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TIME-PERIODIC APPROXIMATIONS OF THE EULER–POISSON SYSTEM NEAR LANE–EMDEN STARS

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We show a long-time validity of the time-periodic linear approximations to the gravitational Euler–Poisson system near Lane–Emden equilibria for all relevant adiabatic exponents. To prove the result, we reformulate the problem in Lagrangian coordinates and use the weighted energy estimates together with Hardy inequalities.

1. Introduction and formulation

One of the simplest fundamental hydrodynamical models for describing the motion of self-gravitating Newtonian inviscid gaseous stars is the compressible Euler–Poisson equations:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) &= 0, \\ \rho(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}) + \nabla p &= -\rho \nabla \Phi, \\ \Delta \Phi &= 4\pi \rho, \end{aligned} \tag{1-1}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and ρ, $\boldsymbol{u}$ and p denote respectively the density, velocity and pressure of gas. Φ is the gravitational potential and it is related to the gas through the Poisson equation. We consider polytropic gases with equation of state given by

$$p = K\rho^\gamma, \tag{1-2}$$

where K is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent. There are many interesting works available on the Euler–Poisson system (1-1); for instance, see [Luo et al. 2014; Makino and Ukai 1987; Nishida 1986] for the existence theory, [Makino 1992] for a nonexistence result and blowup, and [Deng et al. 2002; Jang 2008; 2014; Luo and Smoller 2008; 2009; Rein 2003] for the stability and instability theory. However, some important questions are still waiting to be answered. In this paper, we are interested in long-time radial solutions to (1-1) around the Lane–Emden equilibrium stars.

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The spherically symmetric solutions to the system (1-1) — $\rho(t,x)=\rho(t,r)$ and $\boldsymbol{u}(t,x)=u(t,r)x/r$, where $r=|x|$ — satisfy the equations

$$\begin{aligned}\rho_t+\frac{1}{r^2}(r^2\rho u)_r&=0,\\ \rho u_t+\rho u u_r+p_r+\frac{4\pi\rho}{r^2}\int_0^r \rho s^2\,ds&=0.\end{aligned}\tag{1-3}$$

Static solutions ($\rho_0(r)$, $u_0=0$) of (1-3) satisfy the ordinary differential equation

$$\frac{dp}{dr}+\frac{4\pi\rho}{r^2}\int_0^r \rho s^2\,ds=0,\tag{1-4}$$

which can be transformed into the famous Lane–Emden equation [Chandrasekhar 1938]. Nonnegative solutions of (1-4) can be characterized according to γ as follows [Chandrasekhar 1938; Lin 1997]: Letting $M(\rho)\equiv\int 4\pi s^2\rho(s)\,ds$ be the total mass of a star, if $\gamma>\frac{6}{5}$ and $M>0$ then there exists at least one compactly supported solution ρ such that $M(\rho)=M$. For $\gamma>\frac{4}{3}$, every solution is compactly supported and unique. If $\gamma=\frac{6}{5}$ and $M>0$, there is a unique solution ρ with infinite support. If $1<\gamma<\frac{6}{5}$, there are no stationary solutions with finite total mass. The compactly supported equilibria for $\frac{6}{5}<\gamma<2$ are called the Lane–Emden stars; see also Section 1B.

It is well known [Chandrasekhar 1938; Lin 1997] that the boundary behavior of compactly supported Lane–Emden solutions is characterized as follows:

$$\bar\rho(r)\sim(R-r)^{1/(\gamma-1)}\quad\text{for } r\sim R.\tag{1-5}$$

This boundary behavior is often referred to as physical vacuum [Liu and Yang 2000]. As far as the full dynamics of compressible flows involving physical vacuum is concerned, the degeneracy and the interaction with nonlinearity make the analysis nontrivial. Despite its physical importance, even local-in-time well-posedness of compressible Euler equations in the presence of physical vacuum was established only recently [Coutand and Shkoller 2012; Jang and Masmoudi 2009; 2015]. For more discussion on physical vacuum, we refer to [Jang and Masmoudi 2011] and for other problems involving vacuum see [Jang and Masmoudi 2012; Liu 1996; Liu and Yang 1997; Makino et al. 1986; Sideris 2014].

The goal of this article is to investigate a detailed structure of the solutions to (1-3) near the compactly supported Lane–Emden stars beyond the local existence. More specifically, we will construct the time-periodic linearized solutions and show the validity of such linear approximations in the fully nonlinear setting for large times for all $\frac{6}{5}<\gamma<2$. To this end, we will first introduce suitable Lagrangian coordinates in accordance with the recent advancement of physical vacuum, and formulate the problem in such Lagrangian coordinates.

1A. ***Lagrangian coordinates.*** Let $\eta(t,x)$ be the position of the gas particle x at time t, so that

$$\eta_t=\boldsymbol{u}(t,\eta(t,x))\ \text{ for } t>0\quad\text{and}\quad \eta(0,x)=\eta_0\quad\text{in } \Omega.\tag{1-6}$$

Here Ω is a compact smooth domain and $\eta_0:\Omega\to\Omega$ is a diffeomorphism with positive Jacobian determinant. For the purpose of this article, we take Ω as a ball, which corresponds to the support of a

Lane–Emden solution and the initial density. Our choice of η_0 will depend on the initial density profile and in fact, in our setup, the identity map will correspond to the equilibrium state. The following are the Lagrangian quantities:

$$\begin{aligned} &\boldsymbol{v}(t,x) \equiv \boldsymbol{u}(t,\eta(t,x)), \quad &&\varrho(t,x) \equiv \rho(t,\eta(t,x)), \quad &&\Psi(t,x) \equiv \Phi(t,\eta(t,x)), \\ &A \equiv (D\eta)^{-1}, && J \equiv \det D\eta, && a \equiv JA. \end{aligned}$$

We use Einstein's summation convention and the notation $F,_k$ to denote the k-th partial derivative of F. In this subsection, we use i, j, k, l, r, s to denote 1, 2, 3. The Euler–Poisson equations (1-1) read as

$$\begin{aligned} \varrho_t + \varrho A_i^j \boldsymbol{v}^i,_j &= 0, \\ \varrho \boldsymbol{v}_t^i + K A_i^k \varrho^\gamma,_k &= -\varrho A_i^k \Psi,_k, \\ A_i^k (A_i^l \Psi,_l),_k &= 4\pi \varrho. \end{aligned} \tag{1-7}$$

Since $J_t = J A_i^j v^i,_j$ we find that $\varrho J = \rho(0) J(0) = \rho_{\text{in}} \det D\eta_0$, where ρ_{in} is a given initial density function. For ρ_{in} exhibiting the same boundary behavior as $\bar\rho$ such that $\rho_{\text{in}}/\bar\rho$ is a smooth positive function, we choose η_0 so that

$$\varrho J = \rho_{\text{in}} \det D\eta_0 = \bar\rho, \tag{1-8}$$

where $\bar\rho$ is the equilibrium density profile of the Lane–Emden star given by (1-4). Existence of such an η_0 follows from the Dacorogna–Moser theorem [1990].

By using the relation $A_i^k = J^{-1} a_i^k$, we see that the system (1-7) is reduced to

$$\begin{aligned} \bar\rho \boldsymbol{v}_t^i + K a_i^k (\bar\rho^\gamma J^{-\gamma}),_k &= -\bar\rho A_i^k \Psi,_k, \\ A_i^k (A_i^l \Psi,_l),_k &= 4\pi \bar\rho J^{-1}, \end{aligned} \tag{1-9}$$

along with

$$\eta_t^i = \boldsymbol{v}^i. \tag{1-10}$$

Now we introduce the equilibrium enthalpy

$$w \equiv K \bar\rho^{\gamma-1}. \tag{1-11}$$

We will work with the enthalpy w rather than the density $\bar\rho$, since w behaves like a distance function near the boundary regardless of the values of γ under the physical vacuum condition (1-5). This w will be treated as the weight function. By using the Piola identity $a_i^k,_k = 0$, we see that the system (1-9) takes the form

$$\begin{aligned} w^\alpha \boldsymbol{v}_t^i + (w^{1+\alpha} A_i^k J^{-1/\alpha}),_k &= -w^\alpha A_i^k \Psi,_k, \\ A_i^k (A_i^l \Psi,_l),_k &= 4\pi K^{-\alpha} w^\alpha J^{-1}, \end{aligned} \tag{1-12}$$

where

$$\alpha \equiv \frac{1}{\gamma - 1}. \tag{1-13}$$

Here α has been introduced for notational convenience. We will use both α and γ, which are related through (1-13), in the equations and the estimates throughout the article. For instance, the range of the adiabatic exponents of our interest reads in terms of α as

$$\tfrac{6}{5} < \gamma < 2 \iff 1 < \alpha < 5.$$

For the spherically symmetric Euler–Poisson flows, it is convenient to introduce the expansion and contraction variable ξ as

$$\eta(t,x) \equiv \xi(t,r)x, \tag{1-14}$$

where $r = |x|$. Since $\eta_t = \xi_t\, x = \boldsymbol{v}$, we have $v(t,r) = r\,\xi_t$. Since $\partial_i = (x^i/r)\partial_r$, we can write

$$J = \xi^2(\xi + \xi_r r) \quad \text{and} \quad (D\eta)^{-1} = \frac{1}{\xi} I - \frac{\xi_r}{\xi(\xi+\xi_r r)r}(x^i x^j) \tag{1-15}$$

and hence A_i^k is given by

$$A_i^k = \frac{\delta_i^k}{\xi} - \frac{\xi_r\, x^k x^i}{\xi(\xi+\xi_r r)r}. \tag{1-16}$$

Now, for spherically symmetric functions, the gradient $A_i^k \partial_k$ is given by

$$A_i^k \partial_k = \frac{x^i}{r(\xi+\xi_r r)}\partial_r$$

and the Laplacian $A_i^k \partial_k (A_i^l \partial_l)$ is given by

$$A_i^k \partial_k (A_i^l \partial_l) = \frac{1}{(\xi+\xi_r r)(\xi r)^2}\partial_r\left(\frac{(\xi r)^2}{\xi+\xi_r r}\partial_r\right).$$

Thus the Poisson equation in (1-12) for spherically symmetric flows takes the form

$$\frac{1}{(\xi+\xi_r r)(\xi r)^2}\partial_r\left(\frac{(\xi r)^2}{\xi+\xi_r r}\Psi_r\right) = 4\pi K^{-\alpha} w^\alpha J^{-1}. \tag{1-17}$$

Based on (1-14), (1-23) and (1-16), we see that the momentum equation in (1-12) for spherically symmetric flows can be written as an equation for ξ:

$$w^\alpha \xi_{tt} + \frac{\xi^2}{r}\partial_r\big(w^{1+\alpha}(\xi^2(\xi+\xi_r r))^{-\gamma}\big) + \frac{w^\alpha}{r(\xi+\xi_r r)}\Psi_r = 0 \tag{1-18}$$

for $t \ge 0$ and $0 \le r \le R$, where R is the radius of the Lane–Emden star. We remark that no boundary conditions are necessary to construct smooth solutions for (1-18) due to the degenerate weights [Coutand and Shkoller 2012; Jang and Masmoudi 2015]. More detail on the Lagrangian formulation described in the above can be found in [Jang 2014].

Note that from (1-17) the potential term can be also written as

$$\frac{w^\alpha}{r(\xi+\xi_r r)}\Psi_r = \frac{w^\alpha}{\xi^2 r^3}\int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\, ds = \frac{w^\alpha}{\xi^2 r^3}\int_{B(0,r)} \bar\rho\, dx.$$

This potential term has the right weight w^α and it is of lower order with respect to the differential structure. It looks harmless. However, the potential term plays an important role in the stability theory, as shown in [Jang 2014; Rein 2003]. Not surprisingly, we will show that it also has an impact on the validity time of the time-periodic linear approximations.

1B. ***Lane–Emden star configuration in the Lagrangian formulation.*** In this subsection, we will identify the Lane–Emden stars satisfying (1-4) in our Langrangian formulation. The static equilibria of the Euler–Poisson system under spherical symmetry governed by (1-18) can be found by setting $\xi \equiv 1$. It is clear that w satisfies the ordinary differential equation

$$\frac{1}{r}\partial_r(w^{1+\alpha}) + \frac{w^\alpha}{r^3}\int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\,ds = 0, \tag{1-19}$$

or equivalently

$$w_{rr} + \frac{2}{r}w_r + \frac{4\pi}{(1+\alpha)K^\alpha} w^\alpha = 0. \tag{1-20}$$

This is the so-called Lane–Emden equation, which has been studied extensively. In particular, we recall the well-known existence result from [Chandrasekhar 1938; Lin 1997]: supplemented with the normalized boundary conditions

$$w(0) = 1 \quad \text{and} \quad w_r(0) = 0$$

for a given finite total mass M, there exist a ball-type solution w to the Lane–Emden equation (1-20) and a finite radius R when $1 < \alpha < 5$, or equivalently $\frac{6}{5} < \gamma < 2$, such that (i) $w > 0$ for $0 < r < R$ and $w(R) = 0$; (ii) $-\infty < w_r < 0$ for $0 < r < R$; (iii) w satisfies the physical vacuum condition (1-5). The Lane–Emden configuration w enjoys better regularity. The regularity results of w are summarized in Section 2A.

We next write (1-18) in a perturbation form around the equilibrium state given by $\xi = 1$ and $\xi_t = 0$. Letting $\xi \equiv 1 + \zeta$ with $|\zeta| \ll 1$, we obtain the equation for ζ as

$$w^\alpha \zeta_{tt} + \frac{(1+\zeta)^2}{r}\partial_r\big(w^{1+\alpha}((1+\zeta)^2(1+\zeta+\zeta_r r))^{-\gamma}\big) + \frac{w^\alpha}{(1+\zeta)^2 r^3}\int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\,ds = 0. \tag{1-21}$$

1C. ***ψ formulation.*** We further introduce a variable ψ whose equation displays a better structure for the pressure gradient term in our coordinates. Let

$$\psi \equiv \zeta + \zeta^2 + \tfrac{1}{3}\zeta^3. \tag{1-22}$$

Then, since $d\psi/d\zeta = 1 + 2\zeta + \zeta^2 > 0$, by the inverse function theorem $\zeta = \zeta(\psi)$ can be regarded as a smooth function of ψ. Notice that

$$J = (1+\zeta)^2(1+\zeta+\zeta_r r) = 1 + \frac{1}{r^2}\big(r^3\big(\zeta+\zeta^2+\tfrac{1}{3}\zeta^3\big)\big)_r = 1 + \frac{1}{r^2}(r^3\psi)_r \tag{1-23}$$

and

$$\zeta_t = \frac{\psi_t}{(1+\zeta)^2} \quad \text{and} \quad \zeta_{tt} = \frac{\psi_{tt}}{(1+\zeta)^2} - \frac{2\psi_t^2}{(1+\zeta)^5}. \tag{1-24}$$

Thus (1-21) can be written in terms of ψ as

$$\frac{w^\alpha r^4\psi_{tt}}{(1+\zeta)^4}-\frac{2w^\alpha r^4\psi_t^2}{(1+\zeta)^7}+r^3\partial_r\biggl(w^{1+\alpha}\biggl(\Bigl(1+\frac{1}{r^2}(r^3\psi)_r\Bigr)^{-\gamma}-1\biggr)\biggr)\\ +\frac{1-(1+\zeta)^4}{(1+\zeta)^4}w^\alpha r\int_0^r\frac{4\pi}{K^\alpha}w^\alpha s^2\,ds=0. \tag{1-25}$$

Notice that (1-25) relies on the Lane–Emden equation (1-19).

Throughout the paper, we will use $\mathfrak{A}\lesssim\mathfrak{B}$ to denote that $\mathfrak{A}\le C\mathfrak{B}$ for a generic constant $C>0$. We will use big O notation to describe the leading order of small quantities.

2. Time-periodic linearized solutions and main result

In this section, we study the linearized Euler–Poisson system around compactly supported Lane–Emden stars for $\frac{6}{5}<\gamma<2$ (i.e., $1<\alpha<5$). We will first derive the linearized equation of (1-25). Notice that by Taylor's theorem, for sufficiently small ψ, the nonlinear pressure term in (1-25) can be written as

$$\Bigl(1+\frac{1}{r^2}(r^3\psi)_r\Bigr)^{-\gamma}=1-\frac{\gamma}{r^2}(r^3\psi)_r+h, \tag{2-1}$$

where h is a smooth function of $(1/r^2)(r^3\psi)_r$ and $h=O\bigl(\bigl|(1/r^2)(r^3\psi)_r\bigr|^2\bigr)$. Also, the ζ-related part of the last term in (1-25) can be written as

$$\frac{1-(1+\zeta)^4}{(1+\zeta)^4}=\frac{-4\zeta-6\zeta^2-4\zeta^3-\zeta^4}{(1+\zeta)^4}=\frac{-4\psi-2\zeta^2-\frac{8}{3}\zeta^3-\zeta^4}{(1+\zeta)^4}=-4\psi+f, \tag{2-2}$$

where f is a smooth function of ζ (and hence ψ) and $f=O(|\zeta|^2)=O(|\psi|^2)$ due to (1-22).

Then the linearized equation of (1-25) reads as

$$w^\alpha r^4\psi_{tt}-\gamma r^3\partial_r\Bigl(w^{1+\alpha}\frac{1}{r^2}(r^3\psi)_r\Bigr)+4r^3\partial_r(w^{1+\alpha})\psi=0, \tag{2-3}$$

where we have used (1-19). We will denote the last two terms by $L\psi$. A simple computation shows that

$$\begin{aligned}L\psi&=-\gamma r^3\partial_r\Bigl(w^{1+\alpha}\frac{1}{r^2}(r^3\psi)_r\Bigr)+4r^3\partial_r(w^{1+\alpha})\psi\\&=-\gamma(w^{1+\alpha}r^4\psi_r)_r+(4-3\gamma)r^3\partial_r(w^{1+\alpha})\psi.\end{aligned} \tag{2-4}$$

The associated eigenvalue problem is given by

$$L\psi=\lambda w^\alpha r^4\psi. \tag{2-5}$$

Then L is self-adjoint and hence λ is real. In fact, this eigenvalue problem was considered by Eddington [1918] to explain the luminosity variations of the Cepheid variables and Beyer [1995] studied the spectrum for L in $L^2((0,R),dr)$, which consists of simple eigenvalues $\lambda_1<\cdots<\lambda_n<\lambda_{n+1}<\cdots\to\infty$. See also Proposition 1 in [Makino 2015]. We recall that in [Lin 1997], the stability criterion was introduced based on the eigenvalues: w^α $(\sim\bar\rho)$ is called neutrally stable if $\lambda>0$ for all eigenvalues λ and unstable if

$\lambda < 0$ for some eigenvalue λ, and it was shown that w^α ($\sim\bar{\rho}$) is unstable for any $3 < \alpha < 5$ $\left(\frac{6}{5} < \gamma < \frac{4}{3}\right)$ and stable for $1 < \alpha < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$ in the mass Lagrangian framework. In particular, for $1 < \alpha < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$, the least eigenvalue λ_1 is positive.

Now fix a positive eigenvalue $\lambda = \lambda_n$ for some $\lambda_n > 0$ and an associated eigenfunction $\Psi = \Psi(r)$ of L:

$$L\Psi = \lambda w^\alpha r^4 \Psi. \tag{2-6}$$

We take Ψ that is bounded near both $r \sim 0$ and $r \sim R$, in particular $\Psi \in H$, where H is a Hilbert space with the norm

$$\|\Psi\|_H^2 \equiv \int_0^R w^{1+\alpha} r^4 (\Psi_r)^2\, dr + \int_0^R w^\alpha r^4 \Psi^2\, dr.$$

For more discussion on the existence of such Ψ, see [Makino 2015]. Then, for a given constant θ_0,

$$\psi_1(t, r) := \sin(\sqrt{\lambda} t + \theta_0)\Psi(r) \tag{2-7}$$

is a time-periodic solution to the linearized equation (2-3).

2A. *The behavior of Ψ near the origin and near the boundary.* Notice that Ψ satisfies

$$\lambda w^\alpha r^4 \Psi = -\gamma (w^{1+\alpha} r^4 \Psi')' + (4 - 3\gamma)(w^{1+\alpha})' r^3 \Psi. \tag{2-8}$$

We can deduce the regularity of Ψ from (2-8) based on the behavior of the Lane–Emden solution w. In what follows, we summarize the results from [Jang 2014] regarding w and Ψ.

Lemma 2.1 (regularity of w). *Let $1 < \alpha < 5$ be given and let w be a ball-type solution to the Lane–Emden equation (1-20). Then*:

(1) *w is analytic near the origin. Moreover*,

$$w(r) = 1 - br^2 + O(r^4), \quad r \sim 0,$$

for some positive constant $b > 0$. Also, $(\partial_r^{2k+1} w)(0) = 0$ for any nonnegative integer $k \geq 0$.

(2) *$\partial_r^i w$ is uniformly bounded on $(0, R)$ for each $0 \leq i \leq \alpha + 2$ and also $w^{(k-1)/2} \partial_r^{k+1} w$ is uniformly bounded on $(0, R)$ for each $1 \leq k \leq 2\alpha + 1$. In addition, w enjoys the integral regularity*

$$\int_0^R w^{\alpha+j} r^4 |\partial_r^{j+1} w|^2\, dr < \infty$$

for each $0 \leq j < 3\alpha + 3$.

Lemma 2.2. *Let $\Psi \in H$ be the solution to (2-8). Ψ is analytic at $r = 0$ and, moreover, $\Psi = a + O(r)$ around the origin, where a is a constant.*

Lemma 2.3. *Let Ψ be the solution to (2-8) in H. Then*:

(1) *Ψ has the following integrability: for any $0 \leq \beta \leq \alpha$,*

$$\int_0^R w^{\alpha-\beta} r^4 \Psi^2\, dr + \int_0^R w^{1+\alpha-\beta} r^4 (\Psi')^2\, dr < \infty.$$

Moreover, for any $z > 1$,

$$\int_0^R w^{z-2} r^4 \Psi^2 \, dr < \infty.$$

(2) Ψ *has the following regularity*: *for* $1 \le k \le 2\alpha + 1$,

$$\int_0^R w^{1+\alpha+k} r^4 (\partial_r^{k+1} \Psi)^2 \, dr < \infty.$$

The proofs of Lemmas 2.1, 2.2 and 2.3 can be found in [Jang 2014]. Based on the above lemmas, we deduce that Ψ belongs to the function spaces of interest to us, namely it has a finite total initial energy for $1 < \alpha < 5$; see (2-12) and (2-14).

2B. ***Main result.*** We are interested in solutions (ψ, ψ_t) of (1-25) with the form

$$\psi(t, r; \epsilon) = \epsilon \psi_1(t, r) + \epsilon^2 \varphi(t, r; \epsilon), \tag{2-9}$$

where ψ_1 is a time-periodic linearized solution given in (2-7) and ϵ is a small positive parameter. For given initial data for $(\zeta, \zeta_t)|_{t=0}$ or $(\psi, \psi_t)|_{t=0}$ having a finite energy via (2-14), we can construct local-in-time solutions to (1-21) and hence to (1-25) for $0 < t < T$, where T is independent of ϵ, by the local existence theory [Coutand and Shkoller 2012; Jang and Masmoudi 2015; Luo et al. 2014]. We can set $\epsilon^2 \varphi(t, r; \epsilon) := \psi(t, r; \epsilon) - \epsilon \psi_1(t, r)$ to deduce that $\epsilon^2 \varphi$ is bounded in the corresponding energy norm. However, φ could be very large when ϵ is small. Our aim is to show that this does not happen, namely φ is bounded for all sufficiently small ϵ for all $0 < t < T$. In order to establish $\|\varphi\| = O(1)$, we will derive the uniform-in-ϵ estimates of φ. Let us first derive the equation for φ.

Plugging the ansatz (2-9) into (1-25), using the fact that ψ_1 solves (2-3), and also using (2-2), we obtain

$$\begin{aligned}\frac{w^\alpha r^4 \varphi_{tt}}{(1+\zeta)^4} + \frac{w^\alpha r^4 (\psi_1)_{tt}}{\epsilon}\left(\frac{1}{(1+\zeta)^4} - 1\right) - \frac{2 w^\alpha r^4 |(\psi_1)_t + \epsilon \varphi_t|^2}{(1+\zeta)^7} \\ + \frac{r^3}{\epsilon^2} \partial_r \left(w^{1+\alpha} \left(\left(1 + \frac{1}{r^2} (r^3 (\epsilon \psi_1 + \epsilon^2 \varphi))_r \right)^{-\gamma} - 1 + \gamma \frac{1}{r^2} (r^3 \epsilon \psi_1)_r \right) \right) \\ - 4 w^\alpha r^4 \Phi(r) \varphi + w^\alpha r^4 \Phi(r) \frac{f}{\epsilon^2} = 0,\end{aligned}$$

where $\Phi(r)$ is the prescribed function defined by

$$\Phi(r) \equiv \frac{1}{r^3} \int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2 \, ds = -\frac{(w^{1+\alpha})_r}{r w^\alpha} = -(1+\alpha) \frac{w_r}{r}. \tag{2-10}$$

Notice that $\Phi(r) > 0$ for each $0 < r < R$. By further using $(\psi_1)_{tt} = -\lambda \psi_1$ as well as (2-2), we arrive at

$$\begin{aligned}\frac{w^\alpha r^4 \varphi_{tt}}{(1+\zeta)^4} + 4\lambda w^\alpha r^4 \psi_1^2 + 4\lambda \epsilon w^\alpha r^4 \psi_1 \varphi - \lambda w^\alpha r^4 \psi_1 \frac{f}{\epsilon} - \frac{2 w^\alpha r^4 |(\psi_1)_t + \epsilon \varphi_t|^2}{(1+\zeta)^7} \\ + \frac{r^3}{\epsilon^2} \partial_r \left(w^{1+\alpha} \left(\left(1 + \frac{1}{r^2} (r^3 (\epsilon \psi_1 + \epsilon^2 \varphi))_r \right)^{-\gamma} - 1 + \gamma \frac{1}{r^2} (r^3 \epsilon \psi_1)_r \right) \right) \\ - 4 w^\alpha r^4 \Phi(r) \varphi + w^\alpha r^4 \Phi(r) \frac{f}{\epsilon^2} = 0.\end{aligned} \tag{2-11}$$

We are concerned with the behavior of $(\varphi,\varphi_t)(t,r;\epsilon)$, the solutions of the initial value problem of (2-11) with given initial data $(\varphi,\varphi_t)(0,r;\epsilon)=(\varphi_0(r),\varphi_1(r))$. We remark that the appearance of ϵ in the denominator of the first and third lines in (2-11) is not harmful because $f=f(\psi)=O(|\psi|^2)=O(\epsilon^2)$. For the second line in (2-11) involving the second-order differential operator, at least formally, it is of order 1 with respect to ϵ. To make it rigorous, it needs to be treated very carefully. Notice that we have not decomposed it into the linear and nonlinear parts yet.

Motivated by the work on physical vacuum [Jang 2014; Jang and Masmoudi 2009; 2015], we consider the weighted energy norms: for $j\ge k\ge 0$,

$$\begin{aligned}\bar{\mathcal{E}}^{j,k}&\equiv\int_0^R w^{\alpha+k}r^4|\partial_t^{j-k}\partial_r^k\varphi_t|^2\,dr+\int_0^R w^{1+\alpha+k}r^4|\partial_t^{j-k}\partial_r^k\varphi_r|^2\,dr+\int_0^R w^{\alpha+k}r^4|\partial_t^{j-k}\partial_r^k\varphi|^2\,dr\\&\equiv\bar{\mathcal{E}}_t^{j,k}+\bar{\mathcal{E}}_r^{j,k}+\bar{\mathcal{E}}_0^{j,k}.\end{aligned}\tag{2-12}$$

Notice that the following relations hold:

$$\bar{\mathcal{E}}_t^{j,k}=\bar{\mathcal{E}}_r^{j,k-1}\quad\text{for }\ j\ge k\ge 1;\qquad \bar{\mathcal{E}}_0^{j,k}=\bar{\mathcal{E}}_t^{j-1,k}\quad\text{for }\ j\ge 1,\ j\ge k\ge 0.\tag{2-13}$$

We define the total energy $\bar{\mathcal{E}}$ by

$$\bar{\mathcal{E}}(t)\equiv\sum_{j=0}^{[\alpha]+4}\sum_{k=0}^{j}\bar{\mathcal{E}}^{j,k}(t),\tag{2-14}$$

where $[\alpha]=\max\{N\in\mathbb{Z}:N\le\alpha\}$, so that $0\le\alpha-[\alpha]<1$.

We also introduce the energy space

$$Z_\alpha=\left\{(\varphi_0,\varphi_1)\;\middle|\;\sum_{k=0}^{[\alpha]+5}\int_0^R w^{\alpha+k}r^4|\partial_r^k\varphi_0|^2\,dr+\sum_{k=0}^{[\alpha]+4}\int_0^R w^{\alpha+k}r^4|\partial_r^k\varphi_1|^2\,dr<\infty\right\}.$$

We are now ready to state our main result.

Theorem 2.4. *For given initial data $(\varphi_0,\varphi_1)\in Z_\alpha$ independent of ϵ, let $(\varphi,\varphi_t)=(\varphi,\varphi_t)(t,r;\epsilon)$ be the solution of (2-11) with finite total energy for $0<t\le T$ satisfying $(\varphi,\varphi_t)(0,r;\epsilon)=(\varphi_0(r),\varphi_1(r))$. Then, if $1<\alpha<3$ $\left(\frac{4}{3}<\gamma<2\right)$, there exists an $\epsilon_0=O(1/T)>0$ such that $\sup_{0<t\le T}\bar{\mathcal{E}}(t)=O(1)$ for all $0<\epsilon\le\epsilon_0$, and, if $3\le\alpha<5$ $\left(\frac{6}{5}<\gamma\le\frac{4}{3}\right)$, there exists an $\epsilon_0=O(1/e^{\kappa T})>0$ for some constant $\kappa>0$ such that $\sup_{0<t\le T}\bar{\mathcal{E}}(t)=O(1)$ for all $0<\epsilon\le\epsilon_0$.*

As a direct consequence of Theorem 2.4, we have $\|\psi-\epsilon\psi_1\|_{\bar{\mathcal{E}}}=O(\epsilon^2)$, which asserts the validity of the time-periodic linear approximations ψ_1 defined in (2-7) for the nonlinear solutions ψ to (1-25) having the form of (2-9). In fact, Theorem 2.4 recasts a recent work by Makino [2015], in which the time-periodic linear approximations were shown for γ for which $\gamma/(\gamma-1)$ is an integer and $\frac{6}{5}<\gamma<2$ in a suitable weighted Sobolev space. More importantly, our theorem covers all the relevant exponents γ and it answers an open problem proposed in [Makino 2015]. We take a different approach: while in [Makino 2015], the Nash–Moser–Hamilton theory was used to prove the result, we use the weighted energy estimates that have been proven to be useful to study physical vacuum states of compressible flows [Coutand and Shkoller 2012; Jang and Masmoudi 2015].

The energy inequalities obtained in this article yield a rather concrete upper bound for the total energy involving ϵ, which gives an estimate for an upper bound for ϵ_0 as stated in the theorem. It is noteworthy to observe the qualitative difference on the upper bound ϵ_0 between $\frac{4}{3} < \gamma < 2$ and $\frac{6}{5} < \gamma \leq \frac{4}{3}$. We recall that $\frac{4}{3} < \gamma < 2$ corresponds to the stability regime of Lane–Emden stars and $\frac{6}{5} < \gamma \leq \frac{4}{3}$ to the instability regime [Deng et al. 2002; Jang 2008; 2014; Lin 1997; Rein 2003]. Our result indicates that for a given large time T, a small expansion (approximation) parameter ϵ in the instability regime needs to be taken much smaller than the ϵ in the stability regime in order to guarantee the validity of the expansion (approximation) ansatz (2-9). Even if the same $\lambda > 0$ is allowed to be chosen in (2-7), the set of small parameters ϵ to hold up the validity of such linear approximations could be very different depending on the value of the adiabatic exponent γ. Of course, this comparison and characterization deduced from the energy inequalities may not be optimal.

The estimates of φ obtained in the subsequent sections can be used to establish the existence of the solutions ψ to (1-25) of the form (2-9) with the corresponding initial data of the same expansion form having a finite total energy. We will not pursue this direction in detail in this article, but will make one comment. In this perspective, one can fix a small parameter ϵ first and then derive a lower bound on $T = T(\epsilon)$ that guarantees the existence of the solutions. Then Theorem 2.4 implies that $T = O(1/\epsilon)$ for $\gamma > \frac{4}{3}$ and $T = O(\ln(1/\epsilon))$ for $\frac{6}{5} < \gamma \leq \frac{4}{3}$. We observe that the lifespan of the solutions having finite total energy for a given small $\epsilon > 0$ may depend on whether γ falls into the stability regime or not. Again, this comparison may not be optimal; it would be an interesting problem to study the optimality of such lower bounds.

We can also consider the limit of $\epsilon \to 0$ and the convergence rate. Note that a maximal time T of the convergence of ψ to 0 (0 corresponds to the Lane–Emden stars) goes to infinity as $\epsilon \to 0$, namely the convergence to the equilibrium becomes global. And the rate of convergence may depend on whether the value of γ is in the stability regime or not. It is interesting to point out that a similar question was studied in a completely different context, Hilbert expansion from the Boltzmann theory [Guo et al. 2010; Guo and Jang 2010].

Finally, we remark that by no means does Theorem 2.4 imply a stability result in the usual sense, but it gives a set of initial data having the form (2-9) of which evolutions for later times stay in the same form. In particular, it was shown in [Jang 2014] that for $\frac{6}{5} < \gamma < \frac{4}{3}$ there exists a family of initial data for (1-21) leading to a nonlinear instability for the Lane–Emden equilibrium and thus there's no hope to show the stability result for general initial data. On the other hand, for $\gamma > \frac{4}{3}$, [Rein 2003] gives a nonlinear stability result based on a variational approach. However, the result of [Rein 2003] is conditional, in that the existence of the desired solutions was assumed without a proof. It still remains an interesting open problem to prove a complete stability result for the Euler–Poisson system for $\gamma > \frac{4}{3}$ and we hope that this work provides interesting evidence towards a satisfactory stability theory.

The rest of the paper will be devoted to the proof of Theorem 2.4. The proof consists of three parts. First we give the L^∞ bounds of functions in terms of our energy norms (2-12) by using Hardy inequalities. Then we derive the energy inequalities for nonlinear instant energies (4-1) by the weighed energy method. The estimates of the total energy involving spatial and mixed derivatives are obtained by elliptic estimates.

The embedding results will be used to close the weighted energy estimates as well as the elliptic estimates for the solutions of (2-11). The final step of the proof, solving differential inequalities, will be given in Section 7.

3. L^∞ bounds and embeddings

The goal of this section is to derive the L^∞ bounds of φ and its derivatives with suitable weights by using the energy norms introduced in (2-12) and (2-14). To this end, we will utilize the Hardy inequalities and embedding inequalities.

3A. ***Hardy inequalities.*** We recall the following version of the Hardy inequality:

Lemma 3.1 (Hardy inequality). *Let $k > 1$ be a given real number and let g be a function satisfying $\int_0^1 s^k(g^2+g'^2)\,ds < \infty$. Then we have*

$$\int_0^1 s^{k-2}g^2\,ds \lesssim \int_0^1 s^k(g^2+|g'|^2)\,ds.$$

For the proof of Lemma 3.1, we refer to [Kufner et al. 2007]. Since our energies involve different weights near the origin and near the boundary, we will utilize the localized version of the above Hardy inequalities as in [Jang 2014]. We begin by recalling the following results:

Lemma 3.2 [Jang 2014]. (1) *For any function u satisfying $\int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr < \infty$,*

$$\int_0^{R/2} r^2|u|^2\,dr \lesssim \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr. \tag{3-1}$$

(2) *For any function u satisfying $\int_0^{3R/4} r^4|u_{rr}|^2\,dr + \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr < \infty$,*

$$\int_0^{R/2} |u|^2\,dr \lesssim \int_0^{3R/4} r^4|u_{rr}|^2\,dr + \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr. \tag{3-2}$$

(3) *Let $a > 1$ be given. For any function v satisfying $\int_{R/4}^{R} w^a|v_r|^2\,dr + \int_{R/4}^{R} w^a|v|^2\,dr < \infty$,*

$$\int_{R/2}^{R} w^{a-2}|v|^2\,dr \lesssim \int_{R/4}^{R} w^a|v_r|^2\,dr + \int_{R/4}^{R} w^a|v|^2\,dr. \tag{3-3}$$

We can now derive Hardy embedding inequalities.

Lemma 3.3. *Let m be any nonnegative integer. Then*

$$\|u\|_{L^1}^2 \lesssim \sum_{k=0}^{2}\int_0^{3R/4} r^4|\partial_r^k u|^2\,dr + \sum_{k=0}^{m}\int_{R/4}^{R} w^{\alpha-[\alpha]+2m}|\partial_r^k u|^2\,dr. \tag{3-4}$$

Proof. Consider

$$\int_0^R |u|\,dr = \int_0^{R/2} |u|\,dr + \int_{R/2}^{R} |u_r|\,dr =: \text{(i)} + \text{(ii)}.$$

By Hölder's inequality and (3-2), we obtain

$$\text{(i)} \lesssim \left(\int_0^{R/2} |u|^2\,dr\right)^{\frac{1}{2}} \lesssim \left(\int_0^{3R/4} r^4|u|^2\,dr + \int_0^{3R/4} r^4|\partial_r u|^2\,dr + \int_0^{3R/4} r^4|\partial_r^2 u|^2\,dr\right)^{\frac{1}{2}}.$$

For (ii), we first apply Hölder's inequality to get

$$\text{(ii)} \le \left(\int_{R/2}^{R} w^{-\alpha+[\alpha]}\,dr\right)^{\frac{1}{2}} \left(\int_{R/2}^{R} w^{\alpha-[\alpha]}|u|^2\,dr\right)^{\frac{1}{2}}.$$

Notice that $\int_{R/2}^{R} w^{-\alpha+[\alpha]}\,dr < \infty$, since $0 \le \alpha - [\alpha] < 1$ and $w \sim R - r$ near $r = R$. We then apply the localized Hardy inequality (3-3) to the second term repeatedly to deduce the result. □

Lemma 3.4. *Let m be any nonnegative integer. Then*

$$\|u\|_\infty^2 \lesssim \sum_{k=0}^{3} \int_0^{3R/4} r^4|\partial_r^k u|^2\,dr + \sum_{k=0}^{m+1} \int_{R/4}^{R} w^{\alpha-[\alpha]+2m}|\partial_r^k u|^2\,dr. \tag{3-5}$$

Proof. Notice that, since u is a function on the interval $(0, R)$, u is bounded by the $W^{1,1}$-norm:

$$\|u\|_\infty \lesssim \int_0^R |u|\,dr + \int_0^R |u_r|\,dr.$$

By applying (3-4) to each term, we obtain the desired result. □

3B. *L^∞ bounds.* A direct consequence of the above Hardy embedding inequalities is the validity of the boundedness assumption (4-9) within our energy space.

Lemma 3.5. (1)
$$|\varphi| + |\varphi_t| + |\varphi_{tt}| + \sum_{q=1}^{[\alpha]+2} |r^{\delta(q)} w^{(q-1)/2} \partial_t^{q+2}\varphi| \lesssim \bar{\mathcal{E}}^{1/2},$$

where $\delta(q) = 0$ for $q \le [\alpha]$, $\delta(q) = 1$ for $q = [\alpha]+1$, and $\delta(q) = 2$ for $q = [\alpha]+2$.

(2)
$$|\varphi_r| + |\varphi_{tr}| + \sum_{q=1}^{[\alpha]+2} |r^{\delta(q)} w^{q/2} \partial_t^{q+1}\partial_r\varphi| \lesssim \bar{\mathcal{E}}^{1/2},$$

where $\delta(q) = 0$ for $q \le [\alpha]$, $\delta(q) = 1$ for $q = [\alpha]+1$, and $\delta(q) = 2$ for $q = [\alpha]+2$.

Proof. We will present the details for the terms

$$\partial_t^3\varphi, \quad \partial_t\partial_r\varphi, \quad r^{\delta(2)} w^{1/2}\partial_t^4\varphi, \quad r^2 w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi.$$

Other terms can be treated in the same way. To see the boundedness of $\partial_t^3\varphi$, we apply (3-5) for $u = \partial_t^3\varphi$ with $m = [\alpha]+1$:

$$\|\partial_t^3\varphi\|_\infty^2 \lesssim \sum_{k=0}^{3} \int_0^{3R/4} r^4|\partial_r^k\partial_t^3\varphi|^2\,dr + \sum_{k=0}^{[\alpha]+2} \int_{R/4}^{R} w^{\alpha-[\alpha]+2[\alpha]+2}|\partial_r^k\partial_t^3\varphi|^2\,dr.$$

Then, since w is bounded from below and above on $\left(0, \frac{3}{4}R\right)$ and r is bounded from below and above on $\left(\frac{1}{4}R, R\right)$, we deduce that the right-hand side is bounded by $\bar{\mathcal{E}}$.

To see the boundedness of $\partial_t\partial_r\varphi$, we apply (3-5) for $u=\partial_t\partial_r\varphi$ with $m=[\alpha]+2$:

$$\|\partial_t\partial_r\varphi\|_\infty^2 \lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^{k+1}\partial_t\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+3}\int_{R/4}^{R} w^{\alpha-[\alpha]+2[\alpha]+4}|\partial_r^{k+1}\partial_t\varphi|^2\,dr.$$

It is easy to see that the right-hand side is bounded by $\bar{\mathcal{E}}$.

For the boundedness of $r^{\delta(2)}w^{1/2}\partial_t^3\varphi$, we divide into two cases: $2\le[\alpha]\le 4$ and $[\alpha]=1$. For the first case, $\delta(2)=0$. In this case, it suffices to show the boundedness of $w(\partial_t^4\varphi)^2$. By the Sobolev embedding,

$$\|w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R w(\partial_t^4\varphi)^2\,dr+\int_0^R |(w(\partial_t^4\varphi)^2)_r|\,dr.$$

Since w_r is bounded, by using the Cauchy–Schwarz inequality,

$$\|w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R |\partial_t^4\varphi|^2\,dr+\int_0^R w^2|\partial_r\partial_t^4\varphi|^2\,dr.$$

We now apply Hardy inequalities (3-2) and (3-3) to obtain

$$\begin{aligned}\|w(\partial_t^4\varphi)^2\|_\infty &\lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k\partial_t^4\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+1}\int_{R/4}^{R} w^{2+2[\alpha]}|\partial_r^k\partial_t^4\varphi|^2\,dr\\ &\lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k\partial_t^4\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+1}\int_{R/4}^{R} w^{\alpha+k}|\partial_r^k\partial_t^4\varphi|^2\,dr,\end{aligned}$$

where we have used $w^{[\alpha]+1}\lesssim w^\alpha$. Notice that the right-hand side is bounded by $\bar{\mathcal{E}}$.

When $[\alpha]=1$, we have $\delta(2)=1$. In this case, it suffices to show that $r^2w(\partial_t^4\varphi)^2$ is bounded by $\bar{\mathcal{E}}$. Applying Sobolev embedding, the Cauchy–Schwarz inequality and Hardy inequalities, we obtain

$$\|r^2w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R |\partial_t^4\varphi|^2\,dr+\int_0^R r^4w^2|\partial_r\partial_t^4\varphi|^2\,dr \lesssim \sum_{k=0}^{2}\int_0^R r^4w^{\alpha+k}|\partial_r^k\partial_t^4\varphi|^2\,dr.$$

Since $[\alpha]=1$, the right-hand side is bounded by $\bar{\mathcal{E}}$.

To prove the boundedness of $r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi$, we first apply Sobolev embedding and use the boundedness of w and w_r to obtain

$$\begin{aligned}\|r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi\|_\infty &\lesssim \int_0^R r^2w^{[\alpha]/2}|\partial_t^{[\alpha]+3}\partial_r\varphi|\,dr\\ &\qquad+\int_0^R rw^{([\alpha]+2)/2}|\partial_t^{[\alpha]+3}\partial_r\varphi|\,dr+\int_0^R r^2w^{([\alpha]+2)/2}|\partial_r^2\partial_t^{[\alpha]+3}\varphi|\,dr.\end{aligned}$$

By Hölder's inequality,

$$\|r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi\|_\infty^2 \lesssim \int_0^R r^2w^{\alpha-[\alpha]+[\alpha]}|\partial_t^{[\alpha]+3}\partial_r\varphi|^2\,dr+\int_0^R r^4w^{\alpha-[\alpha]+[\alpha]+2}|\partial_r^2\partial_t^{[\alpha]+3}\varphi|^2\,dr.$$

Notice that the second term in the right-hand side is $\bar{\mathcal{E}}_r^{[\alpha]+4,1}$. For the first term in the right-hand side we apply Hardy inequalities (3-1) and (3-3) to ensure that it is bounded by $\bar{\mathcal{E}}_r^{[\alpha]+3,0}$ and $\bar{\mathcal{E}}_r^{[\alpha]+4,1}$. $\square$

The results can be extended to other quantities involving more spatial derivatives. In the next lemma, we present the weighted L^∞ bounds of φ_{rr} and its time derivatives.

Lemma 3.6. *We have*

$$\sum_{q=0}^{[\alpha]+2} |w^{(q+1)/2} r^{\delta(q)} \partial_t^q \partial_r^2 \varphi| \lesssim \bar{\mathcal{E}}^{1/2}, \tag{3-6}$$

where $\delta(q) = 0$ *for* $q \le [\alpha]$, $\delta(q) = 1$ *for* $q = [\alpha]+1$, *and* $\delta(q) = 2$ *for* $q = [\alpha]+2$.

Proof. The choice of $\delta(q)$ is clear because of (3-5). We will focus on the bound near the boundary. So we will assume that $\delta(q) = 0$ and φ is supported in $\left(\frac{1}{4}R, R\right)$. We will use the $W^{1,1}$ bound for the squared quantity:

$$\begin{aligned}\|w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2\|_\infty &\le \int_0^R w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2\, dr + \int_0^R |(w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2)_r|\, dr \\ &\lesssim \int_0^R w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2\, dr + \int_0^R w^q(\partial_t^q \partial_r^2 \varphi)^2\, dr + \int_0^R w^{q+1} \partial_t^q \partial_r^2 \varphi \partial_t^q \partial_r^3 \varphi\, dr \\ &\lesssim \int_0^R w^q(\partial_t^q \partial_r^2 \varphi)^2\, dr + \int_0^R w^{q+2}(\partial_t^q \partial_r^3 \varphi)^2\, dr,\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the boundedness of w. Applying the Hardy inequality (3-3), we obtain

$$\|w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2\|_\infty \lesssim \sum_{k=0}^{m+1} \int_0^R w^{q+2+2m} |\partial_t^q \partial_r^{k+2} \varphi|^2\, dr.$$

Choose $m = [\alpha] + 2 - q$. Then, since $w^{[\alpha]+2} \lesssim w^{\alpha+1}$ and $0 \le k \le [\alpha] + 3 - q$,

$$\|w^{q+1}(\partial_t^q \partial_r^2 \varphi)^2\|_\infty \lesssim \sum_{k=0}^{[\alpha]+3-q} \int_0^R w^{[\alpha]+2+[\alpha]+4-q} |\partial_t^q \partial_r^{k+2} \varphi|^2\, dr \lesssim \sum_{k=0}^{[\alpha]+3-q} \bar{\mathcal{E}}^{q+k+1,k+1} \lesssim \bar{\mathcal{E}}. \qquad \square$$

Remark 3.7. The strengths of the weights appearing for $\partial_t^{q+2}\varphi$, $\partial_t^{q+1}\partial_r\varphi$ and $\partial_t^q \partial_r^2 \varphi$ in the previous lemmas depend on the number of spatial derivatives as well as the number of time derivatives. This is due to the energy structure of $\bar{\mathcal{E}}$.

4. The instant energy

In this section, we will introduce the various energies and establish the equivalence of the temporal instant energy and the total energy for (φ, φ_t).

Let $T > 0$ be given such that the solutions to (1-21) or (2-11) satisfy the bound

$$\sup_{r \in (0,R)} |(\zeta \circ \psi)(t,r)| = \sup_{r \in (0,R)} \left|\zeta(\epsilon\psi_1(t,r) + \epsilon^2 \varphi(t,r))\right| \le \tfrac{1}{4} \quad \text{for all } 0 \le t \le T.$$

For each time $0 \le t \le T$, we introduce the following instant energies and the total energy for the solutions to the φ equation (2-11). The higher-order (temporal) instant energy is, for $j \ge 0$,

$$\mathcal{E}^j \equiv \int_0^R \frac{w^\alpha r^4 |\partial_t^j \varphi_t|^2}{(1+\zeta)^4}\, dr + \int_0^R \gamma \frac{w^{1+\alpha} J^{-\gamma-1} |(r^3 \partial_t^j \varphi)_r|^2}{r^2}\, dr - a(\gamma) \int_0^R 4 w^\alpha r^4 \Phi(r) |\partial_t^j \varphi|^2\, dr, \quad (4\text{-}1)$$

where J was defined in (1-23), and $a(\gamma) = 1$ for $\gamma > \frac{4}{3}$ and $a(\gamma) = 0$ otherwise. The total instant energy is

$$\mathcal{E}(t) \equiv \sum_{j=0}^{[\alpha]+4} \mathcal{E}^j(t). \quad (4\text{-}2)$$

A simple computation shows — see also the equivalent expressions for L in (2-4) —

$$-r^3 \partial_r \left(w^{1+\alpha} \frac{1}{r^2} (r^3 \psi)_r \right) = -(w^{1+\alpha} r^4 \psi_r)_r - 3 r^3 \partial_r (w^{1+\alpha}) \psi. \quad (4\text{-}3)$$

Multiply this identity by ψ and integrate to obtain

$$\int_0^R \frac{w^{1+\alpha}}{r^2} |(r^3 \psi)_r|^2\, dr = \int_0^R w^{1+\alpha} r^4 |\psi_r|^2\, dr + \int_0^R 3 w^\alpha r^4 \Phi(r) \psi^2\, dr. \quad (4\text{-}4)$$

We observe that (4-4) gives another expression for the spatial part of the instant energy $\mathcal{E}^j$ if $J = 1$ throughout the domain for all time. However, it is not obvious we can guarantee the positiveness of $\mathcal{E}^j$ since J varies in time and radius. In the following lemma, we show the positivity of $\mathcal{E}^j$ and equivalence of the homogeneous energy $\bar{\mathcal{E}}^{j,0}$ for all sufficiently small $\epsilon > 0$.

Lemma 4.1. *Suppose that $\bar{\mathcal{E}}$ given in (2-14) is bounded for all $0 \le t \le T$. Then we have*

$$\mathcal{E}^j = \mathfrak{E}^j + \mathcal{R}^j, \quad (4\text{-}5)$$

where $\mathfrak{E}^j$ and $\mathcal{R}^j$ satisfy the estimates

(1) $(1 + \epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0} \lesssim \mathfrak{E}^j \lesssim (1 + \epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

(2) $|\mathcal{R}^j| \lesssim (\epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

(3) $|d\mathcal{R}^j/dt| \lesssim (\epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

for all sufficiently small $\epsilon > 0$.

Proof. To extract the positive part of $\mathcal{E}^j$, we will rewrite the spatial part similarly as in (4-4). To this end, from (4-3) we first obtain

$$-r^3 \partial_r \left(w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} (r^3 \psi)_r \right) = -(w^{1+\alpha} J^{-\gamma-1} r^4 \psi_r)_r \\ - 3 r^3 J^{-\gamma-1} \partial_r (w^{1+\alpha}) \psi - 3 r^3 \partial_r (J^{-\gamma-1}) w^{1+\alpha} \psi, \quad (4\text{-}6)$$

which in turn yields the integral identity

$$\int_0^R \frac{w^{1+\alpha}J^{-\gamma-1}}{r^2}|(r^3\psi)_r|^2\,dr$$
$$= \int_0^R w^{1+\alpha}J^{-\gamma-1}r^4|\psi_r|^2\,dr + \int_0^R 3w^\alpha J^{-\gamma-1}r^4\Phi(r)\psi^2\,dr - \int_0^R 3r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\psi^2\,dr. \quad (4\text{-}7)$$

By using (4-7), we write $\mathcal{E}^j$ as $\mathcal{E}^j \equiv \mathfrak{E}^j + \mathcal{R}^j$, where

$$\mathfrak{E}^j = \int_0^R \frac{w^\alpha r^4|\partial_t^j\varphi_t|^2}{(1+\zeta)^4}\,dr + \gamma\int_0^R w^{1+\alpha}J^{-\gamma-1}r^4|\partial_t^j\varphi_r|^2\,dr + (3\gamma-4a(\gamma))\int_0^R w^\alpha J^{-\gamma-1}r^4\Phi(r)|\partial_t^j\varphi|^2\,dr,$$

$$\mathcal{R}^j = -3\gamma\int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr + 4a(\gamma)\int_0^R (J^{-\gamma-1}-1)w^\alpha r^4\Phi(r)|\partial_t^j\varphi|^2\,dr. \quad (4\text{-}8)$$

Since $3\gamma-4a(\gamma)>0$ for all γ, we now see that $\mathfrak{E}^j$ is positive for all γ. Moreover, by (1-23), (2-9), Taylor expansion and Lemma 3.5, we deduce the first result, which shows that $\mathfrak{E}^j$ is equivalent to $\bar{\mathcal{E}}^{j,0}$. The estimate of $\mathcal{R}^j$ follows similarly. Here we present the detail for the bound of $d\mathcal{R}^j/dt$. We start with the second term. The time derivative of the second term consists of the two terms

$$\int_0^R J^{-\gamma-2}J_t w^\alpha r^4\Phi(r)|\partial_t^j\varphi|^2\,dr, \qquad \int_0^R (J^{-\gamma-1}-1)w^\alpha r^4\Phi(r)\partial_t^j\varphi\partial_t^j\varphi_t\,dr.$$

Then, since $\Phi(r)<\infty$ and $|J^{-\gamma-2}J_t|\lesssim\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ and $|J^{-\gamma-1}-1|\lesssim\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ by Lemmas 3.6 and 3.5, we obtain the desired bounds in terms of $\bar{\mathcal{E}}^{j,0}$. On the other hand, the time derivative of the first integral of $\mathcal{R}^j$ consists of the two terms

$$\int_0^R r^3\partial_r\partial_t(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr, \qquad \int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\partial_t^j\varphi\partial_t^j\varphi_t\,dr.$$

By Lemmas 3.6 and 3.5, we see that $|w\partial_r\partial_t(J^{-\gamma-1})|\lesssim\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$. Hence, by further using the localized Hardy inequality (3-1) near the origin, we have

$$\left|\int_0^R r^3\partial_r\partial_t(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr\right| \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\int_0^R r^2w^\alpha|\partial_t^j\varphi|^2\,dr \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\bar{\mathcal{E}}^{j,0}.$$

For the second term, we use $|w\partial_r(J^{-\gamma-1})|\lesssim\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ as well as the Cauchy–Schwarz inequality to get

$$\left|\int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\partial_t^j\varphi\partial_t^j\varphi_t\,dr\right| \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\left(\int_0^R r^2w^\alpha|\partial_t^j\varphi|^2\,dr + \int_0^R r^4w^\alpha|\partial_t^j\varphi_t|^2\,dr\right).$$

We apply (3-1) to the first integral to obtain the desired bound. □

Lemma 4.1 implies that, if $\bar{\mathcal{E}}$ is bounded, a nonlinear instant energy $\mathcal{E}^j$ in (4-1) is equivalent to the homogenous energy $\bar{\mathcal{E}}^{j,0}$ given in (2-12) for all sufficiently small $\epsilon>0$.

The next goal is to derive the a priori estimates for $\mathcal{E}$ and $\bar{\mathcal{E}}$ under the assumption

$$|\varphi|+|\varphi_t|+|\varphi_{tt}|+\sum_{q=1}^{[\alpha]+2}|r^{\delta(q)}w^{(q-1)/2}\partial_t^{q+2}\varphi|+|\varphi_r|+|\varphi_{tr}| \\ +\sum_{q=1}^{[\alpha]+2}|r^{\delta(q)}w^{q/2}\partial_t^{q+1}\partial_r\varphi|+\sum_{q=0}^{[\alpha]+2}|w^{(q+1)/2}r^{\delta(q)}\partial_t^q\partial_r^2\varphi|\le M, \tag{4-9}$$

where M is a fixed constant. We recall that the validity of this assumption within the total energy $\bar{\mathcal{E}}$ was provided in Lemmas 3.5 and 3.6. The a priori estimates consist of two parts: the temporal energy estimates for $\mathcal{E}$, and the elliptic estimates to recover all other terms in $\bar{\mathcal{E}}$.

We start with the energy estimates of $\mathcal{E}$.

5. Weighted energy estimates

This section is devoted to the proof of this proposition:

Proposition 5.1. *Suppose that (φ,φ_t) satisfy (2-11) for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume (4-9). Then $\mathcal{E}$ enjoys the energy inequality*

$$\frac{d}{dt}\mathcal{E}\lesssim\sqrt{\mathcal{E}}+(1-a(\gamma))\mathcal{E}+(\epsilon+\epsilon^2M)(\mathcal{E}+\sqrt{\bar{\mathcal{E}}}\sqrt{\mathcal{E}}), \tag{5-1}$$

where $a(\gamma)=1$ for $\gamma>\frac{4}{3}$ and $a(\gamma)=0$ otherwise, and $\epsilon>0$ is small enough.

Remark 5.2. $\mathcal{E}$ is positive for all sufficiently small ϵ due to Lemma 4.1, Hence $\sqrt{\mathcal{E}}$ is well defined in the right-hand side of (5-1).

Lemma 5.3 ($\mathcal{E}^0$). *Suppose that (φ,φ_t) satisfy (2-11) for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume (4-9). Then*

$$\frac{d}{dt}\mathcal{E}^0\lesssim\sqrt{\mathcal{E}^0}+(1-a(\gamma))\mathcal{E}^0+(\epsilon+\epsilon^2M)(\mathcal{E}^0+\mathcal{E}^1), \tag{5-2}$$

where $a(\gamma)$ was introduced in the definition of $\mathcal{E}^0$.

Proof. We begin by multiplying (2-11) by φ and integrating over $(0,R)$:

$$\int_0^R\frac{w^\alpha r^4\varphi_{tt}}{(1+\zeta)^4}\varphi_t\,dr+\int_0^R\biggl(4\lambda w^\alpha r^4\psi_1^2+4\lambda\epsilon w^\alpha r^4\psi_1\varphi-\lambda w^\alpha r^4\psi_1\frac{f}{\epsilon}-\frac{2w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^2}{(1+\zeta)^7}\biggr)\varphi_t\,dr \\ +\int_0^R\frac{r^3}{\epsilon^2}\partial_r\biggl(w^{1+\alpha}\biggl(\Bigl(1+\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\Bigr)^{-\gamma}-1+\gamma\frac{1}{r^2}(r^3\epsilon\psi_1)_r\biggr)\biggr)\varphi_t\,dr \\ -\int_0^R 4w^\alpha r^4\Phi(r)\varphi\varphi_t\,dr+\int_0^R w^\alpha r^4\Phi(r)\frac{f}{\epsilon^2}\varphi_t\,dr=0.$$

We denote the left-hand side by $\sum_{k=1}^5 I_k$. The first term I_1 can be rewritten as

$$I_1=\frac{1}{2}\frac{d}{dt}\int_0^R\frac{w^\alpha r^4|\varphi_t|^2}{(1+\zeta)^4}\,dr+2\int_0^R\frac{w^\alpha r^4|\varphi_t|^2}{(1+\zeta)^5}\frac{(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^2}\,dr,$$

where we have used (1-24). For I_2, we use the boundedness of $(\psi_1)_t$ and $f = O(|\epsilon\psi_1+\epsilon^2\varphi|^2)$ to deduce that

$$|I_2| \lesssim \sqrt{\mathcal{E}^0} + \epsilon\mathcal{E}^0 + \epsilon^2 \sup|\varphi_t|\mathcal{E}^0.$$

For I_3, we integrate by parts and use (2-1):

$$\begin{aligned}
I_3 &= -\int_0^R \frac{w^{1+\alpha}}{\epsilon^2}\left(\left(1+\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right)^{-\gamma} - 1 + \gamma\frac{1}{r^2}(r^3\epsilon\psi_1)_r\right)(r^3\varphi_t)_r\,dr\\
&= -\int_0^R w^{1+\alpha}\left(-\frac{\gamma}{r^2}(r^3\varphi)_r + \frac{h}{\epsilon^2}\right)(r^3\varphi_t)_r\,dr\\
&= -\int_0^R w^{1+\alpha}\left(-\frac{\gamma}{r^2}J^{-\gamma-1}(r^3\varphi)_r + (J^{-\gamma-1}-1)\frac{\gamma}{r^2}(r^3\varphi)_r + \frac{h}{\epsilon^2}\right)(r^3\varphi_t)_r\,dr\\
&= \frac{\gamma}{2}\frac{d}{dt}\int_0^R w^{1+\alpha}J^{-\gamma-1}\frac{|(r^3\varphi)_r|^2}{r^2}\,dr + \frac{\gamma(\gamma+1)}{2}\int_0^R w^{1+\alpha}J^{-\gamma-2}J_t\frac{|(r^3\varphi)_r|^2}{r^2}\,dr\\
&\qquad \underbrace{-\int_0^R w^{1+\alpha}(J^{-\gamma-1}-1)\frac{\gamma}{r^2}(r^3\varphi)_r(r^3\varphi_t)_r\,dr}_{I_3^1} \underbrace{-\int_0^R w^{1+\alpha}\frac{h}{\epsilon^2}(r^3\varphi_t)_r\,dr}_{I_3^2}.
\end{aligned}$$

Since $J_t = 3\psi_t + r\psi_{tr} = 3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t) + r(\epsilon(\psi_1)_{tr}+\epsilon^2\varphi_{tr})$, the commutator involving J_t is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^0$. Notice that $|J^{-\gamma-1}-1| = O\big(\big|(1/r^2)(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\big|\big) \lesssim \epsilon+\epsilon^2M$, so by the Cauchy–Schwarz inequality we see that

$$|I_3^1| \lesssim (\epsilon+\epsilon^2M)(\mathcal{E}^0+\mathcal{E}^1).$$

Since $h = O\big(\big|(1/r^2)(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\big|^2\big)$, we have

$$|I_3^2| \lesssim \sqrt{\mathcal{E}^1} + \epsilon(\mathcal{E}^0+\mathcal{E}^1) + \epsilon^2\sup\left|\frac{(r^3\varphi)_r}{r^2}\right|(\mathcal{E}^0+\mathcal{E}^1).$$

It is easy to see that

$$I_4 = -2\frac{d}{dt}\int_0^R w^\alpha r^4\Phi(r)\varphi^2\,dr \tag{5-3}$$

and also it satisfies

$$|I_4| \lesssim \mathcal{E}^0. \tag{5-4}$$

If $\gamma > \frac{4}{3}$, we will use (5-3) so that I_4 contributes to the energy. If $\gamma \le \frac{4}{3}$, then we will use the estimate (5-4), in which case the contribution of $\mathcal{E}^0$ in the right-hand side of the energy inequality will be of order 1.

For the last term, we obtain

$$|I_5| \lesssim \sqrt{\mathcal{E}^0} + \epsilon\mathcal{E}^0 + \epsilon^2\sup|\varphi|\mathcal{E}^0.$$

This finishes the proof. □

As Lemma 5.3 indicates, the right-hand side of the energy inequality involves higher-order energy due to the nonlinearity and degeneracy, and thus the energy estimates cannot be closed at the physical energy level $\mathcal{E}^0$. This motivates us to go beyond $\mathcal{E}^0$.

The time differentiation of (2-11) yields

$$\begin{aligned}\frac{w^\alpha r^4\varphi_{ttt}}{(1+\zeta)^4}&-\frac{4w^\alpha r^4\varphi_{tt}(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}+8\lambda w^\alpha r^4\psi_1(\psi_1)_t+4\lambda\epsilon w^\alpha r^4(\psi_1)_t\varphi+4\lambda\epsilon w^\alpha r^4\psi_1\varphi_t\\&-\lambda w^\alpha r^4(\psi_1)_t\frac{f}{\epsilon}-\lambda w^\alpha r^4\psi_1\frac{f_t}{\epsilon}-\frac{4w^\alpha r^4((\psi_1)_t+\epsilon\varphi_t)((\psi_1)_{tt}+\epsilon\varphi_{tt})}{(1+\zeta)^7}\\&-\frac{14w^\alpha r^4((\psi_1)_t+\epsilon\varphi_t)^2(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^{10}}\\&-\gamma\frac{r^3}{\epsilon^2}\partial_r\left(w^{1+\alpha}\left(J^{-\gamma-1}\left(\frac{1}{r^2}\big(r^3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)\big)_r\right)-\frac{1}{r^2}(r^3\epsilon(\psi_1)_t)_r\right)\right)\\&\qquad-4w^\alpha r^4\Phi(r)\varphi_t+w^\alpha r^4\Phi(r)\frac{f_t}{\epsilon^2}=0,\end{aligned}\tag{5-5}$$

where we have substituted J for its equivalent expression given in (1-23). We next present the estimates for $\mathcal{E}^1$.

Lemma 5.4 ($\mathcal{E}^1$). *Suppose that* (φ,φ_t) *satisfy* (2-11) *for* $0\le t\le T$ *and the corresponding total instant energy* $\mathcal{E}$ *is bounded. Moreover, we assume* (4-9). *Then*

$$\frac{d}{dt}\mathcal{E}^1\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}+(1-a(\gamma))\mathcal{E}^1+(\epsilon+\epsilon^2M+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1)+\epsilon\sqrt{\bar{\mathcal{E}}^{1,1}}\sqrt{\mathcal{E}^1}.\tag{5-6}$$

Proof. We multiply (5-5) by φ_{tt} and integrate it over $(0,R)$. We denote each integral by I_k for $1\le k\le 12$. We will estimate them term by term. I_1 forms an energy plus a commutator and thus I_1+I_2 can be written as

$$I_1+I_2=\frac12\frac{d}{dt}\int_0^R\frac{w^\alpha r^4|\varphi_{tt}|^2}{(1+\zeta)^4}\,dr-\int_0^R\frac{2w^\alpha r^4\varphi_{tt}^2(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}\,dr,$$

where we have used (1-24). Note that the second term is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^1$ since $(\psi_1)_t$ is bounded and $|\varphi_t|\le M$ due to (4-9).

I_3 is a source term and it is easy to see that

$$|I_3|\lesssim\sqrt{\mathcal{E}^1}$$

due to the boundedness of ψ_1. For I_4 and I_5, we apply the Cauchy–Schwarz inequality to obtain

$$|I_4|+|I_5|\lesssim\epsilon(\mathcal{E}^0+\mathcal{E}^1).$$

In order to estimate I_6 and I_7, we recall (2-2) and that $f=O(|\epsilon\psi_1+\epsilon^2\varphi|^2)$. Then $f/\epsilon=O(\epsilon|\psi_1+\epsilon\varphi|^2)$ and $f_t/\epsilon=O\big(\epsilon(\psi_1+\epsilon\varphi)((\psi_1)_t+\epsilon\varphi_t)\big)$. Hence we deduce that

$$|I_6|+|I_7|\lesssim\epsilon\sqrt{\mathcal{E}^1}+(\epsilon^2+\epsilon^3M)(\mathcal{E}^0+\mathcal{E}^1).$$

I_8 and I_9 can be similarly estimated:

$$|I_8|\lesssim\sqrt{\mathcal{E}^1}+(\epsilon+\epsilon^2M)(\mathcal{E}^0+\mathcal{E}^1)\quad\text{and}\quad|I_9|\lesssim\epsilon\sqrt{\mathcal{E}^1}+(\epsilon^2+\epsilon^3M+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1).$$

We next move onto I_{10}, which will give rise to another energy term. We first rewrite the fourth line in (5-5):

$$-\gamma\frac{r^3}{\epsilon^2}\partial_r\bigg(w^{1+\alpha}\bigg(J^{-\gamma-1}\bigg(\frac{1}{r^2}\big(r^3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)\big)_r\bigg)-\frac{1}{r^2}(r^3\epsilon(\psi_1)_t)_r\bigg)\bigg)$$
$$=-\gamma r^3\partial_r\bigg(w^{1+\alpha}J^{-\gamma-1}\frac{1}{r^2}(r^3\varphi_t)_r\bigg)-\gamma r^3\partial_r\bigg(w^{1+\alpha}\frac{(J^{-\gamma-1}-1)}{\epsilon}\frac{1}{r^2}(r^3(\psi_1)_t)_r\bigg). \quad (5\text{-}7)$$

By replacing the fourth line using (5-7), we have two terms in I_{10}, denoted by I_{10}^1 and I_{10}^2. For I_{10}^1, we integrate by parts to obtain a perfect time derivative plus a commutator:

$$I_{10}^1=\frac{\gamma}{2}\frac{d}{dt}\int_0^R w^{1+\alpha}J^{-\gamma-1}\frac{1}{r^2}|(r^3\varphi_t)_r|^2\,dr+\frac{\gamma(\gamma+1)}{2}\int_0^R w^{1+\alpha}J^{-\gamma-2}J_t\frac{1}{r^2}|(r^3\varphi_t)_r|^2\,dr.$$

Note that $J_t=3\psi_t+r\psi_{tr}=3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)+r(\epsilon(\psi_1)_{tr}+\epsilon^2\varphi_{tr})$. Thus the commutator is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^1$. For I_{10}^2, we first rewrite it as

$$\begin{aligned}I_{10}^2=&-\gamma\int_0^R r^3\varphi_{tt}\partial_r\bigg(w^{1+\alpha}\frac{J^{-\gamma-1}-1}{\epsilon}\frac{1}{r^2}(r^3(\psi_1)_t)_r\bigg)\,dr\\
=&-\gamma\int_0^R r^3\varphi_{tt}(w^{1+\alpha})_r\frac{J^{-\gamma-1}-1}{\epsilon}(3(\psi_1)_t+r(\psi_1)_{tr})\,dr\\
&+\gamma(\gamma+1)\int_0^R r^3\varphi_{tt}w^{1+\alpha}\frac{J^{-\gamma-2}J_r}{\epsilon}(3(\psi_1)_t+r(\psi_1)_{tr})\,dr\\
&-\gamma\int_0^R r^3\varphi_{tt}w^{1+\alpha}\frac{J^{-\gamma-1}-1}{\epsilon}(4(\psi_1)_{tr}+r(\psi_1)_{trr})\,dr\equiv I_{10}^{2,1}+I_{10}^{2,2}+I_{10}^{2,3}.\end{aligned}$$

For $I_{10}^{2,1}$ and $I_{10}^{2,3}$, we note that $(J^{-\gamma-1}-1)/\epsilon=-(\gamma+1)((r^3(\psi_1+\epsilon\varphi))_r)/r^2+\tilde{h}/\epsilon$, where $\tilde{h}=O\big(\big|(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r/r^2\big|^2\big)$, which yields $|(J^{-\gamma-1}-1)/\epsilon|\lesssim 1+\epsilon M$. Then, from Hölder's inequality and the regularity of ψ_1,

$$|I_{10}^{2,3}|\lesssim(1+\epsilon M)\bigg(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\bigg)^{\frac12}\bigg(\int_0^R w^{\alpha+2}\big(r^2|(\psi_1)_{tr}|^2+r^4|(\psi_1)_{trr}|^2\big)\,dr\bigg)^{\frac12}\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}.$$

For $I_{10}^{2,1}$, since $|(w^{1+\alpha})_r|\sim w^\alpha$, we apply the Hardy inequality near the boundary. Then, from the regularity of ψ_1, we obtain

$$|I_{10}^{2,1}|\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}.$$

For $I_{10}^{2,2}$, we first note that $J_r/\epsilon=(4\psi_r+r\psi_{rr})/\epsilon=4((\psi_1)_r+\epsilon\varphi_r)+r((\psi_1)_{rr}+\epsilon\varphi_{rr})$. Then, from the regularity of ψ_1,

$$\begin{aligned}|I_{10}^{2,2}|\lesssim&\bigg(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\bigg)^{\frac12}\bigg(\int_0^R w^{\alpha+2}(r^2|(\psi_1)_r|^2+r^4|(\psi_1)_{rr}|^2)\,dr\bigg)^{\frac12}\\
&+\epsilon\bigg(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\bigg)^{\frac12}\bigg(\int_0^R w^{\alpha+2}(r^2|\varphi_r|^2+r^4|\varphi_{rr}|^2)\,dr\bigg)^{\frac12}\lesssim(1+\epsilon\sqrt{\mathcal{E}^{1,1}})\sqrt{\mathcal{E}^1}.\end{aligned}$$

Next, it is easy to see that

$$I_{11} = -2\frac{d}{dt}\int_0^R w^\alpha r^4 \Phi(r)\varphi_t^2\,dr \quad \text{and} \quad |I_{11}| \lesssim \mathcal{E}^1. \tag{5-8}$$

If $\gamma > \frac{4}{3}$, then I_{11} will contribute to the energy via (5-8). If $\gamma \le \frac{4}{3}$, then we will use the estimate (5-8), in which case the contribution of $\mathcal{E}^1$ in the right-hand side of the energy inequality will be of order 1.

For the last term, since $f_t/\epsilon^2 = O\big((\psi_1+\epsilon\varphi)((\psi_1)_t+\epsilon\varphi_t)\big)$, we obtain

$$|I_{12}| \lesssim \sqrt{\mathcal{E}^1} + (\epsilon+\epsilon^2 M)(\mathcal{E}^0+\mathcal{E}^1).$$

This finishes the proof of the lemma. □

Lemmas 5.3 and 5.4 give rise to the energy inequality for $\mathcal{E}^0+\mathcal{E}^1$. However, the right-hand side involves M from the assumption (4-9) as well as $\bar{\mathcal{E}}^{1,1}$. In order to justify the assumption and to close the estimates, we will carry out the higher-order estimates.

The equations for $\partial_t^i\varphi_t$, $1 \le i \le [\alpha]+4$, can be written in the form

$$\begin{aligned}
&\frac{w^\alpha r^4 \partial_t^i \varphi_{tt}}{(1+\zeta)^4} + \sum_{j=1}^{i} c_{1j} w^\alpha r^4 \partial_t^{i-j}\varphi_{tt}\partial_t^{j-1}\left(\frac{-4(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}\right) \\
&\quad + 8\lambda w^\alpha r^4 \partial_t^{i-1}(\psi_1(\psi_1)_t) + \sum_{j=0}^{i} c_{2j} 4\lambda\epsilon w^\alpha r^4 \partial_t^{i-j}\psi_1 \partial_t^j\varphi - \sum_{j=0}^{i}\lambda w^\alpha r^4 \partial_t^{i-j}\psi_1 \frac{\partial_t^j f}{\epsilon} \\
&\quad - \sum_{j=0}^{i} c_{2j} 2 w^\alpha r^4 \partial_t^{i-j}\big(((\psi_1)_t+\epsilon\varphi_t)^2\big)\partial_t^j\left(\frac{1}{(1+\zeta)^7}\right) \\
&\quad - \gamma r^3 \partial_r\left(w^{1+\alpha} J^{-\gamma-1}\frac{1}{r^2}(r^3\partial_t^i\varphi)_r\right) - 4w^\alpha r^4\Phi(r)\partial_t^i\varphi \\
&\quad - \sum_{j=0}^{i-2} c_{3j}\gamma r^3\partial_r\left(w^{1+\alpha}\partial_t^{i-1-j}(J^{-\gamma-1})\frac{1}{r^2}(r^3\partial_t^j\varphi_t)_r\right) \\
&\quad - \sum_{j=0}^{i-1} c_{3j}\gamma r^3\partial_r\left(w^{1+\alpha}\partial_t^{i-1-j}\left(\frac{J^{-\gamma-1}-1}{\epsilon}\right)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\right) + w^\alpha r^4\Phi(r)\frac{\partial_t^i f}{\epsilon^2} = 0,
\end{aligned} \tag{5-9}$$

where c_{1j}, c_{2j} and c_{3j} are binomial coefficients. Notice that we have used (5-7) to write the elliptic, spatial part.

We record the high-order energy inequalities for the solutions to (5-9):

Lemma 5.5 ($\mathcal{E}^i$, $i \ge 2$). *Suppose that* (φ, φ_t) *satisfy* (2-11) *for* $0 \le t \le T$ *and the corresponding total instant energy* $\mathcal{E}$ *is bounded. Moreover, we assume* (4-9). *Then*

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}^i &\lesssim (1+\epsilon M)\sqrt{\mathcal{E}^i} + (1-a(\gamma))\mathcal{E}^i \\
&\qquad + \sum_{k=1}^{i}(\epsilon+\epsilon^2 M)^k \sum_{j=0}^{i}\mathcal{E}^j + \sum_{k=1}^{i}(\epsilon+\epsilon^2 M)^k\left(\sum_{j=0}^{i}\sum_{l=0}^{j}\sqrt{\bar{\mathcal{E}}^{j,l}}\right)\sqrt{\mathcal{E}^i}.
\end{aligned} \tag{5-10}$$

Proof. We multiply (5-9) by $\partial_t^i\varphi_t$ and integrate it over $(0, R)$. We denote each integral by J_k for $1\le k\le 11$. As before, we will estimate them term by term. As in the case of I_1 in the previous lemma, the first term J_1 forms an energy plus a commutator:

$$J_1=\frac{1}{2}\frac{d}{dt}\int_0^R\frac{w^\alpha r^4|\partial_t^i\varphi_t|^2}{(1+\zeta)^4}\,dr+\int_0^R\frac{2w^\alpha r^4|\partial_t^i\varphi_t|^2(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}\,dr,$$

where we have used (1-24). Note that the second term is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^i$ since $(\psi_1)_t$ is bounded and $|\varphi_t|\le M$ due to (4-9). For J_2, we note that the second factor in the summation of the second term in the first line of (5-9) has the form

$$(\epsilon\partial_t^{j-k}(\psi_1)_t+\epsilon^2\partial_t^{j-k}\varphi_t)(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1},\quad 1\le k\le j;$$

thus, since $|\zeta|\le\frac{1}{4}$, essentially J_2 consists of the following terms: for each $1\le k\le j\le i$,

$$\begin{aligned}\int_0^R & w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j+1}\varphi_t(\epsilon\partial_t^{j-k}(\psi_1)_t+\epsilon^2\partial_t^{j-k}\varphi_t)(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\\ &=\epsilon\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j+1}\varphi_t\partial_t^{j-k}(\psi_1)_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\\ &\qquad\qquad+\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j+1}\varphi_t\partial_t^{j-k}\varphi_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\\ &=J_2^1+J_2^2.\end{aligned}\tag{5-11}$$

For J_2^1, we recall $(\psi_1)_{tt}=-\lambda\psi_1$ and hence $\partial_t^{j-k}(\psi_1)_t$ is a constant multiple of ψ_1 or $(\psi_1)_t$. By further recalling that ψ_1 and $(\psi_1)_t$ are bounded and $|\varphi_t|\lesssim M$, and by using the Cauchy–Schwarz inequality, we see that

$$|J_2^1|\lesssim\epsilon(\epsilon+\epsilon^2M)^{k-1}(\mathcal{E}^i+\mathcal{E}^{i-j+1}).$$

For J_2^2, let $1\le j\le\left[\frac{i}{2}\right]+1$ first. Then

$$\begin{aligned}|J_2^2|&=\epsilon^2\left|\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(\alpha-j+k+1)/2}r^2\partial_t^{i-j+1}\varphi_t w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\right|\\ &\le\epsilon^2\sup\left|w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t\right|(\epsilon+\epsilon^2M)^{k-1}\sqrt{\mathcal{E}^i}\underbrace{\left(\int_0^R w^{\alpha-j+k+1}r^4|\partial_t^{i-j+1}\varphi_t|^2\,dr\right)^{\frac{1}{2}}}_{J_2^{2,1}}.\end{aligned}$$

By (4-9), $\sup\left|w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t\right|\le M$. To estimate $J_2^{2,1}$, since $k\ge1$ we first observe that $J_2^{2,1}\lesssim\mathcal{E}^i$ when $j=1$, and $J_2^{2,1}\lesssim\mathcal{E}^{i-1}$ when $j=2$. Now, when $2\le j\le\left[\frac{i}{2}\right]+1$ we apply the Hardy inequality (3-3) near the boundary $j-2$ times to obtain

$$\int_0^R w^{\alpha-j+k+1}r^4|\partial_t^{i-j+1}\varphi_t|^2\,dr\lesssim\sum_{l=0}^{j-2}\int_0^R w^{\alpha-j+k+1+2(j-2)}r^4|\partial_t^{i-j+1}\partial_r^l\varphi_t|^2\,dr\lesssim\sum_{l=0}^{j-2}\bar{\mathcal{E}}^{i-j+1+l,l}.$$

Now, for J_2^2, when there exist i and j such that $\left[\frac{i}{2}\right]+2\le j\le i$ we write

$$|J_2^2|=\epsilon^2\left|\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t w^{(\alpha-i+j)/2}r^2\partial_t^{j-k}\varphi_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\right|$$
$$\le\epsilon^2\sup\left|w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t\right|(\epsilon+\epsilon^2M)^{k-1}\sqrt{\mathcal{E}^i}\left(\int_0^R w^{\alpha-i+j}r^4|\partial_t^{j-k}\varphi_t|^2\,dr\right)^{\frac{1}{2}}.$$

Note that $\sup\left|w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t\right|\le M$ due to (4-9). Let $J_2^{2,2}$ be the integral in the last term; we apply (3-3) $i-j$ times to get

$$J_2^{2,2}\lesssim\sum_{l=0}^{i-j}\int_0^R w^{\alpha-i+j+2(i-j)}r^4|\partial_t^{j-k}\partial_r^l\varphi_t|^2\,dr\lesssim\sum_{l=0}^{i-j}\bar{\mathcal{E}}^{j-k+l,l}.$$

We summarize the above estimates for J_2:

$$|J_2|\lesssim\sum_{1\le k,j\le i}(\epsilon+\epsilon^2M)^k\mathcal{E}^j+\sqrt{\mathcal{E}^i}\sum_{1\le k\le i}\epsilon^2M(\epsilon+\epsilon^2M)^{k-1}\left(\sum_{0\le l\le j\le i}\bar{\mathcal{E}}^{j,l}\right)^{\frac{1}{2}}.$$

Next, by using $(\psi_1)_{tt}=-\lambda\psi_1$ and the boundedness of ψ_1 and $(\psi_1)_t$, we easily deduce that

$$|J_3|\lesssim\sqrt{\mathcal{E}^i}.$$

Likewise, $\partial_t^{i-j}\psi_1$ in J_4 is a constant multiple of ψ_1 or $(\psi_1)_t$ and hence, by the Cauchy–Schwarz inequality, we obtain

$$|J_4|\lesssim\epsilon\sum_{j=0}^{i}\mathcal{E}^j.$$

To estimate J_5, we observe that $\partial_t^j f/\epsilon$ consists of terms like

$$\epsilon(\partial_t^{j-k}\psi_1+\epsilon\partial_t^{j-k}\varphi)(\partial_t^k\psi_1+\epsilon\partial_t^k\varphi)$$

for $0\le k\le j\le i$. The contribution coming from $\partial_t^{j-k}\psi_1\cdot\partial_t^k\psi_1$, $\partial_t^{j-k}\varphi\cdot\partial_t^k\psi_1$ or $\partial_t^{j-k}\psi_1\cdot\partial_t^k\varphi$ can be bounded by $\epsilon\sqrt{\mathcal{E}^i}+\epsilon^2\sum_{j=0}^i\mathcal{E}^j$. The remaining nonlinear part can be controlled similarly as done for J_2 by using L^∞ bounds and Hardy inequalities. By the boundedness of $\partial_t^{i-j}\psi_1$, it would suffice to estimate

$$\epsilon^3\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{j-k}\varphi\partial_t^k\varphi\,dr.$$

By symmetry of indices, we may assume $0\le k\le\left[\frac{j}{2}\right]$. If k is 0 or 1, then by (4-9) the integral is bounded by $\epsilon^3M(\mathcal{E}^i+\mathcal{E}^{j-k-1})$ with the understanding that $\mathcal{E}^{-1}=\mathcal{E}^0$. Suppose $2\le k\le\left[\frac{j}{2}\right]$. Then we get

$$\epsilon^3\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{j-k}\varphi\partial_t^k\varphi\,dr=\epsilon^3\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(\alpha-k+2)/2}r^2\partial_t^{j-k}\varphi w^{(k-2)/2}\partial_t^k\varphi\,dr$$
$$\lesssim\epsilon^3\sup\left|w^{(k-2)/2}\partial_t^k\varphi\right|\sqrt{\mathcal{E}^i}\underbrace{\left(\int_0^R w^{\alpha-k+2}r^4|\partial_t^{j-k}\varphi|^2\,dr\right)^{\frac{1}{2}}}_{J_5^1}.$$

Due to (4-9), $\sup\left|w^{(k-2)/2}\partial_t^k\varphi\right| \le M$. For J_5^1, we apply the Hardy inequality (3-3) $k-2$ times to obtain

$$J_5^1 \lesssim \sum_{l=0}^{k-2}\int_0^R w^{\alpha-k+2+2(k-2)}r^4|\partial_t^{j-k}\partial_r^l\varphi|^2\,dr \lesssim \sum_{l=0}^{k-2}\bar{\mathcal{E}}^{j-k-1+l,l}.$$

We have derived the estimate of J_5 as

$$|J_5| \lesssim \epsilon\sqrt{\mathcal{E}^i}+\epsilon^2\sum_{j=0}^{i}\mathcal{E}^j+\epsilon^3 M\left(\sum_{j=0}^{i}\mathcal{E}^j+\sqrt{\mathcal{E}^i}\left(\sum_{0\le l\le j\le i-3}\bar{\mathcal{E}}^{j,l}\right)^{\frac12}\right).$$

We next estimate J_6. First let $j=0$. Then the third line of (5-9) essentially takes the following form

$$w^\alpha r^4(\partial_t^{i-k}(\psi_1)_t+\epsilon\partial_t^{i-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t), \quad 0\le k\le i.$$

We may assume $0\le k\le\left[\frac{i}{2}\right]$. As before, it is easy to see that the contribution coming from ψ_1 related terms is bounded by $\sqrt{\mathcal{E}^i}+\epsilon\sum_{j=0}^{i}\mathcal{E}^j$. The remaining nonlinear part can be controlled similarly as in the previous case by using (4-9) and Hardy inequality:

$$\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-k}\varphi_t\partial_t^k\varphi_t\,dr \lesssim \epsilon^2 M\sqrt{\mathcal{E}^i}\left(\sum_{l=0}^{k-1}\bar{\mathcal{E}}^{i-k+l,l}\right)^{\frac12}.$$

Now let $1\le j\le i$. Then the second time-differentiated term $\partial_t^j((1+\zeta)^{-7})$ consists of the terms

$$(\epsilon\partial_t^{j-m}(\psi_1)_t+\epsilon^2\partial_t^{j-m}\varphi_t)(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{m-1}, \quad 1\le m\le j.$$

The term $\epsilon\partial_t^{j-m}(\psi_1)_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{m-1}$ is bounded by $\epsilon(\epsilon+\epsilon^2M)^{m-1}$ and thus, by the same argument as in the previous case, the corresponding integral in J_6 is bounded by

$$\begin{aligned}
&\epsilon(\epsilon+\epsilon^2M)^{m-1}\int_0^R w^\alpha r^4\partial_t^i\varphi_t(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t)\,dr\\
&\quad\lesssim \epsilon(\epsilon+\epsilon^2M)^{m-1}\left(\sqrt{\mathcal{E}^i}+\epsilon(\mathcal{E}^k+\mathcal{E}^{i-j-k})+\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j-k}\varphi_t\partial_t^k\varphi_t\,dr\right)\\
&\quad\lesssim \epsilon(\epsilon+\epsilon^2M)^{m-1}\left(\sqrt{\mathcal{E}^i}+\epsilon(\mathcal{E}^k+\mathcal{E}^{i-j-k})+\epsilon^2M(\sqrt{\mathcal{E}^i}+\sqrt{\mathcal{E}^{i-j}})\left(\sum_{l=0}^{k-1}\bar{\mathcal{E}}^{i-j-k+l,l}\right)^{\frac12}\right),
\end{aligned}$$

where we have expanded $\partial_t^{i-j}\big(((\psi_1)_t+\epsilon\varphi_t)^2\big)$ and assumed $k\le\left[\frac12(i-j)\right]$. The last case is of the form, for $1\le m\le j$ and $k\le i-j$,

$$\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\big(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t\big)\big(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t\big)\partial_t^{j-m}\varphi_t\big(\epsilon(\psi_1)_t+\epsilon^2\varphi_t\big)^{m-1}\,dr,$$

which is bounded by

$$\epsilon^2(\epsilon+\epsilon^2 M)^{m-1}\int_0^R w^\alpha r^4 \partial_t^i\varphi_t(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t)\partial_t^{j-m}\varphi_t\,dr$$
$$\lesssim \epsilon^2(\epsilon+\epsilon^2 M)^{m-1}\bigg(\mathcal{E}^i+\mathcal{E}^{j-m}+\epsilon\underbrace{\int_0^R w^\alpha r^4\partial_t^i\varphi_t(\partial_t^{i-j-k}\varphi_t+\partial_t^k\varphi_t)\partial_t^{j-m}\varphi_t\,dr}_{J_6^1}$$
$$+\epsilon^2\underbrace{\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j-k}\varphi_t\partial_t^k\varphi_t\partial_t^{j-m}\varphi_t\,dr}_{J_6^2}\bigg),$$

where we have used the boundedness of ψ_1 and $(\psi_1)_t$. The estimation of J_6^1 is similar to previous nonlinear terms. First, if $m=j$ then it is clear that $J_6^1\lesssim M(\mathcal{E}^i+\mathcal{E}^{i-j-k}+\mathcal{E}^k)$. So let $1\le m\le j-1$. If $1\le j\le\left[\frac{i}{2}\right]+1$, take the supremum of $w^{(j-m-1)/2}\partial_t^{j-m}\varphi_t$ and apply the Hardy inequality to deduce that

$$J_6^1\lesssim M\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{j-1}\bar{\mathcal{E}}^{i-j-k+l,l}+\bar{\mathcal{E}}^{k+l,l}\bigg)^{\frac{1}{2}}.$$

If $\left[\frac{i}{2}\right]+2\le j\le i$, then take the supremum of $w^{(i-j-1)/2}(\partial_t^{i-j-k}\varphi_t+\partial_t^k\varphi_t)$ when $j<i$, the supremum of φ_t when $j=i$, and apply the Hardy inequality to obtain $J_6^1\lesssim M\sqrt{\mathcal{E}^i}\big(\sum_{l=0}^{j-1}\bar{\mathcal{E}}^{i-j-k+l,l}+\bar{\mathcal{E}}^{k+l,l}\big)^{1/2}$. By the same argument as before, we deduce that

$$J_6^1\lesssim M\big(\mathcal{E}^i+\mathcal{E}^{j-m}+\sqrt{\mathcal{E}^i}\big(\sum_{l=0}^{i-j-1}\bar{\mathcal{E}}^{j-m+l,l}\big)^{1/2}\big).$$

It now remains to estimate J_6^2. Here, not only j but also k will matter. Let us start with $1\le j\le\left[\frac{i}{2}\right]+1$. Due to the symmetry of indices, we can assume that $k\le\left[\frac{1}{2}(i-j)\right]$. Notice that, if $m=j$ or $k=0$, then the last factor or the third factor is bounded by M and thus this reduces to the case that has been treated before. Let $1\le m\le j-1$ and $1\le k\le\left[\frac{1}{2}(i-j)\right]$. We write J_6^2 as

$$J_6^2=\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(\alpha-k-j+m+2)/2}r^2\partial_t^{i-j-k}\varphi_t w^{(k-1)/2}\partial_t^k\varphi_t w^{(j-m-1)/2}\partial_t^{j-m}\varphi_t\,dr.$$

Hence by (4-9) we first see that

$$J_6^2\lesssim M^2\sqrt{\mathcal{E}^i}\bigg(\int_0^R w^{\alpha-k-j+m+2}r^4|\partial_t^{i-j-k}\varphi_t|^2\,dr\bigg)^{\frac{1}{2}}.$$

By applying the Hardy inequality (3-3) $j+k-2$ times to the last term we obtain

$$J_6^2\lesssim M^2\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{j+k-2}\bar{\mathcal{E}}^{i-j-k+l,l}\bigg)^{\frac{1}{2}}.$$

Now let $\left[\frac{i}{2}\right]+2\le j\le i$. If $j=i$ or $j=i-1$, then $k=0$ or $k=1$, and thus $J_6^2\lesssim M^2(\mathcal{E}^i+\mathcal{E}^{j-m})$. If $k=0$ or $k=i-j$, then this reduces to the previous case. So we assume $\left[\frac{i}{2}\right]+2\le j\le i-2$ and

$1 \le k \le i-j-1$. In this case, we have

$$J_6^2 = \int_0^R w^{\alpha/2} r^2 \partial_t^i \varphi_t w^{(i-j-k-1)/2} \partial_t^{i-j-k} \varphi_t w^{(k-1)/2} \partial_t^k \varphi_t w^{(\alpha-i+j+2)/2} r^2 \partial_t^{j-m} \varphi_t \, dr$$
$$\lesssim M^2 \sqrt{\mathcal{E}^i} \Big(\sum_{l=0}^{i-j-2} \bar{\mathcal{E}}^{j-m+l,l} \Big)^{\frac{1}{2}}.$$

We next move onto J_7, which will contribute to the energy. Integration by parts yields

$$J_7 = \gamma \int_0^R \partial_r (r^3 \partial_t^i \varphi_t) w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} (r^3 \partial_t^i \varphi)_r \, dr$$
$$= \frac{\gamma}{2} \frac{d}{dt} \int_0^R w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} |(r^3 \partial_t^i \varphi)_r|^2 \, dr + \frac{\gamma(\gamma+1)}{2} \int_0^R w^{1+\alpha} J^{-\gamma-2} J_t \frac{1}{r^2} |(r^3 \partial_t^i \varphi)_r|^2 \, dr,$$

where the commutator is bounded by $(\epsilon + \epsilon^2 M)\mathcal{E}^i$.

J_8 satisfies

$$J_8 = -2 \frac{d}{dt} \int_0^R w^\alpha r^4 \Phi(r) |\partial_t^i \varphi|^2 \, dr \quad \text{and} \quad |J_8| \lesssim \mathcal{E}^i.$$

If $\gamma > \frac{4}{3}$, the first expression will be used, so that J_8 can contribute to the energy. If $\gamma \le \frac{4}{3}$, then we will use the estimation, so the contribution of $\mathcal{E}^i$ in the right-hand side of the energy inequality will be of order 1.

Next, for J_9, by distributing the spatial derivative we write it as

$$-\frac{J_9}{\gamma} = \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 (w^{1+\alpha})_r \partial_t^{i-1-j} (J^{-\gamma-1}) \frac{1}{r^2} (r^3 \partial_t^j \varphi_t)_r \, dr$$
$$+ \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 w^{1+\alpha} \partial_t^{i-1-j} \partial_r (J^{-\gamma-1}) \frac{1}{r^2} (r^3 \partial_t^j \varphi_t)_r \, dr$$
$$+ \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 w^{1+\alpha} \partial_t^{i-1-j} (J^{-\gamma-1}) (4 \partial_t^j \partial_r \varphi_t + r \partial_t^j \partial_r^2 \varphi_t) \, dr.$$

We denote the integrals in the above three summations by J_9^1, J_9^2, J_9^3. We start with J_9^1. Notice that $\partial_t^{i-1-j} (J^{-\gamma-1})$ consists of $(\partial_t^{i-j-1-k} J)(J_t)^k$ for $0 \le k \le i-j-2$, where

$$\partial_t^{i-j-1-k} J = 3\big(\epsilon \partial_t^{i-j-1-k} (\psi_1) + \epsilon^2 \partial_t^{i-j-1-k} \varphi\big) + r\big(\epsilon \partial_t^{i-j-1-k} (\psi_1)_r + \epsilon^2 \partial_t^{i-j-1-k} \varphi_r\big). \tag{5-12}$$

Let $j+1 \le \left[\frac{i}{2}\right]$. Then $|w^{j/2}(1/r^2)(r^3 \partial_t^j \varphi_t)_r| \lesssim M$ by (4-9), and $|J_t|^k \lesssim (\epsilon + \epsilon^2 M)^k$. We also recall that $(w^{1+\alpha})_r = -r w^\alpha \Phi(r)$, where $\Phi(r)$ is bounded. Thus

$$|J_9^1| \lesssim (\epsilon + \epsilon^2 M)^k M \sqrt{\mathcal{E}^i} \Big(\int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k} J|^2 \, dr \Big)^{\frac{1}{2}}.$$

From (5-12) we use the regularity of ψ_1 and apply the Hardy inequality to obtain

$$\int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k} J|^2\,dr \lesssim \epsilon^2 + \epsilon^4 \int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k}\varphi|^2\,dr + \epsilon^4 \int_0^R w^{\alpha-j} r^6 |\partial_t^{i-j-1-k}\varphi_r|^2\,dr$$
$$\lesssim \epsilon^2 + \epsilon^4 \sum_{l=0}^{j+1} \bar{\mathcal{E}}^{i-j-1-k+l,l}. \tag{5-13}$$

Hence we have $|J_9^1| \lesssim \epsilon(\epsilon+\epsilon^2 M)^k M \sqrt{\mathcal{E}^i}\bigl(1+\epsilon\bigl(\sum_{l=0}^{j+1} \bar{\mathcal{E}}^{i-j-1-k+l,l}\bigr)^{1/2}\bigr)$ for $j+1 \le \bigl[\frac{i}{2}\bigr]$. Now suppose $\bigl[\frac{i}{2}\bigr] \le j \le i-2$. Then $|w^{(i-j-2-k)/2}\partial_t^{i-j-1-k} J| \lesssim \epsilon + \epsilon^2 M$. Therefore, by further applying the Hardy inequality,

$$|J_9^1| \lesssim (\epsilon+\epsilon^2 M)^{k+1}\sqrt{\mathcal{E}^i}\left(\int_0^R w^{\alpha-i+j+2+k}\frac{1}{r^2}|(r^3\partial_t^j\varphi_t)_r|^2\,dr\right)^{\frac12} \lesssim (\epsilon+\epsilon^2 M)^{k+1}\sqrt{\mathcal{E}^i}\left(\sum_{l=0}^{i-j-2}\bar{\mathcal{E}}^{j+1+l,l}\right)^{\frac12}.$$

We next treat J_9^3. Let $j < \bigl[\frac12(i-3)\bigr]$. Then $|w^{(j+2)/2}(4\partial_t^j\partial_r\varphi_t + r\partial_t^j\partial_r^2\varphi_t)| \lesssim M$ by (4-9). Thus

$$|J_9^3| \lesssim (\epsilon+\epsilon^2 M)^k M\sqrt{\mathcal{E}^i}\left(\int_0^R w^{\alpha-j} r^2 |\partial_t^{i-j-1-k} J|^2\,dr\right)^{\frac12},$$

where we have used $|J_t|^k \lesssim (\epsilon+\epsilon^2 M)^k$. Hence, this case is the same as in the previous case of J_9^1 (see (5-13)) except for the factor r^2 instead of r^4. The weight r^4 is recovered by applying the Hardy inequality (3-1) once. Notice that the Hardy inequality near the boundary is used multiple times in (5-13) and thus we obtain the same result as in J_9^1. Now suppose $\bigl[\frac12(i-3)\bigr] \le j \le i-2$. Then $|w^{(i-j-2-k)/2}\partial_t^{i-j-1-k} J| \lesssim \epsilon+\epsilon^2 M$. Therefore, by further applying the Hardy inequality,

$$|J_9^3| \lesssim (\epsilon+\epsilon^2 M)^{k+1} M\sqrt{\mathcal{E}^i}\left(\int_0^R w^{\alpha-i+j+4+k}\bigl(r^2|\partial_t^j\partial_r\varphi_t|^2 + r^4|\partial_t^j\partial_r^2\varphi_t|^2\bigr)\,dr\right)^{\frac12}$$
$$\lesssim (\epsilon+\epsilon^2 M)^{k+1} M\sqrt{\mathcal{E}^i}\left(\sum_{l=0}^{i-j-2}\bar{\mathcal{E}}^{j+2+l,l+1}\right)^{\frac12}.$$

Now J_9^2 can be treated similarly to J_9^3 by considering $j \le \bigl[\frac12(i-3)\bigr]$ and $j > \bigl[\frac12(i-3)\bigr]$, since the nonlinear structure and number of spatial derivatives involved are essentially the same. We omit the details.

We next move onto J_{10}. As in J_9, we first distribute the spatial derivative to write

$$-\frac{J_{10}}{\gamma} = \sum_{j=0}^{i-1} c_{3j}\int_0^R \partial_t^i\varphi_t r^3 (w^{1+\alpha})_r \partial_t^{i-1-j}\left(\frac{J^{-\gamma-1}-1}{\epsilon}\right)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\,dr$$
$$+\sum_{j=0}^{i-1} c_{3j}\int_0^R \partial_t^i\varphi_t r^3 w^{1+\alpha}\partial_t^{i-1-j}\partial_r\left(\frac{J^{-\gamma-1}-1}{\epsilon}\right)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\,dr$$
$$+\sum_{j=0}^{i-1} c_{3j}\int_0^R \partial_t^i\varphi_t r^3 w^{1+\alpha}\partial_t^{i-1-j}\left(\frac{J^{-\gamma-1}-1}{\epsilon}\right)(4\partial_t^{j+1}(\psi_1)_r + \partial_t^{j+1}(\psi_1)_{rr})\,dr.$$

We denote these summands by J_{10}^1, J_{10}^2 and J_{10}^3. Before we discuss further, we remark that, since $\partial_t^{j+1}\psi_1$ is a constant multiple of ψ_1 or $(\psi_1)_t$, the last factor in the integral doesn't lose derivatives at all and it is just a nice function with a desirable regularity in our weighted spaces. We will treat J_{10}^1 and J_{10}^3. Notice that $r^3(w^{1+\alpha})_r \lesssim r^4 w^\alpha$. We first consider $j=i-1$. Then, by recalling $|(J^{-\gamma-1}-1)/\epsilon| \lesssim 1+\epsilon M$ (see the estimation of I_{10}^2 in the previous lemma) and the regularity of ψ_1, we deduce that the integral is bounded by $(1+\epsilon M)\sqrt{\mathcal{E}^i}$. The same argument yields the same bound for the case $j=i-1$ of J_{10}^3. Now let $0\le j\le i-2$. Then $\partial_t^{i-1-j}((J^{-\gamma-1}-1)/\epsilon)$ consists of $(1/\epsilon)(\partial_t^{i-j-1-k}J)(J_t)^k$ for $0\le k\le i-j-2$, where $\partial_t^{i-j-1-k}J$ is given in (5-12). The estimates of J_{10}^1 and J_{10}^3 can be obtained in a similar way as in the previous case. The differences are the presence of $1/\epsilon$ and that the last factor in the integral is a given function in this case, which only makes the argument easier. As can be seen in (5-12) and (5-13), $\partial_t^{i-j-1-k}J/\epsilon$ is bounded by the total energy and the result will be $1/\epsilon$ times the corresponding estimates of J_9^1 and J_9^3. By the same argument, we can obtain the estimate of J_{10}^2 as $1/\epsilon$ times the corresponding estimates of J_9^2. In all cases, the leading order of the bounds is $\sqrt{\mathcal{E}^i}$, while the leading order for J_9 is $\epsilon(M\sqrt{\mathcal{E}^i}+\mathcal{E}^i)$.

Lastly, J_{11} can be estimated in the same way as in the case $j=i$ in J_5. The difference is the order of ϵ:

$$|J_{11}| \lesssim \sqrt{\mathcal{E}^i} + \epsilon\sum_{j=0}^{i}\mathcal{E}^j + \epsilon^2 M\biggl(\sum_{j=0}^{i}\mathcal{E}^j + \sqrt{\mathcal{E}^i}\biggl(\sum_{0\le l\le j\le i-3}\bar{\mathcal{E}}^{j,l}\biggr)^{\frac{1}{2}}\biggr).$$

This finishes the proof of the lemma. □

6. Elliptic estimates

Proposition 6.1. *Suppose that (φ,φ_t) satisfy (2-11) for $0\le t\le T$ and the corresponding total energy $\bar{\mathcal{E}}$ is bounded. Moreover, we assume (4-9). Then $\bar{\mathcal{E}}$ enjoys the estimates*

$$\bar{\mathcal{E}} \lesssim 1+(1+\epsilon^4M^2)\mathcal{E}+\epsilon^2(M^2+\bar{\mathcal{E}}+\epsilon^2M^2\bar{\mathcal{E}}) \tag{6-1}$$

for all sufficiently small $\epsilon>0$.

Notice that (6-1) is trivially obtained for $\bar{\mathcal{E}}^{j,0}$ for $0\le j\le[\alpha]+4$ because $\bar{\mathcal{E}}^{j,0}$ and $\mathcal{E}^j$ are equivalent. Moreover, due to (2-13), it suffices to estimate $\bar{\mathcal{E}}_r^{j,k}$ for $1\le k\le j\le[\alpha]+4$. We start with the simplest case $j=1$ and $k=1$ and then move onto the general case $j\ge 2$.

Lemma 6.2 ($\bar{\mathcal{E}}^{1,1}$). *Suppose that (φ,φ_t) satisfy (2-11) for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume (4-9). Then there exists a constant $C>0$ such that*

$$\bar{\mathcal{E}}_r^{1,1} \lesssim 1+(1+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1)+\epsilon^2(M^2+(1+\epsilon^2M^2)\bar{\mathcal{E}}_r^{1,1}).$$

Proof. In this case, because of (2-13), we only need to show that $\int_0^R w^{2+\alpha}r^4|\varphi_{rr}|^2\,dr$ is bounded by the temporal instant energy. By using (2-1) and (4-3), we rewrite (2-11) in the form

$$\begin{aligned}\gamma\bigl(w^{1+\alpha}r^4\varphi_r\bigr)_r = \frac{w^\alpha r^4\varphi_{tt}}{(1+\zeta)^4} + 4\lambda w^\alpha r^4\psi_1^2 + 4\lambda\epsilon w^\alpha r^4\psi_1\varphi - \lambda w^\alpha r^4\psi_1\frac{f}{\epsilon} + w^\alpha r^4\Phi(r)\frac{f}{\epsilon^2}\\ - \frac{2w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^2}{(1+\zeta)^7} + (3\gamma-4)w^\alpha r^4\Phi(r)\varphi + r^3\biggl(w^{1+\alpha}\frac{h}{\epsilon^2}\biggr)_r.\end{aligned} \tag{6-2}$$

We will exploit the elliptic structure of the term in the left-hand side of (6-2). Square both sides of (6-2), divide them by $w^\alpha r^4$ and integrate the result over $(0, R)$ to get

$$\begin{aligned}\int_0^R \frac{\gamma^2}{w^\alpha r^4}|(w^{1+\alpha}r^4\varphi_r)_r|^2\,dr \\ \lesssim \int_0^R \frac{w^\alpha r^4|\varphi_{tt}|^2}{(1+\zeta)^8}\,dr + \int_0^R w^\alpha r^4\psi_1^4\,dr + \epsilon^2\int_0^R w^\alpha r^4\psi_1^2|\varphi|^2\,dr \\ + \int_0^R w^\alpha r^4\psi_1^2\left|\frac{f}{\epsilon}\right|^2 dr + \int_0^R w^\alpha r^4\left|\Phi(r)\frac{f}{\epsilon^2}\right|^2 dr + \int_0^R \frac{w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^4}{(1+\zeta)^{14}}\,dr \\ + \int_0^R w^\alpha r^4|\Phi(r)\varphi|^2\,dr + \int_0^R \frac{1}{w^\alpha r^4}\left|r^3\left(w^{1+\alpha}\frac{h}{\epsilon^2}\right)_r\right|^2 dr. \qquad (6\text{-}3)\end{aligned}$$

We denote the integral in the left-hand side by I and each integral in the right-hand side by I_k for $1 \le k \le 8$. It is clear that

$$I_1 \lesssim \mathcal{E}^1, \quad I_2 \lesssim 1, \quad I_3 \lesssim \epsilon^2\mathcal{E}^0. \qquad (6\text{-}4)$$

For I_4 and I_5, we recall that $f = O(|\epsilon\psi_1 + \epsilon^2\varphi|^2)$. Then, by using the boundedness of ψ_1 and Φ as well as (4-9), we have

$$I_4 \lesssim \epsilon^2(1+\epsilon^4M^2\mathcal{E}^0), \quad I_5 \lesssim 1+\epsilon^4M^2\mathcal{E}^0.$$

Similarly, we obtain

$$I_6 \lesssim 1+\epsilon^4M^2\mathcal{E}^1, \quad I_7 \lesssim \mathcal{E}^0.$$

The last term involves the full derivatives and it needs to be estimated carefully. Recall that

$$\begin{aligned}h = h\left(\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right) = O\left(\left|\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right|^2\right),\\ \frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r = 3(\epsilon\psi_1+\epsilon^2\varphi) + r(\epsilon(\psi_1)_r+\epsilon^2\varphi_r).\end{aligned}$$

We then see that

$$r^3\left(w^{1+\alpha}\frac{h}{\epsilon^2}\right)_r = r^3w^{1+\alpha}\frac{h^{(1)}}{\epsilon}\cdot\big(4((\psi_1)_r+\epsilon\varphi_r) + r((\psi_1)_{rr}+\epsilon\varphi_{rr})\big) + r^3(w^{1+\alpha})_r\frac{h}{\epsilon^2},$$

where $h^{(1)}$ means the first derivative of h with respect to the argument. By using the notation Φ given in (2-10), we write $(w^{1+\alpha})_r = -rw^\alpha\Phi(r)$ and, hence, we see that I_8 is bounded by

$$\begin{aligned}I_8 \lesssim \int_0^R w^{2+\alpha}r^2\left|\frac{h^{(1)}}{\epsilon}\right|^2|(\psi_1)_r+\epsilon\varphi_r|^2\,dr + \int_0^R w^{2+\alpha}r^4\left|\frac{h^{(1)}}{\epsilon}\right|^2|(\psi_1)_{rr}+\epsilon\varphi_{rr}|^2\,dr \\ + \int_0^R w^\alpha r^4|\Phi(r)|^2\left|\frac{h}{\epsilon^2}\right|^2 dr = I_8^1+I_8^2+I_8^3.\end{aligned}$$

Notice that $|h^{(1)}/\epsilon| \lesssim 1+\epsilon M$ and $|h/\epsilon^2| \lesssim 1+\epsilon^2M^2$. It is easy to see that

$$I_8^2 \lesssim (1+\epsilon M)^2(1+\epsilon^2\bar{\mathcal{E}}_r^{1,1}).$$

For I_8^1 and I_8^3, we further employ the Hardy inequalities near the origin (3-1) and near the boundary (3-3) respectively to deduce that

$$I_8^1 + I_8^3 \lesssim (1+\epsilon^2 M^2)(1+\epsilon^2(\mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1})).$$

We now turn our attention to the integral I in the left-hand side of (6-3). First notice that

$$(w^{1+\alpha} r^4 \varphi_r)_r = w^{1+\alpha} r^4 \varphi_{rr} + 4w^{1+\alpha} r^3 \varphi_r + (w^{1+\alpha})_r r^4 \varphi_r = w^{1+\alpha} r^4 \varphi_{rr} + 4w^{1+\alpha} r^3 \varphi_r - w^\alpha r^5 \Phi(r) \varphi_r.$$

Then I reads as

$$I = \gamma^2 \int_0^R w^\alpha r^4 \left| w\varphi_{rr} + \frac{4w\varphi_r}{r} - r\Phi(r)\varphi_r \right|^2 dr.$$

By expanding out terms, we see that

$$\begin{aligned}\frac{I}{\gamma^2} = \int_0^R w^{2+\alpha} r^4 |\varphi_{rr}|^2 \, dr + 16 \int_0^R w^{2+\alpha} r^2 |\varphi_r|^2 \, dr + \int_0^R w^\alpha r^6 |\Phi(r)|^2 |\varphi_r|^2 \, dr \\ + \underbrace{8 \int_0^R w^{2+\alpha} r^3 \varphi_{rr} \varphi_r \, dr}_{I^1} \underbrace{-2 \int_0^R w^{1+\alpha} r^5 \Phi(r) \varphi_{rr} \varphi_r \, dr}_{I^2} - 8 \int_0^R w^{1+\alpha} r^4 \Phi(r) |\varphi_r|^2 \, dr.\end{aligned}$$

For I^1 and I^2, we integrate by parts to get

$$\begin{aligned} I^1 &= -4 \int_0^R (w^{2+\alpha})_r r^3 |\varphi_r|^2 \, dr - 12 \int_0^R w^{2+\alpha} r^2 |\varphi_r|^2 \, dr \\ &= 4 \frac{2+\alpha}{1+\alpha} \int_0^R w^{1+\alpha} r^4 \Phi(r) |\varphi_r|^2 \, dr - 12 \int_0^R w^{2+\alpha} r^2 |\varphi_r|^2 \, dr, \\ I^2 &= -\int_0^R w^\alpha r^6 |\Phi(r)|^2 |\varphi_r|^2 + 5 \int_0^R w^{1+\alpha} r^4 \Phi(r) |\varphi_r|^2 \, dr + \int_0^R w^{1+\alpha} r^5 \Phi'(r) |\varphi_r|^2 \, dr. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\int_0^R w^{2+\alpha} r^4 |\varphi_{rr}|^2 \, dr + 4 \int_0^R w^{2+\alpha} r^2 |\varphi_r|^2 \, dr \\ &\quad = \frac{I}{\gamma^2} + \underbrace{3 \int_0^R w^{1+\alpha} r^4 \Phi(r) |\varphi_r|^2 \, dr - 4 \frac{2+\alpha}{1+\alpha} \int_0^R w^{1+\alpha} r^4 \Phi(r) |\varphi_r|^2 \, dr - \int_0^R w^{1+\alpha} r^5 \Phi'(r) |\varphi_r|^2 \, dr}_{\lesssim \mathcal{E}^0}. \end{aligned}$$

It is clear that the last three terms in the right-hand side are bounded by the zeroth-order energy $\mathcal{E}^0$. Combining all the estimates, we deduce the result. This finishes the proof for the case of $j = 1$ and $k = 1$. □

We now turn into the cases $[\alpha] + 4 \geq j \geq 2$. As in the case of $j = 1$, we will directly use the equation and take advantage of the elliptic estimates. What is subtle and interesting here is to capture the correct behavior of solutions in the normal direction ∂_r near the boundary.

Lemma 6.3 ($\bar{\mathcal{E}}_r^{j,k}$ for $1 \le k \le j,\ 2 \le j$). *Suppose that (φ, φ_t) satisfy (2-11) for $0 \le t \le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume (4-9). Then there exists a constant $C > 0$ such that*

$$\bar{\mathcal{E}}_r^{j,k} \lesssim (1+\epsilon^4 M^2)\sum_{l=0}^{j}\mathcal{E}^l + \bar{\mathcal{E}}_r^{j-1,k-1} + \epsilon^2\biggl(M^2 + (1+\epsilon^2 M^2)\sum_{m=1}^{j}\sum_{l=1}^{m}\bar{\mathcal{E}}_r^{m,l}\biggr).$$

Proof. Notice that because of (2-13), it suffices to show that each spatial energy term $\mathcal{E}_r^{j,k}$ for $1 \le k \le j$ satisfies the inequality. We will present the detail for $j = 2$; other cases follow by induction on j, k. When $k = 1$, the spatial energy term $\bar{\mathcal{E}}_r^{2,1}$ contains one temporal derivative and two spatial derivatives. The time differentiation of (6-2) is the place to start. Notice that the time derivative does not affect the weights at all since w and r do not change with time. Therefore, following the same procedure for $\bar{\mathcal{E}}_r^{1,1}$ in the previous lemma, we can deduce that

$$\bar{\mathcal{E}}_r^{2,1} \lesssim 1 + (1+\epsilon^4 M^2)(\mathcal{E}^0 + \mathcal{E}^1 + \mathcal{E}^2) + (1+\epsilon^2 M^2)(1+\epsilon^2(\bar{\mathcal{E}}_r^{1,1} + \bar{\mathcal{E}}_r^{2,1})).$$

To deal with $\bar{\mathcal{E}}_r^{2,2}$, which contains three spatial derivatives, we will first derive the equation for φ_{rrr} from (6-2). By following the idea in [Jang 2014], first divide both sides of (6-2) by $r^3 w^\alpha$:

$$\begin{aligned}
&\gamma(wr\varphi_{rr} + (1+\alpha)w_r r\varphi_r + 4w\varphi_r) \\
&\quad = \frac{r\varphi_{tt}}{(1+\zeta)^4} + 4\lambda r\psi_1^2 + 4\lambda\epsilon r\psi_1\varphi - \lambda r\psi_1\frac{f}{\epsilon} + r\Phi(r)\frac{f}{\epsilon^2} - \frac{2r|(\psi_1)_t + \epsilon\varphi_t|^2}{(1+\zeta)^7} + (3\gamma-4)r\Phi(r)\varphi \\
&\qquad + w\frac{h^{(1)}}{\epsilon}\bigl(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\bigr) + (1+\alpha)w_r\frac{h}{\epsilon^2}.
\end{aligned}$$

Then we take ∂_r of both sides of the above equation and move the terms involving φ_r into the right-hand side to get

$$\begin{aligned}
&\gamma(wr\varphi_{rrr} + (2+\alpha)w_r r\varphi_{rr} + 5w\varphi_{rr}) \\
&\quad = -\gamma((5+\alpha)w_r\varphi_r + (1+\alpha)w_{rr}r\varphi_r) \\
&\qquad + \biggl(\frac{r\varphi_{tt}}{(1+\zeta)^4}\biggr)_r + 4\lambda(r\psi_1^2)_r + 4\lambda\epsilon(r\psi_1\varphi)_r - \lambda\biggl(r\psi_1\frac{f}{\epsilon}\biggr)_r + \biggl(r\Phi(r)\frac{f}{\epsilon^2}\biggr)_r \\
&\qquad - \biggl(\frac{2r|(\psi_1)_t + \epsilon\varphi_t|^2}{(1+\zeta)^7}\biggr)_r + (3\gamma-4)(r\Phi(r)\varphi)_r \\
&\qquad + \biggl(w\frac{h^{(1)}}{\epsilon}\bigl(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\bigr) + (1+\alpha)w_r\frac{h}{\epsilon^2}\biggr)_r.
\end{aligned} \tag{6-5}$$

As in the previous lemma, we square both sides of (6-5), multiply by $w^{1+\alpha}r^2$ — here the choice of the multiplier $w^{1+\alpha}$ has been inspired by the analysis carried out in [Jang and Masmoudi 2015] — and integrate it over $(0, R)$ to obtain an integral inequality similar to (6-3). We denote the integral in the left-hand side by I and the integrals in the right-hand side by I_k, $1 \le k \le 9$. For I_1 we apply the Hardy inequality near the origin (3-1) to overcome stronger weights near the origin:

$$I_1 \lesssim \int_0^R w^{1+\alpha} r^2 (|w_r|^2 + |r w_{rr}|^2) |\varphi_r|^2 \, dr \lesssim \mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1}.$$

For I_2, we obtain

$$\begin{aligned} I_2 &\lesssim \int_0^R \frac{w^{1+\alpha} r^4 |\varphi_{ttr}|^2}{(1+\zeta)^8} \, dr + \int_0^R \frac{w^{1+\alpha} r^2 |\varphi_{tt}|^2}{(1+\zeta)^8} \, dr + \int_0^R \frac{w^{1+\alpha} r^4 |\varphi_{tt}(\epsilon(\psi_1)_r + \epsilon^2 \varphi_r)|^2}{(1+\zeta)^{14}} \, dr \\ &\lesssim \bar{\mathcal{E}}_r^{2,0} + (\mathcal{E}^1 + \bar{\mathcal{E}}_r^{2,0}) + (\epsilon^2 + \epsilon^4 M^2) \mathcal{E}^1, \end{aligned}$$

where we have applied the Hardy inequality (3-1) to the second term. Next, by the regularity of ψ_1 and the Hardy inequality (3-1), we observe that

$$I_3 + I_4 \lesssim 1 + \epsilon^2 \mathcal{E}^0.$$

For I_5 and I_6, we note that $f = f(\epsilon\psi_1 + \epsilon^2\varphi) = O(|\epsilon\psi_1 + \epsilon^2\varphi|^2)$ and $f_r = f^{(1)} \cdot (\epsilon(\psi_1)_r + \epsilon^2\varphi_r)$. Hence

$$I_5 + I_6 \lesssim 1 + \epsilon^4 M^2 \mathcal{E}^0.$$

Similarly, by using the Hardy inequality (3-1) and (4-9) we have

$$I_7 \lesssim 1 + \epsilon^4 M^2 (\mathcal{E}^0 + \mathcal{E}^1).$$

Since $(r\Phi(r)\varphi)_r = \Phi(r)\varphi + r\Phi(r)'\varphi + r\Phi(r)\varphi_r$, by (3-1) for the first term again we see that

$$I_8 \lesssim \mathcal{E}^0.$$

For I_9, we note that the last line of (6-5) can be written as follows:

$$\begin{aligned} &w \frac{h^{(1)}}{\epsilon} \big(5((\psi_1)_{rr} + \epsilon\varphi_{rr}) + r((\psi_1)_{rrr} + \epsilon\varphi_{rrr})\big) + w h^{(2)} \big(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\big)^2 \\ &\qquad + (2+\alpha) w_r \frac{h^{(1)}}{\epsilon} \big(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\big) + (1+\alpha) w_{rr} \frac{h}{\epsilon^2}, \end{aligned}$$

where $h^{(2)}$ means the second derivative with respect to the argument. Thus I_9 includes φ_{rrr}, φ_{rr}, φ_r with different weights and it can be treated in a similar way as we did for I_8 of (6-3) in the previous lemma. We expand it out and apply the Hardy inequalities both near the origin (3-1) and near the boundary (3-3) to deduce that

$$I_9 \lesssim (1 + \epsilon M)^2 (1 + \epsilon^2 (\mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1} + \bar{\mathcal{E}}_r^{2,2})). \tag{6-6}$$

What follows now is the elliptic estimate for I coming from the first term in (6-5), which will give rise to the term $\bar{\mathcal{E}}_r^{2,2}$:

$$
\begin{aligned}
I &= \int_0^R w^{1+\alpha} r^2 |w r \varphi_{rrr} + (2+\alpha) w_r r \varphi_{rr} + 5 w \varphi_{rr}|^2 \, dr \\
&= \int_0^R w^{3+\alpha} r^4 |\varphi_{rrr}|^2 \, dr + (2+\alpha)^2 \int_0^R w^{1+\alpha} r^4 |w_r|^2 |\varphi_{rr}|^2 \, dr + 25 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr \\
&\qquad + \underbrace{2(2+\alpha) \int_0^R w^{2+\alpha} r^4 w_r \varphi_{rrr} \varphi_{rr} \, dr}_{I^1} + \underbrace{10 \int_0^R w^{3+\alpha} r^3 \varphi_{rrr} \varphi_{rr} \, dr}_{I^2} \\
&\qquad\quad + 10(2+\alpha) \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr.
\end{aligned}
$$

For I^1 and I^2, we integrate by parts to get

$$
\begin{aligned}
\frac{I^1}{2+\alpha} &= -\int_0^R (w^{2+\alpha})_r r^4 w_r |\varphi_{rr}|^2 \, dr - 4 \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr - \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr \\
&= -(2+\alpha) \int_0^R w^{1+\alpha} r^4 |w_r|^2 |\varphi_{rr}|^2 \, dr - 4 \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr - \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr, \\
I^2 &= -5(3+\alpha) \int_0^R w^{2+\alpha} w_r r^3 |\varphi_{rr}|^2 \, dr - 15 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\int_0^R w^{3+\alpha} r^4 |\varphi_{rrr}|^2 \, dr + 10 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr \\
&\qquad = I - (\alpha - 3) \int_0^R w^{2+\alpha} w_r r^3 |\varphi_{rr}|^2 \, dr + (2+\alpha) \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr.
\end{aligned}
$$

By noting $w_r = -r\Phi(r)/(1+\alpha)$, we see that the last two terms are bounded by $\bar{\mathcal{E}}_r^{1,1}$. By combining with all other estimates, we deduce that

$$
\bar{\mathcal{E}}_r^{2,2} \lesssim 1 + (1+\epsilon^4 M^2)(\mathcal{E}^0 + \mathcal{E}^1) + \mathcal{E}^2 + \bar{\mathcal{E}}_r^{1,1} + (1+\epsilon^2 M^2)(1+\epsilon^2(\mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1} + \bar{\mathcal{E}}_r^{2,2})).
$$

By the previous lemma, the desired result follows and this finishes the proof of the case $j = 2$. Other cases can be done inductively: take ∂_r derivatives of (6-5), square it, multiply by appropriate weights depending on the number of spatial derivatives, and exploit the Hardy inequalities and the elliptic estimates. The procedure and the estimates are similar to the previous cases and we omit the details. □

The inequality (6-1) in Proposition 6.1 now follows from Lemmas 6.2 and 6.3 by considering a suitable linear combination of $\bar{\mathcal{E}}^{j,k}$ to absorb $\bar{\mathcal{E}}^{j-1,k-1}$ and $\epsilon^2 \sum_{m=1}^{j} \sum_{l=1}^{m} \bar{\mathcal{E}}_r^{m,l}$ into the left-hand side.

7. Proof of Theorem 2.4

Since $M \lesssim \bar{\mathcal{E}}^{1/2}$, (6-1) yields

$$\bar{\mathcal{E}} \lesssim 1+\mathcal{E}+\epsilon^2\bar{\mathcal{E}}+\epsilon^4\bar{\mathcal{E}}^2.$$

Therefore, for all sufficiently small $\epsilon > 0$, we deduce that the total energy is bounded by the total temporal energy:

$$\bar{\mathcal{E}} \lesssim 1+\mathcal{E}.$$

Now from the energy inequality (5-1) in Proposition 5.1, we obtain

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1+(1-a(\gamma))\sqrt{\mathcal{E}}+(\epsilon+\epsilon^2M)(\sqrt{\mathcal{E}}+\sqrt{\bar{\mathcal{E}}}) \lesssim 1+(1-a(\gamma))\sqrt{\mathcal{E}}+\epsilon\sqrt{\mathcal{E}}+(\epsilon\sqrt{\mathcal{E}})^2. \tag{7-1}$$

First let $\gamma > \frac{4}{3}$, in which case $a(\gamma)=1$. So the differential inequality becomes

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1+\epsilon\sqrt{\mathcal{E}}+(\epsilon\sqrt{\mathcal{E}})^2,$$

which in turn gives rise to

$$\frac{d}{dt}(\epsilon\sqrt{\mathcal{E}}+1) \lesssim \epsilon(\epsilon\sqrt{\mathcal{E}}+1)^2.$$

Therefore, by solving this differential inequality, we deduce that

$$\sqrt{\mathcal{E}}(t) \lesssim \frac{\sqrt{\mathcal{E}(0)}+(\epsilon\sqrt{\mathcal{E}(0)}+1)t}{1-\epsilon(\epsilon\sqrt{\mathcal{E}(0)}+1)t}.$$

Hence, in the case of $\gamma > \frac{4}{3}$, we conclude that $\sup_{0\le t\le T}\sqrt{\mathcal{E}}(t)$ is bounded for all sufficiently small $\epsilon \le \epsilon_0$, where $\epsilon_0 = O(1/T)$.

Next let $\gamma \le \frac{4}{3}$. Then we need to solve

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1+\sqrt{\mathcal{E}}+\epsilon^2(\sqrt{\mathcal{E}})^2.$$

Equivalently,

$$\frac{d}{dt}(\epsilon^2\sqrt{\mathcal{E}}+1) \lesssim (\epsilon^2\sqrt{\mathcal{E}}+1)^2+\epsilon^2-1.$$

Let $k=\sqrt{1-\epsilon^2}$. Then

$$\left(\frac{1}{\epsilon^2\sqrt{\mathcal{E}}+1-k}-\frac{1}{\epsilon^2\sqrt{\mathcal{E}}+1+k}\right)\frac{d}{dt}(\epsilon^2\sqrt{\mathcal{E}}+1) \lesssim 2k.$$

Thus

$$\sqrt{\mathcal{E}}(t) \lesssim \frac{\sqrt{\mathcal{E}}(0)((1+k)^2e^{2kt}-\epsilon^2)+(1+k)(e^{2kt}-1)}{(1+k)\big(1+k-(1-k)e^{2kt}-\epsilon^2\sqrt{\mathcal{E}}(0)(e^{2kt}-1)\big)}.$$

Notice that $1+k=1+\sqrt{1-\epsilon^2}=O(1)$ and $1-k=\epsilon^2/(1+\sqrt{1-\epsilon^2})=O(\epsilon^2)$. Therefore, we conclude, for $\gamma \le \frac{4}{3}$, that $\sup_{0\le t\le T}\sqrt{\mathcal{E}}(t)$ is bounded for all sufficiently small $\epsilon \le \epsilon_1$, where $\epsilon_1 = O(1/e^{\kappa T})$ for some $\kappa > 0$.

Remark 7.1. If we fix a small $\epsilon > 0$ in the ansatz (2-9) instead of fixing a time T, then the above results would imply that (2-9) can be justified up to $t \le T = O(1/\epsilon)$ for $\gamma > \frac{4}{3}$ and $t \le T = O(|\ln\epsilon|)$ for $\gamma \le \frac{4}{3}$.

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