ANALYSIS & PDE

Volume 9 No. 6 2016

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ANALYSIS AND PDE
Vol. 9, No. 6, 2016
dx.doi.org/10.2140/apde.2016.9.1285

A COMPLETE STUDY OF THE LACK OF COMPACTNESS AND EXISTENCE RESULTS OF A FRACTIONAL NIRENBERG EQUATION VIA A FLATNESS HYPOTHESIS, I

WAEL ABDELHEDI, HICHEM CHTIOUI AND HICHEM HAJAIEJ

Dedicated to the memory of Professor Abbas Bahri who left us on January 10, 2016.

We consider a nonlinear critical problem involving the fractional Laplacian operator arising in conformal geometry, namely the prescribed σ-curvature problem on the standard n-sphere, $n \geq 2$. Under the assumption that the prescribed function is flat near its critical points, we give precise estimates on the losses of the compactness and we provide existence results. In this first part, we will focus on the case $1 < \beta \leq n - 2\sigma$, which is not covered by the method of Jin, Li, and Xiong (2014, 2015).

1. Introduction and main results

Fractional calculus has attracted the interest of a lot of scientists during the last decades. This is essentially due to its numerous applications in various domains: medicine, population modeling, biology, earthquakes, optics, signal processing, astrophysics, water waves, porous media, nonlocal diffusion, image reconstruction problems; see [Hajaiej et al. 2011] and the references [1, 2, 6, 7, 13, 14, 19, 22, 25, 36, 38, 41, 43, 45, 46, 58] therein.

Many important properties of the Laplacian are not inherited, or are only partially satisfied, by its fractional powers. This gave birth to many challenging and rich mathematical problems. However, the literature remained quite silent until the publication of the breakthrough paper of Caffarelli and Silvester [2007]. This seminal work has hugely contributed to unblocking a lot of difficult problems and opening the way for the resolution of many other ones. In this paper, we study another important fractional PDE whose resolution also requires some novelties because of the nonlocal properties of the operator present in it. More precisely, we investigate the existence of solutions for the Nirenberg fractional nonlinear equation

$$P_\sigma u = c(n,\sigma) K u^{(n+2\sigma)/(n-2\sigma)} \quad \text{for } u > 0 \text{ on } \mathbb{S}^n, \tag{1-1}$$

where $\sigma \in (0, 1)$, K is a positive function defined on $(\mathbb{S}^n, g_{\mathbb{S}^n})$,

$$P_\sigma = \frac{\Gamma\left(B+\frac{1}{2}+\sigma\right)}{\Gamma\left(B+\frac{1}{2}-\sigma\right)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2},$$

MSC2010: 35J60, 35B33, 35B99, 35R11, 58E30.
Keywords: fractional Laplacian, critical exponent, σ-curvature, critical points at infinity.

Γ is the gamma function, $c(n,\sigma)=\Gamma\left(\frac{n}{2}+\sigma\right)/\Gamma\left(\frac{n}{2}-\sigma\right)$, and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace–Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. The operator P_σ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. The stereographic projection from $\mathbb{S}^n \setminus \{N\}$ to $\mathbb{R}^n$ is the inverse of $F:\mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$ defined by

$$F(x)=\left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1}\right),$$

where N is the north pole of $\mathbb{S}^n$. For all $f \in C^\infty(\mathbb{S}^n)$, we have

$$(P_\sigma(f))\circ F=\left(\frac{2}{1+|x|^2}\right)^{-(n+2\sigma)/2}(-\Delta)^\sigma\left(\left(\frac{2}{1+|x|^2}\right)^{(n-2\sigma)/2}(f\circ F)\right), \tag{1-2}$$

where $(-\Delta)^\sigma$ is the fractional Laplacian operator (see page 117 of [Stein 1970], for example).

For $\sigma=1$, the classical Nirenberg problem consists of the following question: which function K on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is the scalar curvature of a metric g that is conformal to $g_{\mathbb{S}^n}$? This is equivalent to solving

$$P_1 v+1=-\Delta_{g_{\mathbb{S}^n}} v+1=Ke^{2v} \quad \text{on } \mathbb{S}^2 \tag{1-3}$$

and

$$P_1 w+1=-\Delta_{g_{\mathbb{S}^n}} w+b(n)R_0 w=b(n)Kw^{(n+2)/(n-2)} \quad \text{on } \mathbb{S}^n,\ n\geq 3, \tag{1-4}$$

where $g=e^{2v}g_{\mathbb{S}^n}$, $b(n)=(n-2)/(4(n-1))$, and $w=e^{(n-2)v/4}$, and where $R_0=n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

To our knowledge, the very first contribution to this topic is due to D. Koutroufiotis [1972]. He has been able to solve the above Nirenberg problem (1-3) when K is an antipodally symmetric function which is close to 1. However, his approach only applies to $\mathbb{S}^2$. Following a self-contained method, Moser [1973] has solved the Nirenberg problem on $\mathbb{S}^2$ for all antipodally symmetric functions K which are just positive somewhere. Later on, Chang and Yang [1988] have succeeded in removing the symmetry assumption on K in dimension 2 and Bahri and Coron [1991] have extended these results to dimension 3.

Another important issue related to the study of the classical Nirenberg problem is the compactness of the solutions. This has first been addressed by Chang, Gursky and Yang [Chang et al. 1993], Han [1990] and Schoen and Zhang [1996], for $n=2$ or $n=3$.

Compactness and existence of solutions in higher dimensions have been established in the breakthrough papers of Li [1995; 1996]. Let us point out that the situation is completely different in higher dimensions ($n>3$). More precisely, when $n=2$ or $n=3$, a sequence of solutions of the Nirenberg problem cannot blow up at more than one point. If $n>3$, there could be blow ups at many points, which considerably complicates the study of the problem. Many aspects of this very interesting situation have been addressed in [Ambrosetti et al. 1999; Ben Ayed et al. 1996; Ben Mahmoud and Chtioui 2012; Chen and Lin 2001; Li 1995; 1996].

Another stimulating situation is the study of higher orders and fractional order conformally invariant pseudodifferential operators P_k^g on $(\mathbb{S}^n, g_{\mathbb{S}^n})$, which exist for all positive integers k if n is odd and for $k=\left\{1,\dots,\frac{n}{2}\right\}$ if n is even. These operators were first introduced by Graham, Jenne, Mason and Sparling [Graham et al. 1992]. Beyond the case P_1^g which corresponds to the operator associated to the classical Nirenberg problem discussed above, the operator P_2^g is the well known Paneitz operator; see [Abdelhedi and Chtioui 2006; Djadli et al. 2000; Paneitz 2008; Wei and Xu 2009] and references therein. Up to positive constants, $P_1^g(1)$ is the scalar curvature associated to g and $P_2^g(1)$ is the so-called Q-curvature.

In the last two decades, it has been realized that the conformal Laplacian P_1^g, and more generally P_k^g, play a central role in conformal geometry. As mentioned previously, the classical Nirenberg problem is naturally associated to the conformal Laplacian. Consequently, the higher order Nirenberg problems are associated to Graham, Jenne, Mason and Sparling operators (known as the GJMS operators). Recently, a recursive formula for GJMS operators and Q-curvature has been found by Juhl [2014; 2013] (see also [Fefferman and Graham 2013]). Moreover, Graham and Zworski [2003] have introduced a family of fractional order conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds thanks to a scattering theory.

After this seminal paper, new interpretations of the fractional operators and their associated Q-curvatures have been the subject of many studies; see for example [Chang and González 2011]. For the Q-curvature of order σ on general manifolds, we refer to [Chang and González 2011; González et al. 2012; González and Qing 2013; Graham and Zworski 2003; Qing and Raske 2006] and references therein. Prescribing Q-curvature of order σ on $\mathbb{S}^n$ can be interpreted as a generalization of the Nirenberg problem, called in this context the fractional Nirenberg problem.

For $0 < \sigma < 1$, this challenging problem was first addressed in [Jin et al. 2014; 2015]. In these two groundbreaking papers, the authors were able to show the existence of solutions of (1-1) and to derive some compactness properties. More precisely, thanks to a very subtle approach based on approximation of the solutions of (1-1) by a blow-up subcritical method, they proved the existence of solutions for the critical fractional Nirenberg problem (1-1) (see Theorems 1.1 and 1.2 of [Jin et al. 2014]). Their method is based on tricky variational tools; in particular, they have established many interesting fractional functional inequalities. Their main hypothesis is the so-called flatness condition. Namely, let $K : \mathbb{S}^n \to \mathbb{R}$ be a C^2 positive function. We say that K satisfies the flatness condition $(f)_\beta$ if for each critical point y of K there exist $b_i = b_i(y) \in \mathbb{R}^*$ for $i \le n$, with $\sum_{i=1}^n b_i \neq 0$, such that in some geodesic normal coordinate centered at y we have

$$K(x) = K(y) + \sum_{i=1}^{n} b_i |(x-y)_i|^\beta + R(x-y), \tag{1-5}$$

where $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta-s} = o(1)$ as y tends to zero. Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β. However, they were only able to handle the case $n-2\sigma < \beta < n$ in the flatness hypothesis. This excludes some very interesting functions K. In fact, note that an important class of functions, which is worth including in any results of existence for (1-1), are the Morse functions (C^2 having only nondegenerate critical points). Such functions can be written in the form $(f)_\beta$ with $\beta = 2$. Since Jin, Li and Xiong require $n-2\sigma < \beta < n$ $(0 < \sigma < 1)$, their theorems do not apply to this relevant class of functions. Moreover, they require some additional technical assumptions (K antipodally symmetric in Theorem 1.1 and $K \in C^{1,1}$ positive in Theorem 1.2 of [Jin et al. 2014]).

Motivated by [Jin et al. 2014; 2015] and aiming to include a larger class of functions K in the existence results for (1-1), we develop in this paper a self-contained approach which enables us to include all the plausible cases ($1<\beta<n$). Our method hinges on a readapted characterization of critical points at infinity. The approach is different for $1<\beta\le n-2\sigma$ and $n-2\sigma\le\beta<n$. In this work, we handle the first case.

The spirit of our method goes back to the work of Bahri [1989] and Bahri and Coron [1991]. Nevertheless, the nonlocal properties of the fractional Laplacian involve many additional obstacles and require some novelties in the proofs. Note that in [Abdelhedi and Chtioui 2013], the first two authors have given an existence result for $n=2$, $0<\sigma<1$, through an Euler–Hopf-type formula. In their paper, they assumed that K is a Morse function satisfying the nondegeneracy condition

$$\Delta K(y)\neq 0 \quad \text{whenever } \nabla K(y)=0. \tag{nd}$$

We point out that the criterion of [Abdelhedi and Chtioui 2013] has an equivalent in dimension 3 (see [Abdelhedi and Chtioui $\ge$ 2016]). However, the same method cannot be generalized to higher dimensions $n\ge 4$ under the condition (nd), since the corresponding index counting criteria, when taking into account all the critical points at infinity, are always equal to 1. Recently, Y. Chen, C. Liu and Y. Zheng [Chen et al. 2016] proved an existence result for $n\ge 4$, under the (nd) condition and another topological condition, in the case where the index counting criteria, when taking into account all the critical points at infinity, are equal to 1, but a partial one is not equal to 1.

Convinced that the nondegeneracy assumption would exclude some interesting class of functions K, we opted for the flatness hypothesis used in [Jin et al. 2014; 2015]. But again, in order to include all plausible cases (both $1<\beta\le n-2\sigma$ and $n-2\sigma\le\beta<n$), we need to develop a new line of attack with new ideas. This is essentially due to the structure of the multiple blow-up points, which is much more complicated than in the classical setting. Many new phenomena emerge. More precisely, it turns out that the strong interaction between the bubbles, in the case where $n-2\sigma<\beta<n$, forces all blow-up points to be single, while in the case where $1<\beta<n-2\sigma$ such an interaction of two bubbles is negligible with respect to the self interaction, and if $\beta=n-2\sigma$ there is a phenomenon of balance that is the interaction of two bubbles of the same order with respect to the self interaction. In order to state our results, we need the following notations and assumptions. Let

$$\mathcal{K}=\{y\in\mathbb{S}^n \mid \nabla K(y)=0\}, \qquad \mathcal{K}_{n-2\sigma}=\{y\in\mathcal{K}\mid \beta=\beta(y)=n-2\sigma\},$$
$$\mathcal{K}^+=\Big\{y\in\mathcal{K}\ \Big|\ -\sum_{k=1}^n b_k(y)>0\Big\}, \qquad \tilde{i}(y)=\sharp\{b_k=b_k(y)\mid 1\le k\le n \text{ and } b_k<0\}.$$

For each p-tuple, $1\le p\le\sharp\mathcal{K}$, of distinct points $\tau_p:=(y_{l_1},\dots,y_{l_p})\in(\mathcal{K}_{n-2\sigma})^p$, we define a $p\times p$ symmetric matrix $M(\tau_p)=(m_{ij})$ by

$$\begin{aligned} m_{ii}&=\frac{n-2\sigma}{n}\,\tilde{c}_1\,\frac{-\sum_{k=1}^n b_k(y_{l_i})}{K(y_{l_i})^{n/(2\sigma)}},\\ m_{ij}&=2^{(n-2\sigma)/2}\,c_1\,\frac{-G(y_{l_i},y_{l_j})}{(K(y_{l_i})K(y_{l_j}))^{(n-2\sigma)/(4\sigma)}}, \end{aligned} \tag{1-6}$$

where

$$G(y_{l_i}, y_{l_j}) = \frac{1}{(1-\cos d(y_{l_i}, y_{l_j}))^{(n-2\sigma)/2}},$$
$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+2\sigma)/2}} \quad \text{and} \quad \tilde{c}_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{n-2}}{(1+|x|^2)^n}\, dx. \tag{1-7}$$

Here x_1 is the first component of x in some geodesic normal coordinate system. Let $\rho(\tau_p)$ be the least eigenvalue of $M(\tau_p)$.

$$\text{Assume that } \rho(\tau_p) \neq 0 \text{ for each } \tau_p \in (\mathcal{K}_{n-2\sigma})^p,\ 1 \leq p \leq \sharp\mathcal{K}. \tag{A$_1$}$$

Now, we introduce the following sets:

$$\mathcal{C}^\infty_{n-2\sigma} := \{\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}_{n-2\sigma})^p \mid 1 \leq p \leq \sharp\mathcal{K},\ y_i \neq y_j \text{ for all } i \neq j \text{ and } \rho(\tau_p) > 0\},$$
$$\mathcal{C}^\infty_{<n-2\sigma} := \{\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}^+ \setminus \mathcal{K}_{n-2\sigma})^p \mid 1 \leq p \leq \sharp\mathcal{K} \text{ and } y_i \neq y_j \text{ for all } i \neq j\}.$$

For any $\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K})^p$, we write

$$i(\tau_p)_\infty = p - 1 + \sum_{j=1}^{p} (n - \tilde{i}(y_{l_j})).$$

Theorem 1.1. *Assume that K satisfies (A$_1$) and $(f)_\beta$ with $1 < \beta \leq n - 2\sigma$. If*

$$\sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} + \sum_{\tau'_p \in \mathcal{C}^\infty_{<n-2\sigma}} (-1)^{i(\tau'_p)_\infty} - \sum_{(\tau_p, \tau'_p) \in \mathcal{C}^\infty_{n-2\sigma} \times \mathcal{C}^\infty_{<n-2\sigma}} (-1)^{i(\tau_p)_\infty + i(\tau'_p)_\infty} \neq 1,$$

then (1-1) has at least one solution.

In Part II, we will address the case $n - 2\sigma \leq \beta < n$, following another approach and recovering the main existence results of [Jin et al. 2014; 2015]. More precisely, we will prove:

Theorem 1.2. *Assume that K satisfies (A$_1$) for each $p \geq 1$ and $(f)_\beta$ with $n - 2\sigma \leq \beta < n$. If*

$$\sum_{y \in \mathcal{K}^+ \setminus \mathcal{K}_{n-2\sigma}} (-1)^{i(y)_\infty} + \sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} \neq 1,$$

then (1-1) has at least one solution.

We organize the remainder of our paper as follows. Section 2 is devoted to recalling some preliminary results related to the variational structure associated to problem (1-1). In Section 3, we characterize the critical points at infinity of the associated variational problem. In Section 4, we give the proofs of the main results. The characterization of critical points at infinity requires some technical results, which, for the convenience of the reader, are given in the Appendix.

2. Preliminary results

Problem (1-1) has a variational structure; see Section 3 of [Jin et al. 2015], as well as [Chen and Zheng 2014; 2015; Chen et al. 2016; Jin et al. 2014]. The Euler–Lagrange functional associated to (1-1) is

$$J(u)=\frac{\|u\|^2}{\left(\int_{\mathbb{S}^n} K u^{2n/(n-2\sigma)}\right)^{(n-2\sigma)/n}} \quad \text{for } u\in H^\sigma(\mathbb{S}^n), \tag{2-1}$$

where $H^\sigma(\mathbb{S}^n)$ is the completion of $C^\infty(\mathbb{S}^n)$ by means of the norm

$$\|u\|=\left(\int_{\mathbb{S}^n} P_\sigma u u\right)^{1/2}. \tag{2-2}$$

Problem (1-1) is equivalent to finding the critical points of J subjected to the constraint $u\in\Sigma^+$, where

$$\Sigma^+=\{u\in\Sigma \mid u\geq 0\} \quad \text{and} \quad \Sigma=\{u\in H^\sigma(\mathbb{S}^n)\mid \|u\|=1\}.$$

The exponent $2n/(n-2\sigma)$ is critical for the Sobolev embedding $H^\sigma(\mathbb{S}^n)\to L^q(\mathbb{S}^n)$. This embedding is continuous and not compact. The functional J does not satisfy the Palais–Smale condition on Σ^+, but the sequences which violate the Palais–Smale condition are known. In order to describe them, let us introduce some notation. For $a\in\mathbb{S}^n$ and $\lambda>0$, let

$$\delta_{a,\lambda}(x)=\bar{c}\,\frac{\lambda^{(n-2\sigma)/2}}{\left(1+\frac{1}{2}(\lambda^2-1)\big(1-\cos(d(x,a))\big)\right)^{(n-2\sigma)/2}}, \tag{2-3}$$

where $d(\cdot\,,\cdot)$ is the distance induced by the standard metric of $\mathbb{S}^n$ and $\bar{c}$ is chosen so that $\delta_{a,\lambda}$ is the family of solutions for

$$P_\sigma u=u^{(n+2\sigma)/(n-2\sigma)} \quad \text{for } u>0 \text{ on } \mathbb{S}^n; \tag{2-4}$$

see page 1113 of [Jin et al. 2014]. For $\varepsilon>0$ and $p\in\mathbb{N}^*$, we define the set $V(p,\varepsilon)$ of potential critical points at infinity to be the set of $u\in\Sigma$ for which there exist $a_1,\dots,a_p\in\mathbb{S}^n$, $\alpha_1,\dots,\alpha_p>0$, and $\lambda_1,\dots,\lambda_p>\varepsilon^{-1}$ satisfying

$$\begin{aligned}\Big\|u-\sum_{i=1}^p\alpha_i\delta_{a_i,\lambda_i}\Big\|&<\varepsilon,\\ \big|J(u)^{n/(n-2\sigma)}\alpha_i^{2/(n-2\sigma)}K(a_i)-1\big|&<\varepsilon \quad \text{for all } i,j=1,\dots,p,\\ \varepsilon_{ij}&<\varepsilon \quad \text{for all } i\neq j,\end{aligned}$$

where

$$\varepsilon_{ij}=\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{(2\sigma-n)/2}.$$

Following [Li and Zhu 1995; Brezis and Coron 1985], the failure of the Palais–Smale condition can be described as follows.

Proposition 2.1. *Assume that J has no critical points Σ^+. Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$ which tends to zero, and an extracted subsequence of the u_k, again denoted (u_k), such that $u_k \in V(p, \varepsilon_k)$.*

If u is a function in $V(p, \varepsilon)$, one can find an optimal representation, following the ideas introduced in [Bahri 1996]. Namely, we have:

Proposition 2.2. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \le \varepsilon_p$ and $u \in V(p, \varepsilon)$, then the minimization problem*

$$\min_{\alpha_i>0,\lambda_i>0,a_i\in\mathbb{S}^n} \Big\| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \Big\|$$

has a unique solution (α, λ, a) up to a permutation.

If we denote

$$v := u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)},$$

then v belongs to $H^\sigma(\mathbb{S}^n)$ and, arguing as in page 175 of [Bahri 1989], satisfies the condition

$$\langle v, \varphi_i \rangle = 0 \quad \text{for } \varphi_i = \delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i} \text{ and } i = 1, \dots, p, \tag{V_0}$$

where $\delta_i = \delta_{a_i,\lambda_i}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^\sigma(\mathbb{S}^n)$ defined by

$$\langle u, v \rangle = \int_{\mathbb{S}^n} v P_\sigma u.$$

We say $v \in (V_0)$ if v satisfies (V_0). The following Morse lemma completely gets rid of the v-contributions.

Proposition 2.3. *There is a C^1 map which, to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)}$ belongs to $V(p, \varepsilon)$, associates $\bar v = \bar v(\alpha, a, \lambda)$ such that $\bar v$ is unique and satisfies*

$$J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \bar v\Big) = \min_{v\in(V_0)} \Big\{ J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\Big)\Big\}.$$

Moreover, there exists a change of variables $v - \bar v \to V$ such that

$$J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\Big) = J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \bar v\Big) + \|V\|^2.$$

Furthermore, under the assumption $(f)_\beta$, $1 < \beta \le n$, there exists $c > 0$ such that the following holds:

$$\|\bar v\| \le c \sum_{i=1}^{p} \Big(\frac{1}{\lambda_i^{n/2}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{(n+2\sigma)/(2n)}}{\lambda_i^{(n+2\sigma)/2}} \Big) + c \begin{cases} \sum_{k\neq r} \varepsilon_{kr}^{(n+2\sigma)/(2(n-2\sigma))} (\log \varepsilon_{kr}^{-1})^{(n+2\sigma)/(2n)} & \text{if } n \ge 3, \\ \sum_{k\neq r} \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{(n-2\sigma)/n} & \text{if } n < 3. \end{cases}$$

To conclude this section, we state the definition of critical point at infinity.

Definition 2.4. A critical point at infinity of J on Σ^+ is a limit of a flow-line $u(s)$ of the equation

$$\frac{\partial u}{\partial s} = -\partial J(u(s)), \quad u(0) = u_0,$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$ for $s \geq s_0$. Here $\varepsilon(s) > 0$ and $\to 0$ when $s \to +\infty$. Using Proposition 2.2, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\delta_{(a_i(s),\lambda_i(s))} + v(s).$$

Defining $\tilde{\alpha}_i := \lim_{s\to+\infty} \alpha_i(s)$ and $\tilde{y}_i := \lim_{s\to+\infty} a_i(s)$, we denote a critical point at infinity by

$$\sum_{i=1}^{p} \tilde{\alpha}_i \delta_{(\tilde{y}_i,\infty)} \quad \text{or} \quad (\tilde{y}_1, \dots, \tilde{y}_p)_\infty.$$

3. Characterization of the critical points at infinity for $1 < \beta \leq n - 2\sigma$

This section is devoted to the characterization of the critical points at infinity in $V(p, \varepsilon)$, $p \geq 1$, under the β-flatness condition with $1 < \beta \leq n - 2\sigma$. This characterization is obtained through the construction of a suitable pseudogradient at infinity for which the Palais–Smale condition is satisfied along the decreasing flow-lines, as long as these flow-lines do not enter the neighborhood of a finite number of critical points y_i, $i = 1, \dots, p$, of K such that

$$(y_1, \dots, y_p) \in \mathcal{P}^\infty := \mathcal{C}^\infty_{<n-2\sigma} \cup \mathcal{C}^\infty_{n-2\sigma} \cup (\mathcal{C}^\infty_{<n-2\sigma} \times \mathcal{C}^\infty_{n-2\sigma}).$$

Note that we say $(y_1, \dots, y_p) \in \mathcal{C}^\infty_{<n-2\sigma} \times \mathcal{C}^\infty_{n-2\sigma}$ if there exists $1 \leq s \leq p-1$ such that $(y_1, \dots, y_s) \in \mathcal{C}^\infty_{<n-2\sigma}$ and $(y_{s+1}, \dots, y_p) \in \mathcal{C}^\infty_{n-2\sigma}$. More precisely:

Theorem 3.1. *Assume that K satisfies (A_1) for each $p \geq 1$ and $(f)_\beta$, $1 < \beta \leq n - 2\sigma$. Let*

$$\beta := \max\{\beta(y) \mid y \in \mathcal{K}\}.$$

For each $p \geq 1$, there exists a pseudogradient W in $V(p, \varepsilon)$ and a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p, \varepsilon)$ such that

(i) $$\langle \partial J(u), W(u) \rangle \leq -c\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij} \right),$$

(ii) $$\left\langle \partial J(u+\bar{v}), W(u) + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u)) \right\rangle \leq -c\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij} \right).$$

Furthermore, $|W|$ is bounded in $V(p, \varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i}, \rho)$ with $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$, $(y_{l_1}, \dots, y_{l_p}) \in \mathcal{P}^\infty$ and ρ is a positive constant small enough such that for any $y \in \mathcal{K}$, the expansion $(f)_\beta$ holds in $B(y, \rho)$.

In order to prove Theorem 3.1, we state the following two results, which deal with two specific cases of Theorem 3.1. Let $\delta_i = \delta_{(a_i,\lambda_i)}$ and

$$V_1(p,\varepsilon) = \left\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon) \;\middle|\; a_i \in B(y_{l_i},\rho),\ y_{l_i} \in \mathcal{K} \setminus \mathcal{K}_{n-2\sigma} \text{ for all } i = 1,\dots,p \right\},$$

$$V_2(p,\varepsilon) = \left\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon) \;\middle|\; a_i \in B(y_{l_i},\rho),\ y_{l_i} \in \mathcal{K}_{n-2\sigma} \text{ for all } i = 1,\dots,p \right\}.$$

Proposition 3.2. *For $p \geq 1$, there exists a pseudogradient W_1 in $V_1(p,\varepsilon)$ and $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_1(p,\varepsilon)$ such that*

$$\langle \partial J(u), W_1(u) \rangle \leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right).$$

Furthermore, $|W_1|$ is bounded in $V_1(p,\varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i},\rho)$ with $y_{l_i} \in \mathcal{K}^+$ for all $i = 1,\dots,p$, with $(y_{l_1},\dots,y_{l_p}) \in \mathcal{C}^{\infty}_{<n-2\sigma}$.

Proposition 3.3. *For $p \geq 1$ there exists a pseudogradient W_2 in $V_2(p,\varepsilon)$ and $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_2(p,\varepsilon)$ such that*

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right).$$

Furthermore, $|W_2|$ is bounded in $V_2(p,\varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i},\rho)$ with $y_{l_i} \in \mathcal{K}^+$ for all $i = 1,\dots,p$, with $(y_{l_1},\dots,y_{l_p}) \in \mathcal{C}^{\infty}_{n-2\sigma}$.

In constructing the pseudogradient W, we will use the following notation. Let $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon)$, such that $a_i \in B(y_{l_i},\rho)$ and $y_{l_i} \in \mathcal{K}$ for all $i = 1,\dots,p$. For simplicity, if a_i is close to a critical point y_{l_i}, we will assume that the critical point is at the origin, so we will confuse a_i with $(a_i - y_{l_i})$. Now, let $i \in \{1,\dots,p\}$ and let M_1 be a positive large constant. We say that

$$i \in L_1 \quad \text{if } \lambda_i |a_i| \leq M_1,$$
$$i \in L_2 \quad \text{if } \lambda_i |a_i| > M_1.$$

For each $i \in \{1,\dots,p\}$, we define the vector fields

$$Z_i(u) = \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}, \tag{3-1}$$

$$X_i = \alpha_i \sum_{k=1}^{n} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \int_{\mathbb{R}^n} b_k \frac{|x_k + \lambda_i (a_i)_k|^{\beta}}{(1 + \lambda_i |(a_i)_k|)^{\beta-1}} \frac{x_k}{(1+|x|^2)^{n+1}}\, dx, \tag{3-2}$$

where $(a_i)_k$ is the k-th component of a_i in some geodesic normal coordinate system. We claim that X_i is bounded. Indeed, the claim is trivial if $i \in L_1$. If $i \in L_2$, by elementary computation we have the estimate

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i)_k|^\beta x_k}{(1+|x|^2)^{n+1}}\,dx = (\lambda_i |(a_i)_k|)^\beta \int_{\mathbb{R}^n} \left|1 + \frac{x_k}{\lambda_i ((a_i)_k)}\right|^\beta \frac{x_k}{(1+|x|^2)^{n+1}}\,dx$$
$$= c(\operatorname{sign} \lambda_i (a_i)_k)(\lambda_i |(a_i)_k|)^{\beta-1}(1+o(1)) \tag{3-3}$$

for any k, $1 \le k \le n$, such that $\lambda_i |(a_i)_k| > M_1/\sqrt{n}$. Hence our claim is valid.

Proof of Theorem 3.1. In order to complete the construction of the pseudogradient W suggested in Theorem 3.1, it only remains (using Propositions 3.2 and 3.3) to focus attention on the two following subsets of $V(p, \varepsilon)$.

Subset 1. We consider here the case of $u = \sum_{i=1}^p \alpha_i \delta_i = \sum_{i \in I_1} \alpha_i \delta_i + \sum_{i \in I_2} \alpha_i \delta_i$ such that

$$I_1 \ne \varnothing, \quad I_2 \ne \varnothing, \quad \sum_{i \in I_1} \alpha_i \delta_i \in V_1(\sharp I_1, \varepsilon), \quad \text{and} \quad \sum_{i \in I_2} \alpha_i \delta_i \in V_2(\sharp I_2, \varepsilon).$$

Without loss of generality, we can assume here and in the sequel that

$$\lambda_1 \le \cdots \le \lambda_p.$$

We distinguish three cases.

Case 1: $u_1 := \sum_{i \in I_1} \alpha_i \delta_i \notin V_1^1(\sharp I_1, \varepsilon)$

$$= \left\{ u = \sum_{j=1}^{\sharp I_1} \alpha_j \delta_j \;\middle|\; a_j \in B(y_{l_j}, \rho),\ y_{l_j} \in \mathcal{K}^+ \text{ for } j = 1, \dots, \sharp I_1 \text{ and } y_{l_j} \ne y_{l_k} \text{ for all } j \ne k \right\}.$$

In this case, the pseudogradient $\widetilde{W}_1(u) := W_1(u_1)$, where W_1 is as defined in Proposition 3.2, does not increase the maximum of the λ_i, $i \in I_1$. Using Proposition 3.2, we have

$$\langle \partial J(u), \widetilde{W}_1(u) \rangle \le -c\left(\sum_{i \in I_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{\substack{j \ne i \\ i,j \in I_1}} \varepsilon_{ij} + \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} \right) + O\left(\sum_{i \in I_1,\, j \in I_2} \varepsilon_{ij} \right). \tag{3-4}$$

An easy calculation implies that

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\frac{1}{\lambda_j^{\beta_j}}\right) \quad \text{for all } i \in I_1 \text{ and all } j \in I_2. \tag{3-5}$$

Fixing $i_0 \in I_1$, we define

$$J_1 := \left\{ i \in I_2 \mid \lambda_i^{n-2} \ge \tfrac{1}{2} \lambda_{i_0}^{\beta_{i_0}} \right\} \quad \text{and} \quad J_2 := I_2 \setminus J_1.$$

Using (3-4) and (3-5), we find that

$$\langle \partial J(u), \widetilde{W}_1(u) \rangle \le -c\left(\sum_{i \in I_1 \cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j \ne i \in I_1} \varepsilon_{ij} \right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} \right). \tag{3-6}$$

Let k_i be an index such that

$$|(a_i)_{k_i}| = \max_{1\le j\le n} |(a_i)_j|. \tag{3-7}$$

From Lemma 3.4 we have

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \le c \sum_{j\ne i\in J_1} 2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big). \tag{3-8}$$

Observe that for $i<j$, we have

$$2^i\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}. \tag{3-9}$$

In addition, for $i\in J_1$ and $j\in J_2$ we have $\lambda_j\le\lambda_i$, so by (3-18) we obtain $\lambda_i \dfrac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\varepsilon_{ij}$. These estimates yield

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \\ \le -c \sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i-y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big) + O\Big(\sum_{i\in J_1,\, j\in I_1} \varepsilon_{ij}\Big).$$

Taking $m_1>0$ small enough, using Lemma 3.5, (3-21), and (3-16) we get

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big\rangle \\ \le -c\Big(\sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} + \sum_{i\in J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big),$$

and by (3-6) we obtain

$$\Big\langle \partial J(u), \widetilde{W}_1(u) + m_1\Big(\sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big)\Big\rangle \\ \le -c\Big(\sum_{i\in I_1\cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\ne j\in I_1} \varepsilon_{ij} + \sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} \sum_{i\in I_1\cup J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3-10}$$

We need to add the remaining indices $i\in J_2$. Note that $\tilde u := \sum_{j\in J_2} \alpha_j \delta_j \in V_2(\sharp J_2, \varepsilon)$. Thus, the pseudogradient $\widetilde{W}_2(u) = W_2(\tilde u)$, where W_2 is as defined in Proposition 3.3, satisfies

$$\langle \partial J(u), \widetilde{W}_2(u)\rangle \le -c\Big(\sum_{j\in J_2} \frac{1}{\lambda_j^{\beta_j}} + \sum_{\substack{i\ne j\\ i,j\in J_2}} \varepsilon_{ij} + \sum_{j\in J_2} \frac{|\nabla K(a_j)|}{\lambda_j}\Big) + O\Big(\sum_{i\in J_1,\, j\in J_2} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big), \tag{3-11}$$

since $|a_i - a_j| \ge \rho$ for $i\in I_1$ and $j\in J_2$.

From (3-10) and (3-11), for $W=\widetilde{W}_1+m_1\big(\widetilde{W}_2+\sum_{i\in J_1}-2^i Z_i+m_1\sum_{i\in J_1\cap L_2}X_i\big)$ we obtain

$$\langle \partial J(u),\, W(u)\rangle \le -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\ne j}\varepsilon_{ij}\biggr).$$

Case 2:
$$u_1:=\sum_{i\in I_1}\alpha_i\delta_i\in V_1^1(\sharp I_1,\varepsilon)\quad\text{and}\quad u_2:=\sum_{i\notin I_2}\alpha_i\delta_i\notin V_2^1(\sharp I_2,\varepsilon),$$
where

$$V_2^1(\sharp I_2,\varepsilon):=\biggl\{u=\sum_{j=1}^{\sharp I_2}\alpha_j\delta_j\;\Big|\;a_j\in B(y_{l_j},\rho),\,y_{l_j}\in\mathcal{K}^+\text{ for all } j=1,\dots,\sharp I_2\text{ and }\rho(y_{l_1},\dots,y_{\sharp I_2})>0\biggr\}.$$

Let $V_1(u):=W_2(u_2)$. By Proposition 3.3, we get

$$\langle \partial J(u),\, V_1(u)\rangle \le -c\biggl(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\ne j\\ i,j\in I_2}}\varepsilon_{ij}\biggr)+O\biggl(\sum_{i\in I_2,\,j\in I_1}\varepsilon_{ij}\biggr).\tag{3-12}$$

Observe that $V_1(u)$ does not increase the maximum of the λ_i, $i\in I_2$, since $u_2\notin V_2^1(\sharp I_2,\varepsilon)$. Fix $i_0\in I_2$ and let

$$\widetilde{J}_1=\bigl\{i\in I_1\mid \lambda_i^{\beta_i}\ge\tfrac{1}{2}\lambda_{i_0}^{n-2}\bigr\}\quad\text{and}\quad \widetilde{J}_2=I_1\setminus\widetilde{J}_1.$$

Using (3-12) and (3-5), we get

$$\langle \partial J(u),\, V_1(u)\rangle \le -c\biggl(\sum_{i\in I_2\cup\widetilde{J}_1}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\ne j\\ i,j\in I_2}}\varepsilon_{ij}\biggr)+o\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\biggr).\tag{3-13}$$

We need to add the indices i for $i\in\widetilde{J}_2$. Let $\tilde{u}:=\sum_{j\in\widetilde{J}_2}\alpha_j\delta_j$ and let $V_2(u):=W_1(\tilde{u})$. By Proposition 3.2, we have

$$\langle \partial J(u),\, V_2(u)\rangle \le -c\biggl(\sum_{j\in\widetilde{J}_2}\frac{1}{\lambda_j^{\beta_j}}+\sum_{j\in\widetilde{J}_2}\frac{|\nabla K(a_j)|}{\lambda_j}+\sum_{\substack{i\ne j\\ i,j\in\widetilde{J}_2}}\varepsilon_{ij}\biggr)+O\biggl(\sum_{j\in\widetilde{J}_2,\,i\notin\widetilde{J}_2}\varepsilon_{ij}\biggr).$$

Observe that $I_1=\widetilde{J}_1\cup\widetilde{J}_2$ and we are in the case where for all $i\ne j\in I_1$, we have $|a_i-a_j|\ge\rho$. Thus by (3-16) and (3-5), we get

$$O\biggl(\sum_{j\in\widetilde{J}_2,\,i\notin\widetilde{J}_2}\varepsilon_{ij}\biggr)=o\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\biggr),$$

and hence

$$\langle \partial J(u),\, V_1(u)+V_2(u)\rangle \le -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2\cup\widetilde{J}_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\ne j}\varepsilon_{ij}\biggr).$$

Let in this case $W = V_1 + V_2 + m_1 \sum_{i\in\widetilde{J}_1} X_i(u)$, m_1 small enough. Using the above estimate and Lemma 3.5, we find that

$$\langle \partial J(u), W(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j}\varepsilon_{ij}\Big).$$

Case 3: $$u_1 \in V_1^1(\sharp I_1, \varepsilon) \quad\text{and}\quad u_2 \in V_2^1(\sharp I_2, \varepsilon).$$

For $i = 1, 2$, let $\widetilde{V}_i$ be the pseudogradient in $V(p, \varepsilon)$ defined by $\widetilde{V}_i(u) = W_i(u_i)$ where W_i is the vector field defined by Proposition 3.2 (for $i = 1$) or 3.3 (for $i = 2$) in $V_i^1(\sharp I_i, \varepsilon)$, and let in this case $W = \widetilde{V}_1 + \widetilde{V}_2$. Using Proposition 3.3, Proposition 3.2, and (3-5) we get

$$\langle \partial J(u), W(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j}\varepsilon_{ij}\Big).$$

Notice that in the first and second cases, the maximum of the λ_i, $1 \le i \le p$, is a bounded function and hence the Palais–Smale condition is satisfied along the flow-lines of W. However in the third case all the λ_i, $1 \le i \le p$, will increase and go to $+\infty$ along the flow-lines generated by W.

Subset 2. We consider the case of $u = \sum_{i=1}^{p}\alpha_i\delta_i \in V(p, \varepsilon)$, such that there exist a_i not contained in $\bigcup_{y\in\mathcal{K}} B(y, \rho)$. Let i_1 be such that for any $i < i_1$, we have $a_i \in B(y_{\ell_i}, \rho)$, $y_{\ell_i} \in \mathcal{K}$ and $a_{i_1} \notin \bigcup_{y\in\mathcal{K}} B(y, \rho)$. Let us define

$$u_1 = \sum_{i<i_1}\alpha_i\delta_i.$$

Observe that u_1 must be contained in $V_1(i_1 - 1, \varepsilon)$ or $V_2(i_1 - 1, \varepsilon)$, or else u_1 satisfies the condition of Subset 1. Thus we can apply the associated vector field, which we will denote by Y, and we then have the estimate

$$\langle \partial J(u), Y(u)\rangle \le -c\Big(\sum_{i<i_1}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i<i_1}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{\substack{i\ne j\\ i,j<i_1}}\varepsilon_{ij}\Big) + O\Big(\sum_{i<i_1,\, j\ge i_1}\varepsilon_{ij}\Big).$$

Now we define the vector field

$$Y' = \frac{1}{\lambda_{i_1}}\frac{\partial\delta_{i_1}}{\partial a_{i_1}}\frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} - c'\sum_{i\ge i_1}2^i Z_i.$$

Using Propositions 3.3, 3.2, and the fact that $|\nabla K(a_{i_1})| \ge c > 0$, we derive

$$\langle \partial J(u), Y'(u)\rangle \le -c\frac{1}{\lambda_{i_1}} + O\Big(\sum_{i\ne i_1}\varepsilon_{ij}\Big) - c'\sum_{j\ne i,\, i\ge i_1}\varepsilon_{ij} + o\Big(\sum_{i\ge i_1}\frac{1}{\lambda_i}\Big).$$

Taking $c' > 0$ large enough, we find

$$\langle \partial J(u), Y'(u)\rangle \le -c\Big(\sum_{i=i_1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=i_1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j,\, i\ge i_1}\varepsilon_{ij}\Big).$$

Now let $W := Y' + m_1 Y$, where m_1 is a small positive constant; then we have

$$\langle \partial J(u), W(u)\rangle \le -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\biggr).$$

Finally, observe that our pseudogradient W in $V(p,\varepsilon)$ satisfies Theorem 3.1(i), and it is bounded since $\|\lambda_i\,\partial\delta_i/\partial\lambda_i\|$ and $\|(1/\lambda_i)\,\partial\delta_i/\partial a_i\|$ are bounded. From the definition of W, the λ_i, $1 \le i \le p$, decrease along the flow-lines of W as long as these flow-lines do not enter the neighborhood of a finite number of critical points y_{l_i}, $i = 1, \dots, p$, of $\mathcal{K}$ such that $(y_{l_1}, \dots, y_{l_p}) \in \mathcal{P}^\infty$. Now, arguing as in Appendix 2 of [Bahri 1996], Theorem 3.1(ii) follows from (i) and Proposition 2.3. This complete the proof of Theorem 3.1. □

Proof of Proposition 3.2. In our construction of the pseudogradient W_1, we need the following lemmas. Write 1_A for the characteristic function of a set A.

Lemma 3.4. *Let $u = \sum_{i=1}^{p} \alpha_i\delta_i \in V(p,\varepsilon)$ be such that $a_i \in B(y_{l_i},\rho)$, $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$. We then have*

$$\langle \partial J(u), Z_i(u)\rangle = -2c_2 J(u) \sum_{j\neq i} \alpha_i\alpha_j \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\biggl(\frac{1}{\lambda_i^{\beta_i}}\biggr) + 1_{L_2}(i)\, O\biggl(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\biggr) + o\biggl(\sum_{j\neq i}\varepsilon_{ij}\biggr) + o\biggl(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\biggr),$$

with k_i defined as in (3-7).

Proof. Observe that for $k \in \{1, \dots, n\}$, if $\lambda_i|(a_i - y_{l_i})_k| > M_1/\sqrt{n}$, we have

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i(a_i - y_{l_i})_k|^{\beta_i-1} x_k}{(1+|x|^2)^n}\,dx = O\bigl((\lambda_i|(a_i - y_{l_i})_k|)^{\beta_i-2}\bigr) \tag{3-14}$$

if M_1 is sufficiently large. If not, we have

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i(a_i - y_{l_i})_k|^{\beta_i-1}|x_k|}{(1+|x|^2)^n}\,dx = O(1).$$

Using the fact that the k_i defined in (3-7) satisfies $\lambda_i|(a_i - y_{l_i})_{k_i}| > M_1/\sqrt{n}$ if $i \in L_2$, Lemma 3.4 follows from Proposition A.1. □

Lemma 3.5. *Let $u = \sum_{i=1}^{p} \alpha_i\delta_i \in V(p,\varepsilon)$ be such that $a_i \in B(y_{l_i},\rho)$, $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$. We then have*

$$\langle \partial J(u), X_i(u)\rangle \le O\biggl(\sum_{j\neq i}\frac{1}{\lambda_i}\biggl|\frac{\partial\varepsilon_{ij}}{\partial a_i}\biggr|\biggr) + 1_{L_1}(i)\, O\biggl(\frac{1}{\lambda_i^{\beta_i}}\biggr) - 1_{L_2}(i)\, c\biggl(\frac{1}{\lambda_i^{\beta_i}} + \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-1}}{\lambda_i}\biggr) + o\biggl(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\biggr),$$

with k_i defined as in (3-7).

Proof. Using Proposition A.2, we have

$$\langle \partial J(u), X_i(u)\rangle \le -c\frac{1}{\lambda_i{}^{\beta_i}}\left(\int_{\mathbb{R}^n} b_{k_i}\frac{|x_k+\lambda_i(a_i-y_{l_i})_{k_i}|^{\beta_i}}{(1+\lambda_i|(a_i-y_{l_i})_{k_i}|)^{(\beta_i-1)/2}}\frac{x_{k_i}}{(1+|x|^2)^{n+1}}\,dx\right)^2 + O\left(\sum_{j\neq i}\frac{1}{\lambda_i}\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right|\right)+o\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3-15}$$

Using (3-3) and the fact that

$$\lambda_i|(a_i-y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n}} \quad \text{if } i\in L_2,$$

Lemma 3.5 follows. □

In order to construct the required pseudogradient, we have to divide the set $V_1(p,\varepsilon)$ into four different regions, construct an appropriate pseudogradient in each region, and then glue up through convex combinations. Let Z_1 and Z_2 be two vector fields. A convex combination of Z_1 and Z_2 is given by $\theta Z_1+(1-\theta)Z_2$, where θ is a cutoff function. Let

$$V_1^1(p,\varepsilon) := \left\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\middle|\; y_{l_i}\neq y_{l_j} \text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0, \text{ and } \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p\right\},$$

$$V_1^2(p,\varepsilon) := \left\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\middle|\; y_{l_i}\neq y_{l_j} \text{ for all } i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } -\sum_{k=1}^{n}b_k(y_{l_i})<0 \text{ for some } i\right\},$$

$$V_1^3(p,\varepsilon) := \left\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\middle|\; y_{l_i}\neq y_{l_j} \text{ for all } i\neq j \text{ and } \lambda_j|a_j-y_{l_j}|\ge\frac{\delta}{2} \text{ for some } j\right\},$$

$$V_1^4(p,\varepsilon) := \left\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\middle|\; y_{l_i}= y_{l_j} \text{ for some } i\neq j\right\}.$$

Pseudogradient in $V_1^1(p,\varepsilon)$. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^1(p,\varepsilon)$. For any $i\neq j$, we have $|a_i-a_j|>\rho$; therefore

$$\varepsilon_{ij}=O\left(\frac{1}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}\right)=o\left(\frac{1}{\lambda_i^{\beta_i}}\right)+o\left(\frac{1}{\lambda_j^{\beta_j}}\right), \tag{3-16}$$

since $\beta_i,\beta_j<n-2\sigma$. Let $W_1^1(u)=\sum_{i=1}^{p}Z_i(u)$. Using the fact that $|\nabla K(a_i)|/\lambda_i$ is small with respect to $1/\lambda_i{}^{\beta}$, we obtain from Proposition A.1

$$\langle \partial J(u), W_1^1(u)\rangle \le -c\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right).$$

Pseudogradient in $V_1^2(p,\varepsilon)$. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^2(p,\varepsilon)$. Without loss of generality, we can assume that $i=1,\dots,q$ are the indices which satisfy $-\sum_{k=1}^{n}b_k(y_{l_i})<0$. Let

$$I=\left\{i\in\{1,\dots,p\}\mid \lambda_i^{\beta_i}\le\tfrac{1}{10}\min_{1\le j\le q}\lambda_j^{\beta_j}\right\}.$$

In this region we define $W_1^2(u)=\sum_{i=1}^q(-Z_i)(u)+\sum_{i\in I}Z_i(u)$. Using a calculation similar to [Ben Mahmoud and Chtioui 2012], we obtain

$$\langle\partial J(u),\, W_1^2(u)\rangle\le -c\biggl(\sum_{i=1}^p\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\biggr).$$

Pseudogradient in $V_1^3(p,\varepsilon)$. Let $u=\sum_{i=1}^p\alpha_i\delta_i\in V_1^3(p,\varepsilon)$. Without loss of generality, we can assume that $\lambda_1^{\beta_1}=\min\{\lambda_j^{\beta_j}\mid\lambda_j|a_j-y_{l_j}|\ge\delta\}$. Let

$$J:=\bigl\{i\mid 1\le i\le p\text{ and }\lambda_i^{\beta_i}\ge\tfrac{1}{2}\lambda_1^{\beta_1}\bigr\}.$$

Observe that if $i\notin J$ we have $\lambda_i|a_i-y_{l_i}|\ge\delta$. We write $u=\sum_{i\in J^C}\alpha_i\delta_i+\sum_{i\in J}\alpha_i\delta_i=u_1+u_2$. Observe that u_1 has to satisfy one of the two above cases, that is, $u_1\in V_1^1(\sharp J^C,\varepsilon)$ or $u_1\in V_1^2(\sharp J^C,\varepsilon)$. Let $\widetilde{W}$ be a pseudogradient on $V_1^3(p,\varepsilon)$ defined by $\widetilde{W}(u)=W_1^1(u_1)$ if $u_1\in V_1^1(\sharp J^C,\varepsilon)$, or $\widetilde{W}(u)=W_1^2(u_1)$ if $u_1\in V_1^2(\sharp J^C,\varepsilon)$. In this region let $W_1^3(u)=\widetilde{W}(u)+X_1(u)+\sum_{i\in J\cap L_2}X_i(u)-M_1Z_1(u)$. By Propositions A.1 and A.2, we have

$$\langle\partial J(u),\, W_1^3(u)\rangle\le -c\biggl(\sum_{i=1}^p\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\biggr).$$

Pseudogradient in $V_1^4(p,\varepsilon)$. Finally, let $u=\sum_{i=1}^p\alpha_i\delta_i\in V_1^4(p,\varepsilon)$. Consider

$$B_k=\{j\mid 1\le j\le p\text{ and }a_j\in B(y_{l_k},\rho)\}.$$

In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1,\dots,q$ are the indices such that the set B_k, $1\le k\le q$, contains at least two indices. We will decrease the λ_i for $i\in B_k$ with different speed. For this purpose, let

$$\chi:\mathbb{R}\to\mathbb{R}^+,\qquad t\mapsto\begin{cases}0 & \text{if }|t|\le\tilde{\gamma},\\ 1 & \text{if }|t|\ge 1.\end{cases}$$

Here $\tilde{\gamma}$ is a small constant. For $j\in B_k$, set $\overline{\chi}(\lambda_j)=\sum_{i\neq j,\,i\in B_k}\chi(\lambda_j/\lambda_i)$. Let

$$I_1=\{i\mid 1\le i\le p\text{ and }\lambda_i|a_i-y_{l_i}|\ge\delta\}.$$

We distinguish two cases:

Case 1: $I_1\neq\varnothing$. Let in this case

$$J=\bigl\{j\mid 1\le j\le p\text{ and }\lambda_j^{\beta_j}\ge\tfrac{1}{2}\min_{i\in I_1}\lambda_i^{\beta_i}\bigr\}.$$

Observe that, if $a_i\in B(y_{l_i},\rho)$, we have $|\nabla K(a_i)|\sim\sum_{k=1}^n|b_k||(a_i-y_{l_i})_k|^{\beta_i-1}$. So, if $i\in L_1$ we have $|\nabla K(a_i)|/\lambda_i\le c/\lambda_i^{\beta_i}$, and if $i\in L_2$ we have

$$\frac{|\nabla K(a_i)|}{\lambda_i}\le c\,\frac{|(a_i-y_{l_i})_k|^{\beta_i-1}}{\lambda_i}.$$

Thus by Lemma 3.5 we obtain

$$\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle \le -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\in I_1\cap L_2}\frac{|(a_i-y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big) + O\Big(\sum_{i\neq j,\ i\in I_1}\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big)+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big).$$

Let $\widetilde{C}=\{(i,j)\mid \gamma\le \lambda_i/\lambda_j\le 1/\gamma\}$, where γ is a small positive constant. Observe that

$$\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big| = o(\varepsilon_{ij})\quad \text{for all } (i,j)\in\widetilde{C},\, i\neq j.$$

This with (3-3) yields

$$\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle \le -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\in I_1\cap L_2}\frac{|(a_i-y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big) +o\Big(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ (i,j)\in\widetilde{C},\, i\in I_1}}\varepsilon_{ij}\Big)+O\Big(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ (i,j)\notin\widetilde{C},\, i\in I_1}}\varepsilon_{ij}\Big)+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3-17}$$

For any $k=1,\dots,q$, let $\lambda_{i_k}=\min\{\lambda_i\mid i\in B_k\}$. Define

$$\overline{Z}=-\sum_{k=1}^{q}\sum_{\substack{j\in B_k\\ (i_k,j)\notin\widetilde{C}}}\overline{\chi}(\lambda_j)Z_j-\gamma_1\sum_{k=1}^{q}\sum_{\substack{j\in B_k\\ (i_k,j)\in\widetilde{C}}}\overline{\chi}(\lambda_j)Z_j,$$

where γ_1 is a small positive constant. Using Lemma 3.4, we find that

$$\langle \partial J(u), \overline{Z}(u)\rangle \le c\sum_{k=1}^{q}\sum_{\substack{i\neq j\\ j\in B_k,\,(j,i_k)\notin\widetilde{C}}}\overline{\chi}(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + c\gamma_1\sum_{k=1}^{q}\sum_{\substack{i\neq j\\ j\in B_k,\,(j,i_k)\in\widetilde{C}}}\overline{\chi}(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j}+O\Big(\sum_{k=1}^{q}\sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\notin\widetilde{C}}}\Big(\frac{1}{\lambda_j^{\beta_j}}+\frac{|(a_j-y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Big)\Big) +\gamma_1 O\Big(\sum_{k=1}^{q}\sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\in\widetilde{C}}}\Big(\frac{1}{\lambda_j^{\beta_j}}+\frac{|(a_j-y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Big)\Big).$$

Observe that by using a direct calculation, we have

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\le -c\varepsilon_{ij}\quad \text{if } \lambda_i\ge\lambda_j,\ \lambda_i\sim\lambda_j, \text{ or } |a_i-a_j|\ge\delta_0>0. \tag{3-18}$$

Let $j \in B_k$, $1 \le k \le q$, and let i, $1 \le i \le p$, be such that $i \ne j$. If $i \notin B_k$, or $i \in B_k$ with $(i, j) \in \widetilde{C}$, then we have by (3-18)

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\varepsilon_{ij} \quad \text{and} \quad \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}.$$

In the case where $i \in B_k$ with $(i, j) \notin \widetilde{C}$ (assuming $\lambda_i \ll \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \ge 1$. Thus,

$$\overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}.$$

We therefore have

$$\begin{aligned}\langle \partial J(u), \overline{Z}(u)\rangle \le -c\Bigg(\sum_{k=1}^{q} \sum_{\substack{i\ne j\\ j\in B_k,\,(j,i_k)\notin\widetilde{C}}} \varepsilon_{ij} + \gamma_1 \sum_{k=1}^{q} \sum_{\substack{i\ne j\\ j\in B_k,\,(j,i_k)\in\widetilde{C}}} \varepsilon_{ij}\Bigg) \\ + O\Bigg(\sum_{k=1}^{q} \sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\notin\widetilde{C}}} \Bigg(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Bigg)\Bigg) \\ + \gamma_1 O\Bigg(\sum_{k=1}^{q} \sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\in\widetilde{C}}} \Bigg(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Bigg)\Bigg).\end{aligned} \tag{3-19}$$

Observe that if $j \in B_k$ with $(j, i_k) \in \widetilde{C}$, we have j or $i_k \in I_1$. Thus for M_1 large enough and γ_1 very small, we obtain from (3-17) and (3-19)

$$\begin{aligned}\Big\langle \partial J(u), \sum_{i\in I_1} X_i + M_1\overline{Z}(u)\Big\rangle \\ \le -c\Bigg(\sum_{i\in J} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k=1}^{q} \sum_{\substack{i\ne j\\ j\in B_k}} \varepsilon_{ij}\Bigg) + O\Bigg(\sum_{k=1}^{q} \sum_{\substack{j\in B_k\\ (i_k,j)\notin\widetilde{C}}} \frac{1}{\lambda_j^{\beta_j}}\Bigg),\end{aligned} \tag{3-20}$$

since

$$\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2} = o\Bigg(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-1}}{\lambda_i}\Bigg) \quad \text{for any } i \in L_2 \tag{3-21}$$

(as M_1 is large enough). Now, let in this region

$$W_1^4 := M_1\Bigg(\sum_{i\in I_1} X_i + M_1\overline{Z}\Bigg) + \sum_{i\notin J}\Bigg(-\sum_{k=1}^{n} b_k\Bigg) Z_i.$$

We obtain from the above estimates

$$\langle \partial J(u), W_1^4(u)\rangle \le -c\Bigg(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j} \varepsilon_{ij}\Bigg).$$

Case 2: $I_1 = \varnothing$. Let

$$I_2 = \{1\} \cup \{i \mid 1 \le i \le p \text{ and } \lambda_i \sim \lambda_1\}.$$

We write

$$u=\sum_{i\in I_2}\alpha_i\delta_i+\sum_{i\notin I_2}\alpha_i\delta_i:=u_1+u_2.$$

Observe that, for all $i\neq j\in I_2$ such that $i\neq j$, we have $|a_i-a_j|\geq\delta$. Indeed, if $|a_i-a_j|<\delta$, so $i,j\in B_k$, we get $|a_i-a_j|\leq|a_i-y_{l_i}|+|a_j-y_{l_i}|\leq 2\delta/\lambda_i$, since $I_1=\varnothing$ and $\lambda_i\sim\lambda_j$ for all $i,j\in I_2$. This implies that

$$\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{(n-2\sigma)/2}\leq c_1,$$

and hence $\varepsilon_{ij}\geq c$, which is a contradiction. Thus $u_1\in V_1^j(\sharp I_2,\varepsilon)$, $j=1$ or 2 or 3. Applying the associated pseudogradient denoted by $\overline{W}$, we obtain

$$\langle\partial J(u),\overline{W}(u)\rangle\leq -c\left(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+O\left(\sum_{i\in I_2,\,j\notin I_2}\varepsilon_{ij}\right).$$

Let

$$J_2=\left\{i\mid 1\leq i\leq p,\ \lambda_i^{\beta_i}\geq\min_{j\in I_2}\lambda_j^{\beta_j}\right\}.$$

We can add to the above estimates all indices i such that $i\in J_2$. So, using the estimate (3-16) we obtain

$$\langle\partial J(u),\overline{W}(u)\rangle\leq -c\left(\sum_{i\in J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\right)+O\left(\sum_{\substack{i,j\in B_k\\ i\in I_2,\,j\notin I_2}}\varepsilon_{ij}\right).$$

Let $M_1>0$ be large enough, then the above estimate and (3-19) yields

$$\begin{aligned}&\langle\partial J(u),M_1\overline{Z}(u)+\overline{W}(u)\rangle\\ &\quad\leq -c\left(\sum_{i\in J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{k=1}^{q}\sum_{i\neq j\in B_k}\varepsilon_{ij}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+O\left(\sum_{k=1}^{q}\sum_{\substack{i\in B_k\\ (i_k,i)\notin\widetilde{C}}}\frac{1}{\lambda_i^{\beta_i}}\right).\end{aligned}\tag{3-22}$$

By Step 3 in the proof of Proposition 3.3 below and (3-16), we have

$$\begin{aligned}&\left\langle\partial J(u),\sum_{i\notin J_2}\left(-\sum_{k=1}^{n}b_k\right)Z_i(u)\right\rangle\\ &\quad\leq -c\left(\sum_{i\notin J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\notin J_2}\frac{|\nabla K(a_i)|}{\lambda_i}\right)+O\left(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ i\notin J_2}}\varepsilon_{ij}\right)+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\right).\end{aligned}\tag{3-23}$$

Define

$$W_1^4(u)=M_1(M_1\overline{Z}(u)+\overline{W}(u))+\sum_{i\notin J_2}\left(-\sum_{k=1}^{n}b_k\right)Z_i(u).$$

Using (3-23), we get

$$\langle \partial J(u), W_1^4(u)\rangle \le -c\bigg(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\bigg),$$

since $1/\lambda_i^{\beta_i}=o(1/\lambda_{i_k}^{\beta_{i_k}})$ for all $i\in B_k$ such that $(i,i_k)\notin\widetilde{C}$.

The vector field W_1 in $V_1(p,\varepsilon)$ will be a convex combination of W_1^j, $j=1,\dots,4$. From the definitions of W_1^j, $j=1,\dots,4$, the only case where the maximum of the λ_i increases is when $a_i\in B(y_{l_i},\rho)$, $y_{l_i}\in\mathcal{K}^+$ for all $i=1,\dots,p$, with $y_{l_i}\neq y_{l_j}$ for all $i\neq j$. This concludes the proof of Proposition 3.2. □

Proof of Proposition 3.3. We divide the set $V_2(p,\varepsilon)$ into five sets:

$$V_2^1(p,\varepsilon)=\bigg\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\bigg|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } \rho(y_{l_i},\dots,y_{l_p})>0\bigg\},$$

$$V_2^2(p,\varepsilon)=\bigg\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\bigg|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } \rho(y_{l_i},\dots,y_{l_p})<0\bigg\},$$

$$V_2^3(p,\varepsilon)=\bigg\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\bigg|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p,\\ \text{and there exist } j \text{ such that } -\sum_{k=1}^{n}b_k(y_{l_j})<0\bigg\},$$

$$V_2^4(p,\varepsilon)=\bigg\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\bigg|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\\ \text{and there exist } j \text{ (at least) such that } \lambda_j|a_j-y_{l_j}|\ge\frac{\delta}{2}\bigg\},$$

$$V_2^5(p,\varepsilon)=\bigg\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\bigg|\;\text{such that there exist } i\neq j \text{ satisfying } y_{l_i}=y_{l_j}\bigg\}.$$

We break up the proof into five steps.

Step 1. First, we consider the case $u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2^1(p,\varepsilon)$. We have, for any $i\neq j$, $|a_i-a_j|>\rho$ and therefore,

$$\varepsilon_{ij}=\left(\frac{2}{(1-\cos d(a_i,a_j))\lambda_i\lambda_j}\right)^{(n-2\sigma)/2}(1+o(1))=2^{(n-2\sigma)/2}\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}(1+o(1)).$$

Here $G(a_i,a_j)$ is defined in (1-7). Thus,

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}=-\frac{n-2\sigma}{2}\cdot 2^{(n-2\sigma)/2}\cdot\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}(1+o(1)).$$

Using Proposition A.1 with $\beta = n-2\sigma$ and the fact that $\alpha_i^{4\sigma/(n-2\sigma)} K(a_i) J(u)^{n/(n-2\sigma)} = 1 + o(1)$ for all $i = 1, \dots, p$, we derive that

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = \frac{n-2\sigma}{2} J(u)^{1-n/2} \Biggl(\frac{n-2\sigma}{n} \cdot \tilde{c}_1 \cdot \frac{\sum_{i=1}^p b_k}{K(a_i)^{n/(2\sigma)}} \frac{1}{\lambda_i^{n-2\sigma}} + c_1 2^{(n-2\sigma)/2} \sum_{i \neq j} \frac{G(y_{l_i}, y_{l_j})}{(K(a_i)K(a_j))^{(n-2\sigma)/(4\sigma)}} \frac{1}{(\lambda_i \lambda_j)^{(n-2\sigma)/2}} \Biggr) + o\Biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Biggr).$$

Here $\tilde{c}_1 = c_0^{2n/(n-2\sigma)} \int_{\mathbb{R}^n} \frac{|(x_1)|^{n-2\sigma}}{(1+|x|^2)^n} \, dx$. Hence, using the fact that $|a_i - y_{l_i}| < \delta$ for δ very small, we get

$$\begin{aligned} \left\langle \partial J(u), \sum_{i=1}^p \alpha_i Z_i \right\rangle &\le -c\, {}^t\Lambda M(y_{l_1}, \dots, y_{l_p}) \Lambda + o\Biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Biggr) \\ &\le -c\rho(y_{l_1}, \dots, y_{l_p}) |\Lambda|^2 + o\Biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Biggr), \end{aligned}$$

where $\Lambda = {}^t\bigl(1/\lambda_1^{(n-2\sigma)/2}, \dots, 1/\lambda_p^{(n-2\sigma)/2}\bigr)$. Here $M(y_{l_1}, \dots, y_{l_p})$ is as defined in (1-6) and $\rho(y_{l_1}, \dots, y_{l_p})$ is the least eigenvalue of $M(y_{l_1}, \dots, y_{l_p})$. Using the fact that for all $i \neq j$, we have $\varepsilon_{ij} \le c/(\lambda_i \lambda_j)^{(n-2\sigma)/2}$, since $|a_i - a_j| \ge \delta$, we then obtain

$$\left\langle \partial J(u), \sum_{i=1}^p \alpha_i Z_i \right\rangle \le -c \Biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \Biggr).$$

In addition, for all $i = 1, \dots, p$, if $\lambda_i |a_i| < \delta$ then we have $|\nabla K(a_i)|/\lambda_i \sim |(a_i)_k|^{\beta-1}/\lambda_i \le c/\lambda_i^\beta$. Thus, we derive, for $W_2^1 := \sum_{i=1}^p \alpha_i Z_i$,

$$\langle \partial J(u), W_2^1 \rangle \le -c \Biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \Biggr).$$

Step 2. Secondly, we study the case $u = \sum_{i=1}^p \alpha_i \delta_{a_i \lambda_i} \in V_2^2(p, \varepsilon)$. Since $\rho := \rho(y_{l_1}, \dots, y_{l_p})$ is the least eigenvalue of $M(y_{l_1}, \dots, y_{l_p})$, it satisfies

$$\rho = \inf_{X \in \mathbb{R}^p \setminus \{0\}} \left\{ \frac{{}^tXM(y_{l_1}, \dots, y_{l_p})X}{\|X\|^2} \right\}. \tag{3-24}$$

Therefore, there exists an eigenvector $e = (e_i)_{i=1,\dots,p}$ associated to ρ such that $|e| = 1$ with $e_i > 0$, for all $i = 1, \dots, p$. Indeed,

$$\rho = {}^teM(y_{l_1}, \dots, y_{l_p})e = \sum_{i=1}^p m_{ii} e_i^2 + \sum_{i \neq j} m_{ij} e_i e_j \ge \sum_{i=1}^p m_{ii} |e_i|^2 + \sum_{i \neq j} m_{ij} |e_i| |e_j|, \tag{3-25}$$

since $m_{ij} < 0$ for $i \neq j$. Observe that if there exists $i_0 \neq j_0$ such that $e_{i_0} e_{j_0} < 0$, then the inequality in (3-25) will be strict. This is a contradiction with (3-24). Therefore $e_i e_j \geq 0$ for all $i \neq j$. Hence, we can work with $e = (e_1, \dots, e_p)$ such that $e_i \geq 0$, for all $i = 1 \dots, p$. Now, if there exists i_0 such that $e_{i_0} = 0$, then $M(y_{l_1}, \dots, y_{l_p})e = \rho e$ would imply that $\sum_{j \neq i_0} m_{j i_0} e_j = 0$ and $e_j = 0$, a contradiction. Thus, $e_i > 0$ for all $i = 1, \dots, p$.

Let $\gamma > 0$ such that for any $x \in B(e, \gamma) = \{y \in S^{p-1} \mid |y - e| \leq \gamma\}$, we have

$$ {}^t x M(y_{l_1}, \dots, y_{l_p}) x \leq \tfrac{1}{2} \rho(y_{l_1}, \dots, y_{l_p}). $$

Two cases may occur.

Case 1: $\Lambda/|\Lambda| \in B(e, \gamma)$, where $\Lambda = {}^t\big(1/\lambda_1^{(n-2\sigma)/2}, \dots, 1/\lambda_p^{(n-2\sigma)/2}\big)$.

In this case, we define $W_2^2 = -\sum_{i=1}^p \alpha_i Z_i$. As in Step 1, we find that

$$ \langle \partial J(u), W_2^2(u) \rangle \leq -c \bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \bigg). $$

Case 2: $\Lambda/|\Lambda| \notin B(e, \gamma)$.

In this case, we define

$$ W_2^2 = -\frac{2}{n-2\sigma} |\Lambda| \sum_{i=1}^p \alpha_i \lambda_i^{n/2} \bigg(\frac{|\Lambda| e_i - \Lambda_i}{|\Lambda|} - \frac{\Lambda_i \langle |\Lambda| e - \Lambda, \Lambda \rangle}{|\Lambda|^3} \bigg) \frac{\partial \delta_{a_i \lambda_i}}{\partial \lambda_i}. $$

Using Proposition A.1, we find that

$$ \langle \partial J(u), W_2^2(u) \rangle = -c |\Lambda|^2 \frac{\partial}{\partial t} ({}^t\Lambda(t) M \Lambda(t)) \Big|_{t=0} + o\bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-4}} \bigg) + o\bigg(\sum_{i \neq j} \varepsilon_{ij} \bigg), $$

where $M = M(y_{l_1}, \dots, y_{l_p})$ and $\Lambda(t) = \dfrac{(1-t)\Lambda + t|\Lambda| e}{\big|(1-t)\Lambda + t|\Lambda| e\big|} \Lambda$. Observe that

$$ {}^t\Lambda(t) M \Lambda(t) = \rho + \frac{(1-t)^2}{\big|(1-t)\Lambda + t|\Lambda| e\big|} ({}^t\Lambda M \Lambda - \rho |\Lambda|^2). $$

Thus we obtain $\dfrac{\partial}{\partial t}({}^t\Lambda(t) M \Lambda(t)) < -c$ and therefore,

$$ \langle \partial J(u), W_2^2(u) \rangle \leq -c \bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \bigg). $$

Step 3. Now, we deal with the case $u = \sum_{i=1}^p \alpha_i \delta_{a_i \lambda_i} \in V_2^3(p, \varepsilon)$. Without loss of generality, we can assume that $1, \dots, q$ are the indices which satisfy $-\sum_{k=1}^n b_k(y_{l_i}) < 0$ for all $i = 1, \dots, q$. Let

$$ \widetilde{W}_2^1 = \sum_{i=1}^q -\alpha_i Z_i. $$

By Proposition A.1 and (3-18), we obtain

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \le -c\biggl(\sum_{i=1}^{q} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\,1\le i\le q} \varepsilon_{ij}\biggr).$$

Set

$$I = \bigl\{ i \mid 1 \le i \le p \text{ and } \lambda_i \le \tfrac{1}{10} \min_{1\le j\le q} \lambda_j \bigr\}.$$

It is easy to see that we can add to the above estimates all indices i such that $i \notin I$. Thus

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \le -c\biggl(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\biggr).$$

If $I \neq \varnothing$, in this case, we write

$$u = u_1 + u_2, \qquad u_1 = \sum_{i\in I} \alpha_i \delta_{a_i\lambda_i}, \qquad u_2 = \sum_{i\notin I} \alpha_i \delta_{a_i\lambda_i}.$$

Observe that u_1 must be contained in either $V_2^1(\sharp I, \varepsilon)$ or $V_2^2(\sharp I, \varepsilon)$. Thus we can apply the associated vector field which we denote by $\widetilde{W}_2^2$. We then have

$$\langle \partial J(u), \widetilde{W}_2^2(u)\rangle \le -c\biggl(\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i\in I} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\biggr) + O\biggl(\sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\biggr).$$

Let in this subset $W_2^3 = \widetilde{W}_2^1 + m_1 \widetilde{W}_2^2$ for m_1 a small positive constant. We get

$$\langle \partial J(u), W_2^3(u)\rangle \le -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\biggr).$$

Step 4. We consider next the case $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i\lambda_i} \in V_2^4(p, \varepsilon)$. Let

$$\lambda_{i_1} = \inf\{\lambda_j \mid \lambda_j |a_j| \ge \delta\}.$$

For $m_1 > 0$ small enough, we claim that

$$\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle \le -c\biggl(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \sum_{i=1}^{p} \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\biggr).$$

Indeed, for $i \neq j$, we have $|a_i - a_j| > \rho$, thus in Proposition A.2 the term $\left|\dfrac{1}{\lambda_i} \dfrac{\partial \varepsilon_{ij}}{\partial (a_i)_k}\right|$ is very small with respect to ε_{ij}. Hence,

$$\langle \partial J(u), X_{i_1}(u)\rangle \le -\frac{c}{\lambda_{i_1}^{n-2\sigma}} \biggl(\int_{\mathbb{R}^n} b_{k_{i_1}} \frac{|x_{k_{i_1}} + \lambda_{i_1}(a_{i_1})_{k_{i_1}}|^{\beta}}{(1+\lambda_{i_1}|(a_{i_1})_{k_{i_1}}|)^{(\beta-1)/2}} \frac{x_{k_{i_1}}}{(1+|x|^2)^{n+1}}\, dx\biggr)^2 + o\biggl(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j}\biggr).$$

If $i_1 \in L_1$, in which case $\delta \le \lambda_{i_1}|a_{i_1}| \le M_1$, then an elementary calculation gives

$$\left(\int_{\mathbb{R}^n} b_{k_i} \frac{|x_{k_i} + \lambda_i (a_1)_{k_i}|^\beta}{(1+\lambda_i|(a_1)_{k_i}|)^{(\beta-1)/2}} \frac{x_{k_i}}{(1+|x|^2)^n}\, dx\right)^2 \ge c > 0. \tag{3-26}$$

Using (3-26), we get

$$\langle \partial J(u), X_{i_1}(u)\rangle \le -\frac{c}{\lambda_{i_1}^{n-2\sigma}} + o\Big(\sum_{j\ne i_1} \varepsilon_{i_1 j}\Big) \le -c\sum_{i=i_1}^{p} \frac{1}{\lambda_i^\beta} + o\Big(\sum_{j\ne i_1} \varepsilon_{i_1 j}\Big). \tag{3-27}$$

On the other hand, we have, by Proposition A.1 and (3-18),

$$\langle \partial J(u), Z_{i_1}(u)\rangle \le -c\sum_{j\ne i_1} \varepsilon_{i_1 j} + O\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}}\Big). \tag{3-28}$$

Using (3-27) and (3-28) our claim follows in this case.

If $i_1 \in L_2$, using (3-3), we find

$$\begin{aligned}\langle \partial J(u), X_{i_1}(u)\rangle &\le -c\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\ne i_1} \varepsilon_{i_1 j}\Big)\\ &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\ne i_1} \varepsilon_{i_1 j}\Big),\end{aligned}$$

and by Proposition A.1 and (3-3), we have

$$\langle \partial J(u), -Z_{i_1}(u)\rangle \le -c\sum_{j\ne i_1} \varepsilon_{i_1 j} + O\Big(\frac{|(a_{i_1})_{k_{i_1}}|^{\beta-2}}{\lambda_{i_1}^2}\Big).$$

Now using (3-21), we obtain

$$\begin{aligned}\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\ne i_1} \varepsilon_{i_1 j} + \frac{|(a_{i_1})_k|^{\beta-1}}{\lambda_{i_1}}\Big)\\ &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\ne i_1} \varepsilon_{i_1 j} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big),\end{aligned}$$

since $|\nabla K(a_{i_1})| \sim |(a_{i_1})_{k_i}|^{\beta-1}$. Thus, our claim follows.

Now let

$$I = \big\{ i \mid 1 \le i \le p \text{ and } \lambda_i < \tfrac{1}{10}\lambda_{i_1}\big\}.$$

We have

$$\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle \le -c\Big(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\ne i,\, i\notin I} \varepsilon_{ij} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big).$$

Furthermore, using (3-3), we have

$$\Big\langle \partial J(u), \Big(X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i\Big)(u)\Big\rangle \le -c\Big(\sum_{i\notin I}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\notin I}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j,\, i\notin I}\varepsilon_{ij}\Big),$$

since for $i\notin I$ and $i\in L_1$, we have $|\nabla K(a_i)|/\lambda_i \le c/\lambda_i^{\beta}$. We need to add the remainder terms (if $I\neq\varnothing$). Let $u_1=\sum_{i\in I}\alpha_i\delta_{a_i\lambda_i}$. For all $i\in I$ we have $\lambda_i|a_i|<\delta$. Thus, $u_1\in V_2^j(\sharp I,\varepsilon)$ for $j=1$ or 2 or 3, so we can apply the associated vector field which we will denote $\widetilde{W}_2^4$. We then have

$$\langle \partial J(u), \widetilde{W}_2^4\rangle \le -c\Big(\sum_{i\in I}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i,j\in I}\varepsilon_{ij} + \sum_{i\in I}\frac{|\nabla K(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\in I,\, j\notin I}\varepsilon_{ij}\Big).$$

Let $W_2^4 = X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i + m_2\widetilde{W}_2^4$ for $m_2>0$ small enough. We get

$$\langle \partial J(u), W_2^4(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

<u>Step 5</u>. We study now the case $u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2^5(p,\varepsilon)$. Let

$$B_k = \{ j \mid 1\le j\le p \text{ and } a_j\in B(y_{l_k},\rho)\}.$$

In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1,\dots,q$ are the indices such that the set B_k, $1\le k\le q$, contains at least two indices. We will decrease the λ_i for $i\in B_k$ with different speed. For this purpose, let

$$\chi:\mathbb{R}\to\mathbb{R}^+,\qquad t\mapsto\begin{cases}0 & \text{if } |t|\le\gamma',\\ 1 & \text{if } |t|\ge 1.\end{cases}$$

Here γ' is a small constant.

For $j\in B_k$, set $\overline{\chi}(\lambda_j)=\sum_{i\neq j,\, i\in B_k}\chi(\lambda_j/\lambda_i)$. Define

$$\widetilde{W}_2^5 = -\sum_{k=1}^{q}\sum_{j\in B_k}\alpha_j\overline{\chi}(\lambda_j)Z_j.$$

Using Proposition A.1 and (3-3), we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \le c\sum_{k=1}^{q}\Big(\sum_{i\neq j,\, j\in B_k}\overline{\chi}(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + \sum_{j\in B_k,\, j\in L_1}\overline{\chi}(\lambda_j)O\Big(\frac{1}{\lambda_j^{n-2\sigma}}\Big) + \sum_{j\in B_k,\, j\in L_2}\overline{\chi}(\lambda_j)O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\Big)\Big).$$

For $j\in B_k$, with $k\le q$, if $\overline{\chi}(\lambda_j)\neq 0$, then there exists $i\in B_k$ such that $1/\lambda_j^{n-2\sigma}=o(\varepsilon_{ij})$ (for ρ small enough). Furthermore, for $j\in B_k$, if $i\notin B_k$ (or $i\in B_k$ with $\lambda_i\sim\lambda_j$), then we have, by (3-18),

$$\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j}\le -c\varepsilon_{ij}\quad\text{and}\quad \lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\le -c\varepsilon_{ij}.$$

In the case where $i \in B_k$ (assuming $\lambda_i \ll \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \geq 1$. Thus

$$\overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\varepsilon_{ij}.$$

Thus we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \leq -c \sum_{k=1}^{q} \sum_{j \in B_k} \overline{\chi}(\lambda_j) \biggl(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^{n-2\sigma}} \biggr) + \sum_{k=1}^{q} \sum_{j \in B_k,\, j \in L_2} \overline{\chi}(\lambda_j) O\biggl(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \biggr). \tag{3-29}$$

We need to add the indices $j \in \bigl(\bigcup_{K=1}^{q} B_k\bigr)^C \cup \{ j \in B_k \mid \overline{\chi}(\lambda_j) = 0\}$. Let

$$\lambda_{i_0} = \inf\{\lambda_i \mid i = 1, \dots, p\}.$$

We distinguish two cases.

Case 1: There exists j such that $\overline{\chi}(\lambda_j) \neq 0$, $\lambda_{i_0} \sim \lambda_j$, and $\gamma' \leq \lambda_{i_0}/\lambda_j \leq 1$; then we observe in the above estimate $-1/\lambda_{i_0}^{n-2\sigma}$ and therefore $-\sum_{i=1}^{p} 1/\lambda_i^{n-2\sigma}$ and $-\sum_{k \neq r} \varepsilon_{kr}$. Thus we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \leq -c \biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \biggr) + O\biggl(\sum_{k=1}^{q} \sum_{j \in B_k,\, j \in L_2} \frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \biggr).$$

Now let

$$W_2^5 = \widetilde{W}_2^5 + m_1 \sum_{i=1}^{p} X_i.$$

Using the above estimates with Proposition A.2 and (3-21), we obtain

$$\langle \partial J(u), W_2^5(u)\rangle \leq -c \biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \biggr).$$

Case 2: For each $j \in B_k$, $1 \leq k \leq q$, we have $\lambda_{i_0} \ll \lambda_j$ (i.e., $\lambda_{i_0}/\lambda_j < \gamma'$), or if $\lambda_{i_0} \sim \lambda_j$ we have $\overline{\chi}(\lambda_j) = 0$. In this case we define

$$D = \biggl(\{ i \mid \overline{\chi}(\lambda_i) = 0\} \cup \biggl(\bigcup_{k=1}^{q} B_k \biggr)^C \biggr) \cap \biggl\{ i \biggm| \frac{\lambda_i}{\lambda_{i_0}} < \frac{1}{\gamma'} \biggr\}.$$

It is easy to see that $i_0 \in D$ and if $i \neq j \in \{ i \mid \overline{\chi}(\lambda_i) = 0\} \cup \bigl(\bigcup_{k=1}^{q} B_k\bigr)^C$ we have $a_i \in B(y_{l_i}, \rho)$ and $a_j \in B(y_{l_j}, \rho)$ with $y_{l_i} \neq y_{l_j}$. Let

$$u_1 = \sum_{i \in D} \alpha_i \delta_{a_i \lambda_i}.$$

Then u_1 has to satisfy one of the four subsets above, that is, $u_1 \in V_2^j(\sharp I, \varepsilon)$ for $j = 1, 2, 3$, or 4. Thus we can apply the associated vector field, which we will denote Y, and we have

$$\langle \partial J(u), Y(u)\rangle \leq -c \biggl(\sum_{i \in D} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{\substack{i \neq j \\ i, j \in D}} \varepsilon_{ij} \biggr) + O\biggl(\sum_{i \in D,\, j \notin D} \varepsilon_{ij} \biggr).$$

Observe that in the above estimates, we have the term $-1/\lambda_{i_0}^{n-2\sigma}$, thus we have $-\sum_{i=1}^{p} 1/\lambda_i^{n-2\sigma}$. Concerning the term $-\sum_{i\neq j}\varepsilon_{ij}$ for $i\in D$ and $j\in D^C$, we have

$$D^C=\left\{i \;\middle|\; \frac{\lambda_i}{\lambda_{i_0}}>\frac{1}{\gamma'}\right\}\cup\left(\{i \mid \overline{\chi}(\lambda_i)\neq 0\}\cap\left(\bigcup_{k=1}^{q} B_k\right)\right).$$

If $j\in\{i \mid \overline{\chi}(\lambda_i)\neq 0\}\cap\bigcup_{k=1}^{q} B_k$, then we have $(-\varepsilon_{ij})$ in the estimates (3-29). If $j\in\left\{i \;\middle|\; \frac{\lambda_i}{\lambda_{i_0}}>\frac{1}{\gamma'}\right\}$, we can prove in this case that $|a_i-a_j|\geq\rho$. Thus

$$\varepsilon_{ij}\leq\frac{c}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}<\frac{c\gamma'^{(n-2\sigma)/2}}{(\lambda_{i_0}\lambda_i)^{(n-2\sigma)/2}}=o(\varepsilon_{i_0 i})$$

for γ' small enough. We derive that

$$\begin{aligned}&\left\langle\partial J(u),(\widetilde{W}_2^5+m_1Y)(u)\right\rangle\\&\qquad\leq -c\left(\sum_{i\in D}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}}+\sum_{i\neq j}\varepsilon_{ij}\right)+\sum_{K=1}^{q}\sum_{j\in B_k,\, j\in L_2}\overline{\chi}(\lambda_j)O\left(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\right),\end{aligned}$$

and hence, by (3-21), we get

$$\left\langle\partial J(u),\left(\widetilde{W}_2^5+m_1Y+m_2\sum_{i=1,\, i\in L_2}X_i\right)(u)\right\rangle\leq -c\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2\sigma}}+\sum_{i\neq j}\varepsilon_{ij}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}\right),$$

for m_1 and m_2 two small positive constants. In this case we define

$$W_2^5:=\widetilde{W}_2^5+m_1Y+m_2\sum_{i=1,\, i\in L_2}X_i.$$

The vector field W_2 in $V_2(p,\varepsilon)$ will be a convex combination of W_2^j, $j=1,\dots,5$. This concludes the proof of Proposition 3.3. $\square$

Corollary 3.6. *Let $p\geq 1$. The critical points at infinity of J in $V(p,\varepsilon)$ correspond to*

$$(y_{l_1},\dots,y_{l_p})_\infty:=\sum_{i=1}^{p}\frac{1}{K(y_{l_i})^{(n-2\sigma)/2}}\delta_{(y_{l_i},\infty)},$$

where $(y_{l_1},\dots,y_{l_p})\in\mathcal{P}^\infty$. Moreover, such a critical point at infinity has an index equal to

$$i(y_{l_1},\dots,y_{l_p})_\infty=p-1+\sum_{i=1}^{p}n-\tilde{i}(y).$$

4. Proof of Theorem 1.1

Using Corollary 3.6, the only critical points at infinity associated to problem (1-1) correspond to $w_\infty=(y_{i_1},\dots,y_{i_p})\in\mathcal{P}^\infty$. We prove Theorem 1.1 by contradiction. Therefore, we assume that (1-1) has no solution. For any $w_\infty\in\mathcal{P}^\infty$, let $c(w)_\infty$ denote the associated critical value at infinity. Here we choose to consider a simplified situation where for any $w_\infty \neq w'_\infty$, we have $c(w)_\infty \neq c(w')_\infty$ and thus order the $c(w)_\infty$ with $w_\infty \in \mathcal{P}^\infty$ as

$$c(w_1)_\infty < \cdots < c(w_{k_0})_\infty.$$

For any $\bar{c} \in \mathbb{R}$, let $J_{\bar{c}} = \{u \in \Sigma^+ \mid J(u) \leq \bar{c}\}$. By using a deformation lemma (see [Bahri and Rabinowitz 1991]), we know that if $c(w_{k-1})_\infty < a < c(w_k)_\infty < b < c(w_{k+1})_\infty$, then

$$J_b \simeq J_a \cup W_u^\infty(w_k)_\infty, \tag{4-1}$$

where $W_u^\infty(w_k)_\infty$ denotes the unstable manifolds at infinity of $(w_k)_\infty$ (see [Bahri 1996]) and $\simeq$ denotes retracts by deformation.

Taking the Euler–Poincaré characteristic of both sides of (4-1), we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w_k)_\infty}, \tag{4-2}$$

where $i(w_k)_\infty$ denotes the index of the critical point at infinity $(w_k)_\infty$. Let

$$b_1 < c(w_1)_\infty = \min_{u \in \Sigma^+} J(u) < b_2 < c(w_2)_\infty < \cdots < b_{k_0} < c(w_{k_0})_\infty < b_{k_0+1}.$$

Since we have assumed that (1-1) has no solution, $J_{b_{k_0+1}}$ is a retract by deformation of Σ^+. Therefore $\chi(J_{b_{k_0+1}}) = 1$, since Σ^+ is a contractible set. Now using (4-2), after recalling that $\chi(J_{b_1}) = \chi(\varnothing) = 0$, we derive

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty}. \tag{4-3}$$

So, if (4-3) is violated, then (1-1) has a solution.

If there exists $w_\infty \neq w'_\infty$ such that $a < c(w)_\infty = c(w')_\infty < b$, then

$$J_b \simeq J_a \cup W_u^\infty(w)_\infty \cup W_u^\infty(w')_\infty. \tag{4-4}$$

By taking the Euler–Poincaré characteristic of both sides, we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w)_\infty} + (-1)^{i(w')_\infty}. \tag{4-5}$$

Repeating the same argument used above, we get a contradiction, completing the proof of Theorem 1.1.

Appendix

This appendix is devoted to some useful expansions of the gradient of J near a potential critical point at infinity consisting of p masses. These propositions are proved under some technical estimates of the different integral quantities, extracted from [Bahri 1989] (with some changes).

Proposition A.1. *Assume that K satisfies $(f)_\beta$, $1 < \beta < n$. For any $u = \sum_{j=1}^p \alpha_j \delta_j$ in $V(p, \varepsilon)$, the following expansions hold*:

(i)
$$\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = -2c_2 J(u) \sum_{i\neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\Big(\sum_{i\neq j} \varepsilon_{ij}\Big) + o\Big(\frac{1}{\lambda_i}\Big),$$

where $c_2 = c_0^{2n/(n-2\sigma)} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^{(n+2\sigma)/2}}$.

(ii) *If* $a_i \in B(y_{j_i}, \rho)$, $y_{j_i} \in \mathcal{K}$ *and* ρ *is a positive constant small enough, we have*

$$\begin{aligned}\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle &= 2J(u)\Bigg(-c_2 \sum_{j\neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2\sigma}{2n} c_0^{2n/(n-2\sigma)} \beta \frac{\alpha_i}{K(a_i)} \frac{1}{\lambda_i^\beta} \\ &\qquad \times \sum_{k=1}^n b_k \int_{\mathbb{R}^n} \operatorname{sign}(x_k + \lambda_i (a_i - y_{j_i})_k) |x_k + \lambda_i (a_i - y_{j_i})_k|^{\beta-1} \frac{x_k}{(1+|x|^2)^n}\, dx \\ &\qquad + o\Big(\sum_{j\neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\Big)\Bigg). \qquad \text{(A-1)}\end{aligned}$$

(iii) *Furthermore, if* $\lambda_i |a_i - y_{j_i}| < \delta$, *for* δ *very small, we then have*

$$\begin{aligned}\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle &= 2J(u)\Bigg(\frac{n-2\sigma}{2n} \beta c_3 \frac{\alpha_i}{K(a_i)} \frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} - c_2 \sum_{j\neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\Big(\sum_{j\neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\Big)\Bigg), \qquad \text{(A-2)}\end{aligned}$$

where $c_3 = c_0^{2n/(n-2\sigma)} \int_{S^n} \frac{|x_1|^\beta}{(1+|x|^2)^n}\, dx$.

Proposition A.2. *Under condition* $(f)_\beta$, $1 < \beta < n$, *for each* $u = \sum_{j=1}^p \alpha_j \delta_j \in V(p, \varepsilon)$, *we have*:

(i)
$$\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \right\rangle = -c_5 J(u)^2 \alpha_i^{(n+2\sigma)/(n-2\sigma)} \frac{\nabla K(a_i)}{\lambda_i} + O\Big(\sum_{i\neq j} \frac{1}{\lambda_i} \Big|\frac{\partial \varepsilon_{ij}}{\partial a_i}\Big|\Big) + o\Big(\sum_{i\neq j} \varepsilon_{ij} + \frac{1}{\lambda_i}\Big),$$

where $c_5 = \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^n}$.

(ii) *If* $a_i \in B(y_{j_i}, \rho)$, $y_{j_i} \in \mathcal{K}$, *we have*

$$\begin{aligned}\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \right\rangle &= -2(n-2\sigma) c_0^{2n/(n-2\sigma)} \alpha_i^{(n+2\sigma)/(n-2\sigma)} J(u)^2 \frac{1}{\lambda_i^\beta} \int_{\mathbb{R}^n} b_k |x_k + \lambda_i (a_i - y_{j_i})_k|^\beta \frac{x_k}{(1+|x|^2)^{n+1}}\, dy \\ &\qquad + o\Big(\sum_{i\neq j} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^p \frac{1}{\lambda_i^\beta}\Big) + O\Big(\sum_{i\neq j} \frac{1}{\lambda_i} \Big|\frac{\partial \varepsilon_{ij}}{\partial a_i}\Big|\Big),\end{aligned}$$

where $k = 1, \ldots, n$ *and* $(a_i)_k$ *is the* k-*th component of* a_i *in some geodesic normal coordinate system.*

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Received 17 Mar 2015. Revised 26 Nov 2015. Accepted 12 Apr 2016.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 9 No. 6 2016