ANALYSIS & PDE

Volume 9 No. 3 2016

HAJER BAHOURI, CLOTILDE FERMANIAN-KAMMERER AND ISABELLE GALLAGHER

DISPERSIVE ESTIMATES FOR THE SCHRÖDINGER OPERATOR ON STEP-2 STRATIFIED LIE GROUPS

msp

ANALYSIS AND PDE
Vol. 9, No. 3, 2016
dx.doi.org/10.2140/apde.2016.9.545

DISPERSIVE ESTIMATES FOR THE SCHRÖDINGER OPERATOR ON STEP-2 STRATIFIED LIE GROUPS

HAJER BAHOURI, CLOTILDE FERMANIAN-KAMMERER AND ISABELLE GALLAGHER

The present paper is dedicated to the proof of dispersive estimates on stratified Lie groups of step 2 for the linear Schrödinger equation involving a sublaplacian. It turns out that the propagator behaves like a wave operator on a space of the same dimension p as the center of the group, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form, which suggests a decay rate $|t|^{-(k+p-1)/2}$. We identify a property of the canonical skew-symmetric form under which we establish optimal dispersive estimates with this rate. The relevance of this property is discussed through several examples.

1. Introduction

1A. ***Dispersive inequalities.*** Dispersive inequalities for evolution equations (such as the Schrödinger and wave equations) play a decisive role in the study of semilinear and quasilinear problems which appear in numerous physical applications. Proving dispersion amounts to establishing a decay estimate for the L^∞ norm of the solutions of these equations at time t in terms of some negative power of t and the L^1 norm of the data. In many cases, the main step in the proof of this decay in time relies on the application of a stationary phase theorem on an (approximate) representation of the solution. Combined with an abstract functional analysis argument known as the TT^*-argument, dispersion phenomena yield a range of estimates involving spacetime Lebesgue norms. Those inequalities, called Strichartz estimates, have proved to be powerful in the study of nonlinear equations (for instance, one can consult [Bahouri et al. 2011] and the references therein).

In the $\mathbb{R}^d$ framework, dispersive inequalities have a long history, beginning with [Brenner 1975; Pecher 1976; Segal 1976; Strichartz 1977]. They were subsequently developed by various authors, starting with [Ginibre and Velo 1995] (for a detailed bibliography, we refer to [Keel and Tao 1998; Tao 2006] and the references therein). Bahouri et al. [2000] generalize the dispersive estimates for the wave equation to the Heisenberg group $\mathbb{H}^d$ with an optimal rate of decay of order $|t|^{-1/2}$ (regardless of the dimension d) and prove that no dispersion occurs for the Schrödinger equation. Del Hierro [2005] proved optimal results for the time behavior of the Schrödinger and wave equations on H-type groups: if p is the dimension of the center of the H-type group, Del Hierro establishes sharp dispersive inequalities for the wave equation solution (with a decay rate of $|t|^{-p/2}$) as well as for the Schrödinger equation solution (with a $|t|^{-(p-1)/2}$

MSC2010: 35B40.
Keywords: step-2 stratified Lie groups, Schrödinger equation, dispersive estimates, sublaplacian.

decay). Compared with the $\mathbb{R}^d$ framework, there is an exchange in the rates of decay between the wave and the Schrödinger equations.

Strichartz estimates in other settings have been obtained in a number of works. One can first cite various results dealing with variable coefficient operators (see for instance [Kapitanski 1989; Smith 1998]) or studies concerning domains, such as [Burq et al. 2008; Ivanovici et al. 2014; Smith and Sogge 1995]. One can also refer to the result concerning the full laplacian on the Heisenberg group [Furioli et al. 2007], works in the framework of the real hyperbolic spaces [Anker and Pierfelice 2009; Banica 2007; Tataru 2001], or in the framework of compact and noncompact manifolds [Anton 2008; Banica and Duyckaerts 2007; Burq et al. 2004]; finally, one can mention the quasilinear framework studied in [Bahouri and Chemin 1999; 2003; Klainerman and Rodnianski 2005; Smith and Tataru 2005] and the references therein.

In this paper our goal is to establish optimal dispersive estimates for the solutions of the Schrödinger equation on step-2 stratified Lie groups. We shall emphasize in particular the key role played by the canonical skew-symmetric form in determining the rate of decay of the solutions. It turns out that the Schrödinger propagator on G behaves like a wave operator on a space of the same dimension as the center of G, and like a Schrödinger operator on a space of the same dimension as the radical of the canonical skew-symmetric form associated with the dual of the center. This unusual behavior of the Schrödinger propagator in the case of Lie algebras whose canonical skew-symmetric form is degenerate (known as Lie algebras which are not MW; see [Moore and Wolf 1973; Müller and Ricci 1996], for example) makes the analysis of the explicit representations of the solutions tricky and gives rise to uncommon dispersive estimates. It will also appear from our analysis that the optimal rate of decay is not always in accordance with the dimension of the center: we shall exhibit examples of step-2 stratified Lie groups with center of any dimension for which no dispersion occurs for the Schrödinger equation. We shall actually highlight that the optimal rate of decay in the dispersive estimates for solutions to the Schrödinger equation is, rather, related to the properties of the canonical skew-symmetric form.

1B. ***Stratified Lie groups.*** Let us recall here some basic facts about stratified Lie groups (see [Corwin and Greenleaf 1990; Folland 1989; Folland and Stein 1982; Stein and Weiss 1971] and the references therein for further details). A connected, simply connected, nilpotent Lie group G is called stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed to be real-valued and of finite dimension n) is endowed with a vector space decomposition

$$\mathfrak{g} = \bigoplus_{1\le k\le\infty} \mathfrak{g}_k,$$

where all but finitely many of the $\mathfrak{g}_k$ are $\{0\}$, such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$. If there are p nonzero $\mathfrak{g}_k$ then the group is said to be of step p. Via the exponential map

$$\exp : \mathfrak{g} \to G,$$

which is in that case a diffeomorphism from $\mathfrak{g}$ to G, one identifies G and $\mathfrak{g}$. It turns out that, under this identification, the group law on G (which is generally not commutative) provided by the Campbell–Baker–Hausdorff formula, $(x, y) \mapsto x \cdot y$, is a polynomial map. In the following we shall denote by $\mathfrak{z}$ the center of G, which is simply the last nonzero $\mathfrak{g}_k$, and write

$$G = \mathfrak{v} \oplus \mathfrak{z}, \tag{1-1}$$

where $\mathfrak{v}$ is any subspace of G complementary to $\mathfrak{z}$.

The group G is endowed with a smooth left-invariant measure $\mu(x)$, the Haar measure, induced by the Lebesgue measure on $\mathfrak{g}$, which satisfies the fundamental translation invariance property

$$\int_G f(y)\, d\mu(y) = \int_G f(x \cdot y)\, d\mu(y) \quad \text{for all } \ f \in L^1(G, d\mu),\ x \in G.$$

Note that the convolution of two functions f and g on G is given by

$$f * g(x) := \int_G f(x \cdot y^{-1}) g(y)\, d\mu(y) = \int_G f(y) g(y^{-1} \cdot x)\, d\mu(y) \tag{1-2}$$

and as in the euclidean case we define Lebesgue spaces by

$$\|f\|_{L^p(G)} := \left(\int_G |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}$$

for $p \in [1, \infty[$ with the standard modification when $p = \infty$.

Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can decompose X as $X = \sum X_k$ with $X_k \in \mathfrak{g}_k$, and then

$$\delta_t X := \sum t^k X_k.$$

This allows us to define the dilation δ_t on the Lie group G via the identification by the exponential map:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\delta_t"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\exp \circ \delta_t \circ \exp^{-1}"'] & G \end{tikzcd}$$

To avoid heaviness, we shall still denote by δ_t the map $\exp \circ \delta_t \circ \exp^{-1}$.

Observe that the action of the left-invariant vector fields X_k for X_k belonging to $\mathfrak{g}_k$ changes the homogeneity in the following way:

$$X_k(f \circ \delta_t) = t^k X_k(f) \circ \delta_t,$$

where by definition $X_k(f)(y) := df(y \cdot \exp(sX_k))/ds|_{s=0}$ and the Jacobian of the dilation δ_t is t^Q, where $Q := \sum_{1 \le k \le \infty} k \dim \mathfrak{g}_k$ is called the homogeneous dimension of G:

$$\int_G f(\delta_t y)\, d\mu(y) = t^{-Q} \int_G f(y)\, d\mu(y). \tag{1-3}$$

Let us also point out that there is a natural norm ρ on G, which is homogeneous in the sense that it respects dilations: $x \mapsto \rho(x)$ for $x \in G$ satisfies

$$\rho(x^{-1}) = \rho(x), \quad \rho(\delta_t x) = t\rho(x) \quad \text{for all } \ x \in G; \quad \rho(x) = 0 \iff x = 0.$$

We can define the Schwartz space $\mathcal{S}(G)$ as the set of smooth functions on G such that $x \mapsto \rho^p(x)\mathcal{X}^\alpha f(x)$ belongs to $L^\infty(G)$ for all α in $\mathbb{N}^d$ and p in $\mathbb{N}$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left-invariant vector fields. The Schwartz space $\mathcal{S}(G)$ has properties very similar to those of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, particularly density in Lebesgue spaces.

1C. ***The Fourier transform.*** The group G being noncommutative, its Fourier transform is defined by means of irreducible unitary representations. We devote this section to the introduction of the basic concepts that will be needed in the sequel. From now on, we assume that G is a step-2 stratified Lie group, meaning $\mathfrak{z} = \mathfrak{g}_2$, and we let $\mathfrak{v} = \mathfrak{g}_1$ in (1-1). We choose a scalar product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal.

1C1. *Irreducible unitary representations.* Let us fix some notation, borrowed from [Ciatti et al. 2005] (see also [Corwin and Greenleaf 1990] or [Müller and Ricci 1996]). For any $\lambda \in \mathfrak{z}^\star$ (the dual of the center $\mathfrak{z}$) we define a skew-symmetric bilinear form on $\mathfrak{v}$ by

$$B(\lambda)(U, V) := \lambda([U, V]) \quad \text{for all } U, V \in \mathfrak{v}. \tag{1-4}$$

One can find a Zariski-open subset Λ of $\mathfrak{z}^\star$ such that the number of distinct eigenvalues of $B(\lambda)$ is maximum. We denote by k the dimension of the radical $\mathfrak{r}_\lambda$ of $B(\lambda)$. Since $B(\lambda)$ is skew-symmetric, the dimension of the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{v}$ is an even number, which we shall denote by $2d$. Therefore, there exists an orthonormal basis

$$(P_1(\lambda), \dots, P_d(\lambda), Q_1(\lambda), \dots, Q_d(\lambda), R_1(\lambda), \dots, R_k(\lambda))$$

such that the matrix of $B(\lambda)$ takes the form

$$\begin{pmatrix} 0 & \cdots & 0 & \eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \eta_d(\lambda) & 0 & \cdots & 0 \\ -\eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_d(\lambda) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \dots, \lambda_p)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{v}$ as

$$\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda + \mathfrak{r}_\lambda$$

with

$$\mathfrak{p}_\lambda := \operatorname{Span}(P_1(\lambda), \dots, P_d(\lambda)), \quad \mathfrak{q}_\lambda := \operatorname{Span}(Q_1(\lambda), \dots, Q_d(\lambda)), \quad \mathfrak{r}_\lambda := \operatorname{Span}(R_1(\lambda), \dots, R_k(\lambda)),$$

any element $V \in \mathfrak{v}$ will be written in the following as $P + Q + R$ with $P \in \mathfrak{p}_\lambda$, $Q \in \mathfrak{q}_\lambda$ and $R \in \mathfrak{r}_\lambda$. We then introduce irreducible unitary representations of G on $L^2(\mathfrak{p}_\lambda)$

$$u_X^{\lambda,\nu}\phi(\xi) := \mathrm{e}^{-i\nu(R)-i\lambda(Z+[\xi+P/2,Q])}\phi(P+\xi), \quad \lambda \in \mathfrak{z}^*,\ \nu \in \mathfrak{r}_\lambda^*, \tag{1-5}$$

for any $x = \exp(X) \in G$ with $X = X(\lambda, x) := (P(\lambda, x), Q(\lambda, x), R(\lambda, x), Z(x))$ and $\phi \in L^2(\mathfrak{p}_\lambda)$. In order to shorten notation, we shall omit the dependence on (λ, x) whenever there is no risk of confusion.

1C2. *The Fourier transform.* In contrast with the euclidean case, the Fourier transform is defined on the bundle $\mathfrak{r}(\Lambda)$ above Λ whose fiber above $\lambda \in \Lambda$ is $\mathfrak{r}_\lambda^* \sim \mathbb{R}^k$. It is valued in the space of bounded operators on $L^2(\mathfrak{p}_\lambda)$. More precisely, the Fourier transform of a function f in $L^1(G)$ is defined as follows: for any $(\lambda, \nu) \in \mathfrak{r}(\Lambda)$,

$$\mathcal{F}(f)(\lambda, \nu) := \int_G f(x) u^{\lambda,\nu}_{X(\lambda,x)}\, d\mu(x).$$

Note that, for any (λ, ν), the map $u^{\lambda,\nu}_{X(\lambda,x)}$ is a group homomorphism from G into the group $U(L^2(\mathfrak{p}_\lambda))$ of unitary operators of $L^2(\mathfrak{p}_\lambda)$, so functions f of $L^1(G)$ have a Fourier transform $(\mathcal{F}(f)(\lambda, \nu))_{\lambda,\nu}$ that is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$. One may check that the Fourier transform exchanges convolution, whose definition is recalled in (1-2), and composition:

$$\mathcal{F}(f \star g)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu) \circ \mathcal{F}(g)(\lambda, \nu). \tag{1-6}$$

Further, the Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of two-parameter families $A = \{A(\lambda, \nu)\}_{(\lambda,\nu)\in\mathfrak{r}(\Lambda)}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert–Schmidt for almost every $(\lambda, \nu) \in \mathfrak{r}(\Lambda)$, with $\|A(\lambda, \nu)\|_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))}$ measurable and with norm

$$\|A\| := \left(\int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|A(\lambda, \nu)\|^2_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \right)^{\frac{1}{2}} < \infty,$$

where $|\mathrm{Pf}(\lambda)| := \prod_{j=1}^d \eta_j(\lambda)$ is the Pfaffian of $B(\lambda)$. We have the following Fourier–Plancherel formula: there exists a constant $\kappa > 0$ such that

$$\int_G |f(x)|^2\, dx = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda, \nu)\|^2_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda. \tag{1-7}$$

Finally, we have an inversion formula as stated in the following proposition, proved in the Appendix.

Proposition 1.1. *There exists $\kappa > 0$ such that, for $f \in \mathcal{S}(G)$ and almost all $x \in G$, the following inversion formula holds*:

$$f(x) = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \mathrm{tr}\big((u^{\lambda,\nu}_{X(\lambda,x)})^\star \mathcal{F} f(\lambda, \nu)\big) |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda. \tag{1-8}$$

1C3. *The sublaplacian.* Let $(V_1, \dots, V_m)$ be an orthonormal basis of $\mathfrak{g}_1$. The sublaplacian on G is defined by

$$\Delta_G := \sum_{j=1}^m V_j^2. \tag{1-9}$$

It is a self-adjoint operator which is independent of the orthonormal basis $(V_1, \dots, V_m)$, and homogeneous of degree 2 with respect to the dilations in the sense that

$$\delta_t^{-1} \Delta_G \delta_t = t^2 \Delta_G.$$

To write its expression in Fourier space, we consider the basis of Hermite functions $(h_n)_{n \in \mathbb{N}}$, normalized in $L^2(\mathbb{R})$ and satisfying, for all real numbers ξ,

$$h_n''(\xi) - \xi^2 h_n(\xi) = -(2n+1) h_n(\xi).$$

Then, for any multi-index $\alpha \in \mathbb{N}^d$, we define the functions $h_{\alpha,\eta(\lambda)}$ by

$$\begin{aligned} h_{\alpha,\eta(\lambda)}(\Xi) &:= \prod_{j=1}^{d} h_{\alpha_j,\eta_j(\lambda)}(\xi_j) \quad \text{for all } \Xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \\ h_{n,\beta}(\xi) &:= \beta^{1/4} h_n(\beta^{1/2}\xi) \qquad \text{for all } (n, \beta) \in \mathbb{N} \times \mathbb{R}^+,\ \xi \in \mathbb{R}. \end{aligned} \tag{1-10}$$

The sublaplacian Δ_G defined in (1-9) satisfies

$$\mathcal{F}(-\Delta_G f)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu)(H(\lambda) + |\nu|^2), \tag{1-11}$$

where $|\nu|$ denotes the euclidean norm of the vector ν in $\mathbb{R}^k$ and $H(\lambda)$ is the diagonal operator defined on $L^2(\mathbb{R}^d)$ by

$$H(\lambda) h_{\alpha,\eta(\lambda)} = \sum_{j=1}^{d} (2\alpha_j + 1)\eta_j(\lambda) h_{\alpha,\eta(\lambda)}.$$

In the following we shall denote the "frequencies" associated with $P_j^2(\lambda) + Q_j^2(\lambda)$ by

$$\zeta_j(\alpha, \lambda) := (2\alpha_j + 1)\eta_j(\lambda), \quad (\alpha, \lambda) \in \mathbb{N}^d \times \Lambda, \tag{1-12}$$

and those associated with $H(\lambda)$ by

$$\zeta(\alpha, \lambda) := \sum_{j=1}^{d} \zeta_j(\alpha, \lambda), \quad (\alpha, \lambda) \in \mathbb{N}^d \times \Lambda. \tag{1-13}$$

Note that Δ_G is directly related to the harmonic oscillator via $H(\lambda)$ since eigenfunctions associated with the eigenvalues $\zeta(\alpha, \lambda)$ are the products of 1-dimensional Hermite functions. Also observe that $\zeta(\alpha, \lambda)$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \dots, \lambda_p)$. Moreover, $\zeta(\alpha, \lambda) = 0$ if and only if $B(\lambda) = 0$, or equivalently, by (1-4), $\lambda = 0$.

Notice also that there is a difference in homogeneity in the variables λ and ν. Namely, in the variable ν, the sublaplacian acts as in the euclidean case (homogeneity 2) while in λ, it has the homogeneity 1 of a wave operator.

Finally, for any smooth function Φ, we define the operator $\Phi(-\Delta_G)$ by the formula

$$\mathcal{F}(\Phi(-\Delta_G) f)(\lambda, \nu) := \Phi(H(\lambda) + |\nu|^2)\mathcal{F}(f)(\lambda, \nu), \tag{1-14}$$

which also reads

$$\mathcal{F}(\Phi(-\Delta_G)f)(\lambda,\nu)h_{\alpha,\eta(\lambda)} := \Phi(|\nu|^2+\zeta(\alpha,\lambda))\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}$$

for all $(\lambda,\nu)\in\mathfrak{r}(\Lambda)$ and $\alpha\in\mathbb{N}^d$.

1C4. *Strict spectral localization.* Let us introduce the following notion of spectral localization, which we shall call strict spectral localization and which will be very useful in the following.

Definition 1.2. A function f belonging to $L^1(G)$ is said to be strictly spectrally localized in a set $\mathcal{C}\subset\mathbb{R}$ if there exists a smooth function θ, compactly supported in $\mathcal{C}$, such that, for all $1\le j\le d$,

$$\mathcal{F}(f)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\theta((P_j^2+Q_j^2)(\lambda)) \quad \text{for all } (\lambda,\nu)\in\mathfrak{r}(\Lambda). \tag{1-15}$$

Remark 1.3. One could expect the notion of spectral localization to relate to the laplacian instead of each individual vector field $P_j^2+Q_j^2$, assuming rather the less restrictive condition

$$\mathcal{F}(f)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\theta(H(\lambda)) \quad \text{for all } (\lambda,\nu)\in\mathfrak{r}(\Lambda).$$

The choice we make here is more restrictive due to the anisotropic context (namely the fact that $\eta_j(\lambda)$ depends on j); in the case of the Heisenberg group or, more generally, H-type groups, the notion of "strict spectral localization" in a ring $\mathcal{C}$ of $\mathbb{R}^p$ actually coincides with the more usual definition of "spectral localization" since, as recalled in the next subsection, $\eta_j(\lambda)=4|\lambda|$ (for a complete presentation and more details on spectrally localized functions, we refer the reader to [Bahouri and Gallagher 2001; Bahouri et al. 2012a; 2012b]). Assumption (1-15) guarantees a lower bound, which roughly states that for $\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\lambda}$ to be nonzero we must have

$$(2\alpha_j+1)\eta_j(\lambda)\ge c>0 \quad \text{for all } j\in\{1,\dots,d\}, \tag{1-16}$$

hence each η_j must be bounded away from zero, rather than the sum over j. These lower bounds are important ingredients of the proof (see Section 3C).

1D. ***Examples.*** Let us give a few examples of well-known stratified Lie groups with a step-2 stratification. Note that nilpotent Lie groups which are connected, simply connected and whose Lie algebra admits a step-2 stratification are called Carnot groups.

1D1. *The Heisenberg group.* The Heisenberg group $\mathbb{H}^d$ is defined as the space $\mathbb{R}^{2d+1}$ whose elements can be written $w=(x,y,s)$ with $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, endowed with the product law

$$(x,y,s)\cdot(x',y',s')=(x+x',y+y',s+s'-2(x\mid y')+2(y\mid x')),$$

where $(\cdot\mid\cdot)$ denotes the euclidean scalar product on $\mathbb{R}^d$. In that case the center consists of the points of the form $(0,0,s)$ and is of dimension 1. The Lie algebra of left-invariant vector fields is generated by

$$X_j:=\partial_{x_j}+2y_j\partial_s,\quad Y_j:=\partial_{y_j}-2x_j\partial_s \quad \text{for } 1\le j\le d;\quad S:=\partial_s=\tfrac{1}{4}[Y_j,X_j].$$

The canonical skew-symmetric form $B(\lambda)(U,V)$ defined in (1-4) associated with the frequencies $\lambda\in\mathbb{R}^*$ is proportional to λ, since $[U,V]$ is proportional to ∂_s. Its radical reduces to $\{0\}$ with $\Lambda=\mathbb{R}^*$, and $|\eta_j(\lambda)| = 4|\lambda|$ for all $j \in \{1, \ldots, d\}$. Note in particular that strict spectral localization and spectral localization are equivalent.

1D2. *H-type groups.* These groups are canonically isomorphic to $\mathbb{R}^{m+p}$ and are a multidimensional version of the Heisenberg group. The group law is of the form

$$(x^{(1)}, x^{(2)})\cdot(y^{(1)}, y^{(2)}) := \begin{pmatrix} x_j^{(1)} + y_j^{(1)}, & j = 1, \ldots, m \\ x_k^{(2)} + y_k^{(2)} + \frac{1}{2}\langle x^{(1)}, U^{(k)} y^{(1)}\rangle, & k = 1, \ldots, p \end{pmatrix},$$

where $U^{(j)}$ are $m \times m$ linearly independent, orthogonal, skew-symmetric matrices satisfying the property

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$$

for every $r, s \in \{1, \ldots, p\}$ with $r \neq s$. In that case the center is of dimension p and may be identified with $\mathbb{R}^p$, and the radical of the canonical skew-symmetric form associated with the frequencies λ is again $\{0\}$. For example, the Iwasawa subgroup of semisimple Lie groups of split rank 1 (see [Korányi 1985]) is of this type. On H-type groups, m is an even number, which we denote by $2l$, and the Lie algebra of left-invariant vector fields is spanned by the following vector fields, where we have written $z = (x, y)$ in $\mathbb{R}^l \times \mathbb{R}^l$: for $j = 1, \ldots, l$ and $k = 1, \ldots, p$,

$$X_j := \partial_{x_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2l} z_l U_{l,j}^{(k)} \partial_{s_k}, \quad Y_j := \partial_{y_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2l} z_l U_{l,j+l}^{(k)} \partial_{s_k} \quad \text{and} \quad \partial_{s_k}.$$

In that case, we have $\Lambda = \mathbb{R}^p \setminus \{0\}$ with $\eta_j(\lambda) = \sqrt{\lambda_1^2 + \cdots + \lambda_p^2}$ for all $j \in \{1, \ldots, l\}$ (here, again, strict spectral localization and spectral localization are equivalent).

1D3. *Diamond groups.* These groups, which occur in crystal theory (for more details, consult [Ludwig 1995; Poguntke 1999]), are of the type $\Sigma \ltimes \mathbb{H}^d$, where Σ is a connected Lie group acting smoothly on $\mathbb{H}^d$. One can find examples for which the radical of the canonical skew-symmetric is of any dimension k, $0 \le k \le d$. For example, one can take for Σ the k-dimensional torus, acting on $\mathbb{H}^d$ by

$$\theta(w) := (\theta \cdot z, s) := (\mathrm{e}^{i\theta_1} z_1, \ldots, \mathrm{e}^{i\theta_k} z_k, z_{k+1}, \ldots, z_d, s), \quad w = (z, s),$$

where the element $\theta = (\theta_1, \ldots, \theta_k)$ corresponds to the element $(\mathrm{e}^{i\theta_1}, \ldots, \mathrm{e}^{i\theta_k})$ of $\mathbb{T}^k$. Then the product law on $G = \mathbb{T}^k \ltimes \mathbb{H}^d$ is given by

$$(\theta, w) \cdot (\theta', w') = \big(\theta + \theta', w.(\theta(w'))\big),$$

where $w.(\theta(w'))$ denotes the Heisenberg product of w by $\theta(w')$. As a consequence, the center of G is of dimension 1, since it consists of the points of the form $(0, 0, s)$ for $s \in \mathbb{R}$. Let us choose for simplicity $k = d = 1$; the algebra of left-invariant vector fields is generated by the vector fields ∂_θ, ∂_s, $\Gamma_{\theta,x}$ and $\Gamma_{\theta,y}$, where

$$\Gamma_{\theta,x} = \cos\theta\,\partial_x + \sin\theta\,\partial_y + 2(y\cos\theta - x\sin\theta)\partial_s,$$
$$\Gamma_{\theta,y} = -\sin\theta\,\partial_x + \cos\theta\,\partial_y - 2(y\sin\theta + x\cos\theta)\partial_s.$$

It is not difficult to check that the radical of $B(\lambda)$ is of dimension 1. In the general case, where $k \leq d$, the algebra of left-invariant vector fields is generated by the vector fields ∂_s, the $2(d-k)$ vectors

$$X_l = \partial_{x_l} + 2y_l \partial_s \quad \text{and} \quad Y_l = \partial_{y_l} - 2x_l \partial_s,$$

and the $3k$ vectors defined for $1 \leq j \leq k$ by ∂_{θ_j}, Γ_{θ_j, x_j} and Γ_{θ_j, y_j}, where

$$\begin{aligned}\Gamma_{\theta_j, x_j} &= \cos\theta_j \partial_{x_j} + \sin\theta_j \partial_{y_j} + 2(y_j \cos\theta_j - x_j \sin\theta_j)\partial_s,\\ \Gamma_{\theta_j, y_j} &= -\sin\theta_j \partial_{x_j} + \cos\theta_j \partial_{y_j} - 2(y_j \sin\theta_j + x_j \cos\theta_j)\partial_s,\end{aligned}$$

and this provides an example with a radical of dimension k.

1D4. *The tensor product of Heisenberg groups.* Consider $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$, the set of elements (w_1, w_2) in $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ that can be written as $(w_1, w_2) = (x_1, y_1, s_1, x_2, y_2, s_2)$ in $\mathbb{R}^{2d_1+1} \times \mathbb{R}^{2d_2+1}$, equipped with the product law

$$(w_1, w_2) \cdot (w_1', w_2') = (w_1 \cdot w_1', w_2 \cdot w_2'),$$

where $w_1 \cdot w_1'$ and $w_2 \cdot w_2'$ denote the product in $\mathbb{H}^{d_1}$ and $\mathbb{H}^{d_2}$, respectively. Clearly $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ is a step-2 stratified Lie group with center of dimension 2 and radical index null. Moreover, for $\lambda = (\lambda_1, \lambda_2)$ in the dual of the center, the canonical skew bilinear form $B(\lambda)$ has radical $\{0\}$ with $\Lambda = \mathbb{R}^* \times \mathbb{R}^*$, and one has $\eta_1(\lambda) = 4|\lambda_1|$ and $\eta_2(\lambda) = 4|\lambda_2|$. In that case, strict spectral localization is a more restrictive condition than spectral localization. Indeed, if f is spectrally localized, one has $\lambda_1 \neq 0$ *or* $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$, while one has $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$ if f is strictly spectrally localized.

1D5. *Tensor product of H-type groups.* The group $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$ is easily verified to be a step-2 stratified Lie group with center of dimension $p_1 + p_2$, radical index null and a skew bilinear form $B(\lambda)$ defined on $\mathbb{R}^{m_1+m_2}$ with $m_1 = 2l_1$ and $m_2 = 2l_2$. The Zariski-open set associated with B is given by $\Lambda = (\mathbb{R}^{p_1} \setminus \{0\}) \times (\mathbb{R}^{p_2} \setminus \{0\})$ and, for $\lambda = (\lambda_1, \ldots, \lambda_{p_1+p_2})$, we have

$$\begin{aligned}\eta_j(\lambda) &= \sqrt{\lambda_1^2 + \cdots + \lambda_{p_1}^2} && \text{for all } j \in \{1, \ldots, l_1\},\\ \eta_j(\lambda) &= \sqrt{\lambda_{p_1+1}^2 + \cdots + \lambda_{p_1+p_2}^2} && \text{for all } j \in \{l_1+1, \ldots, l_1+l_2\}.\end{aligned} \tag{1-17}$$

1E. ***Main results.*** The purpose of this paper is to establish optimal dispersive inequalities for the linear Schrödinger equation on step-2 stratified Lie groups associated with the sublaplacian. In view of (1-11) and the fact that the "frequencies" $\zeta(\alpha, \lambda)$ associated with $H(\lambda)$ given by (1-13) are homogeneous of degree 1 in λ, the Schrödinger operator on G behaves like a wave operator on a space of the same dimension p as the center of G, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form. By comparison with the classical dispersive estimates, the expected result would be a dispersion phenomenon with an optimal rate of decay of order $|t|^{-(k+p-1)/2}$. However, as will be seen through various examples, this anticipated rate is not always achieved. To reach this maximum rate of dispersion, we require a condition on $\zeta(\alpha, \lambda)$.

Assumption 1.4. *For each multi-index α in $\mathbb{N}^d$, the Hessian matrix of the map $\lambda \mapsto \zeta(\alpha, \lambda)$ satisfies*

$$\operatorname{rank} D^2_\lambda \zeta(\alpha, \lambda) = p - 1,$$

where p is the dimension of the center of G.

Remark 1.5. As was observed in Section 1C3, $\zeta(\alpha, \lambda)$ is a smooth function, homogeneous of degree 1 on Λ. By homogeneity arguments, one therefore has $D^2_\lambda \zeta(\alpha, \lambda)\lambda = 0$. It follows that

$$\operatorname{rank} D^2_\lambda \zeta(\alpha, \lambda) \leq p - 1$$

always; hence, Assumption 1.4 may be understood as a maximal rank property.

Let us now present the dispersive inequality for the Schrödinger equation. Recall that the linear Schrödinger equation is as follows on G:

$$\begin{cases} (i\partial_t - \Delta_G) f = 0, \\ f|_{t=0} = f_0, \end{cases} \tag{1-18}$$

where the function f with complex values depends on $(t, x) \in \mathbb{R} \times G$.

Theorem 1. *Let G be a step-2 stratified Lie group with center of dimension p with $1 \leq p < n$ and radical index k. Assume that Assumption 1.4 holds. A constant C exists such that, if f_0 belongs to $L^1(G)$ and is strictly spectrally localized in a ring of $\mathbb{R}$ in the sense of Definition 1.2, then the associate solution f to the Schrödinger equation* (1-18) *satisfies*

$$\|f(t, \cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{k/2}(1 + |t|^{(p-1)/2})} \|f_0\|_{L^1(G)} \tag{1-19}$$

for all $t \neq 0$ and the result is sharp in time.

The fact that a spectral localization is required in order to obtain the dispersive estimates is not surprising. Indeed, recall that in the $\mathbb{R}^d$ case, for instance, the dispersive estimate for the Schrödinger equation derives immediately (without any spectral localization assumption) from the fact that the solution $u(t)$ to the free Schrödinger equation on $\mathbb{R}^d$ with Cauchy data u_0 is, for $t \neq 0$,

$$u(t, \cdot) = u_0 * \frac{1}{(-2i\pi t)^{d/2}} \mathrm{e}^{-i|\cdot|^2/(4t)},$$

where $*$ denotes the convolution product in $\mathbb{R}^d$ (for a detailed proof of this fact, see for instance [Bahouri et al. 2011, Proposition 8.3]). However, proving dispersive estimates for the wave equation in $\mathbb{R}^d$ requires more elaborate techniques (including oscillating integrals), which involve an assumption of spectral localization in a ring. In the case of a step-2 stratified Lie group G, the main difficulty arises from the complexity of the expression of a Schrödinger propagator that mixes a wave operator behavior with that of a Schrödinger operator. This explains, on the one hand, the decay rate in the estimate (1-19) and on the other hand the hypothesis of strict spectral localization.

Let us now discuss Assumption 1.4. As mentioned above, there is no dispersion phenomenon for the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$ (see [Bahouri et al. 2000]). Actually the same holds for the tensor product of Heisenberg groups $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ whose center is of dimension $p = 2$ and radical index null, and more generally for the case of step-2 stratified Lie groups, decomposable on nontrivial step-2 stratified Lie groups; indeed, we derive from Theorem 1 the following corollary:

Corollary 1.6. *Let $G = \bigotimes_{1\le m\le r} G_m$ be a decomposable, step-2 stratified Lie group where the groups G_m are nontrivial step-2 stratified Lie groups satisfying Assumption 1.4, of radical index k_m and with centers of dimension p_m. Then the dispersive estimate holds with rate $|t|^{-q}$:*

$$q := \frac{1}{2}\sum_{1\le m\le r}(k_m + p_m - 1) = \frac{1}{2}(k + p - r),$$

where p is the dimension of the center of G and k its radical index. Further, this rate is optimal.

Corollary 1.6 is a direct consequence of Theorem 1 and the simple observation that $\Delta_G = \bigotimes_{1\le m\le r}\Delta_{G_m}$. This result applies, for example, to the tensor product of Heisenberg groups, for which there is no dispersion, and to the tensor product of H-type groups $\mathbb{R}^{m_1+p_1}\otimes\mathbb{R}^{m_2+p_2}$, for which the dispersion rate is $t^{-(p_1+p_2-2)/2}$ (see [Del Hierro 2005]). Corollary 1.6 therefore shows that it can happen that the "best" rate of decay $|t|^{-(k+p-1)/2}$ is not reached, in particular for decomposable Lie groups. This suggests that Assumption 1.4 could be related with decomposability.

More generally, a large class of groups which do not satisfy the Assumption 1.4 is given by step-2 stratified Lie groups G for which $\zeta(0,\lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Of course, the Heisenberg group and any tensor product of Heisenberg groups is of that type. We then have the following result, which illustrates that there exist examples of groups without any dispersion and which do not satisfy Assumption 1.4.

Proposition 1.7. *Consider a step-2 stratified Lie group G whose radical index is null and for which $\zeta(0,\lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Then there exists $f_0 \in \mathcal{S}(G)$, $x \in G$ and $c_0 > 0$ such that*

$$|\mathrm{e}^{-it\Delta_G} f_0(x)| \ge c_0 \quad \textit{for all } t \in \mathbb{R}^+.$$

Finally, we point out that the dispersive estimate given in Theorem 1 can be regarded as a first step towards spacetime estimates of Strichartz type. However, due to the strict spectral localization assumption, the Besov spaces that should appear in the study (after summation over frequency bands) are naturally anisotropic; thus, proving such estimates is likely to be very technical, and is postponed to future works.

1F. ***Strategy of the proof of Theorem 1.*** In the statement of Theorem 1, there are two different results: the dispersive estimate in itself on the one hand, and its optimality on the other. Our strategy of proof is closely related to the method developed in [Bahouri et al. 2000; Del Hierro 2005], with additional, nonnegligible technicalities.

In the situation of [Bahouri et al. 2000], where the Heisenberg group $\mathbb{H}^d$ is considered, the authors prove that there is no dispersion by exhibiting explicitly Cauchy data f_0 for which the solution $f(t,\cdot)$ to the Schrödinger equation (1-18) satisfies

$$\|f(t,\cdot)\|_{L^q(\mathbb{H}^d)} = \|f_0\|_{L^q(\mathbb{H}^d)} \quad \text{for all } q \in [1,\infty]. \tag{1-20}$$

More precisely, they take advantage of the fact that the Kohn laplacian $\Delta_{\mathbb{H}^d}$ can be recast in the form

$$\Delta_{\mathbb{H}^d} = 4\sum_{j=1}^{d}(Z_j\overline{Z}_j + i\partial_s), \tag{1-21}$$

where $\{Z_1, \overline{Z}_1, \dots, Z_d, \overline{Z}_d, \partial_s\}$ is the canonical basis of the Lie algebra of left-invariant vector fields on $\mathbb{H}^d$ (see [Bahouri et al. 2012a] and the references therein for more details). This implies that, for a nonzero function f_0 belonging to $\mathrm{Ker}\big(\sum_{j=1}^{d} Z_j\overline{Z}_j\big)$, the solution of the Schrödinger equation on the Heisenberg group $f(t) = \mathrm{e}^{-it\Delta_{\mathbb{H}^d}} f_0$ actually solves the transport equation

$$f(z,s,t) = \mathrm{e}^{4dt\partial_s} f_0(z,s) = f_0(z, s+4dt)$$

and hence satisfies (1-20). The arguments used in [Del Hierro 2005] for general H-type groups are similar to the ones developed in [Bahouri et al. 2000]: the dispersive estimate is obtained using an explicit formula for the solution, coming from Fourier analysis, combined with a stationary phase theorem. The Cauchy data used to prove the optimality is again in the kernel of an adequate operator, by a decomposition similar to (1-21).

As in [Bahouri et al. 2000; Del Hierro 2005], the first step of the proof of Theorem 1 consists in writing an explicit formula for the solution of the equation by use of the Fourier transform. Let us point out that, in the setting of [Bahouri et al. 2000; Del Hierro 2005], irreducible representations are isotropic with respect to the dual of the center of the group; this isotropy allows us to reduce to a one-dimensional framework and deduce the dispersive effect from a careful use of a stationary phase argument of [Stein 1986]. As we have already seen in Section 1C1, the irreducible representations are no longer isotropic in the general case of stratified Lie groups, and thus we adopt a more technical approach, making use of Schrödinger representation and taking advantage of some properties of Hermite functions appearing in the explicit representation of the solutions derived by Fourier analysis (see Section 3C). The optimality of the inequality is obtained as in [Bahouri et al. 2000; Del Hierro 2005], by an adequate choice of the initial data.

1G. ***Organization of the paper.*** The article is organized as follows. In Section 2, we write an explicit formulation of the solutions of the Schrödinger equation. Then, Section 3 is devoted to the proof of Theorem 1, and in Section 4 we discuss the optimality of the result and prove Proposition 1.7.

Finally, we mention that the letter C will be used to denote a universal constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some constant C.

2. Explicit representation of the solutions

2A. ***The convolution kernel.*** Let f_0 belong to $\mathcal{S}(G)$ and let us consider $f(t,\cdot)$, the solution to the free Schrödinger equation (1-18). In view of (1-11), we have

$$\mathcal{F}(f(t,\cdot))(\lambda,\nu) = \mathcal{F}(f_0)(\lambda,\nu)\mathrm{e}^{it|\nu|^2+itH(\lambda)},$$

which implies easily (arguing as in the Appendix) that $f(t,\cdot)$ belongs to $\mathcal{S}(G)$. Assuming that f_0 is strictly spectrally localized in the sense of Definition 1.2, there exists a smooth function θ compactly supported in a ring $\mathscr{C}$ of $\mathbb{R}$ such that if we define

$$\Theta(\lambda) := \prod_{j=1}^{d} \theta((P_j^2 + Q_j^2)(\lambda))$$

then

$$\mathscr{F}(f(t,\cdot))(\lambda,\nu) = \mathscr{F}(f_0)(\lambda,\nu)\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}.$$

Therefore, by the inverse Fourier transform (1-8), we deduce that the function $f(t,\cdot)$ may be decomposed in the following way:

$$f(t,x) = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \mathrm{tr}\big((u^{\lambda,\nu}_{X(\lambda,x)})^\star \mathscr{F}(f_0)(\lambda,\nu)\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda. \tag{2-1}$$

We set, for $X \in \mathbb{R}^n$,

$$k_t(X) := \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \mathrm{tr}\big(u^{\lambda,\nu}_{X}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda. \tag{2-2}$$

The function k_t plays the role of a convolution kernel in the variables of the Lie algebra and we have the following result:

Proposition 2.1. *If the function k_t defined in* (2-2) *satisfies*

$$\|k_t\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{|t|^{k/2}(1+|t|^{(p-1)/2})} \quad \textit{for all } t\in\mathbb{R}, \tag{2-3}$$

then Theorem 1 holds.

Proof. We write, according to (2-1),

$$\begin{aligned} f(t,x) &= \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G} \mathrm{tr}\big((u^{\lambda,\nu}_{X(\lambda,x)})^* u^{\lambda,\nu}_{X(\lambda,y)}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big) f_0(y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y)\\ &= \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G} \mathrm{tr}\big(u^{\lambda,\nu}_{X(\lambda,y)}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big) f_0(x\cdot y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y). \end{aligned}$$

Note that we have used the property that the map $X \mapsto u^{\lambda,\nu}_X$ is a unitary representation, and the invariance of the Haar measure by translations.

Now we use the exponential law $y \mapsto Y = (P(\lambda,y), Q(\lambda,y), Z, R(\lambda,y))$ and the fact that $d\mu(y) = dY$, the Lebesgue measure; then we perform a linear orthonormal change of variables

$$(P(\lambda,y), Q(\lambda,y), R(\lambda,y)) \mapsto (\widetilde{P}, \widetilde{Q}, \widetilde{R}),$$

so that $d\mu(y) = dY = d\widetilde{P}\,d\widetilde{Q}\,dZ\,d\widetilde{R}$ and we write

$$\begin{aligned} f(t,x) = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{(\widetilde{P},\widetilde{Q},Z,\widetilde{R})\in\mathbb{R}^n} &\mathrm{tr}\big(u^{\lambda,\nu}_{(\widetilde{P},\widetilde{Q},Z,\widetilde{R})}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big)\\ &\times f_0(x\cdot\exp(\widetilde{P},\widetilde{Q},Z,\widetilde{R}))|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\widetilde{P}\,d\widetilde{Q}\,dZ\,d\widetilde{R}. \end{aligned}$$

Thanks to the Fubini theorem and Young inequalities, we can write (dropping the tilde on the variables)

$$\begin{aligned}|f(t,x)| &= \left|\int_{(P,Q,Z,R)\in\mathbb{R}^n} k_t(P,Q,Z,R)\, f_0(x\cdot\exp(P,Q,Z,R))\,dP\,dQ\,dR\,dZ\right|\\ &\le \|k_t\|_{L^\infty(G)}\left|\int_{(P,Q,Z,R)\in\mathbb{R}^n} f_0(x\cdot\exp(P,Q,Z,R))\,dP\,dQ\,dR\,dZ\right|\\ &\le \|k_t\|_{L^\infty(G)}\|f_0\|_{L^1(G)}.\end{aligned}$$

Proposition 2.1 is proved. □

In the next subsections, we make preliminary work by transforming the expression of k_t and reducing the proof to statements equivalent to (2-3).

2B. *Transformation of k_t: expression in terms of Hermite functions.* Decomposing the operator $H(\lambda)$ in the basis of Hermite functions, and recalling notation (1-12) replaces (2-2) with

$$k_t(X) = \kappa \sum_{\alpha\in\mathbb{N}^d}\int_\Lambda\int_{\nu\in\mathfrak{r}_\lambda^*} \mathrm{e}^{it|\nu|^2+it\zeta(\alpha,\lambda)}\prod_{j=1}^d \theta(\zeta_j(\alpha,\lambda))\big(u_X^{\lambda,\nu}h_{\alpha,\eta(\lambda)}\mid h_{\alpha,\eta(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda,\qquad X\in\mathbb{R}^n.$$

Using the explicit form of $u_X^{\lambda,\nu}$ recalled in (1-5), we find the following result:

Lemma 2.2. *There is a constant $\tilde\kappa$ and a smooth function F such that, with the above notation, we have, for $t\neq 0$,*

$$k_t(P,Q,tZ,R) = \frac{\tilde\kappa\mathrm{e}^{-i|R|^2/(4t)}}{t^{k/2}}\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda,$$

where the phase Φ_α is given by

$$\Phi_\alpha(Z,\lambda) := \zeta(\alpha,\lambda)-\lambda(Z)$$

with notation (1-13) *and the function G_α is given by the following formula, for all $(P,Q,\eta)\in\mathbb{R}^{3d}$:*

$$G_\alpha(P,Q,\eta) := \prod_{j=1}^d \theta((2\alpha_j+1)\eta_j)g_{\alpha_j}(\sqrt{\eta_j}P_j,\sqrt{\eta_j}Q_j),\tag{2-4}$$

while, for each (ξ_1,ξ_2,n) in $\mathbb{R}^2\times\mathbb{N}$, using the notation (1-10),

$$g_n(\xi_1,\xi_2) := \mathrm{e}^{-i\xi_1\xi_2/2}\int_{\mathbb{R}} \mathrm{e}^{-i\xi_2\xi}h_n(\xi_1+\xi)h_n(\xi)\,d\xi.\tag{2-5}$$

Notice that $(g_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{R}^2$ thanks to the Cauchy–Schwarz inequality and the fact that $\|h_n\|_{L^2(\mathbb{R})}=1$, and hence the same holds for $(G_\alpha)_{\alpha\in\mathbb{N}^d}$ (in $\mathbb{R}^{3d}$).

Proof. We begin by observing that, for $X=(P,Q,R,Z)$,

$$I := (u_X^{\lambda,\nu}h_{\alpha,\eta(\lambda)}\mid h_{\alpha,\eta(\lambda)}) = \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\int_{\mathbb{R}^d}\mathrm{e}^{-i\lambda([\xi+P/2,Q])}h_{\alpha,\eta(\lambda)}(P+\xi)h_{\alpha,\eta(\lambda)}(\xi)d\xi,$$

with, in view of (1-4),

$$\lambda\big([\xi+\tfrac{1}{2}P,\,Q]\big)=B(\lambda)\big(\xi+\tfrac{1}{2}P,\,Q\big)=\sum_{1\le j\le d}\eta_j(\lambda)Q_j\big(\xi_j+\tfrac{1}{2}P_j\big).$$

As a consequence,

$$I=\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\eta_j(\lambda)(\xi_j+P_j/2)Q_j}h_{\alpha_j,\eta_j(\lambda)}(P_j+\xi_j)h_{\alpha_j,\eta_j(\lambda)}(\xi_j)d\xi_j.$$

The change of variables $\tilde{\xi}_j=\sqrt{\eta_j(\lambda)}\xi_j$ gives, dropping the tilde for simplicity,

$$I=\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\sqrt{\eta_j(\lambda)}Q_j(\xi_j+\sqrt{\eta_j(\lambda)}P_j/2)}h_{\alpha_j}(\xi_j+\sqrt{\eta_j(\lambda)}P_j)h_{\alpha_j}(\xi_j)\,d\xi_j,$$

which implies that

$$k_t(P,Q,tZ,R)=\kappa\sum_{\alpha\in\mathbb{N}^d}\int_{\mathfrak{r}(\Lambda)}\mathrm{e}^{-it\lambda(Z)-i\nu(R)}\mathrm{e}^{it\zeta(\alpha,\lambda)+it|\nu|^2}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda.$$

It is well known (see for instance Proposition 1.28 in [Bahouri et al. 2011]) that, for $t\neq 0$,

$$\int_{\mathbb{R}^k}\mathrm{e}^{-i(\nu|R)+it|\nu|^2}\,d\nu=\Big(\frac{i\pi}{t}\Big)^{\frac{k}{2}}\mathrm{e}^{-i|R|^2/(4t)},\tag{2-6}$$

where $(\cdot\,|\,\cdot)$ denotes the euclidean scalar product on $\mathbb{R}^k$. This implies that, for $t\neq 0$,

$$|k_t(P,Q,tZ,R)|\lesssim\frac{1}{|t|^{k/2}}\bigg|\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda\bigg|,$$

with F the Jacobian of the change of variables $f:\mathfrak{r}^*_\lambda\to\mathbb{R}^k$, which is a smooth function. Lemma 2.2 is proved. □

2C. ***Transformation of the kernel k_t: change of variable.*** We are then reduced to establishing that the kernel $\tilde{k}_t(P,Q,tZ)$, defined by

$$\tilde{k}_t(P,Q,tZ):=\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda,$$

satisfies

$$\|\tilde{k}_t\|_{L^\infty(G)}\le\frac{C}{1+|t|^{(p-1)/2}}\quad\text{for all }\ t\in\mathbb{R}.\tag{2-7}$$

To this end, let us define $m:=|\alpha|=\sum_{j=1}^d\alpha_j$ and, when $m\neq 0$, let us set $\gamma:=m\lambda\in\mathbb{R}^p$. By construction of $\eta(\lambda)$ (which is homogeneous of degree 1), we have

$$\eta(\lambda)=\tilde{\eta}_m(\gamma):=\frac{1}{m}\eta(\gamma)\quad\text{for all }\ m\neq 0.\tag{2-8}$$

Let us check that if λ lies in the support of $\theta(\zeta_j(\alpha,\cdot))$, then γ lies in a fixed ring $\mathscr{C}$ of $\mathbb{R}^p$, independent of α. On the one hand we note that there is a constant $C>0$ such that, on the support of $\theta(\zeta_j(\alpha,\lambda))$, the variable γ must satisfy

$$(2\alpha_j+1)\eta_j(\gamma)\le Cm \quad \text{for all } m\neq 0 \tag{2-9}$$

for all $\alpha\in\mathbb{N}^d$ such that $|\alpha|=m$. Since, for each j, we know that $\eta_j(\gamma)$ is positive and homogeneous of degree 1, we infer that the function $\eta_j(\gamma)$ goes to infinity with $|\gamma|$, so (2-9) implies that γ must remain bounded on the support of $\theta(\zeta_j(\alpha,\lambda))$. Moreover, thanks to (2-9) again, it is clear that the bound may be made uniform in m.

Now let us prove that γ may be bounded from below uniformly. We know that there is a positive constant c such that, for λ in the support of $\theta(\zeta_j(\alpha,\lambda))$, we have

$$\zeta_j(\alpha,\gamma)\ge cm \quad \text{for all } m\neq 0. \tag{2-10}$$

Writing $\gamma=|\gamma|\hat\gamma$ with $\hat\gamma$ on the unit sphere of $\mathbb{R}^p$, we find

$$|\gamma|\ge \frac{cm}{\zeta_j(\alpha,\hat\gamma)}\,.$$

Defining

$$C_j:=\max_{|\hat\gamma|=1}\eta_j(\hat\gamma)<\infty,$$

it is easy to deduce that if (2-10) is satisfied then

$$|\gamma|\ge\frac{cm}{(2m+d)\max_{1\le j\le d}C_j}\,,$$

hence γ lies in a fixed ring of $\mathbb{R}^p$, independent of $\alpha\neq 0$. This fact will turn out to be important to perform the stationary phase argument.

Then we can rewrite the expression of $\tilde k_t(P,Q,tZ)$ in terms of the variable γ, which, in view of the homogeneity of the Pfaffian, produces the formula

$$\begin{aligned}\tilde k_t(P,Q,tZ)=\int_\Lambda &\mathrm{e}^{it\Phi_0(Z,\lambda)}G_0(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda\\ &+\sum_{m\in\mathbb{N}^*}\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=m}}m^{-p-d}\int \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)}G_\alpha(P,Q,\tilde\eta_m(\gamma))|\mathrm{Pf}(\gamma)|F(\gamma/m)\,d\gamma.\end{aligned}$$

Note that the series in m is convergent, since the sum over $|\alpha|=m$ contributes a power m^{d-1}, whence a series of m^{-p-1}, which is convergent since $p\ge 1$. Since the functions G_α are uniformly bounded with respect to $\alpha\in\mathbb{N}^d$ and F is smooth, there is a positive constant C such that

$$\|\tilde k_t\|_{L^\infty(G)}\le C \quad \text{for all } t\in\mathbb{R}.$$

In order to establish the dispersive estimate, it suffices then to prove that

$$\|\tilde k_t\|_{L^\infty(G)}\le \frac{C}{|t|^{(p-1)/2}} \quad \text{for all } t\neq 0. \tag{2-11}$$

3. End of the proof of the dispersive estimate

In order to prove (2-11), we decompose $\tilde{k}_t$ into two parts, writing

$$\tilde{k}_t(P, Q, tZ) = k_t^1(P, Q, tZ) + k_t^2(P, Q, tZ),$$

with, for a constant c_0 to be fixed later on independently of m,

$$k_t^1(P, Q, tZ) := \int_{|\nabla_\lambda \Phi_0(Z,\lambda)|\le c_0} \mathrm{e}^{it\Phi_0(Z,\lambda)} G_0(P, Q, \eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda \\ + \sum_{m\in\mathbb{N}^*} \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=m}} m^{-p-d} \int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|\le c_0} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} G_\alpha(P, Q, \tilde{\eta}_m(\gamma)) \\ \times F(\gamma/m)|\mathrm{Pf}(\gamma)|\,d\gamma. \tag{3-1}$$

In the following subsections, we successively show (2-11) for k_t^1 and k_t^2.

3A. ***Stationary phase argument for k_t^1.*** To establish the estimate (2-11), let us first concentrate on k_t^1. As usual in this type of problem, we define, for each integral of the series defining k_t^1, a vector field that commutes with the phase, prove an estimate for each term and, finally, check the convergence of the series. More precisely, in the case when $\alpha \neq 0$ and $t > 0$ (the case $t < 0$ is dealt with exactly in the same manner), we consider the first-order operator

$$\mathscr{L}_\alpha^1 := \frac{\mathrm{Id} - i\nabla_\gamma(\Phi_\alpha(Z, \gamma/m)) \cdot \nabla_\gamma}{1 + t|\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))|^2}.$$

Clearly we have

$$\mathscr{L}_\alpha^1 \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} = \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)}.$$

Let us accept the next lemma for the time being.

Lemma 3.1. *For any integer N, there is a smooth function θ_N, compactly supported on a ring of $\mathbb{R}^p$, and a positive constant C_N such that, defining*

$$\psi_\alpha(\gamma) := G_\alpha(P, Q, \tilde{\eta}_m(\gamma))F(\gamma/m)|\mathrm{Pf}(\gamma)| \tag{3-2}$$

and recalling (2-8), *we have*

$$|({}^t\mathscr{L}_\alpha^1)^N \psi_\alpha(\gamma)| \le C_N m^N \theta_N(\gamma)\big(1 + \big|t^{1/2}\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))\big|^2\big)^{-N}.$$

Returning to k_t^1, let us define (recalling that γ belongs to a fixed ring $\mathscr{C}$)

$$\mathscr{C}_\alpha(Z) := \big\{\gamma \in \mathscr{C} \,\big|\, |\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))| \le c_0\big\}$$

and let us write, for any integer N and $\alpha \neq 0$ (which we assume to be the case for the rest of the computations),

$$I_\alpha(Z) := \int_{\mathscr{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} \psi_\alpha(\gamma)\,d\gamma = \int_{\mathscr{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} ({}^t\mathscr{L}_\alpha^1)^N \psi_\alpha(\gamma)\,d\gamma, \tag{3-3}$$

where $\psi_\alpha(\gamma)$ has been defined in (3-2). Then, by Lemma 3.1, we find that for each integer N there is a constant C_N such that

$$|I_\alpha(Z)| \leq C_N m^N \int_{\mathscr{C}_\alpha(Z)} \theta_N(\gamma)\big(1+t\big|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big|^2\big)^{-N}\,d\gamma. \tag{3-4}$$

Then the end of the proof relies on three steps:

(1) a careful analysis of the properties of the support of the integral,

(2) a change of variables which leads to the estimate in $t^{-(p-1)/2}$,

(3) a control on m in order to prove the convergence of the sum over m.

Before entering into details for each step, let us observe that, by definition, we have

$$\Phi_\alpha\Big(Z,\frac{\gamma}{m}\Big) = \frac{1}{m}(\zeta(\alpha,\gamma)-\gamma(Z)),$$

with $\gamma(Z) = (A\gamma \mid Z) = (\gamma \mid {}^tAZ)$ for some invertible matrix A. Performing a change of variables in γ if necessary, we can assume without loss of generality that $A = \mathrm{Id}$. Thus we write

$$\nabla_\gamma\Big(\Phi_\alpha\Big(Z,\frac{\gamma}{m}\Big)\Big) = \frac{1}{m}(\nabla_\gamma\zeta(\alpha,\gamma)-Z). \tag{3-5}$$

3A1. *Analysis of the support of the integral defining $I_\alpha(Z)$.* Let us prove the following result:

Proposition 3.2. *One can choose the constant c_0 in* (3-1) *small enough such that, if γ belongs to $\mathscr{C}_\alpha(Z)$, then $\gamma\cdot Z \neq 0$.*

Proof. We write

$$\gamma\cdot Z = \gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma) + \gamma\cdot(Z-\nabla_\gamma\zeta(\alpha,\gamma))$$

and, observing that, thanks to homogeneity arguments, $\gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma) = \zeta(\alpha,\gamma)$, we deduce that, for any $\gamma\in\mathscr{C}_\alpha(Z)$,

$$|\gamma\cdot Z| \geq |\zeta(\alpha,\gamma)| - |\gamma|\,|Z-\nabla_\gamma\zeta(\alpha,\gamma)|.$$

Since, as argued above, γ belongs to a fixed ring and $\zeta(\alpha,\lambda)=0$ if and only if $\lambda=0$ (as noticed in Section 1C3), there is a positive constant c such that, for any $\gamma\in\mathscr{C}_\alpha(Z)$,

$$|\zeta(\alpha,\gamma)| \geq mc,$$

which implies, in view of the definition of $\mathscr{C}_\alpha(Z)$, that there is a positive constant $\tilde{c}$ depending only on the ring $\mathscr{C}$ such that

$$|\gamma\cdot Z| \geq mc - mc_0\tilde{c}.$$

This ensures the desired result, by choosing the constant c_0 in the definition of k_t^1 smaller than $c/\tilde{c}$. Proposition 3.2 is proved. $\square$

3A2. *A change of variables: the diffeomorphism $\mathcal{H}$.* We can assume without loss of generality (if not then the integral is zero) that $\mathcal{C}_\alpha(Z)$ is not empty and, in view of Proposition 3.2, we can write for any $\gamma \in \mathcal{C}_\alpha(Z)$ the orthogonal decomposition (since $Z \neq 0$)

$$\frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) = \widetilde{\Gamma}_1 \widehat{Z}_1 + \widetilde{\Gamma}', \quad \text{with} \quad \widetilde{\Gamma}_1 := \Big(\frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) \,\Big|\, \widehat{Z}\Big) \quad \text{and} \quad \widehat{Z}_1 := \frac{Z}{|Z|}. \tag{3-6}$$

Since $\widetilde{\Gamma}'$ is orthogonal to the vector Z, we infer that

$$|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))| = \frac{1}{m}|Z - \nabla_\gamma \zeta(\alpha,\gamma)| \geq |\widetilde{\Gamma}'|. \tag{3-7}$$

Let us consider an orthonormal basis $(\widehat{Z}_1, \dots, \widehat{Z}_p)$ in $\mathbb{R}^p$. Thanks to Proposition 3.2, we have $\gamma \cdot \widehat{Z}_1 \neq 0$ on the support of the integral defining $I_\alpha(Z)$. Obviously, the vector $\widetilde{\Gamma}'$ defined by (3-6) belongs to the vector space generated by $(\widehat{Z}_2, \dots, \widehat{Z}_p)$. To investigate the integral $I_\alpha(Z)$ defined in (3-3), let us consider the map $\mathcal{H} : \gamma \mapsto \tilde{\gamma}'$ defined by

$$\gamma \mapsto \mathcal{H}(\gamma) := (\gamma \cdot \widehat{Z}_1)\widehat{Z}_1 + \sum_{k=2}^{p} (\widetilde{\Gamma}' \cdot \widehat{Z}_k)\widehat{Z}_k =: \sum_{k=1}^{p} \tilde{\gamma}'_k \widehat{Z}_k \quad \text{for } \gamma \in \mathcal{C}_\alpha(Z). \tag{3-8}$$

Proposition 3.3. *The map $\mathcal{H}$ realizes a diffeomorphism from $\mathcal{C}_\alpha(Z)$ into a fixed compact set of $\mathbb{R}^p$.*

Proof. It is clear that the smooth function $\mathcal{H}$ maps $\mathcal{C}_\alpha(Z)$ into a fixed compact set $\mathcal{K}$ of $\mathbb{R}^p$ and that

$$\tilde{\gamma}'_1 = \gamma \cdot \widehat{Z}_1 \qquad \text{and} \qquad \tilde{\gamma}'_k = \frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) \cdot \widehat{Z}_k \quad \text{for } 2 \leq k \leq p.$$

Now let us prove that, thanks to Assumption 1.4, the map $\mathcal{H}$ constitutes a diffeomorphism. Indeed, by straightforward computations we find that $D\mathcal{H}$, the differential of $\mathcal{H}$, satisfies

$$\begin{aligned}
&\langle D\mathcal{H}(\gamma)\widehat{Z}_1, \widehat{Z}_1\rangle = 1, \\
&\langle D\mathcal{H}(\gamma)\widehat{Z}_1, \widehat{Z}_k\rangle = \Big\langle \frac{1}{m} D^2_\gamma \zeta(\alpha,\gamma)\widehat{Z}_1, \widehat{Z}_k \Big\rangle \quad \text{for } 2 \leq k \leq p, \\
&\langle D\mathcal{H}(\gamma)\widehat{Z}_j, \widehat{Z}_k\rangle = \Big\langle \frac{1}{m} D^2_\gamma \zeta(\alpha,\gamma)\widehat{Z}_j, \widehat{Z}_k \Big\rangle \quad \text{for } 2 \leq j, k \leq p, \\
&\langle D\mathcal{H}(\gamma)\widehat{Z}_j, \widehat{Z}_1\rangle = 0 \qquad\qquad\qquad\qquad\quad \text{for } 2 \leq j \leq p.
\end{aligned}$$

Proving that $\mathcal{H}$ is a diffeomorphism amounts to showing that, for any $\gamma \in \mathcal{C}_\alpha(Z)$, the kernel of $D\mathcal{H}(\gamma)$ reduces to $\{0\}$. In view of the above formulas, if $V = \sum_{j=1}^{p} V_j \widehat{Z}_j$ belongs to the kernel of $D\mathcal{H}(\gamma)$ then $V_1 = V \cdot \widehat{Z}_1 = 0$ and $D^2_\gamma \zeta(\alpha,\gamma)V \cdot \widehat{Z}_k = 0$ for $2 \leq k \leq p$. Thus we can write $D^2_\gamma \zeta(\alpha,\gamma)V = \tau \widehat{Z}_1$ for some $\tau \in \mathbb{R}$. Since the function $\zeta(\alpha,\cdot)$ is homogeneous of degree 1, we have $D^2_\gamma \zeta(\alpha,\gamma)\gamma = 0$. We deduce that

$$0 = D^2_\gamma \zeta(\alpha,\gamma)\gamma \cdot V = \gamma \cdot D^2_\gamma \zeta(\alpha,\gamma)V = \tau\gamma \cdot \widehat{Z}_1.$$

Since $\gamma \cdot \widehat{Z}_1 \neq 0$ for all $\gamma \in \mathcal{C}_\alpha(Z)$, we find that $\tau = 0$ and therefore $D^2_\gamma \zeta(\alpha,\gamma)V = 0$. But Assumption 1.4 states that the Hessian $D^2_\gamma \zeta(\alpha,\gamma)$ is of rank $p-1$, so we conclude that V is collinear to γ. But we have seen that $V \cdot \widehat{Z}_1 = 0$, which contradicts the fact that $\gamma \cdot \widehat{Z}_1 \neq 0$. This entails that V is null and ends the proof of the proposition. ☐

We can therefore perform the change of variables defined by (3-8) in the right-hand side of (3-4), to obtain

$$|I_\alpha(Z)| \le C_N m^N \int_{\mathcal{K}} \frac{1}{(1+t|\tilde\gamma'|^2)^N}\, d\tilde\gamma'\, d\tilde\gamma_1.$$

3A3. *End of the proof: convergence of the series.* Choosing $N = p-1$ implies, by the change of variables $\gamma^\sharp = t^{1/2}\tilde\gamma'$, that there is a constant C such that

$$|I_\alpha(Z)| \le C|t|^{-(p-1)/2} m^{p-1},$$

which gives rise to

$$\left|\int_{\mathscr{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} \psi_\alpha(\gamma)\, d\gamma\right| \le C|t|^{-(p-1)/2} m^{p-1}.$$

We get in exactly the same way that

$$\left|\int_{|\nabla_\lambda \Phi_0(Z,\lambda)|\le c_0} \mathrm{e}^{it\Phi_0(Z,\lambda)} G_0(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\, d\lambda\right| \le C|t|^{-(p-1)/2}.$$

Finally, returning to the kernel k_t^1 defined in (3-1), we get

$$|k_t^1(P,Q,tZ)| \le C|t|^{-(p-1)/2} + C|t|^{-(p-1)/2} \sum_{m\in\mathbb{N}^*} m^{d-1} m^{-d-p} m^{p-1} \le C|t|^{-(p-1)/2},$$

since the series over m is convergent. The dispersive estimate is thus proved for k_t^1.

3B. ***Stationary phase argument for k_t^2.*** We now prove (2-11) for k_t^2, which is easier since the gradient of the phase is bounded from below. We claim that there is a constant C such that

$$|k_t^2(P,Q,tZ)| \le \frac{C}{t^{(p-1)/2}}. \tag{3-9}$$

This can be achieved as above by means of adequate integrations by parts. Indeed, in the case when $\alpha \ne 0$, consider the first-order operator

$$\mathscr{L}_\alpha^2 := -i\,\frac{\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\cdot\nabla_\gamma}{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2}\,.$$

Note that, when $\alpha = 0$, the arguments are the same without performing the change of variable $\lambda = \gamma/m$. The operator $\mathscr{L}_\alpha^2$ obviously satisfies

$$\mathscr{L}_\alpha^2 \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} = t\mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)};$$

hence, by repeated integrations by parts, we get

$$\begin{aligned} J_\alpha(P,Q,tZ) &:= \int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|\ge c_0} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)}\psi_\alpha(\gamma)\, d\gamma \\ &= \frac{1}{t^N}\int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|\ge c_0} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} ({}^t\mathscr{L}_\alpha^2)^N \psi_\alpha(\gamma)\, d\lambda. \end{aligned}$$

Let us accept the following lemma for a while:

Lemma 3.4. *For any integer N, there is a smooth function θ_N compactly supported on a compact set of $\mathbb{R}^p$ such that*

$$|({}^t\mathcal{L}_\alpha^2)^N \psi_\alpha(\gamma)| \leq \frac{\theta_N(\gamma)m^N}{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^N}\,.$$

One then observes that, if γ is in the support of the integral defining k_t^2, the lemma implies

$$|({}^t\mathcal{L}_\alpha^2)^N \psi_\alpha(\gamma)| \leq \frac{\theta_N(\gamma)}{c_0^N} m^N.$$

This estimate ensures the result as in Section 3A by taking $N = p-1$.

3C. ***Proofs of Lemmas 3.1 and 3.4.*** Lemma 3.1 is an obvious consequence of the following Lemma 3.5, taking $(a,b) \equiv (0,0)$. We omit the proof of Lemma 3.4, which consists in a straightforward modification of the arguments developed below.

Lemma 3.5. *For any integer N, one can write*

$$({}^t\mathcal{L}_\alpha^1)^N \psi_\alpha(\gamma) = f_{N,m}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big),$$

with $|\alpha| = m$, and where $f_{N,m}$ is a smooth function supported on $\mathscr{C} \times \mathbb{R}^p$ with $\mathscr{C}$ a fixed ring of $\mathbb{R}^p$, such that for any pair $(a,b) \in \mathbb{N}^p \times \mathbb{N}^p$, there is a constant C (independent of m) such that

$$|\nabla_\gamma^a \nabla_\Theta^b f_{N,m}(\gamma,\Theta)| \leq C m^{N+|a|}(1+|\Theta|^2)^{-N-|b|/2}.$$

Proof of Lemma 3.5. Let us prove the result by induction over N. We start with the case when N is equal to zero. Notice that in that case the function $f_{0,m}(\gamma,\Theta) = \psi_\alpha(\gamma)$ does not depend on the quantity $\Theta = t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))$, so we need to check that, for any $a \in \mathbb{N}^p$, there is a constant C such that

$$|\nabla_\gamma^a \psi_\alpha(\gamma)| \leq C m^{|a|} \tag{3-10}$$

when $|\alpha| = m$. The case when $a = 0$ is obvious thanks to the uniform bound on G_α. To deal with the case $|a| \geq 1$, we state the following technical result, which will be proved at the end of this section.

Lemma 3.6. *For any integer k, there is a constant C such that the following bound holds for the functions g_n, $n \in \mathbb{N}$, defined in* (2-5):

$$|(\xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)| \leq C n^k \quad \textit{for all } (\xi_1,\xi_2) \in \mathbb{R}^2.$$

Let us now compute $\nabla_\gamma^a \psi_\alpha(\gamma)$. Recall that, according to (3-2),

$$\psi_\alpha(\gamma) = G_\alpha(P,Q,\tilde\eta_m(\gamma))F\Big(\frac{\gamma}{m}\Big)|\mathrm{Pf}(\gamma)| = F\Big(\frac{\gamma}{m}\Big)\prod_{j=1}^d \psi_{\alpha,j}(\gamma),$$

where

$$\psi_{\alpha,j}(\gamma) := \eta_j(\gamma)\tilde\theta((2\alpha_j+1)\tilde\eta_{j,m}(\gamma))g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j), \quad \tilde\eta_{j,m}(\gamma) := \frac{1}{m}\eta_j(\gamma).$$

We compute

$$\nabla_\gamma^a \psi_{\alpha,j}(\gamma) = \sum_{\substack{b\in\mathbb{N}^p \\ 0\le|b|\le|a|}} \binom{b}{a} \nabla_\gamma^b\big(\theta((2\alpha_j+1)\tilde\eta_{j,m}(\gamma))\big)\nabla_\gamma^{a-b}\big(\eta_j(\gamma) g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j)\big).$$

Let us assume first that $|a-b|=1$. Then we write, for some $1\le l\le p$,

$$\begin{aligned}\partial_{\gamma_l}\big(\eta_j(\gamma)g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j)\big) \\ = \partial_{\gamma_l}\eta_j(\gamma)g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j) \\ +\eta_j(\gamma)\frac{\partial_{\gamma_l}\tilde\eta_{j,m}(\gamma)}{2\tilde\eta_{j,m}(\gamma)}\times((\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})g_{\alpha_j})(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j).\end{aligned}$$

Next we use the fact that there is a constant C such that, on the support of $\theta((2\alpha_j+1)\tilde\eta_{j,m}(\gamma))$,

$$\tilde\eta_{j,m}(\gamma)\ge\frac{1}{Cm} \quad\text{and}\quad |\partial_{\gamma_l}\tilde\eta_{j,m}(\gamma)|\le\frac{C}{m},$$

so applying Lemma 3.6 gives

$$|\nabla_\gamma\psi_{\alpha,j}(\gamma)|\lesssim\alpha_j.$$

Recalling that $\alpha_j\le m$ and that $\psi_{\alpha,j}$ is uniformly bounded for all $j\in\{1,\dots,d\}$, this easily achieves the proof of the estimate (3-10) in the case $|a|=1$ by taking the product over j. Once we have noticed that

$$\alpha_1^{a_1}\cdots\alpha_d^{a_d}\lesssim(\alpha_1+\cdots+\alpha_d)^{a_1+\cdots+a_d},$$

the general case (when $|a|>1$) is dealt with identically; we omit the details.

Finally let us proceed with the induction: assume that for some integer N one can write

$$({}^t\mathcal{L}_\alpha^1)^{N-1}\psi_\alpha(\gamma) = f_{N-1,m}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big),$$

where $f_{N-1,m}$ is a smooth function supported on $\mathscr{C}\times\mathbb{R}^p$, such that for any pair $(a,b)\in\mathbb{N}^p\times\mathbb{N}^p$ there is a constant C (independent of m) such that

$$|\nabla_\gamma^a\nabla_\Theta^b f_{N-1,m}(\gamma,\Theta)|\le Cm^{N-1+|a|}(1+|\Theta|^2)^{-(N-1)-|b|/2}. \tag{3-11}$$

We compute, for any function $\Psi(\gamma)$,

$$\begin{aligned}{}^t\mathcal{L}_\alpha^1\Psi(\gamma) = i\frac{\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\cdot\nabla_\gamma\Psi(\gamma)}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2}+\frac{1+i\Delta(\Phi_\alpha(Z,\gamma/m))}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2}\Psi(\gamma) \\ -2it\sum_{1\le j,k\le p}\frac{\partial_{\gamma_j}\partial_{\gamma_k}(\Phi_\alpha(Z,\gamma/m))\partial_{\gamma_j}(\Phi_\alpha(Z,\gamma/m))\partial_{\gamma_k}(\Phi_\alpha(Z,\gamma/m))}{\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^2}\Psi(\gamma).\end{aligned}$$

We apply that formula to $\Psi := f_{N-1}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big)$ and, estimating each of the three terms separately, we find (using the fact that $m \geq 1$)

$$\begin{aligned}
&\big|{}^t\mathcal{L}^1_\alpha\big(f_{N-1}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big)\big)\big| \\
&\quad \leq C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-1} m^{N-1+1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)} \\
&\qquad + C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-1} m^{N-1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)} \\
&\qquad\quad + Ct|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-2} m^{N-1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)}
\end{aligned}$$

thanks to the induction assumption (3-11) along with (3-10) and the fact that, on $\mathscr{C}_\alpha(Z)$, all the derivatives of the function $\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))$ are uniformly bounded with respect to α and Z. A similar argument allows us to control derivatives in γ and Θ, so Lemma 3.5 is proved. □

Proof of Lemma 3.6. By definition of g_n and using the change of variable

$$\xi \mapsto \xi - \tfrac{1}{2}\xi_1$$

we recover the Wigner-type formula

$$g_n(\xi_1,\xi_2) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi_2\xi} h_n\big(\xi+\tfrac{1}{2}\xi_1\big) h_n\big(\xi-\tfrac{1}{2}\xi_1\big)\, d\xi.$$

Then an easy computation shows that, for all k,

$$|(\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)| \leq \int_{\mathbb{R}} \big|(\xi_1\partial_{\xi_1}+\xi\,\partial_\xi+1)^k\big(h_n\big(\xi+\tfrac{1}{2}\xi_1\big)h_n\big(\xi-\tfrac{1}{2}\xi_1\big)\big)\big|\, d\xi.$$

By the Cauchy–Schwarz inequality (and a change of variables to transform $\xi+\frac{1}{2}\xi_1$ and $\xi-\frac{1}{2}\xi_1$ into (ξ,ξ')), it remains therefore to check that, for all k,

$$\|(\xi\,\partial_\xi)^k h_n\|_{L^2(\mathbb{R})} \leq C_k n^k.$$

This again reduces to checking that

$$\|\xi^{2k} h_n\|_{L^2(\mathbb{R})} + \|h_n^{(2k)}\|_{L^2(\mathbb{R})} \leq C_k n^k. \tag{3-12}$$

This estimate is a consequence of the identification of the domain of $\sqrt{H}$,

$$D(\sqrt{H}) = \{u \in L^2(\mathbb{R}) \mid \xi u, u' \in L^2(\mathbb{R})\},$$

which classically extends to powers of $\sqrt{H}$ as

$$D(H^{p/2}) = \{u \in L^2(\mathbb{R}) \mid \xi^{p-l}u^{(l)} \in L^2(\mathbb{R}),\ 0 \leq l \leq p\}.$$

Then (3-12) is finally obtained by applying this to $p=2k$, recalling that $H^k h_n = (2n+1)^k h_n$. The lemma is proved. □

4. Optimality of the dispersive estimates

In this section, we first end the proof of Theorem 1 by proving the optimality of the dispersive estimates for groups satisfying Assumption 1.4. Then we prove Proposition 1.7.

4A. ***Optimality for groups satisfying Assumption 1.4.*** Let us now end the proof of Theorem 1 by establishing the optimality of the dispersive estimate (1-19). We use the fact that there always exists $\lambda^* \in \Lambda$ such that

$$\nabla_\lambda \zeta(0, \lambda^*) \neq 0, \tag{4-1}$$

where the function ζ is as defined in (1-12). Indeed, if not, the map $\lambda \mapsto \zeta(0, \lambda)$ would be constant, which is in contradiction with the fact that ζ is homogeneous of degree 1. We prove the following proposition, which yields the optimality of the dispersive estimate of Theorem 1.

Proposition 4.1. *Let $\lambda^* \in \Lambda$ satisfying* (4-1). *There is a function $g \in \mathcal{D}(\mathbb{R}^p)$ compactly supported in a connected open neighborhood of λ^* in Λ such that, for the initial data f_0 defined by*

$$\mathcal{F}(f_0)(\lambda, \nu) h_{\alpha,\eta(\lambda)} = \begin{cases} 0 & \text{if } \alpha \neq 0, \\ g(\lambda) h_{0,\eta(\lambda)} & \text{if } \alpha = 0, \end{cases} \qquad \text{for all } (\lambda, \nu) \in \mathfrak{r}(\Lambda), \tag{4-2}$$

there exists $c_0 > 0$ and $x \in G$ such that

$$|\mathrm{e}^{-it\Delta_G} f_0(x)| \geq c_0 |t|^{-(k+p-1)/2}.$$

Proof. Let g be any smooth, compactly supported function over $\mathbb{R}^p$, and define f_0 by (4-2). For any point $x = \mathrm{e}^X \in G$ in the form $X = (P = 0, Q = 0, Z, R)$, the inversion formula gives

$$\mathrm{e}^{-it\Delta_G} f_0(x) = \kappa \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}^*_\lambda} \mathrm{e}^{it|\nu|^2 + it\zeta(0,\lambda) - i\lambda(Z) - i\nu(R)} g(\lambda) |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda.$$

To simplify notations, we set $\zeta_0(\lambda) := \zeta(0, \lambda)$. Setting $Z = tZ^*$ with $Z^* := \nabla_\lambda \zeta(0, \lambda^*) \neq 0$, we get, as in (2-6),

$$|\mathrm{e}^{-it\Delta_G} f_0(x)| = c_1 |t|^{-k/2} \left| \int_{\lambda \in \mathbb{R}^p} \mathrm{e}^{it(\lambda \cdot Z^* - \zeta_0(\lambda))} g(\lambda) |\mathrm{Pf}(\lambda)| \, d\lambda \right|$$

for some constant $c_1 > 0$. Without loss of generality, we can assume

$$\lambda^* = (1, 0, \ldots, 0)$$

(if not, we perform a change of variables $\lambda \mapsto \Omega\lambda$, where Ω is a fixed orthogonal matrix), and we now shall perform a stationary phase in the variable λ', where we have written $\lambda = (\lambda_1, \lambda')$. For any fixed λ_1, the phase

$$\Phi_{\lambda_1}(\lambda', Z) := Z \cdot \lambda - \zeta_0(\lambda)$$

has a stationary point λ' if and only if $Z' = \nabla_{\lambda'} \zeta_0(\lambda)$ (with the same notation $Z = (Z_1, Z')$). We observe that the homogeneity of the function ζ_0 and the definition of Z^* imply that

$$Z^* = \nabla_\lambda \zeta_0(1, 0, \ldots, 0) = \nabla_\lambda \zeta_0(\lambda_1, 0, \ldots, 0) \quad \text{for all } \lambda_1 \in \mathbb{R};$$

hence, if $\lambda' = 0$, then the phase $\Phi_{\lambda_1}(0, Z^*)$ has a stationary point.

From now on we choose g supported near those stationary points $(\lambda_1, 0)$ and vanishing in the neighborhood of any other stationary point.

Let us now study the Hessian of Φ_{λ_1} in $\lambda' = 0$. Again because of the homogeneity of the function ζ_0, we have

$$[\operatorname{Hess}\zeta_0(\lambda)]\lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^p.$$

In particular, for all $\lambda_1 \neq 0$, $\operatorname{Hess}\zeta_0(\lambda_1, 0, \dots, 0)(\lambda_1, 0, \dots, 0) = 0$ and the matrix $\operatorname{Hess}\zeta_0(\lambda_1, 0, \dots, 0)$ in the canonical basis is of the form

$$\operatorname{Hess}\zeta_0(\lambda_1, 0, \dots, 0) = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{Hess}_{\lambda',\lambda'}\zeta_0(\lambda_1, 0, \dots, 0) \end{pmatrix}.$$

Using that $\operatorname{Hess}\zeta_0(\lambda_1, 0, \dots, 0)$ is of rank $p-1$, we deduce that $\operatorname{Hess}_{\lambda',\lambda'}\zeta_0(\lambda_1, 0, \dots, 0)$ is also of rank $p-1$ and we conclude by the stationary phase theorem [Stein 1993, Chapter VIII.2], choosing g so that the remaining integral in λ_1 does not vanish. □

4B. ***Proof of Proposition 1.7.*** Assume that G is a step-2 stratified Lie group whose radical index is null and for which $\zeta(0, \lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Let g be a smooth nonnegative function supported in one of the connected components of Λ and define f_0 by

$$\mathcal{F}(f_0)(\lambda)h_{\alpha,\eta(\lambda)} = 0 \quad \text{for} \quad \alpha \neq 0 \qquad \text{and} \qquad \mathcal{F}(f_0)(\lambda)h_{0,\eta(\lambda)} = g(\lambda)h_{0,\eta(\lambda)}.$$

By the inverse Fourier formula, if $x = \mathrm{e}^X \in G$ is such that $X = (P = 0, Q = 0, tZ)$, then we have

$$\mathrm{e}^{-it\Delta_G}(x) = \kappa \int \mathrm{e}^{-it\lambda(Z)}\mathrm{e}^{it\zeta(0,\lambda)} g(\lambda)|\mathrm{Pf}(\lambda)|\, d\lambda.$$

Since $\zeta(0, \lambda)$ is a linear form on each connected component of Λ, there exists Z_0 in $\mathfrak{z}$ such that

$$-\lambda(Z_0) + \zeta(0, \lambda) = 0 \quad \text{for all } \lambda \in \mathfrak{z}^* \cap \operatorname{supp} g.$$

As a consequence, choosing $Z = Z_0$, we obtain

$$\mathrm{e}^{-it\Delta_G}(x) = \kappa \int g(\lambda)|\mathrm{Pf}(\lambda)|\, d\lambda \neq 0,$$

which ends the proof of the result.

Appendix: On the inversion formula in Schwartz space

This section is dedicated to the proof of the inversion formula in the Schwartz space $\mathcal{S}(G)$ (Proposition 1.1).

Proof. We first observe that, to establish (1-8), it suffices to prove that

$$f(0) = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(\mathcal{F}(f)(\lambda, \nu))|\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda. \tag{A-1}$$

Indeed, introducing the auxiliary function g defined by $g(x') := f(x \cdot x')$, which obviously belongs to $\mathcal{S}(G)$ and satisfies $\mathcal{F}(g)(\lambda, \nu) = u^{\lambda,\nu}_{X(\lambda,x^{-1})} \circ \mathcal{F}(f)(\lambda, \nu)$, and assuming (A-1) holds, we get

$$f(x) = g(0) = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(\mathcal{F}(g)(\lambda, \nu)) |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda$$
$$= \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(u^{\lambda,\nu}_{X(\lambda,x^{-1})} \mathcal{F}(f)(\lambda, \nu)) |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda,$$

which is the desired result.

Let us now focus on (A-1). In order to compute the right-hand side of (A-1), we introduce

$$A := \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(\mathcal{F}(f)(\lambda, \nu)) |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda$$
$$= \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \sum_{\alpha\in\mathbb{N}^d} \big(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} \bigm| h_{\alpha,\eta(\lambda)}\big) |\mathrm{Pf}(\lambda)| f(x) d\mu(x) \, d\nu \, d\lambda,$$

with the notation of Section 1C. In order to carry on the calculations, we need to resort to a Fubini argument, which comes from the identity

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \|\mathcal{F}(f)(\lambda, \nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda < \infty. \tag{A-2}$$

We postpone the proof of (A-2) to the end of this section. Thanks to (A-2), the order of integration does not matter and we can transform the expression of A: we use the fact that, for any $\alpha \in \mathbb{N}^d$,

$$(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} \mid h_{\alpha,\eta(\lambda)}) = \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \int_{\mathbb{R}^d} \mathrm{e}^{-i \sum_j \eta_j(\lambda)(\xi_j + P_j/2) Q_j} h_{\alpha,\eta(\lambda)}(P + \xi) h_{\alpha,\eta(\lambda)}(\xi) d\xi,$$

where we have identified $\mathfrak{p}_\lambda$ with $\mathbb{R}^d$, and this gives rise to

$$A = \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \int_{\xi\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \mathrm{e}^{-i \sum_j \eta_j(\lambda)(\xi_j + P_j/2) Q_j}$$
$$\times h_{\alpha,\eta(\lambda)}(P + \xi) h_{\alpha,\eta(\lambda)}(\xi) |\mathrm{Pf}(\lambda)| f(x) \, d\mu(x) \, d\xi \, d\nu \, d\lambda,$$

where we recall that

$$h_{\alpha,\eta(\lambda)}(\xi) = \prod_{j=1}^{d} h_{\alpha_j,\eta_j(\lambda)}(\xi_j) \quad \text{with} \quad h_{\alpha_j,\eta_j(\lambda)}(\xi_j) = \eta_j(\lambda)^{1/4} h_{\alpha_j}(\sqrt{\eta_j(\lambda)}\xi_j).$$

Performing the change of variables

$$\tilde{\xi}_j = \sqrt{\eta_j(\lambda)}\xi_j, \quad \widetilde{P}_j = \sqrt{\eta_j(\lambda)} P_j, \quad \widetilde{Q}_j = \sqrt{\eta_j(\lambda)} Q_j$$

for $j \in \{1, \dots, d\}$, we obtain, dropping the tilde on the variables,

$$A = \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{(P,Q,R,Z)\in\mathbb{R}^n} \int_{\xi\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R) - i\lambda(Z)} \mathrm{e}^{-i \sum_l (\xi_l + P_l/2)\cdot Q_l} \prod_{j=1}^{d} h_{\alpha_j}(P_j + \xi_j) h_{\alpha_j}(\xi_j)$$
$$\times f(\eta^{-1/2}(\lambda) P, \eta^{-1/2}(\lambda) Q, R, Z) \, dP \, dQ \, dR \, dZ \, d\xi \, d\nu \, d\lambda,$$

with $\eta^{-1/2}(\lambda)P := (\eta_1^{-1/2}(\lambda)P_1, \dots, \eta_d^{-1/2}(\lambda)P_d)$ and similarly for Q.

Then using the change of variables $\xi'_j = \xi_j + P_j$ for $j \in \{1, \dots, d\}$ gives

$$A = \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{(\xi',Q,R,Z)\in\mathbb{R}^n}\int_{\xi\in\mathbb{R}^d}\sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\mathrm{e}^{-(i/2)\sum_l(\xi_l+\xi'_l)\cdot Q_l}\prod_{j=1}^d h_{\alpha_j}(\xi'_j)h_{\alpha_j}(\xi_j)$$
$$\times f(\eta^{-1/2}(\lambda)\,(\xi'-\xi), \eta^{-1/2}(\lambda)\,Q, R, Z)\,d\xi'\,dQ\,dR\,dZ\,d\xi\,d\nu\,d\lambda\,.$$

Because $(h_\alpha)_{\alpha\in\mathbb{N}^d}$ is a Hilbert basis of $L^2(\mathbb{R}^d)$, we have, for all $\phi \in L^2(\mathbb{R}^d)$,

$$\phi(\xi) = \sum_{\alpha\in\mathbb{N}^d}\int_{\xi'\in\mathbb{R}^d}\phi(\xi')h_\alpha(\xi')\,d\xi'\,h_\alpha(\xi),$$

which leads to

$$A = \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{(Q,R,Z)\in\mathbb{R}^{d+k+p}}\int_{\xi\in\mathbb{R}^d}\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\mathrm{e}^{-i\xi\cdot Q}f(0, \eta^{-1/2}(\lambda)Q, R, Z)\,dQ\,dR\,dZ\,d\xi\,d\nu\,d\lambda.$$

Applying the Fourier inversion formula successively on $\mathbb{R}^d$, $\mathbb{R}^k$ and $\mathbb{R}^p$ (and identifying $\mathfrak{r}(\Lambda)$ with $\mathbb{R}^p\times\mathbb{R}^k$), we conclude that there exists a constant $\kappa > 0$ such that

$$A = \kappa f(0),$$

which ends the proof of (A-1).

Let us conclude the proof by showing (A-2). We choose a nonnegative integer M. From the obvious fact that the function $(\mathrm{Id}-\Delta_G)^M f$ also belongs to $\mathcal{S}(G)$ (hence to $L^1(G)$), we get, in view of (1-11),

$$\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)} = (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M}\mathcal{F}((\mathrm{Id}-\Delta_G)^M f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}.$$

In view of the definition of the Fourier transform on the group G, we thus have

$$\begin{aligned}&\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)}\\ &= (1+|\nu|^2+\zeta(\alpha,\lambda))^{-2M}\\ &\times\int_{\mathfrak{p}_\lambda}\biggl(\int_G(\mathrm{Id}-\Delta_G)^M f(x)u^{\lambda,\nu}_{X(\lambda,x)}h_{\alpha,\eta(\lambda)}(\xi)\,d\mu(x)\biggr)\biggl(\overline{\int_G(\mathrm{Id}-\Delta_G)^M f(x')u^{\lambda,\nu}_{X(\lambda,x')}h_{\alpha,\eta(\lambda)}(\xi)\,d\mu(x')}\biggr)d\xi.\end{aligned}$$

Now, by Fubini's theorem, we get

$$\begin{aligned}&\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)}\\ &= (1+|\nu|^2+\zeta(\alpha,\lambda))^{-2M}\\ &\quad\times\int_G\int_G(\mathrm{Id}-\Delta_G)^M f(x)\overline{(\mathrm{Id}-\Delta_G)^M f(x')}\bigl(u^{\lambda,\nu}_{X(\lambda,x)}h_{\alpha,\eta(\lambda)}\mid u^{\lambda,\nu}_{X(\lambda,x')}h_{\alpha,\eta(\lambda)}\bigr)_{L^2(\mathfrak{p}_\lambda)}\,d\mu(x)\,d\mu(x').\end{aligned}$$

Since the operators $u^{\lambda,\nu}_{X(\lambda,x)}$ and $u^{\lambda,\nu}_{X(\lambda,x')}$ are unitary on $\mathfrak{p}_\lambda$ and the family $(h_{\alpha,\eta(\lambda)})_{\alpha\in\mathbb{N}^d}$ is a Hilbert basis of $\mathfrak{p}_\lambda$, we deduce that

$$\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} \le (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M}\|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)}.$$

Because

$$\operatorname{Card}\bigl(\{\alpha \in \mathbb{N}^d \mid |\alpha| = m\}\bigr) = \binom{m+d-1}{m} \leq C(m+1)^{d-1},$$

this ensures that

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda,\nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \\ \lesssim \|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)} \sum_m (m+1)^{d-1} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda.$$

Hence, taking $M = M_1 + M_2$ with $M_2 > \frac{1}{2}k$ implies that

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda,\nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda \\ \lesssim \|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)} \sum_m (m+1)^{d-1} \int_{\lambda\in\Lambda} (1+\zeta(\alpha,\lambda))^{-M_1} |\mathrm{Pf}(\lambda)|\, d\lambda.$$

Noticing that $\zeta(\alpha,\lambda) = 0$ if and only if $\lambda = 0$ and using the homogeneity of degree 1 of ζ yields that there exists $c > 0$ such that $\zeta(\alpha,\lambda) \geq cm|\lambda|$. Therefore, we can end the proof of (A-2) by choosing M_1 large enough and performing the change of variable $\mu = m\lambda$ in each term of the above series.

Proposition 1.1 is proved. □

Acknowledgements

The authors wish to thank Corinne Blondel, Jean-Yves Charbonnel, Laurent Mazet, Fulvio Ricci and Michèle Vergne for enlightening discussions. They also extend their gratitude to the anonymous referee for numerous remarks which improved the presentation of this paper, and for providing a simpler and more conceptual proof of Lemma 3.6 than our original proof.

References

[Anker and Pierfelice 2009] J.-P. Anker and V. Pierfelice, "Nonlinear Schrödinger equation on real hyperbolic spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:5 (2009), 1853–1869. MR 2566713 Zbl 1176.35166

[Anton 2008] R. Anton, "Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains", *Bull. Soc. Math. France* **136**:1 (2008), 27–65. MR 2415335 Zbl 1157.35100

[Bahouri and Chemin 1999] H. Bahouri and J.-Y. Chemin, "Équations d'ondes quasilinéaires et estimations de Strichartz", *Amer. J. Math.* **121**:6 (1999), 1337–1377. MR 1719798 Zbl 0952.35073

[Bahouri and Chemin 2003] H. Bahouri and J.-Y. Chemin, "Microlocal analysis, bilinear estimates and cubic quasilinear wave equation", pp. 93–141 in *Autour de l'analyse microlocale*, edited by G. Lebeau, Astérisque **284**, Société Mathématique de France, Paris, 2003. MR 2003418 Zbl 1053.35098

[Bahouri and Gallagher 2001] H. Bahouri and I. Gallagher, "Paraproduit sur le groupe de Heisenberg et applications", *Rev. Mat. Iberoamericana* **17**:1 (2001), 69–105. MR 1846091 Zbl 0971.43002

[Bahouri et al. 2000] H. Bahouri, P. Gérard, and C.-J. Xu, "Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg", *J. Anal. Math.* **82** (2000), 93–118. MR 1799659 Zbl 0965.22010

[Bahouri et al. 2011] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Math. Wissenschaften **343**, Springer, Heidelberg, 2011. MR 2768550 Zbl 1227.35004

[Bahouri et al. 2012a] H. Bahouri, C. Fermanian-Kammerer, and I. Gallagher, *Phase-space analysis and pseudodifferential calculus on the Heisenberg group*, Astérisque **342**, Société Mathématique de France, Paris, 2012. MR 2952066 Zbl 1246.35003

[Bahouri et al. 2012b] H. Bahouri, C. Fermanian-Kammerer, and I. Gallagher, "Refined inequalities on graded Lie groups", *C. R. Math. Acad. Sci. Paris* **350**:7–8 (2012), 393–397. MR 2922090 Zbl 1241.22012

[Banica 2007] V. Banica, "The nonlinear Schrödinger equation on hyperbolic space", *Comm. Partial Differential Equations* **32**:10–12 (2007), 1643–1677. MR 2372482 Zbl 1143.35091

[Banica and Duyckaerts 2007] V. Banica and T. Duyckaerts, "Weighted Strichartz estimates for radial Schrödinger equation on noncompact manifolds", *Dyn. Partial Differ. Equ.* **4**:4 (2007), 335–359. MR 2376801 Zbl 1137.35010

[Brenner 1975] P. Brenner, "On L_p–$L_{p'}$ estimates for the wave-equation", *Math. Z.* **145**:3 (1975), 251–254. MR 0387819 Zbl 0321.35052

[Burq et al. 2004] N. Burq, P. Gérard, and N. Tzvetkov, "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds", *Amer. J. Math.* **126**:3 (2004), 569–605. MR 2058384 Zbl 1067.58027

[Burq et al. 2008] N. Burq, G. Lebeau, and F. Planchon, "Global existence for energy critical waves in 3-D domains", *J. Amer. Math. Soc.* **21**:3 (2008), 831–845. MR 2393429 Zbl 1204.35119

[Ciatti et al. 2005] P. Ciatti, F. Ricci, and M. Sundari, "Uncertainty inequalities on stratified nilpotent groups", *Bull. Kerala Math. Assoc.* Special Issue (2005), 53–72. MR 2250035

[Corwin and Greenleaf 1990] L. J. Corwin and F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications, I: Basic theory and examples*, Cambridge Studies in Advanced Mathematics **18**, Cambridge University Press, 1990. MR 1070979 Zbl 0704.22007

[Del Hierro 2005] M. Del Hierro, "Dispersive and Strichartz estimates on H-type groups", *Studia Math.* **169**:1 (2005), 1–20. MR 2139639 Zbl 1076.22010

[Folland 1989] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies **122**, Princeton University Press, 1989. MR 983366 Zbl 0682.43001

[Folland and Stein 1982] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press, 1982. MR 657581 Zbl 0508.42025

[Furioli et al. 2007] G. Furioli, C. Melzi, and A. Veneruso, "Strichartz inequalities for the wave equation with the full Laplacian on the Heisenberg group", *Canad. J. Math.* **59**:6 (2007), 1301–1322. MR 2363068 Zbl 1135.22008

[Ginibre and Velo 1995] J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", *J. Funct. Anal.* **133**:1 (1995), 50–68. MR 1351643 Zbl 0849.35064

[Ivanovici et al. 2014] O. Ivanovici, G. Lebeau, and F. Planchon, "Dispersion for the wave equation inside strictly convex domains I: The Friedlander model case", *Ann. of Math.* (2) **180**:1 (2014), 323–380. MR 3194817 Zbl 1310.35151

[Kapitanski 1989] L. V. Kapitanski, "Некоторые обобщения неравенства Стрихарца–Бреннера", *Alg. Anal.* **1**:3 (1989), 127–159. Translated as "Some generalizations of the Strichartz–Brenner inequality" in *Leningrad Math. J.* **1**:3 (1990), 693–726. MR 1015129 Zbl 0732.35118

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR 1646048 Zbl 0922.35028

[Klainerman and Rodnianski 2005] S. Klainerman and I. Rodnianski, "Rough solutions of the Einstein-vacuum equations", *Ann. of Math.* (2) **161**:3 (2005), 1143–1193. MR 2180400 Zbl 1089.83006

[Korányi 1985] A. Korányi, "Geometric properties of Heisenberg-type groups", *Adv. Math.* **56**:1 (1985), 28–38. MR 782541 Zbl 0589.53053

[Ludwig 1995] J. Ludwig, "Dual topology of diamond groups", *J. Reine Angew. Math.* **467** (1995), 67–87. MR 1355922 Zbl 0833.22011

[Moore and Wolf 1973] C. C. Moore and J. A. Wolf, "Square integrable representations of nilpotent groups", *Trans. Amer. Math. Soc.* **185** (1973), 445–462. MR 0338267 Zbl 0274.22016

[Müller and Ricci 1996] D. Müller and F. Ricci, "Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups", *Ann. of Math.* (2) **143**:1 (1996), 1–49. MR 1370756 Zbl 0841.22006

[Pecher 1976] H. Pecher, "L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen, I", *Math. Z.* **150**:2 (1976), 159–183. MR 0435604 Zbl 0318.35054

[Poguntke 1999] D. Poguntke, "Unitary representations of diamond groups", *Math. Scand.* **85**:1 (1999), 121–160. MR 1707750 Zbl 0955.22010

[Segal 1976] I. E. Segal, "Space-time decay for solutions of wave equations", *Adv. Math.* **22**:3 (1976), 305–311. MR 0492892 Zbl 0344.35058

[Smith 1998] H. F. Smith, "A parametrix construction for wave equations with $C^{1,1}$ coefficients", *Ann. Inst. Fourier* (*Grenoble*) **48**:3 (1998), 797–835. MR 1644105 Zbl 0974.35068

[Smith and Sogge 1995] H. F. Smith and C. D. Sogge, "On the critical semilinear wave equation outside convex obstacles", *J. Amer. Math. Soc.* **8**:4 (1995), 879–916. MR 1308407 Zbl 0860.35081

[Smith and Tataru 2005] H. F. Smith and D. Tataru, "Sharp local well-posedness results for the nonlinear wave equation", *Ann. of Math.* (2) **162**:1 (2005), 291–366. MR 2178963 Zbl 1098.35113

[Stein 1986] E. M. Stein, "Oscillatory integrals in Fourier analysis", pp. 307–355 in *Beijing lectures in harmonic analysis* (Beijing, 1984), edited by E. M. Stein, Annals of Mathematics Studies **112**, Princeton University Press, 1986. MR 864375 Zbl 0618.42006

[Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR 1232192 Zbl 0821.42001

[Stein and Weiss 1971] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, 1971. MR 0304972 Zbl 0232.42007

[Strichartz 1977] R. S. Strichartz, "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations", *Duke Math. J.* **44**:3 (1977), 705–714. MR 0512086 Zbl 0372.35001

[Tao 2006] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics **106**, American Mathematical Society, Providence, RI, 2006. MR 2233925 Zbl 1106.35001

[Tataru 2001] D. Tataru, "Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation", *Trans. Amer. Math. Soc.* **353**:2 (2001), 795–807. MR 1804518 Zbl 0956.35088

Received 22 Mar 2014. Revised 24 Nov 2015. Accepted 30 Jan 2016.

HAJER BAHOURI: hajer.bahouri@math.cnrs.fr
Université Paris Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR CNRS 8050, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France

CLOTILDE FERMANIAN-KAMMERER: clotilde.fermanian@u-pec.fr
Université Paris Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR CNRS 8050, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France

ISABELLE GALLAGHER: isabelle.gallagher@math.univ-paris-diderot.fr
Institut de Mathématiques UMR 7586, Université Paris Diderot (Paris 7), Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US $/year for the electronic version, and $/year (+$, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

ANALYSIS & PDE

Volume 9 No. 3 2016

Local analytic regularity in the linearized Calderón problem 515
JOHANNES SJÖSTRAND and GUNTHER UHLMANN

Dispersive estimates for the Schrödinger operator on step-2 stratified Lie groups 545
HAJER BAHOURI, CLOTILDE FERMANIAN-KAMMERER and ISABELLE GALLAGHER

Obstructions to the existence of limiting Carleman weights 575
PABLO ANGULO-ARDOY, DANIEL FARACO, LUIS GUIJARRO and ALBERTO RUIZ

Finite chains inside thin subsets of $\mathbb{R}^d$ 597
MICHAEL BENNETT, ALEXANDER IOSEVICH and KRYSTAL TAYLOR

Advection-diffusion equations with density constraints 615
ALPÁR RICHÁRD MÉSZÁROS and FILIPPO SANTAMBROGIO

Asymptotic stability in energy space for dark solitons of the Landau–Lifshitz equation 645
YAKINE BAHRI

On the well-posedness of the generalized Korteweg–de Vries equation in scale-critical $\hat{L}^r$-space 699
SATOSHI MASAKI and JUN-ICHI SEGATA

Regularity for parabolic integro-differential equations with very irregular kernels 727
RUSSELL W. SCHWAB and LUIS SILVESTRE

2157-5045(2016)9:3;1-9