}(p)\big) \le k-2\big\}.$$

Proof. Let us first consider a deformation retract that pushes points in $\Sigma \setminus \{p\}$ away from p. Define $H_1(\mathrm{t}, \cdot)$, $\mathrm{t} \in [0, 1]$, as follows: if $\nu = \sum_{i=1}^k t_i \delta_{x_i}$, $x_i \ne p$, then (using normal coordinates around p)

$$H_1(\mathrm{t}, \nu) = \sum_{i=1}^k t_i \delta_{x_{i,\mathrm{t}}}, \quad \text{where } x_{i,\mathrm{t}} = \begin{cases} \dfrac{x_i}{|x_i|}((1-\mathrm{t})|x_i| + \mathrm{t}\delta_2) & \text{if } d(p, x_i) < \delta_2, \\ x_i & \text{if } d(p, x_i) \ge \delta_2. \end{cases}$$

We next introduce two cut-off functions $\chi_1^{\delta_3}$ and $\chi_2^{\delta_3}$ ($\chi_2^{\delta_3}$ corresponds to the dashed graph):



For $\{\boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3\}$, we define the deformation retract $H_2(\mathrm{t}, \cdot)$ as an *interpolation* between the homotopies H_0 and H_1, precisely

$$H_2(\mathrm{t}, \nu) = H_1\left(\mathrm{t}\chi_2^{\delta_3}\left(\boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), H_0\left(\mathrm{t}\chi_1^{\delta_3}\left(\boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), \nu\right)\right).$$

The introduction of the cut-off functions makes the deformation retract continuous with respect to the topology induced by the $\boldsymbol{d}$-distance.

For $\boldsymbol{d}(\nu, \Sigma_{k-1})$ arbitrary, we instead define H as

$$H(\mathrm{t}, \nu) = H_1\big(\mathrm{t}\chi_2^{\delta_3}(\boldsymbol{d}(\nu, \Sigma_{k-1})), H_2\big(\chi_1^{\delta_3}(\boldsymbol{d}(\nu, \Sigma_{k-1})), \nu\big)\big).$$

Again, notice that the cut-off functions in the first argument of H_1 give continuity in ν. □

The main result of this subsection is the following proposition: we retract $\widetilde{Y}_{(1/2,p)}$ to a set of measures $\Sigma_{k,p,\bar{\tau}}$ (see (70)) for which either the support is bounded away from p or for which there are at most $k-2$ points not closest to p. As we will see, these conditions will be helpful to find suitable test functions with low Euler–Lagrange energy; see the next subsections.

Proposition 4.4. *There exist $\bar\tau \gg 1$ and a retraction $\mathcal{R}_p$ of $\widetilde{Y}_{(1/2,p)}$ to the set*

$$\Sigma_{k,p,\bar\tau} = \left\{ \nu = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k : d(x_i, p) \geq \frac{1}{\bar\tau} \text{ for all } i \right\} \\ \cup \left\{ \nu = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k : \operatorname{card}\{x_j : d(x_j, p) > \min_i d(x_i, p)\} \leq k-2 \right\}. \tag{70}$$

Proof. Recall first the definition (68) of χ_{δ_2}. We then extend the result in Lemma 4.3 to arbitrary values of $m_2(\nu) = \int_\Sigma \chi_{\delta_2}\, d\nu$, namely also for $m_2 < \delta_2$, finding a retraction onto $\mathcal{B}$. Consider normal coordinates around p. Define $m(\nu) = \big\| \nu\big(\chi_{\delta_2}(m_2(\nu)) + (1-\chi_{\delta_2}(|x|))(1-\chi_{\delta_2}(m_2(\nu)))\big)\big\|$, and let

$$T(\nu) = \begin{cases} \dfrac{\nu\big(\chi_{\delta_2}(m_2(\nu)) + (1-\chi_{\delta_2}(|x|))(1-\chi_{\delta_2}(m_2(\nu)))\big)}{m(\nu)} & \text{if } m_2(\nu) < 2\delta_2, \\ \nu & \text{if } m_2(\nu) \geq 2\delta_2. \end{cases}$$

We then define the retraction as

$$\widetilde{R}(\nu) = T\big(H(\chi_{\delta_2}(m_2(\nu)), \nu)\big).$$

Let $\nu_H = H(\chi_{\delta_2}(m_2(\nu)), \nu)$. To have $\widetilde{R}$ well-defined, we need to ensure that whenever T is acting, namely for $m_2(\nu_H) < 2\delta_2$, we have $m(\nu_H) > 0$. Clearly, it is enough to show that

$$\int_\Sigma (1-\chi_{\delta_2})\, d\nu_H > 0. \tag{71}$$

We point out that

$$m_2(\nu_H) + \int_\Sigma (1-\chi_{\delta_2})\, d\nu_H = 1.$$

Therefore, by $m_2 < 2\delta_2$, we obtain

$$\int_\Sigma (1-\chi_{\delta_2})\, d\nu_H > 1 - 2\delta_2.$$

Finally, we construct a retraction of $\mathcal{B}$ onto $\Sigma_{k,p,\bar\tau}$. For $\nu \in \mathcal{B}$ with $\|(1-\chi_{\delta_2})\nu\| > 0$, we define a parameter $\tau = \tau(\nu) \in (0, +\infty]$ in the following way:

$$\frac{1}{\tau} = \boldsymbol{d}\left(\frac{(1-\chi_{\delta_2})\nu}{\|(1-\chi_{\delta_2})\nu\|}, \delta_p \right). \tag{72}$$

Consider normal coordinates around p. Let $\bar\tau \gg 1$ be such that $1/\bar\tau \ll \delta_2 \ll 1$, and let $f : \mathcal{B} \times \Sigma \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be two smooth functions such that

$$f(\nu, x) = \begin{cases} 0 & \text{if } \tau = +\infty, \\ 1/\tau & \text{if } \tau < +\infty \text{ and } |x| \leq 1/\bar\tau, \\ |x| & \text{if } \tau < +\infty \text{ and } |x| \geq 2/\bar\tau, \end{cases} \qquad g(t) = \begin{cases} t & \text{if } t \leq 1/\bar\tau, \\ 1 & \text{if } t \geq 2/\bar\tau. \end{cases}$$

For $\nu = \sum_{i=1}^k s_i \delta_{y_i} \in \mathcal{B}$ with $\|(1-\chi_{\delta_2})\nu\| > 0$, we consider $(1-\chi_{\delta_2})\nu = \sum_{i=1}^k t_i \delta_{x_i}$ and then define

$$\tilde\nu = \frac{\sum_{i=1}^k t_i g(|x_i|) \delta_{(x_i/|x_i|) f(\nu, x_i)}}{\sum_{i=1}^k t_i g(|x_i|)}. \tag{73}$$

Observe that, for $d(x_i, p) \le 1/\bar{\tau}$ for all i, (73) reads as

$$\tilde{\nu} = \frac{\sum_{i=1}^k t_i |x_i| \delta_{(x_i/|x_i|)(1/\tau)}}{\sum_{i=1}^k t_i |x_i|}$$

while, for $d(x_i, p) \ge 2/\bar{\tau}$ for all i, we obtain $\tilde{\nu} = \sum_{i=1}^k t_i \delta_{x_i}$.

For a general $\nu \in \mathcal{B}$, the retraction is given by

$$\mathcal{R}_p(\nu) = (1 - m_2)\tilde{\nu} + \chi_{\delta_2} \nu. \tag{74}$$

Observe that, when $\|(1 - \chi_{\delta_2})\nu\| = 0$, τ is not defined. However, the map $\mathcal{R}_p(\nu)$ is well-defined since in this case we have $m_2 = 1$. Notice furthermore that $\mathcal{R}_p(\nu) \in \Sigma_k$ since $\|\mathcal{R}_p(\nu)\| = 1$ and since we do not increase the number of points in the support of ν, due to the fact that the map $\nu \mapsto \tilde{\nu}$ does not affect the points x_i with $d(x_i, p) \ge 2/\bar{\tau}$, which was chosen such that $2/\bar{\tau} \ll \delta_2$. □

Remark 4.5. (i) With the above definitions, letting δ_2 tend to 0, one shows that the map $\mathcal{R}_p$ is homotopic to the identity on its domain.

(ii) The parameter δ_2 is chosen so that $\delta_2 \ll \delta$.

Combining Lemma 4.1 and Proposition 4.4 (applying its proof uniformly in $p \in \Sigma$), we obtain the following result; notice that by construction the retraction $\mathcal{R}_p$ from Proposition 4.4 depends continuously on p.

Corollary 4.6. *There exist $\bar{\tau} \gg 1$ and a continuous deformation $\mathcal{R}$ of $Y \cap \{s = \frac{1}{2}\}$ onto the set*

$$\bigcup_{p \in \Sigma} \{(\nu, \delta_p, \tfrac{1}{2}) : \nu \in \Sigma_{k,p,\bar{\tau}}\},$$

where $\Sigma_{k,p,\bar{\tau}}$ is as in (70).

In the next two subsections, we perform the construction of test functions using the above deformations.

4.2. *Test functions modeled on $\widetilde{Y}_{(1/2,p)} * \delta_p$.* In this subsection, we introduce a class of test functions parametrized on $\widetilde{Y}_{(1/2,p)} * \delta_p \subseteq Y$; see (67) and (51). The latter subset of Y is where the interaction between the two components of (1) is stronger and hence where more refined energy estimates will be needed. The remainder of Y will be taken care of in the next subsection.

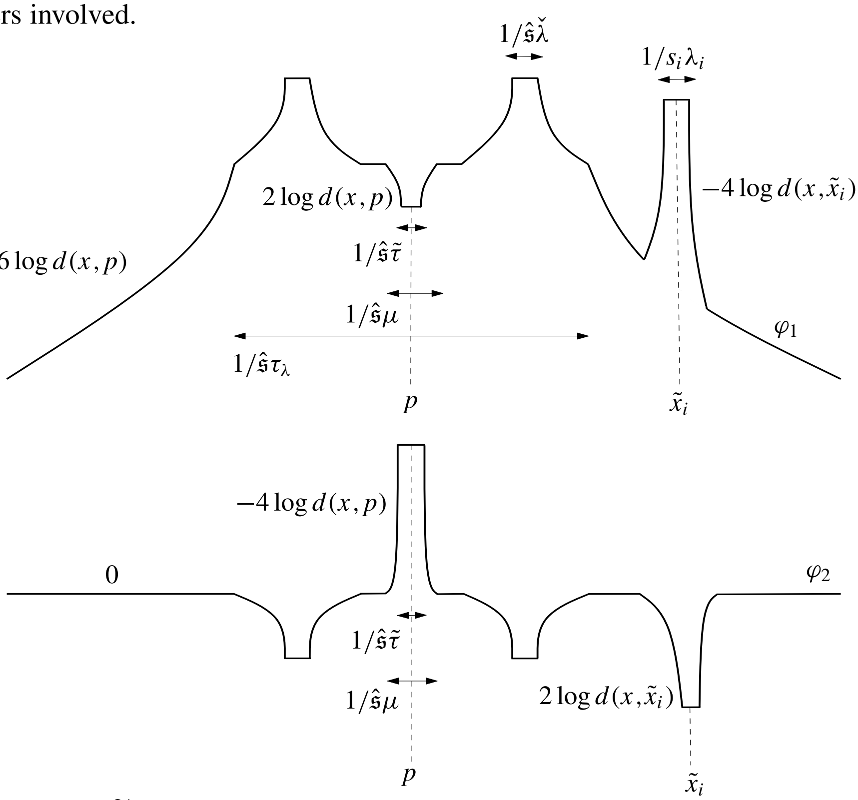
The retraction $\mathcal{R}_p$ defined in Proposition 4.4 will play a crucial role in the construction of the test functions. Indeed, starting from a measure in $\widetilde{Y}_{(1/2,p)}$ we will consider, through the map $\mathcal{R}_p$, a configuration belonging to $\Sigma_{k,p,\bar{\tau}}$; see (70). When considering $\widetilde{Y}_{(1/2,p)} * \delta_p$ and the corresponding join parameter s, our goal is to pass continuously from vector-valued functions (φ_1, φ_2) with $e^{\varphi_1} \simeq \hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ (in the distributional sense) to functions (φ_1, φ_2) with $e^{\varphi_2} \simeq \delta_p$. This needs to be done so that the energy $J_\rho(\varphi_1, \varphi_2)$ stays arbitrarily low.

As the formulas are rather involved, we first discuss the general ideas behind them. Our construction relies on superpositions of *regular bubbles* and *singular bubbles*. Regular bubbles are functions as in (61) that (roughly) optimize inequality (7) in the scalar case. Singular bubbles instead are profiles of solutions to (5) when a Dirac mass is present in the right-hand side: this singular version of (5) *shadows* system (1) when one component has a higher concentration than the other.

From the computational point of view, regular or singular bubbles behave like logarithmic functions of the distance from a point truncated at a proper scale, with coefficient -4 or -6, respectively. By this reason, we sometimes substitute an expression as in (61) (or in the subsequent formula) with truncated logarithms.

Another aspect of the construction is that, at a scale at which the function φ_i dominates, the gradient of the other component φ_j of (1) will behave like $-\frac{1}{2}\nabla\varphi_i$, the reason of which relies on the fact that this choice minimizes $Q(\varphi_1, \varphi_2)$ (see (3)) for φ_i fixed.

We introduce now the test functions (φ_1, φ_2) as in the figure below, starting by motivating the definitions of the parameters involved.



Consider $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$: recalling Proposition 4.4 and defining

$$\hat{\nu} := \mathcal{R}_p(\nu) = \sum_{i=i}^{k} t_i \delta_{x_i} \in \Sigma_{k,p,\bar{\tau}}, \tag{75}$$

let τ be as given in (72). Consider parameters $\tilde{\tau} \gg \mu \gg \lambda \gg 1$, and let $\mathfrak{s} \geq 1$ be a scaling parameter that will be used to deform one component into the other one: this will be chosen to depend on the join parameter s. Roughly speaking, φ_1 is made by a singular bubble at scale $1/\hat{\mathfrak{s}}\tau_\lambda$, where $\hat{\mathfrak{s}}$ is given by (78) (but one can think $\hat{\mathfrak{s}} = \mathfrak{s}$ for the moment) and

$$\tau_\lambda := \min\{\tau, \lambda\}, \tag{76}$$

on top of which we add regular bubbles at scales $1/s_i\lambda_i$ centered at points $\tilde{x}_i$ with $d(\tilde{x}_i, p) \geq 1/\hat{\mathfrak{s}}\tau$ for all i. The parameters s_i and λ_i are defined by (81) and (80) in order to get comparable integrals of e^{φ_1}

near all points $\tilde{x}_i$; we will discuss later why we sometimes take $\hat{\mathfrak{s}} \neq \mathfrak{s}$. The centers $\tilde{x}_i$ of the regular bubbles are defined as follows: letting $\bar{\delta}$ be small but fixed, we set in normal coordinates at p

$$\tilde{x}_i = \frac{1}{\tilde{s}_i} x_i, \qquad \tilde{s}_i = \begin{cases} \hat{\mathfrak{s}} & \text{if } d(x_i, p) \leq \bar{\delta}, \\ 1 & \text{if } d(x_i, p) \geq 2\bar{\delta}. \end{cases} \tag{77}$$

We point out that for $d(x_i, p) \leq \bar{\delta}$ we get $\tilde{x}_i = \frac{1}{\hat{\mathfrak{s}}} x_i$, which gives continuity when x_i approaches the *plateau* $\{d(\,\cdot\,, p) \leq 1/\tau_\lambda\}$. For $d(x_i, p) \geq \bar{\delta}$, instead the position of the points does not depend on $\mathfrak{s}$.

The effect of the increasing parameter $\mathfrak{s}$ depends on the starting configuration $\nu \in \widetilde{Y}_{(1/2, p)}$. In case we have points x_i on the plateau of the singular bubble, i.e., $d(x_i, p) \leq 1/\tau_\lambda$ for some i, the support of the singular and regular bubbles of φ_1 shrinks; moreover, the points $\tilde{x}_i$ approach p. On the other hand, φ_2 is (qualitatively) dilated by a factor of $1/\hat{\mathfrak{s}}$ so that e^{φ_2} loses concentration at the expense of e^{φ_1}.

In case we do not have points on the plateau, namely when $d(\tilde{x}_i, p) \geq 1/\tau_\lambda$ for all i, it is not convenient anymore to develop a singular bubble with center p as $\mathfrak{s}$ increases. To prevent this situation, we give an upper bound on $\hat{\mathfrak{s}}$ depending on τ. For $\tau_1 \geq 1$ large but fixed, we let $\widehat{P} : (0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function defined by

$$\begin{cases} \widehat{P}(t) = 1 & \text{for } t \leq \tau_1, \\ \widehat{P}(t) \to +\infty & \text{for } t \to 2\tau_1. \end{cases}$$

If τ is as in (72), we then define $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}(\mathfrak{s}, \tau)$ as

$$\hat{\mathfrak{s}} = \begin{cases} \min\{\mathfrak{s}, \widehat{P}(\tau)\} & \text{if } \tau < 2\tau_1, \\ \mathfrak{s} & \text{if } \tau \geq 2\tau_1. \end{cases} \tag{78}$$

Notice that by construction of the retraction $\mathcal{R}_p$ (see Proposition 4.4) when there are no points on the plateau $\{d(\,\cdot\,, p) \leq 1/\tau_\lambda\}$, it follows that $\tau \leq C$ and therefore, taking $2\tau_1 > C$, we get $\hat{\mathfrak{s}} \leq \widehat{P}(C) < +\infty$.

In this situation, namely for $\hat{\mathfrak{s}}$ bounded from above, the second component φ_2 remains fixed when we start to concentrate the first component φ_1. To do this, we develop more and more concentrated bubbles around the points $\tilde{x}_i$; we introduce a parameter $\check{\lambda} = \check{\lambda}(\tau)$ so that $\check{\lambda} \to +\infty$ even for $\tau \leq 2\tau_1$ when $\mathfrak{s}$ increases. Let $\check{P} : (0, +\infty) \to (0, +\infty)$ be a nonincreasing continuous function such that

$$\begin{cases} \check{P}(t) \to +\infty & \text{for } t \to 2\tau_1, \\ \check{P}(t) = 1 & \text{for } t \geq 4\tau_1. \end{cases}$$

We then let

$$\check{\lambda} = \check{\mathfrak{s}}\lambda, \qquad \check{\mathfrak{s}} = \begin{cases} \mathfrak{s} & \text{if } \tau \leq 2\tau_1, \\ \min\{\mathfrak{s}, \check{P}(\tau)\} & \text{if } \tau > 2\tau_1. \end{cases} \tag{79}$$

To have a comparable integral of e^{φ_1} at each peak around $\tilde{x}_i$ for $i = 1, \dots, k$, we impose the conditions

$$\begin{cases} \log \lambda_i - \log d(x_i, p) = \log \tau_\lambda + \log \check{\lambda} & \text{if } d(x_i, p) > 1/\tau_\lambda, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i, p) \leq 1/\tau_\lambda \end{cases} \tag{80}$$

and

$$\log s_i + \log \tilde{s}_i = 2 \log \hat{\mathfrak{s}}, \tag{81}$$

which determine λ_i and s_i.

Recall the definitions of $\hat{v}$ in (75): motivated by the above discussion, we define the functions (φ_1, φ_2) as follows (see the figure on page 1995). The positive peaks of φ_1 are given by

$$v_1(x) = v_{1,1}(x) + v_{1,2}(x) = \log \sum_{i=1}^{k} t_i \frac{\max\left\{1, \min\left\{\left(\frac{4}{d(\tilde{x}_i, p)} d(x, \tilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\tilde{x}_i, p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\}}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3},$$

where

$$v_{1,1}(x) = \log \sum_{i=1}^{k} t_i \max\left\{1, \min\left\{\left(\frac{4}{d(\tilde{x}_i, p)} d(x, \tilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\tilde{x}_i, p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\},$$

$$v_{1,2}(x) = \log \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3}.$$

The positive peak of φ_2 is instead defined by

$$v_2(x) = \log\left(\max\left\{1, \min\left\{(\hat{\mathfrak{s}}\mu\, d(x, p))^{-4}, \left(\frac{\mu}{\tilde{\tau}}\right)^{-4}\right\}\right\}\right).$$

We finally set

$$\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2} v_{1,1}(x) + v_2(x) \end{pmatrix}. \tag{82}$$

The main result of this subsection is:

Proposition 4.7. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$, let $\widetilde{\Psi}$ be defined in (29), and let $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}$ be defined in (82), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. Then for suitable values of $\tilde{\tau} \gg \mu \gg \lambda \gg 1$ and for $\mathfrak{s} = 1$, $\widetilde{\Psi}(\varphi_{\lambda,\tilde{\tau},1})$ is valued into the second component of the join $\Sigma_k * \Sigma_1$. Moreover, there is a value $\mathfrak{s}_{p,\nu} > 1$ of $\mathfrak{s}$, which depends continuously on p and ν such that $\widetilde{\Psi}(\varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}})$ is valued into the first component of the join, and such that*

$$J_\rho(\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } \mathfrak{s} \in [1, \mathfrak{s}_{p,\nu}] \text{ and in } p \text{ and } \nu.$$

Proof. As some of the estimates are rather technical, most of the proof is postponed to the Appendix.

Concerning the first statement, when $\mathfrak{s} = 1$, by construction (see in particular Lemma A.2), one can see that most of the integral of e^{φ_2} is concentrated in a ball centered at p with radius of order $1/\tilde{\tau}$ while that of e^{φ_1} near at most k balls of larger scale. From the definitions of scales $\sigma_1(u_1)$ and $\sigma_2(u_2)$ in Section 3.1, it follows that for $\mathfrak{s} = 1$ the quantity $s(\varphi_1, \varphi_2)$ defined in (26) is equal to 1, provided we choose the parameters $\tilde{\tau} \gg \mu \gg \lambda \gg 1$ properly. By the way $\widetilde{\Psi}$ is defined, this implies our first statement.

As $\mathfrak{s}$ increases (see again Lemma A.2), the scale $\sigma_1(\varphi_1)$ (as defined in Section 3.1) decreases while, depending on τ, the scale of $\sigma_2(\varphi_2)$ reaches some positive value bounded away from 0. In particular for $\tau \geq 2\tau_1$ (recall (78)), by the estimates in Lemma A.2, for $\mathfrak{s} \simeq \log \tilde{\tau} - 2 \log \mu$, the scale $\sigma_2(\varphi_2)$ becomes of order 1. In any case, for $\mathfrak{s}$ sufficiently large, $s(\varphi_1, \varphi_2) = 0$, so $\widetilde{\Psi}$ maps the test function into the first component of the joint. As the scales $\sigma_1(\varphi_1)$ and $\sigma_2(\varphi_2)$ vary continuously in φ_1 and φ_2, $\mathfrak{s}_{p,\nu}$ can be chosen to depend continuously on p and ν.

Regarding the energy estimates, the most delicate situation is when τ is large, i.e., when $\hat{\mathfrak{s}} = \mathfrak{s}$; see (78). In this case, $\mathfrak{s}_{p,\nu} \simeq \log \tilde{\tau} - 2 \log \mu$ and the computations are worked out in the Appendix. When τ instead is smaller than the fixed number $2\tau_1$ (see again (78)), the *singular* part of the first component of the test function (with slope $-6 \log d(\,\cdot\,, p)$) has negligible contribution and the support of the measure $\hat{\nu}$ in (75) is bounded away from p by a fixed positive amount. In this case, the interaction between the two components is negligible, and similar estimates as those in Proposition 3.3 of [Battaglia et al. 2015] can be applied. □

We proceed now with parametrizing the above functions via the number s in the topological join. Ideally, one would like to have $\mathfrak{s}$ varying from 1 to $\mathfrak{s}_{p,\nu}$ as s decreases from 1 to 0. However, for this map to be well-defined on the topological join, we will need to eliminate the dependence of the test function on the first and second components of the join when $s = 1$ and $s = 0$, respectively. For this reason, we will need some extra deformations depending on s. The construction goes as follows, depending on three ranges of the join parameter s.

4.2.1. *The case* $s \in [\frac{1}{4}, \frac{3}{4}]$. Let $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}$ be defined in (82), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. We set

$$\Phi_\lambda(\nu, p, s) = \varphi_{\lambda,\tilde{\tau},2(1-\mathfrak{s}_{p,\nu})s+(3/2)\mathfrak{s}_{p,\nu}-1/2} \tag{83}$$

so that $\Phi_\lambda(\nu, p, \frac{1}{4}) = \varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}}$ and $\Phi_\lambda(\nu, p, \frac{3}{4}) = \varphi_{\lambda,\tilde{\tau},1}$.

4.2.2. *The case* $s \in [0, \frac{1}{4}]$. Starting from test functions of the form $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}}$, the goal will be to eliminate the dependence on the second component of the join, namely on the measure δ_p. To this end, we divide the interval $[0, \frac{1}{4}]$ in several subintervals in which we perform different operations on the test functions. Moreover, we want J_ρ to attend arbitrarily low values while doing these procedures. Notice that, in what follows, this range of the join parameter s will correspond to $\mathfrak{s} = \mathfrak{s}_{p,\nu}$, which is given in Proposition 4.7.

Step 1. Let $s \in [\frac{3}{16}, \frac{1}{4}]$. We flatten here the function v_2 in the second component of (82) by considering the deformation

$$\breve{\varphi}^t_{\lambda,\tilde{\tau}}(x) = \begin{pmatrix} \breve{\varphi}^t_1(x) \\ \breve{\varphi}^t_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2}\, t\, v_2(x) \\ -\frac{1}{2} v_{1,1}(x) + t\, v_2(x) \end{pmatrix}, \qquad t \in [0, 1].$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \breve{\varphi}^t_{\lambda,\tilde{\tau}}(x), \qquad t = 16(s - \tfrac{3}{16}). \tag{84}$$

It is easy to see that J_ρ attends arbitrarily low values on this deformation by minor modifications in the proof of Proposition 4.7.

Step 2. Let $s \in [\frac{1}{8}, \frac{3}{16}]$. Starting from $s = \frac{3}{16}$, we deform the test functions introduced in (82) to the standard test functions of the form given as in (63). Roughly speaking, the idea is to modify the profile of the first component φ_1 (see the figure on page 1995) by performing the following two continuous deformations. We first flatten the singular bubble $v_{1,2}$; see (82). On the other hand, we eliminate the dependence of the point p in the regular bubbles $v_{1,1}$. Therefore, we set

$$v^t_1(x) = v^t_{1,1}(x) + v^t_{1,2}(x),$$

where

$$v_{1,1}^t(x) = \log \sum_{i=1}^k t_i \max\left\{1, \min\left\{\left(\left(\frac{4}{d(\tilde{x}_i, p)}\right)^t d(x, \tilde{x}_i)\right)^{-4}, \left(\left(\frac{4}{d(\tilde{x}_i, p)}\right)^t \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\}$$

and $v_{1,2}^t(x) = t\, v_{1,2}(x)$. Finally, recalling that we have flattened v_2 in Step 1, we consider

$$\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t(x) = \begin{pmatrix} \widetilde{\varphi}_1^t(x) \\ \widetilde{\varphi}_2^t(x) \end{pmatrix} := \begin{pmatrix} v_1^t(x) \\ -\frac{1}{2} v_{1,1}^t(x) \end{pmatrix}, \quad t \in [0, 1]. \tag{85}$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \widetilde{\varphi}_{\lambda,\tilde{\tau}}^t(x), \quad t = 16(s - \tfrac{1}{8}). \tag{86}$$

Concerning $\widetilde{\varphi}_1^t$, its peaks around $\tilde{x}_i$ for $i = 1, \dots, k$ are truncated at a scale $1/s_i\lambda_i$, with s_i given by (81) and λ_i to be chosen in the following way in order to have comparable volume at any $\tilde{x}_i$:

$$\begin{cases} \log \lambda_i + \log s_i - t \log d(\tilde{x}_i, p) = (t+1) \log \hat{\mathfrak{s}} + \log \check{\lambda} + t \log \tau_\lambda & \text{if } d(x_i, p) > 1/\tau_\lambda, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i, p) \le 1/\tau_\lambda. \end{cases} \tag{87}$$

Observe that for $t = 0$ we again get (80). The following result holds true:

Proposition 4.8. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Let $\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t$ be defined as in (85), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. Then one has*

$$J_\rho(\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t) \to -\infty \quad \textit{as } \lambda \to +\infty \quad \textit{uniformly in } t \in [0, 1] \textit{ and in } p \textit{ and } \nu.$$

The most delicate case is when the set of the points on the plateau is not empty, i.e., for $I_1 \neq \varnothing$; see (121). We give the proof of the latter result just in this situation, skipping the case $I_1 = \varnothing$ where the singular bubble of the first component of the test function (with slope $-6 \log d(\,\cdot\,, p)$) has negligible contribution and the estimates are rather easy. As observed in Case 1 of the proof of Proposition 4.7 (see (134)), for $I_1 \neq \varnothing$, we deduce $\hat{\mathfrak{s}} = \mathfrak{s}$ and $\check{\lambda} \le C\lambda$. Moreover, for this range of the join parameter s, we have $\mathfrak{s} = \mathfrak{s}_{p,\nu} \gg 1$. The proof will follow from the estimates below, which are obtained exactly as Lemmas A.1, A.2, and A.3 by using (81) and (87).

Lemma 4.9. *For $t \in [0, 1]$, we have that*

$$\fint_\Sigma \widetilde{\varphi}_1^t \, dV_g = O(1), \qquad \fint_\Sigma \widetilde{\varphi}_2^t \, dV_g = O(1).$$

Lemma 4.10. *Recalling the notation in (114), for $t \in [0, 1]$, it holds that*

$$\int_\Sigma e^{\widetilde{\varphi}_1^t} \, dV_g \simeq_C \hat{\mathfrak{s}}^{2+2t} \tau_\lambda^{2t} \check{\lambda}^2, \qquad \int_\Sigma e^{\widetilde{\varphi}_2^t} \, dV_g \simeq_C 1.$$

Lemma 4.11. *Let $I_1, I_2 \subseteq I$ be as in (121). Then for $t \in [0, 1]$, we have*

$$\int_\Sigma Q(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \, dV_g \le 8|I_1|\pi \left(\log \check{\lambda} - t \log \tau_\lambda + (1-t) \log \hat{\mathfrak{s}}\right) + \sum_{i \in I_2} 8\pi \left(\log s_i + \log \lambda_i - t \log d(\tilde{x}_i, p)\right)$$
$$+ 16t\pi \sum_{i \in I_2} \log d(\tilde{x}_i, p) + 24t^2\pi(\log \tau_\lambda + \log \hat{\mathfrak{s}}) + C,$$

for some $C = C(\Sigma)$.

Proof of Proposition 4.8. Using Lemmas 4.9, 4.10, and 4.11, the energy estimate we obtain is

$$J_\rho(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \le 8|I_1|\pi\big(\log\check{\lambda} - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\lambda_i - t\log d(\tilde{x}_i, p)\big) \\ + 16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i, p) + 24t^2\pi(\log\tau_\lambda + \log\hat{\mathfrak{s}}) - \rho_1\big((2+2t)\log\hat{\mathfrak{s}} + 2t\log\tau_\lambda + 2\log\check{\lambda}\big) + C$$

for some constant $C > 0$. Inserting the condition (87), we obtain

$$J_\rho(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \le 8|I_1|\pi\big(\log\check{\lambda} - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big((t+1)\log\hat{\mathfrak{s}} + \log\check{\lambda} + t\log\tau_\lambda\big) \\ + 16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i, p) + 24t^2\pi(\log\tau_\lambda + \log\hat{\mathfrak{s}}) - \rho_1\big((2+2t)\log\hat{\mathfrak{s}} + 2t\log\tau_\lambda + 2\log\check{\lambda}\big) + C.$$

Notice that for $t = 1$ we get exactly the estimate in (134) (recall that we have flattened v_2). The latter estimate can be rewritten as

$$J_\rho(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \le \log\hat{\mathfrak{s}}\big(8(1-t)|I_1|\pi + 8(t+1)|I_2|\pi + 24t^2\pi - (2+2t)\rho_1\big) + \log\check{\lambda}\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) \\ + \log\tau_\lambda\big(8t|I_2|\pi - 8t|I_1|\pi + 24t^2\pi - 2t\rho_1\big) + 16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i, p) + C.$$

As observed in Case 1 of the proof of Proposition 4.7, by construction of $\Sigma_{k,p,\bar{\tau}}$ (see (70)), $|I_2| \le k-2$ whenever $|I_1| \ne \varnothing$. Therefore, we conclude that the latter estimate is uniformly large-negative in $t \in [0, 1]$ since $\rho_1 > 4k\pi$ and by the fact that $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}_{p,\nu} \gg \check{\lambda} \ge \tau_\lambda$. Observe that for $t = 0$ we get

$$J_\rho(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \le \log\hat{\mathfrak{s}}\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + \log\check{\lambda}\big(8(|I_1|+|I_2|)\pi - 2\rho_1\big) + C,$$

which is the estimate one expects by considering standard bubbles as in (63); see for example part (i) of Proposition 4.2 in [Malchiodi and Ndiaye 2007]. □

Recall now the definition of $\hat{\nu}$ given in (75): $\hat{\nu} = \mathcal{R}_p(\nu) = \sum_{i=i}^k t_i\delta_{x_i} \in \Sigma_{k,p,\bar{\tau}}$. Notice that in the construction of the test functions (82), the points x_i are dilated according to (77) so deformed to the points $\tilde{x}_i$. Observe that for $t = 0$ we obtain in (85) standard test functions as in (63). Roughly speaking, the first component resembles the form of $\varphi_{\lambda,\tilde{\nu}}$ (see (63)), where $\tilde{\nu} = \sum_{i=i}^k t_i\delta_{\tilde{x}_i}$.

In what follows, we will skip the energy estimates since they are quite standard for test functions as in (63); see for example part (i) of Proposition 4.2 in [Malchiodi and Ndiaye 2007].

Step 3. Consider $s \in [\frac{1}{16}, \frac{1}{8}]$. We will deform here the points $\tilde{x}_i$ to the original points x_i. Observe that by construction (see (77)) we have $d(x_i, \tilde{x}_i) \le 2\bar{\delta}$ for all i. Hence, there exists a geodesic $\tilde{\gamma}_i$ joining $\tilde{x}_i$ and x_i in unit time, and we set $x_i^t = \tilde{\gamma}_i(t)$ with $t \in [0, 1]$. Denoting by $\widehat{\varphi}_{\lambda,\bar{\tau}}^t = (\widehat{\varphi}_1^t, \widehat{\varphi}_2^t)$ the corresponding test functions, we will then take

$$\Phi_\lambda(\nu, p, s) = \widehat{\varphi}_{\lambda,\bar{\tau}}^t(x), \qquad t = 16(\tfrac{1}{8} - s). \tag{88}$$

Once we have deformed the points $\tilde{x}_i$ to the original ones x_i, i.e., for $t = 1$, we get test functions for which the first component has the form of $\varphi_{\lambda,\mathcal{R}_p(\nu)}$.

Step 4. Consider $s \in [0, \frac{1}{16}]$. In this step, we eliminate the dependence on the map $\mathcal{R}_p$. Observe that $\mathcal{R}_p$ is homotopic to the identity map (see Remark 4.5), and let $\mathcal{H}_{\mathcal{R}_p} : \widetilde{Y}_{(1/2,p)} \times [0,1] \to \widetilde{Y}_{(1/2,p)}$ be a continuous map such that $\mathcal{H}_{\mathcal{R}_p}(\,\cdot\,,0) = \mathcal{R}_p$ and $\mathcal{H}_{\mathcal{R}_p}(\,\cdot\,,1) = \mathrm{Id}_{\widetilde{Y}_{(1/2,p)}}$. We consider then the deformation $\nu_t = \mathcal{H}_{\mathcal{R}_p}(\nu, t)$, and letting $\bar{\varphi}^t_{\lambda,\tilde{\tau}} = (\bar{\varphi}^t_1, \bar{\varphi}^t_2)$ be the corresponding test functions, we set

$$\Phi_\lambda(\nu, p, s) = \bar{\varphi}^t_{\lambda,\tilde{\tau}}(x), \quad t = 16(\tfrac{1}{16} - s). \tag{89}$$

Such a deformation will bring us to test functions that resemble the form of $\varphi_{\lambda,\nu}$.

4.2.3. *The case $s \in [\frac{3}{4}, 1]$.* The goal here will be to continuously deform the initial test functions in (82), with $\mathfrak{s} = 1$, to a configuration that does not depend on the measure ν; see (75). Furthermore, in this procedure, we want J_ρ to attend arbitrarily low values. For this purpose, we flatten v_1 (see (82)) by using the deformation

$$\varphi^t_{\lambda,\tilde{\tau}}(x) = \begin{pmatrix} \varphi^t_1(x) \\ \varphi^t_2(x) \end{pmatrix} := \begin{pmatrix} t v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2} t v_{1,1}(x) + v_2(x) \end{pmatrix}, \quad t \in [0,1]. \tag{90}$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \varphi^t_{\lambda,\tilde{\tau}}(x), \quad t = 4(1-s). \tag{91}$$

Proposition 4.12. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$, and let $\varphi^t_{\lambda,\tilde{\tau}}$ be defined as in (90), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. Then, one has*

$$J_\rho(\varphi^t_{\lambda,\tilde{\tau}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } t \in [0,1] \text{ and in } p \text{ and } \nu.$$

The latter result follows from the next estimates, which are obtained similarly as in Lemmas A.1, A.2, and A.3, using the fact that $\mathfrak{s} = 1$.

Lemma 4.13. *For $t \in [0,1]$, we have that*

$$\fint_\Sigma \varphi^t_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi^t_2 \, dV_g = O(1).$$

Lemma 4.14. *Recalling the notation in (114), there exists a constant $C_1(\tau_\lambda, \lambda)$ such that for $t \in [0,1]$*

$$\int_\Sigma e^{\varphi^t_1} \, dV_g \simeq_C \int_\Sigma e^{t v_1} \, dV_g = C_1(\tau_\lambda, \lambda), \qquad \int_\Sigma e^{\varphi^t_2} \, dV_g \simeq_C \int_\Sigma e^{v_2} \, dV_g \simeq_C \frac{\tilde{\tau}^2}{\mu^4}.$$

Lemma 4.15. *For $t \in [0,1]$, we have that*

$$\int_\Sigma Q(\varphi^t_1, \varphi^t_2) \, dV_g \le 8\pi(\log \tilde{\tau} - \log \mu) + C_2(\tau_\lambda, \lambda)$$

for some constant $C_2(\tau_\lambda, \lambda)$.

Proof of Proposition 4.12. Exploiting Lemmas 4.13, 4.14, and 4.15, we deduce

$$\begin{aligned} J_\rho(\varphi^t_1, \varphi^t_2) &\le 8\pi(\log \tilde{\tau} - \log \mu) - \rho_2(2\log \tilde{\tau} - 4\log \mu) + \widetilde{C}_1(\tau_\lambda, \lambda) + C_2(\tau_\lambda, \lambda) \\ &\le \log \tilde{\tau}(8\pi - 2\rho_2) + \log \mu(4\rho_2 - 8\pi) + \widetilde{C}_1(\tau_\lambda, \lambda) + C_2(\tau_\lambda, \lambda) \end{aligned}$$

for some constant $\widetilde{C}_1(\tau_\lambda, \lambda)$. The latter upper bound is large and negative since $\rho_2 > 4\pi$ and by the choice of the parameters $\tilde{\tau} \gg \mu \gg \lambda \ge \tau_\lambda$. $\square$

4.3. ***The global construction.*** In this subsection, we will perform a global construction of a family of test functions modeled on Y, relying on the estimates of the previous subsection. More precisely, as Y is not compact, we will consider a compact retraction of it.

Letting $(\mathfrak{D}, \frac{1}{2}) \subseteq (\Sigma_k \times \Sigma_1, \frac{1}{2})$ be the domain of the map $\mathfrak{R}$ in Corollary 4.6, we extend it to $\{(\mathfrak{D}, s) : s \in (0, 1)\}$ fixing the second component and considering the same action of $\mathfrak{R}$ on the first one.

Secondly, we retract the set Y to a subset where the (extended) map $\mathfrak{R}$ is well-defined or where $s \in \{0, 1\}$. In order to do this, for $\nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k$, we let

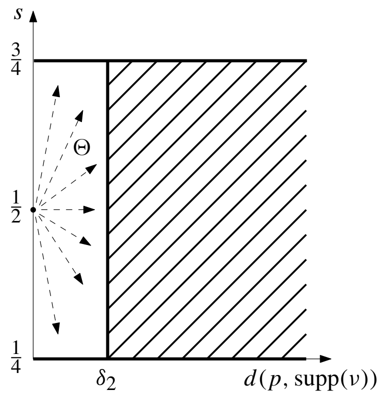
$$\mathscr{D}(\nu) = \min_{i=1,\dots,k,\, i \neq j} \{d(x_i, x_j), t_i, 1 - t_i\}.$$

Moreover, recall the choices of δ and δ_2 given in (52) and (68), respectively. Observe that for $\mathscr{D}(\nu) \leq \delta$ we are in the domain of $\mathfrak{R}$. Moreover, for $\mathscr{D}(\nu) > \delta$ and $d(p, \operatorname{supp}(\nu)) \geq \delta_2$, the map $\mathfrak{R}$ is still well-defined. The idea is then to retract the set Y to a subset where one of the above alternatives holds true or where $s \in \{0, 1\}$. We define now the retraction of Y in three steps.

Step 1. Let $\mathscr{D}(\nu) \geq 2\delta$. In this situation, we can deform a configuration (ν, δ_p, s) to a configuration $(\nu, \delta_{\tilde{p}}, \tilde{s}) \in Y$ (recall (51)) where either $d(\tilde{p}, \operatorname{supp}(\nu)) \geq \delta_2$ or $\tilde{s} \in \{0, 1\}$. Let

$$\Theta = (\Theta_1, \Theta_2) : [0, +\infty) \times [0, 1] \setminus \left\{\left(0, \tfrac{1}{2}\right)\right\} \to [0, +\infty) \times [0, 1] \setminus ((0, \delta_2) \times (0, 1))$$

be the radial projection as in



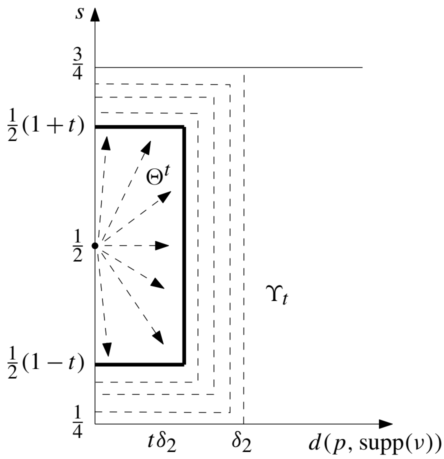
Observe now that by the fact that $\delta_2 \ll \delta$ (recall Remark 4.5), for $\mathscr{D}(\nu) \geq 2\delta$, we get the existence of a unique point $x_{j_p} \in \{x_1, \dots, x_k\}$ such that $d(p, x_{j_p}) \leq \delta_2$. To then get the above-described deformation, we define, in normal coordinates around x_{j_p}, the map

$$(\nu, \delta_p, s) \mapsto \big(\nu, \delta_{\Theta_1(d(p, \operatorname{supp}(\nu)), s))(p/|p|)}, \Theta_2(d(p, \operatorname{supp}(\nu)), s)\big) \in \widetilde{\Upsilon}_\Theta,$$

where

$$\widetilde{\Upsilon}_\Theta = \left\{(\nu, \delta_p, s) : \mathscr{D}(\nu) \geq 2\delta,\ d(p, \operatorname{supp}(\nu)) \geq \delta_2\right\} \\ \cup \left\{(\nu, \delta_p, s) : \mathscr{D}(\nu) \geq 2\delta,\ d(p, \operatorname{supp}(\nu)) \leq \delta_2,\ s \in \{0, 1\}\right\}. \quad (92)$$

Step 2. Let $\mathfrak{D}(\nu) \in [\delta, 2\delta]$. In this range, we interpolate between the deformation Θ and the identity map. Consider the radial projection $\Theta^t = (\Theta_1^t, \Theta_2^t)$ given as in



with $t = (\mathfrak{D}(\nu) - \delta)/\delta$:

$$\Theta^t = (\Theta_1^t, \Theta_2^t) : [0, +\infty) \times [0, 1] \setminus \left\{\left(0, \tfrac{1}{2}\right)\right\} \to \Upsilon_t,$$

where

$$\Upsilon_t = [0, +\infty) \times [0, 1] \setminus \left((0, t\delta_2) \times \left(\tfrac{1}{2}(1-t), \tfrac{1}{2}(1+t)\right)\right).$$

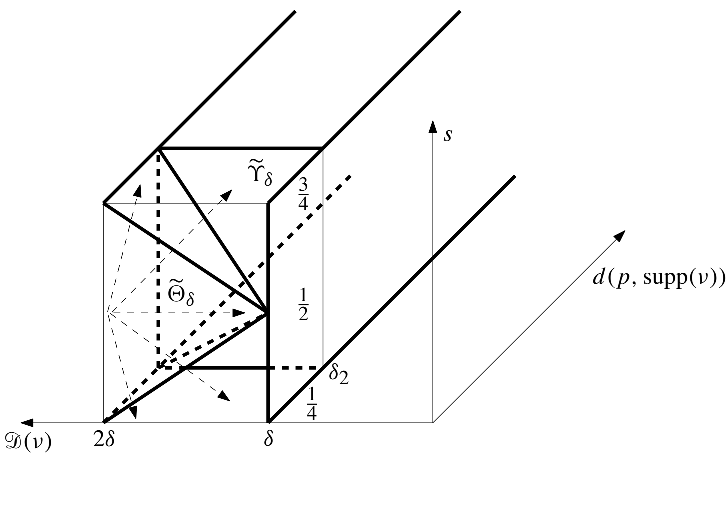
Observe that for $\mathfrak{D}(\nu) = 2\delta$ one gets $\Theta^t = \Theta^1 = \Theta$, while for $\mathfrak{D}(\nu) = \delta$ one deduces $\Theta^t = \Theta^0 = \mathrm{Id}$. We then set

$$(\nu, \delta_p, s) \mapsto \left(\nu, \delta_{\Theta_1^t(d(p,\mathrm{supp}(\nu),s))(p/|p|)}, \Theta_2^t(d(p, \mathrm{supp}(\nu)), s)\right).$$

Step 3. Let us now introduce the set we obtain after the deformation performed in Step 2:

$$\widetilde{\Upsilon}_\delta = \left\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) = t \in [\delta, 2\delta],\ (p, s) \in \Upsilon_t\right\},$$

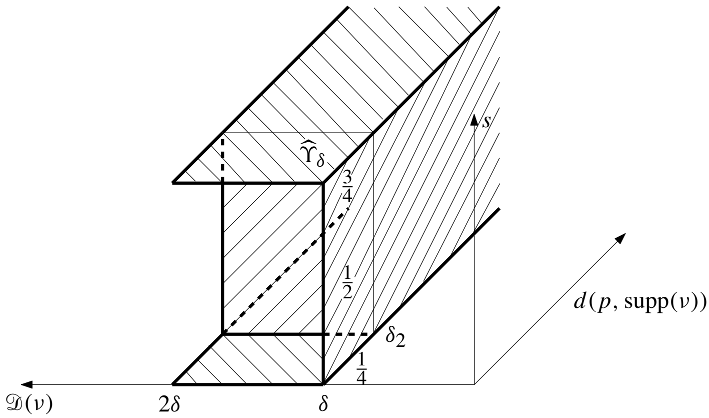
which we will deform using the radial projection $\widetilde{\Theta}_\delta : \widetilde{\Upsilon}_\delta \to \widehat{\Upsilon}_\delta$ given as in



where $\widehat{\Upsilon}_\delta$ is defined by

$$\widehat{\Upsilon}_\delta = \big\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \in [\delta, 2\delta],\ d(p, \mathrm{supp}(\nu)) \le \delta_2,\ s \in \{0, 1\}\big\} \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) = \delta\} \\ \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \in [\delta, 2\delta],\ d(p, \mathrm{supp}(\nu)) \ge \delta_2\}. \quad (93)$$

See the following figure, where $\partial\widehat{\Upsilon}_\delta$ is represented:



Construction of the test functions. Observing that for $\mathfrak{D}(\nu) \le \delta$ we are already in the domain of $\mathfrak{R}$ and recalling the sets (92) and (93), we have found a retraction $\mathcal{F} : Y \to Y_{\mathfrak{R}}$, where

$$\begin{aligned} Y_{\mathfrak{R}} &= \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \le \delta\} \cup \widetilde{\Upsilon}_\delta \cup \widetilde{\Upsilon}_\Theta \\ &= \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \le \delta\} \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \ge \delta,\ d(p, \mathrm{supp}(\nu)) \ge \delta_2\} \\ &\qquad \cup \big\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \ge \delta,\ d(p, \mathrm{supp}(\nu)) \le \delta_2,\ s \in \{0, 1\}\big\}, \end{aligned} \quad (94)$$

on which the map $\mathfrak{R}$ is well-defined or where $s \in \{0, 1\}$.

Remark 4.16. By the way the retraction $\mathcal{F}$ is constructed, it is clear that we have indeed a deformation retract of the set Y onto $Y_{\mathfrak{R}}$, i.e., there exists a continuous map $\mathcal{F}_t : Y \times [0, 1] \to Y$ such that $\mathcal{F}_0 = \mathrm{Id}_Y$, $\mathcal{F}_1 = \mathcal{F} : Y \to Y_{\mathfrak{R}}$, and $\mathcal{F}_1(\xi) = \xi$ for all $\xi \in Y_{\mathfrak{R}}$.

We finally call $\Phi_\lambda = \Phi_\lambda(\nu, p, s)$ the test functions in Sections 4.2.1, 4.2.2, and 4.2.3 (see (83), (84), (86), (88), (89), and (91)) using as parameters $(\nu, p, s) \in Y_{\mathfrak{R}}$ (where we use the identification $p \simeq \delta_p$). By the estimates obtained in Section 4.2, the next result holds true.

Proposition 4.17. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Then we have*

$$J_\rho(\Phi_\lambda(\nu, p, s)) \to -\infty \quad \textit{as } \lambda \to +\infty \quad \textit{uniformly in } (\nu, p, s) \in Y_{\mathfrak{R}}.$$

The definition of Φ_λ reflects naturally the join element (ν, p, s) in the sense that, once composed with the map $\widetilde{\Psi}$ in (29), we obtain a map homotopic to the identity on $Y_{\mathfrak{R}}$; see the next section.

5. Proof of Theorem 1.1

In this section, we introduce the variational scheme that we will use to prove Theorem 1.1. As we already observed, the case of surfaces with positive genus was obtained in [Battaglia et al. 2015]. Therefore, from

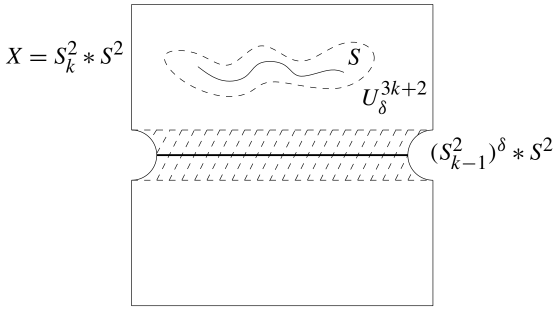


Figure 1. Here $X = S^2_k * S^2$ is the ambient space, $(S^2_{k-1})^\delta * S^2$ is a neighborhood of $S^2_{k-1} * S^2$ in X, S misses this neighborhood, and U_δ is a neighborhood of S in that complement.

now on, we will consider the case when Σ is homeomorphic to S^2. We will first analyze the topological structure of the set Y in (51) and then introduce a suitable min-max scheme.

5.1. ***On the topology of Y when Σ is a sphere.*** In this subsection, we will use the notation $\simeq$ for a homotopy equivalence and $\cong$ for an isomorphism. Consider the topological join $X = S^2_k * S^2$ (observe that $S^2_1 = S^2$), and recall the definition of its subset S given in (52), that is,

$$S = \left\{\left(\nu, \delta_y, \tfrac{1}{2}\right) \in S^2_k * S^2 : \nu \in S^2_k \setminus (S^2_{k-1})^\delta,\ y \in \operatorname{supp}(\nu)\right\},$$

where we have set

$$(S^2_{k-1})^\delta = \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ d(x_i, x_j) < \delta \text{ for some } i \neq j\right\}$$
$$\cup \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ t_i < \delta \text{ for some } i\right\} \cup \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ t_i > 1-\delta \text{ for some } i\right\}.$$

Notice that S is a smooth manifold of dimension $3k-1$, with boundary of dimension $3k-2$.

The key point of this subsection is to prove that the complementary subspace $Y = (S^2_k * S^2) \setminus S$ is not contractible; see Proposition 5.6. Before we do so, we establish some properties of Y and S. Below, U_δ will represent an open neighborhood of S not meeting $(S^2_{k-1})^\delta * S^2$ with the property that $\overline{U}_\delta$ is a manifold with boundary $\partial \overline{U}_\delta$, where both U_δ and $\overline{U}_\delta$ deformation-retract onto S and such that $\overline{U}_\delta \setminus S$ deformation-retracts onto $\partial \overline{U}_\delta$ (see Figure 1).

For a metric space $\mathscr{X}$, throughout this subsection, we use the notation for the k-tuples in $\mathscr{X}$

$$F(\mathscr{X}, k) := \{(x_1, \dots, x_k) \in \mathscr{X}^k : x_i \neq x_j,\ i \neq j\}$$

and $B(\mathscr{X}, n)$ to denote its quotient by the permutation action of the symmetric group. These are the ordered and unordered k-th *configuration spaces* of $\mathscr{X}$, respectively.

Lemma 5.1. *S is up to homotopy equivalence a degree-k covering of $B(S^2, k)$. Its homological dimension is at most k, and its mod-2 homology is completely described by*

$$H_*(S) \cong H_*(S^2) \otimes H_*(B(\mathbb{R}^2, k-1)).$$

Proof. The barycentric set S^2_k is a suitable quotient of

$$\Delta_{k-1} \times_{\mathfrak{S}_k} (S^2)^k,$$

with $\mathfrak{S}_k$ acting diagonally by permutations and $\Delta_{k-1} = \{(t_0, \dots, t_k) : t_i \in [0, 1], \ \sum t_i = 1\}$. The identification occurs when $x_i = x_j$ for some $i \neq j$ or when $t_i = 0$ for some i. When this happens, we are identifying with points in S^2_{k-1}. This means that, if $\dot{\Delta}_{k-1}$ is the open simplex, then

$$S^2_k \setminus S^2_{k-1} = \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k), \tag{95}$$

where $F(S^2, k)$ is the configuration space of k distinct points on S^2. The action of $\mathfrak{S}_k$ on $F(S^2, k)$ is free, so we have a bundle projection

$$\dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k) \to B(S^2, k),$$

where $B(S^2, k) := F(S^2, k)/\mathfrak{S}_k$ is the configuration of k-unordered points on S^2. The preimages, being copies of the simplex, are contractible so that necessarily

$$S^2_k \setminus S^2_{k-1} \simeq B(S^2, k).$$

In fact, $\{1/k\}$ maps to $\dot{\Delta}_{k-1}$ with image $(1/k, \dots, 1/k)$ and the induced map

$$B(S^2, k) = \left\{\frac{1}{k}\right\} \times_{\mathfrak{S}_k} F(S^2, k) \to \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k)$$

is an equivalence. To summarize, S can be deformed onto the subspace

$$W_k = \{([x_1, \dots, x_k], x) \in B(S^2, k) \times S^2 : x = x_i \text{ for some } i\}.$$

By projecting W_k onto $B(S^2, k)$, we get a covering. This implies that the homological dimension hd of W_k is that of $B(S^2, k)$, which is also the homological dimension of its covering space $F(S^2, k)$. We claim that this dimension is at most k. The projection onto the first coordinate $F(S^2, k) \to S^2$ is a bundle map with fiber $F(\mathbb{R}^2, k-1)$, so $\mathrm{hd}(F(S^2, k)) \leq 2 + \mathrm{hd}(F(\mathbb{R}^2, k-1))$. Since we also have a fibration $F(\mathbb{R}^2, k-1) \to F(\mathbb{R}^2, k-2)$ given by projecting onto the first $k-2$ entries, with fiber a copy of $\mathbb{R}^2 \setminus \{x_1, \dots, x_{k-2}\}$ that is a bouquet of circles, the claim follows immediately by induction, knowing that $F(\mathbb{R}^2, 2) \simeq S^1$.

Note that we can identify W_k with the quotient $F(S^2, k)/\mathfrak{S}_{k-1}$ where the symmetric group acts on the first $k-1$ coordinates. In particular in the case $k = 2$, $S \simeq W_2 = F(S^2, 2) \simeq S^2$.

By projecting W_k onto S^2 via the last coordinate, we get a bundle with fiber $B(\mathbb{R}^2, k-1)$. Let us look at the inclusion of the fiber over $\{\infty\} \in S^2 = \mathbb{R}^2 \cup \{\infty\}$ in this bundle

$$B(\mathbb{R}^2, k-1) \hookrightarrow W_k = F(S^2, k)/\mathfrak{S}_{k-1},$$
$$[x_1, \dots, x_{k-1}] \mapsto ([x_1, \dots, x_{k-1}], \infty).$$

Let S^∞ be the direct union of the S^n under inclusion: this is a contractible space. Now S^2 embeds in S^∞ and we have a map of quotients

$$F(S^2,k)/\mathfrak{S}_{k-1} \to F(S^\infty,k)/\mathfrak{S}_{k-1}.$$

The space on the right-hand side projects onto S^∞ with fiber $B(\mathbb{R}^\infty,k-1)$. Since the base space is contractible, there is a homotopy equivalence $F(S^\infty,k)/\mathfrak{S}_{k-1}\simeq B(\mathbb{R}^\infty,k-1)$. Let us consider the composition

$$B(\mathbb{R}^2,k-1)\xrightarrow{\iota} W_k = F(S^2,k)/\mathfrak{S}_{k-1}\to B(\mathbb{R}^\infty,k-1). \tag{96}$$

This composition is homotopic to the map induced on configuration spaces from the inclusion $\mathbb{R}^2\subset\mathbb{R}^\infty$. It is a known useful fact that each embedding $B(\mathbb{R}^n,k)\hookrightarrow B(\mathbb{R}^{n+1},k)$ induces a monomorphism in mod-2 homology.[1] In the case $k=2$ for example, this is $B(\mathbb{R}^n,2)\simeq\mathbb{R}P^{n-1}\to B(\mathbb{R}^{n+1},2)\simeq\mathbb{R}P^n$. This then implies that $B(\mathbb{R}^2,k-1)\hookrightarrow B(\mathbb{R}^\infty,k-1)$ induces in homology mod-2 a monomorphism as well, which then means that the first portion of the composition in (96), which is inclusion of the fiber, injects in homology. Consider the Wang long exact sequence in homology associated to the bundle $W_k\to S^2$ [Mimura and Toda 1991, Theorem 2.5]:

$$H_{q+1}(W_k)\to H_{q-n+1}(B(\mathbb{R}^2,k-1))\to H_q(B(\mathbb{R}^2,k-1))\xrightarrow{\iota_*} H_q(W_k)\to H_{q-n}(B(\mathbb{R}^2,k-1))$$

with $n=2$ in our case. Since ι_* is a monomorphism, the long exact sequence splits into short exact sequences, and because we are working over a field, $H_q(W_k)\cong H_q(B(\mathbb{R}^2,k-1))\oplus H_{q-2}(B(\mathbb{R}^2,k-1))$. Since $H_*(W_k)\cong H_*(S)$, the proof is complete. □

Remark 5.2. The top mod-2 homology group $H_k(S)$ is trivial if $k-1$ is not a binary power and is a copy of $\mathbb{Z}_2$ if $k-1$ is a binary power. This is because $H_{k-2}(B(\mathbb{R}^2,k-1))$ satisfies the same condition [Fuks 1970, p. 146], by Lemma 5.1.

Lemma 5.3. *Suppose $k\ge 3$. The manifold S defined in (52) is not orientable.*

Proof. We first observe that the manifold $S_k^2\setminus S_{k-1}^2$ is not orientable for any $k\ge 2$. From the proof of Lemma 5.1,

$$S_k^2\setminus S_{k-1}^2=\dot{\Delta}_{k-1}\times_{\mathfrak{S}_k}F(S^2,k)$$

is a bundle over $B(S^2,k)$ with fiber the open simplex. Since $B(S^2,k)$ is orientable (because unordered configuration spaces of smooth manifolds are orientable if and only if the dimension of the manifold is even), the orientability of the total space is the same as the orientability of the bundle. But the braid generators of the fundamental group of $B(S^2,k)$ act (after restriction to the open simplex) by transpositions on the vertices of $\overline{\Delta}_{k-1}$ and this is orientation reversing, so the bundle is not orientable.

Now let V_k be the subset of $S_k^2\setminus S_{k-1}^2$ of all sums $\sum t_i\delta_{x_i}$ with $x_i=\infty$ for some i. Again ∞ stands for the north pole of $S^2=\mathbb{R}^2\cup\{\infty\}$. Here $V_k\simeq B(\mathbb{R}^2,k-1)$. Note that $\pi_1(B(\mathbb{R}^2,k-1))$ embeds in $\pi_1(B(S^2,k))$ with similar braid generators. For the exact same reason as for $S_k^2\setminus S_{k-1}^2$, V_k is not orientable.

[1] This follows from the work of Cohen [1976], who first calculated $H_*(B(\mathbb{R}^n,k);\mathbb{F})$ for all n and k and for $\mathbb{F}=\mathbb{Z}_2,\mathbb{Z}_p$, p odd.

Consider finally the manifold

$$\boldsymbol{S} = \big\{\big(\nu, \delta_y, \tfrac{1}{2}\big) \in S^2_k * S^2 : \nu \in S^2_k \setminus S^2_{k-1},\ \ y \in \operatorname{supp}(\nu)\big\}.$$

Then S is a codimension-0 submanifold of $\boldsymbol{S}$ (with boundary) that is also a deformation retract. Both S and $\boldsymbol{S}$ have the same orientation. But there is a bundle map $\boldsymbol{S} \to S^2$ with fiber V_k. It is easy to see now that the orientation of $\boldsymbol{S}$ is that of V_k. Indeed the bundle over the open upper hemisphere D of S^2 is trivial and thus homeomorphic to $V_k \times D$. This is an open subset of $\boldsymbol{S}$ that is nonorientable; thus, $\boldsymbol{S}$ must be nonorientable. □

Lemma 5.4. *Let $k \geq 3$. Then $Y = (S^2_k * S) \setminus S$ has the Euler characteristic of a contractible space*, *i.e.*, $\chi(Y) = 1$.

Proof. By the previous lemma, S is up to homotopy a degree-k covering of $B(S^2, k)$. This gives

$$\chi(S) = k\chi(B(S^2, k)) = k\frac{1}{k!}\chi(F(S^2, k)) = \frac{1}{(k-1)!}\chi(S^2)\chi(F(\mathbb{R}^2, k-1)) = 0.$$

Here what vanishes is $\chi(F(\mathbb{R}^2, k-1)) = 0$ since, letting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, there are homeomorphisms

$$F(\mathbb{R}^2, k-1) = \mathbb{R}^2 \times F(\mathbb{R}^2 \setminus \{(0,0)\}, k-2) = \mathbb{R}^2 \times \mathbb{C}^* \times F(\mathbb{C}^* \setminus \{1\}, k-3)$$

and $\chi(\mathbb{C}^*) = \chi(S^1) = 0$.

On the other hand, S is a smooth $(3k-1)$-dimensional manifold with boundary. A neighborhood of S in $S^2_k * S^2$ is a $(3k+2)$-dimensional open manifold U_δ. This neighborhood is the union of two open subspaces A and B, where A is a fiberwise cone over the interior of S and B is a bundle over ∂S with fiber the cone over a hemisphere. The complement $\overline{U}_\delta \setminus S$ is the union of two subspaces $\widetilde{A}$ and $\widetilde{B}$, where $\widetilde{A}$ retracts onto an S^2-bundle over the interior of S while $\widetilde{B}$ is up to homotopy ∂S. Clearly $\widetilde{A} \cap \widetilde{B}$ retracts onto an S^2-bundle over ∂S. We can then write

$$\begin{aligned}\chi(U_\delta \setminus S) = \chi(\widetilde{A} \cup \widetilde{B}) = \chi(\widetilde{A}) + \chi(\widetilde{B}) - \chi(\widetilde{A} \cap \widetilde{B}) &= 2\chi(S) + \chi(\partial S) - 2\chi(\partial S) \\ &= 2\chi(S) - \chi(\partial S). \qquad (97)\end{aligned}$$

We know that, for a manifold S of dimension m with boundary,

$$\chi(\partial S) = \chi(S) - (-1)^m \chi(S).$$

Since $\chi(S) = 0$, we get $\chi(\partial S) = 0$ and therefore $\chi(U_\delta \setminus S) = 0$ by (97).

Now cover $X = S^2_k * S^2$ by means of $U_\delta \simeq S$ and $Y = X \setminus S$. The inclusion-exclusion property of the Euler characteristic gives that

$$\chi(X) = \chi(U_\delta) + \chi(Y) - \chi(U_\delta \setminus S) = \chi(S) + \chi(Y) = \chi(Y)$$

so that $\chi(Y) = \chi(X)$. But $\chi(X) = 1$ since $\chi(X) = \chi(S^2_k * S^2) = \chi(S^2_k) + \chi(S^2) - \chi(S^2_k)\chi(S^2)$, and $\chi(S^2_k) = 1$ for $k \geq 3$ by the formula

$$\chi(Z_k) = 1 - \frac{1}{k!}(1-\chi)(2-\chi)\cdots(k-\chi)$$

for any surface Z [Malchiodi 2008a] and more generally for any simplicial complex Z [Kallel and Karoui 2011] with $\chi = \chi(Z)$. $\square$

Lemma 5.5. *The set Y is simply connected.*

Proof. Using the same notation as in the proof of the previous lemma, we have the pushout

$$\begin{tikzcd} \widetilde{A} \cap \widetilde{B} \arrow[r] \arrow[d] & \widetilde{A} \arrow[d] \\ \widetilde{B} \arrow[r] & \overline{U}_\delta \setminus S \end{tikzcd}$$

Recall that $\widetilde{A}$ is up to homotopy an S^2-bundle over S, $\widetilde{B} \simeq \partial S$, and $\widetilde{A} \cap \widetilde{B}$ is an S^2-bundle over ∂S. This means that $\pi_1(\widetilde{A} \cap \widetilde{B}) = \pi_1(\partial S)$ and $\pi_1(\widetilde{A}) \cong \pi_1(S)$. We therefore have the pushout in the category of groups (by the van Kampen theorem)

$$\begin{tikzcd} \pi_1(\partial S) \arrow[r] \arrow[d, "\cong"] & \pi_1(S) \arrow[d] \\ \pi_1(\partial S) \arrow[r] & \pi_1(U_\delta \setminus S) \end{tikzcd}$$

which shows that $\pi_1(U_\delta \setminus S) \cong \pi_1(S) \cong \pi_1(U_\delta)$. Observe that we have used the fact that $\overline{U}_\delta \setminus S \simeq U_\delta \setminus S$ since we are removing the boundary from a manifold not intersecting S. On the other hand, we can use the same open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Since X is a join of connected spaces, it is 1-connected. The pushout of groups

$$\begin{tikzcd} \pi_1(U_\delta \setminus S) \arrow[r] \arrow[d, "\cong"] & \pi_1(X \setminus S) \arrow[d] \\ \pi_1(U_\delta) \arrow[r] & 0 \end{tikzcd}$$

implies that, because the left-hand vertical map is an isomorphism, the right-hand vertical map must be an isomorphism as well and $\pi_1(X \setminus S) = \pi_1(Y) = 0$. $\square$

Despite the fact that Y is simply connected and has unit Euler characteristic, it is noncontractible.

Proposition 5.6. *Suppose $k \geq 2$ and $k \neq 4$. Then the subspace*

$$Y = (S^2_k * S^2) \setminus S$$

is not contractible.

Proof. We assume that Y is contractible and derive a contradiction. The main step is to prove that under this condition with mod-2 coefficients we must have

$$H_*(S) \cong H_{3k-1-*}(S^2_k), \quad 0 \leq * \leq k. \tag{98}$$

This will then be shown to be impossible.

The closed subset S has a neighborhood U_δ that is $(3k+2)$-dimensional with $(3k+1)$-dimensional boundary $\partial \overline{U}_\delta$. Using Poincaré's duality with mod-2 coefficients for the closed manifold $\partial \overline{U}_\delta$ gives us

$$H^*(\partial \overline{U}_\delta) \cong H_{3k+1-*}(\partial \overline{U}_\delta).$$

Since $\overline{U}_\delta \setminus S$ retracts onto $\partial \overline{U}_\delta$ and homology is dual to cohomology for finite-type spaces and field coefficients, we can conclude that

$$H_*(\overline{U}_\delta \setminus S) \cong H_{3k+1-*}(\overline{U}_\delta \setminus S), \quad * \geq 0. \tag{99}$$

Next we turn to the open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Using that $Y \cap U_\delta = U_\delta \setminus S$ and $U_\delta \simeq S$, the Mayer–Vietoris sequence for this union takes the form

$$H_*(U_\delta \setminus S) \to H_*(S) \oplus H_*(Y) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \oplus H_{*-1}(Y) \to H_{*-1}(X) \to \cdots .$$

Since Y has trivial reduced homology by assumption, the sequence becomes

$$H_*(U_\delta \setminus S) \to H_*(S) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \to H_{*-1}(X) \to \cdots . \tag{100}$$

But S has homological dimension k (see Lemma 5.1), so for $* > k+1$, we have the isomorphism $H_{*-1}(U_\delta \setminus S) \cong H_*(X)$. Since X is the third suspension of S^2_k, $H_*(X) \cong H_{*-3}(S^2_k)$ and thus

$$H_*(U_\delta \setminus S) \cong H_{*-2}(S^2_k), \quad * > k. \tag{101}$$

It is generally known [Kallel and Karoui 2011] that the barycentric set Z_k is $(2k+r-2)$-connected whenever Z is r-connected, $r \geq 1$. If $Z = S^2$, which is 1-connected, S^2_k is $(2k-1)$-connected and so X is $(2k+2)$-connected. In the range $* \leq 2k+2$, $\widetilde{H}_*(X) = 0$. The Mayer–Vietoris sequence (100) leads in this case to

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad * < 2k+2.$$

Since S has no homology beyond degree k, we can focus on the range below so that

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad 0 \leq * \leq k. \tag{102}$$

We can now combine all previous isomorphisms into one for $0 \leq * \leq k$:

$$H_*(S) \xrightarrow[(102)]{\cong} H_*(U_\delta \setminus S) \xrightarrow[(99)]{\cong} H_{3k+1-*}(U_\delta \setminus S) \xrightarrow[(101)]{\cong} H_{3k-1-*}(S^2_k).$$

This is the claim in (98). Note that S^2_k is $(3k-1)$-dimensional as a CW complex and is $(2k-1)$-connected, so its homology is nonzero only in the range $2k \leq * \leq 3k-1$.

The isomorphism $H_*(S) \cong H_{3k-1-*}(S^2_k)$ cannot hold. First let us check the case $k = 2$. In that case, we pointed out in the proof of Lemma 5.1 that $S \simeq F(S^2, 2) \simeq S^2$. Since $S^2_2 \simeq \Sigma^3 \mathbb{R}P^2$ (the 3-fold suspension of $\mathbb{R}P^2$ [Kallel and Karoui 2011, Corollary 1.6]), the isomorphism obviously cannot hold: in fact, $H_1(S^2) = 0$ but $H_4(\Sigma^3 \mathbb{R}P^2) = H_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

Suppose that $k \geq 3$. According to Theorem 1.3 in [Kallel and Karoui 2011], S_k^2 has the same homology as (one desuspension) of the symmetric smash product $\overline{\mathrm{SP}}^k(S^3) = (S^3)^{\wedge k}/\mathfrak{S}_k$; i.e., $H_*(S_k^2) \cong H_{*+1}(\overline{\mathrm{SP}}^k(S^3))$. Combining this with (98), we get

$$H_*(S) \cong H_{3k-*}(\overline{\mathrm{SP}}^k(S^3)), \quad 0 \leq * \leq k. \tag{103}$$

We will show that this is impossible. To that end, we need to describe the groups on both sides of (103). We work again mod-2. From Lemma 5.1, we have that

$$H_*(S) \cong H_*(B(\mathbb{R}^2, k-1)) \oplus H_{*-2}(B(\mathbb{R}^2, k-1)), \quad * \geq 0.$$

(when $* - 2 < 0$ the corresponding group is zero). The mod-2 homology of $B(\mathbb{R}^2, k-1)$ has been computed by Fuks [1970], and it is best described as a subspace of the polynomial algebra (viewed as an infinite vector space generated by powers of the indicated generators)

$$\mathbb{Z}_2[a_{(1,2)}, a_{(3,4)}, \ldots, a_{(2^i-1,2^i)}, \ldots], \tag{104}$$

where the notation $a_{i,j}$ refers to a generator having homological degree i and a certain *filtration* degree j, both degrees being additive under multiplication of generators. Now the condition for an element $a^{k_1}_{(2^{i_1}-1,2^{i_1})} \cdots a^{k_r}_{(2^{i_r}-1,2^{i_r})} \in H_*(B(\mathbb{R}^2, k-1))$ is that its filtration degree is less than or equal to $k-1$, that is, if and only if $\sum_{i_s} k_{i_s} 2^{i_s} \leq k-1$.

For example, $\widetilde{H}_*(B(\mathbb{R}^2, 2)) = \mathbb{Z}_2\{a_{(1,2)}\}$ (one copy of $\mathbb{Z}_2$ generated by $a_{(1,2)}$ having homological degree 1 and filtration degree 2). Similarly $\widetilde{H}_*(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(1,2)}, a^2_{(1,2)}, a_{(3,4)}\}$ so that

$$H_1(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(1,2)}\}, \qquad H_2(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a^2_{(1,2)}\}, \qquad H_3(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(3,4)}\}.$$

Now $H_*(B(\mathbb{R}^2, 5)) \cong H_*(B(\mathbb{R}^2, 4))$, and this turns out to be a general fact explained in Proposition 5.9 in more geometric terms.

On the other hand, the reduced groups $\widetilde{H}_*(\overline{\mathrm{SP}}^k(S^3))$ form a subvector space of the polynomial algebra

$$\mathbb{Z}_2[\iota_{(3,1)}, f_{(5,2)}, f_{(9,4)}, \ldots, f_{(2^{i+1}+1,2^i)}, \ldots] \tag{105}$$

consisting of those elements of second filtration degree precisely k (see the Appendix in [Kallel and Karoui 2011] and references therein). Here again $f_{(2^{i+1}+1,2^i)}$ denotes an element of homological degree $2^{i+1}+1$ and filtration degree 2^i. For example, (here $\iota = \iota_{(3,1)}$)

$$\widetilde{H}_*(\overline{\mathrm{SP}}^4 S^3) = \mathbb{Z}_2\{\iota^4, \iota^2 f_{(5,2)}, f^2_{(5,2)}, f_{(9,4)}\},$$

which is better listed as

$$H_{12}(\overline{\mathrm{SP}}^4 S^3) = \mathbb{Z}_2\{\iota^4\}, \qquad H_{11}(\overline{\mathrm{SP}}^4 S^3) = \mathbb{Z}_2\{\iota^2 f_{(5,2)}\},$$
$$H_{10}(\overline{\mathrm{SP}}^4 S^3) = \mathbb{Z}_2\{f^2_{(5,2)}\}, \qquad H_9(\overline{\mathrm{SP}}^4 S^3) = \mathbb{Z}_2\{f_{(9,4)}\}.$$

This space $\overline{\mathrm{SP}}^4(S^3)$ is 8-connected, and more generally, $\overline{\mathrm{SP}}^k(S^3)$ is $2k$-connected [Kallel and Karoui 2011].

Let us now compare the groups in (103). When $* = 0$, $H_0(S) = \mathbb{Z}_2$ but so is $H_{3k}(\overline{\mathrm{SP}}^k(S^3))$ generated by the class $\iota^k_{(3,1)}$. Also when $* = 1$ and $k \geq 3$, $H_1(S) = H_1(B(\mathbb{R}^2, k-1)) = \mathbb{Z}_2$ but so is

$H_{3k-1}(\overline{\mathrm{SP}}^k(S^3))$ generated by $\{\iota^{k-2} f_{5,2}\}$. There is no contradiction yet. When $* = 2$, we get the generator $a^2_{(1,2)} \in H_2(B(\mathbb{R}^2, k-1)) \cong \mathbb{Z}_2$ as soon as $k \geq 5$ ($a^2_{(1,2)}$ is in filtration 4). This gives that $H_2(S) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We claim however that $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3)) = \mathbb{Z}_2$, which will give a contradiction in that case. Indeed a generator in filtration degree k in (105) is written as a finite product

$$\iota^{k_0} f^{k_1}_{5,2} \cdots f^{k_i}_{(2^{i+1}+1, 2^i)} \cdots, \qquad \sum_{i \geq 0} k_i 2^i = k.$$

The homological degree of this class is $\sum_{i \geq 0} k_i (2^{i+1} + 1) = 2 \sum_{i \geq 0} k_i 2^i + \sum_{i \geq 0} k_i$. To obtain the rank of H_{3k-2}, we need to find all the possible sequences of integers $(k_0, k_1, k_2, \dots)$ such that $\sum_{i \geq 0} k_i 2^i = k$ and $2 \sum_{i \geq 0} k_i 2^i + \sum_{i \geq 0} k_i = 3k - 2$. We have to solve for

$$\sum_{i \geq 0} k_i 2^i = k = 2 + \sum_{i \geq 0} k_i.$$

This immediately gives that $k_i = 0$, $i \geq 2$. There is one and only one solution: $k_0 = k - 4$ and $k_1 = 2$. And the group $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3)) \cong \mathbb{Z}_2$ is generated by $\iota^{k-4} f^2_{5,2}$.

The isomorphism (103) cannot hold for $k \geq 5$. We are left to consider the case $k = 3$: here $H_3(S) = \mathbb{Z}_2$ but $H_6(\overline{\mathrm{SP}}^3(S^3)) = 0$, giving a contradiction.

In conclusion since the isomorphism (103) (equivalently (98)) cannot hold, Y must have nontrivial mod-2 homology and thus cannot be contractible as we had asserted. □

The next proposition treats the case $k = 4$: in preparation, we need the following lemma. Recall that S is a manifold with boundary embedded in $\overline{U}_\delta \subset S^2_k * S^2$. We can write $\overline{U}_\delta$ as the union of two sets $\bar{A}$ and $\bar{B}$, where $\bar{A}$ is a three-dimensional disk bundle over S and $\bar{A} \cap \bar{B}$ its restriction over ∂S. We refer to this bundle as the *normal disk bundle* and its boundary as the *sphere normal bundle*. Note that, in the proof of Lemma 5.4, we have used $\widetilde{A} = \bar{A} \setminus S$ and $\widetilde{B} = \bar{B} \setminus S$.

Lemma 5.7. *The sphere normal bundle over ∂S is orientable.*

Proof. We will view this bundle as an extension of a normal sphere bundle over the interior $\dot{S} := \mathrm{int}(S)$ that is orientable (in doing so, we give more details on the construction of $\bar{A}$ and $\bar{A} \cap \bar{B}$).

We recall that the join is given by the equivalence relation $X * Y = X \times Y \times I/\sim$, where $\sim$ are identifications at the endpoints of $I = [0, 1]$; see (9). The join contains the open dense subset $X \times Y \times (0, 1)$ (let us call it the *big cell*). This subset is a manifold of dimension $n + m + 1$ if X and Y are manifolds of dimensions n and m, respectively. In our case, S is a subset of the big cell

$$(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1) \subset (S^2_k \setminus (S^2_{k-1})^\delta) * S^2$$

and $\mathrm{int}(S)$ is regularly embedded as a differentiable submanifold. It therefore has a unit normal disk bundle (of dimension 3) in there. This is homeomorphic to a tubular neighborhood V^δ of $\mathrm{int}(S)$. Let us use the same name for the neighborhood and the normal bundle. The normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1)$ is the normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times \{\frac{1}{2}\}$, to which we add a trivial line bundle. We can then consider directly $\dot{S}$ as a subset of $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2$ and show that it has an orientable rank-2 normal bundle there. Write $D_k := S_k^2 \setminus (S_{k-1}^2)^\delta$ and

$$S = \left\{ \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \in D_k \times S^2 : x = x_i \text{ for some } i \right\}.$$

Define V^δ the neighborhood of S in $D_k \times S^2$ as

$$V^\delta = \left\{ \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \in D_k \times S^2 : |x - x_i| < \frac{\delta}{2} \text{ for some and hence unique } x_i \right\}.$$

The choice of x_i is unique as x cannot be strictly within $\delta/2$ of two distinct x_i and x_j since $d(x_i, x_j) \geq \delta$ according to the definition of S. The neighborhood retracts back to S via the map

$$\left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \mapsto \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x_i \right)$$

where $d(x, x_i) < \delta/2$. Consider the projection map $\pi : \dot{S} \to S^2$ sending $(\sum_{i=1}^{k} t_i \delta_{x_i}, x) \mapsto x_i$ if $d(x, x_i) < \delta/2$. We claim that the normal bundle of $\dot{S}$ in $D_k \times S^2$ is isomorphic to the pullback via π of the tangent bundle TS^2 over S^2. We assume δ to be less than the injectivity radius of S^2. Define a homeomorphism between the tubular neighborhood V^δ of $\dot{S}$ and a normal disk bundle of the pullback of TS^2 over $\dot{S}$ by sending $(\sum_{i=1}^{k} t_i \delta_{x_i}, x)$ with $d(x, x_i) < \delta/2$ for some i to the element in the pullback

$$\left(\left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right), \exp_{x_i}^{-1}(x) \right)$$

where $\exp_{x_i}$ is the exponential map at $x_i \in S^2$. This map is a homeomorphism onto its image, and the normal bundle to $\dot{S}$ in $D_k \times S^2$ is isomorphic to TS^2. Since TS^2 is orientable (although nontrivial), the normal bundle over $\dot{S}$ is orientable. This bundle can be extended to S by taking the closure of V^δ in $D_k \times S^2 := (S^2 \setminus (S_{k-1}^2)^\delta) \times S^2 \times \{\frac{1}{2}\}$. This extension is orientable over all of S since it is orientable over the interior. By adding a line bundle, we get the normal bundle over S in the big cell. This bundle is orientable over all of S and in particular over ∂S. This is our claim. □

Proposition 5.8. *The subspace $Y = (S_k^2 * S^2) \setminus S$ is not contractible for all $k \geq 2$.*

Proof. As before, we assume Y is contractible and derive a contradiction. We first show that for any field coefficients $\mathbb{F}$ and $* > k$

$$H_{*+3}(U_\delta \setminus S) \cong H_*(\partial S). \tag{106}$$

Write as before $\overline{U}_\delta \setminus S$ as the union $\widetilde{A} \cup \widetilde{B}$ with $\widetilde{A} \cap \widetilde{B}$ retracting onto the S^2-bundle over ∂S discussed earlier. The Mayer–Vietoris sequence for the union $\widetilde{A} \cup \widetilde{B}$ is given by

$$H_{n+1}(\widetilde{A} \cap \widetilde{B}) \to H_{n+1}(\widetilde{A}) \oplus H_{n+1}(\widetilde{B}) \to H_{n+1}(U_\delta \setminus S) \to H_n(\widetilde{A} \cap \widetilde{B}) \to H_n(\widetilde{A}) \oplus H_n(\widetilde{B}) \to H_n(U_\delta \setminus S).$$

As S has homological dimension at most k and $\widetilde{A}$ is an S^2-bundle over it, $H_n(\widetilde{A})$ vanishes for $n > k+2$. On the other hand, the S^2-bundle over ∂S is orientable (Lemma 5.7) and has a global section; this follows from the fact that the normal bundle over S has a trivial summand and hence there is a nonzero section over all S that we can restrict to ∂S. By the Gysin sequence [Hatcher 2002, §4.D], one has a splitting

$$H_n(\widetilde{A}\cap\widetilde{B}) \cong H_n(\partial S)\oplus H_{n-2}(\partial S), \quad n>2.$$

Replacing in the Mayer–Vietoris sequence gives for $n>k+2$

$$\cdots \to \begin{matrix} H_{n+1}(\partial S)\\ \oplus \\ H_{n-1}(\partial S)\end{matrix} \xrightarrow{\phi_{n+1}} H_{n+1}(\partial S) \to H_{n+1}(U_\delta\setminus S) \to \begin{matrix} H_{n}(\partial S)\\ \oplus \\ H_{n-2}(\partial S)\end{matrix} \xrightarrow{\phi_n} H_n(\partial S) \to \cdots .$$

Now, in every inclusion of $\widetilde{A}\cap\widetilde{B}$ into $\widetilde{B}$, the fibers (i.e., S^2) contract to a point. Therefore, ϕ_n is trivial on the bottom group while restricted to the top group it is a bijection. This map is an epimorphism, and the long exact sequence for $n>k+2$ splits into short exact sequences

$$0\to H_{n+1}(U_\delta\setminus S)\to H_n(\partial S)\oplus H_{n-2}(\partial S)\to H_n(\partial S)\to 0.$$

As vector spaces, we get $H_{n+1}(U_\delta\setminus S)\cong H_{n-2}(\partial S)$, which is our claim. Combined with (101), this yields

$$H_*(\partial S)\cong H_{*+1}(S^2_k), \quad *>k. \tag{107}$$

Next we look at the Mayer–Vietoris sequence for the union $S^2_k=(S^2_k\setminus S^2_{k-1})\cup(S^2_{k-1})^\delta$. It is shown in [Malchiodi 2008a] that $(S^2_{k-1})^\delta\setminus S^2_{k-1}$ retracts onto $\partial(S^2_{k-1})^\delta$ so that the long exact sequence becomes

$$\cdots\to H_{n+1}(\partial(S^2_{k-1})^\delta)\to H_{n+1}(S^2_{k-1})\oplus H_{n+1}(S^2_k\setminus S^2_{k-1})\to H_{n+1}(S^2_k)\to H_n(\partial(S^2_{k-1})^\delta)\to\cdots.$$

Since the inclusion of S^2_{k-1} in S^2_k is contractible and since $S^2_k\setminus S^2_{k-1}\simeq B(S^2,k)$ has homological dimension k (see Lemma 5.1), for $n>k$, the short sequence

$$0\to H_{n+1}(S^2_k)\to H_n(\partial(S^2_{k-1})^\delta)\to H_n(S^2_{k-1})\to 0$$

is exact and we have the splitting

$$H_*(\partial(S^2_{k-1})^\delta)\cong H_*(S^2_{k-1})\oplus H_{*+1}(S^2_k), \quad *>k. \tag{108}$$

Both isomorphisms (107) and (108) cannot hold simultaneously as we now explain.

A key point to observe is that ∂S is a degree-k regular covering of $\partial(S^2_{k-1})^\delta$. A property of a covering $\pi: X\to Y$ is the existence of a transfer morphism $\mathrm{tr}: H_*(Y)\to H_*(X)$ so that $\pi_*\circ\mathrm{tr}$ is multiplication in $H_*(Y)$ by the degree of the covering, i.e., by k [Hatcher 2002, §3.G]. If the characteristic of the field of coefficients is prime to k, then this composite is nontrivial and $H_*(Y)$ injects into $H_*(X)$.

When $k=4$, we have a degree-4 covering $\partial S\to\partial(S^2_3)^\delta$ so that with $\mathbb{F}=\mathbb{F}_3$-coefficients (the finite field with 3 elements) we must have a monomorphism $H_*(\partial(S^2_3)^\delta;\mathbb{F}_3)\hookrightarrow H_*(\partial S;\mathbb{F}_3)$. When $*>4$, upon combining (107) and (108), we get a monomorphism

$$H_*(S^2_3;\mathbb{F}_3)\oplus H_{*+1}(S^2_4;\mathbb{F}_3)\to H_{*+1}(S^2_4;\mathbb{F}_3).$$

This leads immediately to a contradiction if $H_*(S^2_3;\mathbb{F}_3)\neq 0$ in that range of dimensions.

We know that $H_*(S^2_3) \cong H_{*+1}(\overline{\mathrm{SP}}^3(S^3))$. We therefore wish to show that $H_*(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) \neq 0$ for some $* \geq 6$. It turns out that old calculations of Nakaoka [1956] give us precisely the answer. Nakaoka's Theorem 15.5 states that

$$H^r(\mathrm{SP}^3(S^n); \mathbb{F}_3) \cong \mathbb{F}_3$$

for $r = 0, n, n+4k$ with $1 \leq k \leq [n/2]$ and $k \neq [n/4]$, $r = n+4k+1$ with $1 \leq k \leq [(2n-1)/4]$ and $k \neq [(n-1)/4]$, and $r = 2n$ with $n \equiv -2$ or $1 \pmod 4$. In our case $n = 3$, so $H^r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 0, 3, 7, 8$. Dually we obtain the same groups for $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3)$ (since working over a field). But $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3)$ for $r > 3$ for the following three reasons:

- By construction, $H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) = H_r(\mathrm{SP}^3(S^3), \mathrm{SP}^2(S^3); \mathbb{F}_3)$, $r \geq 1$.
- There is a splitting due originally to Steenrod (for any coefficients [Kallel and Karoui 2011]):
$$H_r(\mathrm{SP}^3(S^3)) \cong H_r(\mathrm{SP}^3(S^3), \mathrm{SP}^2(S^3)) \oplus H_r(\mathrm{SP}^2(S^3)).$$
- $H_r(\mathrm{SP}^2(S^3); \mathbb{F}_3) = 0$ if $r > 3$. In fact, from the covering $(S^3)^2 \to \mathrm{SP}^2(S^3)$, by a consequence of the transfer construction, $H_*(\mathrm{SP}^2(S^3); \mathbb{F}_3)$ is the subvector space of invariant cohomology classes in $H_*(S^3 \times S^3)$ under the induced permutation action interchanging the two spheres. Since S^3 is an odd sphere, the involution acts via $\tau_*([S^3] \otimes [S^3]) = -[S^3] \otimes [S^3]$ and the class $[S^3] \otimes [S^3]$ is not invariant so maps to 0 in $H_*(\mathrm{SP}^2(S^3); \mathbb{F}_3)$.

As a consequence, $H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 7, 8$, which gives a contradiction as we had asserted. □

Note that using the transfer property for the homology of a covering used in the proof of Proposition 5.8 we can give an alternative proof of Proposition 5.6 for k odd.

To conclude this topological discussion, it is worthwhile noting that Lemma 5.1 can be used to give a novel proof of the following result on the mod-2 homology of unordered configurations of points in $\mathbb{R}^n$:

Proposition 5.9. *For k odd and $n \geq 2$, one has*

$$H_*(B(\mathbb{R}^n, k); \mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1); \mathbb{Z}_2).$$

Proof. All homology is with mod-2 coefficients. A starting point is the homology splitting

$$H_q(B(S^n, k)) \cong H_q(B(\mathbb{R}^n, k)) \oplus H_{q-n}(B(\mathbb{R}^n, k-1)). \tag{109}$$

One reference to this result is Theorem 18(1) of [Salvatore 2004]. It is also a special case of a similar result of Kallel, where one can replace the sphere by any closed manifold M and $\mathbb{R}^n$ by $M \setminus \{p\}$, its punctured version. Let $W_{n,k} := F(S^n, k)/\mathfrak{S}_{k-1}$ where $\mathfrak{S}_{k-1}$ acts by permutations on the first $k-1$ coordinates. By projecting onto the last coordinate, we obtain a bundle over S^n with fiber $B(\mathbb{R}^n, k-1)$. Precisely as in the proof of Lemma 5.1, we see that

$$H_*(W_{n,k}) \cong H_*(B(\mathbb{R}^n, k-1)) \oplus H_{*-n}(B(\mathbb{R}^n, k-1)). \tag{110}$$

Consider next the degree-k regular covering $\pi : W_{n,k} \to B(S^n, k) := F(S^n, k)/\mathfrak{S}_k$. There is a transfer morphism $\mathrm{tr} : H_*(B(S^n, k)) \to H_*(W_{n,k})$ so that the composite $\pi_* \circ \mathrm{tr}$ is multiplication by k. Since

k is odd and thus prime to the characteristic of the field $\mathbb{Z}_2$, multiplication by k is injective and $H_*(B(S^n, k))$ necessarily embeds in $H_*(W_{n,k})$; that is, (109) embeds into (110). But $H_*(B(\mathbb{R}^n, k-1))$ always embeds into $H_*(B(\mathbb{R}^n, k))$ (in fact for any coefficients as it is relatively easy to see). This means that $H_*(B(\mathbb{R}^n, k); \mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1); \mathbb{Z}_2)$ if k is odd as claimed. It also means that $H_*(B(S^n, k)) \cong H_*(W_{n,k})$. □

5.2. *Min-max scheme.* To prove Theorem 1.1, we will run a min-max scheme based on (a retraction of) the set Y in (51). More precisely, we will consider the set $Y_{\mathcal{R}}$ introduced in (94) on which the test functions Φ_λ are modeled. Some parts are quite standard and follow the ideas of [Ding et al. 1999] (see [Malchiodi 2008a] for a Morse-theoretical point of view). For the specific problem (1), the crucial step is Proposition 5.10, giving information on the topology of the low sublevels of J_ρ; see also the comments after the proof.

Given any $L > 0$, Proposition 4.17 guarantees us the existence of $\lambda > 1$ sufficiently large such that $J_\rho(\Phi_\lambda(\nu, p, s)) < -L$ for any $(\nu, p, s) \in Y_{\mathcal{R}}$. Recalling $\widetilde{\Psi}$ in (29), we take L so large that Corollary 3.8 applies, i.e., such that $\widetilde{\Psi}(J_\rho^{-L}) \subseteq Y$. The crucial step in describing the topology of the low sublevels of J_ρ is:

Proposition 5.10. *Let L and λ be as above, and let $\mathcal{F}$ be the retraction given before (94). Then the composition*

$$Y_{\mathcal{R}} \xrightarrow{\Phi_\lambda} J_\rho^{-L} \xrightarrow{\mathcal{F}\circ\widetilde{\Psi}} Y_{\mathcal{R}}$$

is homotopically equivalent to the identity map on $Y_{\mathcal{R}}$.

Proof. We divide the proof in three cases, depending on the values of the join parameter s.

Case 1. Let $s \in [\frac{3}{4}, 1]$. In this case, the test functions we are considering have the form $(\varphi_1^t, \varphi_2^t)$, $t = t(s)$, as defined in Section 4.2.3. Notice that, as discussed at the beginning of the proof of Proposition 4.7, most of the integral of $e^{\varphi_2^t}$ is localized near p and $\sigma_2(\varphi_2^t) \ll \sigma_1(\varphi_1^t)$ for these values of s, which again implies $s(\varphi_1^t, \varphi_2^t) = 1$; see (26). It turns out that, by the construction in Section 3.1, one has

$$\widetilde{\Psi}(\Phi_\lambda(\nu, p, s)) = \widetilde{\Psi}(\varphi_1^t, \varphi_2^t) = (*, \tilde{p}, 1),$$

where $*$ is an irrelevant element of Σ_k (recall that they are all identified when the join parameter equals 1; see (9)) and where $\tilde{p} \in \Sigma$ is a point close to p. If $p(\mathrm{t}) : [0, 1] \to \Sigma$ is a geodesic joining p to $\tilde{p}$, one can realize the desired homotopy as

$$((\nu, p, s); \mathrm{t}) \mapsto (\nu, p(\mathrm{t}), (1-\mathrm{t})s + \mathrm{t}), \qquad \mathrm{t} \in [0, 1].$$

Case 2. Let $s \in [\frac{1}{4}, \frac{3}{4}]$. The test functions we are considering here are given in Section 4.2.1. For this range of s, the exponential of the first component φ_1 (see (82)) is well-concentrated around the points $\tilde{x}_i$; see (77). The exponential of the second component φ_2, depending on the value of s, will instead either be concentrated near p or will be spread over Σ in the sense that $\sigma_2(\varphi_2)$ might not be small. Recall the maps $\widetilde{\psi}_l$ given in Proposition 2.4 and the definition of $\hat{\nu}$ involved in the construction of the test functions given in (75): $\hat{\nu} = \mathfrak{R}_p(\nu) = \sum_{i=i}^{k} t_i \delta_{x_i}$. We then have

$$\widetilde{\Psi}(\Phi_\lambda(\nu, p, s)) = \widetilde{\Psi}(\varphi_1, \varphi_2) = \begin{cases} (\widetilde{\psi}_k(\varphi_1), \widetilde{\psi}_1(\varphi_2), s(\varphi_1, \varphi_2)) & \text{if } \sigma_2(\varphi_2) \text{ small}, \\ (\widetilde{\psi}_k(\varphi_1), *, 0) & \text{otherwise}, \end{cases}$$

with $\widetilde{\psi}_1(\varphi_2)$ close to p (whenever defined, i.e., for $\sigma_2(\varphi_2)$ small) and $\widetilde{\psi}_k(\varphi_1)$ close to $\sum_{i=1}^{k} t_i \delta_{\tilde{x}_i}$ in the distributional sense. Furthermore, writing $\varphi_1 = \varphi_{1,\lambda}$ to emphasize the dependence on λ, it turns out that

$$\widetilde{\psi}_k(\varphi_{1,\lambda}) \to \sum_{i=1}^{k} t_i \delta_{\tilde{x}_i} \quad \text{as } \lambda \to +\infty,$$

which gives us the homotopy

$$(\nu; \mathfrak{t}) \mapsto \widetilde{\psi}_k(\varphi_{1,\lambda/\mathfrak{t}}), \qquad \mathfrak{t} \in [0, 1].$$

Reasoning as in Step 3 of Section 4.2.2, we get a homotopy that deforms the points $\tilde{x}_i$ to the original ones x_i. Letting $\tilde{\gamma}_i$ be the geodesic joining $\tilde{x}_i$ and x_i in unit time, we consider

$$(\nu; \mathfrak{t}) \mapsto \sum_{i=1}^{k} t_i \delta_{\tilde{\gamma}_i(1-\mathfrak{t})}, \qquad \mathfrak{t} \in [0, 1].$$

Notice that for $\mathfrak{t} = 0$ we get in the above homotopy $(\nu; 0) = \mathfrak{R}_p(\nu)$. Observe now that $\mathfrak{R}_p$ is homotopic to the identity map (see Remark 4.5), and let $\mathcal{H}_{\mathfrak{R}_p}$ be the map introduced in Step 4 of Section 4.2.2, which realizes this homotopy. We then consider

$$(\nu; \mathfrak{t}) \mapsto \mathcal{H}_{\mathfrak{R}_p}(\nu, 1 - \mathfrak{t}), \qquad \mathfrak{t} \in [0, 1].$$

Finally, letting $\mathcal{H}$ be the concatenation of the above homotopies (rescaling the respective domains of definition) and letting $p(\mathfrak{t}) : [0, 1] \to \Sigma$ again be a geodesic joining p to $\widetilde{\psi}_1(\varphi_2)$ (whenever defined), we get the desired homotopy

$$((\nu, p, s); \mathfrak{t}) \mapsto \begin{cases} \big(\mathcal{H}(\nu; \mathfrak{t}), p(\mathfrak{t}), (1-\mathfrak{t})s + \mathfrak{t}s(\varphi_1, \varphi_2)\big), \ \mathfrak{t} \in [0, 1] & \text{if } \sigma_2(\varphi_2) \text{ is small}, \\ \big(\mathcal{H}(\nu; \mathfrak{t}), p, (1-\mathfrak{t})s\big), \ \mathfrak{t} \in [0, 1] & \text{otherwise}. \end{cases} \tag{111}$$

Case 3. Let $s \in [0, \frac{1}{4}]$. In this case, the test functions we are considering are as in Section 4.2.2. Notice that for this range of s we always get $\sigma_2(\widehat{\varphi}_2^t) \ll \sigma_1(\widehat{\varphi}_1^t)$ (see the beginning of the proof of Proposition 4.7) and therefore $s(\widehat{\varphi}_1^t, \widehat{\varphi}_2^t) = 0$. We have to further subdivide this case depending on the values of s due to the construction of the test functions in the Steps 1–4 of Section 4.2.2.

Emphasizing in the test functions the dependence on λ and recalling that $t = t(s)$, for $s \in [\frac{3}{16}, \frac{1}{4}]$, we get the property $\widetilde{\psi}_k(\breve{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^{k} t_i \delta_{\tilde{x}_i}$ (see Step 1). When $s \in [\frac{1}{8}, \frac{3}{16}]$, one has by construction that $\widetilde{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^{k} t_i \delta_{\tilde{x}_i}$ (see Step 2). For $s \in [\frac{1}{8}, \frac{3}{16}]$, we instead get $\widehat{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^{k} t_i \delta_{\tilde{\gamma}_i}$ (see Step 3). Finally, when $s \in [\frac{1}{8}, \frac{3}{16}]$, we obtain $\overline{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \mathcal{H}_{\mathfrak{R}_p}(\nu, t)$ (see Step 4).

In any case, we then proceed analogously as in Step 2 and the desired homotopy is given as in the second part of (111). $\square$

In this situation, one says that the set J_ρ^{-L} *dominates* $Y_{\mathfrak{R}}$ [Hatcher 2002, p. 528]. Recall now that Y is not contractible (see Proposition 5.6); $Y_{\mathfrak{R}}$ being a deformation retract of Y (see Remark 4.16), we get that $Y_{\mathfrak{R}}$ is not contractible too. Therefore, by the latter result, we deduce that

$$\Phi_\lambda(Y_{\mathfrak{R}}) \text{ is not contractible in } J_\rho^{-L}.$$

Moreover, one can take λ large enough so that $\Phi_\lambda(Y_{\mathfrak{R}}) \subset J_\rho^{-2L}$. We next define the topological cone over $Y_{\mathfrak{R}}$ by the equivalence relation

$$\mathscr{C} = Y_{\mathfrak{R}} \times [0, 1]/Y_{\mathfrak{R}} \times \{0\},$$

where $Y_{\mathfrak{R}} \times \{0\}$ is identified to a single point, and we consider the min-max value

$$m = \inf_{h\in\Gamma} \max_{\xi\in\mathscr{C}} J_\rho(h(\xi)),$$

where

$$\Gamma = \left\{h : \mathscr{C} \to H^1(\Sigma) \times H^1(\Sigma) : h(\nu, p, s) = \Phi_\lambda(\nu, p, s) \text{ for all } (\nu, p, s) \in \partial\mathscr{C} \simeq Y_{\mathfrak{R}}\right\}. \tag{112}$$

First, we observe that the map from $\mathscr{C}$ to $H^1(\Sigma) \times H^1(\Sigma)$ defined by $(\,\cdot\,, t) \mapsto t\Phi_\lambda(\,\cdot\,)$ belongs to Γ; hence, this is a nonempty set. Moreover, by the choice of Φ_λ we have

$$\sup_{(\nu,p,s)\in\partial\mathscr{C}} J_\rho(h(\nu, p, s)) = \sup_{(\nu,p,s)\in Y_{\mathfrak{R}}} J_\rho(\Phi_\lambda(\nu, p, s)) \le -2L.$$

The crucial point is to show that $m \ge -L$. Indeed, $\partial\mathscr{C}$ is contractible in $\mathscr{C}$ and hence in $h(\mathscr{C})$ for any $h \in \Gamma$. On the other hand by the fact that $Y_{\mathfrak{R}}$ is not contractible and by Proposition 5.10, $\partial\mathscr{C}$ is not contractible in J_ρ^{-L}, so we deduce that $h(\mathscr{C})$ is not contained in J_ρ^{-L}. This being valid for any $h \in \Gamma$, we conclude that $m \ge -L$ necessarily.

It follows from standard variational arguments [Struwe 2000] that the functional J_ρ admits a Palais–Smale sequence at level m. However, this does not guarantee the existence of a critical point since it is not known whether the Palais–Smale condition holds. To bypass this problem, one needs a different argument, usually named the *monotonicity trick*. This technique was first introduced by Struwe [1985] (see also [Ding et al. 1999; Jeanjean 1999; Lucia 2007]) and has been used intensively, so we will be sketchy.

Let us take $\eta > 0$ such that $[\rho_1 - 2\eta, \rho_1 + 2\eta] \times [\rho_2 - 2\eta, \rho_2 + 2\eta] \subset \mathbb{R}^2 \setminus \Lambda$, where Λ is the set defined in (10). Consider then a parameter $\gamma \in [-\eta, \eta]$. It is easy to see that the above min-max geometry holds uniformly for any $\rho_\gamma = (\rho_1 + \gamma, \rho_2 + \gamma)$. In particular, for any $L > 0$, there exists λ large enough so that

$$\sup_{(\nu,p,s)\in\partial\mathscr{C}} J_{\rho_\gamma}(h(\nu, p, s)) < -2L, \qquad m_\gamma = \inf_{h\in\Gamma} \sup_{\xi\in\mathscr{C}} J_{\rho_\gamma}(h(\xi)) \ge -L. \tag{113}$$

In this setting, the following result is well-known:

Lemma 5.11. *The functional $J_{\rho_{\tilde\gamma}}$ possesses a bounded Palais–Smale sequence $(u_{1,n}, u_{2,n})_n$ at level $m_{\tilde\gamma}$ for almost every $\tilde\gamma \in \Upsilon = [-\eta, \eta]$.*

Standard arguments show that a bounded Palais–Smale sequence yields the existence of a critical point; see, e.g., Proposition 5.4 in [Malchiodi 2008b]. Consider now $\tilde{\gamma}_n \in \Upsilon$ such that $\tilde{\gamma}_n \to 0$, and let $(u_{1,n}, u_{2,n})_n$ denote the corresponding solutions. To conclude, it is then sufficient to apply the compactness result given in Theorem 2.1, which implies convergence of $(u_{1,n}, u_{2,n})$ to a solution of (1).

Appendix: Proof of Proposition 4.7

The energy estimates of Proposition 4.7 will follow from the next three lemmas.

Lemma A.1. *If φ_1 and φ_2 are defined as in (82), we have that*

$$\fint_\Sigma \varphi_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi_2 \, dV_g = O(1).$$

Proof. From elementary inequalities (see also the figure on page 1995), it is easy to show that there exists a constant C so that

$$|\varphi_1| + |\varphi_2| \le C\left(1 + \log \frac{1}{d(\cdot, p)} + \sum_i \frac{1}{d(\cdot, \tilde{x}_i)}\right).$$

As the logarithm of the distance from a fixed point is integrable, the conclusion easily follows. □

In the following, for positive numbers a and b, we will use the notation

$$a \simeq_C b \quad \Longleftrightarrow \quad \text{there exists } C > 1 \text{ such that } \frac{b}{C} \le a \le Cb. \tag{114}$$

Lemma A.2. *Under the above assumptions, one has*

$$\int_\Sigma e^{\varphi_1} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{\varphi_2} \, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1\right\}.$$

Proof. Let $\tau \in (0, +\infty]$ be fixed, and let $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ be as in (75). For simplicity, we may assume that there is only one point in the support of $\hat{\nu}$, i.e., $\hat{\nu} = \delta_{x_j}$. The case of a general $\hat{\nu}$ is then treated in an analogous way. It is not difficult to show that the terms $-\frac{1}{2}v_2$ and $-\frac{1}{2}v_{1,1}$ do not affect the integrals of e^{φ_1} and e^{φ_2}, respectively, and that

$$\int_\Sigma e^{\varphi_1} \, dV_g \simeq_C \int_\Sigma e^{v_1} \, dV_g, \qquad \int_\Sigma e^{\varphi_2} \, dV_g \simeq_C \int_\Sigma e^{v_2} \, dV_g.$$

Therefore, it is enough to prove

$$\int_\Sigma e^{v_1} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{v_2} \, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1\right\}. \tag{115}$$

We start by observing that, by definition, for $d(x_j, p) \le 4/\lambda_j$

$$v_1(x) = \log \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3}.$$

By an elementary change of variables, we find

$$\int_\Sigma e^{v_1} \, dV_g = \int_\Sigma \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^4. \tag{116}$$

By the definition of τ and $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ (see in particular (72) and (73)), recalling that $d(x_j, p) \le 4/\lambda_j$ and that $\lambda_j \ge \lambda$ by construction, we get

$$\frac{1}{\tau} \le d(x_j, p) \le \frac{4}{\lambda_j} \le \frac{C}{\lambda}. \tag{117}$$

By taking λ sufficiently large, we deduce $\tau \gg 1$. It follows that $\check{s} = 1$ and $\check{\lambda} = \lambda$; see (79). Moreover, by (117), we have

$$\frac{C}{\lambda} \le \tau_\lambda \le \lambda.$$

Therefore, we can rewrite (116) as

$$\int_\Sigma e^{v_1}\, dV_g = \int_\Sigma \frac{1}{((\hat{\mathfrak{s}}\tau)^{-2} + d(x,p)^2)^3}\, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2$$

and the proof of the first part of (115) is concluded. Suppose now $d(x_j, p) > 4/\lambda_j$, and divide Σ into three subsets:

$$\mathscr{A} = A_{\tilde{x}_j}\left(\frac{1}{s_j\lambda_j}, \frac{d(\tilde{x}_j, p)}{4}\right), \qquad \mathscr{B} = B_{1/(s_j\lambda_j)}(\tilde{x}_j), \qquad \mathscr{C} = \Sigma \setminus (\mathscr{A} \cup \mathscr{B}).$$

We start by estimating

$$\int_{\mathscr{B}} e^{v_1}\, dV_g = \int_{B_{1/(s_j\lambda_j)}(\tilde{x}_j)} \frac{s_j^4\lambda_j^4 d(\tilde{x}_j, p)^4}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2)^3}\, dV_g.$$

Observe that if in the latter formula we substitute $d(x, p)$ with $d(\tilde{x}_j, p)$ we get negligible errors, which will be omitted. Therefore, we can rewrite it as

$$\begin{aligned}\int_{\mathscr{B}} e^{v_1}\, dV_g &= \int_{B_{1/(s_j\lambda_j)}(\tilde{x}_j)} \frac{s_j^4\lambda_j^4}{d(\tilde{x}_j, p)^2} \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3}\, dV_g \\ &= \frac{s_j^2\lambda_j^2}{d(\tilde{x}_j, p)^2} \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3} = s_j^2\tilde{s}_j^2 \frac{\lambda_j^2}{d(x_j, p)^2} \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3},\end{aligned}$$

where in the last equality we have used (77). Exploiting now the conditions (80) and (81) and the assumption $d(x_j, p) > 4/\lambda_j$ and recalling that $d(x_j, p) \ge 1/\tau$ by definition (73), we conclude that

$$\int_{\mathscr{B}} e^{v_1}\, dV_g = \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3} \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2.$$

It is then not difficult to show that

$$\int_{\mathscr{A}} e^{v_1}\, dV_g \le \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 C, \qquad \int_{\mathscr{C}} e^{v_1}\, dV_g \le \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 C$$

for some $C > 0$. This concludes the proof of the first part of (115).

For the second part of (115), similarly as before, we divide Σ into

$$\widetilde{\mathscr{A}} = A_p\left(\frac{1}{\hat{\mathfrak{s}}\tilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right), \quad \widetilde{\mathscr{B}} = B_{1/(\hat{\mathfrak{s}}\tilde{\tau})}(p), \quad \widetilde{\mathscr{C}} = \Sigma \setminus (\widetilde{\mathscr{A}} \cup \widetilde{\mathscr{B}}).$$

For $x \in \widetilde{\mathcal{B}}$, we have $v_2(x) = \log(\mu/\tilde{\tau})^{-4}$; hence,

$$\int_{\widetilde{\mathcal{B}}} e^{v_2}\, dV_g = \int_{B_{1/(\hat{\mathfrak{s}}\tilde{\tau})}(p)} \left(\frac{\mu}{\tilde{\tau}}\right)^{-4} dV_g = \frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C. \tag{118}$$

Moreover, working in normal coordinates around p, one gets

$$\int_{\widetilde{\mathcal{A}}} e^{v_2}\, dV_g \le \frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C \tag{119}$$

for some $C > 0$. On the other hand, we have

$$\int_{\widetilde{\mathcal{C}}} e^{v_2}\, dV_g \simeq_C 1. \tag{120}$$

From (118), (119), and (120), it follows that

$$\int_{\Sigma} e^{v_2}\, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}, 1\right\},$$

which concludes the proof of the second part of (115). □

Recalling the definition of $\hat{\nu} \in \Sigma_{k,p,\tilde{\tau}}$ in (75), we introduce now the following sets of indices. Let $I \subseteq \{1, \dots, k\}$ be given by

$$I = \left\{ i : d(x_i, p) > \frac{4}{\lambda_i} \right\}.$$

We then subdivide I into two subsets $I_1, I_2 \subseteq I$:

$$I_1 = \left\{ i : d(x_i, p) \le \frac{1}{\tau_\lambda} \right\}, \qquad I_2 = \left\{ i : d(x_i, p) > \frac{1}{\tau_\lambda} \right\}. \tag{121}$$

Lemma A.3. *Under the above assumptions, one has*

$$\int_\Sigma Q(\varphi_1, \varphi_2)\, dV_g \le 8\pi(\log\tilde{\tau} - \log\mu) + 8|I_1|\pi(\log\check{\lambda} - \log\tau_\lambda) + \sum_{i\in I_2} 8\pi(\log s_i + \log\lambda_i - \log d(\tilde{x}_i, p))$$
$$+ 16\pi \sum_{i\in I_2} \log d(\tilde{x}_i, p) + (24\pi \log\tau_\lambda + 24\pi\log\hat{\mathfrak{s}}) + C$$

for some $C = C(\Sigma)$.

Proof. We start by observing that, by definition, $\nabla v_{1,1} = 0$ in $\Sigma \setminus \bigcup_{i\in I} A_{\tilde{x}_i}(1/s_i\lambda_i, d(\tilde{x}_i, p)/4)$ while $\nabla v_2 = 0$ in $\Sigma \setminus A_p(1/\hat{\mathfrak{s}}\tilde{\tau}, 1/\hat{\mathfrak{s}}\mu)$. We next prove the following estimates on the gradients of $v_{1,1}$, $v_{1,2}$, and v_2:

$$|\nabla v_{1,1}(x)| \le \frac{4}{d_{\min}(x)} \quad \text{in} \bigcup_{i\in I} A_{\tilde{x}_i}\left(\frac{1}{s_i\lambda_i}, \frac{d(\tilde{x}_i, p)}{4}\right), \tag{122}$$

$$|\nabla v_2(x)| \le \frac{4}{d(x,p)} \quad \text{in } A_p\left(\frac{1}{\hat{\mathfrak{s}}\tilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right), \tag{123}$$

$$|\nabla v_{1,2}(x)| \le \frac{6}{d(x,p)} \quad \text{for every } x \in \Sigma, \tag{124}$$

where $d_{\min}(x)=\min_{i\in I} d(x,\tilde{x}_i)$ and

$$|\nabla v_{1,2}(x)| \le C\hat{\mathfrak{s}}\tau_\lambda \quad \text{for every } x\in\Sigma, \tag{125}$$

where C is a constant independent of τ_λ and $\hat{\mathfrak{s}}$.

Concerning (122) and (123), we show the inequalities just for $v_{1,1}$ as for v_2 the proof is similar. We have that

$$\nabla v_{1,1}(x) = -4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)}\right)^{-5}\nabla_x\left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)}\right)}{\sum_{j=1}^k t_j\left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)}\right)^{-4}} = -4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)}\right)^{-4}\frac{\nabla_x d(x,\tilde{x}_i)}{d(x,\tilde{x}_i)}}{\sum_{j=1}^k t_j\left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)}\right)^{-4}}$$
$$= -4\frac{\sum_{i=1}^k t_i\left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)}\right)^{-4}\frac{\nabla_x d(x,\tilde{x}_i)}{d_{\min}(x)}}{\sum_{j=1}^k t_j\left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)}\right)^{-4}}.$$

Exploiting the fact that $|\nabla_x d(x,\tilde{x}_i)|\le 1$, we obtain (122). Moreover, by direct computations, one gets (123). We consider now

$$\nabla v_{1,2}(x) = -3\frac{\hat{\mathfrak{s}}^2\tau_\lambda^2\nabla_x(d^2(x,p))}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}.$$

Using the estimate $|\nabla_x(d^2(x,p))|\le 2d(x,p)$, the properties (124) and (125) easily follow by the inequalities

$$\frac{\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}\le 1, \qquad \frac{\hat{\mathfrak{s}}\tau_\lambda d(x,p)}{1+\hat{\mathfrak{s}}^2\tau_\lambda^2 d^2(x,p)}\le 1 \quad \text{for every } x\in\Sigma,$$

respectively. Recalling the definitions of φ_1 and φ_2 in (82) and that $v_1=v_{1,1}+v_{1,2}$, we obtain

$$\begin{aligned}
\int_\Sigma Q(\varphi_1,\varphi_2)\,dV_g &= \frac{1}{3}\int_\Sigma\big(|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+\nabla\varphi_1\cdot\nabla\varphi_2\big)\,dV_g\\
&= \frac{1}{3}\int_\Sigma\big(|\nabla v_1|^2+\tfrac{1}{4}|\nabla v_2|^2-\nabla v_1\cdot\nabla v_2\big)\,dV_g\\
&\qquad\qquad + \frac{1}{3}\int_\Sigma\big(|\nabla v_2|^2+\tfrac{1}{4}|\nabla v_{1,1}|^2-\nabla v_2\cdot\nabla v_{1,1}\big)\,dV_g\\
&\qquad\qquad + \frac{1}{3}\int_\Sigma\big(\nabla v_1-\tfrac{1}{2}\nabla v_2\big)\cdot\big(\nabla v_2-\tfrac{1}{2}\nabla v_{1,1}\big)\,dV_g\\
&= \frac{1}{4}\int_\Sigma|\nabla v_{1,1}|^2\,dV_g+\frac{1}{4}\int_\Sigma|\nabla v_2|^2\,dV_g+\frac{1}{3}\int_\Sigma|\nabla v_{1,2}|^2\,dV_g\\
&\qquad\qquad + \int_\Sigma\big(\tfrac{1}{6}\nabla v_{1,1}\cdot\nabla v_{1,2}-\tfrac{7}{12}\nabla v_{1,1}\cdot\nabla v_2\big)\,dV_g.
\end{aligned} \tag{126}$$

We start by observing that the integral of the mixed terms is uniformly bounded. Indeed, we claim that

$$\nabla v_{1,1}\cdot\nabla v_2 = 0. \tag{127}$$

By the remark before (122), (127) will follow by proving $A_{\tilde{x}_i}(1/s_i\lambda_i, d(\tilde{x}_i,p)/4)\cap A_p(1/\hat{\mathfrak{s}}\tilde{\tau}, 1/\hat{\mathfrak{s}}\mu)=\varnothing$ for all $i\in I$. Recall the constant $\bar{\delta}$ in (77). Clearly, when all the points of the support of $\hat{\nu}$ are bounded away from p, i.e., $d(x_i, p) > \bar{\delta}$ for all i, we get the conclusion. Consider now the case $d(x_i, p) \le \bar{\delta}$ for some i, and observe that in this case $\tilde{s}_i = \hat{\mathfrak{s}}$; see (77). Moreover, by taking $\bar{\delta}$ sufficiently small, one has also $\check{s} \le C$ by the definition (79) (see also (117) and the motivation above it). To prove that the above two subsets are disjoint, one just has to ensure that $d(\tilde{x}_i, p) \gg 1/\hat{\mathfrak{s}}\mu$. We distinguish between two cases. Suppose first that $d(x_i, p) > 1/\tau_\lambda$. By the assumptions we have made and by (80), one gets

$$d(\tilde{x}_i, p) = \frac{1}{\tilde{s}_i} d(x_i, p) = \frac{1}{\hat{\mathfrak{s}}} d(x_i, p) \ge \frac{1}{\hat{\mathfrak{s}}\lambda_i} = \frac{1}{\hat{\mathfrak{s}} d(x_i, p)\tau_\lambda \check{\lambda}} \ge \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda\check{\lambda}} = \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda \check{s}\lambda} \ge \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda\lambda} \gg \frac{1}{\hat{\mathfrak{s}}\mu}$$

by the choice of the parameters μ and λ. The case $d(x_i, p) \le 1/\tau_\lambda$ is treated in the same way with minor modifications. This conclude the proof of (127).

We claim now that

$$\int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2} \, dV_g \le C. \tag{128}$$

We introduce the sets

$$A_i = \Big\{ x \in \Sigma : d(x, \tilde{x}_i) = \min_{j \in I} d(x, x_j) \Big\}. \tag{129}$$

By (122) and (125), we get

$$\begin{aligned} \int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2} \, dV_g &\le \int_\Sigma \frac{C}{d_{\min}(x)\, d(x, p)} \, dV_g \le \sum_{i \in I} \int_{A_i} \frac{C}{d(x, \tilde{x}_i)\, d(x, p)} \, dV_g \\ &\le \sum_{i \in I} \int_{A_{\tilde{x}_i}(1/(s_i\lambda_i), d(\tilde{x}_i, p)/4)} \frac{C}{d(x, \tilde{x}_i)\, d(\tilde{x}_i, p)} \, dV_g \le C, \end{aligned}$$

which proves the claim (128).

Using the estimate (122), one has

$$\begin{aligned} \frac{1}{4} \int_\Sigma |\nabla v_{1,1}|^2 \, dV_g &\le 4 \int_\Sigma \frac{1}{d^2_{\min}(x)} \, dV_g \le 4 \sum_{i \in I} \int_{A_i} \frac{1}{d^2(x, \tilde{x}_i)} \, dV_g \\ &\le 4 \sum_{i \in I} \int_{A_{\tilde{x}_i}(1/(s_i\lambda_i), d(\tilde{x}_i, p)/4)} \frac{1}{d^2(x, \tilde{x}_i)} \, dV_g \\ &\le \sum_{i \in I} 8\pi (\log s_i + \log \lambda_i + \log d(\tilde{x}_i, p)) + C. \end{aligned} \tag{130}$$

Recalling the definitions of $I_1, I_2 \subseteq I$ given in (121), we observe that for $i \in I_1$ we get $\lambda_i = \check{\lambda}$ and $\tilde{s}_i = \hat{\mathfrak{s}}$; see (80) and (77), respectively. Moreover, taking into account (81), we deduce

$$\begin{aligned} \frac{1}{4} \int_\Sigma |\nabla v_{1,1}|^2 \, dV_g &\le 8|I_1|\pi (\log \check{\lambda} - \log \tau_\lambda) + \sum_{i \in I_2} 8\pi (\log s_i + \log \lambda_i + \log d(\tilde{x}_i, p)) + C \\ &= 8|I_1|\pi (\log \check{\lambda} - \log \tau_\lambda) + \sum_{i \in I_2} 8\pi (\log s_i + \log \lambda_i - \log d(\tilde{x}_i, p)) \\ &\qquad + 16\pi \sum_{i \in I_2} \log d(\tilde{x}_i, p) + C. \end{aligned} \tag{131}$$

Similarly as for (130), by (123), we get

$$\frac{1}{4}\int_{\Sigma}|\nabla v_2|^2\,dV_g = 4\int_{A_p(1/(\hat{\mathfrak{s}}\tilde{\tau}),1/(\hat{\mathfrak{s}}\mu))}\frac{1}{d^2(x,p)}\,dV_g \le 8\pi(\log\tilde{\tau}-\log\mu)+C. \tag{132}$$

To estimate the term $|\nabla v_{1,2}|^2$, we consider $\Sigma = B_{1/\hat{\mathfrak{s}}\tau_\lambda}(p)\cup(\Sigma\setminus B_{1/\hat{\mathfrak{s}}\tau_\lambda}(p))$. From (124), we deduce that

$$\int_{B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)}|\nabla v_{1,2}|^2\,dV_g \le C.$$

Then using (124), one finds

$$\frac{1}{3}\int_{\Sigma\setminus B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)}|\nabla v_{1,2}|^2\,dV_g \le 12\int_{\Sigma\setminus B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)}\frac{1}{d^2(x,p)}\,dV_g \le 24\pi(\log\tau_\lambda+\log\hat{\mathfrak{s}})+C. \tag{133}$$

Finally, by (127) and (128) and by inserting (131), (132), and (133) into (126), we get the conclusion. □

Proof of Proposition 4.7. Using Lemmas A.1, A.2, and A.3, the energy estimate we get is

$$\begin{aligned}
J_\rho(\varphi_1,\varphi_2) &\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(\log s_i+\log\lambda_i-\log d(\tilde{x}_i,p))\\
&\qquad +16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}})\\
&\qquad -\rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C\\
&\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(\log s_i+\log\tilde{s}_i+\log\lambda_i\\
&\qquad -\log d(x_i,p))+16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}})\\
&\qquad -\rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C
\end{aligned}$$

for some constant $C>0$. Exploiting the conditions (80) and (81), we obtain

$$\begin{aligned}
J_\rho(\varphi_1,\varphi_2) &\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(2\log\hat{\mathfrak{s}}+\log\check{\lambda}+\log\tau_\lambda)\\
&\qquad +16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}})\\
&\qquad -\rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C.
\end{aligned}\tag{134}$$

Recalling the definitions of I_1 and I_2 in (121), we distinguish between two cases.

Case 1. Suppose first that $I_1\neq\varnothing$. By construction, it follows that $\tau\gg 1$; see (72) and (73). Therefore, by (78), we get $\hat{\mathfrak{s}}=\mathfrak{s}$. On the other hand, using (79) and the definition of $\check{\lambda}$ under it, we deduce $\check{\lambda}\le C\lambda$.

For $\hat{\mathfrak{s}}\ll\tilde{\tau}/\mu^2$, we get in (134)

$$\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}=\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}. \tag{135}$$

In this case, (134) can be rewritten as

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}(8\pi-2\rho_2)+\log\lambda\bigl(8(|I_1|+|I_2|)\pi-2\rho_1\bigr)+\log\hat{\mathfrak{s}}\bigl(24\pi+16|I_2|\pi-4\rho_1+2\rho_2\bigr)$$
$$+\log\tau_\lambda\bigl(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\bigr)+\log\mu(4\rho_2-8\pi)+C. \quad (136)$$

Recalling that $\hat{\mathfrak{s}} \ll \tilde{\tau}/\mu^2$, the latter estimate is negative by the choice of the parameters $\tilde{\tau} \gg \mu \gg \lambda$ and $\rho_2 > 4\pi$.

When instead $\hat{\mathfrak{s}} = \tilde{\tau}/\mu^2 + O(1)$, we have

$$\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}, 1\right\} = 1. \quad (137)$$

Considering now (134) and observing that $\log\hat{\mathfrak{s}} = \log\tilde{\tau} - 2\log\mu + C$, we end up with

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}\bigl(32\pi+16|I_2|\pi-4\rho_1\bigr)+\log\lambda\bigl(8(|I_1|+|I_2|)\pi-2\rho_1\bigr)$$
$$+\log\tau_\lambda\bigl(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\bigr)+\log\mu\bigl(8\rho_1-56\pi-32|I_2|\pi\bigr)+C.$$

The crucial fact is that by construction of $\Sigma_{k,p,\tilde{\tau}}$ (see (70)) $|I_2| \le k-2$ whenever $|I_1| \ne \varnothing$. Hence, we conclude that

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}(16k\pi-4\rho_1)+\log\lambda\bigl(8(|I_1|+|I_2|)\pi-2\rho_1\bigr)+\log\tau_\lambda\bigl(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\bigr)$$
$$+\log\mu\bigl(8\rho_1-56\pi-32|I_2|\pi\bigr)+C,$$

which is large-negative since $\rho_1 > 4k\pi$ and by the choice of the parameters.

Case 2. Suppose now $I_1 = \varnothing$. By construction, we deduce that $\tau \le C$; see (72) and (73). Therefore, using (78), we obtain $\hat{\mathfrak{s}} \le C$. In this case, the equality in (135) always holds true. Moreover, by (79), we have $\check{\lambda} = \mathfrak{s}\lambda$. Hence, (134) can be rewritten as

$$J_\rho(\varphi_1,\varphi_2) \le \log\mathfrak{s}(8|I_2|\pi-2\rho_1)+\log\tilde{\tau}(8\pi-2\rho_2)+\log\lambda(8|I_2|\pi-2\rho_1)$$
$$+\log\tau_\lambda\bigl(8|I_2|\pi+24\pi-2\rho_1\bigr)+\log\mu(4\rho_2-8\pi)+C.$$

Observing that $|I_2| \le k$, we conclude that the latter estimate is large-negative since $\rho_1 > 4k\pi$ and $\rho_2 > 4\pi$ and by the choice of the parameters. □

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