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IN THE LINEARIZED CALDERÓN PROBLEM**

# LOCAL ANALYTIC REGULARITY IN THE LINEARIZED CALDERÓN PROBLEM

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We show that the linearized local Dirichlet-to-Neumann map at a real-analytic potential for measurements made at an analytic open subset of the boundary is injective.

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## 1. Introduction

In this paper, we consider the *linearized* Calderón problem with local partial data and related problems. We first briefly review Calderón's problem including the case of partial data. For a more complete review, see [Uhlmann 2009].

Calderón's problem is, roughly speaking, the question of whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is also called electrical impedance tomography. We describe the problem more precisely below.

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded domain with smooth boundary. The electrical conductivity of  $\Omega$  is represented by a bounded and positive function  $\gamma(x)$ . In the absence of sinks or sources of current, the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega \tag{1-1}$$

since, by Ohm's law,  $\gamma \nabla u$  represents the current flux. Given a potential  $f \in H^{1/2}(\partial\Omega)$  on the boundary, the induced potential  $u \in H^1(\Omega)$  solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{1-2}$$

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The Dirichlet-to-Neumann (DN) map, or voltage-to-current map, is given by

$$\Lambda_\gamma(f) = \left( \gamma \frac{\partial u}{\partial \nu} \right) \Big|_{\partial\Omega}, \quad (1-3)$$

where  $\nu$  denotes the unit outer normal to  $\partial\Omega$ . The inverse problem is to determine  $\gamma$  knowing  $\Lambda_\gamma$ .

The local Calderón problem, or the Calderón problem with partial data, is the question of whether one can determine the conductivity by measuring the DN map on subsets of the boundary for voltages supported in subsets of the boundary. In this paper, we consider the case when the support of the voltages and the induced current fluxes are measured in the same open subset  $\Gamma$ . More conditions on this open set will be stated later. If  $\gamma \in C^\infty(\bar{\Omega})$ , the DN map is a classical pseudodifferential operator of order 1. It was shown in [Sylvester and Uhlmann 1986] that its full symbol computed in boundary normal coordinates near a point of  $\Gamma$  determines the Taylor series of  $\gamma$  at the point giving another proof of the result of Kohn and Vogelius [1984]. In particular, this shows that real-analytic conductivities can be determined by the local DN map. This result was generalized in [Lee and Uhlmann 1989] to the case of anisotropic conductivities using a factorization method related to the methods of this paper. Interior determination was shown in dimension  $n \geq 3$  for  $C^2$  conductivities [Sylvester and Uhlmann 1987]. This was extended to  $C^1$  conductivities in [Haberman and Tataru 2013]. In two dimensions, uniqueness was proven for  $C^2$  conductivities in [Nachman 1996] and for merely  $L^\infty$  conductivities in [Astala and Päiväranta 2006]. The case of partial data in dimension  $n \geq 3$  was considered in [Bukhgeim and Uhlmann 2002; Kenig et al. 2007; Isakov 2007; Kenig and Salo 2013; Imanuvilov and Yamamoto 2013]. The two-dimensional case was solved in [Imanuvilov et al. 2010]. See [Kenig and Salo 2014] for a review. However, it is not known at the present whether one can uniquely determine the conductivity if one measures the DN map on an arbitrarily open subset of the boundary applied to functions supported in the same set. We refer to these types of measurements as the local DN map.

The map  $\gamma \rightarrow \Lambda_\gamma$  is not linear. In this paper, we consider the linearization of the partial-data problem at a real-analytic conductivity for real-analytic  $\Gamma$ . We prove that the linearized map is injective. In fact, we prove a more general statement (see Theorem 1.6)

As in many works on Calderón's problem, one can reduce the problem to a similar one for the Schrödinger equation (see for instance [Uhlmann 2009]). This result uses that one can determine from the DN map the conductivity and the normal derivative of the conductivity. This result is only valid for the local DN map. One can then consider the more general problem of determining a potential from the corresponding DN map. The same is valid for the case of partial data and the linearization. It was shown in [Dos Santos Ferreira et al. 2009] that the linearization of the local DN map at the 0 potential is injective. We consider the linearization of the local DN map at any real-analytic potential assuming that the local DN map is measured on an open real-analytic set. We now describe more precisely our results in this setting.

Consider the Schrödinger operator  $P = \Delta - V$  on the open set  $\Omega \Subset \mathbb{R}^n$ , where the boundary  $\partial\Omega$  is smooth (and later assumed to be analytic in the most interesting region). Assume that 0 is not in the spectrum of the Dirichlet realization of  $P$ . Let  $G$  and  $K$  denote the corresponding Green and Poisson operators. Let  $\gamma : C^\infty(\bar{\Omega}) \rightarrow C^\infty(\partial\Omega)$  be the restriction operator and  $\nu$  the exterior normal. If  $x_0 \in \partial\Omega$ , we

can choose local coordinates  $y = (y_1, \dots, y_n)$ , centered at  $x_0$  so that  $\Omega$  is given by  $y_n > 0$  and  $\nu = -\partial_{y_n}$ . If  $\partial\Omega$  is analytic near  $x_0$ , we can choose the coordinates to be analytic.

The Dirichlet-to-Neumann (DN) operator is

$$\Lambda = \gamma \partial_\nu(x, \partial_x)K. \tag{1-4}$$

Consider a smooth deformation of smooth real-valued potentials

$$\begin{aligned} \text{neigh}(0, \mathbb{R}) \ni t \mapsto P_t &= \Delta - V_t, \\ V_t(x) = V(t, x) &\in C^\infty(\text{neigh}(0, \mathbb{R}) \times \bar{\Omega}; \mathbb{R}). \end{aligned} \tag{1-5}$$

Let  $G_t$  and  $K_t$  be the Green and Poisson kernels for  $P_t$  so that

$$\begin{pmatrix} P_t \\ \gamma \end{pmatrix} : C^\infty(\bar{\Omega}) \rightarrow C^\infty(\bar{\Omega}) \times C^\infty(\partial\Omega)$$

has the inverse

$$(G_t \ K_t).$$

Then, denoting  $t$ -derivatives by dots,

$$(\dot{G}_t \ \dot{K}_t) = - (G_t \ K_t) \begin{pmatrix} \dot{P}_t \\ 0 \end{pmatrix} (G_t \ K_t) = - (G_t \dot{P}_t G_t \ G_t \dot{P}_t K_t);$$

that is,

$$\dot{G} = -G\dot{P}G, \quad \dot{K} = -G\dot{P}K, \tag{1-6}$$

and consequently,

$$\dot{\Lambda} = -\gamma \partial_\nu G \dot{P} K. \tag{1-7}$$

Using the Green formula, we see that

$$\gamma \partial_\nu G = K^t, \tag{1-8}$$

where  $K^t$  denotes the transposed operator.

In fact, write the Green formula,

$$\int_{\Omega} ((Pu_1)u_2 - u_1Pu_2) dx = \int_{\partial\Omega} (\partial_\nu u_1 u_2 - u_1 \partial_\nu u_2) S(dx),$$

put  $u_1 = Gv$  and  $u_2 = Kw$  for  $v \in C^\infty(\bar{\Omega})$  and  $w \in C^\infty(\partial\Omega)$ ,

$$\int_{\Omega} vKw = \int_{\partial\Omega} (\gamma \partial_\nu Gv)wS(dx),$$

and (1-8) follows.

Equation (1-7) becomes

$$\dot{\Lambda} = -K^t \dot{P} K = K^t \dot{V} K. \tag{1-9}$$

The linearized Calderón problem is: if  $V_t = V + tq$ , determine  $q$  from  $\dot{\Lambda}_{t=0}$ . The corresponding partial-data problem is to recover  $q$  or some information about  $q$  from local information about  $\dot{\Lambda}_{t=0}$ . From now on, we restrict the attention to  $t = 0$ . In this paper, we shall study the following linearized

baby problem. Assume that  $V$  and  $\partial\Omega$  are analytic near some point  $x_0 \in \partial\Omega$ . We also assume that  $V$  is smooth. If  $\dot{\Lambda}$  (for  $t = 0$ ) is an analytic pseudodifferential operator near  $x_0$ , can we conclude that  $q$  is analytic near  $x_0$ ? Here,

$$\dot{\Lambda} = K^t q K, \tag{1-10}$$

and we shall view the right-hand side as a Fourier integral operator acting on  $q$ .

Actually this problem is overdetermined in the sense that the symbol of a pseudodifferential operator on the boundary is a function of  $2(n - 1)$  variables while  $q$  is a function on  $n$  variables and  $2(n - 1) \geq n$  for  $n \geq 2$  with equality precisely for  $n = 2$ . In order to have a nonoverdetermined problem, we shall only consider the symbol  $\sigma_{\dot{\Lambda}}(y', \eta')$  of  $\dot{\Lambda}$  along a half-ray in  $\eta'$ ; i.e., we look at  $\sigma_{\dot{\Lambda}}(y', t\eta'_0)$  for some fixed  $\eta'_0 \neq 0$  and for some local coordinates as above. Assuming this restricted symbol to be a classical analytic symbol near  $y' = 0$  and the potential  $V = V_0$  to be analytic near  $y = 0$  (i.e., near  $x_0$ ), we shall show that  $q$  is real-analytic up to the boundary near  $x_0$  (corresponding to  $y = 0$ ).

In order to formulate the result more precisely, we first make some remarks about the analytic singular support of the Schwartz kernels of  $K$  and  $K^t q K$  and then we recall the notion of classical analytic pseudodifferential operators. Assume that  $W \subset \mathbb{R}^n$  is an open neighborhood of  $x_0 \in \partial\mathcal{O}$  and that

$$\partial\Omega \text{ and } V \text{ are analytic in } W. \tag{1-11}$$

For simplicity, we shall use the same symbol to denote operators and their Schwartz kernels. Then:

**Lemma 1.1.** *The Schwartz kernel  $K(x, y')$  is analytic with respect to  $y'$ , locally uniformly on the set*

$$\{(x, y') \in \bar{\Omega} \times (\partial\Omega \cap W) : x \neq y'\}.$$

*Proof.* Using (1-8), we can write  $K(x, y') = \gamma \partial_\nu u(y')$ , where  $u = G(x, \cdot)$  solves the Dirichlet problem

$$(\Delta - V)u = \delta(\cdot - x), \quad \gamma u = 0,$$

and from analytic regularity for elliptic boundary-value problems, we get the lemma. (When  $x \in \partial\Omega$ , we view  $G(x, y)$  away from  $y = x$  as the limit of  $G(x_j, y)$  when  $\Omega \ni x_j \rightarrow x$ .) □

**Lemma 1.2.** *The Schwartz kernel  $(K^t q K)(x', y')$  is analytic on the set*

$$\{(x', y') \in (\partial\Omega \cap W)^2 : x' \neq y'\}. \tag{1-12}$$

*Proof.* Let  $(x'_0, y'_0)$  belong to the set (1-12). After decomposing  $q$  into a sum of two terms, we may assume that  $x'_0 \notin \text{supp}(q)$  or that  $y'_0 \notin \text{supp}(q)$ . In the first case, it follows from Lemma 1.1 that  $(K^t q K)(x', y')$  is analytic in  $x'$  uniformly for  $(x', y')$  in a neighborhood of  $(x'_0, y'_0)$ , and since the kernel is symmetric, we can exchange the roles of  $x'$  and  $y'$  and conclude that  $(K^t q K)(x', y')$  is analytic in  $y'$  uniformly for  $(x', y')$  in a neighborhood of  $(x'_0, y'_0)$ . In the second case, we have the same conclusion about analyticity in  $x'$  and in  $y'$  separately. It then follows that  $(K q K)(x', y')$  is analytic near  $(x'_0, y'_0)$  (by using the Fourier–Bros–Iagolnitzer (FBI) definition of the analytic wave-front set and which can also (most likely) be deduced from a classical result on logarithmic convexity of Reinhardt domains [Hörmander 1990, Theorem 2.4.6]). □

**Remark 1.3.** By the same proof,  $K^t q K(x', y')$  is analytic near

$$\{(x', x') \in (\partial\Omega \cap W)^2 : (x', 0) \notin \text{supp } q\}.$$

We next define the notion of a symbol up to exponentially small contributions. For that purpose, we assume that  $X$  is an analytic manifold and consider an operator

$$A : C_0^\infty(X) \rightarrow C^\infty(X) \tag{1-13}$$

that is also continuous

$$\mathcal{E}'(X) \rightarrow \mathcal{D}'(X). \tag{1-14}$$

Assume (as we have verified for  $K^t q K$  with  $n$  replaced by  $n - 1$  and with  $X = \partial\Omega \cap W$ ) that the distribution kernel  $A(x, y)$  is analytic away from the diagonal. After restricting to a local analytic coordinate chart, we may assume that  $X \subset \mathbb{R}^n$  is an open set. The symbol of  $A$  is formally given on  $T^*X$  by

$$\sigma_A(x, \xi) = e^{-ix \cdot \xi} A(e^{i(\cdot) \cdot \xi}) = \int e^{-i(x-y) \cdot \xi} A(x, y) dy.$$

In the usual case of  $C^\infty$ -theory, we give a meaning to this symbol up to  $\mathcal{O}(|\xi|^{-\infty})$  by introducing a cutoff  $\chi(x, y) \in C^\infty(X \times X)$  that is properly supported and equal to 1 near the diagonal. In the analytic category, we would like to have an exponentially small indeterminacy, and the use of special cutoffs becoming more complicated, we prefer to make a contour deformation.

For  $x$  in a compact subset of  $X$ , let  $r > 0$  be small enough and define for  $\xi \neq 0$

$$\sigma_A(x, \xi) = \int_{x + \Gamma_{r, \xi}} e^{i(y-x) \cdot \xi} A(x, y) dy, \tag{1-15}$$

where

$$\Gamma_{r, \xi} : B(0, r) \ni t \mapsto t + i\chi\left(\frac{t}{r}\right)r \frac{\xi}{|\xi|} \in \mathbb{C}^n$$

and  $\chi \in C^\infty(B(0, 1); [0, 1])$  is a radial function that vanishes on  $B(0, \frac{1}{2})$  and is equal to 1 near  $\partial B(0, 1)$ . Thus, the contour  $x + \Gamma_{r, \xi}$  coincides with  $\mathbb{R}^n$  near  $y = x$  and becomes complex for  $t$  close to the boundary of  $B(0, r)$ . Notice that along this contour

$$|e^{i(y-x) \cdot \xi}| = e^{-\chi(t/r)r|\xi|}$$

is bounded by 1 and for  $t$  close to  $\partial B(0, r)$  it is exponentially decaying in  $|\xi|$ . Thus, from Stokes' formula, it is clear that  $\sigma_A(x, \xi)$  will change only by an exponentially small term if we modify  $r$ . More generally, for  $(x, \xi)$  in a conic neighborhood of a fixed point  $(x_0, \xi_0) \in X \times S^{n-1}$ , we change  $\sigma_A(x, \xi)$  only by an exponentially small term if we replace the contour in (1-15) by  $x_0 + \Gamma_{r, \xi_0}$ , and we then get a function that has a holomorphic extension to a conic neighborhood of  $(x_0, \xi_0)$  in  $\mathbb{C}^n \times (\mathbb{C}^n \setminus \{0\})$ .

**Remark 1.4.** Instead of using contour deformation to define  $\sigma_A$ , we can use an almost-analytic cutoff in the following way. Choose  $C > 0$  so that

$$1 = \int Ch^{-n/2} e^{-(y-t)^2/2h} dt,$$

and put

$$e_t(y) = \tilde{\chi}(y - t)Ch^{-n/2}e^{-(y-t)^2/2h},$$

where  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$  is equal to 1 near 0 and has its support in a small neighborhood of that point. Then if  $\hat{\chi}$  is another cutoff of the same type, we see by contour deformation that

$$\sigma_A(x, \xi) = e^{-ix \cdot \xi} A\left(\int \hat{\chi}(t)e_t e^{(\cdot) \cdot \xi}\right)$$

up to an exponentially decreasing term.

**Definition 1.5.** We say that  $A$  is a classical analytic pseudodifferential operator of order  $m \in \mathbb{R}$  if  $\sigma_A$  is a classical analytic symbol (cl.a.s.) of order  $m$  on  $X \times \mathbb{R}^n$  in the following sense.

There exist holomorphic functions  $p_{m-j}(x, \xi)$  on a fixed complex conic neighborhood  $V$  of  $X \times \mathbb{R}^n$  such that

$$p_k(x, \xi) \text{ is positively homogeneous of degree } k \text{ in } \xi, \tag{1-16}$$

$$\text{for all } K \Subset V \cap \{(x, \xi) : |\xi| = 1\}, \text{ there exists } C = C_K \text{ such that } |p_{m-j}(x, \xi)| \leq C^{j+1} j^j \text{ on } K, \tag{1-17}$$

for all  $K \Subset X$  and every  $C_1 > 0$  large enough, there exists  $C_2 > 0$

$$\text{such that } \left| \sigma_A(x, \xi) - \sum_{0 \leq j \leq |\xi|/C_1} p_{m-j}(x, \xi) \right| \leq C_2 e^{-|\xi|/C_2} \text{ with } (x, \xi) \in K \times \mathbb{R}^n \text{ and } |\xi| \geq 1. \tag{1-18}$$

The formal sum  $\sum_0^\infty p_{m-j}(x, \xi)$  is called a formal cl.a.s. when (1-16) and (1-17) hold. We define cl.a.s. and formal cl.a.s. on open conic subsets of  $X \times \mathbb{R}^n$  and on other similar sets by the obvious modifications of the above definitions. If  $p(x, \xi)$  is a cl.a.s. on  $X \times \mathbb{R}^n$  and if  $\xi_0 \in \mathbb{R}^n$ , then

$$q(x, \tau) := p(x, \tau \xi_0)$$

is a cl.a.s. on  $X \times \mathbb{R}_+$ .

The main result of this work is:

**Theorem 1.6.** *Let  $x_0 \in \partial\Omega$ , and assume that  $\partial\Omega$  and  $V$  are analytic near that point. Let  $q \in L^\infty(\Omega)$ . Choose local analytic coordinates  $y' = (y_1, \dots, y_{n-1})$  on  $\text{neigh}(x_0, \partial\Omega)$ , centered at  $x_0$ , so that the symbol  $\sigma_{\hat{\Lambda}}(y', \eta')$  becomes well defined up to an exponentially small term on  $\text{neigh}(0) \times \mathbb{R}^{n-1}$ . Let  $\eta'_0 \in \mathbb{R}^{n-1}$ .*

*If  $\sigma_{\hat{\Lambda}}(y', \tau \eta'_0)$  is a cl.a.s. on  $\text{neigh}(0, \mathbb{R}^{n-1}) \times \mathbb{R}_+$ , then  $q$  is analytic up to the boundary in a neighborhood of  $x_0$ .*

*We also have the converse statement.*

We have a simpler direct result.

**Proposition 1.7.** *Let  $x_0, \partial\Omega$ , and  $V$  be as in Theorem 1.6, and choose analytic coordinates as done there. If  $q \in L^\infty(\Omega)$  is analytic up to the boundary near  $x_0$ , then  $\hat{\Lambda}$  is an analytic pseudodifferential operator near  $y' = 0$ .*

We get the following immediate consequence.

**Corollary 1.8.** *Under the conditions of the previous theorem, the map*

$$q \rightarrow \dot{\Lambda}$$

*is injective.*

This follows from the previous result since  $q$  must be analytic on  $W$  and, if the Taylor series of  $q$  vanishes on  $W$ , then  $q = 0$  on the set where  $q$  is analytic.

Most of the paper will be devoted to the proof of [Theorem 1.6](#), and in [Section 7](#), we will prove [Proposition 1.7](#).

## 2. Heuristics and some remarks about the Laplace transform

Let us first explain heuristically why some kind of Laplace transform will appear. Assume that  $x_0 \in \partial\Omega$  and that  $V$  and  $\partial\Omega$  are analytic near that point. Choose local analytic coordinates

$$y = (y_1, \dots, y_{n-1}, y_n) = (y', y_n)$$

centered at  $x_0$  such that the set  $\Omega$  coincides near  $x_0$  (i.e.,  $y = 0$ ) with the half-space  $\mathbb{R}_+^n = \{y \in \mathbb{R}^n : y_n > 0\}$ . Assume also (for this heuristic discussion) that we know that  $q(y) = q(y', y_n)$  is analytic in  $y'$  and that the original Laplace operator remains the standard Laplace operator also in the  $y$  coordinates. Then up to a smoothing operator, the Poisson operator is of the form

$$Ku(y) = \frac{1}{(2\pi)^{n-1}} \int e^{i(y'-w') \cdot \eta' - y_n |\eta'|} a(y, \eta') u(w') dw' d\eta',$$

where the symbol  $a$  is equal to 1 to leading order. We can view  $K$ ,  $q$ , and  $K^t$  as pseudodifferential operators in  $y'$  with operator-valued symbols.  $K$  has the operator-valued symbol

$$K(y', \eta') : \mathbb{C} \ni z \mapsto z e^{-y_n |\eta'|} a(y, \eta') \in L^2([0, +\infty[_{y_n}). \tag{2-1}$$

The symbol of multiplication with  $q$  is independent of  $\eta'$  and equals multiplication with  $q(y', \cdot)$ . The symbol of  $K^t$  is

$$K^t(y', \eta') : L^2([0, +\infty[_{y_n}) \ni f(y_n) \mapsto \int_0^\infty e^{-y_n |\eta'|} a(y, -\eta') f(y_n) dy_n \in \mathbb{C}. \tag{2-2}$$

For simplicity, we set  $a = 1$  in the following discussion. To leading order, the symbol of  $\dot{\Lambda}$  is

$$\sigma_{\dot{\Lambda}}(y', \eta') = \int_0^\infty e^{-2y_n |\eta'|} q(y', y_n) dy_n = (\mathcal{L}q(y', \cdot))(2|\eta'|), \tag{2-3}$$

where

$$\mathcal{L}f(\tau) = \int_0^\infty e^{-t\tau} f(t) dt$$

is the Laplace transform.



Now we fix  $\eta'_0 \in \mathbb{R}^{n-1}$  and assume that  $\sigma_{\dot{\Lambda}}(y', \tau \eta'_0)$  is a cl.a.s. on  $\text{neigh}(0, \mathbb{R}^{n-1}) \times \mathbb{R}_+$ :

$$\sigma_{\dot{\Lambda}}(y', \tau \eta'_0) \sim \sum_1^{\infty} n_k(y', \tau), \tag{2-4}$$

where  $n_k$  is analytic in  $y'$  in a fixed complex neighborhood of 0, (positively) homogeneous of degree  $-k$  in  $\tau$ , and satisfies

$$|n_k(y', \tau)| \leq C^{k+1} k^k |\tau|^{-k}. \tag{2-5}$$

More precisely for  $C > 0$  large enough, there exists  $\tilde{C} > 0$  such that

$$\left| \sigma_{\dot{\Lambda}}(y', \tau \eta'_0) - \sum_1^{\lfloor |\eta'_0|/C \rfloor} n_k(y', \tau) \right| \leq \tilde{C} \exp(-\tau/\tilde{C}) \tag{2-6}$$

on the real domain.

From (2-3), we also have

$$\left| (\mathcal{L}q(y', \cdot))(2|\eta'_0|\tau) - \sum_1^{\lfloor |\eta'_0|/C \rfloor} n_k(y', \tau) \right| \leq \exp(-\tau/\tilde{C}) \tag{2-7}$$

for  $y' \in \text{neigh}(0, \mathbb{R}^{n-1})$  and  $\tau \geq 1$ . In this heuristic discussion, we assume that (2-7) extends to  $y' \in \text{neigh}(0, \mathbb{C}^{n-1})$ . It then follows that  $q(y', y_n)$  is analytic for  $y_n$  in a neighborhood of 0, from the following certainly classic result about Borel transforms.

**Proposition 2.1.** *Let  $q \in L^\infty([0, 1])$ , and assume that for some  $C, \tilde{C} > 0$*

$$\left| \mathcal{L}q(\tau) - \sum_0^{\lfloor \tau/C \rfloor} q_k \tau^{-(k+1)} \right| \leq e^{-\tau/\tilde{C}}, \quad \tau > 0, \tag{2-8}$$

$$|q_k| \leq \tilde{C}^{k+1} k^k. \tag{2-9}$$

*Then  $q$  is analytic in a neighborhood of  $t = 0$ . The converse also holds.*

*Proof.* We shall first show the converse statement, namely that, if  $q$  is analytic near  $t = 0$ , then (2-8) and (2-9) hold. We start by computing the Laplace transform of powers of  $t$ .

For  $\tau > 0, a > 0$ , and  $k \in \mathbb{N}$ ,

$$\int_0^\infty e^{-t\tau} t^k dt = \frac{k!}{\tau^{k+1}}. \tag{2-10}$$

In fact, the integral to the left is equal to

$$(-\partial_\tau)^k \left( \int_0^\infty e^{-t\tau} dt \right) = (-\partial_\tau)^k \left( \frac{1}{\tau} \right).$$

Next, for  $a > 0$ , we look at

$$\frac{1}{k!} \int_0^a e^{-t\tau} t^k dt = \frac{1}{\tau^{k+1}} \left( 1 - \frac{\tau^{k+1}}{k!} \int_a^\infty e^{-t\tau} t^k dt \right) = \frac{1}{\tau^{k+1}} \left( 1 - \int_{a\tau}^\infty e^{-s} \frac{s^k}{k!} ds \right). \tag{2-11}$$

First let  $\tau \in ]0, \infty[$  be large. For  $0 < \theta < 1$  to be optimally chosen, we write for  $s \geq 0$

$$\frac{s^k}{k!} e^{-s} = \theta^{-k} \underbrace{\frac{(\theta s)^k}{k!} e^{-\theta s}}_{\leq 1} e^{-(1-\theta)s} \leq \theta^{-k} e^{-(1-\theta)s}.$$

Thus,

$$\int_{a\tau}^{\infty} e^{-s} \frac{s^k}{k!} ds = \theta^{-k} \int_{a\tau}^{\infty} e^{-(1-\theta)s} ds = \frac{\theta^{-k} e^{-(1-\theta)a\tau}}{1-\theta}. \tag{2-12}$$

We will estimate this for  $k \leq a\tau/\mathcal{O}(1)$ . Under the a priori assumption that  $\theta \leq 1 - 1/\mathcal{O}(1)$ , we look for  $\theta$  that minimizes the numerator

$$\theta^{-k} e^{-(1-\theta)a\tau} = e^{-[(1-\theta)a\tau + k \ln \theta]}.$$

Setting the derivative of the exponent equal to 0, we are led to the choice  $\theta = k/(a\tau)$ . Assume that

$$\frac{k}{a\tau} \leq \theta_0 < 1. \tag{2-13}$$

Then,

$$(1-\theta)a\tau + k \ln \theta = a\tau \left(1 - \frac{k}{a\tau} + \frac{k}{a\tau} \ln \frac{k}{a\tau}\right) = a\tau \left(1 - f\left(\frac{k}{a\tau}\right)\right),$$

where

$$f(x) = x + x \ln \frac{1}{x}, \quad 0 \leq x \leq 1.$$

Clearly  $f(0) = 0$  and  $f(1) = 1$ , and for  $0 < x < 1$ , we have  $f'(x) = \ln(1/x) > 0$ , so  $f$  is strictly increasing on  $[0, 1]$ . In view of (2-13),

$$(1-\theta)a\tau + k \ln \theta \geq a\tau(1 - f(\theta_0)),$$

and (2-12) gives

$$\int_{a\tau}^{\infty} e^{-s} \frac{s^k}{k!} ds \leq \frac{e^{-a\tau(1-f(\theta_0))}}{1-\theta_0}. \tag{2-14}$$

Using this in (2-11), we get

$$\frac{1}{k!} \int_0^a e^{-t\tau} t^k dt = \frac{1}{\tau^{k+1}} (1 + \mathcal{O}(1)e^{-a\tau/C(\theta_0)}) \quad \text{for } \frac{k}{a\tau} \leq \theta_0 < 1, \text{ where } C(\theta_0) > 0. \tag{2-15}$$

Now, assume that  $q \in C([0, 1])$  is analytic near  $t = 0$ . Then for  $t \in [0, 2a]$ ,  $0 < a \ll 1$ , we have

$$q(t) = \sum_0^{\infty} \frac{q^{(k)}(0)}{k!} t^k,$$

where

$$\frac{|q^{(k)}(0)|}{k!} \leq \tilde{C} \frac{1}{(2a)^k}, \tag{2-16}$$

so

$$\left| q(t) - \sum_0^{\lceil \tau/C \rceil} \frac{q^{(k)}(0)}{k!} t^k \right| \leq \tilde{C} e^{-\tau/\tilde{C}}, \quad 0 \leq t \leq a.$$

Hence,

$$\mathcal{L}q = \sum_0^{[\tau/C]} \frac{q^{(k)}(0)}{\tau^{k+1}} + \mathcal{O}(e^{-\tau/\tilde{C}}) + \underbrace{\mathcal{L}(1_{[a,1]}q)(\tau)}_{=\mathcal{O}(e^{-\tau/\tilde{C}})}$$

and we obtain (2-8) with  $q_k = q^{(k)}(0)$  while (2-9) follows from (2-16).

We now prove the direct statement in the proposition, so we take  $q \in L^\infty([0, 1])$  satisfying (2-8) and (2-9). For  $a > 0$  small, put

$$\tilde{q}(t) = q(t) - 1_{[0,a]}(t) \sum_0^\infty \frac{q^k}{k!} t^k.$$

The proof of the converse part shows that

$$|\mathcal{L}\tilde{q}(\tau)| \leq e^{-\tau/\tilde{C}}, \tag{2-17}$$

where  $\tilde{C}$  is a new positive constant, and it suffices to show that

$$\tilde{q} \text{ vanishes in a neighborhood of } 0. \tag{2-18}$$

We notice that  $\mathcal{L}\tilde{q}$  is a bounded holomorphic function in the right half-plane. We can therefore apply the Phragmén–Lindelöf theorem in each sector  $\arg \tau \in [0, \frac{\pi}{2}]$  and  $\arg \tau \in [-\frac{\pi}{2}, 0]$  to the holomorphic function

$$e^{\tau/\tilde{C}} \mathcal{L}\tilde{q}(\tau)$$

and conclude that this function is bounded in the right half-plane:

$$|\mathcal{L}\tilde{q}(\tau)| \leq \mathcal{O}(1)e^{-\Re\tau/\tilde{C}}, \quad \Re\tau \geq 0. \tag{2-19}$$

Now,  $\mathcal{L}\tilde{q}(i\sigma) = \mathcal{F}\tilde{q}(\sigma)$ , where  $\mathcal{F}$  denotes the Fourier transform, and the Paley–Wiener theorem allows us to conclude that  $\text{supp } \tilde{q} \subset [1/\tilde{C}, 1]$ . □

### 3. The Fourier integral operator $q \mapsto \sigma_\lambda$

Assume that  $\partial\Omega$  and  $V$  are analytic near the boundary point  $x_0$ . Let  $y' = (y_1, \dots, y_{n-1})$  be local analytic coordinates on  $\partial\Omega$ , centered at  $x_0$ . Then we can extend  $y'$  to analytic coordinates  $y = (y_1, \dots, y_{n-1}, y_n) = (y', y_n)$  in a full neighborhood of  $x_0$ , where  $y'$  is an extension of the given coordinates on the boundary and such that  $\Omega$  is given (near  $x_0$ ) by  $y_n > 0$  and

$$-P = D_{y_n}^2 + R(y, D_{y'}), \tag{3-1}$$

where  $R$  is a second-order elliptic differential operator in  $y'$  with positive principal symbol  $r(y, \eta')$ . (Here we neglect a contribution  $f(y)\partial_{y_n}$ , which can be eliminated by conjugation.) Then there is a neighborhood  $W \subset \mathbb{R}^n$  of  $y = 0$  and a cl.a.s.  $a(y, \xi')$  on  $W \times \mathbb{R}^{n-1}$  of order 0 such that

$$Ku(y) = \frac{1}{(2\pi)^{n-1}} \iint e^{i(\phi(y, \xi') - \tilde{y}' \cdot \xi')} a(y, \xi') u(\tilde{y}') d\tilde{y}' d\xi' + K_a u(y) \tag{3-2}$$

for  $y \in W$  and  $u \in C_0^\infty(W \cap \partial\Omega)$ . The distribution kernel of  $K_a$  is analytic on  $W \times (W \cap \partial\Omega)$ , and we choose a realization of  $a$  that is analytic in  $y$ . Here  $\phi$  is the solution of the Hamilton–Jacobi problem

$$\begin{aligned} (\partial_{y_n}\phi)^2 + r(y, \phi'_{y'}) &= 0, \quad \Im \partial_{y_n}\phi > 0, \\ \phi(y', 0, \xi') &= y' \cdot \xi'. \end{aligned} \tag{3-3}$$

This means that we choose  $\phi$  to be the solution of

$$\partial_{y_n}\phi - ir(y, \phi'_{y'})^{1/2} = 0 \tag{3-4}$$

with the natural branch of  $r^{1/2}$  with a cut along the real negative axis.

To see this, recall (by the analytic Wentzel–Kramers–Brillouin (WKB) method [Sjöstrand 1982, Chapter 9]) that we can construct the first term  $K_{\text{fop}}u$  in the right-hand side of (3-2) such that  $PK_{\text{fop}}$  has analytic distribution kernel and  $\gamma K_{\text{fop}} = 1$ . It then follows from local analytic regularity in elliptic boundary-value problems that the remainder operator  $K_a$  has analytic distribution kernel.

We notice that

$$K(e^{ix' \cdot \xi'}) = e^{i\phi(y, \xi')} a(y, \xi') + \mathcal{O}(e^{-|\xi'|/C}) \tag{3-5}$$

since the first term to the right solves the problem

$$Pu = 0, \quad u|_{y_n=0} = e^{iy' \cdot \xi'},$$

with an exponentially small error in the first equation.  $K$  is a real operator, so  $K(e^{ix' \cdot (-\xi')}) = \overline{K(e^{ix' \cdot \xi'})}$ . It follows that

$$\phi(y, -\xi') = -\overline{\phi(y, \xi')}, \quad a(y, -\xi') = \overline{a(y, \xi')} \tag{3-6}$$

without any error in the last equation when viewing  $a$  as a formal cl.a.s. Notice also that, since  $K$  is real,  $K^t = K^*$ .

We shall now view  $\dot{\Lambda} = K^t q K = K^* q K$  as a pseudodifferential operator in the classical quantization. In this section, we proceed formally in order to study the associated geometry. A more efficient analytic description will be given later for the left composition with an FBI transform in  $x'$ . The symbol becomes

$$\sigma_{\dot{\Lambda}}(x', \xi') = e^{-ix' \cdot \xi'} \mathcal{N}(e^{i(\cdot) \cdot \xi'}) = (2\pi)^{1-n} \iint e^{i(x' \cdot (\eta' - \xi') - \phi^*(y, \eta') + \phi(y, \xi'))} a^*(y, \eta') a(y, \xi') q(y) dy d\eta',$$

where in general we write  $f^*(z) = \overline{f(\bar{z})}$  for the holomorphic extension of the complex conjugate of a function  $f$ .

Actually, rather than letting  $\xi'$  tend to  $\infty$ , we replace  $\xi'$  with  $\xi'/h$ , where the new  $\xi'$  is of length  $\asymp 1$  and  $h \rightarrow 0$ . This amounts to viewing  $\mathcal{N}$  as a semiclassical pseudodifferential operator with semiclassical symbol  $\sigma_{\dot{\Lambda}}(x', \xi'; h) = \sigma_{\dot{\Lambda}}(x', \xi'/h)$ . Thus,

$$\begin{aligned} \sigma_{\dot{\Lambda}}(x', \xi'; h) &= e^{-ix' \cdot \xi'/h} \mathcal{N}(e^{i(\cdot) \cdot \xi'/h}) \\ &= (2\pi h)^{1-n} \iint e^{(i/h)(x' \cdot (\eta' - \xi') - \phi^*(y, \eta') + \phi(y, \xi'))} a^*(y, \eta'; h) a(y, \xi'; h) q(y) dy d\eta', \end{aligned}$$

where  $a(y, \xi'; h) = a(y, \xi'/h)$  and similarly for  $a^*$ .

We have

$$\phi(y, \xi') = y' \cdot \xi' + \psi(y, \xi'), \quad \phi^*(y, \eta') = y' \cdot \eta' + \psi^*(y, \eta'), \tag{3-7}$$

where

$$\Im\psi, \Im\psi^* \asymp y_n, \quad \Re\psi, \Re\psi^* = \mathcal{O}(y_n^2) \tag{3-8}$$

uniformly on every compact set that does not intersect the zero section. Equation (3-6) tells us that  $\Re\psi$  is odd and  $\Im\psi$  is even with respect to the fiber variables  $\xi'$  (and also positively homogeneous of degree 1 of course). Using (3-7) in the formula for the symbol of  $\dot{\Lambda}$ , we get

$$\begin{aligned} \sigma_{\dot{\Lambda}}(x', \xi'; h) &= (2\pi h)^{1-n} \iint e^{(i/h)\Phi_M(x', \xi', y, \eta')} a^*(y, \eta'; h) a(y, \xi'; h) q(y) dy d\eta' \\ &=: Mq(x', \xi'; h), \end{aligned} \tag{3-9}$$

where

$$\Phi_M(x', \xi', y, \eta') = (x' - y') \cdot (\eta' - \xi') + \psi(y, \xi') - \psi^*(y, \eta') \tag{3-10}$$

and  $\eta'$  are the fiber variables. We shall see that this is a nondegenerate phase function in the sense of Hörmander [1971] except for the fact that  $\Phi_M$  is not homogeneous in  $\eta'$  alone, so  $q \mapsto M_h q(x', \xi') := Mq(x', \xi'; h)$  is a semiclassical Fourier integral operator, at least formally.

We fix a vector  $\xi'_0 \in \mathbb{R}^{n-1}$  and consider  $\Phi_M$  in a neighborhood of  $(x', y, \xi', \eta') = (0, 0, \xi'_0, \xi'_0) \in \mathbb{C}^{4(n-1)+1} = \mathbb{C}^{4n-3}$ . The critical set  $C_{\Phi_M}$  of the phase  $\Phi_M$  is given by  $\partial_{\eta'}\Phi_M = 0$ , which means that  $x' - y' - \partial_{\eta'}\psi^*(y, \eta') = 0$  or equivalently

$$x' = y' + \partial_{\eta'}\psi^*(y, \eta'). \tag{3-11}$$

This is a smooth submanifold of codimension  $n - 1$  in  $\mathbb{C}^{4n-3}$  that is parametrized by  $(y, \eta', \xi') \in \text{neigh}((0, \xi'_0, \xi'_0), \mathbb{C}^{3n-2})$ . We also see that  $\Phi_M$  is a nondegenerate phase function in the sense that  $d\partial_{\eta'_1}\Phi_M, \dots, d\partial_{\eta'_{n-1}}\Phi_M$  are linearly independent on  $C_{\Phi_M}$ . Using the above parametrization, we express the graph of the corresponding canonical relation  $\kappa : \mathbb{C}^{2n}_{y, y^*} \rightarrow \mathbb{C}^{4(n-1)}_{x', \xi', x'^*, \xi'^*}$  (where we notice that  $4(n - 1) \geq 2n$  with equality for  $n = 2$  and strict inequality for  $n \geq 3$ ):

$$\begin{aligned} \text{graph}(\kappa) &= \{(x', \xi', \partial_{x'}\Phi_M, \partial_{\xi'}\Phi_M; y, -\partial_y\Phi_M) : (x', \xi', y, \eta') \in C_{\Phi_M}\} \\ &= \{(y' + \partial_{\eta'}\psi^*(y, \eta'), \xi', \eta' - \xi', \partial_{\xi'}\psi(y, \xi') - \partial_{\eta'}\psi^*(y, \eta'); \\ &\quad y, -\partial_{y'}\psi(y, \xi') + \partial_{y'}\psi^*(y, \eta') + \eta' - \xi', -\partial_{y_n}\psi(y, \xi') + \partial_{y_n}\psi^*(y, \eta')\}. \end{aligned} \tag{3-12}$$

The restriction to  $y_n = 0$  of this graph is the set of points

$$(y', \xi', \eta' - \xi', 0; y', 0, \eta' - \xi', -\partial_{y_n}\psi(y', 0, \xi') + \partial_{y_n}\psi^*(y', 0, \eta')). \tag{3-13}$$

It contains the point

$$(0, \xi'_0, 0, 0; 0, 0, -2\partial_{y_n}\psi(0, \xi'_0)) = (0, \xi'_0, 0, 0; 0, 0, -2ir(0, \xi'_0)^{1/2}). \tag{3-14}$$

The tangent space at a point where  $y_n = 0$  is given by

$$\left\{ \left( \delta_{y'} + \psi''_{\eta', y_n} \delta_{y_n}, \delta_{\xi'}, \delta_{\eta'} - \delta_{\xi'}, (\psi''_{\xi', y_n}(y, \xi') - \psi''_{\eta', y_n}(y, \eta')) \delta_{y_n}; \right. \right. \\ \left. \delta_y, (-\psi''_{y', y_n}(y, \xi') + \psi''_{y', y_n}(y, \eta')) \delta_{y_n} + \delta_{\eta'} - \delta_{\xi'}, \right. \\ \left. (-\psi''_{y_n, y}(y, \xi') + \psi''_{y_n, y}(y, \eta')) \delta_y + (-\psi''_{y_n, \xi'} \delta_{\xi'} + \psi''_{y_n, \eta'} \delta_{\eta'}) \right\}. \quad (3-15)$$

From (3-15), we see that, at every point of  $\text{graph}(\kappa)$  with  $y_n = 0$  and with  $\eta' \approx \xi'$ ,

- (1) the projection  $\text{graph}(\kappa) \rightarrow \mathbb{C}_{y, y^*}^{2n}$  has surjective differential and
- (2) the projection  $\text{graph}(\kappa) \rightarrow \mathbb{C}_{x', \xi', x'^*, \xi'^*}^{4(n-1)}$  has injective differential.

In fact, since  $\kappa$  is a canonical relation, (1) and (2) are pointwise equivalent, so it suffices to verify (2). In other words, we have to show that, if

$$\begin{aligned} 0 &= \delta_{y'} + \psi''_{\eta', y_n} \delta_{y_n}, \\ 0 &= \delta_{\xi'}, \\ 0 &= \delta_{\eta'} - \delta_{\xi'}, \\ 0 &= (\psi''_{\xi', y_n}(y, \xi') - \psi''_{\eta', y_n}(y, \eta')) \delta_{y_n}, \end{aligned} \quad (3-16)$$

then  $\delta_{y'} = 0$ ,  $\delta_{y_n} = 0$ ,  $\delta_{\xi'} = 0$ , and  $\delta_{\eta'} = 0$ .

When  $y_n = 0$ , we have  $\psi^* = -\psi$ , and when in addition  $\eta' \approx \xi'$ , we see that the  $(n-1) \times 1$  matrix in the fourth equation is nonvanishing, so this equation implies that  $\delta_{y_n} = 0$ . Then the first equation gives  $\delta_{y'} = 0$ , and from the second and third equations, we get  $\delta_{\xi'} = 0$  and  $\delta_{\eta'} = 0$  and we have verified (2).

As an exercise, let us determine the image under  $\kappa$  of the complexified conormal bundle of the boundary, given by  $y_n = 0$  and  $y^* = 0$ . From (3-13), we see that this image is the set of all points

$$(x', \xi', 0, 0). \quad (3-17)$$

The subset of real points in (3-17) is the image of the set of points  $(y', 0, 0, y_n^*)$  such that  $y'$  is real and  $y_n^* \in -i\mathbb{R}_+$ .

Now restrict  $(x', \xi')$  to the set of  $(x', t\eta'_0)$  with  $x' \in \mathbb{C}^{n-1}$  and  $t \in \mathbb{C}$ , where  $0 \neq \eta'_0 \in \mathbb{R}^{n-1}$ . This means that we restrict the symbol of  $\mathcal{N}$  to the radial direction  $\xi' \in \mathbb{C}\eta'_0$  and consider

$$\begin{aligned} \sigma_{\hat{\Lambda}}(x', t\eta'_0; h) &= Mq(x', t\eta'_0; h) =: M_{\text{new}}q(x', t; h) \\ &= (2\pi h)^{1-n} \iint e^{i\Phi_{M_{\text{new}}}(x', t, y, \eta')/h} a^*(y, \eta'; h) a(y, \xi'; h) q(y) dy d\eta', \end{aligned} \quad (3-18)$$

where

$$\Phi_{M_{\text{new}}}(x', t, y; \eta') = \Phi_M(x', t\eta'_0, y; \eta') = \psi(y, t\eta'_0) - \psi^*(y, \eta') + (x' - y') \cdot (\eta' - t\eta'_0). \quad (3-19)$$

We will soon drop the subscripts “new” when no confusion is possible. This is again a nondegenerate phase function. The new canonical relation  $\kappa_{\text{new}} : \mathbb{C}_{y, y^*}^{2n} \rightarrow \mathbb{C}_{x', t, x'^*, t^*}^{2n}$  has the graph

$$\text{graph}(\kappa_{\text{new}}) = \left\{ (y' + \partial_{\eta'} \psi^*(y, \eta'), t, \eta' - t\eta'_0, \eta'_0 \cdot \partial_{\xi'} \psi(y, t\eta'_0) - \eta'_0 \cdot \partial_{\xi'} \psi^*(y, \eta'); \right. \\ \left. y, -\partial_{y'} \psi(y, t\eta'_0) + \partial_{y'} \psi^*(y, \eta') + \eta' - t\eta'_0, -\partial_{y_n} \psi(y, t\eta'_0) + \partial_{y_n} \psi^*(y, \eta') \right\}. \quad (3-20)$$

This graph is conic with respect to the dilations

$$\mathbb{R}_+ \ni \lambda \mapsto (x', \lambda t, \lambda x'^*, t^*; y, \lambda y^*).$$

The restriction of the graph to  $y_n = 0$  is

$$\{(y', t, \eta' - t\eta'_0, 0; y', 0, \eta' - t\eta'_0, -\partial_{y_n} \psi(y', 0, t\eta'_0) + \partial_{y_n} \psi^*(y', 0, \eta'))\},$$

where

$$\partial_{y_n} \psi(y', 0, \xi') = ir(y', 0, \xi')^{1/2}, \quad \partial_{y_n} \psi^*(y', 0, \xi') = -ir(y', 0, \xi')^{1/2},$$

so the restriction is

$$\{(y', t, \eta' - t\eta'_0, 0; y', 0, \eta' - t\eta'_0, -i(r^{1/2}(y', 0, t\eta'_0) + r^{1/2}(y', 0, \eta')))\}. \quad (3-21)$$

If we take  $\eta = t\eta'_0$  and use that  $r^{1/2}$  is homogeneous of degree 1 in the fiber variables, we get

$$\{(y', t, 0, 0; y', 0, 0, -2itr^{1/2}(y', 0, \eta'_0))\}. \quad (3-22)$$

This is the graph of a diffeomorphism

$$\text{neigh}(0, \partial\Omega) \times (-i\mathbb{R}_{y_n^*}^+) \rightarrow \text{neigh}(0; \partial\Omega) \times \mathbb{R}_t^+.$$

The tangent space at a point where  $y_n = 0$  is given by

$$\left\{ (\delta_{y'} + (\psi^*)''_{\eta', y_n} \delta_{y_n}, \delta_t, \delta_{\eta'} - \delta_t \eta'_0, \eta'_0 \cdot (\psi''_{\xi', y_n} - (\psi^*)''_{\eta', y_n}) \delta_{y_n}; \right. \\ \left. \delta_{y'}, (-\psi''_{y', y_n} + (\psi^*)''_{y', y_n}) \delta_{y_n} + \delta_{\eta'} - \delta_t \eta'_0, (-\psi''_{y_n, y} + (\psi^*)''_{y_n, y}) \delta_y - \psi''_{y_n, \xi'} \delta_t \eta'_0 + (\psi^*)''_{y_n, \eta'} \delta_{\eta'} \right\}. \quad (3-23)$$

The projection onto the first component is injective as can be seen exactly as in the proof of the property (2) stated after (3-15). Now  $\kappa_{\text{new}}$  is a canonical relation between spaces of the same dimension, so we conclude that  $\kappa_{\text{new}}$  is a canonical transformation or more precisely near each point of its graph. Combining this with the observation right after (3-22), we get:

**Proposition 3.1.** *Equation (3-20) is the graph of a bijective canonical transformation*

$$\kappa_{\text{new}} : \text{neigh}((0; 0, -i), \mathbb{C}_y^n \times \mathbb{C}_{y^*}^n) \rightarrow \text{neigh}((0, 1; 0), \mathbb{C}_{x', t}^n \times \mathbb{C}_{x'^*, t^*}^n).$$

The neighborhoods can be taken to be conic with respect to the actions  $\mathbb{R}_+ \ni \lambda \mapsto (y, \lambda y^*)$  and  $\mathbb{R}_+ \ni \lambda \mapsto (x, \lambda t, \lambda x'^*, t^*)$ , and  $\kappa_{\text{new}}$  intertwines the two actions (so  $\kappa_{\text{new}}$  is positively homogeneous of degree 1 with  $y^*$  as the fiber variables on the departure side and with  $t$  and  $x'^*$  as the fiber variables on the arrival side).

Basically, the same exercise as the one leading to (3-17) shows that the image under  $\kappa_{\text{new}}$  of the complexified conormal bundle, given by  $y_n = 0$  and  $(y^*)' = 0$ , is the zero section

$$\{(x', t : (x'^*, t^*) = 0)\}. \tag{3-24}$$

Consider the image of  $T^*\partial\Omega \times i\mathbb{R}_{y_n}^- = \{(y, y^*) : y', (y^*)' \in \mathbb{R}^{n-1}, y_n = 0, y_n^* \in i\mathbb{R}^-\}$  under  $\kappa_{\text{new}}$ . On that image,

$$\begin{aligned} x' &= y' \in \mathbb{R}^{n-1}, \\ \eta' - t\eta'_0 &\in \mathbb{R}^{n-1}, \\ t^* &= \eta'_0 \cdot \partial_{\xi'}\psi(y, t\eta'_0) - \eta'_0 \cdot \partial_{\xi'}\psi^*(y, \eta') = 0. \end{aligned}$$

If we restrict the attention to  $t \in \mathbb{R}_+$  so that  $\eta' = (y^*)' + t\eta'_0 \in \mathbb{R}^{n-1}$ , we see that

$$y_n^* = -\partial_{y_n}\psi(y', 0, t\eta'_0) + \partial_{y_n}\psi(y', 0, \eta') \in i\mathbb{R}^-.$$

Thus, the image contains locally

$$\{(x', t, (x^*)', 0) : x', (x^*)' \in \mathbb{R}^{n-1}, t \in \mathbb{R}^+\},$$

which has the right dimension  $2(n - 1) + 1$

Similarly, the image of  $T^*\partial\Omega \times \text{neigh}(i\mathbb{R}_{y_n}^-, \mathbb{C}_{y_n^*})$  is obtained by dropping the reality condition on  $t$  but keeping that on  $\eta' - t\eta'_0$ , and we get

$$\kappa_{\text{new}}(T^*\partial\Omega \times \text{neigh}(i\mathbb{R}_{y_n}^-, \mathbb{C}_{y_n^*})) = \{(x', t, x'^*, 0) : x', (x^*)' \in \mathbb{R}^{n-1}, t \in \text{neigh}(\mathbb{R}^+, \mathbb{C})\}. \tag{3-25}$$

#### 4. Some function spaces and their FBI transforms

We continue to work locally near a point  $x_0$  where the boundary is analytic, and we use analytic coordinates  $y$  centered at  $x_0$  as specified in the beginning of Section 3.

We start by defining some piecewise-smooth I-Lagrangian manifolds, some of which will be associated with function spaces below.

- The cotangent space  $T^*\Omega$  that we identify with  $(\text{neigh}(0) \cap \mathbb{R}_+^n) \times \mathbb{R}^n$ .
- The real conormal bundle  $N^*\partial\Omega \subset T^*\mathbb{R}^n$ . In the local coordinates  $y$ ,

$$N^*\partial\Omega = \{(y, \eta) \in \mathbb{R}^{2n} : y_n = 0, \eta' = 0\}.$$

It will sometimes be convenient to write  $N^*\partial\Omega = \partial\Omega \times \mathbb{R}^*$ , where of course the second expression appeals to the use of special coordinates as above. More invariantly,  $N^*\partial\Omega$  is the inverse image of the zero-section in  $T^*\partial\Omega$  for the natural projection map  $\pi_{T^*\partial\Omega} : T_{\partial\Omega}^*\mathbb{R}^n \rightarrow T^*\partial\Omega$ .

We will also need some complex sets.

- The complexified zero-section in the complexification  $\widetilde{T^*\mathbb{R}^n} = \mathbb{C}_y^n \times \mathbb{C}_\eta^n$  defined to be

$$\text{neigh}(\bar{0}, \mathbb{C}^n) \times \{\eta = 0\} \subset \mathbb{C}_y^n \times \mathbb{C}_\eta^n.$$

We denote it by  $\mathbb{C}_y^n \times 0_\eta$  for short.



- The complexification  $\widetilde{N^*\partial\Omega}$  of  $N^*\partial\Omega$  defined to be

$$\{(y, \eta) \in \mathbb{C}_y^n \times \mathbb{C}_\eta^n : y \in \text{neigh}(0, \mathbb{C}^n), y_n = 0, \eta' = 0\}.$$

- The space  $\pi^{-1}(T^*\partial\Omega)$ , where  $\pi : T^*_{\partial\Omega}\mathbb{R}^n \otimes \mathbb{C} \rightarrow T^*\partial\Omega \otimes \mathbb{C}$  is the natural projection and  $\otimes \mathbb{C}$  indicates fiberwise complexification. In special coordinates, it is  $\{(y, \eta) : (y', \eta') \in \mathbb{R}^{2(n-1)}, y_n = 0, \eta_n \in \mathbb{C}\}$ . We will denote it by  $T^*\partial\Omega \times \mathbb{C}$  or  $T^*\partial\Omega \times \mathbb{C}_{\eta_n}$  for simplicity. It contains the subset  $T^*\partial\Omega \times \mathbb{C}^-_{\eta_n}$  (easy to define invariantly), where  $\mathbb{C}^-$  is the open lower half-plane. Notice that

$$T^*\partial\Omega \times \partial\mathbb{C}^- = T^*\partial\Omega \times \mathbb{R} = T^*_{\partial\Omega}\mathbb{R}^n.$$

- The piecewise-smooth (Lipschitz) manifold

$$F = \overline{T^*\Omega} \cup (T^*\partial\Omega \times \mathbb{C}^-_{\eta_n}).$$

Notice that the two components to the right have  $T^*_{\partial\Omega}\mathbb{R}^n$  as their common boundary.

- The piecewise-smooth (Lipschitz) manifold  $(\mathbb{C}^n_y \times 0_\eta) \cup \widetilde{N^*\partial\Omega}$ , where the two constituents contain  $\widetilde{\partial\Omega} \times 0_\eta$ . Here  $\widetilde{\partial\Omega}$  denotes a complexification of the boundary (near  $x_0$ ).

Let

$$Tu(z; h) = Ch^{-3n/4} \int_{\mathbb{R}^n} e^{(i/h)\phi(z,y)} u(y) dy, \quad z \in \mathbb{C}^n, \tag{4-1}$$

be a standard FBI transform [Sjöstrand 1982], sending distributions with compact support on  $\mathbb{R}^n$  to holomorphic functions on (in general some subdomains of)  $\mathbb{C}^n$ . For simplicity, we let  $\phi$  be a holomorphic quadratic form so that  $T$  can also be viewed as a generalized Bargmann transform and a metaplectic Fourier integral operator (see for instance [Sjöstrand 1990]). We work under the standard assumptions

$$\Im\phi''_{y,y} > 0, \quad \det\phi''_{z,y} \neq 0. \tag{4-2}$$

We let  $C > 0$  be the unique positive constant for which  $T : L^2(\mathbb{R}^2) \rightarrow H_{\Phi_0}(\mathbb{C}^n)$  is unitary, where

$$\Phi_0(z) = \sup_{y \in \mathbb{R}^n} -\Im\phi(z, y) = -\Im\phi(z, y(z)) \tag{4-3}$$

is a strictly plurisubharmonic (real) quadratic form on  $\mathbb{C}^n$  and  $H_{\Phi_0}$  is the complex Hilbert space  $\text{Hol}(\mathbb{C}^n) \cap L^2(e^{-2\Phi_0/h} L(dz))$  with  $L(dz)$  denoting the Lebesgue measure on  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ . Let

$$\kappa_T : \mathbb{C}^{2n} \ni (y, -\phi'_y(z, y)) \mapsto (z, \phi'_z(z, y)) \in \mathbb{C}^{2n} \tag{4-4}$$

be the complex (linear) canonical transformation associated to  $T$ , and let

$$\Lambda_{\Phi_0} = \left\{ \left( z, \frac{2}{i} \frac{\partial\Phi_0}{\partial z}(z) \right) : z \in \mathbb{C}^n \right\}$$

be the R-symplectic<sup>1</sup> and I-Lagrangian<sup>2</sup> manifold of  $\mathbb{C}^{2n}$ , actually a real-linear subspace since  $\phi$  is quadratic. Then we know that

$$\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n}). \tag{4-5}$$

More explicitly,

$$\kappa_T^{-1}\left(z, \frac{2}{i} \frac{\partial \Phi_0}{\partial z}\right) = (y(z), \eta(z)) \in \mathbb{R}^{2n}, \tag{4-6}$$

where  $y(z)$  appeared in (4-3).

Let

$$\Phi_1^{\text{ext}}(z) = \sup_{y \in \partial \mathbb{R}_+^n} -\Im \phi(z, y) = -\Im \phi(z, \tilde{y}(z)), \tag{4-7}$$

where  $\tilde{y}(z) = (\tilde{y}'(z), 0)$  and  $\tilde{y}'(z)$  is the unique point of maximum in  $\mathbb{R}^{n-1}$  of  $y' \mapsto -\Im \phi(z, y', 0)$ . If  $\text{supp } u \subset \{y \in \mathbb{R}^n : y_n \geq 0\}$ , then  $Tu \in H_{\Phi_1}^{\text{loc}}$ , where

$$\Phi_1(z) = \sup_{y \in \mathbb{R}_+^n} -\Im \phi(z, y) = \begin{cases} \Phi_0(z) & \text{if } y_n(z) \geq 0, \\ \Phi_1^{\text{ext}}(z) & \text{if } y_n(z) \leq 0. \end{cases} \tag{4-8}$$

Notice that

- $-\Im \partial_{y_n} \phi(z, \tilde{y}(z)) \geq 0$  in the first case and
- $-\Im \partial_{y_n} \phi(z, \tilde{y}(z)) \leq 0$  in the second case.

Notice that

$$\frac{2}{i} \frac{\partial \Phi_1}{\partial z}(z) = \frac{2}{i} \left( \frac{\partial}{\partial z} (-\Im \phi) \right) (z, \tilde{y}(z)) = \phi'_z(z, \tilde{y}(z))$$

and  $\tilde{\eta}(z) = -\phi'_y(z, \tilde{y}(z))$  satisfies  $\tilde{\eta}'(z) \in \mathbb{R}^{n-1}$ . When  $\Phi_1(z) = \Phi_1^{\text{ext}}(z)$ ,

$$\tilde{\eta}'(z) \in \mathbb{R}^{n-1}, \quad \Im \tilde{\eta}_n(z) \leq 0. \tag{4-9}$$

This means that

$$\Lambda_{\Phi_1^{\text{ext}}} = \kappa_T(T^* \partial \Omega \times \mathbb{C}_{\eta_n}^*)$$

and that

$$\Lambda_{\Phi_1} = \kappa_T(F), \tag{4-10}$$

where  $F$  was defined above:

$$F = \overline{T^*(\Omega)} \cup \{(y', 0; \eta', \eta_n) : (y', \eta') \in T^* \partial \Omega, \Im \eta_n \leq 0\}. \tag{4-11}$$

It is a Lipschitz manifold. The second component is a union of complex half-lines; consequently in the region where  $\Phi_1 < \Phi_0$ ,  $\Lambda_{\Phi_1}$  is a union of complex half-lines. If we project these lines to the complex  $z$ -space, we get a foliation of  $\mathbb{C}_z^n$  into complex half-lines and the restriction of  $\Phi_1$  to each of these is harmonic.

<sup>1</sup>i.e., symplectic with respect to  $\Re \sigma$ , where  $\sigma = d\zeta \wedge dz$  is the complex symplectic form

<sup>2</sup>i.e., Lagrangian with respect to  $\Im \sigma$

We introduce the real hyperplane

$$H = \pi_z \kappa_T(T^*_{\partial\Omega} \mathbb{R}^n),$$

which is the common boundary of the two half-spaces

$$H_+ = \pi_z \kappa_T(T^* \Omega),$$

$$H_- = \pi_z \kappa_T(\{(y', 0; \eta) : (y', \eta') \in T^* \partial\Omega, \Im \eta_n < 0\}).$$

Here,  $\pi_z : \mathbb{C}^n_z \times \mathbb{C}^n_\zeta \rightarrow \mathbb{C}^n$  is the natural projection. We have

$$\Phi_0 - \Phi_1 \begin{cases} = 0 & \text{in } H_+, \\ \asymp \text{dist}(z, H)^2 & \text{in } H_-. \end{cases} \tag{4-12}$$

Similarly, recall the definition of the complexified normal bundle  $\widetilde{N^* \partial\Omega}$  at the beginning of this section. It is a  $\mathbb{C}$ -Lagrangian manifold.<sup>3</sup> We have  $\kappa_T(\widetilde{N^* \partial\Omega}) = \Lambda_{\Phi_3}$ , where  $\Phi_3$  is pluriharmonic:

$$\Phi_3(z) = \text{vc}_{y' \in \mathbb{C}^{n-1}}(-\Im \phi(z, y', 0)).$$

Similarly  $\kappa_T(\mathbb{C}^n_y \times 0_\eta)$  (with the notation from the beginning of this section) is of the form  $\Lambda_{\Phi_4}$ , where

$$\Phi_4(z) = \text{vc}_{y \in \mathbb{C}^n}(-\Im \phi(z, y)).$$

The complex zero-section  $\mathbb{C}_y \times 0_\eta$  and  $T^* \mathbb{R}^n$  intersect transversally along the real zero-section  $\mathbb{R}^n_y \times 0_\eta$ . Correspondingly, we check that

$$\Phi_0(z) - \Phi_4(z) \asymp \text{dist}(z, \pi_z \circ \kappa_T(\mathbb{R}^n \times 0_\eta))^2. \tag{4-13}$$

Similarly,

$$\Phi_1^{\text{ext}}(z) - \Phi_3(z) \asymp \text{dist}(z, \pi_z \circ \kappa_T((\partial\Omega \times 0) \times \mathbb{C}^*_{\eta_n}))^2, \tag{4-14}$$

where  $\partial\Omega \times 0$  denotes the zero-section in  $T^* \partial\Omega$ , so that

$$(\partial\Omega \times 0) \times \mathbb{C}^*_{\eta_n} = N^* \partial\Omega \otimes \mathbb{C}$$

is the fiberwise complexification of  $N^* \partial\Omega$ . (Here we work locally near  $y = 0$ .)

Let  $u$  be real-analytic in a neighborhood of  $\bar{\Omega}$ , and consider

$$v(z) = T(1_\Omega u)(z), \tag{4-15}$$

where we restrict our attention to  $z \in \mathbb{C}^n$  such that the critical point  $y_{\Phi_4}(z)$  in the definition of  $\Phi_4(z)$  belongs to a small complex neighborhood of  $\bar{\Omega}$  or equivalently to  $z \in \mathbb{C}^n$  in a small neighborhood of  $\kappa_T(\bar{\Omega} \times 0_\eta)$ . By the method of steepest descent, we see that  $v \in H^{\text{loc}}_{\Phi_5}$ , where first of all  $\Phi_5 \leq \Phi_1$  and further

$$\Phi_5(z) = \Phi_4(z) \quad \text{when both} \quad \begin{cases} \Re y_{\Phi_4}(z) \in \Omega, \\ |\Im y_{\Phi_4}(z)| \ll \text{dist}(\Re y_{\Phi_4}(z), \partial\Omega), \end{cases} \tag{4-16}$$

$$\Phi_5(z) = \Phi_3(z) \quad \text{when both} \quad \begin{cases} \Re y_{\Phi_4}(z) \notin \Omega, \\ |\Im y_{\Phi_4}(z)| \ll \text{dist}(\Re y_{\Phi_4}(z), \partial\Omega). \end{cases} \tag{4-17}$$

<sup>3</sup>i.e., a holomorphic manifold that is Lagrangian for the complex symplectic form  $\sigma$

Actually, in the last case, we can relax the condition that  $y_{\Phi_4}(z)$  belongs to a small ( $u$ -dependent) neighborhood of  $\bar{\Omega}$ . The appropriate restriction is then that the critical point  $y_{\Phi_3}(z) \in \widetilde{\partial\Omega}$  in the definition of  $\Phi_3$  belongs to a small ( $u$ -dependent) neighborhood of  $\partial\Omega$ .

### 5. Expressing $M$ with the help of FBI transforms

From now on, we work with  $M_{\text{new}}$ ,  $\Phi_{M_{\text{new}}}$ , and  $\kappa_{\text{new}}$  and we drop the corresponding subscript “new”. Then from (3-18),

$$Mq(x', t) = \frac{1}{(2\pi h)^{n-1}} \iint e^{(i/h)\Phi_M(x', t, y, \eta')} a^*(y, \eta'; h) a(y, t\eta'_0; h) q(y) dy d\eta' \tag{5-1}$$

with  $\Phi_M$  given in (3-19).

We want to express  $Mq$  with the help of  $Tq$ , where  $T$  is as in (4-1), and we start by recalling some general facts about metaplectic Fourier integral operators of this form, following [Sjöstrand 1982] for the local theory and [Sjöstrand 1990] for the simplified global theory in the metaplectic framework (i.e., all phases are quadratic and all amplitudes are constant). To start with, we weaken the assumptions on the quadratic phase in  $T$  and assume only that  $\phi(x, y)$  is a holomorphic quadratic form on  $\mathbb{C}^n \times \mathbb{C}^n$  satisfying the second part of (4-2):

$$\det \phi''_{x,y}(x, y) \neq 0. \tag{5-2}$$

To  $T$  we can still associate a linear canonical transformation  $\kappa_T$  as in (4-4). Let  $\Phi_1$  and  $\Phi_2$  be plurisubharmonic quadratic forms on  $\mathbb{C}^n$  related by

$$\Lambda_{\Phi_2} = \kappa_T(\Lambda_{\Phi_1}). \tag{5-3}$$

Then we can define  $T : H_{\Phi_1} \rightarrow H_{\Phi_2}$  as a bounded operator as in (4-1) with the modification that  $\mathbb{R}^n$  should be replaced by a so-called good contour, which is an affine subspace of  $\mathbb{C}^n$  of real dimension  $n$ , passing through the nondegenerate critical point  $y_c(x)$  the function

$$y \mapsto -\Im\phi(x, y) + \Phi_1(y) \tag{5-4}$$

and along which this function is  $\Phi_2(x) - (\asymp |y - y_c(x)|^2)$ . (Actually in this situation, it would have been better to replace the power  $h^{-3n/4}$  by  $h^{-n/2}$  since we would then get a uniform bound on the norm.)

**Remark 5.1.** Recall also that, if only  $\Phi_1$  is given as above, the existence of a quadratic form  $\Phi_2$  as in (5-3) is equivalent to the fact that (5-4) has a nondegenerate critical point and the plurisubharmonicity of  $\Phi_2$  is equivalent to the fact that the signature of the critical point is  $(n, -n)$  (which represents the maximal number of negative eigenvalues of the Hessian of a plurisubharmonic quadratic form). This in turn is equivalent to the existence of an affine good contour as above.

In this situation,  $T : H_{\Phi_1} \rightarrow H_{\Phi_2}$  is bijective with the inverse

$$Sv(y) = T^{-1}v(y) = \tilde{C}h^{-n/4} \int e^{-(i/h)\phi(z,y)} v(z) dz, \tag{5-5}$$

which can be realized the same way with a good contour, and here the constant  $\tilde{C}$  does not depend on the choice of  $\Phi_j$ ,  $j = 1, 2$ .

**Remark 5.2.** Let us introduce the formal adjoints of  $T$  and  $S$ ,

$$T^t v(y) = Ch^{-3n/4} \int_{\mathbb{R}^n} e^{(i/h)\phi(z,y)} v(x) dx, \quad y \in \mathbb{C}^n,$$

$$S^t u(x) = \tilde{C}h^{-n/4} \int e^{-(i/h)\phi(x,y)} u(y) dy.$$

Let  $\Psi_1$  and  $\Psi_2$  be plurisubharmonic quadratic forms such that  $\kappa_{S^t}(\Lambda_{\Psi_1}) = \Lambda_{\Psi_2}$ . Then as above,  $T^t : H_{\Psi_2} \rightarrow H_{\Psi_1}$  and  $S^t : H_{\Psi_1} \rightarrow H_{\Psi_2}$  are bijective and  $S^t = \text{const}(T^t)^{-1}$ . We claim that  $S^t$  is the inverse of  $T^t$ . In fact, this statement is independent of the choice of  $\Phi_j$  and  $\Psi_j$  as above, and we can choose them to be pluriharmonic in such a way that  $\Lambda_{\Phi_j}$  intersects  $\Lambda_{-\Psi_j}$  transversally for one value of  $j$  and then automatically for the other value. Then for  $j = 1, 2$ , we can define

$$\langle u | v \rangle = \int_{\gamma_j} u(x)v(x) dx$$

for  $u \in H_{\Phi_j}$  and  $v \in H_{\Psi_j}$  (or rather for functions that are  $\mathcal{O}(e^{\Phi_j/h})$  and  $e^{\Psi_j/h}$ , respectively — the space of such functions is of dimension 1, which suffices for our purposes) if we let  $\gamma_j$  be a good contour for  $\Phi_j + \Psi_j$ . For  $u = \mathcal{O}(e^{\Phi_2/h})$  and  $v = \mathcal{O}(e^{\Psi_2/h})$  nonzero,

$$0 \neq \langle u | v \rangle = \langle TSu | v \rangle = \langle Su | T^t v \rangle = \langle u | S^t T^t v \rangle,$$

and knowing already that  $S^t T^t$  is a multiple of the identity, we see that it has to be equal to the identity.

Now return to the discussion of an FBI transform  $T$  whose phase satisfies (4-2). When letting  $T$  act on suitable  $H_\Phi$ -spaces, it has the inverse  $S$  in (5-5). However, if we let  $T$  act on  $L^2(\mathbb{R}^n)$  so that  $Tu \in H_{\Phi_0}$  (with  $\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n})$ ), the best possible contour in (5-5) is

$$\Gamma(y) = \{z \in \mathbb{C}^n : y(z) = y\}.$$

This follows from the property

$$\Phi_0(z) + \Im\phi(z, y) \asymp \text{dist}(z, \Gamma(y))^2 \asymp |y(z) - y|^2, \tag{5-6}$$

so  $\Phi_0(z) + \Im\phi(z, y) = 0$  on  $\Gamma(y)$  and  $e^{-(i/h)\phi(z,y)+(1/h)\Phi_0(z)}$  is bounded there. This is not sufficient for a straightforward definition of  $Sv(y)$ ,  $v \in H_{\Phi_0}$ , since we would need some extra exponential decay along the contour near infinity, but it does suffice to give a precise meaning up to exponentially small errors of the formula

$$\tilde{T}u = (\tilde{T}S)Tu \tag{5-7}$$

in a local situation, where  $\tilde{T} : L^2 \rightarrow H_{\tilde{\Phi}_0}$  is a second FBI transform and where  $\tilde{T}S : H_{\Phi_0} \rightarrow H_{\tilde{\Phi}_0}$  is defined by means of a good contour.

**Proposition 5.3.** *Let  $(y_0, \eta_0) \in \mathbb{R}^{2n}$ ,  $(z_0, \zeta_0) = \kappa_T(y_0, \eta_0)$ , and  $(w_0, \omega_0) = \kappa_{\tilde{T}}(y_0, \eta_0)$ . We realize  $Tu$  and  $\tilde{T}u$  ( $\tilde{T}Su$  modulo exponentially small terms) in  $H_{\Phi_0, z_0}$  and  $H_{\tilde{\Phi}_0, w_0}$  (and  $H_{\tilde{\Phi}_0, w_0}$ ) by choosing good contours restricted to neighborhoods of  $y_0$  and  $y_0$  (and  $z_0$ ), respectively. Then (5-7) holds (modulo an exponentially small error) in  $H_{\tilde{\Phi}_0, w_0}$ . Here  $u \in \mathcal{D}'(\mathbb{R}^n)$  is either independent of  $h$  or of temperate growth in  $\mathcal{D}'(\mathbb{R}^n)$  as a function of  $h$ .*

*Proof.* The left-hand side of (5-7) is

$$\text{const } h^{-3n/4-n} \iiint e^{(i/h)(\tilde{\phi}(w,x)-\phi(z,x)+\phi(z,y))} u(y) \, dy \, dz \, dx,$$

and all good contours being homotopic, we can write it as

$$\tilde{C}h^{-3n/4} \int \left( \text{const } h^{-n} \iint e^{(i/h)(-\phi(z,x)+\phi(z,y))} e^{(i/h)\tilde{\phi}(w,x)} \, dx \, dz \right) u(y) \, dy.$$

The expression in the big parentheses is nothing but  $T^t S^t(e^{(i/h)\tilde{\phi}(w,\cdot)})(y)$ , which by Remark 5.2 is equal to  $e^{(i/h)\tilde{\phi}(w,y)}$ , and (5-7) follows. (In the proof, we have chosen not to spell out the various exponentially small errors due to the fact that the integration contours are confined to various small neighborhoods of certain points.) □

We now return to the operator  $M$  in (5-1). Choose adapted analytic coordinates centered at  $x_0$  as in the beginning of Section 3. In that section (see (3-25)), we have seen that there is a well defined canonical transformation  $\kappa_M$  from a neighborhood of  $(0, 0, -i) \in \mathbb{C}_{y,\eta}^{2n}$  to a neighborhood of  $(0, 1, 0, 0)$  in  $\mathbb{C}_{x'}^{n-1} \times \mathbb{C}_t \times \mathbb{C}_{x'^*}^{n-1} \times \mathbb{C}_{t^*}$  mapping  $T^* \partial \Omega \times i\mathbb{R}_-$  to  $\mathbb{R}_{x'}^{n-1} \times \mathbb{R}_t \times \mathbb{R}_{x'^*}^{n-1} \times \{t^* = 0\}$ . This means that we have a microlocal description of  $Mq$  near  $(0, 1, 0, 0)$  and not a local one near  $x' = 0$  and  $t = 0$ . We shall therefore microlocalize in  $(x', x'^*)$  by means of an FBI transform in the  $x'$  variables.

Let

$$\widehat{T}u(w') = \widehat{C}h^{(1-n)/2} \int_{\mathbb{R}^{n-1}} e^{(i/h)\widehat{\phi}(w',x')} u(x') \, dx', \quad w' \in \mathbb{C}^{n-1}, \tag{5-8}$$

be a second FBI transform as in (4-1) though acting on  $n - 1$  variables and with a different normalization. Assume (for concreteness) that

$$\kappa_{\widehat{T}}(\mathbb{C}_{x'}^{n-1} \times \{0\}) = \mathbb{C}_{w'}^{n-1} \times \{0\}. \tag{5-9}$$

Then

$$\kappa_{\widehat{T}}(T^* \mathbb{R}^{n-1}) = \Lambda_{\widehat{\Phi}_0}, \tag{5-10}$$

where  $\widehat{\Phi}_0$  is a strictly plurisubharmonic quadratic form. In view of (5-9) and the fact that the zero-section  $\mathbb{C}^{n-1} \times \{0\}$  is strictly positive with respect to the real phase space, we also know that

$$\widehat{\Phi}_0(w') \asymp |w'|^2 \tag{5-11}$$

or equivalently that the quadratic form  $\widehat{\Phi}_0$  is strictly convex.

By slight abuse of notation, we also let  $\widehat{T}$  act (as  $\widehat{T} \oplus 0$ ) on functions of  $n$  variables by

$$\widehat{T}(u)(w', t) = (\widehat{T}u(\cdot, t))(w').$$

The presence of  $\widehat{T}$  leads to a formula for  $\widehat{T}M$  that is simpler than the one for  $M$  in (5-1):

$$\begin{aligned} \widehat{T}Mq(w', t) &= \widehat{T}(e^{-(i/h)(\cdot)\cdot t\eta'_0} K^t q K(e^{(i/h)(\cdot)\cdot t\eta'_0}))(w') \\ &= \widehat{C}h^{(1-n)/2} \iiint e^{(i/h)(\widehat{\phi}(w', \tilde{x}') - \tilde{x}'\cdot t\eta'_0)} K(y, \tilde{x}')q(y)K(y, x')e^{(i/h)x'\cdot t\eta'_0} dx' dy d\tilde{x}' \\ &= \int K(e^{(i/h)(\widehat{\phi}(w', \cdot) - (\cdot)\cdot t\eta'_0)}(y))q(y)K(e^{(i/h)(\cdot)\cdot t\eta'_0})(y) dy. \end{aligned}$$

Up to exponentially small errors, we have (see (3-5))

$$K(e^{(i/h)(\cdot)\cdot t\eta'_0})(y) = e^{(i/h)\phi(y, t\eta'_0)} a(y, t\eta'_0; h)$$

and

$$K(e^{(i/h)\widehat{\phi}(w', \cdot)})(y) = e^{(i/h)\tilde{\psi}(w', t\eta'_0, y)} b(w', y, t\eta'_0; h),$$

where  $b$  is an elliptic analytic symbol of order 0 and  $\psi$  is the solution of the eikonal equation in  $y$

$$\partial_{y_n} \tilde{\psi} = ir(y, \partial_{y'} \psi)^{1/2}, \quad \tilde{\psi}|_{y_n=0} = \widehat{\phi}(w', y') - y' \cdot t\eta'_0.$$

Thus, up to exponentially small errors, we get for  $q \in L^\infty(\Omega)$

$$\widehat{T}Mq(w', t) = \int e^{(i/h)\psi(w', t, y)} c(w', t, y; h)q(y) dy, \quad (w', t) \in \text{neigh}((0, 1), \mathbb{C}^{n-1} \times \mathbb{C}), \quad (5-12)$$

where  $c$  is an elliptic analytic symbol of order 0 and

$$\psi(w', t, y) = \tilde{\psi}(w', t, y) + \phi(y, t\eta'_0)$$

satisfies

$$\psi|_{y_n=0} = \widehat{\phi}(w', y'), \quad (5-13)$$

$$\partial_{y_n} \psi|_{y_n=0} = i(r(y', 0, \partial_{y'} \widehat{\phi}(w', y') - t\eta'_0)^{1/2} + r(y', 0, t\eta'_0)^{1/2}). \quad (5-14)$$

Assume for simplicity that  $r(0, 0, \eta'_0) = \frac{1}{4}$ . Then, at the point  $(w' = 0, t = 1, y = 0)$ ,

$$(\partial_{w'} \psi, \partial_t \psi, -\partial_{y'} \psi, -\partial_{y_n} \psi) = (0, 0, 0, -i),$$

so  $\kappa_{\widehat{T}M}(0, 0, -i) = (0, 1, 0, 0)$ .<sup>4</sup> Also,  $\kappa_{\widehat{T}M} = \kappa_{\widehat{T}} \circ \kappa_M$  and

$$\kappa_M(0, 0, -i) = (0, 1, 0, 0),$$

$$\kappa_{\widehat{T}}(0, 1, 0, 0) = (0, 1, 0, 0).$$

Recall from (3-25) that

$$\kappa_M : \text{neigh}((0; 0, -i), T^*\partial\Omega \times \mathbb{C}_{y_n}^-) \rightarrow \text{neigh}((0, 1; 0, 0), \mathbb{R}_{x'}^{n-1} \times \mathbb{C}_t \times \mathbb{R}_{x''}^{n-1} \times \{t^* = 0\}),$$

so

$$\kappa_{\widehat{T}M} : \text{neigh}((0, 0, -i), T^*\partial\Omega \times \mathbb{C}_{y_n}^-) \rightarrow \text{neigh}((0, 1, 0, 0), \Lambda_{\widehat{\phi}_0 \oplus 0}).$$

<sup>4</sup>We can verify directly that  $\det \partial_{w', t} \partial_y \psi \neq 0$ .

On the other hand, we have seen in Section 4 that  $\kappa_T(F) = \Lambda_{\Phi_1}$  and that the part  $T^*\partial\Omega \times \mathbb{C}_{y_n}^-$  of  $F$  is mapped to  $\Lambda_{\Phi_1^{\text{ext}}}$ . More locally,

$$\begin{aligned} \kappa_T &: \text{neigh}((0, 0, -i), T^*\partial\Omega \times \mathbb{C}_{y_n}^-) \rightarrow \text{neigh}(\kappa_T(0, 0, -i), \Lambda_{\Phi_1^{\text{ext}}}) \\ \kappa_S &: \text{neigh}(\kappa_T(0, 0, -i), \Lambda_{\Phi_1^{\text{ext}}}) \rightarrow \text{neigh}((0, 0, -i), T^*\partial\Omega \times \mathbb{C}_{y_n}^-). \end{aligned}$$

Using also (3-25), we get

$$\kappa_{\widehat{TMS}} : \text{neigh}(\pi_z \kappa_T(0, 0, -i), \Lambda_{\Phi_1^{\text{ext}}}) \rightarrow \text{neigh}((0, 1, 0, 0), \Lambda_{\widehat{\Phi}_0 \oplus 0}). \tag{5-15}$$

We then also know that

$$\widehat{\Phi}_0(w') = \text{vc}_{y,z}(-\Im\psi(w', t, y) + \Im\phi_T(z, y)).$$

This means that the formal composition

$$\widehat{TMS}v(w', t) = \widetilde{C}h^{-n/4} \iint e^{(i/h)(\psi(w', t, y) - \phi_T(z, y))} c(w', t, y; h)v(z) dz dy \tag{5-16}$$

gives a well defined operator

$$\widehat{TMS} : H_{\Phi_1^{\text{ext}}, \pi_z \kappa_T(0, 0, -i)} \rightarrow H_{\widehat{\Phi}_0 \oplus 0, (0, 1)} \tag{5-17}$$

that can be realized with the help of a good contour.

We shall next show that

$$\widehat{T}Mu = (\widehat{TMS})Tu \quad \text{in } H_{\widehat{\Phi}_0 \oplus 0, (0, 1)} \tag{5-18}$$

when  $u$  is supported in  $\{y_n \geq 0\}$ . The proof is the same as the one for (5-7). The right-hand side in (5-18) is equal to

$$\text{const } h^{-n} \iiint e^{(i/h)(\psi(w', t, x) - \phi_T(z, x) + \phi_T(z, y))} c(w', t, x; h)u(y) dy dz dx,$$

where the  $y$ -integration is over  $\mathbb{R}_+^n$ , and we may assume without loss of generality that  $u$  has its support in a small neighborhood of  $y = 0$ . The  $dz dx$  integration is, to start with, over the good contour in (5-16). This last integration can be viewed as  $T^t S^t$  acting on  $e^{(i/h)\psi(w', t, \cdot)} c(w', t, \cdot; h)$ , and here  $T^t S^t$  is the identity operator that can be realized with a good contour, so we get

$$(\widehat{TMS})Tu(w', t) = \int e^{(i/h)\psi(w', t, x)} c(w', t, x; h)u(x) dx = \widehat{T}Mu(w', t),$$

and we have verified (5-18).

Above, we have established (5-17) as the quantum version of (5-15). It follows by an easy adaptation of the exercise leading to (3-17) that

$$\kappa_M(\text{neigh}((0, 0, -i), \mathbb{C}^{n-1} \times \{0\} \times \mathbb{C}_{\eta_n}^-)) = \text{neigh}((0, 0, 1, 0), \mathbb{C}_{x'}^{n-1} \times \{x'^* = 0\} \times \mathbb{C}_t \times \{t^* = 0\}), \tag{5-19}$$

and hence,

$$\kappa_{\widehat{TMS}}(\text{neigh}(\kappa_T(0, 0, -i), \Lambda_{\Phi_3})) = \text{neigh}((0, 0, 1, 0), \Lambda_{0 \oplus 0}). \tag{5-20}$$



The quantum version of (5-20) is

$$\widehat{TMS} : H_{\Phi_3, \pi_z(\text{neigh}(\kappa_T(0,0,-i)))}^{\text{loc}} \rightarrow H_{0 \oplus 0, (0,1)}^{\text{loc}} \tag{5-21}$$

We also know that  $\widehat{TMS}$  is an elliptic Fourier integral operator. Consequently, (5-17) and (5-21) have continuous inverses. We also have the following result.

**Proposition 5.4.** *If  $u \in H_{\Phi_1, \pi_z(\text{neigh}(\kappa_T(0,0,-i)))}^{\text{loc}}$  and  $\widehat{TMS}u \in H_{0 \oplus 0, (0,1)}^{\text{loc}}$ , then  $u \in H_{\Phi_3, \pi_z(\text{neigh}(\kappa_T(0,0,-i)))}^{\text{loc}}$ .*

### 6. End of the proof of the main result

We will work with FBI and Laplace transforms of functions that are independent of  $h$  or that have some special  $h$ -dependence. Consider a formal Fourier integral operator  $u \mapsto Tu$ , given by

$$Tu(x; h) = Ch^\alpha \int e^{(i/h)\phi(x,y)} u(y) dy, \tag{6-1}$$

where  $\phi = \phi_T$  is a quadratic form on  $\mathbb{C}_{x,y}^{2n}$  satisfying

$$\det \phi''_{xy} \neq 0 \tag{6-2}$$

and hence generating a canonical transformation that will be used below.

**Proposition 6.1.** *If  $u$  is independent of  $h$ ,*

$$\left( hD_h + \frac{1}{h} P_\alpha(x, hD; h) \right) Tu = 0, \tag{6-3}$$

where

$$P_\alpha = p(x, hD) + ih \left( \alpha + \frac{1}{2} \text{tr}(\phi''_{xx} \phi''_{yx}^{-1} \phi''_{yy} \phi''_{xy}^{-1}) \right), \tag{6-4}$$

$$\begin{aligned} p(x, \xi) &= \frac{1}{2} \phi''_{xx} x \cdot x + x \cdot (\xi - \phi''_{xx} x) + \frac{1}{2} \phi''_{yx}^{-1} \phi''_{yy} \phi''_{xy}^{-1} (\xi - \phi''_{xx} x) \cdot (\xi - \phi''_{xx} x) \\ &= -\frac{1}{2} \phi''_{xx} x \cdot x + \frac{1}{2} \phi''_{xx} \phi''_{yx}^{-1} \phi''_{yy} \phi''_{xy}^{-1} \phi''_{xx} x \cdot x \\ &\quad + x \cdot \xi - \phi''_{yx}^{-1} \phi''_{yy} \phi''_{xy}^{-1} \phi''_{xx} x \cdot \xi + \frac{1}{2} \phi''_{yx}^{-1} \phi''_{yy} \phi''_{xy}^{-1} \xi \cdot \xi. \end{aligned} \tag{6-5}$$

*Proof.* We have

$$\begin{aligned} hD_h(e^{(i/h)\phi(x,y)}) &= -\frac{1}{h} e^{(i/h)\phi(x,y)}, \\ hD_h(h^\alpha) &= \frac{\alpha}{i} h^\alpha, \\ hD_h Tu(x; h) &= -\frac{1}{h} h^\alpha \int e^{(i/h)\phi(x,y)} (ih\alpha + \phi(x, y)) u(y) dy. \end{aligned}$$

Try to write  $\phi(x, y) = p(x, \phi'_x(x, y))$  for a suitable quadratic form  $p(x, \xi)$  (that will turn out to be the one given in (6-5)). We have

$$\phi(x, y) = \frac{1}{2} \phi''_{xx} x \cdot x + \phi''_{xy} y \cdot x + \frac{1}{2} \phi''_{yy} y \cdot y, \tag{6-6}$$

$$\phi'_x = \phi''_{xx} x + \phi''_{xy} y, \quad \text{i.e., } y = \phi''_{xy}^{-1} (\phi'_x - \phi''_{xx} x), \tag{6-7}$$

and using the last relation from (6-7) in (6-6), we get

$$\phi(x, y) = \frac{1}{2}\phi''_{xx}x \cdot x + \phi''_{yx}x \cdot \phi''_{xy}^{-1}(\phi'_x - \phi''_{xx}x) + \frac{1}{2}\phi''_{yy}\phi''_{xy}^{-1}(\phi'_x - \phi''_{xx}x) \cdot \phi''_{xy}^{-1}(\phi'_x - \phi''_{xx}x), \quad (6-8)$$

where the  $\phi''_{yx}$  and  $\phi''_{xy}^{-1}$  in the second term cancel and we get  $p(x, \phi'_x)$  with  $p$  as in (6-5).

To verify (6-4), it suffices to notice that

$$\begin{aligned} e^{-(i/h)\phi(x,y)} p(x, hD_x)(e^{(i/h)\phi(x,y)}) - p(x, \phi'_x) &= \frac{1}{2}\phi''_{yx}^{-1}\phi''_{yy}\phi''_{xy}^{-1}hD_x \cdot (\phi'_x) \\ &= \frac{1}{2}\phi''_{yx}^{-1}\phi''_{yy}\phi''_{xy}^{-1}hD_x \cdot (\phi''_{xx}x) \\ &= \frac{h}{2i}\phi''_{xx}\phi''_{yx}^{-1}\phi''_{yy}\phi''_{xy}^{-1}\partial_x \cdot x \\ &= \frac{h}{2i}\text{tr}(\phi''_{xx}\phi''_{yx}^{-1}\phi''_{yy}\phi''_{xy}^{-1}). \quad \square \end{aligned}$$

**Remark 6.2.** Let  $\kappa_T : (y, -\phi'_y(x, y)) \mapsto (x, \phi'_x(x, y))$  be the canonical transformation associated to  $T$ , which can also be written

$$\kappa_T : (y, -(\phi''_{yx}x + \phi''_{yy}y)) \mapsto (x, \phi''_{xx}x + \phi''_{xy}y)$$

or still  $\kappa_T : (y, \eta) \mapsto (x, \xi)$ , where

$$\begin{aligned} x &= -\phi''_{yx}^{-1}(\eta + \phi''_{yy}y), \\ \xi &= (\phi''_{xy} - \phi''_{xx}\phi''_{yx}^{-1}\phi''_{yy})y - \phi''_{xx}\phi''_{yx}^{-1}\eta. \end{aligned}$$

We see that the following three statements are equivalent.

- $\kappa_T$  maps the Lagrangian space  $\eta = 0$  to  $\xi = 0$ .
- $\phi''_{xy} - \phi''_{xx}\phi''_{yx}^{-1}\phi''_{yy} = 0$ .
- $p(x, 0) = 0$  and  $p'_\xi(x, 0)$  for all  $x$ .

**Example 6.3.** Consider

$$\widehat{T}\mathcal{L}u(x; h) = Ch^{(1-n)/2} \int e^{(i/h)(\phi(x',y')+ix_ny_n)}u(y) dy, \quad \phi = \phi_{\widehat{T}}.$$

If  $P'(x', hD_{x'}; h)$  is the operator associated to  $\widehat{T}$  in  $n - 1$  variables, we get when  $u$  is independent of  $h$

$$\left(hD_h + \frac{1}{h}(P'(x', hD_{x'}; h) + x_nhD_{x_n})\right)\widehat{T}\mathcal{L}u = 0. \quad (6-9)$$

Similarly (though not a direct consequence of Proposition 6.1 but rather of its method of proof), we have for  $\mathcal{L}$  alone that

$$\left(hD_h + \frac{1}{h}x_nhD_{x_n}\right)\mathcal{L}u = 0. \quad (6-10)$$

**Example 6.4.** Let  $T$  be as above, and assume that we are in the situation of Remark 6.2 so that  $p(x, 0) = 0$  and  $p'_\xi(x, 0) = 0$ . Then

$$p(x, hD) = bhD \cdot hD,$$

where  $b$  is a constant symmetric matrix. Then

$$P_\alpha = p(x, hD) + ih(\alpha + f_0), \quad f_0 = \frac{n}{2},$$

and (6-3) reads

$$(hD_h + (hbD \cdot D + i(\alpha + f_0)))Tu = 0. \tag{6-11}$$

If  $Tu = \sum_m^\infty h^k v_k \in H_0$  and  $u$  is independent of  $h$ , we can plug this expression into (6-11) and get the sequence of equations

$$\begin{aligned} \left(\frac{m}{i} + i(\alpha + f_0)\right)v_m &= 0, \\ \left(\frac{m+1}{i} + i(\alpha + f_0)\right)v_{m+1} + bD \cdot Dv_m &= 0, \\ \left(\frac{m+2}{i} + i(\alpha + f_0)\right)v_{m+2} + bD \cdot Dv_{m+1} &= 0, \\ &\vdots \end{aligned}$$

so unless  $v \equiv 0$ , we get  $m = \alpha + f_0$ . We can choose  $v_m \in H_0$  arbitrarily, and  $v_{m+1}, v_{m+2}, \dots$  are then uniquely determined.

Now, consider the situation in Theorem 1.6 and let  $q \in L^\infty(\Omega)$  be independent of  $h$  and such that  $\sigma_{\hat{\Lambda}}(y', t\eta'_0)$  is a cl.a.s. on  $\text{neigh}(\{0\} \times \mathbb{R}_+, \mathbb{R}^{n-1} \times \mathbb{R}_+)$  of order  $-1$  (see (2-4)):

$$\sigma_{\hat{\Lambda}}(y', t\eta'_0) \sim \sum_1^\infty n_k(y', t), \tag{6-12}$$

where  $n_k(y', t)$  is homogeneous of degree  $-k$  in  $t$ .

$$|n_k(y', t)| \leq C^{k+1} k^k |t|^{-k}, \quad y' \in \text{neigh}(0, \mathbb{C}^{n-1}). \tag{6-13}$$

For the moment, we shall only work with formal cl.a.s. and neglect remainders in the asymptotic expansions. The semiclassical symbol of  $\hat{\Lambda}$  is then

$$\sigma_{\hat{\Lambda}}(y', t\eta'_0/h) \sim \sum_1^\infty n_k(y', t/h) = \sum_1^\infty h^k n_k(y', t), \quad (y', t) \in \text{neigh}((0, 1), \mathbb{R}^{n-1} \times \mathbb{R}_+). \tag{6-14}$$

Recall that  $\sigma_{\hat{\Lambda}}(y', t\eta'_0/h) = Mq(y', t; h)$ . From (6-14), we infer that  $\hat{T}Mq$  is a cl.a.s. near  $w' = 0$  and  $t = 1$ :

$$\hat{T}Mq \sim \sum_1^\infty h^k m_k(w', t). \tag{6-15}$$

Formally,

$$\hat{T}M = (\hat{T}M\mathcal{L}^{-1})\mathcal{L}. \tag{6-16}$$

The canonical transformation  $\kappa_{\mathcal{L}}$  is given by

$$(y, \eta) \mapsto (y', i\eta_n, \eta', iy_n).$$

It maps the complex manifold  $\eta' = 0$  and  $y_n = 0$  to the manifold  $\{(z, 0)\}$  and the point  $(0; 0, -i)$  to  $(0, 1; 0)$ , so  $\kappa_{\mathcal{L}^{-1}} = \kappa_{\mathcal{L}}^{-1}$  maps  $\zeta = 0$  to  $\eta' = 0$  and  $y_n = 0$ . We noticed in (3-24) (see (3-22)) that  $\kappa_M$  takes the complexified conormal bundle to the zero-section, and it maps the point  $(0; 0, -i)$  to  $(0, 1; 0)$ . Thus,  $\kappa_{M\mathcal{L}^{-1}}$  maps the zero-section  $\zeta = 0$  to the zero-section and in particular  $(0, 1; 0)$  to  $(0, 1; 0)$ . (We may notice that this is global in the sense that we can extend  $z_n$  to an annulus, and we then get  $t$  in an annulus.) Since  $\kappa_{\widehat{T}}$  maps the zero-section to the zero-section, we have the same facts for  $\kappa_{\widehat{T}M}$ .

From the above, it is clear that  $\widehat{T}M\mathcal{L}^{-1}$  maps formal cl.a.s. to formal cl.a.s.

Recalling (6-14) for  $\sigma_{\lambda}(y', t\eta'_0/h) = Mq(y', t; h)$  and using that  $\widehat{T}M\mathcal{L}^{-1}$  is an elliptic Fourier integral operator whose canonical transformation maps the zero-section to the zero-section, we see that there exists a unique formal cl.a.s.

$$v \sim \sum_1^{\infty} v_k(z', z_n)h^k, \quad z \in \text{neigh}((0, 1), \mathbb{C}^n), \tag{6-17}$$

such that in the sense of formal stationary phase

$$\widehat{T}Mq = \widehat{T}M\mathcal{L}^{-1}v. \tag{6-18}$$

Now  $q$  is independent of  $h$ , so  $Mq$  satisfies a compatibility equation of the form

$$\left(hD_h + \frac{1}{h}P_{\widehat{T}M}\right)Mq = 0. \tag{6-19}$$

This gives rise to a similar compatibility condition for  $v$

$$\left(hD_h + \frac{1}{h}P_{\mathcal{L}M^{-1}\widehat{T}^{-1}\widehat{T}M}\right)v = 0$$

or simply

$$\left(hD_h + \frac{1}{h}P_{\mathcal{L}}\right)v = 0,$$

which is the same as (6-10):

$$(h\partial_h + z_n\partial_{z_n})v = 0. \tag{6-20}$$

Application of this to (6-17) gives

$$(k + z_n\partial_{z_n})v_k = 0, \tag{6-21}$$

i.e.,

$$v_k(z) = q_k(z')z_n^{-k}, \quad |q_k(z')| \leq C^{k+1}k^k. \tag{6-22}$$

Thus,

$$v \sim \sum_1^{\infty} q_k(z')\left(\frac{h}{z_n}\right)^k = \sum_0^{\infty} q_{k+1}(z')\left(\frac{h}{z_n}\right)^{k+1},$$

and we see as in Section 2 that

$$v \sim \mathcal{L}\tilde{q}(z; h), \quad \tilde{q}(y) = 1_{[0,a]}(y_n) \sum_0^{\infty} \frac{q_{k+1}(y')}{k!} y_n^k, \tag{6-23}$$

with  $a > 0$  small enough to assure the convergence of the power series.

More precisely (and now we end the limitation to formal symbols), as in (5-18) and (5-7), we check that

$$\widehat{T}M\tilde{q} \equiv (\widehat{T}M\mathcal{L}^{-1})\mathcal{L}\tilde{q} \quad \text{in } H_{0,(0,1)} \tag{6-24}$$

(up to an exponentially small error). By the construction of  $\tilde{q}$ , the right-hand side is  $\equiv \widehat{T}Mq$  in the same space.

Put  $r = q - \tilde{q}$ . Then

$$\widehat{T}Mr \equiv 0 \quad \text{in } H_{0,(0,1)}. \tag{6-25}$$

Now, we replace  $\mathcal{L}$  with  $T$  and consider in light of (5-18)

$$(\widehat{T}MS)Tr \equiv 0 \quad \text{in } H_{0,(0,1)}, \tag{6-26}$$

which implies that  $Tr \in H_{\Phi_1}$  satisfies

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1^{\text{ext}}, \pi_z \kappa_T(0; 0, -i)}. \tag{6-27}$$

As we saw in Section 4,  $\Lambda_{\Phi_1}$  contains the closure  $\bar{\Gamma}$  of the complex curve

$$\Gamma = \kappa_T(\{(0; 0, \eta_n) : \Im \eta_n < 0\}),$$

and  $\kappa_T((0; 0, -i)) \in \Gamma$ . Consequently,  $\Phi_1|_{\pi_z \Gamma}$  is harmonic and (6-27) and the maximum principle imply that

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1} \text{ on } \pi_z(\bar{\Gamma}). \tag{6-28}$$

In particular,

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1, 0} \tag{6-29}$$

and a fortiori

$$Tr \equiv 0 \quad \text{in } H_{\Phi_0, 0}. \tag{6-30}$$

This implies that  $r = 0$  near  $y = 0$ . Hence,  $q = \tilde{q}$  near  $y = 0$ , which gives the theorem.

### 7. Proof of Proposition 1.7

We choose local coordinates  $y = (y', y_n)$  as in the beginning of Section 2. As in Proposition 1.7, we assume that  $q$  is analytic in a neighborhood of 0. We adopt the alternative definition of symbols in Remark 1.4. It will also be convenient to consider the semiclassical symbol of  $\dot{\Lambda}$ ,  $\sigma_{\dot{\Lambda}}(y', \eta'; h) = \sigma_{\dot{\Lambda}}(y', \eta'/h)$ . For  $y' \in \text{neigh}(0, \mathbb{R}^{n-1})$ ,

$$\sigma_{\dot{\Lambda}}(y', \eta'; h) = -\partial_{y_n} GqK \left( \int \chi(t') e_{t'}(\cdot; h) e^{i(\cdot)\cdot\eta'/h} dt' \right) (y', 0) e^{-iy'\cdot\eta'/h}, \tag{7-1}$$

where  $\chi$  and  $e_t$  were defined in Remark 1.4 with  $n$  there replaced by  $n - 1$ . By analytic WKB (as we already used), we have up to an exponentially small error

$$K(e_{t'}(\cdot; h) e^{i(\cdot)\cdot\eta'/h}) = Ch^{(1-n)/2} a(y, \eta'; h) e^{i\phi(y, t, \eta')/h}, \tag{7-2}$$

where  $\phi$  is the solution of the eikonal problem

$$\partial_{y_n}\phi = ir(y, \partial_{y'}\phi)^{1/2}, \quad \phi|_{y_n=0} = y' \cdot \eta' + \frac{i}{2}(y' - t)^2 \quad (7-3)$$

and  $a$  is an cl.a.s. of order 0 obtained from solving a sequence of transport equations with the “initial” condition  $a(y', 0, \eta'; h) = 1$ .

Using again the analytic WKB method, we can find a cl.a.s.  $b$  of order 0 in  $h$  that solves the following inhomogeneous problem up to exponentially small errors:

$$\begin{cases} (h^2\Delta - h^2V)(h^{(3-n)/2}b(y, t, \eta'; h)e^{(i/h)\phi(y,t,\eta')}) = Ch^{(5-n)/2}ae^{(i/h)\phi}, \\ b(y', 0, t, \eta'; h) = 0. \end{cases}$$

Then up to exponentially small errors,

$$GqK(e_t(\cdot; h)e^{i(\cdot)\eta'/h}) \equiv h^{(3-n)/2}b(y, t, \eta'; h)e^{(i/h)\phi(y,t,\eta')}$$

and similarly for the gradients, so

$$-(\partial_{y_n})_{y_n=0}GqK(e_t(\cdot; h)e^{i(\cdot)\eta'/h}) \equiv -h^{(3-n)/2}(\partial_{y_n}b)(y', 0, t, \eta'; h)e^{(i/h)(y'\eta' + (i/2)(y'-t)^2)}.$$

Multiplying with  $\chi(t')$  and integrating in  $t'$ , we see that  $\sigma_{\tilde{\Lambda}}(y', \eta'; h)$  is a cl.a.s. in the semiclassical sense, and this implies that  $\sigma_{\tilde{\Lambda}}(y', \eta)$  is a cl.a.s.

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