ANALYSIS & PDE

Volume 9 No. 1 2016

ANDREW HASSELL AND JUNYONG ZHANG

GLOBAL-IN-TIME STRICHARTZ ESTIMATES ON NONTRAPPING, ASYMPTOTICALLY CONIC MANIFOLDS

msp

ANALYSIS AND PDE
Vol. 9, No. 1, 2016
dx.doi.org/10.2140/apde.2016.9.151

GLOBAL-IN-TIME STRICHARTZ ESTIMATES ON NONTRAPPING, ASYMPTOTICALLY CONIC MANIFOLDS

ANDREW HASSELL AND JUNYONG ZHANG

We prove global-in-time Strichartz estimates without loss of derivatives for the solution of the Schrödinger equation on a class of nontrapping asymptotically conic manifolds. We obtain estimates for the full set of admissible indices, including the endpoint, in both the homogeneous and inhomogeneous cases. This result improves on the results by Tao, Wunsch and the first author and by Mizutani, which are local in time, as well as results of the second author, which are global in time but with a loss of angular derivatives. In addition, the endpoint inhomogeneous estimate is a strengthened version of the uniform Sobolev estimate recently proved by Guillarmou and the first author. The second author has proved similar results for the wave equation.

1. Introduction

Strichartz estimates are an essential tool for studying the behaviour of solutions to nonlinear Schrödinger equations, nonlinear wave equations and other nonlinear dispersive equations. In particular, global-in-time Strichartz estimates are needed to show global well-posedness and scattering for these equations. The purpose of this article is to prove global-in-time Strichartz estimates for the Schrödinger equation on asymptotically conic, nontrapping manifolds.

Let (M°, g) be a Riemannian manifold of dimension $n \geq 2$ and let $I \subset \mathbb{R}$ be a time interval. Strichartz estimates are a family of dispersive estimates on solutions $u(t, z) : I \times M^\circ \to \mathbb{C}$ to the Schrödinger equation

$$i\,\partial_t u + \Delta_g u = 0, \quad u(0) = u_0(z), \tag{1-1}$$

MSC2010: primary 35Q41; secondary 58J40.
Keywords: Strichartz estimates, asymptotically conic manifolds, spectral measure, Schrödinger propagator.

where Δ_g denotes the Laplace–Beltrami operator on (M°, g). The general Strichartz estimates state that

$$\|u(t,z)\|_{L_t^q L_z^r(I\times M^\circ)} \le C\|u_0\|_{H^s(M^\circ)},$$

where H^s denotes the L^2-Sobolev space over M° and (q,r) is an *admissible pair*, i.e.,

$$2\le q,r\le\infty,\quad \frac{2}{q}+\frac{n}{r}=\frac{n}{2},\quad (q,r,n)\neq(2,\infty,2). \tag{1-2}$$

It is well known that (1-1) holds for $(M^\circ, g)=(\mathbb{R}^n,\delta)$ with $s=0$ and $I=\mathbb{R}$.

In this paper, we continue the investigations carried out in [Hassell et al. 2005; 2006] concerning Strichartz inequalities on a class of non-Euclidean spaces, that is, smooth, complete, noncompact, asymptotically conic Riemannian manifolds (M°, g) which satisfy a nontrapping condition. Here, "asymptotically conic" means that M° has an end of the form $(r_0,\infty)_r\times Y$, with metric asymptotic to dr^2+r^2h as $r\to\infty$, where (Y,h) is a closed Riemannian manifold of dimension $n-1$ (a more precise definition is given below). Hassell, Tao and Wunsch [Hassell et al. 2006] established the local-in-time Strichartz inequalities

$$\|e^{it\Delta_g}u_0\|_{L_t^q L_z^r([0,1]\times M^\circ)} \le C\|u_0\|_{L^2(M^\circ)}. \tag{1-3}$$

Here, we establish the same inequality on the full time interval $\mathbb{R}$. To treat an infinite time interval, the method of [Hassell et al. 2006] no longer works and we take a completely new approach (see Section 1C). Although phrased in terms of asymptotically conic manifolds, we emphasize that our results apply in particular to

- Schrödinger operators $\Delta+V$ on $\mathbb{R}^n$ with V suitably regular and decaying at infinity;
- nontrapping metric perturbations of flat Euclidean space with the perturbation suitably regular and decaying at infinity.

1A. ***Geometric setting.*** Let us recall the asymptotically conic geometric setting, which is the same as in [Guillarmou et al. 2013a; 2013b; Hassell and Wunsch 2005; Hassell et al. 2006]. Let (M°, g) be a complete, noncompact Riemannian manifold of dimension $n\ge 2$ with one end, diffeomorphic to $(0,\infty)\times Y$, where Y is a smooth, compact, connected manifold without boundary. Moreover, we assume (M°, g) is asymptotically conic, which means that M° can be compactified to a manifold M with boundary $\partial M=Y$ such that the metric g becomes a scattering metric on M. That is, in a collar neighbourhood $[0,\epsilon)_x\times\partial M$ of ∂M, g takes the form

$$g=\frac{dx^2}{x^4}+\frac{h(x)}{x^2}=\frac{dx^2}{x^4}+\frac{\sum h_{jk}(x,y)\,dy^j\,dy^k}{x^2}, \tag{1-4}$$

where $x\in C^\infty(M)$ is a boundary defining function for ∂M and h is a smooth family of metrics on Y. Here we use $y=(y_1,\dots,y_{n-1})$ for local coordinates on $Y=\partial M$ and the local coordinates (x,y) on M near ∂M. Away from ∂M, we use $z=(z_1,\dots,z_n)$ to denote the local coordinates. Moreover, if every geodesic $z(s)$ in M reaches Y as $s\to\pm\infty$, we say M is nontrapping. The function $r:=1/x$ near $x=0$ can be thought of as a "radial" variable near infinity and $y=(y_1,\dots,y_{n-1})$ can be regarded as $n-1$

"angular" variables. Rewriting (1-4) using coordinates (r, y), we see that the metric is asymptotic to the exact conic metric $dr^2 + r^2 h(0)$ on $(r_0, \infty)_r \times Y$ as $r \to \infty$.

The Euclidean space $M^\circ = \mathbb{R}^n$, or any compactly supported perturbation of this metric, is an example of an asymptotically conic manifold with Y equal to $\mathbb{S}^{n-1}$ endowed with the standard metric.

Let (M°, g) be an asymptotically conic manifold. The complex Hilbert space $L^2(M^\circ)$ is given by the inner product

$$\langle f_1, f_2 \rangle_{L^2(M^\circ)} = \int_{M^\circ} f_1(z) \overline{f_2(z)} \, dg(z),$$

where $dg(z) = \sqrt{g} \, dz$ is the measure induced by the metric g. Let $\Delta_g = \nabla^* \nabla$ be the Laplace–Beltrami operator on M; our sign convention is that Δ_g is a positive operator. Let V be a real potential function on M such that

$$V \in C^\infty(M), \qquad V(x, y) = O(x^3) \quad \text{as } x \to 0. \tag{1-5}$$

We assume that $n \geq 3$ and that one of two conditions hold: either

$$\boldsymbol{H} := \Delta_g + V \quad \text{has no zero eigenvalue or zero-resonance,} \tag{1-6}$$

or the stronger condition

$$\boldsymbol{H} := \Delta_g + V \quad \text{has no nonpositive eigenvalues or zero-resonance.} \tag{1-7}$$

By a zero-resonance we mean a nontrivial solution u to $\boldsymbol{H} u = 0$ such that $u \to 0$ at infinity. Notice that the second assumption, (1-7), implies that $\boldsymbol{H}$ is a nonnegative operator, so that we can define $\sqrt{\boldsymbol{H}}$. These assumptions allow us to use the results of [Guillarmou et al. 2013a; 2013b].

1B. *Main results.* Now we consider the Schrödinger equation

$$i \, \partial_t u + \boldsymbol{H} u = 0, \quad u(0, \cdot) = u_0 \in L^2(M). \tag{1-8}$$

The main purpose of this paper is to prove the following results. Notice that the endpoint estimate ($q = 2$ and $\tilde{q} = 2$) is included in both cases.

Theorem 1.1 (long-time homogeneous Strichartz estimate). *Let (M°, g) be an asymptotically conic, nontrapping manifold of dimension $n \geq 3$. Let $\boldsymbol{H} = \Delta_g + V$ satisfy* (1-5) *and* (1-7) *and suppose u is the solution to* (1-8). *Then*

$$\|u(t, z)\|_{L^q_t L^r_z(\mathbb{R} \times M^\circ)} \leq C \|u_0\|_{L^2(M^\circ)} \tag{1-9}$$

provided the admissible pair $(q, r) \in [2, \infty]^2$ satisfies (1-2).

Theorem 1.2 (long-time inhomogeneous Strichartz estimate). *Let (M°, g) and $\boldsymbol{H}$ be as in Theorem 1.1. Suppose that u solves the inhomogeneous Schrödinger equation with zero initial data*

$$i \, \partial_t u + \boldsymbol{H} u = F(t, z), \quad u(0, \cdot) = 0. \tag{1-10}$$

Then the inhomogeneous Strichartz estimate

$$\|u(t, z)\|_{L^q_t L^r_z(\mathbb{R} \times M^\circ)} \leq C \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z(\mathbb{R} \times M^\circ)} \tag{1-11}$$

holds for admissible pairs (q, r), $(\tilde{q}, \tilde{r})$.

Remark 1.3. If we make the weaker assumption (1-6), then the statements above still hold, provided that u_0 and $F(t, \cdot)$ lie in the positive spectral subspace of $\boldsymbol{H}$, or in other words that $u_0 = 1_{[0,\infty)}(\boldsymbol{H})(u_0)$, and similarly for $F(t, \cdot)$ for almost every t.

Remark 1.4. We restrict to $n \geq 3$ since the results of [Guillarmou and Hassell 2008] only apply to that case. More recently, Sher [2013] has extended these results to $n = 2$; using his results, one could treat the case $n = 2$ also (noting that the endpoint estimates fail in dimension 2, due to a logarithmic divergence in the resolvent at zero energy occurring in dimension 2). For space reasons, we have not attempted to treat this case in the present paper.

1C. ***Strategy of the proof.*** Our argument here extends to long time and to the endpoint Strichartz estimates of Hassell et al. [2006], who constructed a "local" parametrix for the propagator $e^{it\boldsymbol{H}}$ based on the parametrix from [Hassell and Wunsch 2005]. In that paper, Schrödinger solutions $e^{it\boldsymbol{H}}u_0$ were obtained by applying the parametrix to u_0 and then correcting this approximate solution using Duhamel's formula, using local smoothing estimates to control the correction term. This approach works well on a finite time interval, but cannot be expected to work on an infinite time interval as the errors accumulate over time; certainly they cannot be expected to decay to zero as $t \to \infty$, as would be required to prove L^q estimates in time on an infinite interval.

The main new idea in the current paper is to express the propagator $e^{it\boldsymbol{H}}$ *exactly*, using the spectral measure $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$, exploiting the very precise information on the spectral measure for the Laplacian on asymptotically conic, nontrapping manifolds that has recently become available from the works [Hassell and Vasy 1999; Hassell and Wunsch 2008; Guillarmou et al. 2013a].

After expressing the propagator in terms of an integral of the multiplier $e^{it\lambda^2}$ against the spectral measure, our strategy is to use the abstract Strichartz estimate proved in [Keel and Tao 1998]. Thus, with $U(t)$ denoting the (abstract) propagator, we need to show uniform $L^2 \to L^2$ estimates for $U(t)$, and a $L^1 \to L^\infty$ type dispersive estimate on the $U(t)U(s)^*$ with a bound of the form $O(|t-s|^{-n/2})$. In the flat Euclidean setting, the estimates are obvious because of the explicit formula for the propagator. But in our general setting it turns out to be more complicated. It follows from [Hassell and Wunsch 2005] that the propagator $U(t)(z, z')$ fails to satisfy such a dispersive estimate at any pair of conjugate points $(z, z') \in M^\circ \times M^\circ$ (i.e., pairs (z, z') where geodesics emanating from z focus at z'). Our geometric assumptions allow conjugate points, so we need to modify the propagator such that the failure of the dispersive estimate at conjugate points is avoided.

This is possible due to the TT^* nature of the estimates required by the Keel–Tao formalism. Recall that the dispersive estimate required by Keel and Tao is of the form

$$\|U(t)U(s)^*\|_{L^1 \to L^\infty} \leq C|t-s|^{-n/2}. \tag{1-12}$$

If $U(t)$ is the propagator $e^{it\boldsymbol{H}}$ then the operator on the left-hand side is $e^{i(t-s)\boldsymbol{H}}$. However, nothing in the Keel–Tao formalism requires the $U(t)$ to form a group of operators. Hence we are free to break up $e^{it\boldsymbol{H}} = \sum_j U_j(t)$ and prove the estimate (1-12) for each U_j. Our choice of $U_j(t)$ (sketched directly below) means that $U_j(t)U_j(s)^*$ is essentially the kernel $e^{i(t-s)\boldsymbol{H}}$ localized sufficiently close to the diagonal that we avoid pairs of conjugate points, and hence can prove the dispersive estimate.

Our method of decomposing $e^{it\boldsymbol{H}} = \sum_j U_j(t)$ is motivated by a decomposition used in the proof in [Guillarmou et al. 2013b] of a *restriction estimate* for the spectral measure, that is, an estimate of the form

$$\|dE_{\sqrt{\boldsymbol{H}}}(\lambda)\|_{L^p(M^\circ)\to L^{p'}(M^\circ)} \le C\lambda^{n(1/p-1/p')-1}, \quad 1 \le p \le \frac{2(n+1)}{n+3}.$$

In [Guillarmou et al. 2013b], it was observed that, to prove a restriction estimate for $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$, it suffices (via a TT^* argument) to prove the same estimate for the operators $Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j(\lambda)^*$, where $Q_j(\lambda)$ is a partition of the identity operator in $L^2(M^\circ)$. The operators $Q_j(\lambda)$ used in [Guillarmou et al. 2013b] are pseudodifferential operators (of a certain specific type) serving to localize $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$ in phase space close to the diagonal. Guillarmou et al. [2013b] showed that the localized operators $Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j(\lambda)^*$ satisfy kernel estimates analogous to those satisfied by the spectral measure for $\sqrt{\Delta}$ on flat Euclidean space:

$$\big|(Q_j(\lambda)\, dE^{(l)}_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j(\lambda))(z,z')\big| \le C\lambda^{n-1-l}(1+\lambda d(z,z'))^{-(n-1)/2+l}, \quad l \in \mathbb{N}, \tag{1-13}$$

where $dE^{(l)}_{\sqrt{\boldsymbol{H}}}(\lambda)$ is the l-th derivative in λ of the spectral measure and d is the Riemannian distance on M°.

The authors of [Guillarmou et al. 2013b] hoped that (1-13) could be used as a "black box" in applications of their work. Unfortunately, (1-13) seems inadequate for our present purposes. This is because, in order to obtain the dispersive estimate, we need to efficiently exploit the oscillation of the "spectral multiplier" $e^{it\lambda^2}$, and particularly the discrepancy between the way this function oscillates relative to the oscillations (in λ) of the Schwartz kernel of the spectral measure. The second main innovation of this paper is to improve the estimate (1-13) on the localized spectral measure. We show:

Proposition 1.5. *Let (M°, g) and $\boldsymbol{H}$ be as in Theorem 1.1. Then there exists a λ-dependent operator partition of unity on $L^2(M)$*

$$\mathrm{Id} = \sum_{j=1}^{N} Q_j(\lambda),$$

with N independent of λ, such that for each $1 \le j \le N$ we can write

$$(Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j^*(\lambda))(z,z') = \lambda^{n-1}\Big(\sum_{\pm} e^{\pm i\lambda d(z,z')} a_\pm(\lambda,z,z') + b(\lambda,z,z')\Big), \tag{1-14}$$

with estimates

$$|\partial_\lambda^\alpha a_\pm(\lambda,z,z')| \le C_\alpha \lambda^{-\alpha}(1+\lambda d(z,z'))^{-(n-1)/2}, \tag{1-15}$$

$$|\partial_\lambda^\alpha b(\lambda,z,z')| \le C_{\alpha,M} \lambda^{-\alpha}(1+\lambda d(z,z'))^{-K} \quad \textit{for any } K. \tag{1-16}$$

Here, $d(\cdot\,,\cdot)$ is the Riemannian distance on M°.

Remark 1.6. The estimates (1-15)–(1-16) are easily seen to imply (1-13) (using Lemma 2.3 to estimate the λ-derivatives of the operators $Q_i(\lambda)$). However, (1-15)–(1-16) also capture the oscillatory behaviour of the spectral measure, which is crucial in obtaining sharp dispersive estimates in Section 6.

We now define localized (in phase space) propagators $U_j(t)$ by

$$U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda), \quad 1 \le j \le N. \tag{1-17}$$

Then the operator $U_j(t)U_j(s)^*$ is given, at least formally, by (see Lemma 5.3)

$$U_j(t)U_j(s)^* = \int e^{i(t-s)\lambda^2} Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j(\lambda)^*. \tag{1-18}$$

However, there are subtleties involved in spectral integrals such as (1-17)–(1-18) containing operator-valued functions. Even to show that (1-17) is well-defined as a bounded operator on $L^2(M^\circ)$ is nontrivial. The third main innovation of this paper is to give an effective method for analyzing spectral integrals such as (1-17)–(1-18) with operator-valued multipliers. We use a dyadic decomposition in λ and a Cotlar–Stein almost orthogonality argument to show the well-definedness of (1-17) and prove a uniform estimate on $\|U_j(t)\|_{L^2\to L^2}$, as required by the Keel–Tao formalism.

Having made sense of (1-18), we exploit the oscillations both in the multiplier $e^{i(t-s)\lambda^2}$ and in the localized spectral measure (as expressed by (1-15)–(1-16)) to obtain the required dispersive estimate for $U_j(t)U_j(s)^*$. The homogeneous Strichartz estimate for $e^{it\boldsymbol{H}}$ then follows by applying Keel–Tao to each U_j and summing over j.

Next we consider the inhomogeneous Strichartz estimates. As is well known, the non-endpoint cases of the inhomogeneous estimate follow from the homogeneous estimate and the Christ–Kiselev lemma. The endpoint inhomogeneous estimate requires an additional argument and, in particular, in this case we require estimates on $U_i(t)U_j(s)^*$ for $i \neq j$. This estimate turns out to be very similar to the uniform Sobolev estimate (on asymptotically conic, nontrapping manifolds) of Guillarmou and Hassell [2014]. We use the techniques of that paper, in particular a refined partition of the identity operator. This resemblance to their proof is more than formal: as pointed out to us by Thomas Duyckaerts and Colin Guillarmou, the inhomogeneous endpoint Strichartz estimate implies the uniform Sobolev estimate; we sketch this argument in Section 8. Thus, this part of the paper can be regarded as a time-dependent reformulation of the proof in [Guillarmou and Hassell 2014], leading to a more general result.

1D. ***Previous literature.*** Now we review some classical results about the Strichartz estimates. In the flat Euclidean space, where $M^\circ = \mathbb{R}^n$ and $g_{jk} = \delta_{jk}$, one can take $I = \mathbb{R}$; see [Strichartz 1977; Ginibre and Velo 1985; Keel and Tao 1998] and references therein. The now-classic paper [Keel and Tao 1998] developed an abstract approach to Strichartz estimates, which has become the standard approach in most subsequent literature, including this paper. Strichartz estimates for compact metric perturbations of Euclidean space were proved locally in time by Staffilani and Tataru [2002] and subsequently for asymptotically Euclidean manifolds by Robbiano and Zuily [2005] and Bouclet and Tzvetkov [2007], and in the asymptotically conic setting by Hassell et al. [2006] and Mizutani [2012]. In these works, either the metric is assumed to be nontrapping, or the theorem holds outside a compact set. Burq et al. [2010] proved that Strichartz estimates without loss hold on an asymptotically conic manifold with hyperbolic trapped set. Strichartz estimates have also been studied on exact cones [Ford 2010] and on asymptotically hyperbolic spaces [Bouclet 2011].

There has also been work on Strichartz estimates on compact manifolds and on manifolds with boundary. In the compact case, Strichartz estimates usually are local in time and with some loss of derivatives s (i.e., the right-hand side of (1-9) has to be replaced by the H^s norm of u_0). Estimates for the standard flat 2-torus were shown by Bourgain [1999] to hold for any $s > 0$. For any compact manifold, Burq et al. [2004a] showed that the estimate holds for $s = 1/q$ and that the loss of derivatives, as well as the localization in time, is sharp on the sphere. Manifolds with boundary were studied in [Blair et al. 2008; 2009; 2012; Ivanovici 2010].

Global-in-time Strichartz estimates on asymptotically Euclidean spaces have been proved by Bouclet and Tzvetkov [2008] (but with a low energy cutoff), Metcalfe and Tataru [2012], Marzuola, Metcalfe and Tataru [Marzuola et al. 2008] and Marzuola, Metcalfe, Tataru and Tohaneanu [Marzuola et al. 2010].

The second author has obtained global-in-time Strichartz estimates for the wave equation on asymptotically conic nontrapping manifolds [Zhang 2015b] and for the Schrödinger equation [Zhang 2015a].

As already noted, Strichartz estimates are an essential tool for studying the behaviour of solutions to nonlinear dispersive equations. There is a vast literature on this topic, and it is beyond the scope of this introduction to review it, so we refer instead to Tao's book [2006] and the references therein.

1E. ***Organization of this paper.*** We review the partition of the identity and properties of the microlocalized spectral measure for low energies in Section 2 and for high frequency in Section 3. In Section 4, we prove Proposition 1.5 based on the properties of the microlocalized spectral measure. Section 5 is devoted to the construction of microlocalized propagators and the proof of the L^2 estimates. The dispersive estimates are proved in Section 6. Finally, we prove the homogeneous Strichartz estimates in Section 7 and the inhomogeneous Strichartz estimates in Section 8.

2. Spectral measure and partition of the identity at low energies

The spectral measure for the operator $\boldsymbol{H}$ for low energies was constructed in [Guillarmou and Hassell 2008] on the "low energy space" $M^2_{k,b}$. Here we recall the low energy space $M^2_{k,b}$ and the associated space $M^2_{k,\mathrm{sc}}$. The latter space is needed in order to define the class of pseudodifferential operators in which our operator partition $Q_j(\lambda)$ from Proposition 1.5 lies.

2A. ***Low energy space.*** The low energy space $M^2_{k,b}$, defined in [Guillarmou and Hassell 2008] (based on unpublished work of Melrose and Sá Barreto), is a blown-up version of[1] $[0, \lambda_0] \times M^2$. This space is illustrated in Figure 1. More precisely, we define the codimension-3 corner $C_3 = \{0\} \times \partial M \times \partial M$ and the codimension-2 submanifolds

$$C_{2,L} = \{0\} \times \partial M \times M, \quad C_{2,R} = \{0\} \times M \times \partial M, \quad C_{2,C} = [0, 1] \times \partial M \times \partial M.$$

[1] In [Guillarmou and Hassell 2008], the spectral parameter was denoted by k rather than λ, hence the subscript "k" in $M^2_{k,b}$.

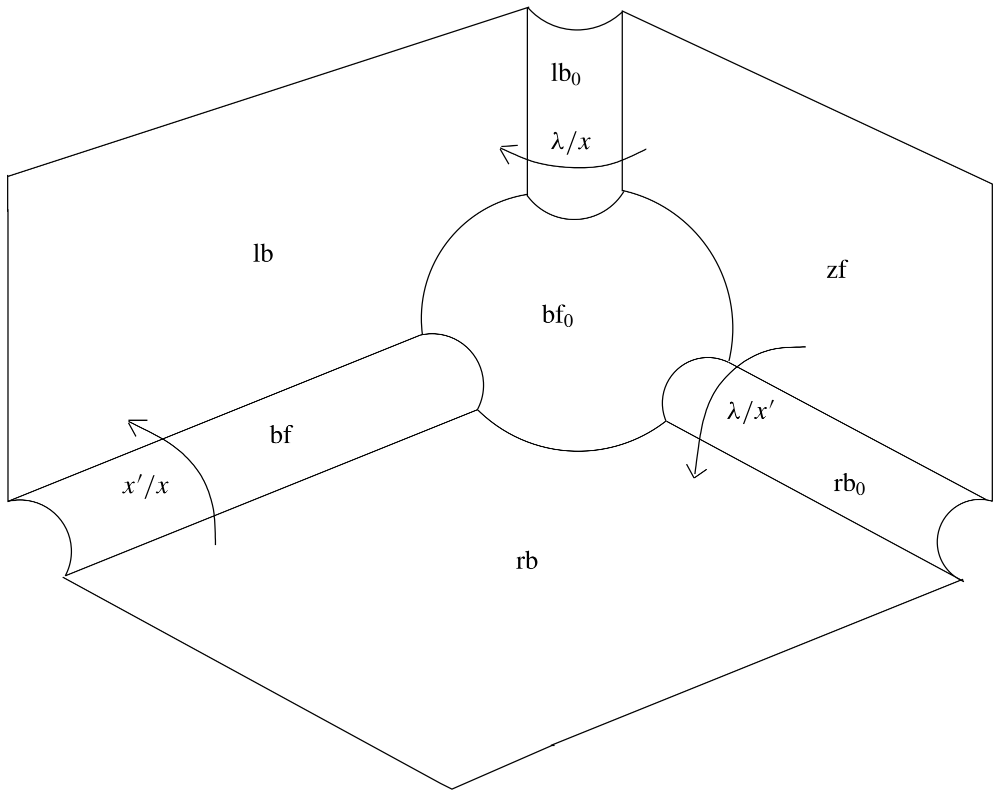

Figure 1. The manifold $M^2_{k,b}$. Arrows show the direction in which the indicated function increases from 0 to ∞.

Without loss of generality, we assume $\lambda_0 = 1$. The space $M^2_{k,b}$ is defined by

$$M^2_{k,b} = [[0, 1] \times M^2; C_3, C_{2,R}, C_{2,L}, C_{2,C}]$$

with blow-down map $\beta_{\mathrm{b}} : M^2_{k,b} \to [0, 1] \times M^2$. Here the notation $[X; Y]$, where X is a manifold with corners and Y a p-submanifold of X,[2] indicates that Y is blown up in X in the real sense; as a set, $[X; Y]$ is the disjoint union of $X \setminus Y$ and the inward-pointing spherical normal bundle SN^+Y of Y. Moreover, $[X; Y_1, Y_2, \dots]$ indicates iterated blow-up. See [Melrose 1994, Section 18] for further details.

The new boundary hypersurfaces created by these blow-ups are labelled by

$$\mathrm{rb} = \operatorname{clos} \beta_{\mathrm{b}}^{-1}([0, 1] \times M \times \partial M), \quad \mathrm{lb} = \operatorname{clos} \beta_{\mathrm{b}}^{-1}([0, 1] \times \partial M \times M), \quad \mathrm{zf} = \operatorname{clos} \beta_{\mathrm{b}}^{-1}(\{0\} \times M \times M),$$

the “b-face” $\mathrm{bf} = \operatorname{clos} \beta_{\mathrm{b}}^{-1}(C_{2,C} \setminus C_3)$ and

$$\mathrm{bf}_0 = \beta_{\mathrm{b}}^{-1}(C_3), \quad \mathrm{rb}_0 = \operatorname{clos} \beta_{\mathrm{b}}^{-1}(C_{2,R} \setminus C_3), \quad \mathrm{lb}_0 = \operatorname{clos} \beta_{\mathrm{b}}^{-1}(C_{2,L} \setminus C_3).$$

[2] We say that Y is a p-submanifold of X if, near every point $p \in Y$, there are local coordinates $x_1, \dots, x_l, y_1, \dots, y_{n-l}$, where $x_i \geq 0$, $y_i \in (-\epsilon, \epsilon)$ and $p = (0, \dots, 0)$, such that Y is given locally by the vanishing of some subset of these coordinates.

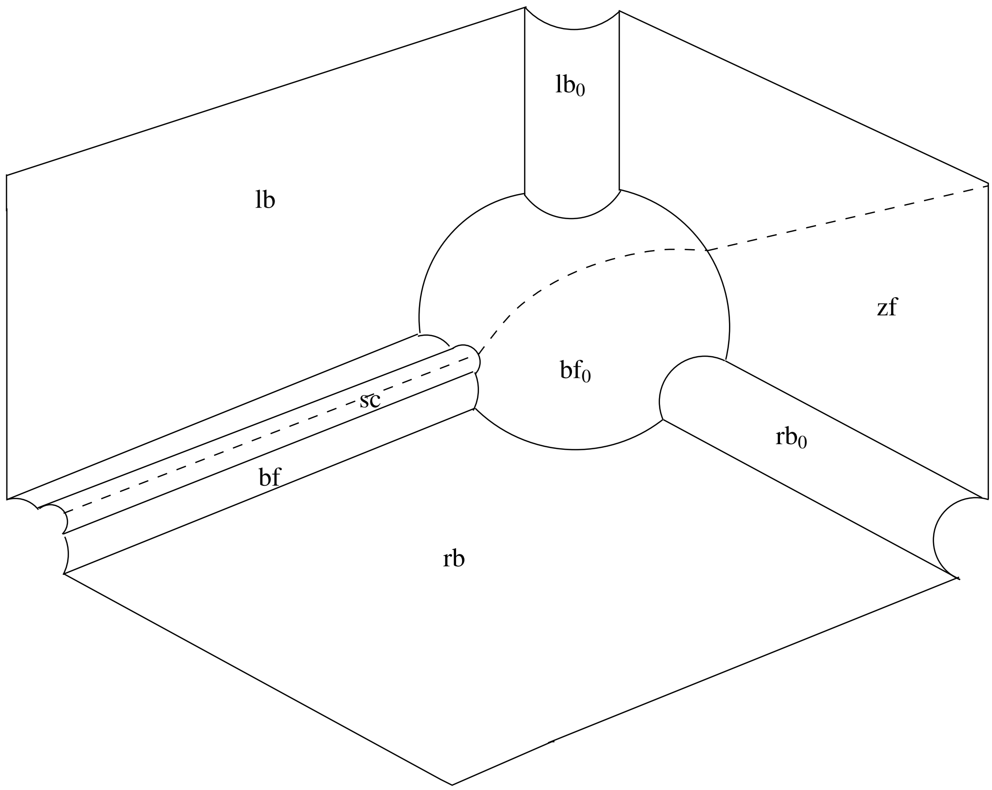

Figure 2. The manifold $M^2_{k,\mathrm{sc}}$; the dashed line is the boundary of the lifted diagonal $\Delta_{k,\mathrm{sc}}$.

The closed lifted diagonal is given by $\mathrm{diag}_{\mathrm{b}} = \mathrm{clos}\, \beta_{\mathrm{b}}^{-1}([0, 1] \times \{(m, m); m \in M^\circ\})$ and its intersection with the face bf is denoted by $\partial_{\mathrm{bf}}\, \mathrm{diag}_{\mathrm{b}}$. We remark that zf is canonically diffeomorphic to the b-double space

$$M^2_{\mathrm{b}} = [M^2; \partial M \times \partial M], \tag{2-1}$$

as is each section $M^2_{k,b} \cap \{\lambda = \lambda_*\}$ for fixed $0 < \lambda_* < 1$.

We further define the space $M^2_{k,\mathrm{sc}}$ to be the blow-up of $M^2_{k,b}$ at $\partial_{\mathrm{bf}}\, \mathrm{diag}_b$. This space is illustrated in Figure 2. The sections $M^2_{k,\mathrm{sc}} \cap \{\lambda = \lambda_*\}$ for fixed $0 < \lambda_* < 1$ are all canonically diffeomorphic to the scattering double space M^2_{sc}, which is the blow-up of M^2_{b} at the boundary of the lifted diagonal

$$M^2_{\mathrm{sc}} = [M^2_{\mathrm{b}}; \partial\, \mathrm{diag}_b].$$

To avoid excessive notation, we denote the diagonal in M^2_{b} and in $M^2_{k,b}$ by the same symbol diag_b. We similarly define $\mathrm{diag}_{\mathrm{sc}}$ to be the closure of the interior of diag_b lifted to M^2_{sc} (or $M^2_{k,\mathrm{sc}}$).

2B. ***Coordinates.*** Let $(x, y) = (x, y_1, \dots, y_{n-1})$ be local coordinates on M near a boundary point, as discussed in Section 1A. We define functions x and y on $M^2_{k,b}$ by lifting from the left factor of M (near ∂M), and x' and y' by lifting from the right factor of M; and similarly z and z' (away from ∂M).

Let $\rho = x/\lambda$, $\rho' = x'/\lambda$ and $\sigma = \rho/\rho' = x/x'$. Then we can use coordinates $(y, y', \sigma, \rho', \lambda)$ near bf and away from rb, while $(y, y', \sigma^{-1}, \rho, \lambda)$ near bf and away from lb.

Next we consider local coordinates on the scattering double space M^2_{sc}. The only difference between this space and M^2_{b} is at the boundary of the diagonal. In local coordinates, near $\partial_{\mathrm{bf}}\,\mathrm{diag}_b$, a boundary defining function for bf is given by x/λ and the diagonal is given by $\sigma = 1,\ y = y'$. Therefore, coordinates on the interior of the new boundary hypersurface, denoted by sc, created by this blow-up are

$$\frac{\lambda(\sigma-1)}{x}, \quad \frac{\lambda(y-y')}{x}, \quad \lambda, \quad y'.$$

We also need to consider coordinates on phase space. As emphasized by Melrose [1994], the appropriate phase space for analyzing the Laplacian with respect to a scattering metric is the scattering cotangent bundle. This is the dual space of the scattering tangent bundle ${}^{\mathrm{sc}}TM$, which is the bundle whose sections are the smooth vector fields over M which are uniformly of finite length with respect to g. Near the boundary, due to the form of the metric (1-4), they are spanned over $C^\infty(M)$ by the vector fields $x^2\,\partial_x$ and $x\,\partial_{y_i}$. Dually, the scattering cotangent bundle is spanned near the boundary by vector fields $dx/x^2 = -d(1/x)$ and dy_i/x; away from the boundary, it is canonically diffeomorphic to the usual cotangent bundle. Thus, a point in the scattering cotangent bundle can be expressed as a linear combination

$$\nu\lambda d\Big(\frac{1}{x}\Big) + \sum_{i=1}^{n-1} \lambda\mu_i\,\frac{dy_i}{x} \tag{2-2}$$

near the boundary, or

$$\sum_{i=1}^{n} \lambda\zeta_i\,dz_i \tag{2-3}$$

away from the boundary, which defines linear coordinates (μ, ν) or ζ on each fibre of the scattering cotangent bundle. Notice that we have introduced a scaling by the spectral parameter λ; as $\lambda = 1/h$, this is essentially the semiclassical scaling, appropriate to our operator $\Delta - \lambda^2 = \lambda^2(h^2\Delta - 1)$, although in this low energy case, we are looking at the limit $h \to \infty$, rather than $h \to 0$ as in the high energy case in Section 3.

The appropriate "compressed cotangent bundle" over $M^2_{k,b}$ is discussed in [Guillarmou et al. 2013a, Section 2.3]. Here, we only describe this for $\lambda > 0$ plus a neighbourhood of the boundary hypersurface bf. In this region, it is given by the lift of the bundle ${}^{\mathrm{sc}}T^*M \times {}^{\mathrm{sc}}T^*M$ to $M^2 \times [0, 1]$ and then to $M^2_{k,b}$. In particular, we use coordinates (μ, ν) lifted from the left factor of M and (μ', ν') lifted from the right factor of M in a neighbourhood of bf. We remark that these coordinates remain valid in a neighbourhood of bf even at $\lambda = 0$, which follows from the fact that (2-2) can be written in the form

$$\nu d\Big(\frac{1}{\rho}\Big) + \sum_{i=1}^{n-1} \mu_i\,\frac{dy_i}{\rho}.$$

The following lemma will be useful in our estimates in Section 4.

Lemma 2.1. *Let* $w = (w_1, \dots, w_n)$ *denote a set of defining functions for* $\mathrm{diag}_b \subset M^2_{k,b}$; *that is, the differentials* dw_i *are linearly independent and* $\mathrm{diag}_b = \{w = 0\}$. *For example, near* bf_0 *or* bf, *we can take* $w = (\sigma - 1, y_1 - y'_1, \dots, y_{n-1} - y'_{n-1})$. *Then* $|w|/x$ *is comparable to* $d(z, z')$ *in a neighbourhood of* diag_b. *Equivalently*, $|w|/\rho$ *is comparable to* $\lambda d(z, z')$.

Proof. Away from $\mathrm{bf}_0 \cup \mathrm{bf}$, $|w|^2$ is a quadratic defining function for diag_b and so is $d(z, z')^2$, hence they are comparable. Now consider what happens near bf_0 or bf. In coordinates $w = (\sigma - 1, y_1 - y'_1, \dots y_{n-1} - y'_{n-1})$, we have

$$\frac{|w|}{x} \sim \left|\frac{\sigma - 1}{x}\right| + \left|\frac{y - y'}{x}\right|.$$

Write $r = 1/x$; then this is

$$|r - r'| + r|y - y'|.$$

Given that the metric takes the form $dr^2 + r^2 h(x, y, dy)$, where h is positive definite, we see that this is comparable to $d(z, z')$. □

Remark 2.2. In the case $M^\circ = \mathbb{R}^n$, with Euclidean coordinates $z = (z_1, \dots, z_n)$, we can take $w = (z_1 - z'_1, \dots, z_n - z'_n)$.

2C. *Pseudodifferential operators on the low energy space.* We use the class of pseudodifferential operators $\Psi^m_k(M; \Omega^{1/2}_{k,b})$ on $M^2_{k,\mathrm{sc}}$ introduced by Guillarmou and Hassell [2008]. By definition, these operators have Schwartz kernels which are half-densities conormal to the diagonal $\mathrm{diag}_{\mathrm{sc}}$, smooth on $M^2_{k,\mathrm{sc}}$ away from the diagonal, and rapidly decreasing at all boundary hypersurfaces not meeting the diagonal, i.e., at lb_0, rb_0, lb and rb. In addition, we will only consider those operators with kernels supported where $\rho, \rho' \le C < \infty$. In this setting we can write the kernel in the form

$$\lambda^n \int e^{i\lambda/x\,((1-\sigma)\nu + (y-y')\cdot\mu)} a(\lambda, \rho, y, \mu, \nu)\, d\mu\, d\nu\, |dg\, dg'|^{1/2}, \tag{2-4}$$

where a is a classical symbol of order m in the (μ, ν) variables, smooth in (λ, ρ, y) and supported where $\rho \le c$. If we write this in the form $A(z, z', \lambda)\, |dg\, dg'|^{1/2}$, then the action on a half-density $f\, |dg|^{1/2}$ is given by

$$\left(\int A(z, z', \lambda) f(z')\, dg(z')\right) |dg(z)|^{1/2}.$$

Given that we have a canonical half-density factor, namely the Riemannian half-density $|dg|^{1/2}$, we will usually omit the half-density factors below.

From the representation (2-4) it is easy to see the following:

Lemma 2.3. *If* $A \in \Psi^m_k(M; \Omega^{1/2}_{k,b})$ *then* $(\lambda\, \partial_\lambda)^N A$ *is also a pseudodifferential operator of order* m, *i.e.*, $(\lambda \partial_\lambda)^N A \in \Psi^m_k(M; \Omega^{1/2}_{k,b})$.

Proof. It suffices to prove this for $N = 1$ and use induction. If $\lambda\, \partial_\lambda$ hits the function a in (2-4), then a is still a symbol of order m in the (μ, ν) variables, smooth in (λ, ρ, y) and supported where $\rho \le c$. (Notice that $\rho = x/\lambda$ depends on λ as well.) On the other hand, if $\lambda\, \partial_\lambda$ hits the phase, this is the same as

$\nu\,\partial_\nu + \mu\,\partial_\mu$ hitting the phase, as it is homogeneous of degree 1 in both λ and (ν, μ). Integrating by parts, we obtain another symbol $\tilde{a}$ of order m. This completes the proof. $\square$

Lemma 2.4. *If $A = A(z, z', \lambda)\,|dg\,dg'|^{1/2} \in \Psi_k^m(M; \Omega_{k,b}^{1/2})$ and $m < -n$, then A satisfies a kernel bound*

$$|A(z, z', \lambda)| \le \lambda^n (1 + \lambda d(z, z'))^{-N} \tag{2-5}$$

for any $N \in \mathbb{N}$.

Proof. If the order m is less than $-n$, then the integral (2-4) is absolutely convergent, showing that the kernel of $\lambda^{-n} A$ is uniformly bounded. Next, we note that the differential operator

$$\frac{1 - \partial_\nu^2 - \sum_i \partial_{\mu_i}^2}{1 + \lambda^2 (x^{-2}(\sigma - 1)^2 + x^{-2}|y - y'|^2)}$$

leaves the exponential in (2-4) invariant. By applying this N times to the exponential and then integrating by parts, we see that the integral is bounded by

$$C_N \big(1 + \lambda^2 (x^{-2}(\sigma - 1)^2 + x^{-2}|y - y'|^2)\big)^{-N}$$

for any N. Finally, as in the proof of Lemma 2.1, the square of the Riemannian distance on M is comparable to

$$\frac{(\sigma - 1)^2}{x^2} + \frac{|y - y'|^2}{x^2},$$

so the integral is bounded by $C_N (1 + \lambda\, d(z, z'))^{-N}$ for any N. $\square$

Corollary 2.5. *If $A \in \Psi_k^m(M; \Omega_{k,b}^{1/2})$ and $m < -n$, then A is bounded $L^2(M^\circ) \to L^2(M^\circ)$ uniformly as $\lambda \to 0$. The same is true for $(\lambda\,\partial_\lambda)^N A$ for any N.*

Proof. This follows from the kernel bound in Lemma 2.4, the volume estimate $cr^n \le V(z, r) \le Cr^n$ for the volume $V(z, r)$ of the ball of radius r centred at $z \in M^\circ$, and Schur's test. $\square$

2D. ***Low energy partition of the identity.*** Recall that, in Proposition 1.5, we employ a partition of the identity. We use essentially the same partition of the identity as in [Guillarmou et al. 2013b]. To define it, we specify the symbols of these operators, which must form a partition of unity on the phase space. We point out that, in our approach, it is crucial to be able to localize in phase space (and hence necessary to use pseudodifferential operators) in order to eliminate difficulties with conjugate points.

For low energies and for a given small positive ϵ, this partition is defined as follows. We first form an open cover $G_0 \cup \dots \cup G_{N_l}$ of the phase space ${}^{sc}T^*M$. The set G_0 consists of all points away from the boundary, that is, the points with $x > \epsilon$. The next set G_1 consists of points near the boundary, say $x < 2\epsilon$, but away from the characteristic variety, that is, satisfying $|\mu|_h^2 + \nu^2 < \frac{1}{2}$ or $|\mu|_h^2 + \nu^2 > \frac{3}{2}$. We then break up the set $\{x < 2\epsilon,\ |\mu|_h^2 + \nu^2 \in [\frac{1}{4}, 2]\}$ into a finite number of sets $G_2, \dots, G_{N_l}$ such that, for each set G_j, the value of ν lies in some interval of length $\le \delta$, where δ is taken to be sufficiently small.

We then form a partition of unity subordinate to this open cover and take these as the principal symbols of pseudodifferential operators Q_j^{low} in the class $\Psi_k^0(M; \Omega_{k,b}^{1/2})$ described above. More precisely, we choose a function $\chi \in C^\infty(\mathbb{R})$ of a real variable with $\chi(t) = 0$ for $t \le \epsilon$ and $\chi(t) = 1$ for $t \ge 2\epsilon$. We define $Q_0^{\text{low}}(\lambda)$ to be multiplication by the function $1-\chi(\rho)$ (recall $\rho = x/\lambda$). Next, we choose $Q_1'(\lambda)$ such that its (full) symbol is equal to 0 for $\frac{1}{2} \le |\mu|_h^2 + \nu^2 \le \frac{3}{2}$ and equal to 1 outside $\frac{1}{4} \le |\mu|_h^2 + \nu^2 \le 2$. Then we define $Q_1^{\text{low}} = \chi(\rho) Q_1'$. This means that the symbol of $\mathrm{Id} - Q_0^{\text{low}} - Q_1^{\text{low}}$ is supported where ρ is small and close to the characteristic variety $|\mu|_h^2 + \nu^2 = 1$. We then decompose this as $Q_2^{\text{low}} + \cdots + Q_{N_l}^{\text{low}}$ so that the symbol of each Q_j^{low}, $j \ge 2$ in supported in G_j, hence supported where ν is contained in an interval of length $\le \delta$.

2E. ***Localized spectral measure.*** The main result of [Guillarmou et al. 2013a] was that the spectral measure for the Laplacian on an asymptotically conic manifold is, for low energies, a Legendre distribution associated to a pair of Legendre submanifolds, the "propagating Legendrian" L^{bf} and the "incoming/outgoing Legendrian" $L^\sharp$. We now explain very briefly what this means. We first have to introduce the contact manifold in which these Legendre submanifolds live. Consider the bundle ${}^\Phi T^* M_{\mathrm{b}}^2$, obtained by lifting ${}^{\mathrm{sc}}T^*M \times {}^{\mathrm{sc}}T^*M$ (viewed as a bundle over M^2) to M_{b}^2. This bundle carries a symplectic structure, but the symplectic form degenerates at the boundary. Nevertheless, it determines a contact structure on this bundle restricted to the boundary hypersurface bf,[3] which we denote by ${}^\Phi T^*_{\mathrm{bf}} M_{\mathrm{b}}^2$. We give this contact structure in local coordinates $(y, y', \sigma, \mu, \mu', \nu, \nu')$ for ${}^\Phi T^*_{\mathrm{bf}} M_{\mathrm{b}}^2$, where $\sigma = x/x'$, (μ, ν) are as in (2-2) and, as above, the unprimed/primed coordinates are lifted from the left/right copies of ${}^{\mathrm{sc}}T^*M$. In these coordinates, the contact form has an expression

$$d\nu - \mu \cdot dy + \sigma(d\nu' - \mu' \cdot dy').$$

A Legendrian submanifold is, by definition, an $2n-1$-dimensional submanifold of this $4n-1$-dimensional space on which the contact form vanishes. The Legendre submanifold $L^\sharp$ is easy to define: it is the submanifold

$$\{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid \mu = \mu' = 0,\ \nu = \nu' = 1\}.$$

The other Legendre submanifold, L^{bf}, is more interesting. It encodes the geodesic flow on the cone over $(\partial M, h)$ where $h = h(0)$ is the metric in (1-4). Let (y, η) be an element of the cosphere bundle $S^*\partial M$ of $T^*\partial M$ and let $\gamma(s) = (y(s), \eta(s))$ be the geodesic with $(y(0), \eta(0)) = (y, \eta)$. Then L^{bf} is given by the union of the leaves $\gamma^2 = \gamma^2(y, \eta)$,

$$\gamma^2 = \mathrm{clos}\biggl\{\Bigl(y, y', \sigma = \frac{x}{x'}, \mu, \mu', \nu, \nu'\Bigr) \biggm| y = y(s),\ y' = y(s'),\ \mu = \eta(s)\sin s,\ \mu' = -\eta(s')\sin s', \\ \nu = -\cos s,\ \nu' = \cos s',\ \sigma = \frac{\sin s}{\sin s'},\ (s, s') \in (0, \pi)^2\biggr\} \tag{2-6}$$

as (y, η) ranges over $S^*\partial M$. We note that this closure includes the sets

$$T_\pm = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid y = y',\ \sigma \in \mathbb{R},\ \mu = \mu' = 0,\ \nu = -\nu' = \pm 1\}, \tag{2-7}$$

corresponding to the limits $s, s' \to 0$ and $s, s' \to \pi$.

[3]We denote the new boundary hypersurface of M_{b}^2, created by the blow-up (2-1), by bf. This is slightly at odds with the way bf is used as a boundary hypersurface of $M_{k,b}^2$ — here it really corresponds to taking a section of $M_{k,b}^2$ at fixed $\lambda_* > 0$ — but hopefully no confusion will be caused.

The statement that the spectral measure is a Legendre distribution with respect to the pair of Legendre submanifolds $(L^{\mathrm{bf}}, L^{\sharp})$ means that the Schwartz kernel of the spectral measure can be expressed as an oscillatory function or oscillatory integral, with a phase function that "parametrizes" the Legendre submanifold. We now state what "parametrizes" means, first in the case of a Legendre submanifold L that projects diffeomorphically to the base bf, in the sense that the projection from ${}^{\Phi}T^*_{\mathrm{bf}}M^2_{\mathrm{b}}$ to bf restricts to a (local) diffeomorphism from L to bf. In this case, there exists a function $\Phi : \mathrm{bf} \to \mathbb{R}$ such that (locally) L is the graph of the differential of the function Φ/x or, in coordinates,

$$L=\{\mu=d_y\Phi(y,y',\sigma),\ \mu'=\sigma^{-1}d_{y'}\Phi(y,y',\sigma),\ \nu=\Phi(y,y',\sigma)-\sigma\, d_\sigma\Phi(y,y',\sigma),\ \nu'=d_\sigma\Phi(y,y',\sigma)\}.$$

We say that Φ, or more accurately Φ/x, (locally) *parametrizes* L. In the general case, there always exist (nonunique) functions $\Phi(y, y', \sigma, v)$, depending on extra variables $(v_1, \dots, v_k)$, that locally parametrize L in the sense that

$$\begin{aligned} L = \big\{\mu = d_y\Phi(y, y', \sigma, v),\ \mu' = \sigma^{-1} d_{y'}\Phi(y, y', \sigma, v),& \\ \nu = \Phi(y, y', \sigma, v) - \sigma\, d_\sigma\Phi(y, y', \sigma, v),\ \nu' = d_\sigma\Phi(y, y', \sigma, v) \,\big|\, d_v\Phi = 0\big\}.& \end{aligned} \tag{2-8}$$

Observe that, if we take the union of the points of (2-6) with $s = s'$, over all $(y, \eta) \in S^*\partial M$, then we get a codimension-1 submanifold of L^{bf}, which is also a codimension-1 submanifold of the conormal bundle of the diagonal $N^*\operatorname{diag}_b$, given by

$$N^*\operatorname{diag}_b = \{(y, y', \sigma, \mu, \mu', \nu, \nu') \mid y = y',\ \sigma = 1,\ \mu = -\mu',\ \nu = -\nu'\}.$$

Claim. *In a deleted neighbourhood of* $N^*\operatorname{diag}_b$, L^{bf} *projects in a* 2:1 *fashion to the base* bf, *i.e.*, $L^{\mathrm{bf}} \setminus N^*\operatorname{diag}_b$ *consists of* 2 *sheets*, *each of which projects diffeomorphically to the base* bf, *that are parametrized by the function* $\pm d_{\mathrm{conic}}$, *where* d_{conic} *is the distance function on the cone over* ∂M.

The conic metric d_{conic} has an explicit expression when $d_{\partial M}(y, y') < \pi$. Writing $r = 1/x$ and $r' = 1/x' = \sigma/x$, it takes the form

$$d_{\mathrm{conic}}(y, y', r, r') = \sqrt{r^2 + r'^2 - 2rr' \cos d_{\partial M}(y, y')} = r\sqrt{1 + \sigma^2 - 2\sigma \cos d_{\partial M}(y, y')}. \tag{2-9}$$

Note that $d_{\mathrm{conic}}(y, y', r, r')/r$ indeed has the form $\Phi(y, y', \sigma)/x$ and is smooth provided that $\cos d_{\partial M}(y, y')$ is smooth, i.e., $d_{\partial M}(y, y')$ is less than the injectivity radius on $(\partial M, h)$.

We next explain why we consider the localized (or more precisely microlocalized) spectral measure, by which we mean any of the operators $Q(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q(\lambda)^*$, where $Q(\lambda)$ is a member of our partition of the identity. The reason is, as shown in [Guillarmou et al. 2013b, Section 5], these terms are also Legendre distributions, but associated only to part of the Legendrian, namely to the subset

$$\{(y, y', \sigma, \mu, \mu', \nu, \nu') \in L \mid (y, \mu, \nu), (y', \mu', \nu') \in WF'(Q)\},$$

where $WF'(Q)$ is the support of the symbol[4] of Q. This is localized close to $N^*\operatorname{diag}_b \cup T_\pm$ (that is, those points in (2-6) corresponding to $s = s'$) if $WF'(Q)$ is well localized. We can then use the Claim above to

[4] The relevant symbol here is the scattering symbol, or boundary symbol, in the scattering calculus, which is a function on ${}^{\Phi}T^*_{\mathrm{bf}}M^2_{\mathrm{b}}$; see [Melrose 1994].

write this piece of the spectral measure using the conic distance function, except near $N^* \operatorname{diag}_b$ itself, where we can express it as an oscillatory integral using a slightly more complicated form of phase function (as in Proposition 2.6(ii)).

We summarize the information we need from [Guillarmou et al. 2013a; 2013b] concerning the spectral measure:

Proposition 2.6. *Let $Q_j^{\mathrm{low}}(\lambda)$ be a member of the partition of the identity defined above. Let $\eta > 0$ be given. Then, for $j, k = 0, 1$, $Q_j^{\mathrm{low}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_k^{\mathrm{low}}(\lambda)^*$ satisfies the estimates on the right-hand side of* (1-16) *and $Q_j^{\mathrm{low}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_j^{\mathrm{low}}(\lambda)^*$, $j \geq 2$, can be written as a finite sum of terms of two types*:

(i) *An oscillatory function of the form*

$$\lambda^{n-1} e^{\pm i\lambda\, d_{\mathrm{conic}}(y, y', 1/x, \sigma/x)} a(y, y', \sigma, x, \lambda), \tag{2-10}$$

where a is supported in $x, x' \leq \eta$ and $d_{\partial M}(y, y') \leq \eta$ and satisfies estimate (1-15).

(ii) *An oscillatory integral of the form*

$$\lambda^{n-1} \int_{\mathbb{R}^{n-1}} e^{i\Phi(y, y', \sigma, v)/\rho} \tilde{a}(y, y', \sigma, v, \rho, \lambda)\, dv, \tag{2-11}$$

where $\tilde{a}$ is smooth in all its arguments and supported in a small neighbourhood of a point $(y_0, y_0, 1, v_0, 0, 0)$ such that $d_v\Phi(y_0, y_0, 1, v_0) = 0$. Moreover, writing $w = (w_1, \ldots, w_n)$ for a set of coordinates defining $\operatorname{diag}_b \subset M^2_{k,b}$, *i.e., $w = (y - y', \sigma - 1)$ and $v = (v_2, \ldots, v_n)$, one can rotate in the w variables so that the function $\Phi = \Phi(y, w, v)$ has the properties*

$$d_{v_j}\Phi = w_j + O(w_1), \tag{2-12a}$$

$$\Phi = \sum_{j=2}^{n} v_j\, d_{v_j}\Phi + O(w_1), \tag{2-12b}$$

$$d^2_{v_j v_k}\Phi = w_1 A_{jk}, \tag{2-12c}$$

$$\frac{\Phi}{x} = \pm d_{\mathrm{conic}}\Big(y, y', \frac{1}{x}, \frac{\sigma}{x}\Big) \quad \text{if } d_v\Phi = 0, \tag{2-12d}$$

where A_{jk} is nondegenerate for all (y, w, v) in the support of b. Here d_{conic} is as in (2-9).

Proof. The statement about $Q_j^{\mathrm{low}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_k^{\mathrm{low}}(\lambda)^*$ for $j, k = 0, 1$, follows from the microlocal support estimates in [Guillarmou et al. 2013b, Section 5]. In fact, $Q_0^{\mathrm{low}}(\lambda)$ has empty wavefront set, while $Q_1^{\mathrm{low}}(\lambda)$ has wavefront set disjoint from the characteristic variety of $\boldsymbol{H} - \lambda^2$, which contains the microlocal support of $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$. It follows that the operators $Q_j^{\mathrm{low}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_k^{\mathrm{low}}(\lambda)^*$, for $j, k = 0, 1$, vanish rapidly at bf, lb and rb. Also, as shown in [Guillarmou et al. 2013a], $dE_{\sqrt{\boldsymbol{H}}}(\lambda)$ is polyhomogeneous at the other boundary hypersurfaces of $M^2_{k,b}$, namely zf, lb_0, rb_0 and bf_0, vanishing to order $n-1$ at each of these faces. Since the $Q_j^{\mathrm{low}}(\lambda)$ are pseudodifferential operators of order zero, the same is true of the composition $Q_j^{\mathrm{low}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q_k^{\mathrm{low}}(\lambda)^*$ for $j, k = 0, 1$ (see [Guillarmou et al. 2013b, Lemma 5.2]). To translate this into an estimate, we observe that λ is a product of boundary defining functions for zf, lb_0, rb_0

and bf_0, while a product of boundary defining functions for bf, lb and rb is $O((1+\lambda d(z,z'))^{-1})$. The estimate (1-16) follows directly.

We next discuss (i) and (ii). Everything in this statement has been proved in [Guillarmou et al. 2013b, Lemma 6.5 and Proposition 6.2] except for the statement that Φ is given by the conic distance function when $d_v\Phi = 0$. To see this, we use the explicit formula (2-9) for the conic distance function, the relation (2-8) and the description of the Legendre submanifold L^{bf} in (2-6). From (2-8), it follows that $\Phi = \nu + \sigma\nu'$. Writing ν and ν' in terms of s and s', using (2-6), we see that

$$d_v\Phi = 0 \implies \Phi = -\cos s + \sigma \cos s'.$$

If we square this then we get

$$d_v\Phi = 0 \implies \Phi^2 = \cos^2 s + \sigma^2\cos^2 s' - 2\sigma \cos s \cos s'.$$

We can write the right-hand side in the form

$$1 - \sin^2 s + \sigma^2(1-\sin^2 s') - 2\sigma(\cos(s-s') - \sin s \sin s').$$

Noting that $\sin^2 s + \sigma^2 \sin^2 s' = 2\sigma \sin s \sin s'$, using the expression for σ in (2-6), we see that

$$d_v\Phi = 0 \implies \Phi^2 = 1 + \sigma^2 - 2\sigma \cos d_{\partial M}(y,y'). \qquad \square$$

Remark 2.7. It might help to give an example to show how (2-12) works. In Euclidean space, the Schwartz kernel of the spectral measure $dE_{\sqrt{\Delta}}(\lambda)$ of $\sqrt{\Delta}$ is given by

$$dE_{\sqrt{\Delta}}(\lambda; z, z') = \frac{\lambda^{n-1}}{(2\pi)^n}\int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z')\cdot\zeta}\, d\zeta$$

and one can find the phase function $(z-z')\cdot\zeta$, where $\zeta \in \mathbb{S}^{n-1}$. Locally near $\zeta = (1,0,\dots,0)$, we can write $\zeta = (\sqrt{1-|v|^2}, v_2, \dots, v_n)$. Write $x = |z|^{-1}$ and $w = (z-z')/|z|$. Then the phase function becomes

$$\Phi = w_1\sqrt{1 - v_2^2 - \cdots - v_n^2} + \sum_{j=2}^{n} w_j v_j,$$

and we can check that properties (2-12) hold in this case.

3. Spectral measure and partition of the identity at high energies

In the previous section we recalled the partition of the identity operator and the structure of the localized spectral measure for low energy, i.e., $0 < \lambda \le \lambda_0$. We now do the same for high energies, $\lambda \in [\lambda_0, \infty)$. For the sake of convenience, we introduce the semiclassical parameter $h = \lambda^{-1}$ (which should not be confused with h in the metric g), so that we pay our attention to the range $h \in (0, h_0]$, where $h_0 = \lambda_0^{-1}$. The spectral measure of the operator $\boldsymbol{H}$ for high energy was constructed in [Hassell and Wunsch 2008] on the high energy space $\boldsymbol{X}$. Our main task is to adapt each of the main results in the previous section to the high energy setting.

3A. ***High energy space.*** The high energy $\boldsymbol{X}$, introduced in [Hassell and Wunsch 2008], is defined by $\boldsymbol{X} = [0, h_0] \times M_{\mathrm{b}}^2$, where $M_{\mathrm{b}}^2 = [M^2; \partial M \times \partial M]$ is as in (2-1). We label the boundary hypersurfaces in $\boldsymbol{X}$ by rb, lb, bf and mf, according as they are the lifts to $\boldsymbol{X}$ of the faces

$$[0, h_0] \times M \times \partial M, \quad [0, h_0] \times \partial M \times M, \quad [0, h_0] \times \partial M \times \partial M \quad \text{or} \quad \{0\} \times M^2$$

of $[0, h_0] \times M^2$, respectively. The labelling of boundary hypersurfaces is consistent with the notations defined in the low energy space, since when $\lambda \in (C^{-1}, C)$ (where $\lambda = 1/h$) the spaces both have the form $(C^{-1}, C) \times M_{\mathrm{b}}^2$. Recall $\sigma = x/x'$; we can use coordinates (y, y', σ, x', h) near bf and away from rb, and coordinates $(y, y', \sigma^{-1}, x, h)$ near bf and away from lb. We use coordinates (z, z', h) away from bf, rb and lb.

3B. ***Semiclassical scattering pseudodifferential operators.*** We recall the space $\Psi_{\mathrm{sc},h}^{m,l,k}(M; {}^{s\Phi}\Omega^{1/2})$ of semiclassical scattering pseudodifferential operators, introduced by Wunsch and Zworski [2000] based on Melrose's scattering calculus [1994]. Such operators are indexed by the differential order m, the boundary order l and the semiclassical order k. One can express this space in terms of the space with $l = k = 0$ by

$$\Psi_{\mathrm{sc},h}^{m,l,k}(M; {}^{s\Phi}\Omega^{1/2}) = x^l h^{-k} \Psi_{\mathrm{sc},h}^{m,0,0}(M; {}^{s\Phi}\Omega^{1/2}).$$

The Schwartz kernel of semiclassical pseudodifferential operator $A \in \Psi_{\mathrm{sc},h}^{m,0,0}(M; {}^{s\Phi}\Omega^{1/2})$ takes the following form on $\boldsymbol{X}$: near the diagonal $\mathrm{diag}_b \subset M_{\mathrm{b}}^2$ and away from bf, it takes the form

$$h^{-n} \int e^{i(z-z')\cdot\zeta/h} a(z, \zeta, h)\, d\zeta\, |dg\, dg'|^{1/2}, \quad n = \dim M, \tag{3-1}$$

while near the boundary of the diagonal, $\mathrm{diag}_b \cap \mathrm{bf}$, it takes the form

$$h^{-n} \int e^{i((y-y')\cdot\mu + (\sigma-1)\nu)/(hx)} a(x, y, \mu, \nu, h)\, d\mu\, d\nu\, |dg\, dg'|^{1/2} \tag{3-2}$$

Here, a is a symbol of order m in the variable ζ or (η, ν) variables and is smooth in the remaining variables. Finally, away from diag_b, the kernel of A is smooth and vanishes to all orders at bf, lb, rb and mf.

Lemma 3.1. *If $A \in \Psi_{\mathrm{sc},h}^{m,0,0}(M; {}^{s\Phi}\Omega^{1/2})$ then $(h\,\partial_h)^N A$ is also a pseudodifferential operator of order m, i.e., $(h\,\partial_h)^N A \in \Psi_{\mathrm{sc},h}^{m,0,0}(M; {}^{s\Phi}\Omega^{1/2})$.*

Proof. Away from the diagonal, the result is trivial, as the kernel is smooth and $O(h^\infty)$. So, consider the representations (3-1)–(3-2). The proof is parallel to the argument in Lemma 2.3. By induction, we only need to consider $N = 1$. If $h\partial_h$ hits the function a in (3-2), then a is still a symbol of order m in the (η, ν) variables, smooth in (h, x, y) and supported in $xh \le c$. On the other hand, if $h\,\partial_h$ hits the phase, this is the same as $\nu\partial_\nu + \eta\cdot\partial_\eta$ hitting the phase, as it brings a factor which is homogeneous of degree -1 in h and degree 1 in (ν, η). Integrating by parts, we obtain another symbol $\tilde{a}$ of order m. The argument for (3-1) is analogous. This completes the proof. □

Lemma 3.2. *If $A = A(z, z', h)\, |dg\, dg'|^{1/2} \in \Psi_{\mathrm{sc},h}^{m,0,0}(M; {}^{s\Phi}\Omega^{1/2})$ and $m < -n$, then A satisfies the kernel bound* (2-5) (*with* $\lambda = h^{-1}$) *for any $N \in \mathbb{N}$.*

Proof. This estimate is straightforward away from the diagonal, as the Schwartz kernel of A vanishes rapidly at all boundaries away from the diagonal. This follows from the nonvanishing of the differential of the phase away from the diagonal. On the other hand, the right-hand side is a positive multiple of $h^{N-n}\rho_{\mathrm{lb}}^N\rho_{\mathrm{bf}}^N\rho_{\mathrm{rb}}^N$ away from the diagonal.

Near the diagonal, we have the representations (3-1)–(3-2). The argument in the case (3-2) is the same as in Lemma 2.4. In the interior case (3-1) we note that the differential operator

$$\frac{1+\Delta_\zeta}{1+h^{-2}|z-z'|^2}$$

leaves the exponential in (3-2) invariant. Applying this differential operator N times and integrating by parts, we see that the integral is bounded by

$$C_N(1+h^{-2}|z-z'|^2)^{-N}$$

for any N. In the interior, the square of the Riemannian distance on M is comparable to $|z-z'|^2$, so the integral is bounded by $C_N(1+h^{-1}d(z,z'))^{-N}$ for any N. □

Corollary 3.3. *If $A\in\Psi_{\mathrm{sc},h}^{m,0,0}(M;{}^{s\Phi}\Omega^{1/2})$ and $m<-n$, then A is bounded $L^2(M^\circ)\to L^2(M^\circ)$ uniformly as $h\to 0$. The same is true for $(h\partial_h)^N A$ for any N.*

Proof. This follows from the kernel bound (2-5) and Schur's test, since there is a uniform volume estimate $cr^n\le V(z,r)\le Cr^n$ for the volume $V(z,r)$ of the ball of radius r centred at $z\in M^\circ$. □

3C. ***High energy partition of the identity.*** We now describe the partition of the identity used in Proposition 1.5 for high energies. Similar to before, these operators are obtained by quantizing symbols which form a partition of unity (independent of h) in the scattering cotangent bundle ${}^{\mathrm{sc}}T^*M$. We modify the open cover $G_0,\dots,G_{N_l}$ from Section 2D by replacing G_0 by a smaller set $\widetilde{G}_0$ given by the points satisfying $x>\epsilon$ and $|\zeta|_g^2\le\frac12$ or $|\zeta|_g^2\ge\frac32$, i.e., the set $\widetilde{G}_0$ is disjoint from the characteristic variety. Then we cover the compact set $\{x\ge\epsilon,|\zeta|_g^2\in[\frac14,2]\}$, which contains $G_0\setminus\widetilde{G}_0$, by a finite number $G_{N_l+1},\dots,G_{N_h}$ of open sets of sufficiently small diameter.

As before, we form a partition of unity subordinate to this refined open cover and take these as the principal symbols of operators Q_j^{high} in the class $\Psi_{\mathrm{sc},h}^{0,0,0}(M;{}^{s\Phi}\Omega^{1/2})$ microsupported in G_j (or $\widetilde{G}_0$ in the case $j=0$). We will assume that $Q_j^{\mathrm{high}}(\lambda)=Q_j^{\mathrm{low}}(\lambda)$ for intermediate energies $\lambda\sim 1$ and $1\le j\le N_l$.

3D. ***Localized spectral measure.*** Hassell and Wunsch [2008] showed that the spectral measure for the Laplacian on this setting is, for high energy, a Legendre distribution associated to a pair of Legendre submanifolds L and $L^\sharp$. We briefly explain the meaning of this statement. The Legendre submanifold $L^\sharp$ has already been defined in Section 2E; it lives in the contact manifold ${}^{\Phi}T^*_{\mathrm{bf}}M^2_{\mathrm{b}}$, living over the boundary hypersurface bf. The new Legendre submanifold L encodes the geodesic flow on T^*M°. It is a submanifold of $\mathbb{R}\times{}^{\Phi}T^*M^2_{\mathrm{b}}$, which has a natural contact form, described as follows. We write α for the contact form on ${}^{\mathrm{sc}}T^*M$ induced by the inclusion of T^*M° into ${}^{\mathrm{sc}}T^*M$, and α and α' for the lift of this contact form to ${}^{\Phi}T^*M^2_{\mathrm{b}}$ by the left and right projections, respectively. Writing τ for the coordinate on the $\mathbb{R}$-factor in

$\mathbb{R} \times {}^{\Phi}T^*M^2_{\mathrm{b}}$, the contact form on this space takes the form

$$\alpha + \alpha' - d\tau.$$

Then L is given as follows: Let Σ denote the characteristic variety of $h^2\Delta_g - 1$, given in local coordinates by $\{|\zeta|_{g(z)} = 1\}$ in the interior or $\{|\mu|^2_{h(x,y)} + \nu^2 = 1\}$ near the boundary. Then L is given in terms of the geodesic flow G_t by

$$L = \{(q, q', \tau) \mid q, q' \in \Sigma,\ q = G_\tau(q')\} \tag{3-3}$$

(this follows from [Guillarmou et al. 2013b, Equation 7.9] and the discussion following). In $\mathbb{R} \times {}^{\Phi}T^*M^2_{\mathrm{b}}$, L can be restricted to $\mathbb{R} \times {}^{\Phi}T^*_{\mathrm{bf}}M^2_{\mathrm{b}}$, i.e., restricted to lie over bf, then, forgetting the τ component, we obtain the Legendre submanifold L^{bf} from Section 2E.[5]

As in Section 2E, the statement that an operator is Legendrian with respect to L means that its Schwartz kernel can be expressed as an oscillatory function or oscillatory integral using a phase function that locally parametrizes L. In the interior of $\boldsymbol{X}$, this means a function $\Psi(z, z', v)$ such that, locally, using coordinates $(z, \zeta, z', \zeta', \tau)$ on $\mathbb{R} \times {}^{\Phi}T^*M^2_{\mathrm{b}}$, we have

$$L = \{(z, d_z\Psi, z', d_{z'}\Psi, \Psi) \mid d_v\Psi = 0\}.$$

In particular, τ is equal to the value of the phase function when $d_v\Psi = 0$. If there are no v variables, the condition $d_v\Psi = 0$ is omitted and then L is (essentially) the graph of the differential of Ψ. Near the boundary bf, we use local coordinates $(x, y, y', \sigma, \mu, \nu, \mu', \nu', \tau)$ and then a local parametrization of L is given by a function $\Psi(x, y, y', \sigma, v)/x$ such that

$$L = \{(x, y, y', \sigma, d_y\Psi, \Psi - x\, d_x\Psi, -\sigma\, d_\sigma\Psi, \sigma^{-1}\, d_{y'}\Psi, d_\sigma\Psi, \Psi) \mid d_v\Psi = 0\}.$$

We give some consequences of this result for the localized spectral measure needed in this paper. As in the low energy case, the localized spectral measure refers to any operator of the form $Q^{\mathrm{high}}(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q^{\mathrm{high}}(\lambda)^*$ where $Q^{\mathrm{high}}(\lambda)$ is a member of the partition of the identity operator from Section 3C. As above, we write $h = 1/\lambda$.

Proposition 3.4. *Let $Q^{\mathrm{high}}_j(\lambda)$ be a member of the partition of the identity defined above. Then, for $j, k = 0, 1$, the operator $Q^{\mathrm{high}}_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q^{\mathrm{high}}_k(\lambda)^*$ satisfies the estimates on the right-hand side of* (1-16) *and $Q^{\mathrm{high}}_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, Q^{\mathrm{high}}_j(\lambda)^*$, $j \geq 2$, can be written as a finite sum of terms of the following three types*:

(i) *An oscillatory function of the form*

$$h^{-(n-1)} e^{\pm i d(z,z')/h} \tilde{a}(z, z', h), \tag{3-4}$$

where $\tilde{a}$ satisfies estimate (1-15).

(ii) *An oscillatory integral supported in $x, x' \geq \epsilon$ of the form*

$$h^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{i\Psi(z,z',v)/h} b(z, z', v, h)\, dv, \tag{3-5}$$

[5] The relation between the various Legendre submanifolds is explained in detail in [Hassell and Wunsch 2008, Part 1].

where b is smooth in all its arguments and supported in a small neighbourhood of a point $(z_0, z_0, v_0, 0)$ *such that* $d_v\Psi(z_0, z_0, v_0) = 0$. *Moreover, writing* $w = z - z'$ *and* $v = (v_2, \dots, v_n)$, *one can rotate in the* w *variables so that the function* $\Psi = \Psi(z, w, v)$ *has the properties*

$$d_{v_j}\Psi = w_j + O(w_1), \tag{3-6a}$$

$$\Psi = \sum_{j=2}^{n} v_j\, d_{v_j}\Psi + O(w_1), \tag{3-6b}$$

$$d^2_{v_j v_k}\Psi = w_1 A_{jk}, \tag{3-6c}$$

$$\Psi(z, z', v) = \pm d(z, z') \quad \text{if } d_v\Psi = 0, \tag{3-6d}$$

where A_{jk} *is nondegenerate at* (z_0, z_0, v_0) *and* $d(z, z')$ *is the Riemannian distance function on* $M^\circ \times M^\circ$.

(iii) *An oscillatory integral supported near* $x = x' = 0$ *of the form*

$$h^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{i\Psi(y,y',\sigma,x,v)/(hx)} b(y, y', \sigma, x, v, h)\, dv, \tag{3-7}$$

where b is smooth in all its arguments and supported in a small neighbourhood of a point $(y_0, y_0, 1, 0, v_0, 0)$ *such that* $d_v\Psi(y_0, y_0, 1, v_0) = 0$. *Moreover, writing* $w = (w_1, \dots, w_n)$ *for a set of coordinates defining* $\mathrm{diag}_b \subset M^2_{\mathrm{b}}$, *i.e.,* $w = (y - y', \sigma - 1)$ *and* $v = (v_2, \dots, v_n)$, *one can rotate in the* w *variables so that the function* $\Psi = \Psi(y, w, x, v)$ *has the properties*

$$d_{v_j}\Psi = w_j + O(w_1), \tag{3-8a}$$

$$\Psi = \sum_{j=2}^{n} v_j\, d_{v_j}\Psi + O(w_1), \tag{3-8b}$$

$$d^2_{v_j v_k}\Psi = w_1 A_{jk}, \tag{3-8c}$$

$$\Psi/x = \pm d(z, z') \quad \text{if } d_v\Psi = 0, \tag{3-8d}$$

where A_{jk} *is nondegenerate at* $(y_0, y_0, 1, 0, v_0, 0)$.

Remark 3.5. Since $\lambda = 1/h$, this is an analogue of Proposition 2.6 for the case $X = [0, h_0] \times M^2_{\mathrm{b}}$.

Proof. The proof is analogous to the proof of Proposition 2.6, with the main difference being that the computation takes place over the whole of M^2_{b} (including the interior), not just at the boundary as in the low energy case. We prove (ii), i.e., we work in the interior of M^2_{b}, using coordinates (z, z'), with z a coordinate on the left copy of M° and z' on the right copy. The proof for (iii) is only notationally different.

As in the low energy case, the Legendre submanifold L has the property that it intersects $N^* \mathrm{diag}_b$ in a codimension-1 submanifold and, in a deleted neighbourhood of $N^* \mathrm{diag}_b$, it projects in a 2:1 fashion down to the base, $\mathrm{mf} = M^2_{\mathrm{b}}$, so that the two sheets are parametrized by the phase functions $\pm d(z, z')$.

We now apply [Guillarmou et al. 2013b, Lemma 7.6 and (ii) of Lemma 7.7]. This tells us that, for any point in the microlocal support of $Q_j^{\mathrm{high}}(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_j^{\mathrm{high}}(\lambda)^*$, either there is a neighbourhood in which L projects diffeomorphically to the base M^2_{b} or the point lies at the conormal bundle to the diagonal, i.e.,

$z = z'$ and $\zeta = -\zeta'$. In the former case, the function $\pm d(z, z')$ can be used directly as the phase function and we obtain the statement (i) in the proposition. In the latter case, a phase function Ψ depending on $n-1$ variables $v_2, \dots, v_n$ can be constructed following the general approach of [Guillarmou et al. 2013b, Proposition 7.5]. Since this was not written down explicitly in the coordinates (z, z') valid in the interior of M_b^2, we sketch briefly how this is done. It follows from the proof of Lemma 7.6 of [Guillarmou et al. 2013b] that we can rotate coordinates so that $w_1, \zeta_2, \dots, \zeta_n, z'$ give coordinates on L locally. (The proof of Lemma 7.6 shows that one can take $(\tau, \zeta_2, \dots, \zeta_n, z')$ but, since it is also shown that $\partial z_1/\partial\tau \neq 0$, one can substitute z_1 for τ and then substitute $w_1 = z_1 - z_1'$ for z_1.) One can therefore express the functions $w_2, \dots, w_n$ and τ on L as smooth functions $W_j(w_1, \zeta_2, \dots, \zeta_n, z')$ and $T(w_1, \zeta_2, \dots, \zeta_n, z')$ of these coordinates. Then the function

$$\Psi(w, z', v) = \sum_{j=2}^{n} (w_j - W_j(w_1, \zeta_2, \dots, \zeta_n, z'))v_j + T(w_1, \zeta_2, \dots, \zeta_n, z')$$

satisfies the requirements of (3-6) and parametrizes L locally. This is shown by adapting the argument of [Guillarmou et al. 2013b, Proof of Proposition 6.2] in a straightforward way (which itself is a minor variation on [Hörmander 1985, Theorem 21.2.18]), so we omit the details. This establishes part (iii) of the proposition. When working close to $x = x' = 0$, we need to use coordinates as in [Guillarmou et al. 2013b, Proposition 7.5] and apply [Guillarmou et al. 2013b, Lemma 7.6 and (i) of Lemma 7.7], and we end up with the statement in part (ii). ☐

Remark 3.6. The Lagrangian L is smooth up to the boundary when viewed as a submanifold in the "scattering-fibred cotangent bundle" described in [Guillarmou et al. 2013a]. The boundary at bf is naturally isomorphic to L^{bf} in Proposition 2.6. Correspondingly, we find that the distance function $d(z, z')$ on M_b^2 satisfies

$$d(z, z') - d_{\mathrm{conic}}\Big(y, y', \frac{1}{x}, \frac{\sigma}{x}\Big) = e(z, z')$$

is a bounded function on M_b^2 or, more precisely, on that part of M_b^2 where $x, x' \le \eta$ and $d_{\partial M}(y, y') \le \eta$ for sufficiently small η (see [Hassell et al. 2005, Lemma 9.4]). From this we see that the results of Propositions 2.6 and 3.4 are compatible, as the factor $\exp(i\lambda e(z, z'))$ — which is the discrepancy between (2-10) and (3-4) and between (2-12d) and (3-6d) — can be absorbed in the symbols $\tilde{a}$ and b, respectively.

Remark 3.7. The results of this paper could be extended to long-range scattering metrics, as treated in [Hassell et al. 2006]. However, this would require an extension of the results of [Hassell and Vasy 2001; Hassell and Wunsch 2008; Guillarmou et al. 2013a] to Lagrangian submanifolds which are only conormal, rather than smooth, at the boundary. If this were done, then the discrepancy $e(z, z')$ between the distance function and the conic distance function would no longer be smooth or even bounded, but rather conormal at the boundary with a bound of the form $(x + x')^{-1+\epsilon}$ at the boundary of M_b^2, i.e., a bit smaller than the distance functions themselves. In this case, the correct description of the localized spectral measure would be with the true distance function $d(z, z')$ as phase function, rather than (2-10), which is only true in the short-range case.

The assumption on the potential could also be weakened; for example, one could assume that V only decays as $x^{2+\epsilon}$ as $x \to 0$, and is only conormal, rather than smooth, as $x \to 0$, instead of (1-5). However, if one assumes only $O(x^2)$ decay then it is not clear whether Theorem 1.1 will hold. For example, if $V \in x^2 C^\infty(M)$ and $V_0 := x^{-2} V|_{\partial M}$ takes values in the range $\bigl(-\frac{1}{4}(n-2)^2, 0\bigr)$, then it follows from [Guillarmou et al. 2013a, Corollary 1.5] that the $L^1 \to L^\infty$ norm of the propagator is at least a constant times $t^{-(\nu_0+1)}$ as $t \to \infty$, where ν_0^2 is the smallest eigenvalue of $\Delta_{\partial M} + V_0 + \frac{1}{4}(n-2)^2$. Under the above assumption on the range of V_0, we see that $\nu_0 < \frac{1}{2}n - 1$. This implies that the *dispersive* estimate (1-12) will no longer be valid as $|t-s| \to \infty$. However, the implications of that for the global-in-time Strichartz estimates are not clear; in the case of inverse-square potentials on $\mathbb{R}^n$, global-in-time Strichartz estimates hold despite the fact that the dispersive estimate is not known to hold for negative inverse-square potentials [Burq et al. 2004b] (for positive inverse-square potentials, the dispersive estimate is proved in [Fanelli et al. 2013]).

The problem, however, is only with the *long*-time Strichartz estimates; for estimates on a finite time interval, the decay condition on V as $x \to 0$ could be weakened considerably.

4. Proof of Proposition 1.5

We now prove Proposition 1.5. We define our partition of unity Q_j by combining the low-energy and high-energy partitions. We choose a cutoff function $\chi(\lambda)$ supported in $[0, 2]$ such that $1-\chi$ is supported in $[1, \infty)$ and define

$$\begin{aligned} Q_1(\lambda) &= \chi(\lambda)(Q_0^{\mathrm{low}} + Q_1^{\mathrm{low}}) + (1-\chi(\lambda))(Q_0^{\mathrm{high}} + Q_1^{\mathrm{high}}), && \\ Q_j(\lambda) &= \chi(\lambda) Q_j^{\mathrm{low}} + (1-\chi(\lambda)) Q_j^{\mathrm{high}} && \text{for } 2 \le j \le N_l, \\ Q_j(\lambda) &= (1-\chi(\lambda)) Q_j^{\mathrm{high}} && \text{for } N_l + 1 \le j \le N. \end{aligned} \tag{4-1}$$

We first note that the term with $Q_1(\lambda)$ satisfies (1-14) (with only the “b” term present) and (1-16), according to Propositions 2.6 and 3.4. (In the case of low energies we also need to use Remark 3.6, which tells us that we can replace the distance function by the conic distance function d_{conic} in (1-14) without affecting the estimates on the amplitudes $a_\pm$.)

Next we prove the proposition for low energies, i.e., for $\lambda \le 2$, and $j \ge 2$. Consider the second type of representation, (2-11), in Proposition 2.6. We break the estimate into various cases. We first observe that estimates of the form (1-15) and (1-16) are unaffected by multiplication by a cutoff function of the form $\chi(\lambda d(z, z'))$, where $\chi \in C_c^\infty(\mathbb{R})$. Therefore, we may treat the cases $\lambda d(z, z') \lesssim 1$ and $\lambda d(z, z') \gtrsim 1$ separately. Consider first the case $\lambda d(z, z') \lesssim 1$ or, equivalently, $|w| \lesssim \rho$. In this case, we show that (2-11) has the form (1-14), where only the “b” term is present, satisfying (1-16). Thus, we need to show that

$$(\lambda\, \partial_\lambda)^\alpha \int_{\mathbb{R}^{n-1}} e^{i\lambda\Phi(y,w,v)/x} \tilde{a}\Bigl(\lambda, \frac{x}{\lambda}, y, w_1, v\Bigr)\, dv$$

is uniformly bounded. For $\alpha = 0$ this is obvious. So consider the effect of applying $\lambda\, \partial_\lambda$. This is harmless when it hits $\tilde{a}$. When it hits the phase, it brings down a factor $i\lambda\Phi/x$. We have $\lambda\Phi/x = \Phi/\rho = v \cdot d_v \Phi/\rho + O(w_1/\rho)$ and, since $|w| \lesssim \rho$, the $O(w_1/\rho)$ is harmless. To treat the $v \cdot d_v \Phi/\rho$ term, we can write, using (2-12b),

$$\frac{v \cdot d_v \Phi}{\rho} e^{i\Phi/\rho} = -iv \cdot d_v e^{i\Phi/\rho},$$

and integrating by parts we see that this term is $O(1)$ after integration. Repeated applications of $\lambda\partial_\lambda$ are treated similarly.

Second, suppose that $|w| \geq C\rho$ for some large C but that $|w_1| \leq \rho$. For large enough C, this means that $d_{v_j}\Phi \neq 0$ for some $j \geq 2$ since, by (2-12a), we have $d_{v_j}\Phi = w_j - O(w_1)$. So, by choosing j so that $|w_j|$ is maximal and then C large enough, we have $|d_{v_j}\Phi| \geq c|w|$. Then we can write

$$e^{i\Phi/\rho} = \left(\frac{\rho\, d_{v_j}}{i\, d_{v_j}\Phi}\right)^N e^{i\Phi/\rho}$$

and integrate by parts. Each integration by parts gains us a factor of $\rho/|w|$. Thus we can estimate (2-11) by $(1+|w|/\rho)^{-K} = (1+\lambda d(z,z'))^{-K}$ for any K. Estimating the terms for $\alpha > 0$ is done just as in the first case above.

Third, suppose that $|w| \geq C|w_1|$ for some large C and that $|w_1| \geq \rho$. Then we can integrate by parts and gain any number of factors of $(1+\lambda d(z,z'))^{-1}$ as in the second case above.

Finally we come to the case where $|w_1| \geq \rho$ and $|w_1|$ is comparable to $|w|$. In this case, we have removed a neighbourhood of $N^* \operatorname{diag}_b$ from the microlocal support of the localized spectral measure. As discussed in Section 2, in this region the Lagrangian L^{bf} is a union of two sheets, each of which projects diffeomorphically to the base bf and is parametrized by the phase function $\pm d_{\mathrm{conic}}$ (in terms of the phase function Φ as in (2-11)–(2-12), this simply corresponds to the sign of w_1). We can thus split this case into two parts, according to the sign of w_1, which give rise to the "$\pm$" terms in (1-14).

In this case, the key is to exploit property (2-12c). Define

$$\widetilde{\Phi}(x,y,w,v) = |w_1|^{-1}(\Phi(y,w,v) \mp x d(z,z')) \tag{4-2}$$

and let $\omega = |w_1|/\rho$; then we need to estimate

$$\lambda^\alpha\, \partial_\lambda^\alpha a(\lambda,z,z') = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\omega^\beta \int_{\mathbb{R}^{n-1}} e^{i\omega\widetilde{\Phi}(x,y,w,v)}\widetilde{\Phi}^\beta(\lambda^\gamma\, \partial_\lambda^\gamma \tilde{a})(\lambda,\rho,y,w_1,v)\, dv.$$

Let $\tilde{b} = \lambda^\gamma\, \partial_\lambda^\gamma \tilde{a}$; then $|\partial_\lambda^\gamma \tilde{b}| \leq C_\gamma \lambda^{-\gamma}$. Thus, noting $\omega \geq 1$, it suffices to show that, for any $0 \leq \beta \leq \alpha$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\omega\widetilde{\Phi}(x,y,w,v)}(\omega\widetilde{\Phi})^\beta \tilde{b}(\lambda,\rho,y,w_1,v)\, dv \right| \leq C\omega^{-(n-1)/2}. \tag{4-3}$$

To proceed, we fix (x,y,w) with $w \neq 0$ (and hence $w_1 \neq 0$ due to our assumption that $|w_1|$ is comparable to $|w|$). We use a cutoff function Υ to divide the v integral into two parts: one on the support of Υ, in which $|d_v\widetilde{\Phi}| \geq \frac{1}{2}\tilde{\epsilon}$, and the other on the support of $1-\Upsilon$, in which $|d_v\widetilde{\Phi}| \leq \tilde{\epsilon}$. On the support of Υ, we integrate by parts in v and gain any power of ω^{-1}, proving (4-3). On the support of $1-\Upsilon$, we make the variable change

$$(v_2,\dots,v_n) \to (\theta_2,\dots,\theta_n), \quad \theta_i = d_{v_i}\widetilde{\Phi}, \quad i = 2,\dots,n.$$

Note that, by property (2-12c),

$$\frac{\partial \theta_j}{\partial v_k} = d^2_{v_j v_k} \widetilde{\Phi} = \pm A_{jk}.$$

The nondegeneracy of A_{jk} shows that this change of variables is locally nonsingular provided $\tilde{\epsilon}$ is sufficiently small. Thus, for each point v in the support of $1 - \Upsilon$, there is a neighbourhood in which we can change variables to θ as above. Using the compactness of the support of b in (2-11), we see that there are a finite number of neighbourhoods covering the intersection of the support of Υ and the v-support of b. For simplicity of exposition, we assume that there is only one such neighbourhood U below.

Let $\mathcal{B}_\delta := \{\theta : |\theta| \le \delta\}$ and choose a C^∞ function $\chi_{\mathcal{B}_\delta}(\theta)$ which equals 1 on the set $\mathcal{B}_\delta$ but equals 0 outside $\mathcal{B}_{2\delta}$, with bounds on the derivatives given by

$$|\nabla^{(j)}_\theta \chi_{\mathcal{B}_\delta}(\theta)| \le C\delta^{-j}.$$

Here δ is a parameter to be chosen later (depending on ω). Consider the integral (4-3) after changing variables and with the cutoff function $\chi_{\mathcal{B}_\delta}(\theta)$ inserted (note that $1 - \Upsilon = 1$ on the support of $\chi_{\mathcal{B}_\delta}(\theta)$, provided $\delta \le \frac{1}{2}\tilde{\epsilon}$):

$$\left| \int e^{i\omega \widetilde{\Phi}(x,y,w,\theta)} (\omega \widetilde{\Phi})^\beta \tilde{b}(\lambda, \rho, y, w_1, \theta) \chi_{\mathcal{B}_\delta}(\theta) \frac{d\theta}{|A^{-1}(y, w, \theta)|} \right|.$$

Using property (2-12d), we see that $\widetilde{\Phi} = 0$ when $\theta = 0$. Also, due to our choice of θ, we have $d_\theta \widetilde{\Phi} = 0$ when $\theta = 0$, so $\widetilde{\Phi} = O(|\theta|^2)$. Hence,

$$\left| \omega^\beta \int e^{i\omega \widetilde{\Phi}(x,y,w,\theta)} \widetilde{\Phi}^\beta \tilde{b}(\lambda, \rho, y, w_1, \theta) \chi_{\mathcal{B}_\delta}(\theta) \frac{d\theta}{|A^{-1}(y, w, \theta)|} \right| \le C(\omega\delta^2)^\beta \delta^{n-1}.$$

It remains to treat the integral with cutoff $1 - \chi_{\mathcal{B}_\delta}(\theta)$ inserted. Notice that $|d_\theta \widetilde{\Phi}|$ is comparable to $|\theta|$ since $d_\theta \widetilde{\Phi} = 0$ when $\theta = 0$, and

$$d^2_{\theta_i \theta_j} \widetilde{\Phi} = \sum_{k,l} (A^{-1})_{il} (A^{-1})_{jk} \, d^2_{v_k v_l} \widetilde{\Phi}$$

is nondegenerate when $\theta = 0$. We define the differential operator L by

$$L = \frac{-i \, d_\theta \widetilde{\Phi} \cdot \partial_\theta}{\omega |d_\theta \widetilde{\Phi}|^2}.$$

Then the adjoint operator is given by

$${}^tL = -L + \frac{i}{\omega} \left(\frac{\Delta_\theta \widetilde{\Phi}}{|d_\theta \widetilde{\Phi}|^2} - 2 \frac{d^2_{\theta_j \theta_k} \widetilde{\Phi} \, d_{\theta_j} \widetilde{\Phi} \, d_{\theta_k} \widetilde{\Phi}}{|d_\theta \widetilde{\Phi}|^4} \right).$$

Since $L e^{i\omega\widetilde{\Phi}} = e^{i\omega\widetilde{\Phi}}$, we integrate by parts N times to obtain

$$\left| \int e^{i\omega \widetilde{\Phi}(x,y,w,\theta)} (\omega \widetilde{\Phi})^\beta \tilde{b}(\lambda, \rho, y, w_1, \theta)(1 - \chi_{\mathcal{B}_\delta}(\theta))(1 - \Upsilon) \, d\theta \right|$$
$$\le C \int \left| ({}^tL)^N \big((\omega\widetilde{\Phi})^\beta \tilde{b}(\lambda, \rho, y, w_1, \theta)(1 - \chi_{\mathcal{B}_\delta}(\theta))(1 - \Upsilon) \big) \right| d\theta.$$

Inductively, we find that

$$\left|({}^tL)^N\big((\omega\widetilde{\Phi})^\beta\tilde{b}(1-\chi_{\mathcal{B}_\delta})(1-\Upsilon)\big)\right| \le C\omega^{-N+\beta}\max\{|\theta|^{2\beta-2N}, |\theta|^{2\beta-N}\delta^{-N}\}.$$

Choosing N large enough, we get

$$\begin{aligned}\left|\int e^{i\omega\widetilde{\Phi}(x,y,w,\theta)}(\omega\widetilde{\Phi})^\beta\tilde{b}(\lambda,\rho,y,w_1,\theta)(1-\chi_{\mathcal{B}_\delta})(1-\Upsilon)\,d\theta\right| &\le \omega^{-N+\beta}\int_{|\theta|\ge\delta}(|\theta|^{2\beta-2N}+|\theta|^{2\beta-N}\delta^{-N})\,d\theta\\ &\le C\omega^{-N+\beta}\delta^{2\beta-2N}\delta^{n-1}.\end{aligned}$$

Choose $\delta=\omega^{-1/2}$ to balance the two parts of the integral (with $\chi_{\mathcal{B}_\delta}$ and with $1-\chi_{\mathcal{B}_\delta}$). We finally obtain

$$\left|\int e^{i\omega\widetilde{\Phi}(x,y,w,\theta)}(\omega\widetilde{\Phi})^\beta\tilde{b}(\lambda,\rho,y,w_1,\theta)(1-\Upsilon)\,d\theta\right| \le C\omega^{-(n-1)/2},$$

which proves (4-3), as desired.

We next sketch how to prove (1-16) in the high-energy case $i > N_l$. In terms of Proposition 3.4, consider a term of type (iii); it suffices to show

$$a(h,z,z') = e^{\mp id(z,z')/h}\int_{\mathbb{R}^{n-1}} e^{i\Psi(y,w,x,v)/(xh)}b(h,x,y,w_1,v)\,dv,$$

satisfies

$$|(h\,\partial_h)^\alpha a(h,z,z')| \le C_\alpha\left(1+\frac{|w|}{xh}\right)^{-\frac{n-1}{2}}.$$

Notice that $\lambda = 1/h$ and Ψ has the same properties (2-12a)–(2-12d) as Φ. Therefore the low energy proof works verbatim, with the argument x of Ψ acting as a smooth parameter, and leads to the desired conclusion. The proof in case (ii) works in exactly the same way, with w given by $z-z'$.

Remark 4.1. To illustrate this theorem, consider the case of the spectral measure on flat $\mathbb{R}^3$, which is

$$dE_{\sqrt{\Delta}}(\lambda)(z,z') = \frac{1}{2\pi^2}\frac{\lambda^2\sin\lambda|z-z'|}{\lambda|z-z'|}\,d\lambda.$$

We decompose this, using the cutoff function χ as in (4-1), according to the size of $\lambda|z-z'|$. Where $\lambda|z-z'|\ge 1$, that is, more than one wavelength from the diagonal, we split the sine factor into exponential terms. Within $O(1)$ wavelengths of the diagonal, however, we keep the sine factor as is, to exploit the cancellation in the difference $e^{+i\lambda|z-z'|}-e^{-i\lambda|z-z'|}$ when $\lambda|z-z'|$ is small. This gives us an expression

$$\frac{\lambda^2}{2\pi^2}\left((1-\chi)(\lambda|z-z'|)\frac{e^{i\lambda|z-z'|}}{2i\lambda|z-z'|}-(1-\chi)(\lambda|z-z'|)\frac{e^{-i\lambda|z-z'|}}{2i\lambda|z-z'|}+\chi(\lambda|z-z'|)\frac{\sin\lambda|z-z'|}{\lambda|z-z'|}\right).$$

This is a decomposition into "±" and "b" terms as in (1-14), where the amplitudes satisfy (1-15) and (1-16). So, we can think of the b term as the near-diagonal term and the other terms as related to the two sheets of the Lagrangian L or L^{bf}, which are separated away from the diagonal. The function of the microlocalizing operators $Q_j(\lambda)$ (which are not required in the case of flat Euclidean space) is to remove parts of the Lagrangian that do not project diffeomorphically to the base.

5. L^2 estimates

In this section, we prove $L^2 \to L^2$ estimates on microlocalized versions of the Schrödinger propagator, using the operator partition of unity Q_j described at the beginning of the previous section, based on [Guillarmou et al. 2013b].

We begin by defining microlocalized propagators. First we give a formal definition. It is not immediately clear that the formal definition is well defined, so our first task is to show this. We do so by showing that each microlocalized propagator is a bounded operator on L^2. This serves both to show the well-definedness of each microlocalized propagator and to establish the $L^2 \to L^2$ estimate needed for the abstract Keel–Tao argument.

We define, as in the introduction,

$$U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda), \tag{5-1}$$

where Q_j is the decomposition defined in (4-1).

Our first task is to make sense of this expression. We do this by showing that each $U_j(t)$ is a bounded operator on $L^2(M^\circ)$. We have:

Proposition 5.1. *For each j, the integral* (5-1) *defining $U_j(t)$ is well defined on each finite interval and converges on $\mathbb{R}_+$ in the strong operator topology to define a bounded operator on $L^2(M^\circ)$. Moreover, the operator norm of $U_j(t)$ on $L^2(M^\circ)$ is bounded uniformly for $t \in \mathbb{R}$. Finally, we have*

$$\sum_j U_j(t) = e^{it\boldsymbol{H}}. \tag{5-2}$$

Proof. Suppose that $A(\lambda)$ is a family of bounded operators on $L^2(M^\circ)$, compactly supported and C^1 in $\lambda \in (0, \infty)$. Integrating by parts,

$$\int_0^\infty A(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)$$

is given by

$$-\int_0^\infty \Big(\frac{d}{d\lambda} A(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda)\, d\lambda.$$

In view of Corollaries 2.5 and 3.3, we can take $A(\lambda)$ to be a smooth function of λ with compact support in $(0, \infty)$ multiplied by $e^{it\lambda^2} Q_j(\lambda)$. This means that the integral (5-1) is well defined over any compact interval in $(0, \infty)$. We need to show that the integral over the whole of $\mathbb{R}_+$ converges in the strong operator topology. To do so, we introduce a dyadic partition of unity on the positive λ axis by choosing $\phi \in C_c^\infty\big(\big[\tfrac{1}{2}, 2\big]\big)$, taking values in [0, 1], such that

$$\sum_{m \in \mathbb{Z}} \phi\Big(\frac{\lambda}{2^m}\Big) = 1.$$

We now define

$$U_{j,m}(t) = -\int_0^\infty \frac{d}{d\lambda}\Big(e^{it\lambda^2} \phi\Big(\frac{\lambda}{2^m}\Big) Q_j(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda). \tag{5-3}$$

We next show that the sum over m of the operators $U_{j,m}(t)$ in (5-3) is well defined. For this we use the Cotlar–Stein lemma, which we recall here (we use the version in [Grafakos 2009, Chapter 8]):

Lemma 5.2 (Cotlar–Stein lemma). *Suppose that $\{A_j\}$ is a sequence of bounded linear operators on a Hilbert space H such that*

$$\|A_m^* A_n\|_{H\to H} \le (\gamma(m-n))^2, \quad \|A_m A_n^*\|_{H\to H} \le (\gamma(m-n))^2, \tag{5-4}$$

where $\{\gamma(m)\}_{m\in\mathbb{Z}}$ is a sequence of positive constants such that $C = \sum_{m\in\mathbb{Z}} \gamma(m) < \infty$. Then, for all $f \in H$, the sequence $\sum_{|m|\le N} A_m f$ converges as $N \to \infty$ to an element $Af \in H$. The operators $A = \sum_m A_m$ and $A^* = \sum_m A_m^*$ so defined (in the strong operator topology) satisfy

$$\|A\|_{H\to H} \le C, \quad \|A^*\|_{H\to H} \le C. \tag{5-5}$$

Moreover, the operator norms of $\sum_{m\in J} A_m$ and $\sum_{m\in J} A_m^*$ are bounded by C for any finite subset J of the integers.

We also use the following lemma:

Lemma 5.3. *Suppose that $A_l(\lambda)$ for $l = 1, 2$ is a family of operators compactly supported in λ in the open interval $(0, \infty)$ with $A_l(\lambda)$ and $\partial_\lambda A_l(\lambda)$ uniformly bounded on $L^2(M^\circ)$. Define*

$$B_l = \int A_l(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda).$$

Then

$$B_1 B_2^* = \int A_1(\lambda)\, dE_{\sqrt{\boldsymbol{H}}}(\lambda)\, A_2(\lambda)^*,$$

where by definition the last expression is equal to

$$\int \Big(-\frac{d}{d\lambda} A_1(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda) A_2(\lambda) - A_1(\lambda) E_{\sqrt{\boldsymbol{H}}}(\lambda) \Big(\frac{d}{d\lambda} A_2(\lambda)\Big). \tag{5-6}$$

Proof. We compute

$$\begin{aligned} B_1 B_2^* &= \iint \Big(\frac{d}{d\lambda} A_1(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda) E_{\sqrt{\boldsymbol{H}}}(\mu) \Big(\frac{d}{d\mu} A_2(\mu)^*\Big)\, d\lambda\, d\mu \\ &= \iint_{\lambda\le\mu} \Big(\frac{d}{d\lambda} A_1(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda) \Big(\frac{d}{d\mu} A_2(\mu)^*\Big)\, d\lambda\, d\mu \\ &\quad + \iint_{\mu\le\lambda} \Big(\frac{d}{d\lambda} A_1(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\mu) \Big(\frac{d}{d\mu} A_2(\mu)^*\Big)\, d\lambda\, d\mu \\ &= \int \Big(\frac{d}{d\lambda} A_1(\lambda)\Big) E_{\sqrt{\boldsymbol{H}}}(\lambda)(-A_2(\lambda)^*)\, d\lambda + \int (-A_1(\mu)) E_{\sqrt{\boldsymbol{H}}}(\mu) \Big(\frac{d}{d\mu} A_2(\mu)^*\Big)\, d\mu \\ &= (5\text{-}6). \end{aligned} \tag{5-7}$$

This concludes the proof. □

Now we show that the sum in (5-3) is well defined. We first note a simplification: since the $Q_j(\lambda)$ are a partition of the identity, we have

$$V_m(t) := \sum_{j=1}^{N} U_{j,m}(t) = \int e^{it\lambda^2} \chi(\lambda)\phi\Big(\frac{\lambda}{2^m}\Big)\, dE_{\sqrt{H}}(\lambda),$$

which is clearly bounded on $L^2(M^\circ)$ with operator norm bounded by 1 using spectral theory. Moreover, the sum of any subset of the V_m converges strongly to an operator with norm bounded by 1. Due to this, we may ignore the case $j = 1$ and prove the L^2 boundedness only for $j \geq 2$.

We have, by Lemma 5.3,

$$\begin{aligned} U_{j,m}(t)U_{j,n}(t)^* &= \int \chi(\lambda)^2 \phi\Big(\frac{\lambda}{2^m}\Big)\phi\Big(\frac{\lambda}{2^n}\Big) Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_j(\lambda)^* \\ &= -\int \frac{d}{d\lambda}\Big(\chi(\lambda)^2 \phi\Big(\frac{\lambda}{2^m}\Big)\phi\Big(\frac{\lambda}{2^n}\Big) Q_j(\lambda)\Big) E_{\sqrt{H}}(\lambda) Q_j(\lambda)^* \\ &\qquad - \int \chi(\lambda)^2 \phi\Big(\frac{\lambda}{2^m}\Big)\phi\Big(\frac{\lambda}{2^n}\Big) Q_j(\lambda) E_{\sqrt{H}}(\lambda) \frac{d}{d\lambda} Q_j(\lambda)^*. \end{aligned} \tag{5-8}$$

We observe that this is independent of t and is identically zero unless $|m-n| \leq 2$. When $|m-n| \leq 2$, we note that the integrand is a bounded operator on L^2, with an operator bound of the form C/λ, where C is uniform, as we see from Corollary 2.5 and the support property of ϕ. The integral is therefore uniformly bounded, as we are integrating over a dyadic interval in λ.

We next consider the operators $U^*_{j,m}(0)U_{j,n}(0)$, just in the case $t = 0$. This has an expression

$$\iint E_{\sqrt{H}}(\lambda) \frac{d}{d\lambda}\Big(\phi\Big(\frac{\lambda}{2^m}\Big) Q_j(\lambda)^*\Big) \frac{d}{d\mu}\Big(Q_j(\mu)\phi\Big(\frac{\mu}{2^n}\Big)\Big) E_{\sqrt{H}}(\mu)\, d\lambda\, d\mu.$$

It is clear that each of these operators is uniformly bounded in m, n in operator norm. To apply Cotlar–Stein, we show a estimate of the form $C2^{-|m-n|}$ for the operator norm of this term. Write $Q^*_{j,m}(\lambda)$ and $Q_{j,n}(\mu)$ for the operators in parentheses above. Consider first the case $2 \leq j \leq N_l$, in which Q_j has Schwartz kernel supported near the boundary of the diagonal. For convenience of exposition, we assume that $\lambda, \mu \leq 2$ (or, equivalently, $m, n \leq 1$). Then, by the construction of Q_j for $2 \leq j \leq N_l$ (see Section 2D and (4-1)), the scattering pseudodifferential operators $Q^*_{j,m}(\lambda)$ and $Q_{j,n}(\mu)$ are smooth and compactly supported in x'/λ and x'/μ, respectively, and are microlocally supported near the characteristic set. More precisely, we see the composition of the two scattering pseudodifferential operators for $j \geq 2$ takes the form

$$\begin{aligned} Q^*_{j,m}(\lambda)Q_{j,n}(\mu) = \int & e^{-i\lambda((y-y')\cdot\eta + (\sigma-1)\nu)/x'} e^{i\mu((y'-y'')\cdot\eta' + (\sigma'-1)\nu')/x'} \\ & \times q_{j,m}\Big(\lambda, y', \frac{x'}{\lambda}, \eta, \nu\Big) q_{j,n}\Big(\mu, y', \frac{x'}{\mu}, \eta', \nu'\Big)\, dx'\, dy'\, d\eta\, d\nu\, d\eta'\, d\nu', \end{aligned}$$

where $\sigma = x'/x$ and $\sigma' = x'/x''$, and $q_{j,m}$ and $q_{j,n}$ are smooth and polyhomogeneous in λ and μ and compactly supported in x'/λ, x'/μ and y'. In addition, we have $\nu^2 + |\eta|^2 \geq \frac{1}{4}$ and $\nu'^2 + |\eta'|^2 \geq \frac{1}{4}$ on the support of $q_{j,m}q_{j,n}$. By symmetry, we assume $\lambda > \mu$ without loss of generality. Let us introduce the operator

$$\mathscr{L} = i[\lambda(|\nu|^2 + |\eta|^2)]^{-1}(x'\eta\,\partial_{y'} - \nu x'^2\,\partial_{x'});$$

then $\mathscr{L}e^{-i\lambda((y-y')\cdot\eta+(\sigma-1)\nu)/x'} = e^{-i\lambda((y-y')\cdot\eta+(\sigma-1)\nu)/x'}$. By using $\mathscr{L}$ to integrate by parts, we gain the factor λ^{-1} since $|\nu|^2 + |\eta|^2$ is uniformly bounded from below; we incur a factor μ if the derivative falls on $e^{i\mu((y'-y'')\cdot\eta'+(\sigma'-1)\nu')/x'}$, or a factor of x' or x'^2/μ if the derivative falls on $q_{j,m}$ or $q_{j,n}$. Since $x' \le \mu$ on the support of $q_{j,m}$, we have an overall gain of $\mu/\lambda \sim 2^{-|m-n|}$. The L^2 boundedness of the spectral projection gives $\|U^*_{j,m}(0)U_{j,n}(0)\|_{L^2\to L^2} \le C2^{-|m-n|}$.

A similar argument works if one or both of m and n are at least 1.

A similar estimate is true in the case $N_l + 1 \le j \le N$, in which case we are automatically in the high-energy case, and with Schwartz kernels supported in the interior of $M^\circ \times M^\circ$. The argument is also almost exactly the same as the previous case. We can write the composition

$$\frac{d}{d\lambda}\biggl(\phi\Bigl(\frac{\lambda}{2^j}\Bigr)Q_j(\lambda)^*\biggr)\frac{d}{d\mu}\biggl(Q_j(\mu)\phi\Bigl(\frac{\mu}{2^k}\Bigr)\biggr)$$

in the form

$$\lambda^n\mu^n\iiint e^{i\lambda(z-z'')\cdot\zeta}q_{j,m}(z'',\zeta,\lambda)e^{i\mu(z''-z')\cdot\zeta'}q_{j,n}(z'',\zeta',\mu)\,d\zeta\,d\zeta'\,dz'', \tag{5-9}$$

where $q_{j,m}$ is supported where $\lambda \sim 2^m$ and $|\zeta|^2 \sim 1$, and is such that $D^\alpha_z D^\beta_\zeta q_{j,m}$ is bounded by $C\lambda^{-1}$. Assume without loss of generality that $m > n$, i.e., $\lambda > \mu$ on the support of the integrand. We note that the differential operator

$$\mathscr{L} = \frac{i\zeta\cdot\partial_{z''}}{\lambda|\zeta|^2}$$

leaves $e^{i\lambda(z-z'')\cdot\zeta}$ invariant, so we can apply it to this phase factor in the integral (5-9). Integrating by parts, the $\partial_{z''}$ derivative either hits the other phase factor $e^{i\mu(z''-z')\cdot\zeta'}$, in which case we incur a factor of μ, or it hits one of the symbols $q_{i,j}$ or $q_{i,k}$, in which case we incur no factor. Thus, we gain a factor of either $\mu/\lambda \sim 2^{-|j-k|}$ or $1/\lambda$ — which is even better since $\mu > 1$ on the support of $q_{j,n}(z'',\zeta',\mu)$. This completes the Cotlar–Stein estimates for $U_i(0)$.

It now follows from the Cotlar–Stein lemma that $U_j(0)^*$, $j = 2,\dots,N$, is well-defined as the strong limit of the sequence of operators

$$\sum_{|m|\le l} U_{j,m}(0)^*.$$

Consider the sequence $\sum_{|m|\le l} U_{j,m}(t)^*$. We claim that this sequence converges strongly and define $U_j(t)^*$ to be this limit. To prove this claim, choose an arbitrary $f \in L^2(M^\circ)$. We have shown that

$$\lim_{l\to\infty}\sup_{L>l}\biggl\|\sum_{l\le|m|\le L} U_{j,m}(0)^* f\biggr\|_2^2 = 0.$$

This is equivalent to

$$\lim_{l\to\infty}\sup_{L>l}\sum_{l\le|m|,|m'|\le L}\langle U_{j,m}(0)U_{j,m'}(0)^* f, f\rangle = 0.$$

But we saw in (5-8) that $U_{j,m}(0)U_{j,m'}(0)^* = U_{j,m}(t)U_{j,m'}(t)^*$. Hence we have

$$\lim_{l\to\infty} \sup_{L>l} \sum_{l\le|m|,|m'|\le L} \langle U_{j,m}(t)U_{j,m'}(t)^* f, f\rangle = 0,$$

which implies that

$$\lim_{l\to\infty} \sup_{L>l} \Bigl\| \sum_{l\le|m|\le L} U_{j,m}(t)^* f \Bigr\|_2^2 = 0.$$

Hence the sequence $\sum_{|m|\le l} U_{j,m}(t)^* f$ converges for every $f \in L^2(M^\circ)$ as $l \to \infty$, i.e., the sequence $\sum_{|m|\le l} U_{j,m}(t)^*$ converges strongly. We see from this that the integral

$$\int e^{-it\lambda^2}\, dE_{\sqrt{H}}(\lambda)\, Q_j(\lambda)^*$$

converges in the strong topology, hence defines $U_j(t)^*$. Finally we show that the operator norm of $U_j(t)^*$ is bounded uniformly in t. Since $\sum_{|m|\le l} U_{j,m}(t)^*$ converges in the strong operator topology, we have

$$\|U_j(t)^*\| \le \sup_{l\to\infty} \Bigl\| \sum_{|m|\le l} U_{j,m}(t)^* \Bigr\|.$$

But we have

$$\Bigl\| \sum_{|m|\le l} U_{j,m}(t)^* \Bigr\|^2 = \Bigl\| \sum_{|m|,|m'|\le l} U_{j,m}(t)U_{j,m'}(t)^* \Bigr\| = \Bigl\| \sum_{|m|,|m'|\le l} U_{j,m}(0)U_{j,m'}(0)^* \Bigr\| = \Bigl\| \sum_{|m|\le l} U_{j,m}(0)^* \Bigr\|^2$$

and the operator norm of $\sum_{|m|\le l} U_{j,m}(0)^*$ is bounded uniformly in l by the estimates proved above using the Cotlar–Stein lemma.

This completes the proof of Proposition 5.1. □

Remark 5.4. This argument allows us to avoid using a Littlewood–Paley-type decomposition in this setting. Littlewood–Paley-type estimates were established in [Bouclet 2010] for asymptotically conic manifolds in the form of

$$\|f\|_{L^p} \lesssim \Bigl(\sum_{k\ge 0} \|\phi(2^{-2k}\Delta_g) f\|_{L^p}^2 \Bigr)^{\frac{1}{2}} + \Bigl\| \sum_{k\le 0} \phi(2^{-2k}\Delta_g) f \Bigr\|_{L^p}.$$

6. Dispersive estimates

In this section, we use stationary phase and Proposition 1.5 to establish the microlocalized dispersive estimates.

Proposition 6.1 (microlocalized dispersive estimates). *Let $Q_j(\lambda)$ be as defined in* (4-1). *Then, for all integers $j \ge 1$, the kernel estimate*

$$\Bigl| \int_0^\infty e^{it\lambda^2} (Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_j^*(\lambda))(z, z')\, d\lambda \Bigr| \le C|t|^{-n/2} \tag{6-1}$$

holds for a constant C independent of points $z, z' \in M^\circ$.

Proof. The key to the proof is to use the estimates in Proposition 1.5. We first consider $j=1$. Since the term with $Q_1(\lambda)$ satisfies (1-14) with only the "b" term, then we can use the estimate (1-16) to obtain

$$\left|\left(\frac{d}{d\lambda}\right)^N (Q_1(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_1^*(\lambda))(z,z')\right| \le C_N\lambda^{n-1-N} \quad \text{for all } N\in\mathbb{N}. \tag{6-2}$$

Let δ be a small constant to be chosen later. Recall that we chose $\phi\in C_c^\infty\big(\big[\frac{1}{2},2\big]\big)$ with $\sum_{m\in\mathbb{Z}}\phi(2^{-m}\lambda)=1$; we write $\phi_0(\lambda)=\sum_{m\le -1}\phi(2^{-m}\lambda)$. Then

$$\left|\int_0^\infty e^{it\lambda^2}(Q_1(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_1^*(\lambda))(z,z')\phi_0\left(\frac{\lambda}{\delta}\right)d\lambda\right| \le C\int_0^\delta \lambda^{n-1}\,d\lambda \le C\delta^n.$$

We use integration by parts N times to obtain, using (6-2),

$$\begin{aligned}
&\left|\int_0^\infty e^{it\lambda^2}\sum_{m\ge 0}\phi\left(\frac{\lambda}{2^m\delta}\right)(Q_1(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_1^*(\lambda))(z,z')\,d\lambda\right| \\
&\qquad \le \sum_{m\ge 0}\left|\int_0^\infty \left(\frac{1}{2\lambda t}\frac{\partial}{\partial\lambda}\right)^N (e^{it\lambda^2})\phi\left(\frac{\lambda}{2^m\delta}\right)(Q_1(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_1^*(\lambda))(z,z')\,d\lambda\right| \\
&\qquad \le C_N|t|^{-N}\sum_{m\ge 0}\int_{2^{m-1}\delta}^{2^{m+1}\delta}\lambda^{n-1-2N}\,d\lambda \\
&\qquad \le C_N|t|^{-N}\delta^{n-2N}.
\end{aligned}$$

Choosing $\delta=|t|^{-1/2}$, we have thus proved

$$\left|\int_0^\infty e^{it\lambda^2}(Q_1(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_1^*(\lambda))(z,z')\,d\lambda\right| \le C_N|t|^{-n/2}. \tag{6-3}$$

Now we consider the case $j\ge 2$. Let $r=d(z,z')$ and $\bar r = rt^{-1/2}$. In this case, we write the kernel using Proposition 1.5 as

$$\begin{aligned}
&\int_0^\infty e^{it\lambda^2}\big(Q_j(\lambda)\,dE_{\sqrt{H}}(\lambda)\,Q_j^*(\lambda)\big)(z,z')\,d\lambda \\
&\qquad = \sum_{\pm}\int_0^\infty e^{it\lambda^2}e^{\pm ir\lambda}\lambda^{n-1}a_\pm(\lambda,z,z')\,d\lambda + \int_0^\infty e^{it\lambda^2}\lambda^{n-1}b(\lambda,z,z')\,d\lambda \\
&\qquad = t^{-n/2}\sum_{\pm}\int_0^\infty e^{i\lambda^2}e^{\pm i\bar r\lambda}\lambda^{n-1}a_\pm(t^{-1/2}\lambda,z,z')\,d\lambda + \int_0^\infty e^{it\lambda^2}\lambda^{n-1}b(\lambda,z,z')\,d\lambda,
\end{aligned} \tag{6-4}$$

where $a_\pm$ satisfies

$$|\partial_\lambda^\alpha a_\pm(\lambda,z,z')| \le C_\alpha\lambda^{-\alpha}(1+\lambda d(z,z'))^{-(n-1)/2},$$

and therefore

$$|\partial_\lambda^\alpha (a_\pm(t^{-1/2}\lambda,z,z'))| \le C_\alpha\lambda^{-\alpha}(1+\lambda\bar r)^{-(n-1)/2}. \tag{6-5}$$

By (1.16), the above term with $b(\lambda,z,z')$ can be estimated by using the same argument as for Q_1. Now we consider first term in the right-hand side of (6-4). We divide it into two pieces using the partition of unity above. It suffices to prove that there exists a constant C independent of $\bar{r}$ such that

$$\begin{aligned} I^{\pm} &:= \left| \int_0^\infty e^{i\lambda^2} e^{\pm i\bar{r}\lambda} \lambda^{n-1} a_{\pm}(t^{-1/2}\lambda, z, z') \phi_0(\lambda)\, d\lambda \right| \le C, \\ II^{\pm} &:= \left| \sum_{m\ge 0} \int_0^\infty e^{i\lambda^2} e^{\pm i\bar{r}\lambda} \lambda^{n-1} a_{\pm}(t^{-1/2}\lambda, z, z') \phi\left(\frac{\lambda}{2^m}\right) d\lambda \right| \le C. \end{aligned}$$

The estimate for $I^{\pm}$ is obvious, since $\lambda \le 1$. For II^+, we use integration by parts. Notice that

$$L^+(e^{i\lambda^2 + i\bar{r}\lambda}) = e^{i\lambda^2 + i\bar{r}\lambda}, \quad L^+ = \frac{-i}{2\lambda + \bar{r}} \frac{\partial}{\partial\lambda}.$$

Writing

$$e^{i\lambda^2 + i\bar{r}\lambda} = (L^+)^N (e^{i\lambda^2 + i\bar{r}\lambda})$$

and integrating by parts, we gain a factor of λ^{-2N} thanks to (6-5). Thus II^+ can be estimated by

$$\sum_{m\ge 0} \int_{\lambda\sim 2^m} \lambda^{n-1-2N}\, d\lambda \le C.$$

To treat II^-, we introduce a further decomposition, based on the size of $\bar{r}\lambda$. We write $II^- = II_1^- + II_2^-$, where (dropping the $-$ superscripts and subscripts from here on)

$$\begin{aligned} II_1 &= \left| \sum_{m\ge 0} \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z') \phi\left(\frac{\lambda}{2^m}\right) \phi_0(4\bar{r}\lambda)\, d\lambda \right|, \\ II_2 &= \left| \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z') (1-\phi_0(\lambda))(1-\phi_0(4\bar{r}\lambda))\, d\lambda \right|. \end{aligned}$$

Let $\Phi(\lambda, \bar{r}) = \lambda^2 - \bar{r}\lambda$. We first consider II_1. Since the integral for II_1 is supported where $\lambda \le (4\bar{r})^{-1}$ and $\lambda \ge \frac{1}{2}$, the integrand is only nonzero when $\bar{r} \le \frac{1}{2}$. Therefore, $|\partial_\lambda \Phi| = 2\lambda - \bar{r} \ge \frac{1}{2}\lambda$. Define the operator $L = L(\lambda, \bar{r}) = (2\lambda - \bar{r})^{-1}\, \partial_\lambda$. By (6-5) and using integration by parts, we obtain, for $N > \frac{1}{2}n$,

$$\begin{aligned} II_1 &\le \sum_{m\ge 0} \left| \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z') \phi\left(\frac{\lambda}{2^m}\right) \phi_0(4\bar{r}\lambda)\, d\lambda \right| \\ &= \sum_{m\ge 0} \left| \int L^N(e^{i(\lambda^2 - \bar{r}\lambda)}) \left[\lambda^{n-1} a(t^{-1/2}\lambda, z, z') \phi\left(\frac{\lambda}{2^m}\right) \phi_0(4\bar{r}\lambda) \right] d\lambda \right| \\ &\le C_N \sum_{m\ge 0} \int_{|\lambda|\sim 2^m} \lambda^{n-1-2N}\, d\lambda \\ &\le C_N. \end{aligned}$$

Finally, we consider II_2. Here, we replace the decomposition $\sum_m \phi(2^{-m}\lambda)$ with a different decomposition, based on the size of $\partial_\lambda \Phi$:

$$\begin{aligned} II_2 &\leq \left| \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z')(1-\phi_0(\lambda))\phi_0(2\lambda-\bar{r})(1-\phi_0(4\bar{r}\lambda))\, d\lambda \right| \\ &\qquad + \sum_{m\geq 0} \left| \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z')(1-\phi_0(\lambda))\phi\left(\frac{2\lambda-\bar{r}}{2^m}\right)(1-\phi_0(4\bar{r}\lambda))\, d\lambda \right| \\ &:= II_2^1 + II_2^2. \end{aligned}$$

If $\bar{r} \leq 10$, then for the integrand of II_2^1 to be nonzero we must have $\lambda \leq 10$, due to the ϕ_0 factor. Then it is easy to see that II_2^1 is uniformly bounded. If $\bar{r} \geq 10$, we have $\bar{r} \sim \lambda$ since $|2\lambda - \bar{r}| \leq 1$. Hence, using (6-5) with $\alpha = 0$,

$$II_2^1 \leq \int_{|2\lambda-\bar{r}|\leq 1} \lambda^{n-1}(1+\bar{r}\lambda)^{-(n-1)/2}\, d\lambda \leq C.$$

Now we consider the second term. Integrating by parts, we show by (6-5) that

$$\begin{aligned} II_2^2 &\leq \sum_{m\geq 0} \left| \int e^{i\lambda^2} e^{-i\bar{r}\lambda} \lambda^{n-1} a(t^{-1/2}\lambda, z, z')(1-\phi_0(\lambda))\phi\left(\frac{2\lambda-\bar{r}}{2^m}\right)(1-\phi_0(4\bar{r}\lambda))\, d\lambda \right| \\ &= \sum_{m\geq 0} \left| \int L^N(e^{i(\lambda^2-\bar{r}\lambda)}) \left[\lambda^{n-1} a(t^{-1/2}\lambda, z, z')(1-\phi_0(\lambda))\phi\left(\frac{2\lambda-\bar{r}}{2^m}\right)(1-\phi_0(4\bar{r}\lambda)) \right] d\lambda \right| \\ &\leq C_N \sum_{m\geq 0} 2^{-mN} \int_{|2\lambda-\bar{r}|\sim 2^m} \lambda^{n-1}(1+\bar{r}\lambda)^{-(n-1)/2}\, d\lambda. \end{aligned}$$

If $\bar{r} \leq 2^{m+1}$, then $\lambda \leq 2^{m+2}$ on the support of the integrand. The m-th term can then be estimated by $C_N 2^{-mN} 2^{(m+2)n}$, which is summable for $N > n$. Otherwise, we have $\lambda \sim \bar{r}$, which means the integrand is bounded and we estimate the m-th term by $C_N 2^{-mN} 2^m$, which is summable for $N > 1$. Therefore, we have completed the proof of Proposition 6.1. □

7. Homogeneous Strichartz estimates

We use the L^2 estimates and the microlocalized dispersive estimates to conclude the proof of Theorem 1.1. By Proposition 5.1, we have, for all $t \in \mathbb{R}$ and all $u_0 \in L^2$,

$$\|U_j(t)u_0\|_{L^2(M^\circ)} \lesssim \|u_0\|_{L^2(M^\circ)}.$$

By Lemma 5.3,

$$U_j(s)U_j^*(t)f = \int_0^\infty e^{i(s-t)\lambda^2} Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_j^*(\lambda) f.$$

Hence we have the decay estimates, by Proposition 6.1,

$$\|U_j(s)U_j^*(t)f\|_{L^\infty} \lesssim |t-s|^{-n/2}\|f\|_{L^1}.$$

As a consequence of the Keel–Tao abstract Strichartz estimate [1998], we have

$$\|U_j(t)u_0\|_{L^q(\mathbb{R};L^r(M^\circ))} \lesssim \|u_0\|_{L^2(M^\circ)}, \tag{7-1}$$

where (q, r) is sharp $\frac{n}{2}$-admissible, that is, $q, r \geq 2$, $(q, r, n) \neq (2, \infty, 2)$ and $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$. By the definition of $U_j(t)$ based on the construction of Q_j, we see that

$$e^{it\boldsymbol{H}} = \sum_{j=1}^{N} U_j(t). \tag{7-2}$$

Combining (7-1) and (7-2) proves the long-time homogeneous Strichartz estimate.

8. Inhomogeneous Strichartz estimates

In this section, we prove Theorem 1.2, including at the endpoint $(q, r) = (\tilde{q}, \tilde{r}) = (2, 2n/(n-2))$ for $n \geq 3$. Let $\boldsymbol{U}(t) = e^{it\boldsymbol{H}} : L^2 \to L^2$. We have already proved that

$$\|\boldsymbol{U}(t)u_0\|_{L^q_t L^r_z} \lesssim \|u_0\|_{L^2}$$

holds for all (q, r) satisfying (1-2). By duality, the estimate is equivalent to

$$\left\| \int_{\mathbb{R}} \boldsymbol{U}(t)\boldsymbol{U}^*(s)F(s)\, ds \right\|_{L^q_t L^r_z} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z},$$

where both (q, r) and $(\tilde{q}, \tilde{r})$ satisfy (1-2). By the Christ–Kiselev lemma [2001], we obtain, for $q > \tilde{q}'$,

$$\left\| \int_{s<t} \boldsymbol{U}(t)\boldsymbol{U}^*(s)F(s)\, ds \right\|_{L^q_t L^r_z} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_z}. \tag{8-1}$$

Notice that $\tilde{q}' \leq 2 \leq q$; therefore, we have proved all inhomogeneous Strichartz estimates except the endpoint $(q, r) = (\tilde{q}, \tilde{r}) = (2, 2n/(n-2))$. To treat the endpoint, we need to show the bilinear form estimate

$$|T(F, G)| \leq \|F\|_{L^2_t L^{r'}_z} \|G\|_{L^2_t L^{r'}_z}, \tag{8-2}$$

where $r = 2n/(n-2)$ and $T(F, G)$ is the bilinear form

$$T(F, G) = \iint_{s<t} \langle \boldsymbol{U}(t)\boldsymbol{U}^*(s)F(s), G(t)\rangle_{L^2}\, ds\, dt. \tag{8-3}$$

Theorem 1.2 follows from:

Proposition 8.1. *There exists a partition of the identity $Q_j(\lambda)$ on $L^2(M^\circ)$ such that, with $U_j(t)$ defined as in* (5-1), *there exists a constant C such that, for each pair (j, k), either*

$$\iint_{s<t} \langle U_j(t)U_k^*(s)F(s), G(t)\rangle_{L^2}\, ds\, dt \leq C\|F\|_{L^2_t L^{r'}_z} \|G\|_{L^2_t L^{r'}_z} \tag{8-4}$$

$$\textit{or} \quad \iint_{s>t} \langle U_j(t)U_k^*(s)F(s), G(t)\rangle_{L^2}\, ds\, dt \leq C\|F\|_{L^2_t L^{r'}_z} \|G\|_{L^2_t L^{r'}_z}. \tag{8-5}$$

Proof of Theorem 1.2 assuming Proposition 8.1. We have proved that, for all $1 \le j \le N$,

$$\|U_j(t)u_0\|_{L^2_t L^r_z} \lesssim \|u_0\|_{L^2};$$

hence it follows by duality that, for all $1 \le j, k \le N$,

$$\iint_{\mathbb{R}^2} \langle U_j(t)U_k^*(s)F(s), G(t)\rangle_{L^2}\, ds\, dt \le C\|F\|_{L^2_t L^{r'}_z} \|G\|_{L^2_t L^{r'}_z}. \tag{8-6}$$

Subtracting (8-5) from (8-6) shows that (8-4) holds for every pair (j, k). Then, by summing over all j and k, we obtain (8-2). □

To prove Proposition 8.1 we use the following lemma, proved in [Guillarmou and Hassell 2014, Lemmas 5.3 and 5.4].

Lemma 8.2. *The partition of the identity $Q_j(\lambda)$ can be chosen so that the pairs of indices (j, k), $1 \le j, k \le N$, can be divided into three classes,*

$$\{1, \ldots, N\}^2 = J_{\text{near}} \cup J_{\text{not-out}} \cup J_{\text{not-inc}},$$

such that

- *if $(j, k) \in J_{\text{near}}$, then $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k(\lambda)^*$ satisfies the conclusions of Proposition 1.5;*
- *if $(j, k) \in J_{\text{non-inc}}$, then $Q_j(\lambda)$ is not incoming-related to $Q_k(\lambda)$, in the sense that no point in the operator wavefront set (microlocal support) of $Q_j(\lambda)$ is related to a point in the operator wavefront set of $Q_k(\lambda)$ by backward bicharacteristic flow;*
- *if $(j, k) \in J_{\text{non-out}}$, then $Q_j(\lambda)$ is not outgoing-related to $Q_k(\lambda)$, in the sense that no point in the operator wavefront set of $Q_j(\lambda)$ is related to a point in the operator wavefront set of $Q_k(\lambda)$ by forward bicharacteristic flow.*

We exploit the not-incoming or not-outgoing property of $Q_j(\lambda)$ with respect to $Q_k(\lambda)$ in the following two lemmas.

Lemma 8.3. *Let $Q_j(\lambda)$ and $Q_k(\lambda)$ be such that Q_j is not outgoing-related to Q_k. Then, for $\lambda \le 2$, as a multiple of $|dg\, dg'|^{1/2}\, |d\lambda|$ the Schwartz kernel of $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k(\lambda)^*$ can be expressed as the sum of a finite number of terms of the form*

$$\lambda^{n-1} \int_{\mathbb{R}^k} e^{i\lambda\Phi(y,y',\sigma,v)/x} \left(\frac{x'}{\lambda}\right)^{\frac{n-1}{2}-\frac{k}{2}} a\left(\lambda, y, y', \sigma, \frac{x'}{\lambda}, v\right) dv \tag{8-7}$$

$$\text{or} \quad \lambda^{n-1} \int_{\mathbb{R}^{k-1}} \int_0^\infty e^{i\lambda\Phi(y,y',\sigma,v,s)/x} \left(\frac{x'}{\lambda s}\right)^{\frac{n-1}{2}-\frac{k}{2}} s^{n-2} a\left(\lambda, y, y', \sigma, \frac{x'}{\lambda}, v, s\right) ds\, dv \tag{8-8}$$

in the region $\sigma = x/x' \le 2$, $x'/\lambda \le 2$, or

$$\lambda^{n-1} a\left(\lambda, y, y', \sigma, \frac{x'}{\lambda}\right) \tag{8-9}$$

in the region $\sigma = x/x' \le 2$, $x'/\lambda \ge 1$, where, in each case, $\Phi < -\epsilon < 0$ and a is a smooth function, compactly supported in the v and s variables (where present), such that $|(\lambda\, \partial_\lambda)^N a| \le C_N$ for all $N \in \mathbb{N}$.

In each case, we may assume that $k \le n-1$; *if* $k=0$ *in* (8-7) *or* $k=1$ *in* (8-8) *then there is no variable* v *and no* v *integral. The key point is that, in each expression, the phase function is strictly negative.*

If, instead, Q_j *is not incoming-related to* Q_k, *then the same conclusion holds with the reversed sign: the Schwartz kernel can be written as a finite sum of terms with a strictly positive phase function.*

Remark 8.4. For $\sigma \ge \frac{1}{2}$, the Schwartz kernel has a similar description, as follows immediately from the symmetry of the kernel under interchanging the left and right variables.

Proof. The statement that the Schwartz kernel has the indicated forms above follows immediately from the description of the spectral measure in [Guillarmou et al. 2013a, Theorem 3.10] as a Legendre distribution in the class $I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}}}(M^2_{k,b}, (L^{\mathrm{bf}}, L^\sharp); \Omega^{1/2}_{k,b})$, where $m=-\frac{1}{2}$, $p=\frac{1}{2}(n-2)$ and $r_{\mathrm{lb}}=r_{\mathrm{rb}}=\frac{1}{2}(n-1)$. The bound on k follows from the fact that k can be taken as the drop in rank of the projection from L^{bf} to the base $(\partial M)^2 \times (0,\infty)_\sigma$, which is the front face (that is, the face created by blow-up) of M^2_{b}. We claim that the drop in rank is at most $n-1$, which proves that we may assume that $k \le n-1$. To prove this claim, we show that the differentials $dy_1, \dots dy_{n-1}$ and at least one of $d\sigma, dy'_1, \dots, dy'_{n-1}$ are linearly independent on L. This can be seen from the description of L as the flowout from the set

$$\{(y, y, 1, \mu, -\mu, \nu, -\mu) \mid \nu^2 + h(\mu) = 1\}, \tag{8-10}$$

using the coordinates of (2-6), by the flow of the vector field V_r, which is the vector field given by x^{-1} multiplied by the Hamilton vector field of the principal symbol of Δ acting in the right variables on $M^2_{k,b}$. In fact, $V_r = \sin s'\, \partial_{s'}$ in the coordinates (s, s') on the leaves γ^2 of (2-6) and takes the form (see [Hassell and Vasy 2001, Equation (2.26)] or [Guillarmou et al. 2013a, Equation (3.5)])

$$2\nu'\sigma \frac{\partial}{\partial\sigma} - 2\nu'\mu' \cdot \frac{\partial}{\partial\mu'} + h'\frac{\partial}{\partial\nu'} + \left(\frac{\partial h'}{\partial\mu'}\frac{\partial}{\partial y'} - \frac{\partial h'}{\partial y'}\frac{\partial}{\partial\mu'}\right), \quad h' = h(y', \mu') = \sum_{i,j} h^{ij}(y')\mu'_i\mu'_j.$$

It is clear that $dy_1, \dots, dy_{n-1}$ are linearly independent at the initial set (8-10). Moreover, their Lie derivative with respect to V_r vanishes, so they are linearly dependent on all of L^{bf}. Also, since $h' + \nu'^2 = 1$ on L^{bf}, either the ∂_σ or the $\partial_{y'}$ component of the vector field V_r does not vanish, unless $\sigma = 0$, showing that either $d\sigma$ or one of the dy'_i do not vanish at each point of L^{bf} for $\sigma \ne 0$. But it was shown in [Hassell and Vasy 2001] that L^{bf} is transversal to the boundary at $\sigma = 0$, which means that $d\sigma \ne 0$ on L^{bf} when σ is small. This proves the claim.

We next show that Φ can be taken to be strictly negative. We use the microlocal support estimates from [Guillarmou et al. 2013b]. Applying [Guillarmou et al. 2013b, Corollary 5.3], we find that the microlocal support of $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k(\lambda)^*$ is contained in that part of L^{bf} where, in the notation of (2-6), $s < s'$ (since the initial set (8-10) corresponds to $s = s'$, and ∂_s and $\partial_{s'}$ move in the outgoing and incoming directions, respectively, along the flow). Repeating the calculation following (2-6), we see that the value of Φ "on the Legendrian" is $\Phi = -\cos s + \sigma \cos s' = (\sin s')^{-1} \sin(s - s')$, which is strictly negative. By restricting the support of the amplitude a in (8-7)–(8-9), we can assume that Φ is negative everywhere on the support of the integrand. □

Lemma 8.5. *Let $Q_j(\lambda)$ and $Q_k(\lambda)$ be such that Q_j is not outgoing-related to Q_k. Then, for $\lambda \geq 1$, and as a multiple of $|dg\, dg'|^{1/2}\, |d\lambda|$, the Schwartz kernel of $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k(\lambda)^*$ can be written in terms of a finite number of oscillatory integrals of the form*

$$\int_{\mathbb{R}^k} e^{i\lambda\Phi(y,y',\sigma,x,v)/x}\lambda^{n-1+k/2}x^{(n-1)/2-k/2}a(\lambda, y, y', \sigma, x, v)\, dv \tag{8-11}$$

$$\text{or}\quad \int_{\mathbb{R}^{k-1}}\int_0^\infty e^{i\lambda\Phi(y,y',\sigma,x,v,s)/x}\lambda^{n-1+k/2}\Big(\frac{x}{s}\Big)^{(n-1)/2-k/2}s^{n-2}a(\lambda, y, y', \sigma, x, v, s)\, ds\, dv \tag{8-12}$$

in the region $\sigma = x/x' \leq 2$, $x \leq \delta$, or

$$\int_{\mathbb{R}^k} e^{i\lambda\Phi(z,z',v)}\lambda^{n-1+k/2}a(\lambda, z, z', v)\, dv \tag{8-13}$$

in the region $x \geq \delta$, $x' \geq \delta$, where, in each case, $\Phi < -\epsilon < 0$ and a is a smooth function compactly supported in the v and s variables (where present) such that $|(\lambda\, \partial_\lambda)^N a| \leq C_N$. In each case, we may assume that $k \leq n-1$; if $k = 0$ in (8-11) *or* (8-13), *or $k = 1$ in* (8-12), *then there is no variable v and no v integral. Again, the key point is that, in each expression, the phase function is strictly negative.*

If, instead, Q_j is not incoming-related to Q_k, then the same conclusion holds with the reversed sign: the Schwartz kernel can be written as a finite sum of terms with a strictly positive phase function.

Proof. The proof is essentially identical to that of Lemma 8.3. The form of the oscillatory integrals comes from the fact that the spectral measure, for high energies, is a Legendre distribution in the class $I^{m,p;r_{\mathrm{lb}},r_{\mathrm{rb}}}(X, (L, L^\sharp); \Omega^{s\Phi}\Omega^{1/2})$, where the Lagrangian L is given by (3-3). The non-outgoing relation implies, via the microlocal support estimates of [Guillarmou et al. 2013b, Section 7], that $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k(\lambda)^*$ is microsupported where $\tau < 0$ in the coordinates of (3-3). Since $\Phi = \tau$ when $d_v\Phi = 0$, this implies that $\Phi < 0$ when $d_v\Phi = 0$. By restricting the support of the amplitude close to the set where $d_v\Phi = 0$, we can assume that $\Phi < 0$ everywhere on the support of the integrand. ☐

Lemma 8.6. *We have the following dispersive estimates on $U_j(t)U_k(s)^*$:*

- *If $(j, k) \in J_{\mathrm{near}}$, then for all $t \neq s$ we have*

$$\|U_j(t)U_k^*(s)\|_{L^1\to L^\infty} \leq C|t-s|^{-n/2}. \tag{8-14}$$

- *If (j, k) is such that Q_j is not outgoing-related to Q_k, and $t < s$, then*

$$\|U_j(t)U_k^*(s)\|_{L^1\to L^\infty} \leq C|t-s|^{-n/2}. \tag{8-15}$$

- *Similarly, if (j, k) is such that Q_j is not incoming-related to Q_k and $s < t$, then*

$$\|U_j(t)U_k^*(s)\|_{L^1\to L^\infty} \leq C|t-s|^{-n/2}. \tag{8-16}$$

Proof. The estimate (8-14) is essentially proved in Proposition 6.1, since we can use Proposition 1.5. Assume that Q_j is not incoming-related to Q_k and consider (8-16). By Lemma 5.3, $U_j(t)U_k(s)^*$ is given by

$$\int_0^\infty e^{i(t-s)\lambda^2}(Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k^*(\lambda))(z, z'). \tag{8-17}$$

Then we need to show that, for $s < t$,

$$\left| \int_0^\infty e^{i(t-s)\lambda^2} (Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k^*(\lambda))(z, z')\, d\lambda \right| \le C|t-s|^{-n/2}. \tag{8-18}$$

Case 1: $t - s \ge 1$**.** We introduce a dyadic partition of unity in λ. Let $\phi \in C_c^\infty\big(\big[\tfrac{1}{2}, 2\big]\big)$ be as in Section 5 with $\sum_m \phi(2^{-m}\sqrt{t-s}\lambda) = 1$, define

$$\phi_0(\sqrt{t-s}\lambda) = \sum_{m \le 0} \phi(2^{-m}\sqrt{t-s}\lambda)$$

and insert

$$1 = \phi_0(\sqrt{t-s}\lambda) + \sum_{m \ge 1} \phi_m(\sqrt{t-s}\lambda), \quad \phi_m(\lambda) := \phi(2^{-m}\lambda),$$

into the integral (8-17). In addition, we substitute for $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k^*(\lambda)$ one of the expressions in Lemmas 8.3 and 8.5. Since $t - s \ge 1$, for the ϕ_0 term only the low energy expressions are relevant. The estimate follows immediately from noticing that these expressions are pointwise bounded by $C\lambda^{n-1}$, using the fact that $k \le n-1$ in these expressions.

To treat the ϕ_m terms for $m \ge 1$, we again substitute one of the expressions in Lemmas 8.3 and 8.5. For notational simplicity we consider the expression (8-13), but the argument is similar in the other cases. We scale the λ variable and obtain the expression

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^k} e^{i(t-s)\lambda^2} e^{i\lambda\Phi(z,z',v)} \lambda^{n-1+k/2} a(\lambda, z, z', v) \phi_m(\sqrt{t-s}\lambda)\, dv\, d\lambda \\ &\quad = (t-s)^{-n/2-k/4} \int_0^\infty \int_{\mathbb{R}^k} e^{i(\bar\lambda^2 + \bar\lambda\Phi(z,z',v)/\sqrt{t-s})} \bar\lambda^{n-1+k/2} a\bigg(\frac{\bar\lambda}{\sqrt{t-s}}, y, y', \sigma, v\bigg) \phi_m(\bar\lambda)\, dv\, d\bar\lambda, \end{aligned} \tag{8-19}$$

where $\bar\lambda = \sqrt{t-s}\lambda$. We observe that the overall exponential factor is invariant under the differential operator

$$L = \frac{-i}{2\bar\lambda^2 + \bar\lambda\Phi/\sqrt{t-s}} \bar\lambda \frac{\partial}{\partial\bar\lambda}.$$

The adjoint of this is

$$L^t = -L + \frac{i}{2\bar\lambda^2 + \bar\lambda\Phi/\sqrt{t-s}} - i\frac{4\bar\lambda^2 + \bar\lambda\Phi/\sqrt{t-s}}{(2\bar\lambda^2 + \bar\lambda\Phi/\sqrt{t-s})^2}.$$

We apply L^N to the exponential factors and integrate by parts N times. Since $\Phi \ge 0$ according to Lemma 8.5, and since we have an estimate $|(\bar\lambda\, \partial_{\bar\lambda})^N a| \le C_N$, each time we integrate by parts we gain a factor $\bar\lambda^{-2} \sim 2^{-2m}$. It follows that the integral with $\phi(2^{-m}\bar\lambda)$ inserted is bounded by $(t-s)^{-n/2} 2^{-m(2N-n-k/2)}$ uniformly for $t - s \ge 1$. Hence we prove (8-16) by summing over $m \ge 0$. The argument to prove (8-15) is analogous.

Case 2: $t - s \le 1$**.** In this case, we use a dyadic decomposition in terms of the original variable λ. We consider the integral (8-17), insert the dyadic decomposition

$$1 = \sum_{m \ge 0} \phi_m(\lambda),$$

and substitute for $Q_j(\lambda)\, dE_{\sqrt{H}}(\lambda)\, Q_k^*(\lambda)$ one of the expressions in Lemmas 8.3 and 8.5.

For the case $m = 0$, the estimate follows immediately from the uniform boundedness of (8-7)–(8-9). For the cases $m \geq 1$, we use the expressions in Lemma 8.5 and observe that the overall exponential factor is invariant under the differential operator

$$L = \frac{-i}{2(t-s)\lambda^2 + \lambda\Phi} \lambda \frac{\partial}{\partial\lambda}.$$

The adjoint of this is

$$L^t = -L + \frac{i}{2(t-s)\lambda^2 + \lambda\Phi} - i\frac{4(t-s)\lambda^2 + \lambda\Phi}{(2(t-s)\lambda^2 + \lambda\Phi)^2}.$$

We apply L to the exponential factors N times and integrate by parts. Since $\Phi \geq \epsilon > 0$ according to Lemma 8.5, and since we have an estimate $|(\lambda\, \partial_\lambda)^N a| \leq C_N$, each time we integrate by parts we gain a factor $\lambda^{-1} \sim 2^{-m}$. It follows that the integral with $\phi(2^{-m}\lambda)$ inserted is bounded by $2^{-m(N-n-k/2)}$ uniformly for $t - s \leq 1$. Hence we prove (8-16) by summing over $m \geq 0$. The argument to prove (8-15) is analogous. □

Remark 8.7. Notice that, in the cases (8-15) and (8-16), there is a lot of "slack" in the estimates. This is because the sign of $t - s$ has the favourable sign relative to the sign of the phase function, so that the overall phase in integrals such as (8-19) are never stationary. Then integration by parts give us more decay than needed to prove the estimates. This is important because it overcomes the growth of the spectral measure as $\lambda \to \infty$ at conjugate points: at pairs of conjugate points we have $k > 0$ and we see from, say, (8-13) that the spectral measure will not obey the localized (near the diagonal) estimates of Proposition 1.5, by a factor $\lambda^{k/2}$. The geometric meaning of k is the drop in rank of the projection from L down to M_b^2, hence is positive precisely at pairs of conjugate points.

We now complete the proof of Theorem 1.2 by proving Proposition 8.1.

Proof of Proposition 8.1. We use a partition of the identity as in Lemma 8.2. In the case that $(j, k) \in J_{\text{near}}$, we have the dispersive estimate (8-14). This allows us to apply the argument of [Keel and Tao 1998, Sections 4–7] to obtain (8-4). In the case that $(j, k) \in J_{\text{non-out}}$, we obtain (8-4) following the argument in [Keel and Tao 1998] since we have the dispersive estimate (8-16) when $s < t$. Finally, in the case that $(j, k) \in J_{\text{non-inc}}$, we obtain (8-5) since we have the dispersive estimate (8-15) for $s > t$. □

Remark 8.8. The endpoint inhomogeneous Strichartz estimate is closely related to the uniform Sobolev estimate

$$\|(\boldsymbol{H} - \alpha)^{-1}\|_{L^r \to L^{r'}} \leq C, \quad r = \frac{2n}{n+2}, \tag{8-20}$$

where C is independent of $\alpha \in \mathbb{C}$. This estimate was proved by [Kenig et al. 1987] for the flat Laplacian and by [Guillarmou and Hassell 2014] for the Laplacian on nontrapping asymptotically conic manifolds (it was also shown in [Guillarmou and Hassell 2014] that (8-20) holds for $r \in [2n/(n+2), 2(n+1)/(n+3)]$ with a power of α on the right-hand side). In fact, it was pointed out to the authors by Thomas Duyckaerts and Colin Guillarmou that the endpoint inhomogeneous Strichartz estimate implies the uniform Sobolev estimate (8-20). To see this, we choose $w \in C_c^\infty(M^\circ)$ and $\chi(t)$ equal to 1 on $[-T, T]$ and zero for $|t| \geq T+1$ and let $u(t,z) = \chi(t)e^{i\alpha t}w(z)$. Then

$$(i\,\partial_t + \boldsymbol{H})u = F(t,z), \quad F(t,z) := \chi(t)e^{i\alpha t}(\boldsymbol{H}-\alpha)w(z) + i\chi'(t)e^{i\alpha t}w(z).$$

Applying the endpoint inhomogeneous Strichartz estimate, we obtain

$$\|u\|_{L_t^2 L_z^{r'}} \leq C\|F\|_{L_t^2 L_z^r}.$$

From the specific form of u and F we have

$$\|u\|_{L_t^2 L_z^{r'}} = \sqrt{2T}\|w\|_{L^{r'}} + O(1), \quad \|F\|_{L_t^2 L_z^r} = \sqrt{2T}\|(\boldsymbol{H}-\alpha)w\|_{L^r} + O(1).$$

Taking the limit $T \to \infty$ we find that

$$\|w\|_{L^{r'}} \leq C\|(\boldsymbol{H}-\alpha)w\|_{L^r},$$

which implies the uniform Sobolev estimate.

In the other direction, suppose that the uniform Sobolev estimate holds. If u and F satisfy (1-10), then taking the Fourier transform in t we find that

$$(\boldsymbol{H}-\alpha)\hat{u}(\alpha, z) = \widehat{F}(\alpha, z). \tag{8-21}$$

Suppose for a moment that the following statement were true: "Fourier transformation in t is a bounded linear map from $L^2(\mathbb{R}_t; L^p(M^\circ))$ to $L^2(\mathbb{R}_\alpha; L^p(M^\circ))$ for $p = r', r$". Using this and the uniform Sobolev inequality, applied to (8-21), we would obtain the inhomogeneous Strichartz estimate. Unfortunately, the statement in quotation marks is known to be false, so this argument is purely heuristic. Nevertheless, it illustrates the close relation between the two estimates. It would be interesting to know if there are general conditions under which the two estimates are equivalent.

Acknowledgements

We thank Colin Guillarmou, Adam Sikora, Jean-Marc Bouclet, Thomas Duyckaerts and Pierre Portal for helpful conversations. This research was supported by Future Fellowship FT0990895 and Discovery Grants DP1095448 and DP120102019 from the Australian Research Council. Zhang was supported by Beijing Natural Science Foundation (1144014) and National Natural Science Foundation of China (11401024).

References

[Blair et al. 2008] M. D. Blair, H. F. Smith, and C. D. Sogge, "On Strichartz estimates for Schrödinger operators in compact manifolds with boundary", *Proc. Amer. Math. Soc.* **136**:1 (2008), 247–256. MR 2008k:35386 Zbl 1169.35012

[Blair et al. 2009] M. D. Blair, H. F. Smith, and C. D. Sogge, "Strichartz estimates for the wave equation on manifolds with boundary", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:5 (2009), 1817–1829. MR 2010k:35500 Zbl 1198.58012

[Blair et al. 2012] M. D. Blair, G. A. Ford, S. Herr, and J. L. Marzuola, "Strichartz estimates for the Schrödinger equation on polygonal domains", *J. Geom. Anal.* **22**:2 (2012), 339–351. MR 2891729 Zbl 1252.35237

[Bouclet 2010] J.-M. Bouclet, "Littlewood–Paley decompositions on manifolds with ends", *Bull. Soc. Math. France* **138**:1 (2010), 1–37. MR 2011e:42020 Zbl 1198.42013

[Bouclet 2011] J.-M. Bouclet, "Strichartz estimates on asymptotically hyperbolic manifolds", *Anal. PDE* **4**:1 (2011), 1–84. MR 2012i:58022 Zbl 1230.35027

[Bouclet and Tzvetkov 2007] J.-M. Bouclet and N. Tzvetkov, "Strichartz estimates for long range perturbations", *Amer. J. Math.* **129**:6 (2007), 1565–1609. MR 2009d:58035

[Bouclet and Tzvetkov 2008] J.-M. Bouclet and N. Tzvetkov, "On global Strichartz estimates for non-trapping metrics", *J. Funct. Anal.* **254**:6 (2008), 1661–1682. MR 2009d:35039 Zbl 1168.35005

[Bourgain 1999] J. Bourgain, *Global solutions of nonlinear Schrödinger equations*, American Mathematical Society Colloquium Publications **46**, American Mathematical Society, Providence, RI, 1999. MR 2000h:35147 Zbl 0933.35178

[Burq et al. 2004a] N. Burq, P. Gérard, and N. Tzvetkov, "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds", *Amer. J. Math.* **126**:3 (2004), 569–605. MR 2005h:58036 Zbl 1067.58027

[Burq et al. 2004b] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay", *Indiana Univ. Math. J.* **53**:6 (2004), 1665–1680. MR 2005k:35241 Zbl 1084.35014

[Burq et al. 2010] N. Burq, C. Guillarmou, and A. Hassell, "Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics", *Geom. Funct. Anal.* **20**:3 (2010), 627–656. MR 2012f:58068 Zbl 1206.58009

[Christ and Kiselev 2001] M. Christ and A. Kiselev, "Maximal functions associated to filtrations", *J. Funct. Anal.* **179**:2 (2001), 409–425. MR 2001i:47054 Zbl 0974.47025

[Fanelli et al. 2013] L. Fanelli, V. Felli, M. A. Fontelos, and A. Primo, "Time decay of scaling critical electromagnetic Schrödinger flows", *Comm. Math. Phys.* **324**:3 (2013), 1033–1067. MR 3123544 Zbl 1293.35266

[Ford 2010] G. A. Ford, "The fundamental solution and Strichartz estimates for the Schrödinger equation on flat Euclidean cones", *Comm. Math. Phys.* **299**:2 (2010), 447–467. MR 2011i:35204 Zbl 1198.35062

[Ginibre and Velo 1985] J. Ginibre and G. Velo, "Scattering theory in the energy space for a class of nonlinear Schrödinger equations", *J. Math. Pures Appl.* (9) **64**:4 (1985), 363–401. MR 87i:35171 Zbl 0535.35069

[Grafakos 2009] L. Grafakos, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics **250**, Springer, New York, 2009. MR 2011d:42001 Zbl 1158.42001

[Guillarmou and Hassell 2008] C. Guillarmou and A. Hassell, "Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds, I", *Math. Ann.* **341**:4 (2008), 859–896. MR 2009j:58043 Zbl 1141.58017

[Guillarmou and Hassell 2014] C. Guillarmou and A. Hassell, "Uniform Sobolev estimates for non-trapping metrics", *J. Inst. Math. Jussieu* **13**:3 (2014), 599–632. MR 3211800 Zbl 1321.58013

[Guillarmou et al. 2013a] C. Guillarmou, A. Hassell, and A. Sikora, "Resolvent at low energy, III: The spectral measure", *Trans. Amer. Math. Soc.* **365**:11 (2013), 6103–6148. MR 3091277 Zbl 1303.58010

[Guillarmou et al. 2013b] C. Guillarmou, A. Hassell, and A. Sikora, "Restriction and spectral multiplier theorems on asymptotically conic manifolds", *Anal. PDE* **6**:4 (2013), 893–950. MR 3092733 Zbl 1293.35187

[Hassell and Vasy 1999] A. Hassell and A. Vasy, "The spectral projections and the resolvent for scattering metrics", *J. Anal. Math.* **79** (1999), 241–298. MR 2001d:58034 Zbl 0981.58025

[Hassell and Vasy 2001] A. Hassell and A. Vasy, "The resolvent for Laplace-type operators on asymptotically conic spaces", *Ann. Inst. Fourier* (*Grenoble*) **51**:5 (2001), 1299–1346. MR 2002i:58037

[Hassell and Wunsch 2005] A. Hassell and J. Wunsch, "The Schrödinger propagator for scattering metrics", *Ann. of Math.* (2) **162**:1 (2005), 487–523. MR 2006k:58048 Zbl 1126.58016

[Hassell and Wunsch 2008] A. Hassell and J. Wunsch, "The semiclassical resolvent and the propagator for non-trapping scattering metrics", *Adv. Math.* **217**:2 (2008), 586–682. MR 2009b:58070 Zbl 1131.58018

[Hassell et al. 2005] A. Hassell, T. Tao, and J. Wunsch, "A Strichartz inequality for the Schrödinger equation on nontrapping asymptotically conic manifolds", *Comm. Partial Differential Equations* **30**:1–3 (2005), 157–205. MR 2006i:58045 Zbl 1068.35119

[Hassell et al. 2006] A. Hassell, T. Tao, and J. Wunsch, "Sharp Strichartz estimates on nontrapping asymptotically conic manifolds", *Amer. J. Math.* **128**:4 (2006), 963–1024. MR 2007d:58053 Zbl 1177.58019

[Hörmander 1985] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren der Math. Wissenschaften **274**, Springer, Berlin, 1985. MR 87d:35002a Zbl 0601.35001

[Ivanovici 2010] O. Ivanovici, "On the Schrödinger equation outside strictly convex obstacles", *Anal. PDE* **3**:3 (2010), 261–293. MR 2011j:58037 Zbl 1222.35186

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR 2000d:35018 Zbl 0922.35028

[Kenig et al. 1987] C. E. Kenig, A. Ruiz, and C. D. Sogge, "Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators", *Duke Math. J.* **55**:2 (1987), 329–347. MR 88d:35037 Zbl 0644.35012

[Marzuola et al. 2008] J. Marzuola, J. Metcalfe, and D. Tataru, "Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations", *J. Funct. Anal.* **255**:6 (2008), 1497–1553. MR 2011c:35485 Zbl 1180.35187

[Marzuola et al. 2010] J. Marzuola, J. Metcalfe, D. Tataru, and M. Tohaneanu, "Strichartz estimates on Schwarzschild black hole backgrounds", *Comm. Math. Phys.* **293**:1 (2010), 37–83. MR 2010m:58043 Zbl 1202.35327

[Melrose 1994] R. B. Melrose, "Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces", pp. 85–130 in *Spectral and scattering theory* (Sanda, 1992), edited by M. Ikawa, Lecture Notes in Pure and Appl. Math. **161**, Dekker, New York, 1994. MR 95k:58168

[Metcalfe and Tataru 2012] J. Metcalfe and D. Tataru, "Global parametrices and dispersive estimates for variable coefficient wave equations", *Math. Ann.* **353**:4 (2012), 1183–1237. MR 2944027 Zbl 1259.35006

[Mizutani 2012] H. Mizutani, "Strichartz estimates for Schrödinger equations on scattering manifolds", *Comm. Partial Differential Equations* **37**:2 (2012), 169–224. MR 2876829 Zbl 1244.35020

[Robbiano and Zuily 2005] L. Robbiano and C. Zuily, *Strichartz estimates for Schrödinger equations with variable coefficients*, Mém. Soc. Math. Fr. (N.S.) **101–102**, Société Mathématique de France, Paris, 2005. MR 2006i:35047 Zbl 1097.35002

[Sher 2013] D. A. Sher, "The heat kernel on an asymptotically conic manifold", *Anal. PDE* **6**:7 (2013), 1755–1791. MR 3148066 Zbl 1296.58015

[Staffilani and Tataru 2002] G. Staffilani and D. Tataru, "Strichartz estimates for a Schrödinger operator with nonsmooth coefficients", *Comm. Partial Differential Equations* **27**:7–8 (2002), 1337–1372. MR 2003f:35248 Zbl 1010.35015

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 58 #23577 Zbl 0372.35001

[Tao 2006] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics **106**, American Mathematical Society, Providence, RI, 2006. MR 2008i:35211 Zbl 1106.35001

[Wunsch and Zworski 2000] J. Wunsch and M. Zworski, "Distribution of resonances for asymptotically Euclidean manifolds", *J. Differential Geom.* **55**:1 (2000), 43–82. MR 2002e:58062 Zbl 1030.58024

[Zhang 2015a] J. Zhang, "Linear restriction estimates for Schrödinger equation on metric cones", *Comm. Partial Differential Equations* **40**:6 (2015), 995–1028. MR 3321069 Zbl 1326.42029

[Zhang 2015b] J. Zhang, "Strichartz estimates and nonlinear wave equation on nontrapping asymptotically conic manifolds", *Adv. Math.* **271** (2015), 91–111. MR 3291858

Received 12 Apr 2015. Revised 24 Sep 2015. Accepted 28 Oct 2015.

ANDREW HASSELL: Andrew.Hassell@anu.edu.au
Department of Mathematics, Australian National University, Canberra ACT 2601, Australia

JUNYONG ZHANG: zhang_junyong@bit.edu.cn
Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, China
and
Department of Mathematics, Australian National University, Canberra ACT 2601, Australia

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US $235/year for the electronic version, and $430/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 9 No. 1 2016

2157-5045(2016)9:1;1-B